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# Finitely additive measures and their decompositions

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Abstract: We define the notion of a finitely additive measure on a $\sigma$-algebra. We prove that a bounded finitely additive measure can be uniquely represented as a sum of a "$\sigma$-additive part" and a "purely finitely additive part" and that it also has a decomposition similar to the Lebesgue decomposition for $\sigma$-additive measures. Bounded finitely additive measures defined on the Borel $\sigma$-algebra form a normed linear space and those that are zero on Lebesgue null sets form its subspace. We show that the former one is isometrically isomorphic to the dual space of the space of bounded Borel functions and the latter one is isometrically isomorphic to the dual space of the space of essentially bounded functions.

Keywords: finitely additive measure, Yosida-Hewitt decomposition, Lebesgue decomposition, space of finitely additive measures

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# Contents

# Introduction

The primary topic of this thesis are finitely additive measures, which we define to be real-valued finitely additive set functions on a $\sigma$-algebra of sets. In Chapter 1, we introduce the total variation and show that a bounded finitely additive measure can be written as a difference of two nonnegative finitely additive measures, which is an analogue of the Jordan decomposition of a $\sigma$-additive measure.

Bounded finitely additive measures are of crucial importance in our investigations. In Chapters 2 and 3, we study two of their decompositions. The first one is a decomposition into a "$\sigma$-additive part" and a "purely finitely additive part". It was originally presented by Yosida and Hewitt in the article [2]. Our proof of their result is different and completely self-contained. The second decomposition is an analogue of the Lebesgue decomposition for $\sigma$-additive measures.

Bounded finitely additive measures defined on the Borel $\sigma$-algebra form a normed linear space. When dealing with the Lebesgue decomposition, we define its subspace of bounded finitely additive measures which are zero on Lebesgue null sets. This subspace plays an important role in the final chapter of this thesis. We use it to characterize the dual space of the space of essentially bounded functions in Theorem 38. The proof is based on Theorem 35, which states that the dual space of the space of bounded Borel functions is isometrically isomorphic to the space of bounded finitely additive measures on the Borel $\sigma$-algebra.

It is a well-known fact that a $\sigma$-additive measure which takes real values must be bounded. However, it was not known to us whether the same is true for finitely additive measures defined on a $\sigma$-algebra. We used the techniques developed in Chapter 4 to show that there exists an unbounded finitely additive measure (we refer the reader to Proposition 36). Although the proof is not very difficult, we are not aware of any appearance of this result in the literature.

All the other proofs in this thesis were devised according to the instructions of the thesis supervisor independently of those that can be found in the literature.

Let us discuss the above mentioned results in a broader context. In the definition of a finitely additive measure, it would be sufficient to assume that the uderlying structure of sets is an algebra and not necessarily a $\sigma$-algebra. This concept is thoroughly treated in the monograph [1] by Rao. The reason why we worked with finitely additive measures defined on a $\sigma$-algebra is that they allow us to characterize dual spaces of certain spaces of functions as described in one of the previous paragraphs.

Yosida and Hewitt proved their decomposition theorem for any bounded finitely additive measure defined on an algebra of sets. Hence their approach is more general. On the other hand, it heavily relies on lattice theory and some of the proofs are very technical (the reader is referred to Chapter 1 of [2]). Our proof covers only the case of finitely additive measures defined on a $\sigma$-algebra, but it is elementary and we believe it to be more readable.

One must be careful when formulating an analogue of the Lebesgue decomposition theorem for bounded finitely additive measures on an algebra. The definitions of absolute continuity and singularity for a general finitely additive measure given in [1, Definition 6.1.1 nad Definition 6.1.14] are different from those for $\sigma$-additive measures. The Lebesgue decomposition theorem in which these alternative notions are used is true for any bounded finitely additive measure on an algebra [1, 6.2.4].

Let us conclude this section with a comment regarding the content of Chapter 4, where we characterize the dual space of the space of bounded Borel functions and the dual space of the space of essentially bounded functions. In each case, we find a linear isometry from the dual space onto a space of finitely additive measures. The same results are proved in [4, Theorem IV.5.1 and Theorem IV.8.16], except that the isometric isomorphisms produced in [4] are inverse to the ones provided by us. The proofs we present require only basics of functional analysis.

# 1. Total variation and the Jordan decomposition

In this chapter we present basic properties of the total variation and the positive and negative variations of a finitely additive measure.

## 1.1 Total variation

**Definition 1.** *Let $X$ be an abstract set and $\mathcal{A} \subset \mathcal{P}(X)$ a $\sigma$-algebra. A finitely additive measure is a mapping $\mu : \mathcal{A} \to \mathbb{R}$ such that for any finite collection $\{A_i\}_{i=1}^n \subset \mathcal{A}$ of pairwise disjoint sets*

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

*The total variation $|\mu| : \mathcal{A} \to [0, \infty]$ is defined by setting*

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{n} |\mu(A_i)| : \ A_1, \ldots, A_n \in \mathcal{A} \text{ pairwise disjoint subsets of } A\right\}$$

*for $A \in \mathcal{A}$.*

Note that we defined finitely additive measure on a $\sigma$-algebra. It would be possible to define it more generally on an algebra of sets. However, our convention simplifies the technical details and enables us to focus on the essential parts of the theory we want to develop. Unless otherwise stated, we presume every finitely additive measure to be defined on a $\sigma$-algebra $\mathcal{A}$.

**Theorem 2.** *If $\mu$ is a $\sigma$-additive measure, then $|\mu|$ is $\sigma$-additive.*

*Proof.* Assume that $\mu$ is $\sigma$-additive. Let $E_j \in \mathcal{A}$, $j = 1, 2, \ldots$, be a sequence of pairwise disjoint sets and define $E = \bigcup\limits_{j=1}^{\infty} E_j$. We want to prove

$$|\mu|(E) = \sum_{j=1}^{\infty} |\mu|(E_j).$$

First, we prove the inequality $\sum\limits_{j=1}^{\infty} |\mu|(E_j) \leq |\mu|(E)$. For each $j = 1, 2, \ldots$ take $n_j \in \mathbb{N}$ and $A_{ji} \in \mathcal{A}$, $i = 1, 2, \ldots, n_j$, pairwise disjoint sets such that $\bigcup\limits_{i=1}^{n_j} A_{ji} \subset E_j$. All the sets $A_{ji}$ are pairwise disjoint and for $n \in \mathbb{N}$ we have $\bigcup\limits_{j=1}^{n} \bigcup\limits_{i=1}^{n_j} A_{ji} \subset E$. It follows that

$$\sum_{j=1}^{n} \sum_{i=1}^{n_j} |\mu(A_{ji})| \leq |\mu|(E).$$

Taking the supremum of each summand of the outer sum on the left side we obtain

$$\sum_{j=1}^{n} |\mu|(E_j) \leq |\mu|(E).$$

The last inequality holds for arbitrary $n \in \mathbb{N}$, so the desired inequality is proved.

Now we will prove $|\mu|(E) \leq \sum\limits_{j=1}^{\infty} |\mu|(E_j)$. Take $m \in \mathbb{N}$ and $A_i \in \mathcal{A}$, $i = 1, \ldots, m$, pairwise disjoint sets such that $\bigcup\limits_{i=1}^{m} A_i \subset E$. Then we have

$$\sum_{i=1}^{m} |\mu(A_i)| = \sum_{i=1}^{m} \left| \mu\left( \bigcup_{j=1}^{\infty} (A_i \cap E_j) \right) \right| = \sum_{i=1}^{m} \left| \sum_{j=1}^{\infty} \mu(A_i \cap E_j) \right|$$
$$\leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} |\mu(A_i \cap E_j)| \leq \sum_{j=1}^{\infty} \sum_{i=1}^{m} |\mu(A_i \cap E_j)| \leq \sum_{j=1}^{\infty} |\mu|(E_j),$$

where in the second equality we used the $\sigma$-additivity of $\mu$. This amounts to

$$\sum_{i=1}^{m} |\mu(A_i)| \leq \sum_{j=1}^{\infty} |\mu|(E_j).$$

The proof is completed by taking the supremum of the left side.

□

**Theorem 3.** *If $\mu$ is a finitely additive measure, then $|\mu|$ is finitely additive.*

*Proof.* We proceed analogously as in the proof of Theorem 2. Let $E_j \in \mathcal{A}$, $j = 1, 2, \ldots, n$, be a finite sequence of pairwise disjoint sets. Denote by $E$ their union. The theorem states that $|\mu|(E) = \sum\limits_{j=1}^{n} |\mu|(E_j)$. Keeping the definition of the sets $A_{ji}$ from the previous proof, we get the same inequalities as before:

$$\sum_{j=1}^{n} \sum_{i=1}^{n_j} |\mu(A_{ji})| \leq |\mu|(E)$$
$$\sum_{j=1}^{n} |\mu|(E_j) \leq |\mu|(E).$$

We obtain the proof of the remaining inequality from the second part of the proof of Theorem 2 if we replace the symbol $\infty$ by $n$ and use finite additivity instead of $\sigma$-additivity.

□

The following elementary result is proved in [4, Lemma III.1.5].

