FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Bc. Jaroslav Šedina

Multicriteria and robust extension of news-boy problem

Department of Probability and Mathematical Statistics

Supervisor of the master thesis: doc. RNDr. Ing. Miloš Kopa, Ph.D.
Study programme: Mathematics
Study branch: Probability, Mathematical Statistics and Econometrics

Prague 2018

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In Prague date 11.05.2018 signature of the author

I thank doc. RNDr. Ing. Miloš Kopa, Ph.D. for his great supervision of this thesis, rapid and subtle comments and suggestions for improvements.

Furthermore, I thank my family and friends for their support.

To my family, friends, and in loving memory of my father.

Title: Multicriteria and robust extension of news-boy problem

Author: Bc. Jaroslav Šedina

Department: Department of Probability and Mathematical Statistics

Supervisor: doc. RNDr. Ing. Miloš Kopa, Ph.D., Department of Probability and Mathematical Statistics

Abstract: This thesis studies a classic single-period stochastic optimization problem called the newsvendor problem. A news-boy must decide how many items to order under the random demand. The simple model is extended in the following ways: endogenous demand in the additive and multiplicative manner, objective function composed of the expected value and Conditional Value at Risk (CVaR) of profit, multicriteria objective with price-dependent demand, multiproduct extension under dependent and independent demands, distributional robustness. In most cases, the optimal solution is provided. The thesis concludes with the numerical study that compares results of two models after applying the Sample Average Approximation (SAA) method. This study is conducted on the real data.

Keywords: newsvendor problem, stochastic optimization, multicriteria optimization, robustness, endogenous randomness

Contents

Introduction

This thesis studies the single-period newsvendor problem and its modifications whilst searching for optimal solutions. The newsvendor problem, or alternatively the news-boy problem (Choi, 2012), is one of the classical problems in the stochastic programming/inventory management literature. In its basic form it is a very simple optimization problem that features stochastic demand (Khouja, 1999). Due to its versatility and simplicity, it is a very popular model among researchers.

Simply put, every morning a newsvendor has to decide how many items to order. However, the demand is random and hence the newsvendor must in some way decide the number of products to be ordered. The problem formulation differs upon the circumstances and newsvendor's thinking. The aim is to find the optimal solution of the given problem that maximizes the newsvendor's expected profit.

In the first chapter the single-period newsvendor problem (NP) with known demand distribution is introduced and the optimal solution for both discrete and continuous random demand are found.

The second chapter contains extensions of the single-period newsvendor model with the assumption that the demand depends on the selling price or the amount of money put into advertising or both combined. We distinguish whether the endogenous variable (customer demand) is interrelated in the additive or multiplicative manner in all cases. The considered response functions are commonly used in literature and could be possibly switched to others (Hrabec, 2016). Furthermore, we find relations between the optimal solutions in the additive and multiplicative demand cases via the riskless case, i.e. the problem formulation with absence of uncertainty.

The third chapter extends the classical single-period newsvendor model in terms of multiple criteria and multiple product. The multicriteria extensions begin with a definition of the multicriteria optimization problem and follow with the definition of Conditional Value at Risk (CVaR) that is used as a measure of risk and serves as the other decision criterion for the newsvendor. The CVaR measure was first introduced by Rockafellar and Uryasev (2002) as a risk measure of loss. In this paper, however, we use the CVaR of profit (Pflug and Römisch, 2007). Thereafter the optimal solution of only CVaR objective is derived so that it might be used in various reformulations of the multicriteria newsvendor problem. Lastly, we combine the multicriteria newsvendor problem with dependency of the customer demand on selling price. For multiproduct extensions two cases are assumed: the case where demands are independent and the case where they are dependent. In addition, we derive an algorithm to find the optimal solution under the independent demands in case the solutions of single-item newsvendor problems are infeasible.

In the fourth chapter we provide the distributionally robust single-period newsvendor model (Wiesemann et al., 2014) that makes the newsvendor immune to the worst possible distribution of demand that is drawn from the set of all distributions featuring certain characteristics. The considered ambiguity set contains distributions that come from the discrete support, have given mean and are restricted above by the variance. Moreover, we try to find the best and the worst distribution for the given ambiguity set.

Using the given real data, the last chapter contains the comparison of solutions of two models derived in the theoretical part of the thesis. Specifically, we compare the classical newsvendor model with the model with pricing. To approximate the objective function we use the Sample Average Approximation (SAA) method that is described by Levi et al. (2015).

Chapter 1

The classical newsvendor problem

Every morning a news-boy can order his daily supply of newspaper for the price c per item (cost). Hence news-boy buys x pieces of newspaper where x is a fixed number. Selling price of each newspaper is p for which holds $p > c$. Although, number of newspaper which could be sold each day is not fixed. It is given by random demand ω which probability distribution P is known. In case there are newspaper left at the end of the selling period they are salvaged for the value v per item. On the other hand, in case there is lack of newspaper the shortage penalty s is executed for every item that could have been sold. Note that v may be negative, in which case it represents per-unit disposal value. Moreover, to avoid trivial solution we assume that $p > c > v$. This reflects real case scenarios where selling price is greater than buying price (i.e. the newsvendor behaves reasonably and his only goal is to maximized its profit) and buying price is greater than salvage value (i.e. otherwise the newsvendor would buy infinitely many items).

Denote $\pi(x;\omega)$ as the news-boy's profit in case he ordered x newspaper and real demand was ω in the given day. From the task description we obtain that

$$\pi(x;\omega) = \begin{cases} p\omega - cx + v(x-\omega) & \text{for} \quad \omega \leq x, \\ px - cx - s(\omega - x) & \text{for} \quad \omega > x. \end{cases} \tag{1.1}$$

This utility function is continuous from the problem description. Then the objective is to maximize (1.1) and the optimization problem is to

$$\begin{aligned} &\underset{x}{\text{maximize}} \quad p\min\{\omega, x\} - cx + v\max\{x-\omega, 0\} - s\max\{\omega - x, 0\} \\ &\text{for} \quad x \geq 0. \end{aligned}$$

Since the demand is not realized at the beginning of the selling period, the newsvendor cannot observe the actual profit. Thus, the traditional approach to analyze the problem is based on assuming a risk neutral newsvendor who makes the optimal quantity decision at the beginning of the selling period.

Actually, researchers have followed two approaches to solving the single-period problem. In the first approach, the expected costs of overestimating and underestimating demand are minimized. The second approach is based on maximizing the total expected profit. Both approaches lead to the same optimal solution

(Qin et al., 2011). We perform second approach in this paper where the optimal quantity is evaluated by maximizing the total expected profit.

Denote mean profit

$$\Pi(x) = \mathsf{E}_\mathsf{P}[\pi(x;\omega)].$$

Since demand cannot be negative in real situations, suppose that $\omega \geq 0$. Hence for any probability distribution P with cumulative distribution function (cdf) F we obtain

$$\Pi(x) = \int_0^\infty \pi(x;\omega)\ \mathrm{d}F(\omega). \tag{1.2}$$

Suppose that ω has mean value $\mathsf{E}_\mathsf{P}\,\omega$. Then, according to (1.2), holds

$$\begin{aligned}\Pi(x) &= \int_0^x (p\,\omega - cx + vx - v\,\omega)\ \mathrm{d}F(\omega) + \int_x^\infty (px - cx - s\,\omega + sx)\ \mathrm{d}F(\omega)\\ &= (p-c)\int_x^\infty x\ \mathrm{d}F(\omega) + p\int_0^x \omega\ \mathrm{d}F(\omega) - c\int_0^x x\ \mathrm{d}F(\omega)\\ &\quad + v\int_0^x (x-\omega)\ \mathrm{d}F(\omega) - s\int_x^\infty (\omega - x)\ \mathrm{d}F(\omega).\end{aligned}$$

Since $(p-c)\int_x^\infty x\ \mathrm{d}F(\omega) + p\int_0^x \omega\ \mathrm{d}F(\omega) - c\int_0^x x\ \mathrm{d}F(\omega)$ can be rewritten as

$$(p-c)x - p\int_0^x (x-\omega)\ \mathrm{d}F(\omega),$$

we finally obtain

$$\Pi(x) = (p-c)x - \left[s\int_x^\infty (\omega - x)\ \mathrm{d}F(\omega) + (p-v)\int_0^x (x-\omega)\ \mathrm{d}F(\omega)\right], \tag{1.3}$$

where $\int_x^\infty (\omega - x)\ \mathrm{d}F(\omega)$ represents the expected shortages in case x items is ordered and $\int_0^x (x-\omega)\ \mathrm{d}F(\omega)$ the expected leftovers. The expected shortages can be written as $\int_x^\infty (\omega - x)\ \mathrm{d}F(\omega) = \mathsf{E}_\mathsf{P}(\omega - x)^+$, where $y^+ = \max\{0, y\}$ is called the positive part function. Analogically for the expected leftovers $\int_0^x (x-\omega)\ \mathrm{d}F(\omega) = \mathsf{E}_\mathsf{P}(x-\omega)^+$. Therefore, from (1.3), the optimization problem with the maximal expected profit criterion is formulated as:

$$\begin{aligned}&\underset{x}{\text{maximize}} \quad (p-c)x - \left[s\int_x^\infty (\omega - x)\ \mathrm{d}F(\omega) + (p-v)\int_0^x (x-\omega)\ \mathrm{d}F(\omega)\right]\\ &\text{for} \quad x \geq 0.\end{aligned} \tag{1.4}$$

Optimality

Analogically to Šedina (2015), the model (1.4) is easily solvable optimization problem. Notice the shape of function $\Pi(x)$ under assumption that the random demand ω is bounded on a finite interval $[\,\underline{b}, \overline{b}\,]$:

$$\Pi(x) = \begin{cases} (p-c)x - s\,\mathsf{E_P}(\omega - x) & \text{if } x < \underline{b}, \\ (p-c)x - \left[s\,\mathsf{E_P}(\omega - x)^+ + (p-v)\,\mathsf{E_P}(x-\omega)^+\right] & \text{if } \underline{b} \le x \le \overline{b}, \\ p\,\mathsf{E_P}\,\omega - cx + v\,\mathsf{E_P}(x - \omega) & \text{if } x > \overline{b}. \end{cases}$$

If ω has a mean value, we can similarly derive $\Pi(x)$ in case $\underline{b} = -\infty$ or $\overline{b} = \infty$. As we stated in task description, $p > c > v$ holds. Moreover, it is reasonable to assume that $p > c > 0$ and $s \ge 0$. Under such assumptions is

$$\Pi(x) = (p-c)x - \mathsf{E_P}[\varphi(x;\omega)]$$

a concave function in x, since it is the difference of linear function and mean of function

$$\varphi(x;\omega) = (p-v)(x-\omega)^+ + s(\omega-x)^+ = (p-v)\max\{x-\omega, 0\} + s\max\{\omega - x, 0\}$$

that is a convex function of x given ω because the sum of convex functions is convex and maximum multiplied by nonnegative parameter is obviously convex (see figure 1.1).

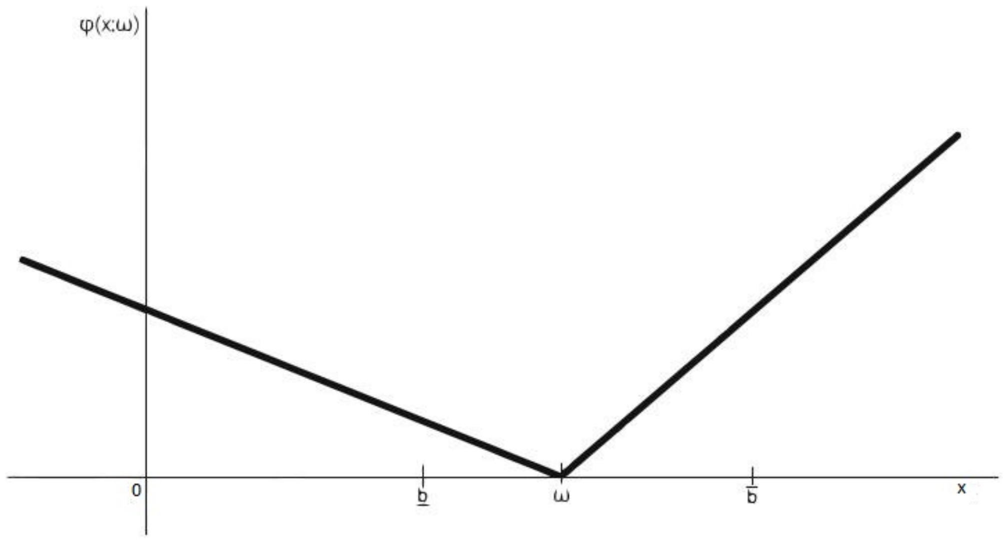

Figure 1.1: Graph of function $\varphi(x;\omega)$ for given ω, where $p - v > s$.

Next part, evaluation of the optimal solution, is inspired by Dupačová (1986). If $x < \underline{b}$ then $\Pi(x)$ is linear and $\Pi'(x) = p - c + s > 0$. Similarly if $x > \overline{b}$ then $\Pi'(x) = v - c < 0$. The derivative of $\Pi(x)$ may not exist in all points of the range $x \in [\,\underline{b}, \overline{b}\,]$. Although, existence of subdifferential $\partial\Pi(x)$ is guaranteed. Then

$$\begin{aligned} \partial\Pi(x) &= \{\ \eta : \Pi(l) \le \Pi(x) + \eta(l - x) \quad \forall\, l\ \} \\ &= \{\ \eta : (p-c+s) - (p-v+s)F(x) \le \eta \le (p-c+s) - (p-v+s)F^-(x)\}. \end{aligned}$$

Endpoints of interval $\partial\Pi(x)$ are left hand and right hand derivatives of function Π in point x and $F^-(x) = \lim_{y \to x^-} F(y)$.

If $p > c > v$, $p > c > 0$ and $s \geq 0$, then the maximum of $\Pi(x)$ is obviously reached for $x^0 \in [\underline{b}, \overline{b}]$ that satisfies condition

$$0 \in \partial\Pi(x^0).$$

Thus

$$(p-c+s) - (p-v+s)F(x^0) \leq 0 \leq (p-c+s) - (p-v+s)F^-(x^0)$$

which leads to

$$F^-(x^0) \leq \frac{p+s-c}{p+s-v} \leq F(x^0). \tag{1.5}$$

Let the subscript * denote optimality. If $x^0 \geq 0$, then the optimal order quantity of (1.4) is $x^* = x^0$. Otherwise, the optimal decision is apparently $x^* = 0$.

Continuous distribution case

For absolutely continuous distribution is $F^-(x) = F(x)$ and $\partial\Pi(x)$ is single-point set. Therefore the function Π is differentiable

$$\frac{\partial\Pi(x)}{\partial x} = p - c - [s(F(x)-1) + (p-v)F(x)]$$

for $x \in [\underline{b}, \overline{b}]$ and the optimal decision x^* is given by condition

$$F(x^*) = \frac{p+s-c}{p+s-v}, \tag{1.6}$$

alternatively

$$F(x^*) = 1 - \frac{c-v}{p+s-v},$$

or eventually by condition $x^* = 0$.

Thanks to our assumptions on the news-boy problem, we are able to express x^* as a quantile of a distribution. Hence the optimal order quantity for $x^* \in [\underline{b}, \overline{b}]$ and F invertible is

$$x^* = F^{-1}\left(\frac{p+s-c}{p+s-v}\right). \tag{1.7}$$

Example. Suppose that ω has continuous uniform distribution on interval $[A, B]$, where $A \geq 0$. For $A \leq x \leq B$ is the total expected profit

$$\Pi(x) = (p-c)x - \frac{p-v}{B-A}\int_A^x (x-\omega)\,\mathrm{d}\omega - \frac{s}{B-A}\int_x^B (\omega - x)\,\mathrm{d}\omega.$$

Straight computation leads to quadratic function

$$\Pi(x) = (p-c)x - \frac{(p-v)(x-A)^2 + s(B-x)^2}{2(B-A)},$$

which reaches its maximum in the point

$$x^* = A + \frac{(B-A)(p-c+s)}{p-v+s} \tag{1.8}$$

that belongs to the interval $[A, B]$.

Discrete distribution case

For a discrete distribution, the inequality (1.5) may be satisfied by multiple points. Suppose that the random variable ω takes values $\omega_1, \ldots, \omega_k$ with probabilities $q_1, \ldots, q_k$, where $\forall j \in \{1, \ldots, k\} : q_j > 0$ and $\sum_{j=1}^{k} q_j = 1$. Moreover $\omega_1 < \omega_2 < \ldots < \omega_k$ holds. Recall that the cdf of discrete distribution is piece-wise linear, right hand continuous and nondecreasing function. If exists $x^0 \in [\omega_1, \omega_k]$ for which $F(x^0) = \sum_{j=1}^{i} q_j = \frac{p+s-c}{p+s-v}$ for some $i \in \{1, \ldots, k\}$, then the optimal decision is to buy $x^* = x^0$ units. This corresponds to the optimal ratio $\frac{p+s-c}{p+s-v}$ being equal to the one step of cdf. Otherwise $F(\omega_{i-1}) < \frac{p+s-c}{p+s-v} < F(\omega_i)$ holds for some $i \in \{2, \ldots, k\}$, which corresponds to the optimal ratio $\frac{p+s-c}{p+s-v}$ being in between steps of cdf. Therefore the optimal order quantity is $x^* = \omega_i$.

The results from previous paragraph could be summarized and we obtain that for the discrete distribution the optimal order quantity is

$$x^* = \inf \left\{ x_j : F(x_j) \geq \frac{p+s-c}{p+s-v} \right\}, \tag{1.9}$$

which is again $\frac{p+s-c}{p+s-v}$-quantile of distribution F that is generalized for discrete distributions.

Chapter 2

Endogenous extensions of the classical newsvendor model

The customer demand is typically treated as an exogenous parameter in the prior analysis. However, depending upon the context, it could be argued that the customer demand ω and price p are interrelated or customer demand ω could be influenced by marketing effort a expended by the newsboy. We examine both cases in the following sections and, furthermore, add section where both the price p and the advertising a together influence the customer demand ω. In these cases, the endogenous randomness is considered.

2.1 Model with pricing

This section is mainly inspired by Petruzzi and Dada (1999). In a classic newsvendor problem, selling price is considered as exogenous, over which the vendor has absolutely no control. This is true in a perfectly competitive market where buyers are mere price-takers. In practice, however, newsvendor can adjust the current selling price in order to increase or reduce demand. This model is mostly called as the newsvendor problem with pricing (NPP). The first mathematical formulation of the price effect in inventory control problems was provided by Whitin (1955).

Consider a firm that stocks a single product. However, this firm faces a price-dependent demand function, and has an objective of determining jointly the optimal ordering quantity x and the selling price p to maximize the total expected profit. The expected profit function is now a bivariate function with price and order quantity as the decision variables. However, from the managerial perspective, we deal with problems where the decision-maker does not know the real demand. Therefore, we further model the demand as a function, which can be affected by the price chosen and which somehow depends on a random element. Randomness in demand is independent of price and can be modeled by either multiplicative form or additive form. Specifically, the multiplicative form of demand, first introduced by Karlin and Carr (1962), can be expressed as

$$\omega^M(p, \epsilon) = d^M(p)\epsilon, \tag{2.1}$$

and the additive form, first introduced by Mills (1959), as

$$\omega^A(p, \epsilon) = d^A(p) + \epsilon, \tag{2.2}$$

where both $d^M(p)$ and $d^A(p)$ are response functions that describe relation between price and demand, and ϵ is a continuous random variable defined on the range $[A, B]$. Furthermore, let the response functions be continuous, positive, twice differentiable and strictly decreasing on its domain $[p_l, p_u]$ in the pricing policy p (Khouja, 1999). Positiveness is required due to the upcoming substitution where the response function is a denominator. Otherwise, only nonnegativity of the response function would be sufficient because setting the price too high might lead to zero demand. Limitations on the price reflect real situations where retailer cannot set price unrealistically and without any bounds ($p \in [p_l, p_u]$ where $p_u > p_l > c$).

Since the exact form of $d^M(p)$ and $d^A(p)$ influences the exact form of (2.1) and (2.2), we denote

$$d^M(p) = \alpha p^{-\beta} \quad (\alpha > 0, \beta > 1) \tag{2.3}$$

in the multiplicative case and

$$d^A(p) = \alpha - \beta p \quad (\alpha > 0, \beta > 0) \tag{2.4}$$

in the additive case. Both representations of $d^M(p)$ and $d^A(p)$ are common in the economics literature. Additive case corresponds to a linear demand curve and multiplicative case represents an isoelastic demand curve. One interpretation of the model could be that the demand curve shape is deterministic and random scaling parameter corresponds to the size of the market. Figure 2.1 illustrates both of the response functions mentioned above.

