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Optimality of function spaces for classical integral operators

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Abstract: We investigate optimal partnership of rearrangement-invariant Banach function spaces for the Hilbert transform and the Riesz potential. We establish sharp theorems which characterize optimal action of these operators on such spaces. These results enable us to construct optimal domain (i.e. the largest) and optimal range (i.e. the smallest) partner spaces when the other space is given. We illustrate the obtained results by non-trivial examples involving Generalized Lorentz–Zygmund spaces with broken logarithmic functions. The method is presented in such a way that it should be easily adaptable to other appropriate operators.

Keywords: rearrangement–invariant spaces, integral operators, optimality, Hilbert transform, Riesz potential

I would like to express my sincere gratitude to Luboš Pick for his patience, enthusiasm, friendly attitude, and helpfulness, which helped me immeasurably. Above all, if it had not been for him, I might have never discovered the beauty of the theory of function spaces.

Contents

Introduction

In this thesis, we address the question of an optimal partnership of function spaces with respect to classical operators of harmonic analysis. We shall work in the general setting of so-called rearrangement-invariant function spaces. Our main goal is to establish sharp theorems guaranteeing boundedness of given classical operators on such function spaces, which are optimal in the sense that they cannot be essentially improved within this category of function spaces. We shall treat in detail namely the Hilbert transform and the Riesz potential. Both these operators have been known for very long time to be indispensable in various parts of mathematics including the theory of PDEs, spectral theory, complex analysis, Fourier analysis, mathematical physics, probability theory and more. While the action of these operators on Lebesgue spaces is well known, it is known that the scale of Lebesgue spaces, although clearly still playing a primary role in many parts of mathematical analysis, is not rich enough in order to describe all important cases, in particular in various limiting situations. Over the years, it has become evident that other (finer) scales of function spaces are also of interest, in fact necessary, when certain critical or limiting cases of the action of operators are in question. These new scales include as pivotal examples two-parametric Lorentz spaces, Orlicz spaces and Zygmund classes, but various specific tasks enforced the authors to study yet finer function spaces. Let us mention as particular examples the Lorentz-Zygmund spaces (together with their various generalizations), the Lorentz-Karamata spaces, the Gould space, function spaces obtained via interpolation, or Lorentz and Marcinkiewicz endpoint spaces.

It turns out that rather than face the specific difficulties connected to each of the spaces, it is more rewarding to work in the fairly general setup of rearrangement-invariant function spaces, which is a category which covers all of the above-mentioned types of function spaces (and more), providing thereby a general framework for a thorough and comprehensive study.

The key question studied in this thesis can be formulated as follows. Given an operator T and a rearrangement-invariant space X (or Y), find another such space Y (or X) with the property that T is defined on X, bounded from X to Y, and if there is another rearrangement-invariant space Z such that T is bounded from X to Z (or from Z to Y), then Y is contained in Z (or Z is contained in X). The latter property is what we call an "optimal partnership". If we are able to find such a space Y (or X), then it automatically becomes the smallest range space (or the largest domain space) which renders the boundedness of the operator true in the scope of rearrangement-invariant spaces. Let us note that such a result provides us with extremely useful information that there is no point in trying to find a better range (or domain) partner rearrangement-invariant space for the given operator because there cannot exist any.

Even though the subject of our interest is different, techniques of proofs are up to some extent built upon those that have been developed mostly during the last two decades for similar results concerning Sobolev embeddings, trace embeddings and logarithmic Sobolev inequalities (cf. e.g. [9, 4, 16, 5, 6]), with one important difference: our operators are defined on functions whose domain is $\mathbb{R}^n$, that is, a space of infinite measure. We utilize these methods combined with some new arguments developed here and some classical arguments from the theory of rearrangement-invariant spaces and also with inequalities in the spirit of Herz-Wiener inequality, as sharp as possible, and appropriately chosen for each operator in question. The technique is up to some extent explained in [23]. Let us also note that some partial results concerning optimal function spaces for classical operators were established in [26].

The thesis is structured as follows. In Chapter 1, we collect all the necessary background theory to be used throughout the text. Even though comprehensive standard references ([2, 24]) are available, we give this preliminary part in detail. Our main results are then contained in Chapter 2, where optimal partner spaces are characterized for the Hilbert transform (namely Theorem 2.13, Theorem 2.18 and Theorem 2.19), and Chapter 3, where basically the same things are established for the Riesz potential (namely Theorem 3.14 and Theorem 3.19). We illustrate the results with examples (namely Examples 2.16, Example 2.21 and Examples 3.17) in which the principal role is played by the Generalized Lorentz-Zygmund spaces with broken logarithmic functions. This choice was made because of the great versatility and applicability of such spaces.

1. Preliminaries

Conventions. Throughout the entire thesis, (R, μ) denotes a $\sigma-$finite measure space if not stated otherwise. As usual, we identify any two functions which coincide $\mu-$a.e. Furthermore, we follow the rule that

$$0 \cdot \infty = \infty \cdot 0 = 0.$$

If M, N are two quantities, we write $M \lesssim N$ if there exists a positive constant C independent of "all important parameters" such that

$$M \leq C \cdot N.$$

Obviously, one can use this abbreviation only when it is perfectly clear what the "all important quantities" are. We also write $M \gtrsim N$ with the obvious meaning. We write $M \approx N$ if simultaneously $M \lesssim N$ and $M \gtrsim N$.

Notation 1.1. *We denote*

$$\mathfrak{M}\,(R, \mu) = \{f; f \textit{ is a} \mu\textit{-measurable function on } R \textit{ whose values are in } [-\infty, \infty]\},$$
$$\mathfrak{M}^+\,(R, \mu) = \{f \in \mathfrak{M}\,(R, \mu); f \geq 0\}.$$

If no confusion is possible, we will simply write $\mathfrak{M}$ *or* $\mathfrak{M}^+$*. We also denote*

$$\mathfrak{M}_0 = \{f \in \mathfrak{M}; f \textit{ is finite } \mu\textit{-a.e.}\},$$
$$\mathfrak{M}_0^+ = \{f \in \mathfrak{M}_0; f \geq 0\}.$$

1.1 Banach Function Spaces

Our goal is to find optimal function spaces for some classical integral operators but one needs to specify what we mean by function spaces at the very beginning. More precisely, we must decide what class of function spaces we consider and then we can pursue the question of optimality in the scope of the given class. The class of function spaces which we would like to consider should, on the one hand, come with theory which is nice enough so that we could conduct all the necessary steps for constructing optimal function spaces, while, on the other hand, the class should be as general as possible containing many classical function spaces. It turns out that a suitable choice is the class of rearrangement-invariant Banach function spaces.

We begin by providing some fundamentals of the theory of (rearrangement-invariant) Banach function spaces. Proofs of the facts listed here can be found in [2, Chapter 1 and Chapter 2] or [24, Chapter 6 and Chapter 7].

Definition 1.2. We say that a mapping $\varrho : \mathfrak{M}^+(R, \mu) \longrightarrow [0, \infty]$ is a *Banach function norm* (shortly a *function norm*) if the following seven conditions hold:

1. $\varrho(f) = 0 \Leftrightarrow f = 0\ \mu -$ a.e.,
2. $\varrho(\alpha f) = \alpha \varrho(f)$,

3. $\varrho(f+g) \leq \varrho(f) + \varrho(g)$,
4. $f \leq g \ \mu -$ a.e. $\Rightarrow \varrho(f) \leq \varrho(g)$,
5. $f_n \uparrow f \ \mu -$ a.e. $\Rightarrow \varrho(f_n) \uparrow \varrho(f)$,
6. $\mu(E) < \infty \Rightarrow \varrho(\chi_E) < \infty$,
7. $\mu(E) < \infty \Rightarrow \int_E f \, d\mu \leq C_E \varrho(f)$,

where $n \in \mathbb{N}$, $f, g, f_n \in \mathfrak{M}^+(R, \mu)$, $\alpha \in [0, \infty)$, E is a $\mu-$measurable subset of R, and C_E is a positive constant which may depend on E (and ϱ) but does not depend on f.

Definition 1.3. For a given function norm ϱ on (R, μ), we set

$$X = X(\varrho) = \{f \in \mathfrak{M}(R, \mu); \varrho(|f|) < \infty\}$$

and define

$$\|f\|_X = \varrho(|f|), \quad \forall f \in X.$$

The pair $(X, \|\cdot\|_X)$ is then called a *Banach function space.*

Remark 1.4. We shall refer to the mere set X from the above definition as a Banach function space. Furthermore, if no confusion is possible, we shall simply write $\|\cdot\|$ instead of $\|\cdot\|_X$.

As the name itself indicates, Banach function spaces are Banach spaces in the functional-analytic sense. Moreover, they always contain simple functions and consequently they are non-trivial (unless the measure space itself is trivial).

Theorem 1.5. *Let ϱ be a function norm on (R, μ) and let X be as in Definition 1.3. Then*

$$S \subseteq X \subseteq \mathfrak{M}_0(R, \mu),$$

where S is the set of all simple functions on R, and $(X, \|\cdot\|_X)$ is a complete vector space (under the natural vector operations). In particular, $\|\cdot\|_X$ is a norm on X.

Remark 1.6. By Definition 1.2, a Banach function norm is supposed to be defined on all functions from $\mathfrak{M}^+(R, \mu)$. Theorem 1.5 shows, however, that functions which are not finite a.e. are of no interest in the scope of Banach function spaces. Therefore, one can define a Banach function norm only on $\mathfrak{M}_0^+(R, \mu)$ and assume that it equals ∞ on $\mathfrak{M}^+(R, \mu) \setminus \mathfrak{M}_0^+(R, \mu)$. We shall follow this convention throughout this thesis.

Theorem 1.7. *Let $X = X(\varrho)$ be a Banach function space. Then for $f, g \in \mathfrak{M}(R, \mu)$ and μ-measurable $E \subseteq R$, it holds that:*

1. *If $|f| \leq |g| \ \mu-a.e.$ and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$. In particular, $f \in X$ if and only if $|f| \in X$ and in that case $\|f\|_X = \||f|\|_X$.*
2. *There exists a positive constant C_E independent of f such that*

$$\int_E |f| \, d\mu \leq C_E \|f\|_X.$$

The way function spaces are embedded in each other is of great interest in the theory of function spaces. The following theorem shows that the set-theoretic inclusion is enough for Banach function spaces to be continuously embedded.

Theorem 1.8. *Let X and Y be Banach function spaces over the same measure space. If $X \subseteq Y$, then $X \hookrightarrow Y$, that is, there exists a positive constant C such that*

$$\|f\|_Y \le C\|f\|_X$$

for each $f \in X$.

Example 1.9. The class of Banach function spaces contains several important function spaces. Namely, Lebesgue spaces ($p \in [1, \infty]$), Lorentz spaces, Orlicz spaces (see [24, Chapter 4]), Lorentz-Zygmund spaces, endpoint spaces, Morrey spaces (see [24, Chapter 5]) and more. Lebesgue spaces, Lorentz spaces as well as Lorentz-Zygmund spaces will be treated as special instances of so-called *Generalized Lorentz-Zygmund spaces* in Section 1.6. Endpoint spaces are covered in Section 1.5.

Unfortunately, not all important function spaces are Banach function spaces. For example, spaces of smooth functions $C^k(\overline{\Omega})$ and $C^{k,\alpha}(\overline{\Omega})$ (see [24, Chapter 2]) are Banach spaces which are not Banach function spaces. Further, whereas Morrey spaces are Banach function spaces, quite similar Campanato spaces (see [24, Chapter 5]) are not. Probably the most important example of function spaces which are not Banach function spaces are classical Sobolev spaces $W^{m,p}(\Omega)$ (see [1, Chapter 3]), which are an indispensable tool for the modern theory of partial differential equations.

1.2 The Associate Space

As Banach function spaces are Banach spaces, one can talk about their dual spaces. However, the dual space of a Banach space (even of a Banach function space) can behave badly. A classical example is the space L^∞. Fortunately, when dealing with Banach function spaces, we can often substitute the dual space with the associate space, which we are about to define. The associate space behaves, in general, better than the dual space and simultaneously it is always canonically isometric to a closed norm-fundamental subspace of the dual space (see [2, Chapter 1, Theorem 2.9]), thus emerging often as a convenient substitute of the dual space in many argumentations.

Definition 1.10. Let ϱ be a function norm. We define its *associate norm* ϱ' as

$$\varrho'(g) = \sup\{\int_R fg\,d\mu; f \in \mathfrak{M}^+(R,\mu), \varrho(f) \le 1\}, \quad g \in \mathfrak{M}^+(R,\mu).$$

Theorem 1.11. *The associate norm ϱ' of a function norm ϱ is itself a function norm.*

Definition 1.12. Let $X = X(\varrho)$ be a Banach function space. Then its *associate space* is the Banach function space $X' = X'(\varrho')$.

Remark 1.13. Let X be a Banach function space. The norm in the associate space can be written as

$$\|g\|_{X'} = \sup\{\int_R |fg|\ d\mu; f \in X, \|f\|_X \leq 1\}, \quad g \in X'.$$

Example 1.14. Assume that $p \in [1, \infty]$. Then $(L^p)' = L^{p'}$ where p' is the Hölder conjugate index of p. We stress that the case $p = \infty$ is not excluded. In particular, L^∞ is an example of a space whose associate space is strictly smaller than its dual space.

The following inequality reminds us of the classical Hölder inequality for L^p spaces. In fact, as the previous example illustrates, this inequality extends the classical Hölder inequality. Thus it is reasonable to call it *Hölder inequality for Banach function spaces.* Moreover, it immediately shows that the associate space (with the obvious identification) is always a subspace of the dual space.

Theorem 1.15. *Let ϱ be a function norm. Then*

$$\int_R |fg|\ d\mu \leq \varrho(|f|)\varrho'(|g|)$$

for every $f, g \in \mathfrak{M}(R, \mu)$. In particular, if $f \in X$ and $g \in X'$, then

$$\int_R |fg|\ d\mu \leq \|f\|_X \|g\|_{X'} < \infty.$$

The following two theorems are of immense importance for our purpose as we shall see in the next chapters. Even though a Banach function space need not be reflexive (again, L^1 or L^∞ serves as an immediate example), if we replace the dual space with the associate, which is often possible, Banach function spaces are always reflexive in the following sense.

Theorem 1.16. *Let X be a Banach function space. Then $X = X'' = (X')'$. More precisely, we have for every $f \in \mathfrak{M}(R, \mu)$ that*

$$f \in X \Leftrightarrow f \in X''$$

and

$$\|f\|_X = \|f\|_{X''}.$$

Theorem 1.17. *Let X and Y be Banach function spaces. Then $X \hookrightarrow Y$ if and only if $Y' \hookrightarrow X'$.*

1.3 Rearrangement-invariant Banach Function Spaces

Thus far, we have considered general Banach function spaces. We shall introduce their important subclass in this section. We motivate the reason for it by two examples.

Consider the ℓ^p norm ($p \in [1, \infty]$) on $\mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$. Clearly, if we rearrange the coordinates of x and denote the new vector by $\tilde{x}$, then $\|x\|_{\ell^p} = \|\tilde{x}\|_{\ell^p}$ because it is obvious from the very definition of the norm that the ℓ^p norm does not depend on the order of the coordinates. It only depends on the magnitude of the coordinates.