**Proposition 4.** *Let $\mu$ be a finitely additive measure. If $\mu$ is bounded, then $|\mu|$ is also bounded.*

*Remark.* Theorem 3 combined with Proposition 4 implies that if $\mu$ is a bounded finitely additive measure, so is $|\mu|$.

**Definition 5.** *We denote by $\mathcal{M}(X, \mathcal{A})$ the linear space of all bounded finitely additive measures and by $\mathcal{M}_\sigma(X, \mathcal{A})$ the linear space of all real valued $\sigma$-additive measures. For $\mu \in \mathcal{M}(X, \mathcal{A})$, we define $\|\mu\| = |\mu|(X)$.*

**Proposition 6.** $(\mathcal{M}(X, \mathcal{A}), \|\cdot\|)$ *is a normed linear space.*

*Proof.* We must verify that $\|\cdot\|$ has the properties of norm.

With the use of the inequality

$$|\mu|(A) \geq |\mu(A)|, \; A \in \mathcal{A}, \tag{1.1}$$

which is an immediate consequence of Definition 1, it is easy to show that $\mu$ is identically zero if and only if $|\mu|$ is identically zero. Now let $a \in \mathbb{R}$. From the definition of $|\mu|$, it follows that $|a\mu| = |a| \cdot |\mu|$. Finally, for two finitely additive measures $\mu_1$ and $\mu_2$, it holds that

$$|\mu_1 + \mu_2| \leq |\mu_1| + |\mu_2|. \tag{1.2}$$

Indeed, if $A \in \mathcal{A}$ and $A_i \in \mathcal{A}$, $i = 1, 2, \dots, n$, is a finite collection of pairwise disjoint subsets of $A$, then

$$\sum_{i=1}^{n} |(\mu_1 + \mu_2)(A_i)| \leq \sum_{i=1}^{n} |\mu_1(A_i)| + \sum_{i=1}^{n} |\mu_2(A_i)| \leq |\mu_1|(A) + |\mu_2|(A)$$

and taking the supremum on the left side yields $|\mu_1 + \mu_2|(A) \leq |\mu_1|(A) + |\mu_2|(A)$. For $A = X$, this becomes the triangle inequality $\|\mu_1 + \mu_2\| \leq \|\mu_1\| + \|\mu_2\|$.

□

The next theorem is similar to a statement taught in the course in the theory of measure and integral. For the proof we refer the reader to [4, Lemma III.4.4].

**Theorem 7.** *Every real valued $\sigma$-additive measure is bounded, i. e. $\mathcal{M}_\sigma(X, \mathcal{A})$ is a subspace of $\mathcal{M}(X, \mathcal{A})$.*

In contrast, we establish the existence of a finitely additive measure which is not bounded in Proposition 36.

## 1.2 Jordan decomposition

**Definition 8.** *Let $\mu$ be a finitely additive measure. We define the positive and the negative variation of $\mu$ by setting $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$, respectively.*

**Theorem 9** (Jordan decomposition)**.** *Let $\mu$ be a bounded finitely additive measure. Then $\mu^+$ and $\mu^-$ are nonnegative bounded finitely additive measures satisfying the relations $\mu^+ + \mu^- = |\mu|$, $\mu^+ - \mu^- = \mu$.*

*Proof.* By Proposition 4, $|\mu|$ is bounded. This fact together with the inequality (1.1) implies that $\mu^+$ and $\mu^-$ are nonnegative and bounded.

Because $|\mu|$ is finitely additive by Theorem 3, the finite additivity of $\mu^+$ and $\mu^-$ is obvious. The remaining relations follow directly from Definition 8.

□

**Proposition 10.** *Let $\mu$ be a finitely additive measure and $A$ a measurable set. Then we have*

$$\mu^+(A) = \sup\{\mu(F) : \; F \subset A, F \in \mathcal{A}\}, \quad \mu^-(A) = -\inf\{\mu(F) : \; F \subset A, F \in \mathcal{A}\}.$$

*Proof.* From the definitions of $\mu^+$ and $|\mu|$, we obtain

$$\begin{aligned}\mu^+(A) = \frac{1}{2}(\mu(A) + |\mu|(A)) &= \frac{1}{2}\left(\mu(A) + \sup\left\{\sum_{i=1}^{n} |\mu(A_i)|\right\}\right) \\ &= \sup\left\{\frac{1}{2}\left(\mu(A) + \sum_{i=1}^{n} |\mu(A_i)|\right)\right\},\end{aligned}$$

where the supremum is taken over all $n \in \mathbb{N}$ and all collections $A_1, \dots, A_n \in \mathcal{A}$ of pairwise disjoint subsets of $A$. We ought to prove

$$\sup\{\mu(F) :\ F \subset A, F \in \mathcal{A}\} = \sup\left\{\frac{1}{2}\left(\mu(A) + \sum_{i=1}^{n} |\mu(A_i)|\right)\right\}.$$

It is sufficient to show

1. For each $F \subset A$, $F \in \mathcal{A}$, there exists a value $\frac{1}{2}(\mu(A) + \sum\limits_{i=1}^{n} |\mu(A_i)|)$ greater or equal to $\mu(F)$: Fix such $F$ and set $n = 2$, $A_1 = F$, $A_2 = A \setminus F$. Then we have

$$\begin{aligned}\frac{1}{2}(\mu(A) + |\mu(F)| + |\mu(A \setminus F)|) &= \frac{1}{2}(\mu(F) + \mu(A \setminus F) + \\ +|\mu(F)| + |\mu(A \setminus F)|) &\geq \frac{1}{2}(\mu(F) + |\mu(F)|) \geq \mu(F).\end{aligned}$$

2. For each value $\frac{1}{2}(\mu(A) + \sum\limits_{i=1}^{n} |\mu(A_i)|)$, there exists $F \subset A$, $F \in \mathcal{A}$ such that $\mu(F)$ is greater or equal to $\frac{1}{2}(\mu(A) + \sum\limits_{i=1}^{n} |\mu(A_i)|)$: Take an arbitrary $\frac{1}{2}(\mu(A)+\sum\limits_{i=1}^{n} |\mu(A_i)|)$. Without loss of generality we can assume that $\bigcup\limits_{i=1}^{n} A_i = A$ (otherwise, we would add the set $A \setminus \bigcup\limits_{i=1}^{n} A_i$ to the collection $\{A_i\}_{i=1}^{n}$). Define

$$I_+ = \{i \in \{1, \dots, n\} :\ \mu(A_i) \geq 0\}, \quad I_- = \{i \in \{1, \dots, n\} :\ \mu(A_i) < 0\}.$$

Then we obtain

$$\begin{aligned}&\frac{1}{2}\left(\mu(A) + \sum_{i=1}^{n} |\mu(A_i)|\right) \\ &= \frac{1}{2}\left(\sum_{i \in I_+} \mu(A_i) + \sum_{i \in I_-} \mu(A_i) + \sum_{i \in I_+} \mu(A_i) - \sum_{i \in I_-} \mu(A_i)\right) \\ &= \mu\left(\bigcup_{i \in I_+} A_i\right).\end{aligned}$$

Hence we can take $F = \bigcup\limits_{i \in I_+} A_i$.

To prove the second claim, we observe that $\mu^- = (-\mu)^+$. Using the first part of the proposition, we get

$$\begin{aligned}\mu^-(A) = (-\mu)^+(A) &= \sup\{-\mu(F) :\ F \subset A, F \in \mathcal{A}\} \\ &= -\inf\{\mu(F) :\ F \subset A, F \in \mathcal{A}\},\end{aligned}$$

which concludes the proof. $\square$

**Proposition 11.** *Let $\mu$ be a bounded finitely additive measure. Then we have $\mu^+ = \sup\{\mu, 0\}$ and $\mu^- = -\inf\{\mu, 0\}$ (the partial ordering on the set of measures is defined as follows: $\mu \leq \nu$ if and only if $\mu(A) \leq \nu(A)$ for every $A \in \mathcal{A}$).*

*Proof.* First, we show that $\mu^+ = \sup\{\mu, 0\}$. By Theorem 9, $\mu^+$ and $\mu^-$ are nonnegative. Hence $0 \leq \mu^+$ and the equality $\mu = \mu^+ - \mu^-$ implies $\mu \leq \mu^+$.

Let $\nu$ be a finitely additive measure such that $\mu \leq \nu$ and $0 \leq \nu$. We want to prove that $\mu^+ \leq \nu$. If $A, F \in \mathcal{A}$, $F \subset A$, then $\nu(A) \geq \nu(F)$ because $\nu$ is nonnegative. Since $\mu \leq \nu$, we have

$$\sup\{\mu(F) : \ F \subset A, F \in \mathcal{A}\} \leq \sup\{\nu(F) : \ F \subset A, F \in \mathcal{A}\}.$$

The supremum on the left side is $\mu^+(A)$ by Proposition 10 and the supremum on the right side is $\nu(A)$ because $\nu$ is monotone. This proves $\mu^+(A) \leq \nu(A)$.

The second claim of the proposition follows from the identity $\mu^- = (-\mu)^+$. $\square$

# 2. Yosida-Hewitt decomposition

The aim of this chapter is to show that every bounded finitely additive measure can be written as a sum of a $\sigma$-additive part and a purely finitely additive part (see Definition 14). Furthermore, the decomposition is unique.