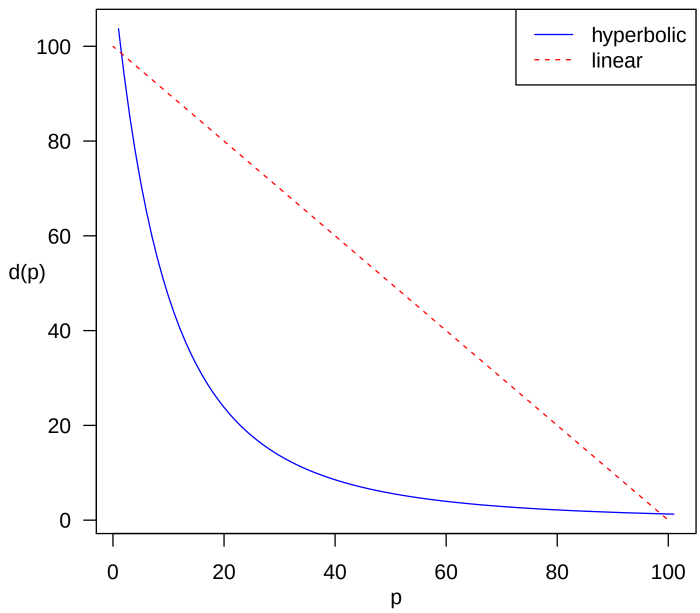

Figure 2.1: Shape of response functions considered in the newsvendor problem with pricing. The hyperbolic function is given by (2.1) and the linear by (2.2).

However, we have to still keep in mind that the demand cannot be negative. Therefore, we suppose that the lower endpoint of ϵ satisfies $A > 0$ in the multiplicative case and $A > \beta p_u - \alpha$ in the additive case. All the assumptions combined now guarantee that nonnegative demand is observed for all allowed ranges of p. Let F represent the cumulative distribution function (cdf) of ϵ, and f the probability density function. Likewise, we define μ and σ as the mean and standard deviation of ϵ, respectively. Researchers usually assume that μ is equal to zero and one in the additive demand case and the multiplicative demand case, respectively (Yao et al. (2006),Hrabec (2016)). This basically says that the demand can be on average fully represented by the appropriate response function. Precisely that both $\mathsf{E}_\epsilon[\omega^A(p,\epsilon)] = d^A(p)$ and $\mathsf{E}_\epsilon[\omega^M(p,\epsilon)] = d^M(p)$. We skip this assumption in order to deduce more general results and models.

Hence $\mathsf{E}_\epsilon[\omega^A(p,\epsilon)] = d^A(p) + \mu$ and $\mathsf{E}_\epsilon[\omega^M(p,\epsilon)] = d^M(p)\mu$ are both strictly decreasing in price as mean value in multiplicative demand case is always positive ($A > 0$). Monotonicity of the expect demand is satisfied for all common items, except only special luxury goods exhibiting the Veblen paradox (Yao et al., 2006).

2.1.1 Additive demand case

In the additive demand case the demand function is $\omega^A(p,\epsilon) = d^A(p) + \epsilon$ where $d^A(p) = \alpha - \beta p$ (see (2.2) and (2.4)). The single-period problem's task description is similar as in Chapter 1 with the only difference that the random demand ω is now represented by demand function $\omega^A(p,\epsilon)$. Thus, if we substitute $\omega = \omega^A(p,\epsilon)$ in (1.1), we obtain the utility function

$$\pi^A(x;p;\omega) = \begin{cases} p\omega^A(p,\epsilon) - cx + v[x - \omega^A(p,\epsilon)] & \text{for} \quad \omega^A(p,\epsilon) \le x, \\ px - cx - s[\omega^A(p,\epsilon) - x] & \text{for} \quad \omega^A(p,\epsilon) > x. \end{cases} \tag{2.5}$$

Again, a newsboy stocks x units at the beginning of selling period for cx. Depending upon he underestimated or overestimated demand, his income for sales is $p\min\{\omega^A(p,\epsilon), x\}$. Term $v[x - \omega^A(p,\epsilon)]$ is added to overall profit if a newsvendor stocked more units than demand, where v is per-unit disposal (salvage) value and $x - \omega^A(p,\epsilon)$ is number of units left. And penalty $s[\omega^A(p,\epsilon) - x]$ is subtracted from overall profit in case demand is not fulfilled, where s is per-unit penalty value and $\omega^A(p,\epsilon) - x$ is number of units missing.

Therefore, the optimization problem with the maximal expected profit criterion for additive price-dependent demand case is formulated as:

$$\begin{aligned} \underset{x;p}{\text{maximize}} \quad (p-c)x - \Bigg[\; & s\int_x^\infty (\omega^A(p,\epsilon) - x)\ \mathrm{d}F(\omega^A(p,\epsilon)) \\ & + (p - v)\int_0^x (x - \omega^A(p,\epsilon))\ \mathrm{d}F(\omega^A(p,\epsilon))\Bigg] \end{aligned} \tag{2.6}$$

$$\text{for} \quad x \ge 0,\ p \in [p_l, p_u].$$

As described in Petruzzi and Dada (1999) a convenient expression for this profit function is obtained by substituting $\omega^A(p,\epsilon) = d^A(p) + \epsilon$ to (2.5) and defining $z = x - d^A(p)$ where $z \in \mathbb{R}$. The substitution was first established by

Whitin (1955) and he introduced the sequential method of finding the optimal z given p performed on the following pages. Therefore, we get from (2.5) that

$$\pi^A(z;p;\epsilon) = \begin{cases} p[d^A(p)+\epsilon] - c[d^A(p)+z] + v[z-\epsilon] & \text{for} \quad \epsilon \le z, \\ p[d^A(p)+z] - c[d^A(p)+z] - s[\epsilon - z] & \text{for} \quad \epsilon > z. \end{cases} \tag{2.7}$$

For better understanding we unify notation for both multiplicative and additive cases and the same symbols π are also used for objective functions involving either x or z.

This transformation to z provides an alternative interpretation of the quantity decision: If the z is larger than the realized value of ϵ, then leftovers occur; if the amount of z is smaller than the realized value of ϵ, then shortages occur. Let z^* and p^* maximize the total expected profit. Hence the optimal stocking and pricing policy is to stock $x^* = d^A(p^*) + z^*$ units to sell at price p^* for one paper.

Remember that random variable ϵ is defined on the region $[A, B]$. From (2.7), the total expected profit is

$$\begin{aligned} \Pi^A(z;p) = \mathsf{E}_\epsilon[\pi^A(z;p;\epsilon)] = & \int_A^z (p[d^A(p)+\epsilon] + v[z-\epsilon]) \,\mathrm{d}F(\epsilon) \\ & + \int_z^B (p[d^A(p)+z] - s[\epsilon - z]) \,\mathrm{d}F(\epsilon) \\ & - c[d^A(p)+z]. \end{aligned}$$

That could rewritten as follows:

$$\begin{aligned} \Pi^A(z;p) = & \int_A^B pd^A(p) \,\mathrm{d}F(\epsilon) + \int_A^z p\epsilon \,\mathrm{d}F(\epsilon) + \int_z^B pz \,\mathrm{d}F(\epsilon) \\ & - cd^A(p) - c\int_A^B (z - \epsilon + \epsilon) \,\mathrm{d}F(\epsilon) \\ & + v\int_A^z (z-\epsilon) \,\mathrm{d}F(\epsilon) - s\int_z^B (\epsilon - z) \,\mathrm{d}F(\epsilon) \\ = & (p-c)d^A(p) + p\mu - \int_z^B p\epsilon \,\mathrm{d}F(\epsilon) + \int_z^B pz \,\mathrm{d}F(\epsilon) \\ & - c\int_A^z (z-\epsilon) \,\mathrm{d}F(\epsilon) + c\int_z^B (\epsilon - z) \,\mathrm{d}F(\epsilon) - c\mu \\ & + v\int_A^z (z-\epsilon) \,\mathrm{d}F(\epsilon) - s\int_z^B (\epsilon - z) \,\mathrm{d}F(\epsilon) \\ = & (p-c)[d^A(p) + \mu] \\ & - (c-v)\int_A^z (z-\epsilon) \,\mathrm{d}F(\epsilon) - (p+s-c)\int_z^B (\epsilon - z) \,\mathrm{d}F(\epsilon). \end{aligned} \tag{2.8}$$

By denoting

$$\Lambda(z) = \int_A^z (z-\epsilon) \,\mathrm{d}F(\epsilon) \tag{2.9}$$

and

$$\Theta(z) = \int_z^B (\epsilon - z) \, \mathrm{d}F(\epsilon) \tag{2.10}$$

we can rewrite the total expected profit as

$$\Pi^A(z; p) = \Psi^A(p) - L^A(z, p), \tag{2.11}$$

where

$$\Psi^A(p) = (p - c)[d^A(p) + \mu] \tag{2.12}$$

and

$$L^A(z, p) = (c - v)\Lambda(z) + (p + s - c)\Theta(z). \tag{2.13}$$

Equation (2.12) represents riskless profit function, where profit margin $(p-c)$ per sold item is multiplied by the sum of price-dependent demand $d^A(p)$ and mean μ of the random variable ϵ. Equation (2.13) is the loss function that captures loss in case demand is higher or lower than quantity stocked. Specifically, it assesses the overage cost $(c - v)$ for each of the $\Lambda(z)$ expected leftovers, represented by (2.9), when z is too high and the underage cost $(p + s - c)$ for each of the $\Theta(z)$ expected shortages, represented by (2.10), when z is too low. Equation (2.11) corresponds to the total expected profit since it is the subtraction of the riskless profit, which would occur in the absence of uncertainty, by the overall loss that occurs as a result of the presence of uncertainty.

The objective is to maximize the total expected profit. Thus from (2.11) we obtain the optimization problem as:

$$\begin{aligned} \underset{z;p}{\text{maximize}} \quad & \Pi^A(z; p) \\ \text{subject to} \quad & p \in [p_l, p_u]. \end{aligned} \tag{2.14}$$

We use the first and the second order derivatives of $\Pi^A(z; p)$, given by (2.11), with respect to z and p to find the optimal solution. Hence

$$\frac{\partial \Pi^A(z; p)}{\partial z} = p - c - [s(F(x) - 1) + (p - v)F(x)], \tag{2.15}$$

$$\frac{\partial^2 \Pi^A(z; p)}{\partial z^2} = -(p + s - v)f(z), \tag{2.16}$$

$$\frac{\partial \Pi^A(z; p)}{\partial p} = 2\beta(p^A_\Psi - p) - \Theta(z), \tag{2.17}$$

where $p^A_\Psi = \frac{\alpha+\beta c+\mu}{2\beta}$ is the optimal riskless price, which is the price that maximizes $\Psi^A(z)$ (2.12). And the last derivative is

$$\frac{\partial^2 \Pi^A(z; p)}{\partial p^2} = -2\beta. \tag{2.18}$$

Notice that $\Pi^A(z; p)$ is concave in z for a given price p from (2.16). Thus, it is possible to reduce (2.14) to an optimization problem of the single variable p if we first solve the problem in z as a function of p and then substitute result back into $\Pi^A(z;p)$. This method is described in detail in Whitin (1955) and yields the rule for determining z,

$$1 - F(z^*) = \frac{c - v}{p + s - v}, \tag{2.19}$$

which is the same formula as if p would be fixed (see (1.6)). Thus it corresponds to the standard NP solution in case p is fixed. Similarly, we can notice that $\Pi^A(z;p)$ is concave in p for a given z from (2.18). Hence we can get the optimal solution by first optimizing p for a given z and then searching over the resulting optimal space to maximize $\Pi^A(z;p^*)$. Both approaches yield the same conclusions, but only the latter procedure is presented. See Hrabec (2016) for the other approach.

Lemma 1 follows directly from (2.17) and (2.18).

Lemma 1 (Petruzzi and Dada (1999))**.** *For a fixed z, the optimal price is determined uniquely as a function of z:*

$$p^* = p(z) = p_\Psi^A - \frac{\Theta(z)}{2\beta}. \tag{2.20}$$

Since the expected shortage $\Theta(z)$ (2.10) is nonnegative, then $p^* \leq p_\Psi^A$ from (2.20). Consequently, the optimal price p^* must satisfy $c < p^* \leq p_\Psi^A$ and from (2.14) yields the boundary condition that is given by

$$c < p_l \leq p^* \leq \min\{p_\Psi^A, p_u\}.$$

We can now substitute $p^* = p(z)$ to (2.14) and the optimization problem with two decision variables becomes the maximization over a single variable z:

$$\underset{z}{\text{maximize}} \quad \Pi^A(z;p(z)).$$

Therefore, the optimal quantity decision and pricing policy depends on the shape of $\Pi^A(z;p(z))$. However, as shown in Theorem 2, $\Pi^A(z;p(z))$ might have more than one point that satisfy the first order optimality condition depending on the value of parameters of the problem.

Theorem 2 (Petruzzi and Dada (1999))**.** *The single-period optimal stocking and pricing policy for the additive demand case is to stock $x^* = d^A(p^*) + z^*$ units to sell at the unit price p^*, where p^* is specified by Lemma 1 and z^* is determined according to the following:*

(a) If F is an arbitrary distribution function, then an exhaustive search over all values of z in the region [A,B] will determine z^*.

(b) If F is a distribution function satisfying the condition $2r(z)^2 + \partial r(z)/\partial z > 0$ for $A \leq z \leq B$, where $r(\cdot) = f(\cdot)/[1 - F(\cdot)]$ is the hazard rate, then z^* is the largest z in the region [A,B] that satisfies

$$\frac{\partial \Pi^A(z;p(z))}{\partial z} = 0.$$

(c) If the condition for (b) is met AND $\alpha - \beta(c - 2s) + A > 0$, then z^* is the only z in the region [A,B] that satisfies

$$\frac{\partial \Pi^A(z;p(z))}{\partial z} = 0.$$

Proof. From the chain rule and Lemma 1 follows that

$$\frac{\partial \Pi^A(z;p(z))}{\partial z} = -(c-v) + \left(p_\Psi^A + s - v - \frac{\Theta(z)}{2\beta}\right)[1-F(z)].$$

The rest of proof is conducted in Petruzzi and Dada (1999).

❑

In case F is a distribution function that satisfies $2r(\cdot)^2 + r^{\cdot}(\cdot) > 0$, then the second condition in (c) causes that $\Pi^A(z;p(z))$ is unimodal in z (Petruzzi and Dada, 1999). Condition (b) is satisfied by all nondecreasing hazard rate distributions, which include PF_2 distributions and the log-normal distribution (Barlow and Proschan, 1987, Theorem 4.1.). In particular, the exponential, the reflected exponential (if X is exponential, then $-X$ is reflected exponential), the uniform, the Erlang, the normal, the truncated normal, and all translations ($g+X$ is a translation for scalar g) and convolutions of such distributions (Porteus, 2002, Page 135).

Zhan and Shen (2005) deal with the solution based on the relation between the equations (2.19) and (2.20), and give a geometrical interpretation. They also developed an iterative algorithm and simulation based algorithm to solve the system of equations (2.19) and (2.20).

Example. This example is inspired by Hrabec et al. (2012). Assume the maximal expected profit criterion (2.6) before substitution. Suppose that ϵ has continuous uniform distribution on interval $[A,B]$, thus $\epsilon \sim U(A,B)$. Hence price-dependent demand $\omega^A(p,\epsilon)$ can be represented as $\omega^A(p)$ that has uniform distribution with bounds $[A(p),B(p)]$ where $A(p) = \alpha - \beta p + A$ and $B(p) = \alpha - \beta p + B$. That is $\omega^A(p) \sim U(A(p),B(p))$. Moreover $B > A > \beta p_u - \alpha$ ($p \in [p_l, p_u]$) so that we always observe nonnegative demand. Therefore the random demand ω^A is a linear nonnegative function of bounded price p.

Hrabec et al. (2012) states that the uniform distribution is suitable for cases where bounds of uncertainty are known, otherwise there is a lack of information about uncertainty. We may think that this linear dependency does not approximate some real situations very well. Hyperbolic dependency might be used, as shown in next section 2.1.2, that can be piece-wise approximated. So, thinking about Taylor expansion features, we assume that linear approximation can be acceptable.

Denote $\Pi_{\omega^A}(x;p) = \mathsf{E}_{\omega^A}[\pi^A(x;p;\omega^A(p))]$. We know that $\omega^A(p) \in [A(p),B(p)]$ and thus, we may rewrite the model (2.6) as follows

$$\Pi_{\omega^A}(x;p) = \begin{cases} (p-c)x - s\,\mathsf{E}[\omega^A(p)-x], & x < A(p), \\ (p-c)x - s\,\mathsf{E}[\omega^A(p)-x]^+ - (p-v)\mathsf{E}[x-\omega^A(p)]^+, & x \in [A(p),B(p)] \\ p\,\mathsf{E}\,\omega^A(p) - cx + v\,\mathsf{E}[x-\omega^A(p)], & x > B(p). \end{cases}$$

Thus for uniformly distributed random demand $\omega^A(p)$ we get

$$\begin{aligned}\Pi_{\omega^A}(x;p) =& (p-c)x - \frac{p-v}{B(p)-A(p)} \int_{A(p)}^{x} (x-\omega^A(p))\,\mathrm{d}\omega^A(p) \\ &- \frac{s}{B(p)-A(p)} \int_{x}^{B(p)} (\omega^A(p)-x)\,\mathrm{d}\omega^A(p).\end{aligned}$$

Straight computation leads to quadratic function

$$\Pi_{\omega^A}(x;p) = (p-c)x - \frac{(p-v)(x-A(p))^2 + s(B(p)-x)^2}{2(B(p)-A(p))}.$$

Then, under assumptions given, model (2.6) may be rewritten as:

$$\begin{aligned} &\underset{x;p}{\text{maximize}} \quad (p-c)x - \frac{(p-v)(x-A(p))^2 + s(B(p)-x)^2}{2(B(p)-A(p))} \\ &\text{for} \quad x \geq 0,\ p \in [p_l, p_u]. \end{aligned} \tag{2.21}$$

Notice that the objective function of model (2.21) is concave in x with fixed p since it is the difference of linear function and sum of convex functions that is again convex. Hence the necessary optimality condition is sufficient for optimality and by taking $\frac{\partial \Pi_{\omega^A}(x;p)}{\partial x} = 0$ we get

$$x^*(p) = A(p) + \frac{(B-A)(p-c+s)}{p-v+s} \tag{2.22}$$

that belongs to the interval $[A(p), B(p)]$. Note that we obtained similar optimal ordering as in NP, compare (1.8) and (2.22). The optimal stocking decision depends on the choice of pricing, i.e. x^* is a function of p.

The problem of maximizing (2.21) over two variables is reduced to a maximization problem over the single variable p by substituting back (2.22) to (2.21), i.e. $x = x^*(p)$. We obtain

$$\Pi_{\omega^A}(x^*(p);p) = (p-c)x^*(p) - \frac{B-A}{2}\left[(p-v)\left(\frac{p-c+s}{p-v+s}\right)^2 + s\left(\frac{c-v}{p-v+s}\right)^2\right].$$

2.1.2 Multiplicative demand case

As stated before, in the multiplicative demand case $\omega^M(p,\epsilon) = d^M(p)\epsilon$ and $d^M(p) = \alpha p^{-\beta}$ (see (2.1) and (2.3)). We perform the same type of substitution as in section 2.1.1. Specifically, $\omega^M(p,\epsilon) = d^M(p)\epsilon$ replaces ω and $x = d^M(p)z$ in equation (1.1). Hence $z \geq 0$ (note that when $x = 0$ then $z = 0$ since $d^M(p) > 0$). The single-period total profit function can be rewritten as follows:

$$\pi^M(z;p;\epsilon) = \begin{cases} pd^M(p)\epsilon - cd^M(p)z + vd^M(p)[z-\epsilon] & \text{for} \quad \epsilon \leq z, \\ pd^M(p)z - cd^M(p)z - sd^M(p)[\epsilon - z] & \text{for} \quad \epsilon > z. \end{cases}$$

Although z is defined differently compared to the additive demand case, the effect is the same. Thus leftovers occur if z is larger than the realized value of ϵ and shortages occur if z is smaller than the realized value of ϵ. A managerial interpretation of z is provided in the next section 2.1.3.