More generally, the well-known *layer cake formula* (here $p \in [1, \infty)$)

$$\|f\|_{L^p(R,\mu)}^p = p \int_0^\infty t^{p-1} \mu\left(\{x \in R; |f(x)| > t\}\right) dt, \quad f \in L^p(R, \mu),$$

shows that the L^p norm depends only on the measure of level sets of a given function, that is, the set where the function is greater than the given value, but it is independent of the way the function is arranged. This is also true for L^∞ as we can write

$$\|f\|_{L^\infty(R,\mu)} = \inf\{\lambda \geq 0; \mu\left(\{x \in R; |f(x)| > \lambda\}\right) = 0\}, \quad f \in L^\infty(R, \mu).$$

Rearrangement-invariant Banach function spaces generalize this concept.

Definition 1.18. Let $f \in \mathfrak{M}_0(R, \mu)$. We define its *distribution function* μ_f as

$$\mu_f(\lambda) = \mu\left(\{x \in R; |f(x)| > \lambda\}\right), \quad \lambda \geq 0.$$

Remark 1.19. The distribution function of a function clearly depends only on the absolute value of the function and it is allowed to attain the value ∞.

Definition 1.20. We say that functions $f \in \mathfrak{M}_0(R, \mu)$ and $g \in \mathfrak{M}_0(S, \nu)$ are *equimeasurable*, where (R, μ) and (S, ν) are $\sigma-$finite measure spaces, if their distribution functions are equal, that is, $\mu_f(\lambda) = \nu_g(\lambda)$ for every $\lambda \geq 0$.

For a given function f, there are, in general, many functions which are equimeasurable with f. There is, however, one particular function which is equimeasurable with f and has some special properties (see Theorem 1.23).

Definition 1.21. Let $f \in \mathfrak{M}_0(R, \mu)$. The function $f^* : (0, \infty) \longrightarrow [0, \infty]$ defined by

$$f^*(t) = \inf\{\lambda \geq 0; \mu_f(\lambda) \leq t\}, \quad t \in (0, \infty),$$

is called the *non-increasing rearrangement* of f.

The non-increasing rearrangement of f is a generalized inverse function to its distribution function μ_f. If μ_f happens to be continuous and decreasing, then f^* is its ordinary inverse.

Remarks 1.22.

1. In the preceding definition, we use the convention that $\inf \emptyset = \infty$.
2. If $\mu_f(\lambda) > t$ for every $\lambda \geq 0$, then $f^*(t) = \infty$.
3. If $\mu(R) < \infty$, then $f^*(t) = 0$ for every $t \geq \mu(R)$. In this case, we may regard f^* as a function defined only on $(0, \mu(R))$.
4. It follows immediately from the very definition of the non-increasing rearrangement that $f^* = g^*$ whenever f and g are equimeasurable.

Although all the basic properties of the non-increasing rearrangement listed below are important, we emphasize the fact that f^* is monotone, namely non-increasing as indicated by the terminology.

Theorem 1.23. *Suppose that* $n \in \mathbb{N}$, $f, g, f_n \in \mathfrak{M}_0(R,\mu)$, *and* $\alpha \in \mathbb{R}$. *The non-increasing rearrangement* f^* *is non-negative, non-increasing and right-continuous function on* $(0,\infty)$. *Furthermore, we have that:*

1. $|f| \le |g| \Rightarrow f^* \le g^*$.
2. $(\alpha f)^* = |\alpha|\, f^*$.
3. $(f+g)^*(t_1+t_2) \le f^*(t_1) + g^*(t_2)$ *for every* $t_1, t_2 > 0$.
4. $|f_n| \uparrow |f|$ μ*−a.e.* $\Rightarrow f_n^* \uparrow f^*$.
5. *The functions* f *and* f^* *are equimeasurable.*

Remark 1.24. From the very definition of the non-increasing rearrangement, one can easily compute that $\chi_E^* = \chi_{(0,\mu(E))}$ for every $\mu-$measurable $E \subseteq R$.

Remark 1.25. Even though the operator $f \mapsto f^*$ is not subadditive, a partial remedy can be sometimes provided by the inequality $(f+g)^*(t) \le f^*(\frac{t}{2}) + g^*(\frac{t}{2})$. In order to see that $f \mapsto f^*$ is not subadditive, consider $\mathbb{R}$ with the standard Lebesgue measure, $f = \chi_{(0,1)}$ and $g = \chi_{[1,2)}$.

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$ having non-negative coordinates. It is known due to G. H. Hardy and J. E. Littlewood (see [14, Chapter X]) that the quantity $\sum_{j=1}^{n} x_j y_j$ is maximized when the vectors are arranged in the non-increasing order, that is,

$$\sum_{j=1}^{n} x_j y_j \le \sum_{j=1}^{n} x_j^* y_j^*, \tag{1.1}$$

where $x^* = (x_1^*, \ldots, x_n^*), y = (y_1^*, \ldots, y_n^*)$ are their non-increasing rearrangements. This inequality can be generalized to the following integral form.

Theorem 1.26. *Let* $f, g \in \mathfrak{M}_0(R,\mu)$. *Then*

$$\int_R |fg|\, d\mu \le \int_0^\infty f^*(t) g^*(t)\, dt. \tag{1.2}$$

Remark 1.27. For $f \in \mathfrak{M}_0(R,\mu)$ and $g = \chi_E$ where $E \subseteq R$ is $\mu-$measurable, (1.2) turns into (see also Remark 1.24)

$$\int_E |f|\, d\mu \le \int_0^{\mu(E)} f^*(t)\, dt, \tag{1.3}$$

which we shall find useful later.

It follows immediately from the Hardy-Littlewood inequality (1.2) that

$$\int_R |f\tilde{g}|\, d\mu \le \int_0^\infty f^*(t)g^*(t)\, dt$$

whenever $\tilde{g}$ is a function on R equimeasurable with g. When (R,μ) is the counting measure on a finite set, that is, f, g are finite sequences and (1.2) reads as (1.1), it is obvious that we can find $\tilde{g}$ such that the equality is attained. However, for a general measure space, it is not obvious whether there exists such a $\tilde{g}$. The following example shows that it is not true for a general measure space.

Let (R,μ) consist of two atoms a, b such that $\mu(\{a\}) = 1$ and $\mu(\{b\}) = 2$. Consider $f = \chi_{\{a\}}$ and $g = \chi_{\{b\}}$. Then $\int_0^\infty f^*(t)g^*(t)\, dt = 1$ but it is easy to see that $\int_R |f\tilde{g}|\, d\mu = 0$ whenever $\tilde{g}$ is equimeasurable with g.

Not only has the example shown that the equality need not be attained, but it even shows that the inequality need not be saturated by taking the supremum over all $\tilde{g}$. The measure spaces for which the inequality is saturated plays an important role and are called *resonant*.

Definition 1.28. We say that a $\sigma-$finite measure space (R,μ) is *resonant* if

$$\int_0^\infty f^*(t)g^*(t)\, dt = \sup_{\tilde{g}} \int_R |f\tilde{g}|\, d\mu$$

for every $f, g \in \mathfrak{M}_0(R,\mu)$, where the supremum is taken over all functions $\tilde{g} \in \mathfrak{M}_0(R,\mu)$ equimeasurable with g.

Fortunately, resonant measure spaces have the following simple characterization.

Theorem 1.29. *A $\sigma-$finite measure space (R,μ) is resonant if and only if it is either non-atomic or completely atomic with all atoms having equal measure.*

It is sometimes convenient to work with the integral mean of f^* instead of just f^* as it dominates the non-increasing rearrangement and possesses some better properties (in particular subadditivity).

Definition 1.30. Let $f \in \mathfrak{M}_0(R,\mu)$. We define the *maximal function* f^{**} (of f^*) by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds, \quad t > 0.$$

Remark 1.31. It follows from the very definition of the maximal function that $f^{**} = g^{**}$ whenever f and g are equimeasurable (recall Remarks 1.22).

Theorem 1.32. *Suppose that $n \in \mathbb{N}$, $f, g, f_n \in \mathfrak{M}_0(R,\mu)$, and $\alpha \in \mathbb{R}$. The maximal function f^{**} is non-negative, non-increasing and continuous on $(0,\infty)$. Furthermore, the following properties hold:*

1. $f^{**} \equiv 0 \Leftrightarrow f = 0$ *$\mu-a.e.$*
2. $f^* \le f^{**}$.
3. $|f| \le |g| \Rightarrow f^{**} \le g^{**}$.

4. $(\alpha f)^{**} = |\alpha| \, f^{**}$

5. $|f_n| \uparrow |f| \Rightarrow f_n^{**} \uparrow f^{**}$.

6. $(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \quad t > 0$.

Now, we finally define what rearrangement-invariant Banach function spaces are.

Definition 1.33. A function norm ϱ over (R, μ) is said to be *rearrangement-invariant* if

$$\varrho(f) = \varrho(g)$$

whenever $f, g \in \mathfrak{M}_0^+(R, \mu)$ are equimeasurable. In this case, the corresponding Banach function space $X = X(\varrho)$ is called a *rearrangement-invariant Banach function space.*

Example 1.34. Some examples of Banach function spaces have been provided in Example 1.9. All of them but Morrey spaces are also rearrangement invariant. Morrey spaces are an example of Banach function spaces which are not rearrangement invariant. Another such example is provided by weighted Lebesgue spaces or variable-exponent Lebesgue spaces (see [24, Chapter 11]).

We also note here that even though classical Sobolev spaces $W^{m,p}(\Omega)$ are not Banach function spaces, let alone rearrangement invariant, Sobolev spaces can be built upon rearrangement-invariant Banach function spaces. For more details, see e.g. [16].

Theorem 1.35. *Let ϱ be a rearrangement-invariant function norm over a resonant measure space (R, μ). Then the associate norm ϱ' is also rearrangement-invariant and it holds that*

$$\varrho'(g) = \sup\{\int_0^{\mu(R)} f^*(t)g^*(t)\,dt; \varrho(f) \leq 1\}, \quad g \in \mathfrak{M}_0^+(R, \mu),$$

and

$$\varrho(f) = \sup\{\int_0^{\mu(R)} f^*(t)g^*(t)\,dt; \varrho'(g) \leq 1\}, \quad f \in \mathfrak{M}_0^+(R, \mu).$$

The following expressions of the associate norm are often useful in computations.

Corollary 1.36. *Let ϱ be a rearrangement-invariant function norm over a resonant measure space (R, μ). If $f, g \in \mathfrak{M}_0^+(R, \mu)$, then*

$$\int_R fg\,d\mu \leq \int_0^{\mu(R)} f^*(t)g^*(t)\,dt \leq \varrho(f)\varrho'(g).$$

Corollary 1.37. *Assume that X is a Banach function space over a resonant measure space (R, μ). Then X is rearrangement-invariant if and only if X' is, and in this case, it holds that*

$$\|g\|_{X'} = \sup\{\int_0^{\mu(R)} f^*(t)g^*(t)\,dt; \|f\|_X \leq 1\}, \quad g \in X', \tag{1.4}$$

and

$$\|f\|_X = \sup\{\int_0^{\mu(R)} f^*(t)g^*(t)\,dt; \|g\|_{X'} \leq 1\}, \quad f \in X. \tag{1.5}$$

Corollary 1.38. *Let X be a rearrangement-invariant Banach function space over a resonant measure space (R, μ). It holds that*

$$\int_R |fg| \, d\mu \le \int_0^{\mu(R)} f^*(t) g^*(t) \, dt \le \|f\|_X \|g\|_{X'},$$

for every $f \in X$ and $g \in X'$.

1.4 Representation Space

Let (R, μ) be an arbitrary $\sigma-$finite measure space and assume that τ is a rearrangement-invariant Banach function norm over $((0, \infty), \lambda)$. Then the functional $\tilde{\tau}$ defined as

$$\tilde{\tau}(f) = \tau(f^*), \quad f \in \mathfrak{M}_0^+(R, \mu),$$

is a rearrangement-invariant Banach function norm over (R, μ) (see [2, Chapter 2, Theorem 4.9]).

On the other hand, the following theorem shows that every rearrangement-invariant Banach function norm over a resonant measure space arises in this way. This sometimes enables us to work with $(0, \infty)$ equipped with the standard Lebesgue measure instead of a possibly more complicated measure space (R, μ).

Theorem 1.39. *Let X be a rearrangement-invariant Banach function space over a resonant measure space (R, μ).*

Firstly, there exists a (not necessarily unique) rearrangement-invariant Banach function space $\overline{X}$ over $((0, \mu(R)), \lambda_1)$ such that

$$\|g\|_X = \|g^*\|_{\overline{X}}, \quad g \in \mathfrak{M}_0^+(R, \mu). \tag{1.6}$$

Secondly, if $\overline{X}$ is any rearrangement-invariant Banach function space over $((0, \mu(R)), \lambda_1)$ which represents X in the sense of (1.6)*, then*

$$\|g\|_{X'} = \|g^*\|_{\overline{X}'}, \quad g \in \mathfrak{M}_0^+(R, \mu).$$

Remarks 1.40.

1. Even though $\overline{X}$ is not unique in general, if (R, μ) happens to be non-atomic and of infinite measure (say $\mathbb{R}^n$ equipped with the Lebesgue measure), it can be shown that the representation $X \to \overline{X}$ is unique.

2. The second part of Theorem 1.39 shows that $\overline{X}' = \overline{X'}$ if the representation $X \to \overline{X}$ is unique.

Definition 1.41. Let X be a rearrangement-invariant Banach function space over a non-atomic measure space (R, μ) of infinite measure. The (unique) rearrangement-invariant Banach function space $\overline{X}$ which represents X in the sense of Theorem 1.39 is called the *representation space* (of X) and we denote the representation space of X by $X(0, \infty)$.

The non-increasing rearrangement of a function is a non-negative, non-increasing, right-continuous function on $(0, \infty)$. The natural question is whether every function on $(0, \infty)$ having these properties is the non-increasing rearrangement of some function on (R, μ). It turns out that such a function can be constructed by means of measure-preserving transformations (see [2, Chapter 2, Section 7] for details). Precisely, we have the following result.

Theorem 1.42. *Let (R, μ) be a non-atomic $\sigma-$finite measure space. Then every non-negative, non-increasing, right-continuous function h on $(0, \infty)$ is the non-increasing rearrangement of some $f \in \mathfrak{M}_0(R, \mu)$.*

Remark 1.43. The preceding theorem in particular shows that if $h \in X(0, \infty)$, then there exists a function $f \in X$ such that $f^* = h^*$ (recall Theorem 1.23). Furthermore, it also implies (recall Corollary 1.37) that the norm of $h \in X(0, \infty)$ can be expressed as

$$\|h\|_{X(0,\infty)} = \sup_{g \in X', \|g\|_{X'} \leq 1} \int_0^{\mu(R)} h^*(t) g^*(t)\, dt.$$

1.5 Endpoint Spaces

Endpoint spaces play an important role in the theory of rearrangement-invariant Banach function spaces. Endpoint spaces are, in some sense, the largest and the smallest rearrangement-invariant Banach function spaces (see [24, Chapter 7, Theorem 7.10.18]) satisfying certain restrictions. The theory of Lorentz and Marcinkiewicz endpoint spaces can be found in [24, Chapter 7, Section 10].

Convention. Throughout this section, we assume that (R, μ) is resonant.

We begin with Lorentz endpoint spaces, which we shall need later on.