## 2.1 Decomposition of a nonnegative measure

**Definition 12.** *Let $\mu$ be a bounded nonnegative finitely additive measure. We define*

$$\mu_c(A) = \inf\left\{\sum_{i=1}^{\infty}\mu(A_i) :\ A_i \in \mathcal{A}, \bigcup_{i=1}^{\infty} A_i \supset A\right\},\ A \in \mathcal{A}.$$

**Proposition 13.** *Let $\mu$ be a bounded nonnegative finitely additive measure. Then $\mu_c$ is $\sigma$-additive and $0 \leq \mu_c \leq \mu$ (where $0$ denotes the measure which is identically zero). If $\mu$ is $\sigma$-additive, then $\mu = \mu_c$.*

*Proof.* Assume that $E_j \in \mathcal{A}$, $j = 1, 2, \ldots$, are pairwise disjoint sets and $E = \bigcup\limits_{j=1}^{\infty} E_j$. As a first step we shall prove $\mu_c(E) \leq \sum\limits_{j=1}^{\infty}\mu_c(E_j)$. For each $j = 1, 2, \ldots$ let $A_{ji} \in \mathcal{A}$, $i = 1, 2, \ldots$, be a sequence of sets such that $\bigcup\limits_{i=1}^{\infty} A_{ji} \supset E_j$. Then we have $\bigcup\limits_{j=1}^{\infty}\bigcup\limits_{i=1}^{\infty} A_{ji} \supset E$, and so by the definition of $\mu_c$

$$\mu_c(E) \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\mu(A_{ji}).$$

All that remains is to take the infimum of each sum $\sum\limits_{i=1}^{\infty}\mu(A_{ji})$ on the right side.

The second step is to prove $\mu_c(E) \geq \sum\limits_{j=1}^{\infty}\mu_c(E_j)$. Take $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$, such that $\bigcup\limits_{i=1}^{\infty} A_i \supset E$. Since $\mu$ is nonnegative, for a fixed $n \in \mathbb{N}$, we get

$$\mu(A_i) \geq \mu\left(\bigcup_{j=1}^{n}(A_i \cap E_j)\right).$$

Consequently, using the finite additivity of $\mu$, we obtain

$$\sum_{i=1}^{\infty}\mu(A_i) \geq \sum_{i=1}^{\infty}\mu\left(\bigcup_{j=1}^{n}(A_i \cap E_j)\right) = \sum_{i=1}^{\infty}\sum_{j=1}^{n}\mu(A_i \cap E_j) =$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{\infty}\mu(A_i \cap E_j) \geq \sum_{j=1}^{n}\mu_c(E_j).$$

We derived

$$\sum_{i=1}^{\infty}\mu(A_i) \geq \sum_{j=1}^{n}\mu_c(E_j).$$

Taking the infimum of the left side and letting $n$ approach $\infty$ gives the second inequality. Hence the $\sigma$-additivity of $\mu_c$ is proved.

Clearly, $\mu_c$ is nonnegative. The inequality $\mu_c \leq \mu$ follows at once if we observe that for a given $A \in \mathcal{A}$, $\mu(A)$ is in the set whose infimum is $\mu_c(A)$.

Finally, assume that $\mu$ is $\sigma$-additive. To prove that $\mu$ equals $\mu_c$, it is enough to show $\mu \leq \mu_c$. For $A \in \mathcal{A}$, let $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$, be a sequence of sets satisfying $\bigcup\limits_{i=1}^{\infty} A_i \supset A$. We have

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

because $\mu$ is $\sigma$-additive and nonnegative. Taking the infimum of the right side yields $\mu(A) \leq \mu_c(A)$.

□

**Definition 14.** *Let $\mu$ be a bounded nonnegative finitely additive measure. We say that $\mu$ is purely finitely additive if every $\sigma$-additive $\lambda : \mathcal{A} \to [0, +\infty)$ such that $\lambda \leq \mu$ is identically zero.*

The following proposition states the existence of the decomposition we seek in the special case of a nonnegative bounded finitely additive measure.

**Proposition 15.** *Let $\mu$ be a bounded nonnegative finitely additive measure and define $\mu_f = \mu - \mu_c$. Then $\mu_f$ is purely finitely additive.*

*Proof.* The inequality $0 \leq \mu_c \leq \mu$ from Proposition 13 shows that $\mu_f$ is nonnegative and that $\mu_c$ is bounded, hence $\mu_f$ is bounded as well. By Proposition 13 again, $\mu_c$ is $\sigma$-additive, therefore finitely additive, which implies $\mu_f$ is finitely additive.

Let $\lambda : \mathcal{A} \to [0, +\infty)$ be a $\sigma$-additive measure such that $\lambda \leq \mu - \mu_c$. The proof will be completed if we show $0 \geq \lambda$. Take $A \in \mathcal{A}$ and $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$, such that $\bigcup\limits_{i=1}^{\infty} A_i \supset A$ and $\sum\limits_{i=1}^{\infty} \mu(A_i) < +\infty$. Since $\mu_c$ and $\lambda$ are nonnegative and $\sigma$-additive, we have $\mu_c(A) \leq \sum\limits_{i=1}^{\infty} \mu_c(A_i)$ and $\lambda(A) \leq \sum\limits_{i=1}^{\infty} \lambda(A_i)$. Consequently,

$$\begin{aligned} \sum_{i=1}^{\infty} \mu(A_i) - \mu_c(A) &\geq \sum_{i=1}^{\infty} \mu(A_i) - \sum_{i=1}^{\infty} \mu_c(A_i) \geq \\ &\geq \sum_{i=1}^{\infty} (\mu(A_i) - \mu_c(A_i)) \geq \sum_{i=1}^{\infty} \lambda(A_i) \geq \lambda(A). \end{aligned}$$

The expressions in the inequalities above are defined because of the assumption $\sum\limits_{i=1}^{\infty} \mu(A_i) < +\infty$. Taking the infimum of the left side of $\sum\limits_{i=1}^{\infty} \mu(A_i) - \mu_c(A) \geq \lambda(A)$ yields $0 \geq \lambda(A)$.

□

**Corollary 16.** *Let $\mu$ be bounded nonnegative finitely additive measure. Then $\mu$ is purely finitely additive if and only if $\mu_c = 0$.*

*Proof.* To prove the first implication, assume that $\mu$ is purely finitely additive. By Proposition 13, $\mu_c$ is nonnegative, $\sigma$-additive and $\mu_c \leq \mu$. Therefore $\mu_c$ satisfies the requirements on $\lambda$ in Definition 14, which implies $\mu_c = 0$.

Conversely, if $\mu_c = 0$, then Proposition 15 implies $\mu_f = \mu - \mu_c = \mu$ is purely finitely additive.

$\square$

**Proposition 17.** *Let $\mu$ be a bounded nonnegative finitely additive measure. Let $\nu_1$, $\nu_2$ be nonnegative finitely additive measures such that $\nu_1$ is $\sigma$-additive, $\nu_2$ is purely finitely additive and $\mu = \nu_1 + \nu_2$. Then $\nu_1 = \mu_c$ and $\nu_2 = \mu_f$.*

*Proof.* Take $A \in \mathcal{A}$ and a sequence of sets $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$, satisfying $\bigcup\limits_{i=1}^{\infty} A_i \supset A$. We have $0 \le \nu_1 \le \mu$ because $\nu_2$ is nonnegative. Therefore

$$\nu_1(A) \le \sum_{i=1}^{\infty} \nu_1(A_i) \le \sum_{i=1}^{\infty} \mu(A_i),$$

where we also used that $\nu_1$ is $\sigma$-additive. By taking the infimum of the right side, we obtain $\nu_1(A) \le \mu_c(A)$.

From Proposition 15 we get that $\mu$ decomposes into $\mu_c + \mu_f$. Comparison of the two decompositions gives

$$\mu_c(A) - \nu_1(A) = \nu_2(A) - \mu_f(A) \le \nu_2(A).$$

We already showed that $\mu_c - \nu_1$ is nonnegative. It is also $\sigma$-additive and less than or equal to the purely finitely additive measure $\nu_2$, thus it must be identically zero. Hence $\mu_c = \nu_1$ and from the equality of the two decompositions we conclude that $\mu_f = \nu_2$.

$\square$

Proposition 17 alternatively follows from a more general Theorem 22, which is proved in the next section.

## 2.2 Decomposition in the general case

**Lemma 18.** *Let $\mu_1$ and $\mu_2$ be nonnegative purely finitely additive measures. Then $\mu_1 + \mu_2$ is purely finitely additive.*

*Proof.* First, we observe that in Definition 12, it is enough to consider only disjoint sets $A_i$ whose union is $A$. Indeed, if $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$, are sets satisfying $\bigcup\limits_{i=1}^{\infty} A_i \supset A$, we can find disjoint sets $B_i$, $i = 1, 2, \ldots$, such that $B_i \subset A_i \cap A$ and $\bigcup\limits_{i=1}^{\infty} B_i = A$. For those sets, we have $\sum\limits_{i=1}^{\infty} \mu(B_i) \le \sum\limits_{i=1}^{\infty} \mu(A_i)$.