Analogous to the additive demand case, the optimal order quantity and pricing policy is to buy $x^* = d^M(p^*)z^*$ items at the unit price p^*, where z^* and p^* jointly maximize the total expected profit. Similarly to the additive demand case (see (2.8)), the expected profit can be written as:

$$\Pi^M(z;p) = \mathsf{E}_\epsilon[\pi^M(z;p;\epsilon)] = \Psi^M(p) - L^M(z,p). \tag{2.23}$$

But this time

$$\Psi^M(p) = (p-c)d^M(p)\mu, \tag{2.24}$$

and

$$L^M(z,p) = d^M(p)\Big[(c-v)\Lambda(z) + (p+s-c)\Theta(z)\Big] = d^M(p)L^A(z,p), \tag{2.25}$$

where $L^A(z,p)$ is denoted by (2.13). Consequently, $\Psi^M(p)$ is again interpreted as riskless profit and $L^M(z,p)$ as expected loss due to uncertainty. However, this time the expected leftovers are represented by $\Lambda(z)d^M(p)$ and the expected shortages by $\Theta(z)d^M(p)$.

To obtain the optimal solution of $\Pi^M(z;p)$ (2.23) we have to follow the same procedure that was described in previous section in detail (section 2.1.1). Initially, the optimal selling price p^* is denoted as a function of z ($p^* = p(z)$). Then the optimal price is substituted back into the total expected profit function and therefore the problem is reduced to a single variable problem. The opposite approach where optimal $z^* = z(p)$ is substituted back into $\Pi^M(z;p)$ is conducted in Hrabec (2016).

Note. If we derive the necessary optimal condition with respect to the price p, i.e. $\frac{\partial \Pi^M(z;p)}{\partial p} = 0$, we obtain

$$\frac{\partial d^M(p)}{\partial p}[(p-c)\mu - L^A(z,p)] + d^M(p)[\mu - \Theta(z)] = 0. \tag{2.26}$$

This condition is further used to compare the optimal price of the NPP with the optimal price of the newsvendor problem with joint pricing and advertising (NPPA) in section 2.3.

Lemma 3 (Petruzzi and Dada (1999))**.** *For a fixed z, the optimal price is determined uniquely as a function of z:*

$$p^* = p(z) = p^M_\Psi + \frac{\beta}{\beta-1}\left[\frac{(c-v)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right],$$

where $p^M_\Psi = \frac{\beta c}{\beta - 1}$ *and* $\Lambda(z)$ *are expected leftovers given by* (2.9) *and* $\Theta(z)$ *are expected shortages given by* (2.10)*.*

Proof. See Petruzzi and Dada (1999) for proof. ❑

By assumption $\beta > 1$ and $A > 0$. Moreover, it can be shown that $\Theta(z)$ is nonincreasing in z, and thus $\mu - \Theta(z) \geq \mu - \Theta(A) = A > 0$. Consequently, it holds that $p^* \geq p^M_\Psi$, where p^M_Ψ represents the optimal riskless price. Note that in this case p^M_Ψ does not depend on the random variable ϵ, whereas in the additive demand case p^A_Ψ is a linear function of the mean of ϵ. From inequality $p^* \geq p^M_\Psi$ yield the boundary condition

$$\max\{p^M_\Psi, p_l\} \leq p^* \leq p_u.$$

Similarly to the additive case, the shape of $\Pi^M(z;p(z))$ depends on the parameters of the problem

Theorem 4 (Petruzzi and Dada (1999)). *The single-period optimal stocking and pricing policy for the multiplicative demand case is to stock $x^* = d^M(p^*)z^*$ units to sell at the unit price p^*, where p^* is specified by Lemma 3 and z^* is determined according to the following:*

(a) If F is an arbitrary distribution function, then an exhaustive search over all values of z in the region [A,B] will determine z^*.

(b) If F is a distribution function satisfying the condition $2r(z)^2 + \partial r(z)/\partial z > 0$ for $A \leq z \leq B$, and $\beta \geq 2$, then z^* is the only z in the region [A,B] that satisfies

$$\frac{\partial \Pi^M(z;p(z))}{\partial z} = 0.$$

Proof. The proof is similar to that of Theorem 2. From the chain rule and equations (2.23) - (2.25) we get

$$\frac{\partial \Pi^M(z;p(z))}{\partial z} = d^M(p(z))[1 - F(z)]\left[p(z) + s - v - \frac{c - v}{1 - F(z)}\right].$$

The rest of the proof is conducted in Petruzzi and Dada (1999). ❏

Condition $\beta \geq 2$ implies that the product has to be elastic enough.

2.1.3 Comparison and unified framework

It is essential to see the difference in how pricing decision contributes to demand uncertainty in the additive and multiplicative demand case. Let the random variable $\omega(p, \epsilon)$ stands for the random demand in both the additive demand case and the multiplicative demand case. And let $\omega(p, \epsilon)$ has the first and the second order moments. Then the unified expected demand is

$$\mathsf{E}[\omega(p, \epsilon)] = \begin{cases} d^A(p) + \mu & \text{for the additive demand case,} \\ d^M(p)\mu & \text{for the multiplicative demand case,} \end{cases}$$

and the unified variance of the demand satisfies

$$\text{var}[\omega(p, \epsilon)] = \begin{cases} \sigma^2 & \text{for the additive demand case,} \\ d^M(p)^2\sigma^2 & \text{for the multiplicative demand case.} \end{cases}$$

Hence, the variance of the demand is independent of price in the additive demand case. However, the demand variance is a decreasing function of price in the multiplicative demand case due to the shape of $d^M(p)$ (2.3). And

$$\frac{\sqrt{\text{var}[\omega(p, \epsilon)]}}{\mathsf{E}[\omega(p, \epsilon)]},$$

the demand coefficient of variation, is an increasing function of price in the additive demand case and is independent of price in the multiplicative demand case.

In case we assume a deterministic setting, thus $\epsilon = \mu$, the optimal decision on price is to choose the riskless price as there is no risk of understocking or overstocking. In case we assume uncertainty, there is the risk of understocking and overstocking. Luckily, price can be used to reduce this risk. In ideal scenario, price can be used to reduce both the variance and the coefficient of variation, as they are two common measures of uncertainty. But that is not possible in either the additive or the multiplicative case. Nevertheless, it is possible to decrease the demand coefficient of variation without affecting the variance by lowering the price in the additive demand case, and to decrease the demand variance without affecting the coefficient of variation by the price raise in the multiplicative demand case. Therefore it is obvious that $p^* \leq p_\Psi^A$ in the additive demand case and $p^* \geq p_\Psi^M$ in the multiplicative demand case. Thus the pricing strategy differs according to how the randomness is included in the demand.

Let us consider that we do not know anything about the form of demand uncertainty. We try to develop an unified framework for joint stocking and pricing problem given both the additive and the multiplicative demand case. First, we provide a managerial interpretation for z and, then, we define a pricing benchmark that we refer to as the base price.

Despite the fact that z is defined differently in both the additive and the multiplicative demand case, its meaning is similar for both. As stated in Petruzzi and Dada (1999), it is a stocking factor that we define as a surrogate for safety factor (SF).

Definition 1 (safety factor, Silver and Peterson (1985))**.** *Safety factor, SF, is the number of standard deviations that stocking quantity deviates from the expected demand:*

$$SF = \frac{x - \mathsf{E}[\omega(p, \epsilon)]}{sd[\omega(p, \epsilon)]}, \tag{2.27}$$

where $sd[\omega(p, \epsilon)] = \sqrt{var[\omega(p, \epsilon)]}$.

This definition provides the basis for the following theorem.

Theorem 5 (Petruzzi and Dada (1999))**.** *For both the additive and the multiplicative demand cases, the variable z represents the stocking factor, defined as follows:*

$$z = \mu + SF\sigma.$$

Since z represents the stocking factor, our problem can be then transformed into an equivalent optimization problem, where we have to choose the optimal pricing policy and the stocking factor, regardless to the problem being formulated as the additive demand or the multiplicative demand case. This is important because substituting z for x provides analytical tractability.

Base price is developed as it is a convenient tool in case the optimal pricing strategy depends on the choice between the additive and the multiplicative model. From equations (2.11)-(2.13) and (2.23)-(2.25), the expected profit for the NPP can be expressed as

$$\begin{aligned} \Pi(z; p) = (p - c)\, \mathsf{E}_\epsilon[\omega(p, \epsilon)] \\ - \Big((c - v)\, \mathsf{E}_\epsilon[\text{Left}(z, p, \epsilon)] + (p - c + s)\, \mathsf{E}_\epsilon[\text{Short}(z, p, \epsilon)]\Big), \end{aligned} \tag{2.28}$$

where

$$\mathsf{E}_\epsilon[\text{Left}(z,p,\epsilon)] = \begin{cases} \Lambda(z) & \text{for the additive demand case,} \\ d^M(p)\Lambda(z) & \text{for the multiplicative demand case,} \end{cases}$$

are the unified expected leftovers and

$$\mathsf{E}_\epsilon[\text{Short}(z,p,\epsilon)] = \begin{cases} \Theta(z) & \text{for the additive demand case,} \\ d^M(p)\Theta(z) & \text{for the multiplicative demand case.} \end{cases}$$

are the unified expected shortages. We interpret equation (2.28) as a difference between the riskless profit and the loss function. This expression is convenient for further analysis. When we fix p, then the task to find the order quantity that maximizes the expected profit is equivalent to the task to find the order quantity that minimizes the loss function. However, we will apply the identity

$$\mathsf{E}[\text{Sales}] = \mathsf{E}[\text{Demand}] - \mathsf{E}[\text{Shortages}] = \mathsf{E}_\epsilon[\omega(p,\epsilon)] - \mathsf{E}_\epsilon[\text{Short}(z,p,\epsilon)],$$

because the price is not fixed in the NPP. The total expected profit can be then expressed as

$$\begin{aligned} \Pi(z;p) = (p-c)\,\mathsf{E}_\epsilon[\text{Sales}(z,p,\epsilon)] \\ - \Big((c-v)\,\mathsf{E}_\epsilon[\text{Left}(z,p,\epsilon)] + s\,\mathsf{E}_\epsilon[\text{Short}(z,p,\epsilon)]\Big). \end{aligned} \tag{2.29}$$

We interpret (2.29) similarly to (2.28): the total expected profit is the difference between the total expected sales profit and the total expected loss caused by inevitable occurrence of either shortages or leftovers. This allows us to define the base price.

Definition 2 (base price, Petruzzi and Dada (1999))**.** *For a given value of z, we define the base price, $p_B(z)$, as the price that maximizes the function $J(z,p) = (p-c)\,\mathsf{E}_\epsilon[Sales(z,p,\epsilon)]$, which represents the expected sales contribution.*

Lemma 6 (Petruzzi and Dada (1999))**.** *For both the additive and the multiplicative demand cases, $p_B(z)$ is the unique value of p, given z, that satisfies*

$$p = c + \left(- \frac{\mathsf{E}_\epsilon[Sales(z,p,\epsilon)]}{\partial\,\mathsf{E}_\epsilon[Sales(z,p,\epsilon)]/\partial p} \right).$$

Proof. Petruzzi and Dada (1999).

❑

Lemmas 1 and 6 imply that for a given z in the additive demand case the following holds $p_\Psi^A \geq p^* = p_B(z)$. On the other hand, Lemmas 3 and 6 imply that for a given z in the multiplicative demand case the following holds $p^* \geq p_\Psi^M = p_B(z)$. Hence for both the additive and the multiplicative cases, $p^* \geq p_B(z)$. Thus, given z, p^* can be interpreted as the sum of the base price and a premium, where amount of the premium depends on the risk of overstocking and understocking. And since $p^* = p_B(z)$ in the additive demand case, the premium associated with the optimal selling price is equal to zero for this case. Thus it is consistent with the fact that the demand variance is independent of price in the additive demand case. In the multiplicative demand case, the premium can be expressed from Lemma 3 as

$$\text{Premium} = p^* - p_B(z) = p^* - p_\Psi^M = \frac{\beta}{\beta - 1}\left[\frac{(c-v)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right],$$

and, if we multiply both the numerator and the denominator by $d^M(p)$, we obtain

$$\text{Premium} = \frac{\beta}{\beta - 1}\left[\frac{(c-v)\,\mathsf{E}_\epsilon[\text{Left}(z,p,\epsilon)] + s\,\mathsf{E}_\epsilon[\text{Short}(z,p,\epsilon)]}{\mathsf{E}_\epsilon[\text{Sales}(z,p,\epsilon)]}\right].$$

The premium in selling price is based on a formula where the sum of the total expected leftover cost ($(c-v)\,\mathsf{E}_\epsilon[\text{Left}(z,p,\epsilon)]$) and the total expected shortage cost ($s\,\mathsf{E}_\epsilon[\text{Short}(z,p,\epsilon)]$) is spread over the total expected sales and multiplied by the weighting $\frac{\beta}{\beta-1}$ that depends on the form of the demand function and is related to its price elasticity.

Under reasonable technical assumptions such as the market being large enough, when this concept is developed as a multiple period problem or we let the price change by the end of the selling period, the temporary sales prices can be used to either boost sales or sell out stocks. One example could be a bakery that places discount prices on all products an hour prior to closing in order to reduce leftovers as much as possible. Another example could be a newsvendor who sells weekly magazines and on the last day he decides to reduce the price in order to empty his stocks.

2.2 Model with advertising

The effect exhibited by advertising on the sales represents an important aspect of demand-based problems. When a newsvendor faces a demand of the stochastic advertising-sensitive type, he is forced to make a decision concerning advertising and inventory prior to the demand being met. Furthermore, the advertising policy is a crucial aspect of marketing and many businesses try to find the optimal policy where the effort put into advertising is most effective. This effort is usually represented money-wise as various marketing channels have different advertising response functions. The advertising response function is a function that describes the relation between the effort put into advertising and the relative increase in demand. A major issue is how to measure effectiveness of advertising. In other words, which response function shall be chosen. To capture a real situation/dependency between the advertising expenditure and the demand, Hrabec et al. (2016) suggests three particular response functions, that are convenient thanks to their illustrative-suitable behaviour: a concave function without threshold, a concave function with threshold and a S-shaped function with threshold. This newsvendor problem modification is usually referred as the newsvendor problem with advertising (NPA).

We consider two cases how the response function incorporate the demand, the additive and the multiplicative case, similarly to the price-dependent demand in the NPP (section 2.1). Again, we assume that the advertising-related randomness is independent of the demand.

Advertising response function

The response function describes the sales effect of costs spent on advertising. Let the response function $d(a)$ be continuous, positive, twice differentiable and increasing on its domain $[0, a_{max}]$ in the advertising expenditure (Khouja, 1999). Positiveness is required due to the upcoming substitution where the response function is a denominator.

The advertising function without threshold in demand is a function with diminishing returns, that is given by

$$d_1(a) = d_0 + \beta a^{\alpha}, \tag{2.30}$$

where $\alpha \in [0, 1]$ and $\beta > 0$ are empirically determined constants indicating the effectiveness of advertising and $d_0 > 0$ represents the initial demand for $a = 0$. Note that if we allow $\beta = 0$ then the demand is independent of the advertising expenditure. Since α is in the exponent, the larger the value of α is, the more effective advertising is. The response function (2.30) represents an idea that from the beginning every amount spent on advertising has enormous impact on the increase in sales and this impact decreases with more money spent. However, this function does not have an upper bound. Therefore every increase in advertising expenditure has perceptible increase in sales.

The advertising function with threshold in demand is an asymptotic function with an upper bound, that is given by

$$d_2(a) = d_0 + \theta \left[1 - \frac{1}{(a+1)^{\delta}} \right], \tag{2.31}$$

where θ and δ are positive real numbers and d_0 is an initial demand. The upper bound (horizontal asymptote) is defined by value $d_0 + \theta$ since $1 - \frac{1}{(a+1)^\delta}$ is always less than one. The larger the value of δ is, the faster function speeds towards the upper bound. The response function (2.31) is similar to (2.30) with only one difference. It is that the advertising amount once reaches a point where the relative sales increase is almost monotonous with more effort put into advertising. This means that the advertising do not have almost any impact on buyers anymore.

The S-shaped function with threshold in demand is a logistic function that is represented by a typical "S" shape graph. The function is given by

$$d_3(a) = d_0 + \frac{\theta}{1 + (\frac{\theta - \theta_l}{\theta_l})e^{-\gamma a}}, \tag{2.32}$$

where θ specifies an upper asymptote, γ is a coefficient of growth and θ_l defines a lower asymptote. The S-function is a bounded real function with a positive derivative at each point which is first convex and then concave. It means that in the beginning the sales do not respond to the advertising because the advertising budget is too low. It supposedly takes time for the advertising to wear in. After the advertising budget exceeds some minimum critical-level threshold, sales start to respond to the increased advertising. Eventually, the curve begins to slope downward again, when the diminishing returns phase appears.

Figure 2.2 illustrates three particular functions (2.30)-(2.32). In this section all three response functions are considered in both the additive and the multiplicative demand case. Hence we refer to the advertising response function as $d(a)$ without specifying the exact shape.

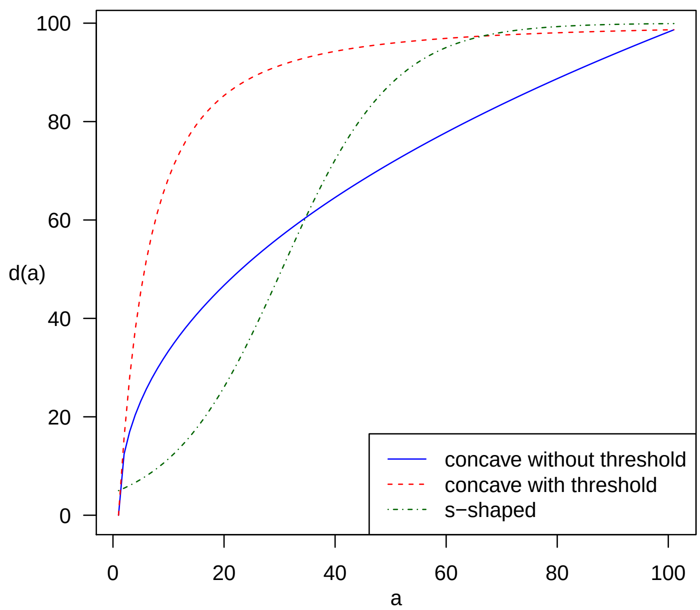

Figure 2.2: Shape of the considered response functions in the newsvendor problem with advertising given by (2.30)-(2.32).

2.2.1 Additive demand case

Similarly to the price-dependent demand case (2.2), the demand function is defined as $\omega^A(a,\epsilon) = d(a) + \epsilon$, where ϵ is a continuous random variable. Let ϵ be defined on the domain $[A, B]$. Again, we could require $\mathsf{E}\,\epsilon = 0$ alike in Hrabec et al. (2016). But we do not need that because results can be evaluated in general form. Moreover, suppose that $A > -d(a)$ so that we guarantee nonnegative demand at all times.

The problem description and assumptions on parameters are similar to the NP (section 1). Only difference is that we have to include the advertising expenditure a into the overall profit function (1.1). Thus, if we substitute $\omega = \omega^A(a,\epsilon)$ and add the advertising amount cost a to (1.1), we obtain the utility function

$$\pi^A(x;a;\omega) = \begin{cases} p\omega^A(a,\epsilon) - cx + v[x - \omega^A(a,\epsilon)] - a & \text{for} \quad \omega^A \le x, \\ px - cx - s[\omega^A(a,\epsilon) - x] - a & \text{for} \quad \omega^A > x. \end{cases} \tag{2.33}$$

The decision variables are the order quantity x and the amount spent on advertising a.

We perform similar transformation as in section 2.1.1 (the additive NPP). Let us denote $z = x - d(a)$, where $z \in \mathbb{R}$ is the stocking factor, and we get from (2.33) that

$$\pi^A(z;a;\epsilon) = \begin{cases} p[d(a)+\epsilon] - c[d(a)+z] + v[z-\epsilon] - a & \text{for} \quad \epsilon \le z, \\ p[d(a)+z] - c[d(a)+z] - s[\epsilon - z] - a & \text{for} \quad \epsilon > z. \end{cases}$$

Interpretation of z is the same as in section 2.1.1. Since there are many similarities with the price-dependent demand case, we use the same notation.

The expected profit can be expressed by

$$\begin{aligned} \Pi^A(z;a) = \mathsf{E}_\epsilon[\pi^A(z;a;\epsilon)] =& \int_A^z (p[d(a)+\epsilon] + v[z-\epsilon]) \,\mathrm{d}F(\epsilon) \\ &+ \int_z^B (p[d(a)+z] - s[\epsilon - z]) \,\mathrm{d}F(\epsilon) \\ &- c[d(a)+z] - a \\ =& \Psi^A(a) - L^A(z), \end{aligned} \tag{2.34}$$

where

$$\Psi^A(a) = (p-c)[d(a)+\mu] - a \tag{2.35}$$

is the riskless profit and

$$L^A(z) = (c-v)\Lambda(z) + (p+s-c)\Theta(z) \tag{2.36}$$

is the expected loss. Series of equations (2.34) is evaluated similarly as in the NPP case (see (2.8)).