Definition 1.44. Assume that $\varphi : [0, \mu(R)) \to [0, \infty)$ is a non-negative, non-decreasing and concave function. We define the *Lorentz endpoint space* $\Lambda_\varphi(R, \mu)$ as

$$\Lambda_\varphi(R, \mu) = \{f \in \mathfrak{M}_0(R, \mu); \|f\|_{\Lambda_\varphi(R,\mu)} < \infty\}$$

where the functional $\|\cdot\|_{\Lambda_\varphi(R,\mu)}$ is defined as

$$\|f\|_{\Lambda_\varphi(R,\mu)} = \int_0^{\mu(R)} f^*(t)\, d\varphi(t), \quad f \in \mathfrak{M}_0(R, \mu), \tag{1.7}$$

where the integral on the right-hand side is the classical Lebesgue-Stieltjes integral.

Remark 1.45. We note that the integral in (1.7) is well-defined as the function φ is non-decreasing. Moreover, as φ is non-decreasing and concave, the integral may be rewritten as

$$f^*(0_+)\varphi(0_+) + \int_0^{\mu(R)} f^*(t)\phi(t)\, dt$$

where $\phi(t) : (0, \mu(R)) \to [0, \infty)$ is a non-negative and non-increasing function. Furthermore, if φ is smooth enough, say $\mathcal{C}^1$, then $\phi(t) = \varphi'(t)$ for $t > 0$.

Theorem 1.46. *Let $\varphi : [0, \mu(R)) \to [0, \infty)$ be a non-decreasing and concave function. Then the functional $\|\cdot\|_{\Lambda_\varphi(R,\mu)}$ defined by* (1.7) *is a rearrangement-invariant Banach function norm. Hence, $\left(\Lambda_\varphi(R, \mu), \|\cdot\|_{\Lambda_\varphi(R,\mu)}\right)$ is a rearrangement-invariant Banach function space.*

Before we define Marcinkiewicz endpoint spaces, we need to define what *quasi-concave* functions are.

Definition 1.47. Let $b \in (0, \infty]$. We say that a function $\varphi : [0, b) \to [0, \infty)$ is *quasi-concave* on $[0, b)$ if φ is non-decreasing, $\frac{\varphi(t)}{t}$ is non-increasing on $(0, b)$ and

$$\varphi(t) = 0 \text{ if and only if } t = 0.$$

Definition 1.48. Let φ be a quasi-concave function on $[0, \mu(R))$. We define *Marcinkiewicz endpoint space* $M_\varphi(R, \mu)$ as

$$M_\varphi(R, \mu) = \{f \in \mathfrak{M}_0(R, \mu); \|f\|_{M_\varphi(R,\mu)} < \infty\},$$

where

$$\|f\|_{M_\varphi(R,\mu)} = \sup_{t\in(0,\mu(R))} f^{**}(t)\varphi(t), \quad f \in \mathfrak{M}_0(R, \mu). \tag{1.8}$$

Theorem 1.49. *Let φ be a quasi-concave function on $[0, \mu(R))$. Then the functional $\|\cdot\|_{M_\varphi(R,\mu)}$ defined by* (1.8) *is a rearrangement-invariant Banach function norm. Hence, $\left(M_\varphi(R, \mu), \|\cdot\|_{M_\varphi(R,\mu)}\right)$ is a rearrangement-invariant Banach function space.*

From now on, we assume in this section that $\mu(R) = \infty$. The following modification of $M_\varphi(R, \mu)$ will prove useful for us later on.

Definition 1.50. Let $\varphi : (0, \infty) \to (0, \infty)$ be a non-decreasing function. We define the function space $m_\varphi(R, \mu)$ by

$$m_\varphi(R, \mu) = \{f \in \mathfrak{M}_0(R, \mu); \|f\|_{m_\varphi(R,\mu)} < \infty\},$$

where the functional $\|\cdot\|_{m_\varphi(R,\mu)}$ is defined by

$$\|f\|_{m_\varphi(R,\mu)} = \sup_{t>0} f^*(t)\varphi(t), \quad f \in \mathfrak{M}_0(R, \mu).$$

We also define its associate space $(m_\varphi)'(R, \mu)$ by

$$(m_\varphi)'(R, \mu) = \{g \in \mathfrak{M}_0(R, \mu); \|g\|_{(m_\varphi)'(R,\mu)} < \infty\},$$

where the functional $\|\cdot\|_{(m_\varphi)'(R,\mu)}$ is defined by

$$\|g\|_{(m_\varphi)'(R,\mu)} = \sup_{\|f\|_{m_\varphi(R,\mu)}\leq 1} \int_0^\infty f^*(t)g^*(t)\, dt, \quad g \in \mathfrak{M}_0.$$

Remark 1.51. Despite the used notation, the functional $\|\cdot\|_{m_\varphi}$ need not be a Banach function norm because it may lack the property of triangle inequality. Worse still, even the word "space" may be misleading here because it indicates that the structure being considered is at least linear. It need not, however, be the case for m_φ and a general φ. To see this, set $\varphi(t) = \mathrm{e}^t - 1$. The fact that m_φ with this particular choice of φ is not a linear set, let alone a Banach function space, then easily follows from [24, Theorem 10.2.6], which characterizes linearity of more general *weak classical Lorentz spaces* $\Lambda^{p,\infty}(w)$ (see [24, Chapter 10]). For this reason, the term *function class* or *function cone* might have been more suitable.

Nevertheless, even when m_φ is not a Banach function space, we still may define another function space (possibly trivial) $(m_\varphi)'$, which we call its associate space. If m_φ happens to be a Banach function space, then $(m_\varphi)'$ is indeed its associate space in the sense of Definition 1.12 (cf. Theorem 1.35). However, one must be careful when dealing with the associate space of a space which is not a (rearrangement-invariant) Banach function space, because some properties of associate spaces of (rearrangement-invariant) Banach function spaces may no longer be true. In particular, (1.5) or Theorem 1.16 need not be true. Moreover, it can be easily verified that if m_φ violates the sixth (or the seventh) property of Definition 1.2, then $(m_\varphi)'$ violates the seventh (or the sixth) one, respectively. Nevertheless, it follows from [12, Theorem 3.1] that if m_φ satisfies these two axioms, then $(m_\varphi)'$ is a (rearrangement-invariant) Banach function space even though m_φ itself need not be.

When φ is not just a non-decreasing function but a quasi-concave function, a sufficient condition for m_φ to be equivalent to a Banach function space, is provided by the following theorem.

Theorem 1.52. *Let φ be a quasi-concave function on $[0,\infty)$. Assume that there exists a positive constant C such that*

$$\frac{1}{t}\int_0^t \frac{1}{\varphi(s)}\,ds \le \frac{C}{\varphi(t)}$$

for each $t > 0$. Then

$$\|f\|_{M_\varphi(R,\mu)} \le C\|f\|_{m_\varphi(R,\mu)}$$

for every $f \in \mathfrak{M}_0(R,\mu)$. In particular, $m_\varphi(R,\mu)$ is equivalent to $M_\varphi(R,\mu)$.

Even though the next two propositions are rather known, we are not aware of any reference which could be cited. For this reason, the propositions are proven here.

Proposition 1.53. *Let $\varphi : (0,\infty) \to (0,\infty)$ be a non-decreasing function and $g \in \mathfrak{M}_0(R,\mu)$. Then*

$$\|g\|_{(m_\varphi)'} = \int_0^\infty g^*(t)\frac{1}{\varphi(t)}\,dt.$$

Proof. On the one hand, since $\|\frac{1}{\varphi}\|_{m_\varphi} = 1$, we have that

$$\int_0^\infty g^*(t)\frac{1}{\varphi(t)}\,dt = \int_0^\infty g^*(t)\left(\frac{1}{\varphi}\right)^*(t)\,dt \le \sup_{\|f\|_{m_\varphi}\le 1}\int_0^\infty f^*(t)g^*(t)\,dt = \|g\|_{(m_\varphi)'}.$$

On the other hand, for every $f \in m_\varphi$ such that $\|f\|_{m_\varphi} \le 1$, we compute that

$$\begin{aligned}\int_0^\infty f^*(t)g^*(t)\,dt &= \int_0^\infty f^*(t)\varphi(t)g^*(t)\frac{1}{\varphi(t)}\,dt \\ &\le \|f\|_{m_\varphi}\int_0^\infty g^*(t)\frac{1}{\varphi(t)}\,dt \le \int_0^\infty g^*(t)\frac{1}{\varphi(t)}\,dt.\end{aligned}$$

Taking the supremum over all $f \in m_\varphi$ such that $\|f\|_{m_\varphi} \le 1$, we obtain that

$$\|g\|_{(m_\varphi)'} \le \int_0^\infty g^*(t)\frac{1}{\varphi(t)}\,dt.$$

❑

Proposition 1.54. *Let X be a rearrangement-invariant Banach function space and let $\varphi : (0,\infty) \to (0,\infty)$ be a non-decreasing function. Then the following three statements are equivalent:*

$$\frac{1}{\varphi} \in X'(0,\infty), \tag{1.9}$$

$$m_\varphi \hookrightarrow X', \tag{1.10}$$

and

$$X \hookrightarrow (m_\varphi)'. \tag{1.11}$$

Proof. Firstly, assume that (1.9) holds. Fix $f \in m_\varphi$. Clearly, we have that

$$f^*(t) = \frac{1}{\varphi(t)}f^*(t)\varphi(t) \le \frac{1}{\varphi(t)}\sup_{s>0} f^*(s)\varphi(s) = \frac{1}{\varphi(t)}\|f\|_{m_\varphi},$$

for each $t > 0$. Thus

$$\|f\|_{X'} = \|f^*\|_{X'(0,\infty)} \le \|\frac{1}{\varphi}\|_{X'(0,\infty)}\|f\|_{m_\varphi}$$

as $X'(0,\infty)$ is a rearrangement-invariant Banach function space (see Theorem 1.35) and f and f^* are equimeasurable. Hence (1.10) holds since we assume that $\|\frac{1}{\varphi}\|_{X'(0,\infty)} < \infty$.

Secondly, assume that (1.10) holds, that is, there exists a positive constant $C > 0$ such that

$$\|h\|_{X'} \le C\|h\|_{m_\varphi}$$

for every $h \in m_\varphi$. Fix $f \in X$. Then, using Corollary 1.38,

$$\int_0^\infty f^*(t)h^*(t)\,dt \le \|f\|_X\|h\|_{X'} \le C\|h\|_{m_\varphi}\|f\|_X \le C\|f\|_X$$

for every $h \in m_\varphi$ such that $\|h\|_{m_\varphi} \le 1$. Taking the supremum over all $h \in m_\varphi$ such that $\|h\|_{m_\varphi} \le 1$, we obtain that

$$\|f\|_{(m_\varphi)'} \le C\|f\|_X,$$

which is nothing else than (1.11), as $f \in X$ was chosen arbitrarily.

Finally, we shall prove that (1.11) implies (1.9). There exists a positive constant $C > 0$ such that

$$\|f\|_{(m_\varphi)'} \leq C\|f\|_X$$

for every $f \in X$. Fix $h \in X(0,\infty)$ such that $\|h\|_{X(0,\infty)} = \|h^*\|_{X(0,\infty)} \leq 1$. By Remark 1.43, there exists $f \in X$ such that $f^* = h^*$. Then by Proposition 1.53

$$\begin{aligned}\int_0^\infty h^*(t)\frac{1}{\varphi(t)}\,dt &= \int_0^\infty f^*(t)\left(\frac{1}{\varphi}\right)^*(t)\,dt = \|f\|_{(m_\varphi)'} \leq C\|f\|_X \\ &= C\|f^*\|_{X(0,\infty)} = C\|h^*\|_{X(0,\infty)} \leq C,\end{aligned}$$

as $\|\frac{1}{\varphi}\|_{m_\varphi} = 1$. Therefore, $\|\frac{1}{\varphi}\|_{X'(0,\infty)} < \infty$. Hence (1.9) holds.

❑

1.6 Generalized Lorentz-Zygmund Spaces

In the following chapters, we shall provide a complete, yet rather abstract, characterization of the optimal range partner (or domain) for some integral operators. The abstract nature of the results, however, appeals for some concrete examples. *Generalized Lorentz-Zygmund spaces*, whose theory we shall outline in this section, are definitely a good choice of function spaces for concrete examples, because not only is their theory fairly nice, but they are also reasonably general.

Conventions. In this section, we assume that (R, μ) is a non-atomic σ-finite measure space.

For $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$ and a scalar $r \in \mathbb{R}$, we consider the component-wise addition and multiplication, that is,

$$\mathbb{A} + r = r + \mathbb{A} = (\alpha_0 + r, \alpha_\infty + r)$$

and

$$\mathbb{A}r = r\mathbb{A} = (\alpha_0 r, \alpha_\infty r).$$

The theory of Generalized Lorentz-Zygmund spaces is covered in [22] or [24, Chapter 9]. We list here just these properties of Generalized Lorentz-Zygmund spaces which we will directly need later on.

Definition 1.55. We define functions ℓ and $\ell\ell$ as

$$\begin{aligned}\ell(t) &= 1 + |\log t|, & t > 0,\\ \ell\ell(t) &= 1 + \log(\ell(t)), & t > 0.\end{aligned}$$

For $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, we define a *broken-logarithmic function* $\ell^{\mathbb{A}}(t)$ as

$$\ell^{\mathbb{A}}(t) = \begin{cases}\ell^{\alpha_0}(t) & \text{if } 1 < t \leq 1,\\ \ell^{\alpha_\infty}(t) & \text{if } t > 1.\end{cases}$$

We also analogously define $\ell\ell^{\mathbb{A}}$.

Definition 1.56. Let $p, q \in [1, \infty]$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. We define the *Generalized Lorentz-Zygmund space* $L^{p,q;\mathbb{A},\mathbb{B}}(R, \mu)$ as

$$L^{p,q;\mathbb{A},\mathbb{B}}(R, \mu) = \{f \in \mathfrak{M}_0(R, \mu); \|f\|_{p,q;\mathbb{A},\mathbb{B}} < \infty\},$$

where the functional $\|\cdot\|_{p,q;\mathbb{A},\mathbb{B}}$ is defined as

$$\|f\|_{p,q;\mathbb{A},\mathbb{B}} = \|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) f^*(t)\|_q, \quad f \in \mathfrak{M}_0(R, \mu).$$

If $\mathbb{B} = (0, 0)$, we simply write $L^{p,q;\mathbb{A}} = L^{p,q;\mathbb{A}}(R, \mu)$ instead of $L^{p,q;\mathbb{A},\mathbb{B}} = L^{p,q;\mathbb{A},\mathbb{B}}(R, \mu)$. We also write $L^{p,q} = L^{p,q}(R, \mu)$ instead of $L^{p,q;\mathbb{A},\mathbb{B}} = L^{p,q;\mathbb{A},\mathbb{B}}(R, \mu)$ if $\mathbb{A} = \mathbb{B} = (0, 0)$.