Define $\mu = \mu_1 + \mu_2$. Then $\mu$ is nonnegative and finitely additive. By Corollary 16, it is sufficient to show that $\mu_c = 0$. We will prove that for each $A \in \mathcal{A}$:

$$\mu_c(A) \le (\mu_1)_c(A) + (\mu_2)_c(A). \tag{2.1}$$

Let $\sum\limits_{i=1}^{\infty} \mu_1(A_i)$ be an element of the set whose infimum is $(\mu_1)_c(A)$ and $\sum\limits_{j=1}^{\infty} \mu_2(B_j)$ an element of the set whose infimum is $(\mu_2)_c(A)$. Because of the observation at the beginning of this proof, we can presume that the sets $A_i$ are disjoint and their union is $A$. The same can be presumed for the sets $B_j$.

Denote by $E_{ij}$ the intersection $A_i \cap B_j$ for $i, j \in \mathbb{N}$. Then $E_{ij}$ are pairwise disjoint measurable sets and their union is $A$, hence $\sum_{i,j=1}^{\infty} \mu(E_{ij})$ is an element of the set whose infimum is $\mu_c(A)$. We have

$$A_i = \bigcup_{j=1}^{\infty} E_{ij}, \quad B_j = \bigcup_{i=1}^{\infty} E_{ij}.$$

Since $\mu_1$ is nonnegative, we get for $n \in \mathbb{N}$:

$$\mu_1(A_i) \geq \mu_1\left(\bigcup_{j=1}^{n} E_{ij}\right) = \sum_{j=1}^{n} \mu_1(E_{ij}).$$

Letting $n \to \infty$, this becomes

$$\mu_1(A_i) \geq \sum_{j=1}^{\infty} \mu_1(E_{ij}). \tag{2.2}$$

Analogously to (2.2), we derive

$$\mu_2(B_j) \geq \sum_{i=1}^{\infty} \mu_2(E_{ij}). \tag{2.3}$$

Finally, using (2.2) and (2.3), we obtain

$$\mu_c(A) \leq \sum_{i,j=1}^{\infty} \mu(E_{ij}) = \sum_{i,j=1}^{\infty} (\mu_1(E_{ij}) + \mu_2(E_{ij})) \leq \sum_{i=1}^{\infty} \mu_1(A_i) + \sum_{j=1}^{\infty} \mu_2(B_j).$$

Taking the infimum of both sums on the right hand side yields (2.1).

If $\mu_1$ and $\mu_2$ are purely finitely additive, Proposition 16 implies $(\mu_1)_c = 0$ and $(\mu_2)_c = 0$. It follows from (2.1) that $\mu_c = 0$, which was to be proved.

□

**Definition 19.** *Let $\mu$ be a bounded finitely additive measure. We say that $\mu$ is purely finitely additive if $\mu^+$ and $\mu^-$ are purely finitely additive measures.*

**Proposition 20.** *Let $\mu$ be a bounded finitely additive measure. Then $\mu$ is purely finitely additive if and only if $|\mu|$ is purely finitely additive.*

*Proof.* If $\mu$ is purely finitely additive, then $\mu^+$ and $\mu^-$ are purely finitely additive by Definition 19. Theorem 9 states that $|\mu| = \mu^+ + \mu^-$ and this sum is purely finitely additive by Lemma 18.

To prove the converse implication, assume that $|\mu|$ is purely finitely additive. If $\lambda : \mathcal{A} \to [0, +\infty)$ is $\sigma$-additive and $\lambda \leq \mu^+$, then $\lambda \leq |\mu|$. From Definiton 19, it follows that $\lambda = 0$, which proves that $\mu^+$ is purely finitely additive. In the same way, we can show that $\mu^-$ is purely finitely additive. Hence $\mu$ is purely finitely additive by definition.

□

**Theorem 21.** *Purely finitely additive measures form a linear subspace of the space $\mathcal{M}(X, \mathcal{A})$.*

*Proof.* Proposition 20 tells us that if we want to prove a measure is purely finitely additive, it is sufficient to show that its total variation is purely finitely additive. In the following, we use this several times without stating it again explicitly.

The set of purely finitely additive measures contains the zero measure, so it remains to prove that it is closed under addition and scalar multiplication.

First, we prove that if $\mu_1$ and $\mu_2$ are purely finitely additive measures, then $\mu_1 + \mu_2$ is purely finitely additive. Proposition 20 implies that $|\mu_1|$ and $|\mu_2|$ are purely finitely additive. Because total variation is nonnegative, $|\mu_1| + |\mu_2|$ is purely finitely additive by Lemma 18. The inequality (1.2) from the proof of Proposition 6 implies that a $\sigma$-additive measure $\lambda : \mathcal{A} \to [0, +\infty)$ such that $\lambda \leq |\mu_1 + \mu_2|$ satisfies $\lambda \leq |\mu_1| + |\mu_2|$ and is therefore zero. Hence $|\mu_1 + \mu_2|$ is purely finitely additive.

Now we prove that if $\mu$ is purely finitely additive and $a \in \mathbb{R}$, then $a\mu$ is also purely finitely additive. For $a = 0$, the claim is obviously true.

In the case $a > 0$, we show that $|a\mu| = a|\mu|$ is purely finitely additive. Take a $\sigma$-additive measure $\lambda : \mathcal{A} \to [0, +\infty)$ satisfying $\lambda \leq |a\mu|$, which is equivalent to $\frac{1}{a}\lambda \leq |\mu|$. This implies $\frac{1}{a}\lambda = 0$, hence $\lambda = 0$, which shows that $|a\mu|$ is purely finitely additive.

To prove the claim in the case $a < 0$, it is enough to prove it for $a = -1$. That is easily done, because the total variations of $\mu$ and $-\mu$ are the same.

□

**Theorem 22** (Yosida-Hewitt decomposition)**.** *Let $\mu$ be a bounded finitely additive measure. Then there exists a unique decomposition $\mu = \mu_c + \mu_f$, where $\mu_c$ is a $\sigma$-additive measure and $\mu_f$ is a purely finitely additive measure.*

*Proof.* Because $\mu^+$ and $\mu^-$ are nonnegative finitely additive measures, we know from Proposition 15 that $\mu^+ = (\mu^+)_c + (\mu^+)_f$ and $\mu^- = (\mu^-)_c + (\mu^-)_f$. We use these decompositions to express $\mu$:

$$\mu = \mu^+ - \mu^- = ((\mu^+)_c - (\mu^-)_c) + ((\mu^+)_f - (\mu^-)_f).$$

By Proposition 13, $(\mu^+)_c$ and $(\mu^-)_c$ are $\sigma$-additive, hence their difference is also $\sigma$-additive. By Proposition 15, $(\mu^+)_f$ and $(\mu^-)_f$ are purely finitely additive and their difference is purely finitely additive because purely finitely additive measures form a linear space by Theorem 21. We define $\mu_c = (\mu^+)_c - (\mu^-)_c$ and $\mu_f = (\mu^+)_f - (\mu^-)_f$ to obtain the required decomposition.

Next, we prove uniqueness. Let $\mu = \nu_1 + \nu_2$, where $\nu_1$ is $\sigma$-additive and $\nu_2$ is purely finitely additive. By comparing the two decompositions, we get

$$\mu_c - \nu_1 = \nu_2 - \mu_f.$$

The difference $\nu_2 - \mu_f$ is purely finitely additive by Theorem 21. In Proposition 20, we showed that $|\mu_c - \nu_1| = |\nu_2 - \mu_f|$ must be purely finitely additive. Theorem 2 claims that $|\mu_c - \nu_1|$ is also $\sigma$-additive. In other words, it is a nonnegative purely finitely additive measure which is equal to a $\sigma$-additive measure. As a consequence of Definition 14, we obtain that $|\mu_c - \nu_1| = 0$, which implies $\mu_c - \nu_1 = 0$. This completes the proof.

□

**Proposition 23.** *Let $\mu$ be a bounded finitely additive measure and let $\mu = \mu_c + \mu_f$ be the decomposition of $\mu$ from Theorem 22. Then we have*

$$|\mu| = |\mu_c| + |\mu_f|.$$

*Proof.* The inequality in one direction is a special case of (1.2). It remains to prove $|\mu_c| + |\mu_f| \leq |\mu|$. In the proof of Theorem 22, we defined $\mu_c$ as $(\mu^+)_c - (\mu^-)_c$ and $\mu_f$ as $(\mu^+)_f - (\mu^-)_f$, from which we have

$$|\mu_c| + |\mu_f| = |(\mu^+)_c - (\mu^-)_c| + |(\mu^-)_f - (\mu^-)_f|.$$

Again by (1.2), the expression on the right side is less than or equal to

$$|(\mu^+)_c| + |(\mu^-)_c| + |(\mu^+)_f| + |(\mu^-)_f|.$$

The finitely additive measures $(\mu^+)_c$, $(\mu^-)_c$, $(\mu^+)_f$ and $(\mu^-)_f$ are nonnegative, therefore equal to their total variations. Hence the expression further simplifies to

$$(\mu^+)_c + (\mu^-)_c + (\mu^+)_f + (\mu^-)_f = \mu^+ + \mu^- = |\mu|.$$

This proves the second inequality.