We can see that the decision on a and z are made independently from (2.34). Therefore, we can handle the problem of finding the optimal decision independently. Thus the optimal amount of advertising a^* must satisfy the necessary optimal condition (first order condition for the riskless profit $\Psi^A(a)$), that is given by

$$\frac{\partial d(a)}{\partial a} = \frac{1}{p-c}. \tag{2.37}$$

The exact formulas of the optimal advertising expenditure a^* for considered response functions will be provided at the end of the next section (section 2.2.2). The optimal stocking quantity z^* is determined by solving the first order condition since we know that $L^A(z)$ given by (2.36) is concave (follows from (2.16) because the result of the second order derivative of $L^A(z)$ is identical). The optimal z can be then expressed by

$$F(z^*) = \frac{p+s-c}{p+s-v}. \tag{2.38}$$

2.2.2 Multiplicative demand case

For the multiplicative demand case, the demand function $\omega^M(a,\epsilon)$ is defined as $\omega^M(a,\epsilon) = d(a)\epsilon$. In order to assure that the demand is positive, we require that $A > 0$. The utility function is (2.33) and by substituting $z = \frac{x}{d(a)}$, where $z \geq 0$ is the stocking factor, we obtain

$$\pi^M(z;a;\epsilon) = \begin{cases} pd(a)\epsilon - cd(a)z + vd(a)[z-\epsilon] - a & \text{for} \quad \epsilon \leq z, \\ pd(a)z - cd(a)z - sd(a)[\epsilon - z] - a & \text{for} \quad \epsilon > z. \end{cases}$$

Analogically to the additive demand case, the expected profit is

$$\Pi^M(z;a) = \mathsf{E}_\epsilon[\pi^M(z;a;\epsilon)] = \Psi^M(a) + L^M(a,z), \tag{2.39}$$

where

$$\Psi^M(a) = (p-c)d(a)\mu - a \tag{2.40}$$

is the riskless profit and

$$L^M(a,z) = d(a)[(c-v)\Lambda(z) + (p+s-c)\Theta(z)] = d(a)L^A(z) \tag{2.41}$$

is the expected loss.

In order to find the optimal solution with respect to x and a we perform two consequent steps. Thanks to the form of the expected loss function (2.41) we can obtain the optimal stocking factor z^* independently on the optimal value of a. After that we can determine the optimal value of a using a suitable response function $d(a)$. After that we can determine the optimal ordering quantity x^* by substituting back to $x^* = d(a^*)z^*$.

Analogically to the finding of the optimal stocking factor z^* in the additive demand case, we obtain that the optimal z satisfies (2.38) ($\Pi^M(z;a)$ is concave in z). Moreover, when F is invertible, we can express the optimal and unique z^* as

$$z^* = F^{-1}\left(\frac{p+s-c}{p+s-v}\right), \tag{2.42}$$

which corresponds to the standard NP result (see (1.7)) and the optimal z^* derived for the additive demand case (see (2.38)).

Notice that (2.39) can be rewritten as

$$\Pi^M(z;a) = \Psi^M(a) + d(a)L^A(z) = d(a)[(p-c)\mu - L^A(z)] - a, \tag{2.43}$$

where $L^A(z)$ is given by (2.36). And by substituting (2.42) into (2.43) we get the following expression:

$$\Pi^M(z^*;a) = d(a)[(p-c)\mu - L^A(z^*)] - a. \tag{2.44}$$

Notice in (2.44) that, since we assume $d(a) > 0$, it could happen for $(p-c)\mu - L^A(z^*) < 0$ that the expected profit $\Pi^M(z^*;a)$ is negative and strictly decreasing in a. Similarly when $(p-c)\mu - L^A(z^*) = 0$. But that does not capture any real situation because the expected per-unit profit would be negative. In that case, the only good strategy is to do nothing and buy zero items ($x = 0$, $a = 0$ and thus $z = 0$). Therefore we introduce another assumption that verifies, that the expected per-unit profit is greater than zero. Thus we assume that $(p-c)\mu - L^A(z^*) > 0$. This assumption could be easily violated mainly due to the shape of distribution F, i.e. high variance causes large expected loss as there would be high expected shortages and leftovers that cannot be covered by expected gain. However, all parameters could possibly cause violation of this assumption too, e.g. small per-unit profit margin $p - c$.

Note (Assumption 1). Denote the inequality $(p-c)\mu - L^A(z^*) > 0$ as Assumption 1.

Now solving the first order condition of $\Pi^M(z^*;a)$ (2.44) with respect to a determines the optimal advertising expenditure a^*. Hence a^* must satisfy the necessary optimality condition

$$\frac{\partial d(a)}{\partial a} = \frac{1}{(p-c)\mu - L^A(z^*)}. \tag{2.45}$$

Optimal advertising for given response functions

We can now express the exact form of the optimal advertising expenditure a^* with respect to the advertising response function $d(a)$ used. The following expressions are actual for the multiplicative demand case. While to get expressions for the additive demand case, substitute $\mu = 1$ and $L^A(z^*) = 0$ (see/compare (2.45) and (2.37)).

For the advertising response function without threshold in demand $d_1(a)$ given by (2.30) we get that the optimal a is

$$a^* = \sqrt[1-\alpha]{\alpha\beta[(p-c)\mu - L^A(z^*)]}. \tag{2.46}$$

For the advertising response function with threshold in demand $d_2(a)$ given by (2.31) we get

$$a^* = \sqrt[\delta+1]{\theta\delta[(p-c)\mu - L^A(z^*)]} - 1.$$

And the advertising optimal for the S-shaped response function $d_3(a)$ given by (2.32) is expressed by

$$a^* = -\frac{\ln\left(\frac{\left\{1-\frac{1}{2}[(p-c)\mu-L^A(z^*)]\theta\gamma+\frac{1}{2}\sqrt{-4[(p-c)\mu-L^A(z^*)]\theta\gamma+[(p-c)\mu-L^A(z^*)]^2\theta^2\gamma^2}\right\}\theta_l}{\theta_l-\theta}\right)}{\gamma}$$

2.2.3 Monotonicity

Let us investigate some properties of the expected profit $\Pi^M(z^*;a)$ (2.44). If we compute the second order derivative of the expected profit given by (2.44), we get

$$\frac{\partial^2 \Pi^M(z^*;a)}{\partial a^2} = \frac{\partial^2 d(a)}{\partial a^2}[(p-c)\mu - L^A(z^*)]. \tag{2.47}$$

This is the fundamental for the following proposition.

Proposition 7 (Hrabec et al. (2016))**.** *Under Assumption 1 the intervals where the expected profit* $\Pi^M(z^*;a)$ *is convex or concave with respect to the advertising* a *are equal to the intervals where the advertising response function* $d(a)$ *is convex or concave.*

Proof. The direct implication of (2.47) and Assumption 1.

❑

We need to add another assumption that would together with Assumption 1 help us guarantee that the optimal solution is unique for selected types of demand functions (i.e. concave and S-shaped function). From equation (2.45) and the fact that $d(a)$'s domain is $[0, a_{max}]$ yields that we must assume that $\frac{\partial d(0)^+}{\partial a} > \frac{1}{(p-c)\mu - L^A(z^*)}$ and $\frac{\partial d(a_{max})^-}{\partial a} < \frac{1}{(p-c)\mu - L^A(z^*)}$ to obtain a unique solution. In case $d(a)$ is defined on a greater interval than $[0, a_{max}]$, the assumption can be expressed as $\frac{\partial d(0)}{\partial a} > \frac{1}{(p-c)\mu - L^A(z^*)}$ and $\frac{\partial d(a_{max})}{\partial a} < \frac{1}{(p-c)\mu - L^A(z^*)}$.

Note (Assumption 2)*.* Denote the inequalities $\frac{\partial d(0)^+}{\partial a} > \frac{1}{(p-c)\mu - L^A(z^*)}$ and $\frac{\partial d(a_{max})^-}{\partial a} < \frac{1}{(p-c)\mu - L^A(z^*)}$ as Assumption 2.

We can deduce the following theorem for the strictly concave response function in the multiplicative demand case.

Theorem 8 (Hrabec et al. (2016))**.** *If the response function* $d(a)$ *is strictly concave, then, under Assumptions 1 and 2, the expected profit* $\Pi^M(z^*;a)$ *is strictly concave in* a *and so the globally optimal advertising expenditure* a^* *is unique and is given by solution of* (2.45) *with respect to the decision variable* a*.*

Proof. Since the response function $d(a)$ is considered to be strictly concave in its domain then from Proposition 7 follows that also the expected profit $\Pi^M(z^*;a)$ is strictly concave in a. Moreover, Assumption 2 guarantees that the expected profit $\Pi^M(z^*;a)$ is increasing at its initial point and decreasing at its endpoint. Then, the critical point determined from the optimality condition (2.45) is unique and is the optimal advertising amount a^*.

❑

Similar theorem holds for the additive demand case.

Theorem 9 (Hrabec (2016))**.** *If the response function* $d(a)$ *is strictly concave, the expected profit* $\Pi^A(z^*;a)$ *is strictly concave in* a *and, under Assumption 2, the optimal advertising amount is unique and is given by* (2.37)*.*

Proof. The proof is much the same as the one of Theorem 8. ❑

And for the S-shaped response function we get the following theorem in case of the multiplicative demand case.

Theorem 10 (Hrabec et al. (2016))**.** *If the response function $d(a)$ is S-shaped, then, under Assumptions 1 and 2, the expected profit $\Pi^M(z^*; a)$ is strictly quasi-concave in a and so the globally optimal advertising expenditure is unique and is given by* (2.45)*.*

Proof. Since the response function is S-shaped, we know that the expected profit function $\Pi^M(z^*; a)$ is first convex and then concave in a from Proposition 7. Moreover, Assumption 2 guarantees that the expected profit $\Pi^M(z^*; a)$ is increasing at its initial point and so it increases until it reaches its maximum. In other words, $\Pi^M(z^*; a)$ is strictly quasi-concave in a. Then, from the optimality condition (2.45), we can obtain a single critical point a^*, the optimal advertising amount, which always lies in the concave range. ❑

Again, for the additive demand case, we obtain the similar result.

Theorem 11 (Hrabec (2016))**.** *If the response function $d(a)$ is S-shaped, the expected profit $\Pi^A(z^*; a)$ is strictly quasi-concave in a and, under Assumption 2, the optimal advertising amount is unique and is given by* (2.37)*.*

Proof. The proof is much the same as the one of Theorem 10. ❑

Since we derived the unique optimal solution for $\Pi^M(z; a)$ with respect to both a and z, we can now solve the original problem of maximizing the expected profit given by the objective function (2.33) with respect to the order quantity x. This can be done easily by substituting back with the formula $x^* = \frac{z^*}{d(a^*)}$. The pair $[a^*, x^*]$ then represents the optimal solution of the newsvendor problem with pricing in case of the multiplicative demand.

Similarly, we obtain the optimal ordering quantity in case of the additive demand as $x^* = z^* + d(a^*)$.

Comparison with riskless problem

Let us take a closer look on the riskless problem and compare it with the NPP results. Suppose that the advertising problem does not contain any uncertainty. Problem can be then reduced only to a deterministic problem and its objective function is the riskless profit $\Psi^A(a) = (p-c)d(a) - a$ for the additive demand case and $\Psi^M(a) = (p-c)d(a)\mu - a$ for the multiplicative demand case (see (2.35) and (2.40)). Solving the first order condition of $\Psi^M(a)$, we get the following necessary optimality condition:

$$\frac{\partial d(a)}{\partial a} = \frac{1}{(p-c)\mu}, \tag{2.48}$$

that must be satisfied by the optimal riskless advertising a^M_Ψ in the multiplicative demand case. If the response function $d(a)$ is either concave or S-shaped, then

Proposition 7 can be applied adequately (Assumption 1 holds thanks to absence of uncertainty) and, under Assumption 2, the necessary optimal condition (2.48) is also sufficient for the optimal riskless advertising a_Ψ^M.

The following theorem summarizes relation between the optimal decision of a in absence and presence of uncertainty for the multiplicative demand case.

Theorem 12 (Hrabec et al. (2016))**.** *Based on the optimality condition* (2.48)*, considering concave and S-shaped functions, and under Assumptions 1 and 2, we can derive that for the multiplicative demand model the optimal advertising a^* is always less than or equal to the optimal riskless advertising a_Ψ^M.*

Proof. We know that with presence of uncertainty the function $L^A(z^*)$ given by (2.36) is always positive. Hence if we compare equations (2.45) and (2.48), we obtain inequality $\frac{1}{(p-c)\mu} \leq \frac{1}{(p-c)\mu - L^A(z^*)}$ which is equivalent with $\frac{\partial d(a_\Psi^M)}{\partial a} \leq \frac{\partial d(a^*)}{\partial a}$. For the response functions considered, concave and S-shaped, the optimal advertising a^*, if it exists and is greater than zero, belongs to the concave part of $d(a)$. Then, for the concave part of $d(a)$, we know that the derivative of $d(a)$ is a decreasing function of a. So from $\frac{\partial d(a_\Psi^M)}{\partial a} \leq \frac{\partial d(a^*)}{\partial a}$ follows that $a_\Psi^M \geq a^*$. ❑

Recall that for the NPP with the multiplicative demand case the optimal price is not less than the riskless price. Despite the expected profit functions of the NPA and the NPP are similar (see (2.23) and (2.43)), the demand response functions are defined differently: $d(p)$ is decreasing in p whereas $d(a)$ is increasing in a. Therefore, there is no surprise that we derived the opposite effect of uncertainty in the NPA given by Theorem 12 and in the NPP case (see Lemma 3 from which follows $p^* \geq p_\Psi^M$).

In the additive demand case the decisions on a and z are made independently from equation (2.34) unlike in the multiplicative demand case (see (2.43)). Therefore, the optimal advertising a^* is always equal to the riskless optimal advertising a_Ψ^A, which is obtained by taking $\frac{\partial \Psi^A(a)}{\partial a} = 0$ that leads to

$$\frac{\partial d(a)}{\partial a} = \frac{1}{(p-c)}.$$

However, this result arises doubts. Can indeed be the optimal advertising a^* resistant to the demand uncertainty? An analysis that answers this question might help the interested manager to choose a suitable model.

Similarly to the NPP (section 2.1.3), it is essential to see the difference in how advertising contributes to the demand uncertainty, precisely the variance and the coefficient of variation, in the additive and the multiplicative demand case. While in the additive case the variance of the demand is constant, i.e. $\text{var}[\omega^A(a,\epsilon)] = \sigma^2$, in the multiplicative case the variance is a function of the response function, i.e. $\text{var}[\omega^M(a,\epsilon)] = d(a)\sigma^2$. Thus in the multiplicative demand case the variance is increasing function of advertising due to the shape of $d(a)$. The demand coefficient of variation is independent of advertising in the multiplicative case, i.e. $\frac{\sqrt{\text{var}[\omega^M(a,\epsilon)]}}{\mathsf{E}[\omega^M(a,\epsilon)]} = \frac{\sigma}{\mu}$, thus constant. On the other hand, in the additive case the demand coefficient of variation is a decreasing function of the advertising amount.

Hence an enlargement of advertising amount a in the additive demand case helps to reduce risk because the variance is kept constant and the coefficient of variation decreases. On the other hand, an increment in advertising amount a for the multiplicative demand case causes that the coefficient of variation is kept constant but the variance increases. Therefore, it is sensible to keep the advertising expenses low in the multiplicative demand case of the NPA if we want to avoid leftovers or shortages.

Example. Let the random variable ϵ has continuous uniform distribution, i.e. $\epsilon \sim U(A, B)$, and consider the multiplicative demand case. Then, from (2.42), we obtain the optimal $z^* = A + \frac{p+s-c}{p+s-v}(B - A)$. If we want to derive Assumption 1, that states $(p - c)\mu - L^A(z^*) > 0$, we must substitute z^* into $L^A(z)$ (2.36). Then $L^A(z^*) = (z^* - A)\frac{c-v}{2} = \frac{p+s-c}{p+s-v}(c - v)\frac{B-A}{2}$. Hence we get that Assumption 1 must satisfy

$$\frac{B - A}{2}\left(p - c - (c - v)\frac{p + s - c}{p + s - v}\right) > 0.$$

Assumption 1 then crucially depends on all defined parameters apart from the advertising expenditure a.

2.3 Model with joint pricing and advertising

The newsvendor problem with joint pricing and advertising (NPPA) combines all of the operational and marketing strategies introduced in sections 1, 2.1 and 2.2. Therefore, the NPPA is a problem with three decision variables: ordering (stocking), pricing, advertising, and one random variable influencing the demand.

Setting where a firm wants to optimize both pricing and advertising is most common in business. Here, a decision-maker may in most cases adjust the selling price in order to lower or elevate the demand and he has also the benefit of influencing the final demand by choosing and investing into the right marketing activities, e.g. promotional displays, advertising, and other demand-enhancing activities.

This section follows Hrabec (2016) that summarizes two papers on the NPPA: Dai and Meng (2015) and Wang and Hu (2011). The NPPA model is in its formulation similar to the NPA model (section 2.2). The crucial difference is that the newsvendor faces stochastic demand $\omega(a, p, \epsilon)$, where p is the decision variable as well. Thus we define the joint advertising and pricing response function $d(a, p)$ that is assumed to be separable, nonnegative, twice differentiable, strictly concave, and is defined on $[0, \infty) \times [0, \infty)$. Conceivably, $d(a, p)$ is strictly increasing and concave in the advertising amount and is strictly decreasing and convex in the selling price. Thanks to the separability the response function satisfies $d(a, p) = d_1(a)d_2(p)$. The newsvendor simultaneously decides on: the advertising amount a, the selling price p, and the amount x to be stocked and sold. Replacing $\omega^A(a, \epsilon)$ with $\omega(a, p, \epsilon)$ in the NPA model utility function (2.33), the NPPA model utility function is expressed as

$$\pi(x; a; p; \omega) = \begin{cases} p\omega(a, p, \epsilon) - cx + v[x - \omega(a, p, \epsilon)] - a & \text{for} \quad \omega \le x, \\ px - cx - s[\omega(a, p, \epsilon) - x] - a & \text{for} \quad \omega > x. \end{cases} \tag{2.49}$$

In the paper by Dai and Meng (2015) two demand cases are investigated: the marketing-dependent price-multiplicative case (MDPM) and the marketing-dependent price-additive case (MDPA). Let the demand function for MDPM case is denoted as $\omega^M(a, p, \epsilon)$ and let it satisfy

$$\omega^M(a, p, \epsilon) = d_1(a)d_2(p)\epsilon. \tag{2.50}$$

The demand function for MDPA case is denoted as $\omega^A(a, p, \epsilon)$ and is given by $\omega^A(a, p, \epsilon) = d_1(a)[d_2(p) + \epsilon]$. Moreover, let the random variable ϵ be defined on the domain $[A, B]$ and satisfy $A > 0$ for MDPM and $A > -d_2(p)$ for MDPA.

In this thesis, the MDPM case is further investigated.

2.3.1 Marketing-dependent price-multiplicative demand model

Let $\epsilon \in [A, B]$, where $A > 0$, and let $d(a, p)$ denote the general response function that could be expressed as $d_1(a)d_2(p)$. Then, the demand is in the MDPM form like (2.50). The utility function (2.49) may be rewritten by substituting (2.50) into $\omega(a, p, \epsilon)$ and defining the stocking factor as $z = \frac{x}{d(a,p)}$:

$$\pi(z; a; p; \epsilon) = \begin{cases} pd(a, p)\epsilon - czd(a, p) + vd(a, p)[z - \epsilon)] - a & \text{for} \quad \epsilon \le z, \\ pzd(a, p) - czd(a, p) - s[\epsilon - z] - a & \text{for} \quad \epsilon > z. \end{cases} \tag{2.51}$$

The objective is to maximize the expected profit. Let us denote $\Pi(a,p,z) = \mathsf{E}_\epsilon[\pi(z;a;p;\epsilon)]$. Then the expected profit is expressed as

$$\Pi(a,p,z) = \Psi(a,p) - L(a,p,z) = \Psi(a,p) - d(a,p)l(p,z), \tag{2.52}$$

where $\Psi(a,p) = (p-c)d(a,p)\mu - a$ is the riskless profit and $l(p,z) = (c-v)\Lambda(z) + (p+s-c)\Theta(z)$. Recall that $d(a,p)\Lambda(z)$ are the expected leftovers and $d(a,p)\Theta(z)$ are the expected shortages.