Furthermore, we define the space $L^{(p,q;\mathbb{A},\mathbb{B})}(R, \mu)$ as

$$L^{(p,q;\mathbb{A},\mathbb{B})}(R, \mu) = \{f \in \mathfrak{M}_0(R, \mu); \|f\|_{(p,q;\mathbb{A},\mathbb{B})} < \infty\},$$

where the functional $\|\cdot\|_{(p,q;\mathbb{A},\mathbb{B})}$ is defined as

$$\|f\|_{(p,q;\mathbb{A},\mathbb{B})} = \|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) f^{**}(t)\|_q, \quad f \in \mathfrak{M}_0(R, \mu).$$

If $\mathbb{B} = (0, 0)$, we simply write $L^{(p,q;\mathbb{A})} = L^{(p,q;\mathbb{A})}(R, \mu)$ instead of $L^{(p,q;\mathbb{A},\mathbb{B})} = L^{(p,q;\mathbb{A},\mathbb{B})}(R, \mu)$. We also write $L^{(p,q)} = L^{(p,q)}(R, \mu)$ instead of $L^{(p,q;\mathbb{A},\mathbb{B})} = L^{(p,q;\mathbb{A},\mathbb{B})}(R, \mu)$ if $\mathbb{A} = \mathbb{B} = (0, 0)$.

Remark 1.57. Despite the notation, neither $\|\cdot\|_{p,q;\mathbb{A},\mathbb{B}}$ nor $\|\cdot\|_{(p,q;\mathbb{A},\mathbb{B})}$ is always a rearrangement-invariant Banach function norm (or at least equivalent to one). In fact, these spaces can be even trivial, that is, containing just the zero function. Fortunately, the following two theorems characterize when it is the case.

We also note that if $\mathbb{A} = \mathbb{B} = (0, 0)$, then $L^{p,q;\mathbb{A},\mathbb{B}}$ and $L^{(p,q;\mathbb{A},\mathbb{B})}$ coincide with *two-parametric Lorentz spaces* $L^{p,q}$ and $L^{(p,q)}$ respectively (see e.g. [24, Chapter 8]). Therefore, the shortened notation introduced here is consistent with the common notation.

Theorem 1.58. *Let $p, q \in [1, \infty]$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. The space $L^{p,q;\mathbb{A},\mathbb{B}}$ is not trivial if and only if one of the following conditions holds:*

$$\begin{cases} p < \infty; \\ p = \infty, \alpha_0 + \frac{1}{q} < 0; \\ p = \infty, \alpha_0 + \frac{1}{q} = 0, \beta_0 + \frac{1}{q} < 0; \\ p = q = \infty, \alpha_0 = 0, \beta_0 = 0. \end{cases} \tag{1.12}$$

The space $L^{(p,q;\mathbb{A},\mathbb{B})}$ is not trivial if and only if one of the following conditions holds:

$$\begin{cases} 1 < p < \infty; \\ p = \infty, \alpha_0 + \frac{1}{q} < 0; \\ p = \infty, \alpha_0 + \frac{1}{q} = 0, \beta_0 + \frac{1}{q} < 0; \\ p = q = \infty, \alpha_0 = 0, \beta_0 = 0; \\ p = 1, \alpha_\infty + \frac{1}{q} < 0; \\ p = 1, \alpha_\infty + \frac{1}{q} = 0, \beta_\infty + \frac{1}{q} < 0; \\ p = 1, q = \infty, \alpha_\infty = 0, \beta_\infty = 0. \end{cases} \tag{1.13}$$

Theorem 1.59. *Let $p, q \in [1, \infty]$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. The space $L^{p,q;\mathbb{A},\mathbb{B}}$ is equivalent to a rearrangement-invariant Banach function space if and only if one of the following conditions holds:*

$$\begin{cases} 1 < p < \infty; \\ p = q = 1, \alpha_0 > 0, \alpha_\infty < 0; \\ p = q = 1, \alpha_0 > 0, \alpha_\infty = 0, \beta_\infty \leq 0; \\ p = q = 1, \alpha_0 = 0, \beta_0 \geq 0, \alpha_\infty < 0; \\ p = q = 1, \alpha_0 = 0, \beta_0 \geq 0, \alpha_\infty = 0, \beta_\infty \leq 0; \\ p = \infty, \alpha_0 + \frac{1}{q} < 0; \\ p = \infty, \alpha_0 + \frac{1}{q} = 0, \beta_0 + \frac{1}{q} < 0; \\ p = q = \infty, \alpha_0 = 0, \beta_0 = 0. \end{cases} \tag{1.14}$$

Furthermore, the space $L^{(p,q;\mathbb{A},\mathbb{B})}$ is a rearrangement-invariant Banach function space if and only if one of the conditions (1.13) *holds.*

As one would expect, there is a close relation between $L^{p,q;\mathbb{A},\mathbb{B}}$ and $L^{(p,q;\mathbb{A},\mathbb{B})}$ spaces. This connection between them is the content of the following theorem.

Theorem 1.60. *Let $p, q \in [1, \infty]$, $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$ and assume that one of the conditions* (1.13) *is satisfied. Then:*

1. *If $p \in (1, \infty]$, then $L^{(p,q;\mathbb{A})}$ is equivalent to $L^{p,q;\mathbb{A}}$.*
2. *The space $L^{(1,1;\mathbb{A})}$ is equivalent to:*

$$\begin{cases} L^{1,1;\mathbb{A}+1} & \text{if } \alpha_\infty + 1 < 0, \alpha_0 + 1 > 0; \\ L^{1,1;(0,\alpha_\infty+1),(1,0)} & \text{if } \alpha_\infty + 1 < 0, \alpha_0 + 1 = 0; \\ L^{1,1;(0,\alpha_\infty+1)} & \text{if } \alpha_\infty + 1 < 0, \alpha_0 + 1 < 0; \end{cases}$$

We have seen that Generalized Lorentz-Zygmund spaces are not always Banach function spaces. Yet, it makes sense to define their associate spaces in the same way as we did in Definition 1.50 for m_φ. Again, the following definition of the associate space of a Generalized Lorentz-Zygmund space coincides with the definition of the associate space of a rearrangement-invariant Banach function space whenever the given space is a rearrangement-invariant Banach function space. Hence the notation used here is consistent.

Definition 1.61. Let $p, q \in [1, \infty]$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. We define the *associate space* of $L^{p,q;\mathbb{A},\mathbb{B}}(R, \mu)$ as

$$(L^{p,q;\mathbb{A},\mathbb{B}}(R, \mu))' = \{g \in \mathfrak{M}_0(R, \mu); \|g\|_{(L^{p,q;\mathbb{A},\mathbb{B}}(R,\mu))'} < \infty\}$$

where the functional $\|\cdot\|_{(L^{p,q;\mathbb{A},\mathbb{B}}(R,\mu))'}$ is defined for $g \in \mathfrak{M}_0(R, \mu)$ as

$$\|g\|_{(L^{p,q;\mathbb{A},\mathbb{B}}(R,\mu))'} = \sup\{\int_0^{\mu(R)} f^*(t)g^*(t)\,dt; f \in L^{p,q;\mathbb{A},\mathbb{B}}(R, \mu), \|f\|_{L^{p,q;\mathbb{A},\mathbb{B}}(R,\mu)} \leq 1\}.$$

We also define the *associate space* of $L^{(p,q;\mathbb{A},\mathbb{B})}(R, \mu)$ in the obvious way.

As will be made obvious in the following chapters, the associate spaces of $L^{p,q;\mathbb{A},\mathbb{B}}$ (or $L^{(p,q;\mathbb{A},\mathbb{B})}$) are of great importance for us. Luckily for us, we have the following two theorems at our disposal.

Theorem 1.62. *Let $p,q\in[1,\infty]$ and $\mathbb{A}\in\mathbb{R}^2$. Assume that one of the conditions (1.14) holds. Then the associate space of $L^{p,q;\mathbb{A}}$ is equivalent to:*

$$\begin{cases} L^{1,1;-\mathbb{A}} & \text{if } p=q=\infty \text{ and } \alpha_\infty\geq 0;\\ \mathscr{L}=\{f\in\mathfrak{M}_0;\|f\|_{\mathscr{L}}=\int_0^1(1+\log\frac{1}{t})^{-\alpha_0}f^*(t)\,dt+\|f\|_1<\infty\} & \text{if } p=q=\infty \text{ and } \alpha_\infty<0;\\ L^{(1,q';-\mathbb{A}-1)} & \text{if } p=\infty, q\in[1,\infty),\\ & \alpha_0<-\frac{1}{q} \text{ and } \alpha_\infty>-\frac{1}{q};\\ L^{(1,q';(-\alpha_0-1,-\frac{1}{q'}),(0,-1))} & \text{if } p=\infty, q\in[1,\infty),\\ & \alpha_0<-\frac{1}{q} \text{ and } \alpha_\infty=-\frac{1}{q};\\ \{f\in\mathfrak{M}_0;\|f\|_{\mathscr{L}}=\|t^{\frac{1}{q}}\ell(t)^{-\alpha_0-1}f^{**}(t)\|_{L^{q'}(0,1)}+\|f\|_1<\infty\} & \text{if } p=\infty, q\in[1,\infty),\\ & \alpha_0<-\frac{1}{q} \text{ and } \alpha_\infty<-\frac{1}{q};\\ L^{p',q';-\mathbb{A}} & \text{if } p\in[1,\infty), q\in[1,\infty]; \end{cases}$$

The associate space of $L^{1,1;\mathbb{A},\mathbb{B}}$ is equivalent to $L^{\infty,\infty;-\mathbb{A},-\mathbb{B}}$.

Theorem 1.63. *Let $q\in(1,\infty]$ and $\mathbb{A}\in\mathbb{R}^2$. Assume that one of the conditions (1.13) holds. Then the associate space of $L^{(1,q;\mathbb{A})}$ is equivalent to:*

$$\begin{cases} L^{\infty,q';-\mathbb{A}-1} & \text{if } \alpha_0+\frac{1}{q}>0, \alpha_\infty+\frac{1}{q}<0;\\ L^{\infty,q';(-\frac{1}{q'},-\alpha_\infty-1),(-1,0)} & \text{if } \alpha_0+\frac{1}{q}=0, \alpha_\infty+\frac{1}{q}<0;\\ \mathscr{L}=\{f\in\mathfrak{M}_0;\|f\|_{\mathscr{L}}=\|f\|_\infty+N_1(f)<\infty\} & \text{if } \alpha_0+\frac{1}{q}<0, \alpha_\infty+\frac{1}{q}<0;\\ \mathscr{L}=\{f\in\mathfrak{M}_0;\|f\|_{\mathscr{L}}=N_2(f)<\infty\} & \text{if } q=\infty \text{ and } \alpha_0>0, \alpha_\infty=0;\\ L^{\infty,1;(-1,-\alpha_\infty-1),(-1,0),(-1,0)} & \text{if } q=\infty \text{ and } \alpha_0=0, \alpha_\infty<0;\\ L^\infty & \text{if } q=\infty \text{ and } \alpha_0\leq 0, \alpha_\infty=0; \end{cases}$$

The functionals $N_1(f), N_2(f)$ are defined as

$$N_1(f)=\|t^{-\frac{1}{q'}}(1+\log t)^{-\alpha_\infty-1}f^*(t)\|_{L^{q'}(1,\infty)},\qquad f\in\mathfrak{M}_0,$$

$$N_2(f)=\|t^{-1}\left(1+\log\frac{1}{t}\right)^{-\alpha_0-1}f^*(t)\|_{L^1(0,1)},\qquad f\in\mathfrak{M}_0.$$

Hardy inequalities with power-logarithmic weights play a crucial role in the theory of Generalized Lorentz-Zygmund spaces. More of them than we list here can be found in [11, Chapter 4]. Hardy-type inequalities with general weights are exhaustively covered in [21] or [17].

Theorem 1.64. *Let $1\leq r\leq s\leq\infty$, $\nu\neq 0$ and $\mathbb{A},\mathbb{B}\in\mathbb{R}^2$.*

1. *The inequality*

$$\|t^{\nu-\frac{1}{s}}\ell^{\mathbb{A}}(t)\int_0^t g(u)\,du\|_s\lesssim\|t^{\nu+\frac{1}{r'}}\ell^{\mathbb{B}}(t)g(t)\|_r$$

holds for every $g\in\mathfrak{M}_0^+(0,\infty)$ if and only if

$$\nu<0 \text{ and } \mathbb{A}\leq\mathbb{B}.$$

2. *The inequality*

$$\|t^{\nu-\frac{1}{s}}\ell^{\mathbb{A}}(t)\int_t^\infty g(u)\,du\|_s \lesssim \|t^{\nu+\frac{1}{r'}}\ell^{\mathbb{B}}(t)g(t)\|_r$$

holds for every $g \in \mathfrak{M}_0^+(0,\infty)$ *if and only if*

$$\nu > 0 \text{ and } \mathbb{A} \leq \mathbb{B}.$$

Theorem 1.65. *Let* $1 \leq r \leq s \leq \infty$ *and* $\mathbb{A} \in \mathbb{R}^2$.

1. *The inequality*

$$\|t^{-\frac{1}{s}}\ell^{\mathbb{A}-\frac{1}{s}}(t)\int_0^t g(u)\,du\|_s \lesssim \|t^{\frac{1}{r'}}\ell^{\mathbb{A}+\frac{1}{r'}}(t)g(t)\|_r$$

holds for every $g \in \mathfrak{M}_0^+(0,\infty)$ *if and only if either*

$$\alpha_\infty < 0 < \alpha_0$$

or

$$r = 1, s = \infty \text{ and } \alpha_\infty \leq 0 \leq \alpha_0.$$

2. *The inequality*

$$\|t^{-\frac{1}{s}}\ell^{\mathbb{A}-\frac{1}{s}}(t)\int_t^\infty g(u)\,du\|_s \lesssim \|t^{\frac{1}{r'}}\ell^{\mathbb{A}+\frac{1}{r'}}(t)g(t)\|_r$$

holds for every $g \in \mathfrak{M}_0^+(0,\infty)$ *if and only if either*

$$\alpha_0 < 0 < \alpha_\infty$$

or

$$r = 1, s = \infty \text{ and } \alpha_0 \leq 0 \leq \alpha_\infty.$$

2. Hilbert Transform

The Hilbert transform is a textbook example of a singular integral of convolution type. Indeed, even [25], which is arguably the bible of the theory of singular integrals, uses it as an illustrative example (see [25, Chapter II]). It is one of the classical operators of harmonic analysis. It has a close connection to boundary values of analytic functions in the upper-plane (see [13, Chapter 4, Section 4.1.2]). It also has practical applications. The Hilbert transform is an instrumental tool in the theory of signal processing, which leads to further applications in physics.

Convention. In this chapter, we assume that $(R, \mu) = (\mathbb{R}, \lambda)$ where λ is the Lebesgue measure over $\mathbb{R}$. We shall write $|E|$ instead of $\lambda(E)$ for $\lambda-$measurable $E \subseteq \mathbb{R}$.

Definition 2.1. Let $f \in L^1_{loc}(\mathbb{R})$. We define its *Hilbert transform* Hf by

$$Hf(x) = \lim_{\varepsilon \to 0_+} \frac{1}{\pi} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} \, dt$$

provided that the limit exists for a.e. $x \in \mathbb{R}$. The operator $H : f \mapsto Hf$ is also referred to as the *Hilbert transform.*

Remark 2.2. Whenever we say that the Hilbert transform is bounded from a function space X to a function space Y, we implicitly assume that Hf is well-defined for every $f \in X$, that is, $f \in L^1_{loc}(\mathbb{R})$ and the limit $\lim\limits_{\varepsilon \to 0_+} \frac{1}{\pi} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} \, dt$ exists for a.e. $x \in \mathbb{R}$. We also note the obvious fact that if X is a Banach function space, then $X \subseteq L^1_{loc}(\mathbb{R})$.