□

## 2.3 Projection onto the space $\mathcal{M}_\sigma(X, \mathcal{A})$

**Definition 24.** *We define the mapping $P : \mathcal{M}(X, \mathcal{A}) \to \mathcal{M}_\sigma(X, \mathcal{A})$ by setting $P(\mu) = \mu_c$, where $\mu_c$ is the $\sigma$-additive part of $\mu$ from Theorem 22.*

**Proposition 25.** *The mapping $P$ is a projection onto $\mathcal{M}_\sigma(X, \mathcal{A})$. The kernel of $P$ consists of purely finitely additive measures.*

*Proof.* Let $\mu_1$ and $\mu_2$ be elements of $\mathcal{M}(X, \mathcal{A})$ and $a, b \in \mathbb{R}$. Then the finitely additive measure $a\mu_1 + b\mu_2$ decomposes as

$$a\mu_1 + b\mu_2 = (a(\mu_1)_c + b(\mu_2)_c) + (a(\mu_1)_f + b(\mu_2)_f).$$

We have that $a(\mu_1)_c + b(\mu_2)_c$ is $\sigma$-additive and $a(\mu_1)_f + b(\mu_2)_f$ is purely finitely additive because $\sigma$-additive as well as purely finitely additive measures form a vector space. Since the decomposition into a $\sigma$-additive part and a purely finitely additive part is unique, we obtained

$$(a\mu_1 + b\mu_2)_c = a(\mu_1)_c + b(\mu_2)_c.$$

It follows from the definition of $P$ that $P$ is linear.

We next show that $P$ is idempotent. If $\mu \in \mathcal{M}(X, \mathcal{A})$, then $\mu_c = \mu_c + 0$ and the uniqueness of the decomposition implies $(\mu_c)_c = \mu_c$. From this we have

$$P(P(\mu)) = P(\mu_c) = \mu_c = P(\mu).$$

The projection $P$ is onto $\mathcal{M}_\sigma(X, \mathcal{A})$ because if we take $\mu \in \mathcal{M}_\sigma(X, \mathcal{A})$, then $P(\mu) = \mu$.

If $\mu \in \mathcal{M}(X, \mathcal{A})$ satisfies $P(\mu) = 0$, then $\mu_c = 0$, therefore $\mu$ is purely finitely additive. Conversely, if $\mu$ is purely finitely additive, then the uniqueness of the decomposition implies $P(\mu) = \mu_c = 0$. This completes the proof.

□

*Remark.* Let $\mu \in \mathcal{M}(X, \mathcal{A})$. The identity from Proposition 23 is equivalent to $|\mu| = |P(\mu)| + |\mu - P(\mu)|$. Consequently, $\|\mu\| = \|P(\mu)\| + \|\mu - P(\mu)\|$, from which it follows that $\|P(\mu)\| \leq \|\mu\|$. This estimate together with Proposition 25 shows that $P$ is a bounded linear operator of norm less than or equal to 1 ($P$ maps every bounded $\sigma$-additive measure to itself, so the norm is in fact 1).

Combining this with Theorem 22, we obtain that the linear space $\mathcal{M}(X, \mathcal{A})$ is a topological sum of the subspace of real valued $\sigma$-additive measures and the subspace of purely finitely additive measures.

**Proposition 26.** *The projection $P$ satisfies $P(\mu^+) = P(\mu)^+$, $\mu \in \mathcal{M}(X, \mathcal{A})$.*

*Proof.* Let $\mu$ be a bounded finitely additive measure. Proposition 23 states that $|\mu|$ decomposes into the sum $|\mu_c| + |\mu_f|$. By Theorem 2, $|\mu_c|$ is $\sigma$-additive and by Proposition 20, $|\mu_f|$ is purely finitely additive. From uniqueness of the decomposition we have $|\mu|_c = |\mu_c|$. Next, we use the linearity of $P$ to get

$$P(\mu) = P(\mu^+) - P(\mu^-), \quad |P(\mu)| = |\mu_c| = |\mu|_c = P(|\mu|) = P(\mu^+) + P(\mu^-).$$

Thus the definition of the positive variation implies

$$P(\mu)^+ = \frac{1}{2}(P(\mu) + |P(\mu)|) = P(\mu^+),$$

which was to be proved. $\square$

# 3. Lebesgue decomposition

**Definition 27.** *Let $G$ be an open subset of $\mathbb{R}^d$ and $\mathcal{B}(G)$ the $\sigma$-algebra of Borel subsets of $G$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}^d$. We define*

$$\mathcal{M}_{ac}(G, \mathcal{B}(G)) = \{\mu \in \mathcal{M}(G, \mathcal{B}(G)) : \ \forall A \in \mathcal{B}(G) : \lambda(A) = 0 \Rightarrow \mu(A) = 0\},$$
$$\mathcal{M}_{s}(G, \mathcal{B}(G)) = \{\mu \in \mathcal{M}(G, \mathcal{B}(G)) : \ \exists B \in \mathcal{B}(G) : \lambda(B) = 0 \wedge |\mu|(G \setminus B) = 0\}.$$

It is well known that every $\sigma$-additive real valued measure $\mu$ on $\mathcal{B}(G)$ uniquely decomposes into the sum $\mu_{ac} + \mu_s$, where $\mu_{ac}$ is absolutely continuous with respect to $\lambda$ and $\mu_s$ and $\lambda$ are mutually singular. This decomposition is called the Lebesgue decomposition. For a precise formulation of this statement and the definitions involved, see [3, 13.8. and 13.10]. The conditions defining $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ and $\mathcal{M}_s(G, \mathcal{B}(G))$ correspond to the conditions defining absolutely continuous and singular $\sigma$-additive measures, respectively.

In the following, we prove an exact analogue of the Lebesgue decomposition theorem for finitely additive measures.

**Proposition 28.** *The sets $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ and $\mathcal{M}_s(G, \mathcal{B}(G))$ are linear subspaces of $\mathcal{M}(G, \mathcal{B}(G))$.*

*Proof.* Both of these sets contain the zero measure. It remains to prove that they are closed under addition and scalar multiplication. The case of $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ is trivial, so we treat only the case of $\mathcal{M}_s(G, \mathcal{B}(G))$. Let $\mu, \nu \in \mathcal{M}_s(G, \mathcal{B}(G))$. Then there exist $B_1, B_2 \in \mathcal{B}(G)$ such that $\lambda(B_1) = \lambda(B_2) = 0$ and $|\mu|(G \setminus B_1) = |\nu|(G \setminus B_2) = 0$. If we set $B = B_1 \cup B_2$, then $\lambda(B) \leq \lambda(B_1) + \lambda(B_2) = 0$ and

$$|\mu + \nu|(G \setminus B) \leq |\mu|(G \setminus B) + |\nu|(G \setminus B) \leq |\mu|(G \setminus B_1) + |\nu|(G \setminus B_2) = 0.$$

This proves $\mu + \nu \in \mathcal{M}_s(G, \mathcal{B}(G))$. For $a \in \mathbb{R}$, we have

$$|a\mu|(G \setminus B_1) = |a| \cdot |\mu|(G \setminus B_1) = 0,$$

which proves $a\mu \in \mathcal{M}_s(G, \mathcal{B}(G))$.

□

**Theorem 29** (Lebesgue decomposition)**.** *Let $\mu \in \mathcal{M}(G, \mathcal{B}(G))$. Then $\mu$ can be written as $\mu = \mu_{ac} + \mu_s$, where $\mu_{ac} \in \mathcal{M}_{ac}(G, \mathcal{B}(G))$ and $\mu_s \in \mathcal{M}_s(G, \mathcal{B}(G))$, in exactly one way.*

*Proof.* First, we prove the existence of the decomposition. For a nonnegative $\mu \in \mathcal{M}(G, \mathcal{B}(G))$, define $\mu_s$ by setting

$$\mu_s(E) = \sup\{\mu(F) : \ \mathcal{B}(G) \ni F \subset E, \lambda(F) = 0\}, \ E \in \mathcal{B}(G).$$

Step 1: We will show that $\mu_s \in \mathcal{M}(G, \mathcal{B}(G))$. Because $\mu$ is bounded, it follows from the definition of $\mu_s$ that $\mu_s$ is bounded. Let $A, B \in \mathcal{B}(G)$ be disjoint and $F \in \mathcal{B}(G), F \subset A \cup B$ such that $\lambda(F) = 0$. Define $F_1 = A \cap F$ and $F_2 = B \cap F$. Then we have $\mathcal{B}(G) \ni F_1 \subset A$, $\lambda(F_1) = 0$ and $\mathcal{B}(G) \ni F_2 \subset B$, $\lambda(F_2) = 0$. From the definition of $\mu_s$, we obtain

$$\mu(F) = \mu(F_1) + \mu(F_2) \leq \mu_s(A) + \mu_s(B).$$

Taking the supremum of the left side yields $\mu_s(A \cup B) \leq \mu_s(A) + \mu_s(B)$.