Optimality

To maximize the expected profit $\Pi(a,p,z)$ (2.52) we use the following sequence of steps: first the optimal stocking quantity is evaluated, after that the optimal pricing policy and finally the optimal advertising expenditure is found. Taking $\frac{\partial \Pi(a,p,z)}{\partial z} = 0$, it leads to the optimal stocking quantity z^*, if F is invertible, that is expressed as

$$z^* = F^{-1}\left(\frac{p+s-c}{p+s-v}\right).$$

Notice that we derived once again the solution that corresponds to the standard NP optimal quantity (1.7) and to the optimal stocking factor of the NPP (2.19) as well as of the NPA (2.42).

The next optimal value of a decision variable to be found is price p. If we set the derivative of the expected profit w.r.t. p to zero, i.e. $\frac{\partial \Pi(a,p,z)}{\partial p} = 0$, it can be observed that $\frac{\partial d(a,p)}{\partial p}[(p-c)\mu - l(z,p)] + d(a,p)[\mu - \Theta(z)] = 0$. Substitute $d(a,p) = d_1(a)d_2(p)$ into the derived formula and we obtain

$$\frac{\partial d_2(p)}{\partial p}[(p-c)\mu - l(z,p)] + d_2(p)[\mu - \Theta(z)] = 0. \tag{2.53}$$

Notice that the necessary optimal pricing condition of price p does not contain advertising amount a and is identical to the optimal pricing condition in the NPA (2.26). In order to find the unique optimal price p^*, the following definitions are needed (see Yao et al. (2006)).

Definition 3 (GSIFR, IFR)**.** *We say that the distribution $F(\cdot)$ has generalized strict increasing failure rate (GSIFR) if*

$$g'(\epsilon) > 0 \quad \forall \epsilon,$$

where $g(\epsilon) = \epsilon \cdot r(\epsilon)$ is generalized failure rate function and $r(\epsilon) = \frac{f(\epsilon)}{1-F(\epsilon)}$ is hazard rate (or so called failure rate function) used in Theorems 2 and 4.

Moreover, we say that the distribution $F(\cdot)$ has increasing failure rate (IFR) if

$$r'(\epsilon) \geq 0 \quad \forall \epsilon.$$

GSIFR class of distributions include PF_2 distributions as well as log-normal distribution with parameter restrictions. GSIFR is generalization of IFR and thus include more distribution used in literature. IFR does not include all gamma and Weibull distributions that are present in GSIFR. For more discussion on GSIFR and IFR, see Yao et al. (2006), Barlow and Proschan (1987) or Chen et al. (2007).

Definition 4 (IPE)**.** *We say that the expectation of ω, i.e. $\mathsf{E}_\epsilon[\omega(p,\epsilon)] = y(p)$, belongs to a class of demand functions with an increasing pricing elasticity (IPE) if*

$$\frac{\partial e}{\partial p} \geq 0,$$

where $e = -\frac{py'(p)}{y(p)}$ denotes the price elasticity of $y(p)$.

The price elasticity gives the percentage change in demand in response to a one percent change in price. In our case $\mathsf{E}_\epsilon[\omega(p,\epsilon)] = d(p)\mu$. Thus if $d(p)$ has IPE, then $\mathsf{E}_\epsilon[\omega(p,\epsilon)]$ has IPE as well because $e = -\frac{pd'(p)}{d(p)}$ for both. For more details on IPE, see Yao et al. (2006).

Theorem 13 (Hrabec (2016))**.** *If $d_2(p)$ has IPE and the cdf $F(\cdot)$ has GSIFR, then $\Pi(a,p,z)$ is quasi-concave in p, and optimal price of the MDPM model is unique and is always equal to that of the multiplicative form in the NPP.*

Proof. Consider the NPP model with the multiplicative demand form (section 2.1.2). As stated in Hrabec (2016), it can be shown that if the mean demand $\mathsf{E}_\epsilon[\omega^M(p,\epsilon)]$ has IPE and the distribution F has GSIFR, then Π is quasi-concave in p in the range $[p_l, p_u]$ and thus the first order condition $\frac{\partial \Pi^M(z(p),p)}{\partial p} = 0$ has a unique solution (see Yao et al. (2006)).

Moreover, $\mathsf{E}_\epsilon[\omega^M(p,\epsilon)] = d^M(p)\mu$ and if $d^M(p)$ has IPE, then $\mathsf{E}_\epsilon[\omega^M(p,\epsilon)]$ has IPE.

The proof is then obvious comparing two optimal price conditions, the NPP condition (2.26) and the NPPA condition (2.53).

❑

Example. Let $d_2(p) = d^M(p) = \alpha p^{-\beta}$, which corresponds to the isoelastic pricing function (2.3). Assume that $\alpha > 0$ and $\beta > 1$. Then, after substituting $d_2(p)$ into (2.53), we obtain

$$\frac{\partial \Pi(a,p,z)}{\partial p} = (\beta - 1)\frac{d_2(p)}{p}[\mu - \Theta(z)]\left\{p_\Psi^M + \frac{\beta}{\beta - 1}\left[\frac{(c-v)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right] - p\right\},$$

where p_Ψ^M is the optimal riskless profit. Then, the optimal price expression is equal to the optimal price of the NPP (see Lemma 3) and satisfies

$$p^* = p_\Psi^M + \frac{\beta}{\beta - 1}\left[\frac{(c-v)\Lambda(z) + s\Theta(z)}{\mu - \Theta(z)}\right].$$

If we set $\frac{\partial \Pi(a,p^*,z^*)}{\partial a} = 0$ and substitute $d(a,p^*) = d_1(a)d_2(p^*)$ we get the necessary optimal condition for advertising a as follows:

$$\frac{\partial d_1(a)}{\partial a} = \frac{1}{d_2(p^*)[(p^* - c)\mu - l(z^*,p^*)]}. \tag{2.54}$$

The optimal advertising condition (2.54) depends on the choice of price p as well as on the shape of the function $d_2(p)$. Moreover, with increasing p (or p^*, respectively) the value of $d_2(p)$ decreases. Therefore, the optimal advertising for the NPPA model depends both on $d_1(a)$ and $d_2(p)$.

Example. Consider $d_1(a) = d_0 + \beta a^\alpha$, which corresponds to the concave advertising response function without threshold (2.30). Then the optimal advertising a^* can be expressed by substituting $d_1(a)$ into (2.54):

$$a^* = \sqrt[1-\alpha]{\alpha\beta d_2(p^*)[(p^* - c)\mu - l(z^*, p^*)]},$$

which is equal to the NPA result (2.46) if we multiply (2.46) by $\sqrt[1-\alpha]{d_2(p^*)}$.

Remember that the optimal stocking quantity x^* is determined as $x^* = z^* d(a^*, p^*)$.

Comparison with riskless problem

Let Ψ be the the riskless problem of the MDPM for which $\Psi(a, p) = (p - c)d_1(a)d_2(p)\mu - a$.

Then taking $\frac{\partial\Psi(a,p)}{\partial p} = 0$ gives the optimality condition for the riskless price p_Ψ that is

$$d_2(p) + (p - c)\frac{\partial d_2(p)}{\partial p} = 0. \tag{2.55}$$

Example. If we set $d_2(p)$ to be the isoelastic function, i.e. $d_2(p) = \alpha p^{-\beta}$, then the optimal riskless pricing is $p_\Psi = \frac{\beta c}{\beta - 1}$ which is the optimal riskless pricing p_Ψ^M in the multiplicative NPP, see Lemma 3.

Then we can derive an equivalent relation between the optimal pricing in absence and presence of uncertainty as in the NPP. See the first paragraph after Lemma 3 for comparison.

Theorem 14 (Hrabec (2016))**.** *For the isoelastic response function is the optimal riskless pricing p_Ψ in the MDPM model, given by condition* (2.55)*, always greater or equal to the optimal pricing p^*.*

Proof. We found the same property in section 2.1.2 (between Lemma 3 and Theorem 4). The proof is similar.

❑

And taking $\frac{\partial\Psi(a,p)}{\partial a} = 0$ gives the optimality condition for the riskless advertising a_Ψ that is

$$\frac{\partial d_1(a)}{\partial a} = \frac{1}{d_2(p)(p - c)\mu}. \tag{2.56}$$

Then we can derive an equivalent relation between the optimal advertising in absence and presence of uncertainty as in the NPA if we adjust appropriately assumptions for the NPPA.

Note. (Assumption 2.2) Assumption 2 needs adjustment, compare Assumption 2 and (2.54). Then Assumption 2.2 is $\frac{\partial d_1(0)^+}{\partial a} > \frac{1}{d_2(p^*)[(p^*-c)\mu - l(z^*,p^*)]}$ and $\frac{\partial d_1(a_{max})^-}{\partial a} < \frac{1}{d_2(p^*)[(p^*-c)\mu - l(z^*,p^*)]}$, where $a \in [0, a_{max}]$.

If the response function $d_1(a)$ is either concave or S-shaped and Assumption 2.2 holds, the necessary optimal condition given by (2.56) is also sufficient optimal condition for the riskless optimal advertising a_Ψ. Notice that Assumption 1 is always satisfied because $d_2(p^*)$ is always positive in (2.54).

Theorem 15 (Hrabec (2016))**.** *For the MDPM, based on the optimality condition* (2.56)*, if* $d_1(a)$ *is either concave or S-shaped, then, under Assumptions 1 and 2.2, the optimal advertising* a^* *is always less than or equal to the optimal riskless advertising* a_Ψ*.*

Proof. Analogically to the proof of Theorem 12. ❑

Example. If we consider $d_1(a) = d_0 + \beta a^\alpha$, then the optimal riskless advertising is

$$a_\Psi = \sqrt[1-\alpha]{\alpha\beta d_2(p)(p-c)\mu}.$$

Chapter 3

Multicriteria and multiproduct extensions of the newsvendor model

3.1 Multicriteria extensions

Until now we dealt with the single-criteria optimization problems. It is due to the fact that we maximized the expected profit that was represented by a single objective function. Extending the stochastic problem for multiple criteria mean that we add another objective functions capturing problems that we want to handle. This additional objective functions can be either maximized or minimized. Generally speaking, the multicriteria optimization problem with random element is defined as

$$\begin{aligned} \text{"max"} \quad & [f_1(x,\omega),\dots,f_K(x,\omega)] \\ \text{subject to} \quad & \text{"}g(x,\omega) \geq 0\text{"}, x \in X, \end{aligned}$$

where $X \subset \mathbb{R}^n$ is non-empty set, ω is s-dimensional random vector from given probability space $(\Omega, \mathcal{A}, P)$, $f_k : \mathbb{R}^n \times \Omega \to \mathbb{R}$, $k = 1,\dots,K$ and $g : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$ are given real functions.

We show addition of a single objective function in this paper. It has been pointed out in the literature that maximizing the expected profit is not satisfactory from practical point of view, and managers are usually more concerned with other types of objectives.

For instance, the risk-sensitive newsvendor might be interested in lowering leftovers and shortages that often occur while still maximizing the expected profit. Such a scenario could happen in case the newsvendor's financial resources are not ample enough to survive a streak of several above-average losses. One of the ways how to handle the demand fluctuations is to establish an appropriate risk measure and add it to the problem. Therefore, a condition specifying that the riskiness of the profit does not exceed a predetermined risk level is added to the problem. In this paper, we use the Conditional Value at Risk (CVaR) at level $\eta \in (0,1]$ to quantify the riskiness. CVaR is widely used risk measure that is popular mainly in financial risk management. For the CVaR definition and properties see section 3.1.1 and for the problem formulation see section 3.1.2.

On the other hand, other newsvendors try to reach a predetermined fixed target profit. However, this is still insufficient criterion as it might lead to the significant loss. To reduce such a risk arising from the variation of the profit one could minimize the riskiness of the profit. Such a scenario could happen in case the newsvendor needs to cover on average his daily out-of-business costs at any cost. For the problem formulation see section 3.1.3.

Note. (Jammernegg and Kischka, 2007) The classical newsvendor maximizes its expected profit. Within the expected utility theory, this is equivalent to the assumption of risk-neutral behaviour: the expected profit derived from the optimal order quantity $\mathsf{E}_\omega[\pi(x^*;\omega)]$ is considered indifferent to the random profit $\pi(x^*;\omega)$. At the setting with the CVaR, the newsvendor is considered risk-averse. In the decision theory the risk-aversion is characterized by the fact that the expected value $\mathsf{E}_\omega[\pi(x^*;\omega)]$ is prefered to the random variable $\pi(x^*;\omega)$.

3.1.1 Conditional Value at Risk

We briefly describe the CVaR performance measure and show some most important properties. The CVaR maximizes the average profit of the profit falling below a certain quantile level (or VaR) which is defined as the maximum profit at a specified confidence level. Formally, CVaR of profit $\pi(x;\omega)$ given by (1.1) is defined as (see Pflug and Römisch (2007) for the definition that is derived from Rockafellar and Uryasev (2002))

$$\mathrm{CVaR}_\eta[\pi(x;\omega)] = \max_{\phi\in\mathbb{R}} \Gamma_\eta(\pi(x;\omega),\phi), \tag{3.1}$$

where

$$\Gamma_\eta(\pi(x;\omega),\phi) = \phi - \frac{1}{\eta}\,\mathsf{E}_\omega[\phi-\pi(x;\omega)]^+.$$

The parameter $\eta \in (0,1]$ reflects the degree of risk aversion for the newsvendor. The smaller the η is, the more risk-averse newsvendor is.

The CVaR is also known as a risk measure that is coherent (Artzner et al., 1999), and consistent with the second (or higher) order stochastic dominance (Ogryczak and Ruszczýnski, 2002). A coherent risk measure has better computational properties than a non-coherent risk measure. The consistency with the stochastic dominance implies that maximizing the CVaR never conflicts with maximizing the expectation of any risk-averse utility function (Ogryczak and Ruszczýnski, 2002). Furthermore, notice that, by (3.1), the CVaR of $\eta = 1$ equals to the expected profit.

CVaR only objective

Some researchers address the newsvendor problem as minimizing the downside risk of the profit. Hence the objective of the risk-averse newsvendor is to maximize the CVaR, i.e.

$$\max_x \ \mathrm{CVaR}_\eta[\pi(x;\omega)]. \tag{3.2}$$

The following theorem gives a closed form solution of the unconstrained problem (3.2).

Theorem 16 (Gotoh and Takano, 2007, Proposition 3.3)**.** *Assume that there exists the inverse of the distribution function of the product demand. Then,* (3.2) *with* $\eta \in (0, 1]$ *has an optimal solution* (x^*_{CVaR}, ϕ^*) *defined by*

$$\begin{aligned} x^*_{CVaR} &= \frac{p-v}{p-v+s} F^{-1}\left(\eta \frac{p-c+s}{p-v+s}\right) + \frac{s}{p-v+s} F^{-1}\left(\frac{(p-v+s)-\eta(c-v)}{p-v+s}\right), \\ \phi^* &= \frac{(p-c+s)(p-v)}{p-v+s} F^{-1}\left(\eta \frac{p-c+s}{p-v+s}\right) - \frac{(c-v)s}{p-v+s} F^{-1}\left(\frac{(p-v+s)-\eta(c-v)}{p-v+s}\right). \end{aligned} \tag{3.3}$$

Proof. The proof is conducted in Gotoh and Takano (2007) with different notation or in Xu and Li (2010, Theorem 2) with similar notation.

❑

In case the shortage penalty is set to zero, i.e. $s = 0$, we obtain the following simplier solution.

Corollary. (Gotoh and Takano, 2007, Corollary 3.4) Under the same assumptions as in Theorem 16 with $s = 0$, the optimal ordering quantity of the risk-averse newsvendor is

$$x^*_{\text{CVaR}\backslash\text{s}} = F^{-1}\left(\eta \frac{p-c}{p-v}\right). \tag{3.4}$$

Notice that (3.4) is similar to the classical NP problem solution (1.7) with only difference in the coefficient in the argument of the inverse F^{-1}. Consequently, the difference between the optimal ordering given by (3.3) or (3.4) and the classic one (1.7) depends only on two parameters s and η. The higher the degree of risk aversion is in (3.4), i.e. the smaller the η is, the less items is order by the newsvendor. We kindly refer reader to Gotoh and Takano (2007) or Xu and Li (2010) for the sensitivity analysis of the optimal ordering x^*_{CVaR} and $x^*_{\text{CVaR}\backslash\text{s}}$.

Wu et al. (2013) note that although the CVaR has better computational characteristics, as a coherent risk measure, and is often used in financial management, there exists a limitation when the CVaR is applied to this specific newsvendor modification. If v converges to c, the newsvendor should order as many items as possible since there is almost no risk of overstocking. However, under the CVaR only criterion (3.2) with $s = 0$, the newsvendor orders only $F^{-1}(\eta)$ from (3.4). Interpretation of this issue is that the CVaR only criterion is too conservative in some cases because only the worst outcome is considered. Generally speaking, for small η, the CVaR only criterion describes the risk-aversion and neglects a large part of the profit distribution; whereas for large η, the CVaR only criterion comprises a large part of the profit distribution and does not reflect the real newsvendor's risk attitude. To address issues mentioned, the decision makers dealing with the portfolio optimization/inventory problems usually put their interest into the tradeoff between the expected profit and the risk. This concept is defined and further analyzed in section 3.1.4.

General formulation and efficient solution

Symbolically, the multicriteria optimization problem of maximizing the expected profit and minimizing the variation of the profit using CVaR is as follows

$$\begin{array}{ll} \text{"maximize"}_{x} & \Big[\mathsf{E}_\omega[\pi(x;\omega)], \mathrm{CVaR}_\eta[\pi(x;\omega)]\Big] \\ \text{subject to} & x \geq 0. \end{array} \tag{3.5}$$

In order to find the optimal solution with respect to the multiobjective programming problem (3.5) we introduce a concept of solution x which is acceptable from the point of view of "maximization" of all considered objective functions. The definition is to be found in Dupačová et al. (2003).

Definition 5 (efficient solution)**.** *A solution $x^* \geq 0$ is an efficient solution of the multicriteria optimization problem* (3.5) *if there is no element $x \geq 0$ for which*

$$\Big[\mathsf{E}_\omega[\pi(x;\omega)], CVaR_\eta[\pi(x;\omega)]\Big] \geq \Big[\mathsf{E}_\omega[\pi(x^*;\omega)], CVaR_\eta[\pi(x^*;\omega)]\Big]$$

and

$$\Big[\mathsf{E}_\omega[\pi(x;\omega)], CVaR_\eta[\pi(x;\omega)]\Big] \neq \Big[\mathsf{E}_\omega[\pi(x^*;\omega)], CVaR_\eta[\pi(x^*;\omega)]\Big].$$

Interpretation is clear: no other feasible decision is uniformly better with respect to all considered criteria.

3.1.2 Expected profit in the objective

Consider a newsvendor who wants to maximize its expected profit while keeping the risk below some reasonable level. Denote R as a maximal risk that can the newsvendor take. Then the optimization problem is to

$$\begin{array}{ll} \underset{x}{\text{maximize}} & \mathsf{E}_\omega[\pi(x;\omega)] \\ \text{subject to} & \mathrm{CVaR}_\eta[\pi(x;\omega)] \geq R, \\ & x \geq 0, \end{array} \tag{3.6}$$

where $\pi(x;\omega)$ is the NP profit function given by (1.1) and the CVaR is considered on the confidence level η and is defined in section 3.1.1.

3.1.3 Conditional value at risk in the objective

Consider a newsvendor who wants to minimize its risk while keeping the return at least as large as a predetermined target profit. Denote G as a minimal profit that the newsvendor wants to achieve. Then the optimization problem is to

$$\begin{array}{ll} \underset{x}{\text{maximize}} & \mathrm{CVaR}_\eta[\pi(x;\omega)] \\ \text{subject to} & \mathsf{E}_\omega[\pi(x;\omega)] \geq G, \\ & x \geq 0, \end{array} \tag{3.7}$$

where $\pi(x;\omega)$ is the NP profit function given by (1.1) and the CVaR is considered on the confidence level η and is defined in section 3.1.1.

Gotoh and Takano (2007) provide a linear program (LP) transformation in case the demand distribution is given by a finite number of scenarios. The advantages of such LP transformation are significant especially when the newsvendor deals with the multiproduct problem and many constraints are imposed since LP can handle efficiently huge number of constraints and variables. Apart from that, when we are not able to compute any closed form solution, we can find an (approximating) optimal solution and find the optimal distribution of any related random variable in an approximate manner.