Even for very nice functions, say compactly supported smooth functions, it is by no means obvious that the principal value integral defining the Hilbert transform exists. It is known that the Hilbert transform is well-defined, that is, the principal value integral (absolutely) converges for a.e. $x \in \mathbb{R}$, for $f \in L^p(\mathbb{R})$ where $p \in [1, \infty)$. Moreover, the Hilbert transform is of *strong type* (p, p) for $1 < p < \infty$ and of *weak type* $(1, 1)$ (see [2, Chapter 3, Theorem 4.9]). Nevertheless, we need yet another sufficient condition for the Hilbert transform to be well-defined, which is more suitable for our purposes.

Definition 2.3. For the purpose of this chapter, we define the operator Q by

$$Qh(t) = \int_t^\infty \frac{h(s)}{s} \, ds, \quad t > 0, h \in \mathfrak{M}_0^+. \tag{2.1}$$

Theorem 2.4. *Assume that $f \in L^1_{loc}(\mathbb{R})$ satisfies that*

$$Qf^{**}(1) < \infty. \tag{2.2}$$

Then the limit $\lim\limits_{\varepsilon \to 0_+} \frac{1}{\pi} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} \, dt$ exists for a.e. $x \in \mathbb{R}$, that is, the Hilbert transform Hf of f exists.

Proof. See [2, Chapter 3, Theorem 4.8]. ❑

We shall see shortly that the operator Q plays a key role.

Theorem 2.5. *Assume that $f \in L^1_{loc}(\mathbb{R})$ satisfies (2.2). Then there exists a positive constant C independent of f such that*

$$(Hf)^*(t) \leq CQf^{**}(t)$$

for every $t > 0$.

Proof. See [2, Chapter 3, Theorem 4.8]. ❑

Theorem 2.6. *Assume that $f \in L^1_{loc}(\mathbb{R})$ satisfies (2.2). Then there exists a function g equimeasurable with f such that*

$$Qf^{**}(t) \leq 2\pi\,(Hg)^*(t)$$

for every $t > 0$.

Proof. See [2, Chapter 3, Proposition 4.10]. ❑

Remark 2.7. We note that, in the preceding theorem, g is defined as

$$g(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ f^*(-x) & \text{if } x < 0. \end{cases}$$

The assumption that f satisfies (2.2) is used just for the fact that Hg exists as $Qg^{**}(1) = Qf^{**}(1) < \infty$. Therefore, if we assume that Hg exists, we can conclude that

$$Qf^{**}(t) \leq 2\pi\,(Hg)^*(t)$$

even without assuming that f satisfies (2.2).

What follows is the key lemma of this chapter, which also demonstrates the prominent role of the operator Q, because it characterizes boundedness of the (fairly complicated) Hilbert transform by means of boundedness of (far simpler) operator Q.

Lemma 2.8. *Let X and Y be rearrangement-invariant Banach function spaces. Assume that (2.2) is satisfied for every $f \in X$. Then the Hilbert transform $H : X \to Y$ is bounded if and only if there exists a positive constant C such that*

$$\|Qg^{**}\|_{X'(0,\infty)} \leq C\|g^*\|_{Y'(0,\infty)} \tag{2.3}$$

for every $g \in Y'$.

Proof. We note that the Hilbert transform is well-defined on X since we assume that (2.2) is satisfied for every $f \in X$ (recall Theorem 2.4).

The Hilbert transform H is bounded from X to Y if and only if there exists a positive constant $C_1 > 0$ such that

$$\|Hf\|_Y \leq C_1 \|f\|_X$$

for every $f \in X$, which is equivalent to the fact that

$$\|(Hf)^*\|_{Y(0,\infty)} \leq C_1 \|f^*\|_{X(0,\infty)} \tag{2.4}$$

for every $f \in X$ (recall Definition 1.41). It is easy to see that (2.4) is equivalent to

$$\|Qf^{**}\|_{Y(0,\infty)} \leq C_2 \|f^*\|_{X(0,\infty)} \tag{2.5}$$

for every $f \in X$. Indeed, on the one hand, assume (2.4). Then

$$\|Qf^{**}\|_{Y(0,\infty)} \leq 2\pi \|(Hg)^*\|_{Y(0,\infty)} \leq 2\pi C_1 \|g^*\|_{X(0,\infty)} = C_2 \|f^*\|_{X(0,\infty)}$$

by Theorem 2.6, where g is a function equimeasurable with f. On the other hand, if (2.5) holds, then

$$\|(Hf)^*\|_{Y(0,\infty)} \leq C \|Qf^{**}\|_{Y(0,\infty)} \leq \|f^*\|_{X(0,\infty)}$$

by Theorem 2.5.

It is a matter of a straightforward computation involving the Fubini theorem to verify that

$$\int_0^\infty Qf^{**}(t) g^*(t)\, dt = \int_0^\infty f^*(t) Qg^{**}(t)\, dt \tag{2.6}$$

for every $f, g \in \mathfrak{M}_0^+$. Using this last identity, we compute that

$$\begin{aligned}
\sup_{0 \neq f \in X} \frac{\|Qf^{**}\|_{Y(0,\infty)}}{\|f^*\|_{X(0,\infty)}} &= \sup_{\substack{0 \neq f \in X \\ 0 \neq g \in Y'(0,\infty)}} \frac{\int_0^\infty Qf^{**}(t) g^*(t)\, dt}{\|f^*\|_{X(0,\infty)} \|g^*\|_{Y'(0,\infty)}} \\
&= \sup_{\substack{0 \neq f \in X \\ 0 \neq g \in Y'(0,\infty)}} \frac{\int_0^\infty f^*(t) Qg^{**}(t)\, dt}{\|f^*\|_{X(0,\infty)} \|g^*\|_{Y'(0,\infty)}} \\
&= \sup_{\substack{0 \neq f \in X(0,\infty) \\ 0 \neq g \in Y'(0,\infty)}} \frac{\int_0^\infty f^*(t) Qg^{**}(t)\, dt}{\|f^*\|_{X(0,\infty)} \|g^*\|_{Y'(0,\infty)}} \\
&= \sup_{0 \neq g \in Y'(0,\infty)} \frac{\|Qg^{**}\|_{X'(0,\infty)}}{\|g^*\|_{Y'(0,\infty)}} \\
&= \sup_{0 \neq g \in Y'} \frac{\|Qg^{**}\|_{X'(0,\infty)}}{\|g^*\|_{Y'(0,\infty)}},
\end{aligned}$$

where the third and the last equality follow from Remark 1.43. Hence (2.5) is equivalent to (2.3) with $C = \sup_{0 \neq g \in Y'} \frac{\|Qg^{**}\|_{X'}}{\|g\|_{Y'}}$, which completes the proof. ❑

The following function space, which is about to be defined, is, in some sense, the biggest function space for which we can find a rearrangement-invariant target space for Hilbert transform. The precise meaning shall be made clear after the main theorem of the chapter (Theorem 2.13).

Definition 2.9. For the purpose of this chapter, we denote by Λ the rearrangement-invariant Banach function space $\Lambda = \Lambda_\varphi(\mathbb{R}, \lambda)$ where φ is defined as

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0, \\ t(2 + \log \frac{1}{t}) & \text{if } t \in (0, 1], \\ 2 + \log t & \text{if } t > 1. \end{cases}$$

We note that

$$\varphi'(t) = (1 + \log \frac{1}{t})\chi_{(0,1]}(t) + \frac{1}{t}\chi_{(1,\infty)}(t), \quad t > 0,$$

and $\varphi(0_+) = 0$. Hence (recall Remark 1.45)

$$\|f\|_\Lambda = \int_0^\infty f^*(t) \left((1 + \log \frac{1}{t})\chi_{(0,1]}(t) + \frac{1}{t}\chi_{(1,\infty)}(t) \right) dt, \quad f \in \Lambda.$$

It turns out that it is of great interest for us to know when the embedding $X \hookrightarrow \Lambda$, where X is a given rearrangement-invariant Banach function space, holds true. The following lemma provides a condition that is easy to verify in concrete applications.

Lemma 2.10. *Let X be a rearrangement-invariant Banach function space. Then the following three statements are equivalent.*

1. *It holds that*

$$\left(1 + \log\left(\frac{1}{t}\right)\right) \chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t) \in X'(0, \infty). \tag{2.7}$$

2. *For every $K > 0$, it holds that*

$$\left(1 + \log\left(\frac{K}{t}\right)\right) \chi_{(0,K)}(t) + \frac{K}{t}\chi_{(K,\infty)}(t) \in X'(0, \infty). \tag{2.8}$$

3. $$X \hookrightarrow \Lambda. \tag{2.9}$$

Proof. Assume that (2.7) holds. Fix $K > 0$. If $K \in (0, 1)$, then

$$\begin{aligned} \left(1 + \log\left(\frac{K}{t}\right)\right) \chi_{(0,K)}(t) + \frac{K}{t}\chi_{(K,\infty)}(t) &= \left(1 + \log\left(\frac{1}{t}\right)\right) \chi_{(0,K)}(t) + K\frac{1}{t}\chi_{(1,\infty)}(t) \\ &+ \log K \chi_{(0,K)}(t) + \frac{K}{t}\chi_{(K,1]}(t) \\ &\le \left(1 + \log\left(\frac{1}{t}\right)\right) \chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t) \\ &+ \log K \chi_{(0,K)}(t) + \chi_{(K,1]}(t) \end{aligned}$$

for every $t > 0$. Thus

$$\begin{aligned}
&\left\| \left(1+\log\left(\frac{K}{t}\right)\right)\chi_{(0,K)}(t) + \frac{K}{t}\chi_{(K,\infty)}(t) \right\|_{X'(0,\infty)} \\
&\le \left\| \left(1+\log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t) + \log K \chi_{(0,K)}(t) + \chi_{(K,1]}(t) \right\|_{X'(0,\infty)} \\
&\le \left\| \left(1+\log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t) \right\|_{X'(0,\infty)} + |\log K| \, \|\chi_{(0,K)}\|_{X'(0,\infty)} + \|\chi_{(K,1]}\|_{X'(0,\infty)} \\
&< \infty,
\end{aligned}$$

as $X'(0,\infty)$ is a (rearrangement-invariant) Banach function space. If $K \in (1,\infty)$, we proceed in the same way, using the fact that

$$\begin{aligned}
&\left(1+\log\left(\frac{K}{t}\right)\right)\chi_{(0,K)}(t) + \frac{K}{t}\chi_{(K,\infty)}(t) \\
&\le \left(1+\log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + K\frac{1}{t}\chi_{(1,\infty)}(t) + \log K \chi_{(0,1)}(t) + (1+\log K)\,\chi_{[1,K)}(t).
\end{aligned}$$

Hence (2.8) holds.

On the other hand, (2.8) clearly implies (2.7).

In order to complete the proof, we show that (2.7) and (2.9) are equivalent. We set

$$\varphi(t) = \frac{1}{\left(1+\log\left(\frac{1}{t}\right)\right)\chi_{(0,1]}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)}, \quad t > 0,$$

which is a positive increasing function as one can readily verify. By Proposition 1.54, (2.7) is equivalent to the fact that

$$X \hookrightarrow (m_\varphi)'.$$

Finally, we note that Proposition 1.53 implies that

$$(m_\varphi)' = \Lambda.$$

Hence the assertion follows.

❑

Assume that we are given a rearrangement-invariant Banach function space X. After all the work which we have done, it is not hard to see, and we shall do it in the proof of Theorem 2.13, that if we set

$$\varrho_{Y'_X}(g) = \|Qg^{**}\|_{X'(0,\infty)}, \quad g \in \mathfrak{M}_0^+, \tag{2.10}$$

then the space $Y'_X = Y'_X(\varrho_{Y'_X})$ is the associate space of the optimal rearrangement-invariant target space of X for the Hilbert transform, provided that $\varrho_{Y'_X}$ is a rearrangement-invariant function norm. The following lemma characterizes when it is the case. We shall see that every property of a (rearrangement-invariant) Banach function norm is automatically satisfied from the very definition of the functional $\varrho_{Y'_X}$ except for the non-triviality, that is, the sixth property of Definition 1.2.

If the norm of X' is too strong, then it may happen that the functional $\varrho_{Y'_X}$ is finite only on a zero function. Indeed, consider $X = L^1(\mathbb{R})$. Then

$$\varrho_{Y'_X}(g) = \|Qg^{**}\|_{L^\infty(0,\infty)} = \int_0^\infty \frac{g^{**}(s)}{s}\, ds,$$

which cannot be finite unless $g = 0$ as the function $\frac{1}{s}$ is not integrable near 0. More precisely, assume that $\varrho_{Y'_X}(g) < \infty$. Then

$$\infty > \int_0^\infty \frac{g^{**}(s)}{s}\, ds \geq \int_0^a \frac{g^{**}(s)}{s}\, ds \geq g^{**}(a) \int_0^a \frac{1}{s}\, ds = g^{**}(a) \cdot \infty$$

for each $a > 0$. Hence $g^{**} = 0$ and consequently $g(t) = 0$ for a.e. $t > 0$.

Lemma 2.11. *Let X be a rearrangement-invariant Banach function space and define*

$$\varrho_{Y'_X}(g) = \|Qg^{**}\|_{X'(0,\infty)}, \quad g \in \mathfrak{M}_0^+. \tag{2.11}$$

Then $\varrho_{Y'_X}(\cdot)$ is a rearrangement-invariant Banach function norm (over $(\mathbb{R}, \lambda)$) if and only if

$$X \hookrightarrow \Lambda. \tag{2.12}$$

Proof. Firstly, we show the sufficiency.

The first five properties (recall Definition 1.2) of a Banach function norm can be easily verified using the very definition of the operator Q, the fact that $X'(0,\infty)$ is itself a Banach function space, and the basic properties of the maximal function listed in Theorem 1.32.