To prove the opposite inequality, take $F_1 \in \mathcal{B}(G)$, $F_1 \subset A$ such that $\lambda(F_1) = 0$ and $F_2 \in \mathcal{B}(G)$, $F_2 \subset B$ such that $\lambda(F_2) = 0$. Denote by $F$ the union $F_1 \cup F_2$. Then we have $\mathcal{B}(G) \ni F \subset A \cup B$, $\lambda(F) = \lambda(F_1) + \lambda(F_2) = 0$. Again from the definition of $\mu_s$, we obtain

$$\mu(F_1) + \mu(F_2) = \mu(F) \leq \mu_s(A \cup B).$$

Taking the supremum of both summands on the left side yields $\mu_s(A) + \mu_s(B) \leq \mu_s(A \cup B)$. From the equality $\mu_s(A) + \mu_s(B) = \mu_s(A \cup B)$ the finite additivity of $\mu_s$ follows by induction.

Step 2: Next we prove that $\mu_s \in \mathcal{M}_s(G, \mathcal{B}(G))$. By the definition of $\mu_s(G)$, for each $n \in \mathbb{N}$, there exists a set $F_n \in \mathcal{B}(G)$ such that $\mu(F_n) \geq \mu_s(G) - \frac{1}{n}$ and $\lambda(F_n) = 0$. If we set $F = \bigcup\limits_{n=1}^{\infty} F_n$, then the nonnegativity of $\mu$ implies

$$\mu(F) \geq \mu(F_n) \geq \mu_s(G) - \frac{1}{n}$$

for each $n \in \mathbb{N}$, hence $\mu(F) \geq \mu_s(G)$. Because $F$ is a countable union of Lebesgue null sets, it is also a Lebesgue null set. Hence it is one of the sets over which we take supremum in the definition of $\mu_s(G)$, and so $\mu_s(G) \geq \mu(F)$. This proves $\mu_s(G) = \mu(F)$ and since $\lambda(F) = 0$, $\mu(F)$ is equal to $\mu_s(F)$. The finite additivity of $\mu_s$, which was proved in Step 1, implies $\mu_s(G \setminus F) = \mu_s(G) - \mu_s(F) = 0$. A nonnegative finitely additive measure is equal to its total variation. Applying this to $\mu_s$, we obtain $|\mu_s|(G \setminus F) = 0$, from which we conclude that $\mu_s \in \mathcal{M}_s(G, \mathcal{B}(G))$.

Step 3: Define $\mu_{ac} = \mu - \mu_s$. If $A \in \mathcal{B}(G)$ satisfies $\lambda(A) = 0$, then it follows from the definition of $\mu_s(A)$ that $\mu_s(A) = \mu(A)$, hence $\mu_{ac}(A) = 0$. This shows that $\mu_{ac}$ is an element of $\mathcal{M}_{ac}(G, \mathcal{B}(G))$. The decomposition for a nonnegative finitely additive measure was found.

For a general $\mu \in \mathcal{M}(G, \mathcal{B}(G))$, we define $\mu_s = (\mu^+)_s - (\mu^-)_s$ and $\mu_{ac} = (\mu^+)_{ac} - (\mu^-)_{ac}$. By Proposition 28, $\mu_s \in \mathcal{M}_s(G, \mathcal{B}(G))$ and $\mu_{ac} \in \mathcal{M}_{ac}(G, \mathcal{B}(G))$. Also, $\mu_{ac} + \mu_s$ is a decomposition of $\mu$ because

$$\mu = \mu^+ - \mu^- = (\mu^+)_s + (\mu^+)_{ac} - (\mu^-)_s - (\mu^-)_{ac} = \mu_s + \mu_{ac}.$$

This completes the proof of the existence of the decomposition.

To prove that the decomposition is unique, it is sufficient to show

$$\mathcal{M}_{ac}(G, \mathcal{B}(G)) \cap \mathcal{M}_s(G, \mathcal{B}(G)) = \{0\}.$$

Let $\mu$ be an element of the intersection of $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ and $\mathcal{M}_s(G, \mathcal{B}(G))$. Then there exists $B \in \mathcal{B}(G)$ such that $\lambda(B) = 0$ and $|\mu|(G \setminus B) = 0$. For an arbitrary set $E \in \mathcal{B}(G)$, we have $\lambda(B \cap E) = 0$, which implies $\mu(B \cap E) = 0$. Additionally,

$$|\mu((G \setminus B) \cap E)| \leq |\mu|((G \setminus B) \cap E) = 0,$$

hence $\mu((G \setminus B) \cap E) = 0$. We obtain the following identity for the value of $\mu(E)$:

$$\mu(E) = \mu(B \cap E) + \mu((G \setminus B) \cap E) = 0.$$

Since $E \in \mathcal{B}(G)$ was arbitrary, $\mu$ is the zero measure, which was to be proved.

$\square$

**Proposition 30.** *Let $\mu \in \mathcal{M}(G, \mathcal{B}(G))$ and let $\mu = \mu_{ac} + \mu_s$ be the Lebesgue decomposition of $\mu$ from Theorem 29. Then it holds that*

$$|\mu| = |\mu_{ac}| + |\mu_s|.$$

*Proof.* The proof is similar to that of Proposition 26. The inequality $|\mu| \leq |\mu_{ac}| + |\mu_s|$ follows from the relation (1.2). By the same relation and the definitions of $\mu_{ac}$ and $\mu_s$ from Step 3 of the preceding proof, we have

$$|\mu_{ac}| = |(\mu^+)_{ac} - (\mu^-)_{ac}| \leq |(\mu^+)_{ac}| + |(\mu^-)_{ac}| = (\mu^+)_{ac} + (\mu^-)_{ac},$$
$$|\mu_s| = |(\mu^+)_s - (\mu^-)_s| \leq |(\mu^+)_s| + |(\mu^-)_s| = (\mu^+)_s + (\mu^-)_s.$$

These relations together imply

$$|\mu_{ac}| + |\mu_s| \leq (\mu^+)_{ac} + (\mu^-)_{ac} + (\mu^+)_s + (\mu^-)_s = \mu^+ + \mu^- = |\mu|.$$

This proves the inequality $|\mu| \geq |\mu_{ac}| + |\mu_s|$.

□

**Definition 31.** *We define the mapping $S : \mathcal{M}(G, \mathcal{B}(G)) \to \mathcal{M}_{ac}(G, \mathcal{B}(G))$ by setting $S(\mu) = \mu_{ac}$, where $\mu_{ac}$ is as in Theorem 29.*

*Remark.* The mapping $S$ is a projection onto $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ and the kernel of $S$ is $\mathcal{M}_s(G, \mathcal{B}(G))$. The proof of this statement is almost identical to the proof of Proposition 25 so it will be omitted. Proposition 30 implies that $S$ has the property that $\|\mu\| = \|S(\mu)\| + \|\mu - S(\mu)\|$ for $\mu \in \mathcal{M}(G, \mathcal{B}(G))$.

As in the case of the projection $P$, we obtain that $S$ is a bounded operator of norm 1 and $\mathcal{M}(G, \mathcal{B}(G))$ is a topological sum of the subspaces $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ and $\mathcal{M}_s(G, \mathcal{B}(G))$. The reader might want to compare this remark to the one after Proposition 25.

**Proposition 32.** *The projection $S$ satisfies $S(\mu)^+ = S(\mu^+)$, $\mu \in \mathcal{M}(G, \mathcal{B}(G))$.*

*Proof.* Let $\mu$ be a bounded finitely additive measure. By Proposition 30, $|\mu| = |\mu_{ac}| + |\mu_s|$. Because $\mu_s \in \mathcal{M}_s(G, \mathcal{B}(G))$, there exists $B \in \mathcal{B}(G)$ such that $\lambda(B) = 0$ and $|\mu_s|(G \setminus B) = 0$. This condition also guarantees that $|\mu_s|$ is an element of $\mathcal{M}_s(G, \mathcal{B}(G))$.

Let $A \in \mathcal{B}(G)$ be a Lebesgue null set. Then each Borel subset $A_1$ of $A$ is also Lebesgue null, which implies $\mu_{ac}(A_1) = 0$. From the definition of the total variation of $\mu_{ac}$, it follows that $|\mu_{ac}|(A) = 0$, hence $|\mu_{ac}| \in \mathcal{M}_{ac}(G, \mathcal{B}(G))$.

The uniqueness of the Lebesgue decomposition yields $|\mu|_{ac} = |\mu_{ac}|$. If we use the linearity of the projection $S$, then we can complete the proof similarly to the proof of Proposition 26.

□

# 4. Dual spaces isometrically isomorphic to spaces of measures

## 4.1 Dual space of the space of bounded Borel functions

**Definition 33.** *Let $G \subset \mathbb{R}^d$ be an open set. We define $(\mathcal{B}_b(G), \|\cdot\|_\infty)$ to be the linear space of all bounded Borel functions on $G$ equipped with the supremum norm (which is defined as $\|f\|_\infty = \sup\{|f(x)| : x \in G\}$ for $f \in \mathcal{B}_b(G)$).*

*The symbol $\mathcal{B}_s(G)$ will denote the set of simple functions in $\mathcal{B}_b(G)$.*

*Remark.* The normed linear space $(\mathcal{B}_b(G), \|\cdot\|_\infty)$ is a Banach space. The proof of this fact is a standard exercise in functional analysis and we omitt it. The set $\mathcal{B}_s(G)$ is clearly a linear subspace of $\mathcal{B}_b(G)$. Moreover, it is dense in $\mathcal{B}_b(G)$. The proof of this classical result is also omitted.