3.1.4 Expected profit and CVaR in the objective

As stated above, the CVaR only criterion (3.2) has some weaknesses and thus, in order to overcome them, researchers propose a more general performance measure. Specifically, an objective function that is a combination of the expected profit and the CVaR of the profit. Such an objective reflects the desire of the risk-averse newsvendor to minimize the downside risk of the profit at one hand, and to maximize the expected profit on the other hand. Hence, the newsvendor seeks for a balance/tradeoff between the profit and the risk. The objective function of the mean-CVaR model can be expressed as

$$\lambda\,\mathsf{E}_\omega[\pi(x;\omega)] + (1-\lambda)\mathrm{CVaR}_\eta[\pi(x;\omega)], \tag{3.8}$$

where $\lambda \in [0,1]$ is a weight that represents the relative importance of the expected profit compared to the CVaR. Note that if $\lambda = 0$ then (3.8) turns into the CVaR only criterion (3.2), i.e. the newsvendor cares only about the risk of the profit and does not consider the expected profit into his decision. On the other hand, if $0 < \lambda < 1$, then the newsvendor's decision criterion is a convex combination of the expected profit and the CVaR, i.e. the newsvendor is also risk-averse with both the expected profit and the CVaR in the objective. And lastly, if $\lambda = 1$, then the decision criterion becomes the expected profit. For any $\lambda \in [0,1]$, when $\eta = 1$, the objective (3.8) is reduced to the classical NP model (1.4).

We know from section 1 that the newsvendor's profit function $\pi(x;\omega)$ is jointly concave in x and ω. The objective function (3.8) is therefore equivalent to the following concave single-stage stochastic program

$$\underset{x\in\mathbb{R}_+,\phi\in\mathbb{R}}{\text{maximize}} \quad (1-\lambda)\phi + \mathsf{E}_\omega\left[\lambda\pi(x;\omega) - \frac{1-\lambda}{\eta}(\phi - \pi(x;\omega))^+\right].$$

The following theorem gives a closed form optimal ordering or an equation that the optimal ordering solves depending upon a specific condition.

Theorem 17 (Xu and Li, 2010, Theorem 3)**.** *Suppose that the newsvendor is risk-averse with the tradeoff objective function given by* (3.8)*. If* $(p-c+s) - \lambda(p-v+s)F(s\cdot F^{-1}(1-\eta)/(p-v+s)) \le 0$*, then the optimal order quantity is*

$$x^*_{E\text{-}CVaR} = F^{-1}\left(\frac{p-c+s}{\lambda(p-v+s)}\right), \tag{3.9}$$

otherwise, the optimal order quantity $x^*_{E\text{-}CVaR}$ *solves*

$$\begin{aligned}(p-v+s)x = (p-v)F^{-1}&\left(\eta\frac{(p-c+s)-\lambda(p-v+s)F(x)}{(1-\lambda)(p-v+s)}\right)\\ &+ sF^{-1}\left(\frac{\eta(p-c+s)+(p-v+s)[(1-\lambda)(1-\eta)-\eta\lambda F(x)]}{(1-\lambda)(p-v+s)}\right).\end{aligned} \tag{3.10}$$

Proof. The proof is conducted in Xu and Li (2010).

❑

Notice that when $\lambda = 0$, the optimal order quantity $x^*_{\text{E-CVaR}}$ given by (3.10) is reduced to the optimal solution of the CVaR only criterion x^*_{CVaR} (3.3). Furthermore, when $\lambda = 1$, the optimal order quantity $x^*_{\text{E-CVaR}}$ given by (3.9) is reduced to the optimal solution of the classical NP problem x^* (1.7). We refer to Xu and Li (2010) for further details, solution properties and the sensitivity analysis.

Gotoh and Takano (2007) propose a slightly different optimization model compared to (3.8), i.e. the objective is defined as

$$\mathsf{E}_\omega[\pi(x;\omega)] + \lambda \text{CVaR}_\eta[\pi(x;\omega)].$$

They provide a numerical procedure to find the optimal ordering solution in case $s > 0$ and give the closed form solution for the ordering policy in case $s = 0$. Moreover, they provide an equivalent LP model of the constrained mean-CVaR model for the multiple products case.

Wu et al. (2013) warn that although the mean-CVaR criterion (3.8) can capture the tradeoff between the expected profit and the risk of the profit and, on top of that, the criterion is a coherent risk measure which means it has better computational characteristics, it may not be easy for the managers to determine the appropriate value of the weight λ. And because the optimal ordering quantity heavily depends on the choice of λ, an effective mechanism for the selection of λ should be developed in order to make the mean-CVaR criterion implementable in practice.

Note. Apart from so called ϵ-constrained programming approach represented by section 3.1.2 and 3.1.3, and the weighted sum programming approach for which section 3.1.4 gives the typical example, there exists a method called the goal programming that is not present in this paper. These methods and their combinations are typical approaches of solving the multicriteria stochastic optimization problems.

3.1.5 Model with pricing and CVaR

Suppose that a risk-averse newsvendor faces the price-dependent demand. Hence, in this section, we combine results on the NPP (section 2.1) with the risk-averse newsvendor results performed in previous sections of this chapter. In this paper, we analyze the mean-risk model that is studied in section 3.1.4 with additional pricing decision. Other formulations in section 3.1.2 and 3.1.3 could be considered as well. However, due to their simplicity and analogous solving procedure we skip them.

Let us refresh the notation used in the NPP sense and add an assumption on the expected value of the random element. Let $\omega(p,\epsilon)$ is the price-dependent product demand that is decreasing in p and strictly increasing in ϵ, where $\epsilon \in [A, B]$ is the random price-independent component. Next, we assume that the mean demand $\mathsf{E}_\epsilon[\omega(p,\epsilon)]$ is a continuous, strictly decreasing, nonnegative, twice-differentiable function and defined on a closed interval $[p_l, p_u]$, where p_l and p_u are the minimal and maximal admissible price, respectively, satisfying $c < p_l < p_u$. Moreover, let $\mathsf{E}_\epsilon[\omega(p,\epsilon)]$ has an increasing price elasticity (IPE; see definition 4) and let the cdf $F(\cdot)$ of the random element ϵ has either generalized strict increasing failure rate or increasing failure rate depending upon context (GSIFR or IFR; see definition 3). Similarly to the NPP, the additive and the multiplicative demand functions are considered. In case of the additive demand see (2.2) for the demand function and (2.4) for the response function. In case of the multiplicative demand see (2.1) and (2.3), analogically.

Due to the simplicity, we assume that the shortage penalty is zero in this section, i.e. $s = 0$. The procedure for $s > 0$ is analogous and is not covered in this paper. Moreover, we suppose that $\mathsf{E}[\epsilon] = 0$ in the additive demand case and $\mathsf{E}[\epsilon] = 1$ in the multiplicative demand case.

General model

Consider a general demand model, i.e. we do not distinguish between the additive and the multiplicative demand cases yet. Then the profit function $\pi(x; p; \omega)$ can be obtained by substituting $\omega(p, \epsilon)$ into the NP profit function (1.1). Obviously, the general model includes both the additive and the multiplicative models. Chen et al. (2009) suggest the mean-CVaR criterion to solve the newsvendor model with pricing incorporating the CVaR. Consequently, we get an objective function as

$$\lambda \, \mathsf{E}_\omega[\pi(x; p; \omega)] + (1 - \lambda)\mathrm{CVaR}_\eta[\pi(x; p; \omega)],$$

that is similar to (3.8) with only differences that the random element is included in other manner and the price p becomes a decision variable together with the order amount x. Chen et al. (2007) states that it is extremely difficult or even impossible to find structural properties of the optimal ordering and pricing policy under such a general performance measure. Therefore, they suggest utilizing the numerical analysis to investigate the tradeoff between the expected profit and the risk of the profit, and to fully understand the two extreme cases. The case when $\lambda = 1$ is extensively analyzed in section 2.1 for both the additive and the multiplicative demand case. We focus on the joint optimal ordering and pricing when $\lambda = 0$ in this section.

We showed that substituting the stocking factor z in the NPP and finding the optimal solution of z leads to the classical NP solution (see (2.19) for the additive case; z^* for the multiplicative case can be expressed by similar steps as in the additive case and is identical to the additive demand case optimal stocking factor, i.e. (2.19)). Analogically, after substituting z in $\pi(x; p; \omega)$ and fixing the price p, we obtain from corollary of Theorem 16 that

$$z^* = z^*(p) = F^{-1}\left(\eta \frac{p - c}{p - v}\right).$$

Next, we consider the case where the price p is also a decision variable. From the results of Theorem 16 we know that for any fixed p the optimal value of ϕ in the general definition of CVaR (3.1) satisfies $\phi^*(p) = (p - c)z^*(p)$. Hence, the objective function can be converted to a single variable function,

$$g(p) = \Gamma_\eta(\pi(z^*(p); p; \omega), \phi^*(p)) = \frac{p - v}{\eta} \int\limits_A^{F^{-1}(\eta \frac{p-c}{p-v})} \omega(p, \epsilon) \, \mathrm{d}F(\epsilon). \qquad (3.11)$$

The problem simplifies to the task of finding a price $p^* \in [p_l, p_u]$ that maximizes (3.11). From that we easily get the stocking factor $z^*(p)$ and the order quantity x^* by substituting back.

Before investigating the optimal pricing decision under the CVaR only criterion, we first find the optimal price without demand uncertainty. In the riskless problem the demand is simply $d(p)$, that is given either by (2.3) or (2.4), and the order quantity should be $d(p)$. The profit function becomes

$$\pi_d(p) = (p - c)d(p).$$

The following lemma is easily verifiable compilation of Theorem 13 and Theorem 14 (and its surrounding calculations). The lemma is thus given without proof.

Lemma 18 (Yao et al. (2006))**.** *If the demand function $d(p)$ has the IPE, $\pi_d(p)$ is quasi-concave (or unimodal) in p on the interval $[p_l, p_u]$, then the optimal riskless price p_d^* is determined by the first order condition, i.e. $(p - c)d'(p) + d(p) = 0$.*

Additive demand case

Consider the demand captured by

$$\omega(p, \epsilon) = d^A(p) + \epsilon = \alpha - \beta p + \epsilon, \tag{3.12}$$

where $\mathsf{E}[\epsilon] = 0$ and $d^A(p_u) + A \geq 0$. Thus, price affects the location of the demand distribution, but not the demand variance. From (3.11), the newsvendor's objective in the additive case is to maximize

$$\begin{aligned} g_a(p) &= \frac{p - v}{\eta} \int_A^{F^{-1}(\eta \frac{p-c}{p-v})} d^A(p) + \epsilon \; \mathrm{d}F(\epsilon) \\ &= \frac{p - v}{\eta} \int_A^{F^{-1}(\eta \frac{p-c}{p-v})} \epsilon \; \mathrm{d}F(\epsilon) + (p - c)(\alpha - \beta p). \end{aligned}$$

Lemma 19 (Chen et al. (2009))**.** *The optimal selling price in the additive demand case, denoted by p_a^*, is less than or equal to the optimal riskless price p_d^*.*

Proof. See Chen et al. (2007, Lemma 4) for proof.

❑

The following theorem gives a sufficient condition for the uniqueness of the optimal price p.

Theorem 20 (Chen et al. (2009))**.** *For the additive demand model, if distribution of $F(\cdot)$ has the IFR and $d^A(p)$ has the IPE, then $g_a(p)$ is quasi-concave in p on the range $[p_l, p_u]$. Therefore, there is a unique maximizer of $g_a(p)$ in $[p_l, p_u]$ which is determined by the first order condition $\frac{\partial g_a(p)}{\partial p} = 0$.*

Proof. See Chen et al. (2007, Theorem 2) for proof.

❑

Thus, under the given conditions on the distribution and the response function, we found again that the optimal solution must satisfy the first order condition due to the quasi-concavity of $g_a(p)$.

Multiplicative demand case

Consider the demand represented by

$$\omega(p,\epsilon) = d^M(p)\epsilon = \alpha p^{-\beta}\epsilon, \tag{3.13}$$

where $\mathsf{E}[\epsilon] = 1$ and $A \geq 0$. Therefore, price influences the scale of the distribution, but not the coefficient of variation. From (3.11), the newsvendor's objective in the multiplicative case is to maximize

$$g_m(p) = \frac{(p-v)d^M(p)}{\eta} \int\limits_{A}^{F^{-1}(\eta\frac{p-c}{p-v})} \epsilon \, \mathrm{d}F(\epsilon).$$

Lemma 21 (Chen et al. (2009))**.** *The optimal selling price in the multiplicative demand case, denoted by p_m^*, is greater than or equal to the optimal riskless price p_d^*.*

Proof. See Chen et al. (2007, Lemma 3) for proof.

❑

From Lemma 19 and 21, we know that $p_a^* \leq p_d^* \leq p_m^*$. This result is consistent with results obtained in the NPP (section 2.1) if we would assume $\mathsf{E}[\epsilon] = 1$ and $\mathsf{E}[\epsilon] = 0$ in the multiplicative and additive demand case, respectively, and the response functions are identical for both demand cases.

The following theorem gives a sufficient condition for the uniqueness of the optimal price p.

Theorem 22 (Chen et al. (2009))**.** *For the multiplicative demand model, if distribution of $F(\cdot)$ has the GSIFR and $d^M(p)$ has the IPE, then $g_m(p)$ is quasi-concave in p in the range $[p_l, p_u]$. Therefore, there is a unique maximizer of $g_m(p)$ in $[p_l, p_u]$ which is determined by the first order condition $\frac{\partial g_m(p)}{\partial p} = 0$.*

Proof. See Chen et al. (2007, Theorem 2) for proof.

❑

Once again, the solution satisfying the first order condition is optimal, if we enforce specific properties on the demand distribution and the response function, since then the function $g_m(p)$ is quasi-concave. The sensitivity analysis of both the additive and multiplicative demand case is performed in Chen et al. (2009, 2007) and hence we refer reader there for further analysis.

3.2 Multiproduct extensions

Assume that the newsboy faces uncertain demand for multiple products. The problem description is similar to the NP model. However, we have to introduce new notation in order to capture the multiple product NP problem. See section 1 for description. Denote $i = 1, \ldots, n$ index of products, where n is the total number of products. Then parameters are defined as

- p_i the unit selling price of product i,
- c_i the unit buying cost of product i,
- v_i the unit salvage value (in case of overstocking) of product i,
- s_i the unit shortage penalty (in case of understocking) of product i,
- ω_i the random demand for product i,
- x_i the quantity of product i purchased, a decision variable.

3.2.1 Independent demands model

Suppose that the demand do not influence each other, i.e. $\forall i$ ω_i are independent, and that the total profit is sum of the individual product's profit. Let M be the budget, i.e. the amount of money that the newsboy allocated for the selling period. Thus the objective of maximizing the total expected profit for multiple independent products is formulated as:

$$\underset{x_i}{\text{maximize}} \quad \sum_{i=1}^{n} \left((p_i - c_i)x_i - \left[s_i \int_{x_i}^{\infty} (\omega_i - x_i) \, \mathrm{d}F_i(\omega_i) + (p_i - v_i) \int_{0}^{x_i} (x_i - \omega_i) \, \mathrm{d}F_i(\omega_i) \right] \right) \tag{3.14}$$

$$\text{subject to} \quad x_i \geq 0, \quad i = 1, \dots, n,$$

$$\sum_{i=1}^{n} c_i x_i \leq M. \tag{3.15}$$

The budget constraint (3.15) was added to the problem to capture real situations where the newsvendor's access to money is limited and he cannot order infinitely many items.

When we skip the budget constraint (3.15) from the model (3.14)-(3.15), the problem becomes separable and, under general assumptions on the NP, the problem can be treated as n independent problems. It is then obvious that the optimal decision is to order

$$x_i^* = F_i^{-1}\left(\frac{p_i - c_i + s_i}{p_i - v_i + s_i} \right), \tag{3.16}$$

where F_i is the cdf of demand ω_i. See Choi (2012, Chapter 1) for details. In case ω_i are independent and identically distributed (iid) then the cdf of demand ω_i satisfies $F_i = F$ $\forall i$.

Optimal seeking algorithm

Consider the full model (3.14)-(3.15) with zero shortage penalty, i.e. $s_i = 0 \, \forall i$. Then a simple heuristic/algorithm could be developed in order to obtain a feasible solution that is optimal as well. In the previous paragraph we found the optimal ordering x_i^* given by (3.16) without the budget constraint (3.15). Therefore, under the full model, the solution (3.16) might not be feasible. The following algorithm gives optimal solution of the full model (3.14)-(3.15).

Remember that the NP is considered as a continuous problem in this thesis. Moreover we can naturally assume that $\mathsf{E}_{\omega_i}[\pi_i(x_i^*;\omega_i)] > 0 \; \forall \, i$ which yields $x_i^* > 0 \; \forall \, i$. Other solutions are not profitable for the newsvendor and hence they are excluded from the algorithm (obviously their optimal ordering and expected profit are both zero).

1. If the budget constraint (3.15) is not violated for x_i^* by (3.16), then x_i^* is the optimal solution. Otherwise proceed to the next step with x_i^* given by (3.16) with $s_i = 0$.

2. Order items x_i with non-zero ordering x_i^* according to its profit relative per-item margin m_i , i.e. the ratio between the optimal expected profit and the products purchasing cost,

$$m_i = \frac{\mathsf{E}_{\omega_i}[\pi_i(x_i^*;\omega_i)]}{c_i x_i^*}. \tag{3.17}$$

   If $x_i^* = 0$ we set $m_i = 0$. Hence $m_1 \leq m_2 \leq \ldots \leq m_n$ and ordering $(m_1, \ldots, m_n)$ corresponds to the items ordering $(x_1, \ldots, x_n)$.

3. Take the item j with the lowest margin m_j and non-zero optimal ordering x_j^* given by (3.16). Set x_j^* to zero.

4. Check whether the budget constraint (3.15) is satisfied. If yes, proceed to the next step. Otherwise go back to step 2.

5. Set the optimal ordering of product $k \in \{1, \ldots, j\}$ to

$$x_k^* = \frac{M - \sum_{i=j+1}^n c_i x_i^*}{c_k} = \frac{M - \sum_{i=j+1}^n c_i F_i^{-1}\left(\frac{p_i - c_i}{p_i - v_i}\right)}{c_k} \tag{3.18}$$

   for which

$$k = \operatorname*{argmax}_{h=1,\ldots,j} \mathsf{E}_{\omega_h}\left[\pi_h\left(\frac{M - \sum_{i=j+1}^n c_i x_i^*}{c_h}, \omega_h\right)\right].$$

   We found the feasible optimal ordering policy. Furthermore, the budget constraint (3.15) is satisfied with equality.

Note. In the last iteration of step 3 we found the item j that makes the problem feasible. This, however, does not guarantee that ordering more items of j makes the solution optimal due to the different distributions of the random demands ω_i.

It is worth noting that ordering (3.18) units of item k is profitable due to the shape of the expected profit function $\mathsf{E}_\omega[\pi(x;\omega)]$. As we can see in figure 3.1 and proved in section 1, the expected profit with respect to ordering x is concave and, under zero shortage penalty, does not allow negative expected profit for $x \in (0, F^{-1}\left(\frac{p-c}{p-v}\right))$.

The following theorem summarizes the output of the algorithm.

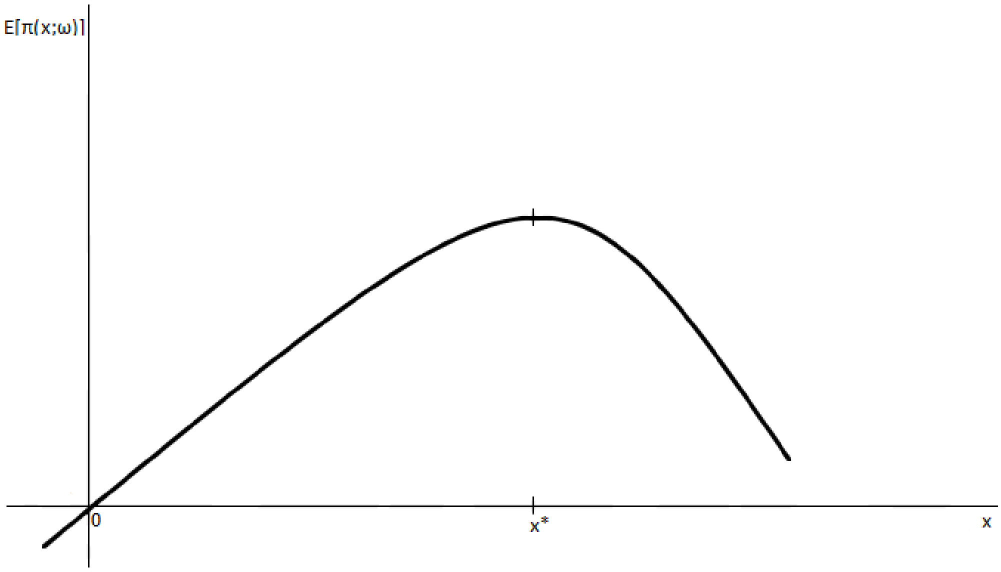

Figure 3.1: Graph of the expected profit $\mathsf{E}_\omega[\pi(x;\omega)]$ where shortage penalty $s = 0$ and demand is continuous.