In order to prove the sixth property of a Banach function norm, fix measurable $E \subseteq \mathbb{R}$ such that $|E| < \infty$. We straightforwardly compute that

$$\begin{aligned} Q\chi_E^{**}(t) &= \int_t^\infty \frac{1}{s^2} \int_0^s \chi_{(0,|E|)}(u)\, du\, ds \\ &= \int_t^{|E|} \frac{1}{s^2} \int_0^s \chi_{(0,|E|)}(u)\, du\, ds + \int_{|E|}^\infty \frac{1}{s^2} \int_0^s \chi_{(0,|E|)}(u)\, du\, ds \\ &= \int_t^{|E|} \frac{1}{s}\, ds + |E| \int_{|E|}^\infty \frac{1}{s^2}\, ds = 1 + \log\left(\frac{|E|}{t}\right) \end{aligned}$$

if $0 < t \leq |E|$, and

$$Q\chi_E^{**}(t) = \int_t^\infty \frac{1}{s^2} \int_0^s \chi_{(0,|E|)}(u)\, du\, ds = |E| \int_t^\infty \frac{1}{s^2}\, ds = \frac{|E|}{t}$$

if $t > |E|$. By Lemma 2.10, the assumption that $X \hookrightarrow \Lambda$ implies that

$$\varrho_{Y'_X}(\chi_E) = \|Q\chi_E^{**}\|_{X'(0,\infty)} = \|\left(1 + \log\left(\frac{|E|}{t}\right)\right)\chi_{(0,|E|)}(t) + \frac{|E|}{t}\chi_{(|E|,\infty)}(t)\|_{X'(0,\infty)} < \infty.$$

In order to prove the last property of a Banach function norm, fix measurable $E \subseteq \mathbb{R}$ such that $|E| < \infty$. Then, using (1.3), Theorem 1.15, and the fact that $f^* \leq f^{**}$ (see Theorem 1.32),

$$\begin{aligned} \int_E f(x)\, dx &\leq \int_0^{|E|} f^*(t)\, dt = \int_0^{|E|} \frac{f^*(t)}{t} \int_0^t ds\, dt = \int_0^{|E|} \int_s^{|E|} \frac{f^*(t)}{t}\, dt\, ds \\ &\leq \int_0^\infty \chi_{(0,|E|)}(s) Qf^{**}(s)\, ds \leq \|\chi_{(0,|E|)}\|_{X(0,\infty)} \|Qf^{**}\|_{X'(0,\infty)} \leq C_E \varrho_{Y'_X}(f), \end{aligned}$$

for every $f \in \mathfrak{M}_0^+$, where $C_E = \|\chi_{(0,|E|)}\|_{X(0,\infty)} + 1 < \infty$ as $X(0,\infty)$ is a Banach function space.

Hence $\varrho_{Y'_X}$ is a Banach function norm. It remains to prove that $\varrho_{Y'_X}$ is a rearrangement-invariant function norm. This, however, follows immediately from the very definition of $\varrho_{Y'_X}$ as

$$\varrho_{Y'_X}(f) = \|Qf^{**}\|_{X'(0,\infty)} = \|Qg^{**}\|_{X'(0,\infty)} = \varrho_{Y'_X}(g),$$

whenever $f, g \in \mathfrak{M}_0^+$ are equimeasurable (recall Remark 1.31).

Finally, the computations above also prove the necessity. ❑

Remark 2.12. We note the obvious fact that if $X \hookrightarrow \Lambda$, then (2.2) is satisfied for every $f \in X$. Hence the Hilbert transform H is well-defined on X by Theorem 2.4.

At last, we have prepared everything that we need to state and prove the main theorem of this chapter.

Theorem 2.13. *Let X be a rearrangement-invariant Banach function space such that*

$$X \hookrightarrow \Lambda. \tag{2.13}$$

Then $Y_X = \big(Y'(\varrho_{Y'_X})\big)'$, where $\varrho_{Y'_X}$ is defined by (2.11), is the optimal range partner of X for the Hilbert transform H in the class of rearrangement-invariant Banach function spaces, that is, $H : X \to Y_X$ is bounded and whenever $H : X \to Z$ is bounded where Z is a rearrangement-invariant Banach function space, then $Y_X \hookrightarrow Z$.

Moreover, the condition (2.13) is necessary in the sense that should there exist any rearrangement-invariant Banach function space Z such that $H : X \to Z$ is bounded, then $X \hookrightarrow \Lambda$.

Proof. Firstly, we observe that Y_X is really a rearrangement-invariant Banach function space. Indeed, $Y'(\varrho_{Y'_X})$ is a rearrangement-invariant Banach function space by Lemma 2.11, which means that $Y_X = \big(Y'(\varrho_{Y'_X})\big)'$ is a rearrangement-invariant Banach function space (see Theorem 1.35). Furthermore, we note that even the chosen notation is consistent since $(Y_X)' = \big(Y'(\varrho_{Y'_X})\big)'' = Y'(\varrho_{Y'_X})$ by Theorem 1.16, that is, the associate space of Y_X is indeed $Y'(\varrho_{Y'_X})$ as the notation indicates.

Secondly, we wish to prove the optimality of Y_X. Clearly,

$$\|Qg^{**}\|_{X'(0,\infty)} = \|g\|_{Y'_X} = \|g^*\|_{Y'_X(0,\infty)}$$

for every $g \in Y'_X$. Hence $H : X \to Y_X$ is bounded by Lemma 2.8 (recall Remark 2.12). Now, assume that $H : X \to Z$ is bounded where Z is a rearrangement-invariant Banach function space. By Lemma 2.8, there exists a positive constant C such that

$$\|g\|_{Y'_X} = \|Qg^{**}\|_{X'(0,\infty)} \leq C\|g^*\|_{Z'(0,\infty)} = C\|g\|_{Z'}$$

for every $g \in Z'$. This is, however, nothing else than $Z' \hookrightarrow Y'_X$. Hence $Y_X \hookrightarrow Z$ (recall Theorem 1.17 and Theorem 1.16).

Finally, we shall prove that the condition (2.13) is indeed necessary in the appropriate sense. Assume that there exists a rearrangement-invariant Banach function space Z such that $H : X \to Z$ is bounded. In particular, the Hilbert transform H is well-defined on X. Therefore, if we check the proof of Lemma 2.8 and recall Remark 2.7, we obtain that there exists a positive constant C such that

$$\|Qg^{**}\|_{X'(0,\infty)} \leq C\|g^*\|_{Z'(0,\infty)}$$

for every $g \in Z'$. Set $g = \chi_E$ where E is a measurable subset of $\mathbb{R}$ such that $|E| = 1$. Then $g \in Z'$, $g^* = \chi_{(0,1)}$, and (see the proof of Lemma 2.11)

$$Qg^{**}(t) = \left(1 + \log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t). \tag{2.14}$$

Thus $\left(1 + \log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t) \in X'(0,\infty)$ as

$$\|\left(1 + \log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)\|_{X'(0,\infty)} = \|Qg^{**}\|_{X'(0,\infty)} \leq C\|\chi_{(0,1)}\|_{Z'(0,\infty)} < \infty.$$

Hence (2.13) holds by Lemma 2.10.

❑

Remark 2.14. We stress the "moreover" part of the preceding theorem. It shows that the condition (2.13) imposed on X is, in fact, not restrictive, as should X violate it, there is no rearrangement-invariant target for the Hilbert transform at all, which makes the question of the optimal one irrelevant. The condition consequently provides an upper bound how large the domain space X can be in order to have any rearrangement-invariant target partner space with respect to the Hilbert transform.

Having proven Theorem 2.13, we have answered our question what the optimal rearrangement-invariant target space of a given rearrangement-invariant space for the Hilbert transform is. Nevertheless, the answer, which the theorem gives us, is fairly abstract. Firstly, we need to know what the associate space of the given function space is. Fortunately, this was extensively studied for common function spaces. Therefore, this usually does not pose a problem for us. However, the theorem does not provide us directly with the optimal target space itself but only with its associate space. This means that we have to be able to find the associate space of the function space Y'_X, which is defined by the means of the operator Q and the associate space of the given function space, in order to obtain the desired optimal target space. Thus it is advisable to provide some concrete examples which illustrate a possible usage of the theorem.

We begin with a simple but useful point-wise estimate of the operator Q.

Proposition 2.15. *Let $g \in \mathfrak{M}_0$ be arbitrary. Then*

$$g^{**}(t) \leq Qg^{**}(t)$$

for each $t > 0$.

Proof. Clearly

$$Qg^{**}(t) = \int_t^\infty \frac{g^{**}(s)}{s}\, ds = \int_t^\infty \frac{1}{s^2} \int_0^s g^*(u)\, du\, ds$$
$$\geq \int_t^\infty \frac{1}{s^2} \int_0^t g^*(u)\, du\, ds = g^{**}(t).$$

❑

Examples 2.16.

1. Assume that $p \in (1, \infty)$ and $q \in [1, \infty]$ and set $X = L^{p,q;\mathbb{A}}$. Then X is equivalent to a rearrangement-invariant Banach function space, X satisfies (2.13) and Y_X is equivalent to $L^{p,q;\mathbb{A}}$.

2. Set $X = L^{1,1;\mathbb{A}}$. Then X is equivalent to a rearrangement-invariant Banach function space if and only if $\alpha_0 \geq 0$ and $\alpha_\infty \leq 0$. Assume that $\alpha_0 \geq 0$ and $\alpha_\infty \leq 0$. Then X satisfies (2.13) if and only if $\alpha_0 \geq 1$. If $\alpha_0 \geq 1$ and $\alpha_\infty < 0$, then Y_X is equivalent to $L^{1,1;\mathbb{A}-1}$.

   Moreover, there is no rearrangement-invariant target space if $\alpha_0 \in [0, 1)$ and $\alpha_\infty \leq 0$.

3. Set $X = L^{\infty,\infty;\mathbb{A}}$. Then X is equivalent to a rearrangement-invariant Banach function space if and only if $\alpha_0 \leq 0$. Assume that $\alpha_0 \leq 0$. Then X satisfies (2.13) if and only if $\alpha_\infty > 1$. If $\alpha_0 \leq 0$ and $\alpha_\infty > 1$, then Y_X is equivalent to $L^{\infty,\infty;\mathbb{A}-1}$.

   Moreover, there is no rearrangement-invariant target space if $\alpha_0 \leq 0$ and $\alpha_\infty \leq 1$.

4. Set $X = L^{\infty,q;\mathbb{A}}$ where $q \in (1, \infty)$. Then X is equivalent to a rearrangement-invariant Banach function space if and only if $\alpha_0 < -\frac{1}{q}$. Assume that $\alpha_0 < -\frac{1}{q}$. Then X satisfies (2.13) if and only if $\alpha_\infty > \frac{1}{q'}$. If $\alpha_0 < -\frac{1}{q}$ and $\alpha_\infty > \frac{1}{q'}$, we can use Theorem 2.13 in order to obtain the optimal range partner of X for the Hilbert transform, but we do not know an explicit description of Y_X.

   Moreover, there is no rearrangement-invariant target space if $\alpha_0 < -\frac{1}{q}$ and $\alpha_\infty \leq \frac{1}{q'}$.

5. Set $X = L^{\infty,1;\mathbb{A}}$. Then X is equivalent to a rearrangement-invariant Banach function space if and only if $\alpha_0 < -1$. Assume that $\alpha_0 < -1$. Then X satisfies (2.13) if and only if $\alpha_\infty \geq 0$. If $\alpha_0 < -1$ and $\alpha_\infty \geq 0$, we can use Theorem 2.13 in order to obtain the optimal range partner of X for the Hilbert transform, but we do not know an explicit description of Y_X.

   Moreover, there is no rearrangement-invariant target space if $\alpha_0 < -1$ and $\alpha_\infty < 0$.

Proof. We note that the equivalence of X to a rearrangement-invariant Banach function space is characterized by Theorem 1.59.

We shall prove just the second example because the remaining ones are similar and easier. The last two examples involve lengthy but straightforward verifications of convergence of integrals. For the reader's convenience, we note here that if $f(t) = (1 + \log(\frac{1}{t}))\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)$, then

$$f^{**}(t) = \left(2 + \log\left(\frac{1}{t}\right)\right)\chi_{(0,1]}(t) + \frac{2 + \log t}{t}\chi_{(1,\infty)}(t).$$

Firstly, we need to verify that (2.13) holds, that is by Lemma 2.10 and Theorem 1.62, we need to check that

$$\|\ell^{-\mathbb{A}}(t)\left(\left(1 + \log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)\right)\|_{\infty} < \infty.$$

It is easy to see that this quantity is finite if and only if $\alpha_0 \geq 1$. Hence there is no rearrangement-invariant target space if $\alpha_0 \in [0, 1)$ and $\alpha_\infty \leq 0$ by the moreover part of Theorem 2.13.

On the one hand, we have

$$\begin{aligned}\|Qg^{**}\|_{\infty,\infty;-\mathbb{A}} &= \|\ell^{-\mathbb{A}}(t)\int_t^\infty \frac{g^{**}(u)}{u}\,du\|_\infty \lesssim \|\ell^{-\mathbb{A}+1}(t)g^{**}(t)\|_\infty \\ &= \|g\|_{(\infty,\infty;-\mathbb{A}+1)} \approx \|g\|_{\infty,\infty;-\mathbb{A}+1},\end{aligned}$$

where the inequality follows from Theorem 1.65 and the last equality from Theorem 1.60.

On the other hand, we need to show that

$$\|g\|_{\infty,\infty;-\mathbb{A}+1} \lesssim \|Qg^{**}\|_{\infty,\infty;-\mathbb{A}}.$$

It is sufficient to show that $\|g\|_{\infty,\infty;-\mathbb{A}+1}$ is finite whenever $\|Qg^{**}\|_{\infty,\infty;-\mathbb{A}}$ is finite by virtue of Theorem 1.8. Assume that $\|Qg^{**}\|_{\infty,\infty;-\mathbb{A}} < \infty$. Then we compute that

$$\begin{aligned}\|Qg^{**}\|_{\infty,\infty;-\mathbb{A}} &\geq \sup_{t\in(0,\frac{1}{4}]}\left(1 + \log\frac{1}{t}\right)^{-\alpha_0}\int_t^\infty \frac{g^*(u)}{u}\,du \\ &= \sup_{t\in(0,\frac{1}{2}]}\left(1 + \log\frac{1}{t^2}\right)^{-\alpha_0}\int_{t^2}^\infty \frac{g^*(u)}{u}\,du \\ &\geq \sup_{t\in(0,\frac{1}{2}]}\left(1 + \log\frac{1}{t^2}\right)^{-\alpha_0}\int_{t^2}^t \frac{g^*(u)}{u}\,du \\ &\geq \sup_{t\in(0,\frac{1}{2}]}\left(1 + \log\frac{1}{t^2}\right)^{-\alpha_0} g^*(t)\log\frac{1}{t} \\ &\approx \sup_{t\in(0,\frac{1}{2}]}\left(1 + \log\frac{1}{t}\right)^{-\alpha_0+1} g^*(t).\end{aligned}$$

Similarly, we compute that

$$\begin{aligned}\|Qg^{**}\|_{\infty,\infty;-\mathbb{A}} &\geq \sup_{t\in[\sqrt{e},\infty)} (1+\log t)^{-\alpha_\infty} \int_t^\infty \frac{g^*(u)}{u}\,du \\ &= \sup_{t\in[e,\infty)} \left(1+\log\sqrt{t}\right)^{-\alpha_\infty} \int_{\sqrt{t}}^\infty \frac{g^*(u)}{u}\,du \\ &\geq \sup_{t\in[e,\infty)} \left(1+\log\sqrt{t}\right)^{-\alpha_\infty} \int_{\sqrt{t}}^t \frac{g^*(u)}{u}\,du \\ &\geq \sup_{t\in[e,\infty)} \left(1+\log\sqrt{t}\right)^{-\alpha_\infty} g^*(t)\log\sqrt{t} \\ &\approx \sup_{t\in[e,\infty)} (1+\log t)^{-\alpha_\infty+1}\, g^*(t).\end{aligned}$$

It just remains to observe that

$$\sup_{t\in[\frac{1}{2},e]} \ell^{-\mathbb{A}+1}(t)g^*(t) \leq g^*(e) \sup_{t\in[\frac{1}{2},e]} \ell^{-\mathbb{A}+1}(t) < \infty$$

because $g^*(t)$ is finite for each $t>0$ since $Qg^*(t)$ must be finite for each $t>0$ whenever $\|Qg^{**}\|_{\infty,\infty;-\mathbb{A}} < \infty$. Hence

$$\|g\|_{\infty,\infty;-\mathbb{A}+1} < \infty.$$

Overall, we have shown that

$$\|g\|_{\infty,\infty;-\mathbb{A}+1} \approx \|Qg^{**}\|_{\infty,\infty;-\mathbb{A}}.$$

Hence the optimal range space Y_X is indeed equivalent to $L^{1,1;\mathbb{A}-1}$ by Theorem 2.13 and Theorem 1.62.