The purpose of this section is to characterize the dual space of $(\mathcal{B}_b(G), \|\cdot\|_\infty)$. To do so, we will need to apply the following result of functional analysis.

**Lemma 34.** *Let $(X, |\cdot|)$ be a normed linear space, $Y$ a dense linear subspace of $X$ and $y^*$ a bounded linear functional on $Y$. Then $y^*$ can be uniquely extended to a bounded linear functional $x^*$ on $X$. The extension satisfies $\|x^*\| = \|y^*\|$.*

*Proof.* This statement is taught in the introductory course of functional analysis, hence the proof will be only indicated briefly. For $x \in X \setminus Y$, find a sequence $\{y_n\}_{n=1}^\infty$ of points in $Y$ such that $\lim_{n\to\infty} y_n = x$. If there exists $x^*$ with the required properties, it must satisfy

$$x^*(x) = \lim_{n\to\infty} y^*(y_n). \tag{4.1}$$

On the other hand, formula (4.1) correctly defines a linear functional on $X$. The inequality $\|x^*\| \geq \|y^*\|$ holds because $x^*$ is an extension of $y^*$. The opposite inequality follows from the estimation

$$|x^*(x)| = \lim_{n\to\infty} |y^*(y_n)| \leq \lim_{n\to\infty} \|y^*\| \cdot |y_n| = \|y^*\| \cdot |x|.$$

□

**Theorem 35.** *The dual space of $(\mathcal{B}_b(G), \|\cdot\|_\infty)$ is isometrically isomorphic to the space $\mathcal{M}(G, \mathcal{B}(G))$. The operator $T$ defined by setting*

$$T(x^*) = \mu,\ x^* \in \mathcal{B}_b(G)^*,$$

*where*

$$\mu(A) = x^*(\chi_A),\ A \in \mathcal{B}(G)$$

*is a linear isometry mapping $\mathcal{B}_b(G)^*$ onto $\mathcal{M}(G, \mathcal{B}(G))$.*

*Proof.* Step 1: $T$ maps $\mathcal{B}_b(G)^*$ into $\mathcal{M}(G, \mathcal{B}(G))$. Let $x^*$ be a bounded linear functional on $\mathcal{B}_b(G)$ and set $\mu = T(x^*)$. We must prove that $\mu$ is a bounded finitely additive measure. For two disjoint sets $A$ and $B$ in $\mathcal{B}(G)$, we have

$$\mu(A \cup B) = x^*(\chi_{A\cup B}) = x^*(\chi_A + \chi_B) = x^*(\chi_A) + x^*(\chi_B) = \mu(A) + \mu(B),$$

from which the finite additivity of $\mu$ follows by induction. Now take an arbitrary set $A \in \mathcal{B}(G)$. Because $\|\chi_A\|_\infty \leq 1$, the boundedness of $x^*$ implies

$$|\mu(A)| = |x^*(\chi_A)| \leq \|x^*\|.$$

This shows that $\mu$ is bounded.

Step 2: $T$ is a linear isometry. It is easy to check the linearity of $T$. To show that $T$ is an isometry, let $x^* \in \mathcal{B}_b(G)^*$ and $\mu = T(x^*)$. We must prove $\|x^*\| = \|\mu\|$. Consider the functional $y^* = x^*|_{\mathcal{B}_s(G)}$ (the restriction of $x^*$ to the subspace of simple functions). By the remark after Definition 33, $\mathcal{B}_s(G)$ is dense in $\mathcal{B}_b(G)$. Lemma 34 implies $\|x^*\| = \|y^*\|$, therefore it suffices to show $\|y^*\| = \|\mu\|$.

Let $s \in \mathcal{B}_b(G)$ be a simple function such that $\|s\|_\infty \leq 1$. It can be written as

$$s = \sum_{i=1}^{n} c_i \chi_{E_i}, \tag{4.2}$$

where $n \in \mathbb{N}$, $E_1, \dots E_n \in \mathcal{B}(G)$ are disjoint sets and $c_1, \dots, c_n \in \mathbb{R}$. We can assume that all the sets $E_i$ are nonempty. Then $\|s\|_\infty \leq 1$ implies $|c_i| \leq 1$ for $i = 1, \dots, n$. Using the linearity of $y^*$, we obtain

$$|y^*(s)| = \left| y^* \left( \sum_{i=1}^{n} c_i \chi_{E_i} \right) \right| \leq \sum_{i=1}^{n} |c_i| \cdot |y^*(\chi_{E_i})| \leq \sum_{i=1}^{n} |\mu(E_i)| \leq \|\mu\|. \tag{4.3}$$

We proved that $|y^*(s)| \leq \|\mu\|$ for an arbitrary simple function $s \in \mathcal{B}_b(G)$ such that $\|s\|_\infty \leq 1$, or equivalently, $\|y^*\| \leq \|\mu\|$.

To show the opposite inequality, take a finite collection $\{E_i\}_{i=1}^n$ of disjoint sets in $\mathcal{B}(G)$. For $i = 1, \dots, n$, define

$$c_i = \begin{cases} 1, & \text{if } y^*(\chi_{E_i}) \geq 0, \\ -1, & \text{if } y^*(\chi_{E_i}) < 0. \end{cases}$$

Then the simple function $s = \sum\limits_{i=1}^{n} c_i \chi_{E_i}$ satisfies $\|s\|_\infty \leq 1$. By the definition of $\mu$, $\mu(E_i) = x^*(\chi_{E_i}) = y^*(\chi_{E_i})$ for $i = 1, \dots, n$, which yields

$$\sum_{i=1}^{n} |\mu(E_i)| = \sum_{i=1}^{n} |y^*(\chi_{E_i})| = \sum_{i=1}^{n} c_i y^*(\chi_{E_i}) = y^*(s) \leq |y^*(s)| \leq \|y^*\|.$$

Taking the supremum of the left side, we get $\|\mu\| \leq \|y^*\|$.

Step 3: The mapping $T$ is onto $\mathcal{M}(G, \mathcal{B}(G))$. We want to show that for a fixed $\mu \in \mathcal{M}(G, \mathcal{B}(G))$, there is a linear functional $x^* \in \mathcal{B}_b(G)^*$ such that $T(x^*) = \mu$. First, we find a suitable functional $y^*$ on the set of simple functions and then we extend it to obtain $x^*$. If $\chi_A$ is a characteristic function of $A \in \mathcal{B}(G)$, we define

$$y^*(\chi_A) = \mu(A). \tag{4.4}$$

Now let $s \in \mathcal{B}_b(G)$ be a simple function of the form (4.2). We will check that $y^*$ is well-defined by the formula

$$y^*(s) = \sum_{i=1}^{n} c_i y^*(\chi_{E_i}).$$

Denote by $r_1, \dots, r_m$ the values of $s$. The function $s$ attains a given value $r_j$ exactly on those sets $E_i$ for which $c_i = r_j$ because the sets $E_i$ are pairwise disjoint. Using this observation, we get

$$y^*(s) = \sum_{i=1}^{n} c_i y^*(\chi_{E_i}) = \sum_{j=1}^{m} \sum_{\substack{i \in \{1,\dots,n\}, \\ c_i = r_j}} c_i y^*(\chi_{E_i}) = \sum_{j=1}^{m} r_j y^*(\chi_{s^{-1}(r_j)}),$$

where the last equality holds because of the relation (4.4) defining $y^*$ on characteristic functions. The rightmost expression defines the value $y^*(s)$ unambiguously.

The linearity of $y^*$ follows directly from the definition. The same argument as the one used in Step 2 shows that $\|y^*\| \le \|\mu\|$, hence $y^*$ is bounded. By Lemma 35, $y^*$ has a unique extension $x^* \in \mathcal{B}_b(G)^*$. It follows from (4.4) that the functional $x^*$ satisfies $T(x^*) = \mu$.

□

We often required that a finitely additive measure $\mu$ be bounded. The next proposition shows that this condition does not hold automatically.

**Proposition 36.** *There exists an unbounded finitely additive measure.*

*Proof.* First we establish the existence of an unbounded linear functional $y^*$ on the space $\mathcal{B}_s(G)$. Take an algebraic basis $B$ of $\mathcal{B}_s(G)$ whose elements have norm 1. Choose a sequence $e_1, e_2, \dots$ of elements in $B$ and define $y^*(e_i) = i$ for all $i \in \mathbb{N}$. On the remaining elements of $B$, set the value of $y^*$ arbitrarily (for example as 0). Then $y^*$ can be extended to an unbounded linear functional on the space $\mathcal{B}_s(G)$.

Define $\mu$ by setting $\mu(A) = y^*(\chi_A)$ for $A \in \mathcal{B}(G)$. If $A, B \in \mathcal{B}(G)$ are disjoint, then we have $y^*(\chi_{A \cup B}) = y^*(\chi_A) + y^*(\chi_B)$, hence $\mu$ is a finitely additive measure. To obtain a contradiction, assume that $\mu$ is bounded. Let $s \in \mathcal{B}_s(G)$ satisfy $\|s\|_\infty \le 1$. By an estimation similar to (4.3), we obtain $|y^*(s)| \le \|\mu\|$, which implies $y^*$ is bounded, a contradiction.