Theorem 23. *Let the items are ordered according to the per item relative profit margin m_i (3.17) and let the unconstrained optimal ordering x_i^* is given by (3.16). Then the ordering decision*

$$\boldsymbol{x}^* = \Big(0, \dots, 0, \frac{1}{c_k}(M - \sum_{i=j+1}^{n} c_i x_i^*), 0, \dots, 0, x_{j+1}^*, \dots, x_n^*\Big) \tag{3.19}$$

with the index j found in step 5 of the algorithm is the optimal decision for the multiproduct independent demand model (3.14)-(3.15) *with $s_i = 0\ \forall i$.*

Proof. Let $\boldsymbol{y}^*$ is a better solution of (3.14)-(3.15) than $\boldsymbol{x}^*$. Then

$$\mathsf{E}_{\boldsymbol{\omega}}[\pi(\boldsymbol{y}^*;\boldsymbol{\omega})] > \mathsf{E}_{\boldsymbol{\omega}}[\pi(\boldsymbol{x}^*;\boldsymbol{\omega})], \tag{3.20}$$

where $\boldsymbol{x}^*$ is given by (3.19). Moreover, we can allocate $y_i^* \in [0, F_i^{-1}(\frac{p_i - c_i}{p_i - v_i})]$ since, due to the concavity of the expected profit, ordering more items than optimal leads to decrease in the expected profit (see figure 3.1) and increase in overall costs. This move would hence make the solution more infeasible.

If $\boldsymbol{y}^* \leq \boldsymbol{x}^*$ & $\boldsymbol{y}^* \neq \boldsymbol{x}^*$, i.e. $\exists\, i \in \{1, \dots, n\} : y_i^* < x_i^*$ and for the other items $y_k^* \leq x_k^*\ \forall\, k \in \{1, \dots, n\} \setminus \{i\}$, then, due to the concavity of expected profit, $\mathsf{E}_{\boldsymbol{\omega}}[\pi(\boldsymbol{y}^*;\boldsymbol{\omega})] < \mathsf{E}_{\boldsymbol{\omega}}[\pi(\boldsymbol{x}^*;\boldsymbol{\omega})]$.

Therefore, since in the algorithm we decrease the ordering quantity of items that have the worst contribution to the profit with respect to the purchasing cost, the following must hold in order to inequality (3.20) being satisfied: $\boldsymbol{y}^* \geq \boldsymbol{x}^*$ & $\boldsymbol{y}^* \neq \boldsymbol{x}^*$, i.e. $\exists\, i \in \{1, \dots, n\} : y_i^* > x_i^*$ and for the other items $y_k^* \geq x_k^*\ \forall\, k \in \{1, \dots, n\} \setminus \{i\}$.

From step 5 of the algorithm we know that the index j identifies an item that makes (3.14) feasible with respect to the solution $\boldsymbol{x}^*$. Thus $\sum_{i=j+1}^{n} c_i x_i^* \leq M$. If we add $c_k x_k^*$ to the left-side of inequality, we obtain

$$\sum_{i=j+1}^{n} c_i x_i^* + c_k x_k^* = \sum_{i=j+1}^{n} c_i x_i^* + c_k \cdot \frac{1}{c_k}\Big(M - \sum_{i=j+1}^{n} c_i x_i^*\Big) = M.$$

Hence the budget constraint (3.15) is satisfied with equality for $\boldsymbol{x}^*$ because the optimal ordering of items $1\ldots,k-1,k+1,\ldots,j$ is zero. Therefore, since $\boldsymbol{y}^* \geq \boldsymbol{x}^*$ & $\boldsymbol{y}^* \neq \boldsymbol{x}^*$ and $c_i > 0\ \forall i$, we get that

$$\sum_{i=1}^{n} c_i y_i^* > \sum_{i=1}^{n} c_i x_i^* = M.$$

That is, however, contradiction as solution $\boldsymbol{y}^*$ becomes infeasible. Thanks to the products' ordering being determined with respect to the highest profit-cost ratio, the ordering policy $\boldsymbol{x}^*$ is optimal.

❑

Note. The algorithm provided could be easily modified for the NP with a discrete random demand.

3.2.2 Dependent demands model

Suppose now that the newsvendor's product portfolio is composed of products whose demands are correlated (dependent). E.g. $\mathrm{cor}(\omega_1, \omega_2) = \rho \neq 0$. Therefore, the objective function is not decomposable anymore and a multidimensional joint distribution of demand $F(\boldsymbol{\omega}) = F(\omega_1, \ldots, \omega_n)$ must be considered with the variance matrix $\boldsymbol{\Sigma}$ that has some nonzero off-diagonal elements. Denote bold symbols as vectors; for instance $(p_1, \ldots, p_n)^\top = \boldsymbol{p}$. Furthermore, let $T = [x_1, \infty) \times [x_2, \infty) \times \ldots \times [x_n, \infty)$ and $C = [0, x_1) \times [0, x_2) \times \ldots \times [0, x_n)$ be n-dimensional half-closed domains. Then the objective of maximizing the total expected profit for multiple dependent products is formulated as:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & (\boldsymbol{p} - \boldsymbol{c})^\top \boldsymbol{x} - \Bigg[\int_T \boldsymbol{s}^\top (\boldsymbol{\omega} - \boldsymbol{x}) \ \mathrm{d}F(\boldsymbol{\omega}) \\ & \qquad + \int_C (\boldsymbol{p} - \boldsymbol{v})^\top (\boldsymbol{x} - \boldsymbol{\omega}) \ \mathrm{d}F(\boldsymbol{\omega}) \Bigg] \\ \text{subject to} \quad & \boldsymbol{x} \geq 0, \\ & \boldsymbol{c}^\top \boldsymbol{x} \leq M, \end{aligned}$$

where condition $\boldsymbol{x} \geq 0$ is meant componentwise.

Chapter 4

Distributional robustness of the newsvendor model

In stochastic optimization, similarly to the previous chapters, we usually assume that the demand distribution or the random element distribution P is precisely known from the data and we neglect the fact that data contain a noise (uncertainty). Such a noise might originate in an inaccurate measurement or an incomplete knowledge about the underlying demand distribution at the moment when the ordering decision is being made, for instance. Such problems motivate us to use optimization methods that are immune to the noise in historical data. Small change in the demand should lead into a small change in the result and a great change should not cause an enormous loss. In practice, a newsvendor has to estimate the demand distribution P using only a limited structural information about the demand and historical data, or even expert opinions. Therefore, the assumption of complete and accurate information about P is unrealistic.

On the other hand, solving the stochastic problem with respect to all possible demand distributions, i.e. the robust optimization approach, results in overly conservative decision. Problem is that the newsvendor does not take the benefit of knowing a partial information about the demand in the robust optimization models. Decision makers can typically deduce specific properties of the demand distribution from existing domain knowledge (e.g., bounds of the customer demand or symmetry in the deviations) or from statistical analysis (e.g., estimation of mean and covariance from historical data).

In cases where the newsvendor is risk-averse and/or he has some (but not full) knowledge of underlying demand distribution, an alternative approach is to use distributionally robust stochastic programs. Hence, we study the distributionally robust newsvendor problem (DRNP) in this paper. In the DRNP, the goal is to find a decision that maximizes the worst-case expected profit, where worst-case refers to a set of distributions called ambiguity sets.

The classical NP problem, where the demand distribution is known, and the robust optimization approach are special cases of the DRNP, since we get the classical NP if the ambiguity set contains only one distribution and we get the classical robust optimization if the ambiguity set involves all possible demand distributions with the same support. Hence, the DRNP lies between these two approaches. Moreover, Wiesemann et al. (2014) show that distributionally robust optimization problems are computationally tractable.

For deeper introduction into the distributionally robust optimization we recommend paper by Wiesemann et al. (2014) or newer paper by Hanasusanto et al. (2015).

4.1 Ambiguity set

As stated above, the ambiguity set is a set of distributions that follow a specific properties. Properties could range from the appropriate family distribution, exact mean and varying variance, to the bounds of the domain for which the demand falls within the predetermined bounds with 95% confidence and many more. Consequently, there are numerous ways how to define the ambiguity set. For instance, Wiesemann et al. (2014) give the general definition of the ambiguity sets that are highly expressive and involve many ambiguity sets from recent literature as special cases. However, their framework does not cover ambiguity sets that impose infinitely many moment restrictions, that would be necessary to describe symmetry, independence or unimodality characteristics. Paper by Hanasusanto et al. (2015) provide wider list of ambiguity sets that are not covered in Wiesemann et al. (2014). Moreover, they provide convenient tractable reformulations and properties.

In this paper we use the ambiguity set as follows. Let $\mathcal{P}$ denote the set of all distributions on $\mathbb{N}^0$ that are consistent with the known properties of demand, such as its first and second order moment and its support. Since demand can acquire only nonnegative values and is usually bounded from above, we assume that support of $\mathcal{P}$ belongs to $\mathbb{N}^0_n$, where $\mathbb{N}^0_n = \{0, 1, \dots, n\}$ has $n+1$ elements. Hence ω is a discrete random variable. Moreover, let $\mu \in \mathbb{N}^0_n$ is the mean and $\sigma^2 \in \mathbb{R}_+$ is the variance of random demand ω under the true distribution $\mathbb{Q}$. Thus, we implicitly assume finite second order moments of $\mathbb{Q}$. Specifically, we assume that $\mathcal{P}$ is the ambiguity set of all distributions on $\mathbb{N}^0_n$ with the same first order moment as $\mathbb{Q}$ and second order moment bounded above by σ^2, that is,

$$\mathcal{P} = \left\{ \mathsf{P} \in \mathbb{Q}(\mathbb{N}^0_n) : \mathsf{E}_\mathsf{P}[\omega] = \mu, \mathsf{E}_\mathsf{P}\left[(\omega-\mu)^2\right] \le \sigma^2 \right\}. \tag{4.1}$$

4.2 General model

Assume the classical NP defined in section 1 with zero shortage penalty, i.e. $s = 0$. Furthermore, we assume that the probability distribution P describing the product demand ω is unknown. Hence the DRNP with the profit function $\pi(x;\omega)$ given by (1.1) is formulated as:

$$\begin{aligned} \max_x \quad \inf_\mathsf{P} \quad & \mathsf{E}_\mathsf{P}[\pi(x;\omega)] \\ \text{subject to} \quad & x \ge 0, \\ & \mathsf{P} \in \mathcal{P}, \end{aligned} \tag{4.2}$$

where $\mathcal{P}$, given by (4.1), is an ambiguity set built up from the newsvendor's knowledge about the product demand.

Let us denote W as the worst possible distribution from $\mathcal{P}$ and suppose that W somehow depends on the optimal choice of ordering x. If we denote $F_{W(x)}$

as the cdf of distribution W and substitute the expected profit $\mathsf{E}_{\mathsf{P}}[\pi(x;\omega)]$ as in (1.2) and (1.3), we obtain the reformulation of (4.2) as

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & (p-c)x - (p-v)\int_0^x (x-\omega)\, \mathrm{d}F_{W(x)}(\omega) \\ \text{subject to} \quad & x \geq 0, \end{aligned} \tag{4.3}$$

where

$$W(x) = \underset{\mathsf{P}\in\mathcal{P}}{\text{argmin}}\, \mathsf{E}_{\mathsf{P}}[\pi(x;\omega)].$$

The problem (4.3) is easily solvable convex optimization problem given we know the worst cdf $F_{W(x)}$.

4.2.1 Best and worst distribution

A newsvendor might be interested in the fact which distribution from the ambiguity set $\mathcal{P}$ gives him the largest and lowest expected profit. Hence, in this section, we try to find the best and the worst possible distribution of the model (4.3).

Best distribution

Consider the model (4.3). In the following theorem we provide the best possible distribution B that belongs to $\mathcal{P}$, i.e. the distribution with fixed support, the same first order moment and second order moment within the bounds set, that leads to the largest profit that the newsvendor can achieve.

Proposition 24. *The best possible distribution B of the problem* (4.3)*, where B is drawn from the ambiguity set $\mathcal{P}$ given by* (4.1)*, is degenerate distribution with the single value that equals to the mean μ with probability one. Hence the cdf of B is*

$$F_B(\omega) = \begin{cases} 0 & \text{if } \omega < \mu, \\ 1 & \text{if } \omega \geq \mu. \end{cases}$$

Proof. From the definition of the optimal ordering for the discrete NP (1.9) and concavity of the expected profit obviously follows that the optimal ordering is $x^* = \mu$. Optimality is guaranteed thanks to the presence of the single possible demand value. ❑

Worst distribution

We introduce a simulation in order to find the worst possible distribution $W(x)$ with respect to the ordering x. Then, from the results, we might find a closed form of the worst cdf $F_{W(x)}$. Moreover, we assume a special case $n = 10$.

Since we assume the discrete random demand ω, we can rewrite the expected profit as

$$\Pi(x) = \mathsf{E}_{\mathsf{P}}[\pi(x;\omega)] = \sum_{i=0}^{10} \pi(x;\omega_i)q_i = \sum_{i=0}^{10} \pi(x;i)q_i, \tag{4.4}$$

where ω acquires values from $\omega_0 = 0$ to $\omega_{10} = 10$ with probabilities $q_0, \ldots, q_{10}$ for which hold $\forall i \in \mathbb{N}_0^{10} : q_i \geq 0$ and $\sum_{i=0}^{10} q_i = 1$. Allowing probabilities q_i to attain zero value yield that the worst possible distribution $W(x)$ might have less than 11 atoms. Equation (4.4) may be reformulated as

$$\Pi(x) = (p-c)x - (p-v) \sum_{\{i:\omega_i \leq x\}} (x-\omega_i)q_i = (p-c)x - (p-v) \sum_{\{i \leq x\}} (x-i)q_i. \quad (4.5)$$

Hence, the objective of finding the worst distribution for given x is determined from (4.5) as

$$\min_{q_0,\cdots,q_{10}} (p-c)x - (p-v) \sum_{\{i \leq x\}} (x-i)q_i, \quad (4.6)$$

where probabilities $q_0, \ldots, q_{10}$ are the decision variables because all the other parameters are known (e.g. the mean and values of ω). The objective (4.6) can be then simplified to

$$\max_{q_0,\cdots,q_{10}} \sum_{\{i \leq x\}} (x-i)q_i. \quad (4.7)$$

Now, we have to add constraints to (4.7). More precisely, the properties of the discrete random variable probabilities:

$$q_i \geq 0, \ \forall i = 0, \ldots, 10 \quad \& \quad \sum_{i=0}^{10} q_i = 1; \quad (4.8)$$

the properties of the distribution drawn from the ambiguity set $\mathcal{P}$ given by (4.1):

$$\mu = \sum_{i=0}^{10} i q_i \quad \& \quad \sigma^2 \geq \sum_{i=0}^{10} \left(i - \mu\right)^2 q_i; \quad (4.9)$$

and, lastly, conditions to guarantee that x is the optimal decision with respect to the discrete random variable given by (1.9):

$$\sum_{i=0}^{x} q_i \geq \frac{p-c}{p-v} \quad \& \quad \sum_{i=0}^{x-1} q_i < \frac{p-c}{p-v}. \quad (4.10)$$

Combining the objective function (4.7) and constraints (4.8)-(4.10) gives the following deterministic linear optimization program for given x:

$$\begin{aligned} \underset{q_0,\cdots,q_{10}}{\text{maximize}} \quad & \sum_{\{i \leq x\}} (x-i)q_i \\ \text{subject to} \quad & q_i \geq 0, \quad i = 0, \ldots, 10, \\ & \sum_{i=0}^{10} q_i = 1, \\ & \sum_{i=0}^{10} i q_i = \mu, \\ & \sum_{i=0}^{10} i^2 q_i \leq \sigma^2 + \mu^2, \\ & \sum_{i=0}^{x} q_i \geq \frac{p-c}{p-v}, \\ & \sum_{i=0}^{x-1} q_i < \frac{p-c}{p-v}. \end{aligned} \quad (4.11)$$

Example. Consider the model (4.11) with parameters $p = 10, c = 7, v = 1, \mu = 5$ and $\sigma^2 = 16$. Then $\frac{p-c}{p-v} = \frac{1}{3} \approx 0.333$. Let $\tilde{\sigma}^2$ is the variance of the optimal solution and let Q^x and Q_x correspond to the sums $\sum_{i=0}^{x} q_i$ and $\sum_{i=0}^{x-1} q_i$, respectively. The table 4.1 gives results for orderings $x = 0, \dots, 10$.

Firstly, as we can see in table 4.1, for $x \geq 8$ we cannot find a feasible solution under the given conditions. Even increasing the variance σ^2 of $\mathcal{P}$ would not help as the problem lies in the support size and the mean value of $\mathcal{P}$. An increment in variance would only change optimal probabilities and increase the objective value for items $x \in \{1 \dots, 5\}$ since the variance limitation constraint is active in these cases, i.e. $\sum_{i=0}^{10} i^2 q_i = \sigma^2 + \mu^2$.

Furthermore, it is obvious from the optimal distribution probabilities and values of Q_x that the rounding error is present in the solution. For instance the optimal (worst) demand distribution in case the newsvendor orders $x = 3$ items is approximately

$$W = \begin{cases} 0 & \text{with probability } q_0 \to \frac{1}{3}^-, \\ 3 & \text{with probability } q_3 \to 0^+, \\ 5 & \text{with probability } q_5 = \frac{4}{15}, \\ 6 & \text{with probability } q_6 = \frac{1}{9}, \\ 10 & \text{with probability } q_{10} = \frac{13}{45}, \end{cases}$$

where $y \to g^+ = \lim_{y \to g^+} y$ and $y \to g^- = \lim_{y \to g^-} y$ are one-sided limits. Probability q_3 must be positive and q_0 less than $\frac{1}{3}$ in order to the constraint $\sum_{i=0}^{x-1} q_i < \frac{p-c}{p-v} \leq \sum_{i=0}^{x} q_i$ being satisfied.

The R code for finding the optimal solution of model (4.11) is provided in the attachment (electronic version only).

Example. Again consider the model (4.11). The parameters are $p = 5, c = 3, v = 2, \mu = 6$ and $\sigma^2 = 12$ this time. Thus $\frac{p-c}{p-v} = \frac{2}{3} \approx 0.667$. The notation used is the same as in the previous example. The table 4.2 gives results for orderings $x = 0, \dots, 10$.

Again, orderings $x \in \{0, \dots, 3\}$ are not solvable under given parameters. This time the rounding problem does not occur.

The optimal (worst) demand distribution in case the newsvendor orders $x = 6$ items is

$$W = \begin{cases} 1 & \text{with probability } q_1 = \frac{4}{15}, \\ 6 & \text{with probability } q_6 = \frac{4}{10}, \\ 10 & \text{with probability } q_{10} = \frac{1}{3}. \end{cases}$$

This time the probabilities of W are given precisely.

Tables 4.1 and 4.2 does not indicate there would exist a closed form representation of the worst demand distribution $W(x)$ with or without respect to x or any other variable.

x	sol[a]	q_0	q_4	q_5	q_6	q_7	q_8	q_{10}	$\tilde{\sigma}^2$	Q^x	Q_x
0	y	.333	.083	0	0	0	.583	0	13.7	.333	0
1	y	.333	0	.267	0	.111	0	.289	16	.333	.333
2	y	.333	0	.267	0	.111	0	.289	16	.333	.333
3	y	.333	0	.267	0	.111	0	.289	16	.333	.333
4	y	.333	.083	0	.292	0	0	.292	16	.417	.333
5	y	.333	0	.267	0	.111	0	.289	16	.6	.333
6	y	.333	0	0	.417	0	0	.25	15	.75	.333
7	y	.333	0	0	0	.556	0	.111	13.3	.889	.333
8	n										
9	n										
10	n										

Note: [a] Indicator whether the model (4.11) has feasible solution (y=yes;n=no).