❑

Remark 2.17. Assume that $\mathbb{A}=(\alpha_0,0)$ where $\alpha_0\geq 1$. Then $L^{1,1;\mathbb{A}}$ is equivalent to a rearrangement-invariant Banach function space and (2.13) is satisfied. Hence one can use Theorem 2.13 to construct the optimal rearrangement-invariant target space. Unfortunately, we have not been able to find an explicit description of the associate space of Y'_X. One needs to find the associate space of the space given by the rearrangement-invariant norm $\|g\|_{Y'_X} = \|\ell^{-\mathbb{A}}(t)Qg^{**}(t)\|_\infty$. A similar type of spaces, namely so-called *Copson-Lorentz spaces*, has been recently introduced in [18]. However, the case $q=\infty$ is not covered there and what is even more important is that Copson-Lorentz spaces are defined by means of the non-increasing rearrangement not the maximal function, which further complicates the matter.

A similar problem arises when one considers the case $p=\infty$ and $q\in[1,\infty)$. This time, even the description of the associate space of $L^{\infty,q;\mathbb{A}}$ is quite complicated (see [22, Theorem 6.2 and Theorem 6.6]).

We stress here the important case of Lebesgue spaces. There is no rearrangement-invariant target space for L^1 even though the Hilbert transform is known to be of weak type $(1,1)$ (see [2, Chapter 3, Theorem 4.9]). These two facts are, however, consistent with each other because $L^{1,\infty}$ is not equivalent to a rearrangement-invariant Banach function space (recall Theorem 1.59). If $p\in(1,\infty)$, then L^p is the optimal range partner of itself. If $p=\infty$, then the Hilbert transform, as defined here, is not well-defined on L^∞ (to see this, consider $f\equiv 1$).

Now, we shall answer the opposite question than Theorem 2.13. This time, we do not describe the optimal target space of a given domain space, but instead we describe the optimal domain space of a given target space. We note that there are obvious parallels with Theorem 2.13. One can also observe that the description of the optimal domain space is simpler than the description of the optimal range space, since there are no associate spaces involved.

Theorem 2.18. *Let Y be a rearrangement-invariant Banach function space such that*

$$m_\varphi \hookrightarrow Y, \tag{2.15}$$

where

$$\varphi(t) = \frac{1}{\left(1 + \log\left(\frac{1}{t}\right)\right) \chi_{(0,1]}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)}, \quad t > 0.$$

Then $X_Y = X_Y(\varrho_{X_Y})$, where ϱ_{X_Y} is defined by

$$\varrho_{X_Y}(f) = \|Qf^{**}\|_{Y(0,\infty)}, \quad f \in \mathfrak{M}_0^+,$$

is the optimal domain partner of Y for the Hilbert transform H in the class of rearrangement-invariant Banach function spaces, that is, $H : X_Y \to Y$ is bounded and whenever $H : Z \to Y$ is bounded where Z is a rearrangement-invariant Banach function space, then $Z \hookrightarrow X_Y$.

Moreover, the condition (2.15) *is necessary in the sense that should there exist any rearrangement-invariant Banach function space W such that $H : W \to Y$ is bounded, then $m_\varphi \hookrightarrow Y$.*

Proof. Firstly, we observe that $Y' \hookrightarrow \Lambda$. Indeed, it comes from Proposition 1.54 and Lemma 2.10 applied to Y' (recall Theorem 1.16). By Lemma 2.11 applied to Y', ϱ_{X_Y} is a rearrangement-invariant Banach function norm. Hence X_Y is indeed a rearrangement-invariant Banach function space.

Secondly, we observe that (2.2) is satisfied by each $f \in X_Y$. Hence, in particular, the Hilbert transform H is well-defined on X_Y by Theorem 2.4. Indeed, for every $f \in X_Y$, we have that

$$\|Qf^{**}\|_{Y(0,\infty)} = \|f\|_{X_Y} < \infty.$$

Hence Qf^{**} is finite a.e. (recall Theorem 1.5) for every $f \in X_Y$. Assume for a contradiction that there exists a function $f \in X_Y$ such that $Qf^{**}(1) = \infty$. Then $Qf^{**}(t) = \infty$ for every $t \in (0, 1]$ as Qf^{**} is non-increasing. However, this means that Qf^{**} attains the value ∞ on a set of positive measure, which contradicts the fact that Qf^{**} is finite a.e. Hence $Qf^{**}(1) < \infty$ for each $f \in X_Y$.

Now, we shall prove the optimality of X_Y. On the one hand, $H : X_Y \to Y$ is bounded. Indeed, by Theorem 2.5, there exists a positive constant C such that

$$\|Hf\|_Y = \|(Hf)^*\|_{Y(0,\infty)} \leq C\|Qf^{**}\|_{Y(0,\infty)} = C\|f\|_{X_Y}$$

for every $f \in X_Y$, which is nothing else than the desired boundedness. On the other hand, assume that $H : Z \to Y$ is bounded where Z is a rearrangement-invariant Banach function space. We claim that $Z \hookrightarrow X_Y$. Indeed, there exists a positive constant C such that

$$\|Hf\|_Y \leq C\|f\|_Z$$

for every $f \in Z$. Fix $f \in Z$. By Theorem 2.6, we have that

$$\|f\|_{X_Y} = \|Qf^{**}\|_{Y(0,\infty)} \le 2\pi \|(Hg)^*\|_{Y(0,\infty)} = 2\pi \|Hg\|_Y \le 2\pi C \|g\|_Z = 2\pi C \|f\|_Z$$

where g is a function equimeasurable with f. Hence $Z \hookrightarrow X_Y$ as $f \in Z$ was chosen arbitrarily.

Lastly, we wish to prove the necessity of (2.15) in the appropriate sense. Assume that $H : W \to Y$ is bounded. Set $f = \chi_{(0,1)}$. Then (see the proof of Lemma 2.11) $Qf^{**}(1) = 1 < \infty$ and by Theorem 2.6 we have that

$$\|\left(1+\log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)\|_{Y(0,\infty)} = \|Qf^{**}\|_{Y(0,\infty)} \le C\|\chi_{(0,1)}\|_W < \infty.$$

Hence $m_\varphi \hookrightarrow Y$ by Proposition 1.54 applied to Y'.

❑

The main results of this chapter, namely Theorem 2.13 and Theorem 2.18, can be combined to obtain the following theorem.

Theorem 2.19. *Let X be a rearrangement-invariant Banach function space. Then there exists a positive constant C such that*

$$\|Qf^{**}\|_{X(0,\infty)} \le C\|f\|_X, \quad \forall f \in \mathfrak{M}_0, \tag{2.16}$$

if and only if (X, X) is an optimal pair of rearrangement-invariant spaces for the Hilbert transform, that is, X is the optimal range partner of X for the Hilbert transform in the sense of Theorem 2.13 and simultaneously X is the optimal domain partner of X for the Hilbert transform in the sense of Theorem 2.18.

Proof. Assume that (2.16) holds. Firstly, we observe that also

$$\|Qf^{**}\|_{X'(0,\infty)} \le C\|f\|_{X'}$$

holds for each $f \in \mathfrak{M}_0$. Indeed, using (2.6) and Corollary 1.36, we obtain that

$$\begin{aligned}\|Qf^{**}\|_{X'(0,\infty)} &= \sup_{\|g\|_X \le 1} \int_0^\infty Qf^{**}(t)g^*(t)\,dt = \sup_{\|g\|_X \le 1} \int_0^\infty f^*(t)Qg^{**}(t)\,dt \\ &\le \sup_{\|g\|_X \le 1} \|f\|_{X'}\|Qg^{**}\|_{X(0,\infty)} \le C \sup_{\|g\|_X \le 1} \|f\|_{X'}\|g\|_X = C\|f\|_{X'}.\end{aligned}$$

Consequently, recalling Proposition 2.15, we have that

$$\|Qf^{**}\|_{X(0,\infty)} \approx \|f\|_X \text{ and } \|Qf^{**}\|_{X'(0,\infty)} \approx \|f\|_{X'}. \tag{2.17}$$

Secondly, using (2.14), we compute that

$$\|\left(1+\log\left(\frac{1}{t}\right)\right)\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)\|_{X'(0,\infty)} = \|Q\chi_{(0,1)}^{**}\|_{X'(0,\infty)} \le C\|\chi_{(0,1)}\|_{X'} < \infty,$$

which is nothing else than $X \hookrightarrow \Lambda$ by Lemma 2.10. Hence, X is the optimal range partner of X for the Hilbert transform by Theorem 2.13 and (2.17).

Likewise, it can be shown that X satisfies (2.15) using Proposition 1.54 instead of Lemma 2.10. Hence, X is the optimal domain partner of X for the Hilbert transform by Theorem 2.18 and (2.17).

Finally, assume that (X, X) is an optimal pair in the sense of this theorem. In particular, this means that the Hilbert transform is bounded from X to X. By the "moreover" part of the Theorem 2.18, X must satisfy (2.15). Now, we can apply Theorem 2.18 to $Y = X$ to obtain X_Y. However, X_Y must be equivalent to X, which means that

$$\|Qf^{**}\|_{X(0,\infty)} = \|f\|_{X_Y} \approx \|f\|_X.$$

❑

Remark 2.20. The validity of (2.16) has an intimate connection with boundedness of the Hilbert transform on the associate space (recall Lemma 2.8). The boundedness of the Hilbert transform on Orlicz spaces, which are an important type of a function space not covered here, is characterized in [3].

Example 2.21. Let $X = L^{p,q;\mathbb{A}}$ where $p \in (1,\infty)$, $q \in [1,\infty]$ and $\mathbb{A} \in \mathbb{R}^2$. Then (X, X) is an optimal pair for the Hilbert transform in the sense of the preceding theorem. Indeed, it follows from the first example in Examples 2.16 that the optimal range partner of $L^{p',q';-\mathbb{A}}$ is equivalent to $L^{p',q';-\mathbb{A}}$ itself, whence (recall Theorem 1.62)

$$\|Qf^{**}\|_{p,q;\mathbb{A}} = \|f\|_{(Y_{L^{p',q';-\mathbb{A}}})'} \approx \|f\|_{p,q;\mathbb{A}}.$$

The statement then follows immediately from the preceding theorem. In particular, (L^p, L^p) is an optimal pair for the Hilbert transform for $p \in (1,\infty)$.

We note that in the other cases mentioned in Examples 2.16, the Hilbert transform H is not bounded on their associate spaces.

3. Riesz Potential

Another classic singular integral operator is the Riesz potential, which we shall focus on now. The Riesz potential has an intimate connection with the theory of the $p-$*Laplacian*, that is, a fractional power of the Laplacian operator. For more details, see for example [25, Chapter V].

Convention. In this chapter, we assume that $(R, \mu) = (\mathbb{R}^n, \lambda)$ where λ is the Lebesgue measure over $\mathbb{R}^n$ and n is a (fixed) dimension. We shall write $|E|$ instead of $\lambda(E)$ for $\lambda-$measurable $E \subseteq \mathbb{R}^n$.

Definition 3.1. Assume that $f \in L^1_{loc}(\mathbb{R}^n)$ and $\gamma \in (0, n)$. We define its *Riesz potential (of order γ)* by

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy, \quad x \in \mathbb{R}^n,$$

provided that the integral exists for a.e. $x \in \mathbb{R}^n$. The operator $I_\gamma : f \mapsto I_\gamma f$ itself is also referred to as the *Riesz potential.*

Remark 3.2. As with the Hilbert transform, the Riesz potential is not well-defined for arbitrary $f \in L^1_{loc}(\mathbb{R}^n)$ as the integral need not exist. Therefore, whenever we say that the Riesz potential is bounded from a function space X to a function space Y, we implicitly mean that the Riesz potential is well-defined for every $f \in X$. Later on, we shall provide a sufficient condition ensuring that the Riesz potential is well-defined on X.

We also note that our definition of the Riesz potential differs from the definition usually found in the literature by a multiplicative constant. We omit the constant because it is completely irrelevant for our purpose.

As with the Hilbert transform, we start with reducing the complicated problem involving the Riesz potential to another problem involving a simpler one-dimensional operator T_γ, which is in some sense equivalent.

Definition 3.3. Let $\gamma \in (0, n)$. For the purpose of this chapter, we define the operator T_γ by

$$T_\gamma f(t) = \int_t^\infty f(s) s^{\frac{\gamma}{n}-1}\, ds, \quad t \in (0, \infty), f \in \mathfrak{M}_0^+.$$

Theorem 3.4. *Let $\gamma \in (0, n)$ and assume that $f \in L^1_{loc}(\mathbb{R}^n)$. Assume that $I_\gamma f$ is well-defined. Then there exists a positive constant C independent of f such that*

$$(I_\gamma f)^*(t) \le C T_\gamma f^{**}(t)$$

for every $t > 0$.

Proof. It follows from the O'Neil inequality (see [20, Lemma 1.5]). ❑

Theorem 3.5. *Let $\gamma \in (0,n)$ and assume that $f \in L^1_{loc}(\mathbb{R}^n)$. Then there exist a function g equimeasurable with f and a positive constant C independent of f such that*

$$(I_\gamma g)^*(t) \geq C T_\gamma f^{**}(t)$$

for every $t > 0$.

Proof. We set

$$g(y) = f^*(\omega_n |y|^n), \quad y \in \mathbb{R}^n,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$, and observe that g is equimeasurable with f (cf. [2, Chapter 2, Proposition 7.2]). The rest of the proof follows immediately from [8, Lemma 3.4].

❑

Lemma 3.6. *Let X and Y be rearrangement-invariant Banach function spaces and assume that $\gamma \in (0,n)$. Furthermore, assume that the Riesz potential I_γ is well-defined on X. Then the Riesz potential $I_\gamma : X \to Y$ is bounded if and only if there exists a positive constant C such that*

$$\|T_\gamma g^{**}\|_{X'(0,\infty)} \leq C \|g^*\|_{Y'(0,\infty)}$$

for each $g \in Y'$.

Proof. We omit the proof here because this lemma can be proven in the same way as Lemma 2.8. We simply use Theorem 3.4 and Theorem 3.5 instead of Theorem 2.5 and Theorem 2.6.

❑

The function $t \in (0,\infty) \mapsto t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t)$ and especially its non-increasing rearrangement will prove to be useful for use. Hence we state this auxiliary computation as a separate lemma.

Lemma 3.7. *Let $\gamma \in (0,n)$. Set*

$$v(t) = t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t), \quad t \in (0,\infty).$$

Then

$$v^*(t) = (t+1)^{\frac{\gamma-n}{n}}, \quad t \in (0,\infty).$$

Proof. The proof is just a matter of simple computation. We straightforwardly compute, using the very definition of the distribution function of v, that

$$\lambda_v(t) = (t^{\frac{n}{\gamma-n}} - 1)\chi_{(0,1)}(t)$$

for $t \in (0,\infty)$. Hence, using the definition of the non-increasing rearrangement,

$$v^*(t) = (t+1)^{\frac{\gamma-n}{n}}.$$

❑

The following function space is the biggest function space for which there exists any rearrangement-invariant target space for the Riesz potential. We shall see the precise meaning of it in the main theorem of this chapter (Theorem 3.14).