□

## 4.2 Dual space of $L_\infty(G)$

**Definition 37.** *Let $G$ be an open subset of $\mathbb{R}^d$. We denote by $(L_\infty(G), \|\cdot\|_{L_\infty})$ the normed linear space of essentially bounded functions, where the norm $\|\cdot\|_{L_\infty}$ is defined by setting*

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in G} |f(x)|, \; f \in L_\infty(G).$$

*Remark.* As in the case of $\mathcal{B}_b(G)$ in the previous section, it is well known that $L_\infty(G)$ is a Banach space and that characteristic functions form a dense linear subspace of $L_\infty(G)$.

Recall that in Definition 27, we introduced the space $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ of finitely additive measures in $\mathcal{M}(G, \mathcal{B}(G))$ which are zero on Lebesgue null sets.

**Theorem 38.** *The dual space of $(L_\infty(G), \|\cdot\|_{L_\infty})$ is isometrically isomorphic to the space $\mathcal{M}_{ac}(G, \mathcal{B}(G))$. The operator $T$ defined by setting*

$$T(x^*) = \mu,\ x^* \in L_\infty(G)^*,$$

*where*

$$\mu(A) = x^*(\chi_A),\ A \in \mathcal{B}(G)$$

*is a linear isometry mapping $L_\infty(G)^*$ onto $\mathcal{M}_{ac}(G, \mathcal{B}(G))$.*

*Proof.* Fix $x^* \in L_\infty(G)^*$ and let $y^* : \mathcal{B}_b(G) \to \mathbb{R}$ be a functional given by the equality

$$y^*(f) = x^*(f),\ f \in \mathcal{B}_b(G),$$

where on the right hand side, we regard $f$ as an element of $L_\infty(G)$. Because $x^*$ is linear, $y^*$ is also linear. We will show that $y^*$ satisfies $\|y^*\| = \|x^*\|$. If $f \in \mathcal{B}_b(G)$, then the inequality $|f(x)| \le \|f\|_\infty$, which holds for every $x \in G$, implies $\|f\|_{L_\infty} \le \|f\|_\infty$. We obtain

$$|y^*(f)| = |x^*(f)| \le \|x^*\| \cdot \|f\|_{L_\infty} \le \|x^*\| \cdot \|f\|_\infty$$

and because $f$ was arbitrary, it follows that $\|y^*\| \le \|x^*\|$. Now take $f \in L_\infty(G)$ such that $\|f\|_{L_\infty} \le 1$ and define $g$ by setting

$$g(x) = \begin{cases} f(x), & \text{if } |f(x)| \le \|f\|_{L_\infty}, \\ 0, & \text{if } |f(x)| > \|f\|_{L_\infty}. \end{cases} \tag{4.5}$$

The function $g$ is equal to $f$ a. e. and satisfies $\|g\|_\infty \le \|f\|_{L_\infty} \le 1$. We have

$$\|y^*\| \ge |y^*(g)| = |x^*(g)| = |x^*(f)|,$$

where we used the fact that $x^*$ does not distinguish functions that are equal almost everywhere. Taking the supremum on the right side yields $\|y^*\| \ge \|x^*\|$.

We proved that $y^* \in \mathcal{B}_b(G)^*$. Let $\mu$ denote $T(x^*)$. By the definition of $y^*$,

$$\mu(A) = y^*(\chi_A),\ A \in \mathcal{B}(G).$$

It immediatelly follows from Theorem 35 that $T$ is an isometric isomorphism into $\mathcal{M}(G, \mathcal{B}(G))$ (because the norms of $x^*$ and $y^*$ are equal).

It remains to prove that the finitely additive measure $\mu$ is an element of $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ and $T$ maps $L_\infty(G)$ onto $\mathcal{M}_{ac}(G, \mathcal{B}(G))$. Take $A \in \mathcal{B}(G)$ such that $\lambda(A) = 0$. The characteristic function of $A$ is zero almost everywhere, hence

$$\mu(A) = x^*(\chi_A) = x^*(0) = 0.$$

We verified that $\mu$ satisfies the condition defining $\mathcal{M}_{ac}(G, \mathcal{B}(G))$.

Let $\mu$ be a finitely additive measure in $\mathcal{M}_{ac}(G, \mathcal{B}(G))$. By Theorem 35, there exists $y^* \in \mathcal{B}_b(G)^*$ such that $\mu(A) = y^*(\chi_A)$ for each set $A \in \mathcal{B}(G)$. We want to use $y^*$ to define a functional $x^* \in L_\infty(G)^*$ whose image under $T$ is $\mu$. Assume that $s$ is a simple function in $\mathcal{B}_b(G)$ and set

$$x^*(s) = y^*(s), \tag{4.6}$$

where on the left side, we consider $s$ an element of $L_\infty(G)$, therefore it actually represents a whole class of functions, which are equal almost everywhere. We must show that $x^*$ is well-defined. If two simple functions $s_1, s_2 \in \mathcal{B}_b(G)$ are equal a. e., then $\tilde{s} = s_1 - s_2$ is a simple function which is zero almost everywhere. This allows us to express $\tilde{s}$ as a finite sum $\sum\limits_{i=1}^{n} c_i \chi_{A_i}$, where $c_i \in \mathbb{R}$ are nonzero and $A_i \in \mathcal{B}(G)$ satisfy $\lambda(A_i) = 0$ for $i = 1, \ldots, n$. From the fact that $\mu \in \mathcal{M}_{ac}(G, \mathcal{B}(G))$, it follows that $\mu(A_i) = 0$ for $i = 1, \ldots, n$. We obtain

$$y^*(\tilde{s}) = y^* \left( \sum_{i=1}^{n} c_i \chi_{A_i} \right) = \sum_{i=1}^{n} c_i y^*(\chi_{A_i}) = \sum_{i=1}^{n} c_i \mu(A_i) = 0,$$

which implies $y^*(s_1) = y^*(s_2)$. Hence $x^*(s)$ is well-defined by (4.6).

The functional $x^*$ is linear because $y^*$ is linear. We will prove that $x^*$ is bounded. Let $s \in L_\infty(G)$ be a simple function such that $\|s\|_{L_\infty} \leq 1$. Analogously to (4.5), we can define a simple bounded function $t$ such that $t$ is equal to $s$ a. e. and $\|t\|_\infty \leq \|s\|_{L_\infty} \leq 1$. These properties of $t$ yield

$$|x^*(s)| = |x^*(t)| = |y^*(t)| \leq \|y^*\|.$$

We proved that $x^*$ is a bounded linear functional on the subspace of simple functions in $L_\infty(G)$ which is dense in $L_\infty(G)$ by the remark preceding this theorem. Lemma 34 now implies that $x^*$ can be extended to the whole space $L_\infty(G)$. For $A \in \mathcal{B}(G)$, we have $\mu(A) = y^*(\chi_A) = x^*(\chi_A)$. This proves that $\mu$ is the image of $x^*$ under $T$, therefore the mapping $T$ is onto $\mathcal{M}_{ac}(G, \mathcal{B}(G))$.

□

*Remark.* Let us denote by $L_1(G)$ the space of integrable functions defined on an open set $G \subset \mathbb{R}^d$. As a consequence of the previous theorem, we obtain the following interesting result: The space $L_1(G)$ is complemented in its second conjugate. We only sketch the proof briefly.

Since $L_1(G)^*$ is isometrically isomorphic to $L_\infty(G)$, we can identify $L_1(G)^{**}$ with $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ by Theorem 38. Then the natural embedding maps $L_1(G)$ into $\mathcal{M}_{ac}(G, \mathcal{B}(G))$.

Let $\nu$ be a $\sigma$-additive measure in $\mathcal{M}_{ac}(G, \mathcal{B}(G))$. Radon-Nikodým theorem implies there exists a function $f \in L_1(G)$ such that $\nu(E) = \int_E f d\lambda$ for all $E \in \mathcal{B}(G)$. It is possible to show that the natural embedding discussed above maps $f$ to $\nu$. Consequently, the embedding maps $L_1(G)$ onto the space $\mathcal{M}_\sigma(G, \mathcal{B}(G)) \cap \mathcal{M}_{ac}(G, \mathcal{B}(G))$ of $\sigma$-additive measures in $\mathcal{M}_{ac}(G, \mathcal{B}(G))$.

In Definition 24, we introduced the projection $P$ which sends a bounded finitely additive measure $\mu$ to its $\sigma$-additive part. It is not very difficult to prove that if $P$ is defined on $\mathcal{M}(G, \mathcal{B}(G))$, then the restriction of $P$ to $\mathcal{M}_{ac}(G, \mathcal{B}(G))$ is a projection onto the space of $\sigma$-additive measures in $\mathcal{M}_{ac}(G, \mathcal{B}(G))$. Because $P$ is bounded, $\mathcal{M}_\sigma(G, \mathcal{B}(G)) \cap \mathcal{M}_{ac}(G, \mathcal{B}(G))$ is complemented in $\mathcal{M}_{ac}(G, \mathcal{B}(G))$. It follows that $L_1(G)$ is complemented in its second conjugate.

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