Table 4.1: The table of results of model (4.11) with given parameters ($p = 10, c = 7, v = 1, \mu = 5$ and $\sigma^2 = 16$). Probabilities q_1, q_2, q_3 and q_9 are omitted as they are equal to zero.

x	sol[a]	q_1	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}	$\tilde{\sigma}^2$	Q^x	Q_x
0	n												
1	n												
2	n												
3	n												
4	y	0	0	.667	0	0	0	0	0	.333	8	.667	0
5	y	.167	0	0	.5	0	0	0	0	.333	10	.667	.167
6	y	.267	0	0	0	.4	0	0	0	.333	12	.667	.267
7	y	.167	.25	0	0	0	.25	0	0	.333	12	.667	.417
8	y	.19	0	.333	0	0	0	.143	0	.333	12	.667	.524
9	y	.167	0	.4	0	0	0	0	.1	.333	12	.667	.567
10	y	.148	0	.444	0	0	0	0	0	.407	12	1	.593

Note: [a] Indicator whether the model (4.11) has feasible solution (y=yes;n=no).

Table 4.2: The table of results of model (4.11) with given parameters ($p = 5, c = 3, v = 2, \mu = 6$ and $\sigma^2 = 12$). Probabilities q_0 and q_2 are omitted as they are equal to zero.

Chapter 5

Numerical example

In this chapter we show the data-driven NP reformulation. More precisely, we give reformulation of the basic single-period newsvendor problem (NP; see section 1) and the newsvendor problem with pricing (NPP; see section 2.1). At the end, we compare results of the NP and the NPP approach. The method used to apply the real data on the NP and the NPP is called Sample Average Approximation (SAA).

For purpose of the practical example we use the data about the sales of the specific model and brand of wet wipes from a major Czech e-commerce retailer in the period from the mid-October of 2014 to the mid-April of 2018. The wet wipes appear to be a product whose overall demand is constant over the year and advertising or price policy are main drivers for changes in the sales numbers. The collected data contain information about the selling price per item p_t, the purchasing cost per item c and the number of products that were sold ω_t, where $t = 1, \ldots, 182$ represents the week in the given period. In order to get a complete weekly observation we require to have availability of the sales numbers for every day of the week. The selling price p_t is aggregated via the weighted average, where weight is the order quantity, since the daily and other discounts are present in the data. On the other hand, the purchasing cost c is the average value of costs over the whole period as the number of items bought for given cost is not provided. The weekly demand ω_t is simply the sum of order volumes for the given week. Moreover, we know that ω_t is indeed the demand since it never happened that the retailer would run out of stock.

In figure 5.1 we can see how the per-item price effects the demand. It is obvious that wet wipes are price sensitive product which is probably caused by vast number of competitive products and substitutes.

NP reformulation

Consider the NP as in section 1. In case the distribution of demand ω is known, we can easily use cdf F to compute the optimal solution x^*. However, in many practical applications the demand distribution is not known. We have already shown one approach to use real data for finding the optimal solution, the DRNP in section 4. Another approach is to apply the SAA method that is commonly used in such cases.

Suppose we have a random sample $\{\chi_1, \ldots, \chi_T\}$ where $\chi_1, \ldots, \chi_T$ are independently drawn from the distribution of ω and T is the sample size. The main

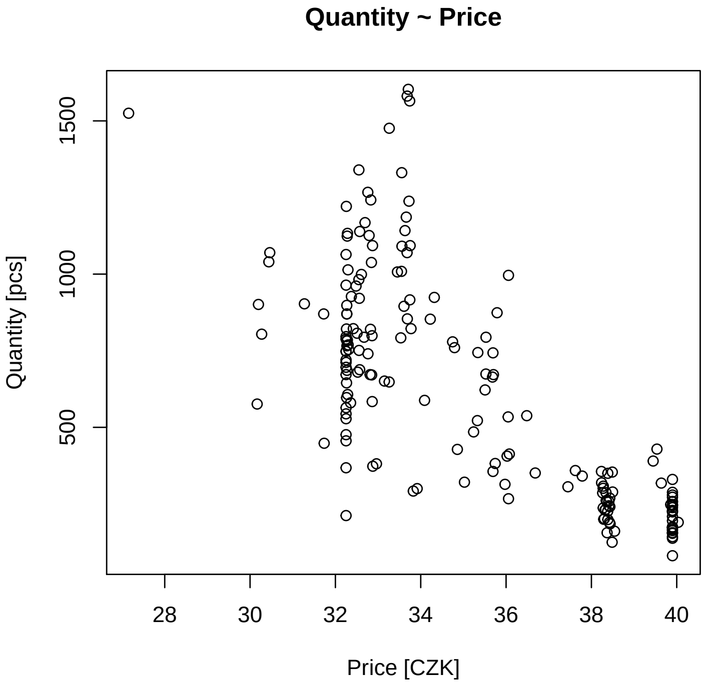

Figure 5.1: The demand and price scatter plot for wet wipes.

idea of the SAA method is to approximate the real distribution with the empirical cumulative distribution function. Based on the particular sample, the empirical distribution is formed by putting a weight of $\frac{1}{N}$ on each of the demand data. Formally, we denote the empirical cdf as $\hat{F}_T(x) = \frac{1}{T}\sum_{k=1}^{T} \mathbb{1}_{[\chi_k \le x]}$, where $\mathbb{1}_{[\cdot]}$ is the indicator function. Denote the profit expected expressed by the empirical cdf as $\hat{\Pi}_T$. Then, the resulting objective function is maximized.

Let us have T independent realizations of the random sample from the real distribution $\mathbb{Q}$ which we denote $\{\chi_1, \dots, \chi_T\}$. Then, by applying the SAA method, we obtain

$$\max_{x \ge 0} \hat{\Pi}_T(x) = \frac{1}{T}\sum_{t=1}^{T}[(p-c)x - s(\chi_t - x)^+ - (p-v)(x-\chi_t)^+]. \tag{5.1}$$

The optimal solution of (5.1) is again $\frac{p-c+s}{p-v+s}$-quantile. Thus, x_T^* is the $\frac{p-c+s}{p-v+s}$-quantile of the random sample:

$$x_T^* = \max\left\{x : \hat{F}_T(x) \le \frac{p-c+s}{p-v+s}\right\}. \tag{5.2}$$

Note that x_T^* is a random variable since its value depends on the particular realization of the random sample.

The reason why the SAA method is suitable for the newsvendor type problems is because the solution can be found effectively by ordering the random sample. Moreover, as it shown in (Šedina, 2015, Theorem 5), the optimal quantile of empirical distribution (5.2) converges almost surely (a.s.) to the optimal quantile of true distribution $\mathbb{Q}$ of the original NP.

Due to the manner of our data we modify (5.1) so that it fits our case the most. Firstly, we set the shortage penalty and salvage value to zero, i.e. $s = 0$ and $v = 0$. Then, since we need a fixed selling price for the whole period, we set $p = \frac{p_1\omega_1+\ldots+p_T\omega_T}{\omega_1+\ldots+\omega_T}$ which is the weighted average. Moreover, the per-item cost is $c = 24.02$. Then for $\chi_t = \omega_t$ the model (5.1) turns into

$$\max_{x\geq 0} \frac{1}{T}\sum_{t=1}^{T}[(p-c)x - p(x-\omega_t)^+]. \tag{5.3}$$

We do not execute the model (5.3) in this paper since the optimal ordering decision is the quantile of the empirical cdf (5.2). However, the model (5.3) is still not implementable because it contains the function of positive part of number $(\cdot)^+$. See the next subsection on how to turn (5.3) into the model that could be solved in GAMS software - the optimization solving tool.

NPP reformulation

Consider the NPP as in section 2.1. We have to fit the data in figure 5.1 in order to obtain a demand function $\omega(p;\epsilon)$ with respect to the price. We can pick from an additive demand function (2.2) and a multiplicative demand function (2.1).

The additive model $\omega(p,\epsilon) = \alpha - \beta p + \epsilon$ derived in (2.2) and (2.4) performs poorly and many assumptions on the linear model residuals are not heavily satisfied (e.g. normality, homoscedasticity, independence). The performance measure R^2 has a value of 0.57.

The multiplicative model $\omega(p,\epsilon) = \alpha p^{-\beta}\epsilon$ derived in (2.1) and (2.3), that is equivalent with the linear model

$$\log(\omega(p,\epsilon)) = \tilde{\alpha} - \tilde{\beta}\log(p) + \tilde{\epsilon} \tag{5.4}$$

using the log-transformation, performs better than the additive model. The performance measure R^2 attains a value of 0.68. The normality of errors is probably satisfied since the Shapiro-Wilk test's p-value is 0.33 and hence we do not reject the null hypothesis of normally distributed errors at the 5% confidence level. Hereafter, the heteroscedasticity is most likely not present in the errors since both the studentized and nonstudentized Breusch-Pagan test's null hypothesis is not rejected at the 5% level.

However, the errors are correlated. For instance, the Durbin-Watson test statistic that measures the first order correlation is equal to 0.96. Moreover, the autoregressive function reveals the fourth order correlation of the errors as well. Therefore, an addition of lagged response variables to (5.4) solves the dependence of errors. Nevertheless, it also causes the violation of the normality assumption (although we could rely on the asymptotic properties). Despite the fact we would probably obtain a better model, we use the model (5.4) mindful of the presence of error's correlation because it is consistent with the theoretical results derived in section 2.1 (NPP). The description of tests used to validate the model is in Komárek (2017).

After fitting the model (5.4) we obtain that the scale parameter

$$\alpha = e^{\tilde{\alpha}} = e^{28.71} \approx 2.94 \cdot 10^{12}$$

and the power parameter for price p is

$$\beta = \tilde{\beta} = 6.32.$$

With respect to the figure 5.1 we set the lower and upper bound on price as $p_l = 28$ and $p_u = 40$

Then the model (2.6) with zero shortage penalty and salvage value after applying the SAA method changes to

$$\begin{aligned} &\max_{x;p} \frac{1}{T}\sum_{t=1}^{T}\left[(p-c)x - p\left(x - \alpha p^{-\beta}\epsilon_t\right)^+\right] \\ &\text{s.t.} \quad x \geq 0,\ p \in [p_l, p_u], \end{aligned} \tag{5.5}$$

where $\epsilon_t = e^{\tilde{\epsilon}_t}$.

However, the model (5.5) still contains the positive number function $(\cdot)^+$. In order to obtain a model without the positive part function we have to introduce a series of real valued decision variables τ_t. Hence, the model (2.21) changes to the nonlinear programming (NLP) problem as follows:

$$\begin{aligned} \underset{x;p;\tau_1,\dots,\tau_T}{\text{maximize}} \quad & \frac{1}{T}\sum_{t=1}^{T}[(p-c)x - p\tau_t] \\ \text{subject to} \quad & x \geq 0, \\ & p \leq p_u, \\ & p_l \leq p, \\ & \tau_t \geq x - \alpha p^{-\beta}\epsilon_t, \qquad t = 1,\dots,T, \\ & \tau_t \geq 0, \qquad t = 1,\dots,T. \end{aligned} \tag{5.6}$$

Decision variables τ_t serve as the decision whether the $x - \alpha p^{-\beta}\epsilon_t$ is greater than zero for given t. We select the larger value due to the maximization type of the programming problem. The model (5.6) is now implementable in the GAMS software and the code is provided in the attachment (electronic version only).

Optimal ordering comparison

Results of models (5.3) and (5.6) for given parameters are visualized in the table 5.1.

The resulting optimal prices differ significantly. However, the difference between the optimal ordering is even greater. The optimal solution of the NPP model (5.6) suggests the newsvendor to order more than three times more items than the NP model (5.3) and sell them for roughly 10 % lower price. If we compare the result of the NP model (5.3) with the figure 5.1 we can conclude that the NP optimal ordering is fairly very conservative for the given price. The riskless price for the multiplicative NPP is 28.54.

	x^*	p^*
NP	300.46	33.82
NPP	1064.23	29.45

Table 5.1: The table of results of models (5.3) and (5.6). Note that the price p is fixed in the NP.

Conclusion

This thesis provides an overview of the newsvendor problem models. In the first chapter the classical newsvendor problem is analyzed. We assume the continuous and discrete demand distribution and for both of them the closed form optimal solution is derived. In the rest of the thesis the continuous demand is assume if not specified otherwise.

The next chapter adds to the problem description a parameter-dependent demand for which the optimal ordering and parameter are jointly seeked. Specifically, the parameter is price, advertising or both together. In each case, the additive and the multiplicative dependency of the parameter on the demand is assumed. For each case the relation to the riskless problem is provided. Moreover, we show how would the variance and the coefficient of variation of profit change in case the parameter is shifted.

In the third chapter we assume a risk-averse newsvendor whose aversion is represented by the CVaR of the profit. We form several models that are common in the multicriteria stochastic optimization and give a closed form solution in case the objective is tradeoff between the expected profit and the riskiness. However, finding the weight between the two terms is crucial and an effective mechanism should be developed. The chapter continues with the section where we assume the mean-CVaR model which demand is price-dependent. The last part is devoted to the multiproduct extension depending upon the dependency between random demands. For the independent demand case we provide an algorithm to find an optimal solution of the budget constrained model from the optimal ordering of the unconstrained newsvendor model. Moreover, the solution of the algorithm is proven to be optimal.

The fourth chapter deals with the distributionally robust newsvendor problem, where the ambiguity set is given by its support, first order moment and limitation on the second order moment. For the given ambiguity set we provide the best possible distribution, that is degenerate and its only value is equal to the given mean. First, we assume that the worst distribution depends in some way on the ordering amount and run couple of simulations to see whether there is any pattern in the distribution giving the lowest profit for given ordering. However, the results do not indicate any systematic behaviour.

The last chapter devotes to the practical example with given dataset from a major Czech e-commerce company. The observations are included in the model with the SAA method which idea is to approximate the true distribution with the empirical cdf. The classical newsvendor problem and the model with pricing are investigated. For the NP model, the optimal solution is a quantile of the empirical cdf. However, the optimal solution of the NPP model must be found using the optimization software. We find out, interestingly, that the ability to choose the pricing policy causes that the optimal decision is to lower the price and increase significantly the number of ordered items compared to the classical model.

Throughout the paper we discover that the results are consistent among the models and a more complex formulation of the model is a generalization of the simpler model of which it is derived from.

Applicability of the newsvendor problem and its modifications is wide. The newsvendor type models might be very helpful tool in production planning (e.g. production of vaccines where shortage might cause death and leftovers are destroyed for large expenses) or retail with perishable products (e.g. fashion, food with a short shelf life, etc.).

Further research might contain e.g. involving the price-dependency in the distributionally robust model and/or extending the framework for multiple periods.

Bibliography

P. Artzner, F. Delbaen, J.M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9:203–228, 1999.

R. E. Barlow and F. Proschan. *Mathematical theory of reliability.* Classics in applied mathematics 17. Society for Industrial and Applied Mathematics, 1987. ISBN 0-898-71369-2.

Y.F. Chen, M. Xu, and Z.G. Zhang. A Risk-Averse Newsvendor Model Under the CVaR Criterion. Technical report, 2007. URL `https://pdfs.semanticscholar.org/f2b3/91ab22787892dc36ef62ce6c3bb6552e5938.pdf`.

Y.F. Chen, M. Xu, and Z.G. Zhang. Technical Note - A Risk-Averse Newsvendor Model Under the CVaR Criterion. *Operations Research*, 57, 07-08 2009.

T.M. Choi, editor. *Handbook of Newsvendor Problems : Models, Extensions and Applications*, volume 176 of *International Series in Operations Research & Management Science*. Springer, 2012.

J. Dai and W. Meng. A risk-averse newsvendor model under marketing-dependency and price-dependency. *International Journal of Production Economics*, 160:220–229, 2015.

J. Dupačová. *Stochastické programování.* První vydání. Ministerstvo školství ČSR, Praha, 1986. (In Czech).

J. Dupačová, J. Hurt, and J. Štěpán. *Stochastic Modeling in Economics and Finance*, volume 75 of *Applied Optimization.* Kluwer Academic Publishers, 1st edition, 2003. ISBN 1-4020-0840-6.

J. Gotoh and Y. Takano. Newsvendor solutions via conditional value-at-risk minimization. *European Journal of Operational Research*, 179, 2007.

G. A. Hanasusanto, V. Roitch, D. Kuhn, and W. Wiesemann. A Distributionally Robust Perspective on Uncertainty Quantification and Chance Constrained Programming. *Mathematical Programming*, 151, 6 2015.

D. Hrabec. Mathematical Programs for Dynamic Pricing - Demand Based Management. Ph.d thesis, Brno University of Technology, Brno, 2016.

D. Hrabec, P. Popela, J. Novotný, and A. Olstad. A note on the newsvendor problem with pricing. In *Proceedings of 18th International Conference on Soft Computing MENDEL*, pages 410–415, 2012. ISBN 978-80-214-4540-6.

D. Hrabec, K. K. Haugen, and P. Popela. The newsvendor problem with advertising: an overview with extensions. *Review of Managerial Science*, 7 2016.

W. Jammernegg and P. Kischka. Risk-averse and risk-taking newsvendors: a conditional expected value approach. *Review of Managerial Science*, 1, 04 2007.

S. Karlin and S. R. Carr. Prices and Optimal Inventory Policy. *Studies in Applied Probability and Management Science*, pages 299–336, 1962.

M. Khouja. The single-period (news-vendor) problem: literature review and suggestions for future research. *Omega*, 27, 1999.

A. Komárek. Linear Regression Course Notes, December 2017. URL `http://www.karlin.mff.cuni.cz/~komarek/vyuka/2017_18/nmsa407/2017-NMSA407-notes.pdf`.

R. Levi, G. Perakis, and J. Uichanco. The Data-Driven Newsvendor Problem: New Bounds and Insights. *Operations Research*, 10 2015.

E. S. Mills. Uncertainty and price theory. *The Quarterly Journal of Economics*, 73:116–130, 1959.

W. Ogryczak and A. Ruszczýnski. Dual stochastic dominance and related mean-risk models. *SIAM Journal of Optimization*, 13:60–78, 2002.

N. C. Petruzzi and M. Dada. Pricing and the newsvendor problem: a review with extensions. *Operations Research*, 47:183–194, 1999.

G. Ch. Pflug and W. Römisch. *Modeling, Measuring and Managing Risk*. World Scientific, 2007. ISBN 978-981-270-740-6.

E. L. Porteus. *Foundations of stochastic inventory theory*. Stanford University Press, 2002. ISBN 0-804-74399-1.

Y. Qin, R. Wang, A. J. Vakharia, Y. Chen, and M. M. H. Seref. The newsvendor problem: Review and directions for future research. *European Journal of Operational Research*, 213, 2011.

R. Rockafellar and S. Uryasev. Conditional value-at-risk for general loss distribution. *Journal of Banking & Science*, 26, 2002.

E. A. Silver and R. Peterson. *Decision Systems for Inventory Management and Production Planning*. John Wiley, New York, 1985. ISBN 0-471-86782-9.

J. Šedina. Problém prodavače novin. Bachelor thesis, Charles University, Prague, 2015. (In Czech).

T. Wang and Q. Hu. Coordination of supply chain with advertise-and-price-setting newsvendor. In *Control and decision conference (CCDC), Shenyang, China*, pages 885–890, 2011.

T. M. Whitin. Inventory control and price theory. *Management Science*, 2:61–68, 1955.

W. Wiesemann, D. Kuhn, and M. Sim. Distributionally Robust Convex Optimization. *Operations Research*, 62, 12 2014.

M. Wu, S.X. Zhu, and R.H. Teunter. The risk-averse newsvendor problem with random capacity. *European Journal of Operational Research*, 231, 2013.

M. Xu and J. Li. Optimal decisions when balancing expected profit and conditional value-at-risk in newsvendor models. *Journal of Systems Science and Complexity*, 23, 12 2010.

L. Yao, Y.F. Chen, and H. Yan. The newsvendor problem with pricing: Extensions. *International Journal of Management Science and Engineering Management*, 1:3–16, 2006.

R.L. Zhan and Z.-J.M. Shen. Newsvendor Problem with Pricing: Properties, Algorithms, and Simulation. In *Proceedings of the Winter Simulation Conference*, Orlando, 2005. ISBN 0-7803-9519-0.

List of Figures

List of Tables

List of Abbreviations

cdf cumulative distribution function

CVaR conditional value at risk

DRNP distributionally robust newsvendor problem

GSIFR generalized strict increasing failure rate

IFR increasing failure rate

IPE increasing price elasticity

LP linear programming

MDPM marketing-dependent price-multiplicative

NP newsvendor problem

NPA newsvendor problem with advertising

NPP newsvendor problem with pricing

NPPA newsvendor problem with pricing and advertising

SAA sample average approximation