Definition 3.8. For the purpose of this chapter, we denote (for fixed $\gamma \in (0,n)$) by Λ the rearrangement-invariant Banach function space $\Lambda = \Lambda_\varphi(\mathbb{R}^n, \lambda)$ where φ is defined as

$$\varphi(t) = \frac{n}{\gamma}\left((t+1)^{\frac{\gamma}{n}} - 1\right), \quad t \geq 0.$$

We note that

$$\varphi'(t) = (t+1)^{\frac{\gamma-n}{n}}, \quad t > 0,$$

and $\varphi(0_+) = 0$. Hence (recall Remark 1.45)

$$\|f\|_\Lambda = \int_0^\infty f^*(t)\,(t+1)^{\frac{\gamma-n}{n}}\,dt, \quad f \in \Lambda.$$

Lemma 3.9. *Let X be a rearrangement-invariant Banach function space and $\gamma \in (0,n)$. The following three statements are equivalent.*

1. *It holds that*
$$t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t) \in X'(0,\infty). \tag{3.1}$$
2. *For each $K > 0$, it holds that*
$$t^{\frac{\gamma}{n}-1}\chi_{[K,\infty)}(t) \in X'(0,\infty). \tag{3.2}$$
3. *It holds that*
$$X \hookrightarrow \Lambda. \tag{3.3}$$

Proof. Firstly, in order to prove that (3.1) implies (3.2), we can proceed in a similar way to the proof of Lemma 2.10.

Secondly, assume that (3.2) holds and we shall prove that (3.3) holds. Fix $f \in X$. Then, using Theorem 1.15 and Lemma 3.7,

$$\begin{aligned}\|f\|_\Lambda &= \int_0^\infty f^*(t)(t+1)^{\frac{\gamma-n}{n}}\,dt \leq \|f^*\|_{X(0,\infty)}\|(t+1)^{\frac{\gamma-n}{n}}\|_{X'(0,\infty)} \\ &= \|t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t)\|_{X'(0,\infty)}\|f\|_X < \infty.\end{aligned}$$

Hence, $X \hookrightarrow \Lambda$.

Finally, we shall prove that (3.3) implies (3.1). Recalling Lemma 3.7 and Remark 1.43, we compute that

$$\begin{aligned}\|t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}\|_{X'(0,\infty)} &= \sup_{f\in X, \|f\|_X \leq 1} \int_0^\infty f^*(t)(t+1)^{\frac{\gamma-n}{n}}\,dt \\ &= \sup_{f\in X, \|f\|_X\leq 1} \|f\|_\Lambda \leq C < \infty\end{aligned}$$

where C is a positive constant independent of f which exists as we assume that (3.3) holds. Hence (3.1) holds, which completes the proof. ❑

It will be of particular interest for us when the functional

$$\varrho^{(\gamma)}_{Y'_X}(g) = \|T_\gamma g^{**}\|_{X'(0,\infty)}, \quad g \in \mathfrak{M}_0^+,$$

defines a rearrangement-invariant function norm for a fixed X. We characterize it by means of an embedding into the space Λ.

Lemma 3.10. *Let X be a rearrangement-invariant Banach function space. The functional $\varrho^{(\gamma)}_{Y'_X}$ defined by*

$$\varrho^{(\gamma)}_{Y'_X}(g) = \|T_\gamma g^{**}\|_{X'(0,\infty)}, \quad g \in \mathfrak{M}_0^+, \tag{3.4}$$

is a rearrangement-invariant Banach function norm if and only if

$$X \hookrightarrow \Lambda.$$

Proof. Proving the sufficiency, as in the case of the Hilbert transform (see Lemma 2.11), one can readily verify that the first five properties of a Banach function norm are satisfied. In order to prove the sixth property, assume that $E \subseteq \mathbb{R}^n$ is a $\lambda-$measurable set of finite measure. For $t > 0$, we easily see that

$$T_\gamma \chi_E^{**}(t) = \int_t^\infty s^{\frac{\gamma}{n}-2} \int_0^s \chi_{(0,|E|)}(u)\,du\,ds. \tag{3.5}$$

Assume that $|E| \le t$. Then, using (3.5), we compute that

$$T_\gamma \chi_E^{**}(t) = |E| \int_t^\infty s^{\frac{\gamma}{n}-2}\,ds = \frac{|E|\,n}{n-\gamma} t^{\frac{\gamma}{n}-1}. \tag{3.6}$$

Now, assume that $t < |E|$. Then

$$\begin{aligned} T_\gamma \chi_E^{**}(t) &= \int_t^{|E|} s^{\frac{\gamma}{n}-2} \int_0^s \chi_{(0,|E|)}(u)\,du\,ds + \int_{|E|}^\infty s^{\frac{\gamma}{n}-2} \int_0^s \chi_{(0,|E|)}(u)\,du\,ds \\ &= \int_t^{|E|} s^{\frac{\gamma}{n}-1}\,ds + |E| \int_{|E|}^\infty s^{\frac{\gamma}{n}-2}\,ds = \frac{n}{\gamma}\left(|E|^{\frac{\gamma}{n}} - t^{\frac{\gamma}{n}}\right) + \frac{|E|^{\frac{\gamma}{n}}\,n}{n-\gamma}. \end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we arrive at

$$T_\gamma \chi_E^{**}(t) = \left(\frac{n}{\gamma}\left(|E|^{\frac{\gamma}{n}} - t^{\frac{\gamma}{n}}\right) + \frac{|E|^{\frac{\gamma}{n}}\,n}{n-\gamma}\right)\chi_{(0,|E|)}(t) + \frac{|E|\,n}{n-\gamma} t^{\frac{\gamma}{n}-1}\chi_{[|E|,\infty)}(t) \tag{3.8}$$

for each $t > 0$. Hence

$$\begin{aligned} \varrho^{(\gamma)}_{Y'_X}(\chi_E) = \|T_\gamma \chi_E^{**}(t)\|_{X'(0,\infty)} &\le \frac{|E|\,n}{n-\gamma}\|t^{\frac{\gamma}{n}-1}\chi_{[|E|,\infty)}(t)\|_{X'(0,\infty)} \\ &+ \left(\frac{n}{\gamma} + \frac{n}{n-\gamma}\right)|E|^{\frac{\gamma}{n}}\,\|\chi_{(0,|E|)}(t)\|_{X'(0,\infty)} < \infty \end{aligned}$$

since the assumption that $X \hookrightarrow \Lambda$ is equivalent to the fact that $t^{\frac{\gamma}{n}-1}\chi_{[|E|,\infty)}(t) \in X'(0,\infty)$ by Lemma 3.9.

Finally, we shall verify the last property of a Banach function norm. Assume that $E \subseteq \mathbb{R}^n$ is a $\lambda-$measurable set of finite measure and $g \in \mathfrak{M}_0^+$. We may assume that $|E| > 0$ because there is nothing to prove when $|E| = 0$. One can easily verify the following chain of (in)equalities. For every $t > 0$,

$$\begin{aligned}T_\gamma g^{**}(t) &= \int_t^\infty g^{**}(s)s^{\frac{\gamma}{n}-1}\,ds = \int_t^\infty s^{\frac{\gamma}{n}-2}\int_0^s g^*(u)\,du\,ds\\ &\geq \chi_{(0,|E|)}(t)\int_{|E|}^\infty s^{\frac{\gamma}{n}-2}\int_0^s g^*(u)\,du\,ds\\ &\geq \chi_{(0,|E|)}(t)\int_{|E|}^\infty s^{\frac{\gamma}{n}-2}\int_0^{|E|} g^*(u)\,du\,ds\\ &= \frac{n}{n-\gamma}\,|E|^{\frac{\gamma}{n}-1}\,\chi_{(0,|E|)}(t)\int_0^{|E|} g^*(u)\,du\\ &\geq \frac{n}{n-\gamma}\,|E|^{\frac{\gamma}{n}-1}\,\chi_{(0,|E|)}(t)\int_E g(x)\,dx,\end{aligned}$$

where the last inequality is just (1.3). Hence

$$\int_E g(x)\,dx \leq C_E\,\varrho^{(\gamma)}_{Y'_X}(g)$$

with $C_E = \frac{n-\gamma}{n}\,|E|^{1-\frac{\gamma}{n}}\,\frac{1}{\|\chi_{(0,|E|)}\|_{X'(0,\infty)}}$.

Clearly $\varrho^{(\gamma)}_{Y'_X}(g) = \varrho^{(\gamma)}_{Y'_X}(h)$ whenever g and h are equimeasurable. Hence $\varrho^{(\gamma)}_{Y'_X}$ is a rearrangement-invariant Banach function norm as we wished to prove.

The necessity of the embedding is obvious from the computations above (namely (3.8)) as

$$\varrho^{(\gamma)}_{Y'_X}(\chi_E) = \|T_\gamma \chi_E^{**}(t)\|_{X'(0,\infty)} \geq \frac{|E|\,n}{n-\gamma}\|t^{\frac{\gamma}{n}-1}\chi_{[|E|,\infty)}(t)\|_{X'(0,\infty)},$$

where the left-hand side is finite for each $E \subseteq \mathbb{R}^n$ of finite measure, provided that $\varrho^{(\gamma)}_{Y'_X}$ is a (rearrangement-invariant) Banach function norm. Hence $X \hookrightarrow \Lambda$ by Lemma 3.9.

❑

As with the Hilbert transform, the Riesz potential need not exist for every locally integrable function. Indeed, set

$$f(x) = \chi_{\{y=(y_1,\ldots,y_n)\in\mathbb{R}^n;y_1>0\}}(x) - \chi_{\{y=(y_1,\ldots,y_n)\in\mathbb{R}^n;y_1<0\}}(x), \quad x \in \mathbb{R}^n.$$

Then $I_\alpha f$ does not exist as $I_\alpha f^+(x) = I_\alpha f^-(x) = \infty$ for each $x \in \mathbb{R}^n$. Hence we would like to have a suitable condition that ensures that the Riesz potential is well-defined. The following conditions are well-known and we utilize them in order to prove that the Riesz potential is well-defined on X whenever $X \hookrightarrow \Lambda$ holds true.

Proposition 3.11. *Let $\gamma \in (0,n)$. Assume that $f \in L^p$ for $1 \leq p < \frac{n}{\gamma}$ or $f \in L^{\frac{n}{\gamma},1}$. Then the Riesz potential $I_\gamma f(x)$ is finite for a.e. $x \in \mathbb{R}^n$. In particular, I_γ is well-defined.*

Proof. It is just a matter of straightforward computations to verify that

$$h^*(t) = \omega_n^{1-\frac{\gamma}{n}} t^{\frac{\gamma}{n}-1}, \quad t > 0,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and the function h is defined by

$$h(y) = \frac{1}{|x-y|^{n-\gamma}}, \quad y \in \mathbb{R}^n,$$

for a fixed point $x \in \mathbb{R}^n$.

Suppose that $f \in L^{\frac{n}{\gamma},1}$. Then, using Theorem 1.26, we have that

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\gamma}}\, dy \le \omega_n^{1-\frac{\gamma}{n}} \int_0^\infty f^*(t) t^{\frac{\gamma}{n}-1}\, dt = \omega_n^{1-\frac{\gamma}{n}} \|f\|_{L^{\frac{n}{\gamma},1}}.$$

Hence, the Riesz potential $I_\gamma f(x)$ is finite for every $x \in \mathbb{R}^n$ provided that $f \in L^{\frac{n}{\gamma},1}$.

The fact that the Riesz potential $I_\gamma f(x)$ is finite for a.e. $x \in \mathbb{R}^n$ provided that $f \in L^p$ for $1 \le p < \frac{n}{\gamma}$ is more involved. A proof can be found in [15] (even for a generalized Riesz potential) or in the classical book on the matter [25, page 119-121].

❑

Remark 3.12. In fact, far more can be said. The Riesz potential is known to be of *strong type* $(p, \frac{np}{n-\gamma p})$ for $1 < p < \frac{n}{\gamma}$ and of *weak type* $(1, \frac{n}{n-\gamma})$. For details, see again [25, page 119-121].

Proposition 3.13. *Assume that $\gamma \in (0,n)$ and let X be a rearrangement-invariant Banach function space such that*

$$X \hookrightarrow \Lambda.$$

Then the Riesz potential is well-defined on X.

Proof. Fix $f \in X$, $x \in \mathbb{R}^n$, and set $h(y) = \frac{\chi_{\{z\in\mathbb{R}^n;|z-x|>1\}}(y)}{|y-x|^{n-\gamma}}$ for $y \in \mathbb{R}^n$. It is just a matter of a straightforward computation to check that

$$\lambda_h(t) = \begin{cases} \omega_n \left(\frac{1}{t^{\frac{n}{n-\gamma}}} - 1 \right) & \text{if } 0 < t < 1, \\ 0 & \text{if } t \ge 1, \end{cases}$$

where ω_n is the volume of the $n-$dimensional unit ball, and consequently

$$h^*(t) = \omega_n^{1-\frac{\gamma}{n}} (t+\omega_n)^{\frac{\gamma}{n}-1}, \quad t > 0.$$

Combining $X \hookrightarrow \Lambda$ with Theorem 1.26, we obtain that

$$\begin{aligned} \int_{\mathbb{R}^n} |f(y)|\, h(y)\, dy &\le \omega^{1-\frac{\gamma}{n}} \int_0^\infty f^*(t)\, (t+\omega_n)^{\frac{\gamma}{n}-1}\, dt \\ &\le \omega^{1-\frac{\gamma}{n}} \int_0^\infty f^*(t)\, (t+1)^{\frac{\gamma}{n}-1}\, dt \\ &= \omega^{1-\frac{\gamma}{n}} \|f\|_\Lambda < \infty. \end{aligned}$$

Hence $I_\gamma f \chi_{\{y\in\mathbb{R}^n;|y-x|>1\}}(x)$ is finite.

On the other hand, $f\chi_{\{z\in\mathbb{R}^n;|z-x|\le 1\}}(y) \in L^1(\mathbb{R}^n)$ as X is a Banach function space and $\{z \in \mathbb{R}^n; |z-x| \le 1\}$ is a set of finite measure. Hence $I_\gamma f\chi_{\{z\in\mathbb{R}^n;|z-x|\le 1\}}(x)$ is finite for a.e. $x \in \mathbb{R}^n$ by virtue of Proposition 3.11.

Therefore, $I_\gamma f$ is well-defined as we can write

$$I_\gamma f(x) = I_\gamma f\chi_{\{y\in\mathbb{R}^n;|y-x|>1\}}(x) + I_\gamma f\chi_{\{y\in\mathbb{R}^n;|y-x|\le 1\}}(x)$$

and the right-hand side is finite for a.e. $x \in \mathbb{R}^n$.

❑

Now, we can proceed with the main theorem of this chapter.

Theorem 3.14. *Assume that $\gamma \in (0,n)$ and let X be a rearrangement-invariant Banach function space such that*

$$X \hookrightarrow \Lambda. \tag{3.9}$$

Then the rearrangement-invariant Banach function space $Y = \left(Y'(\varrho^{(\gamma)}_{Y'_X})\right)'$, where $\varrho^{(\gamma)}_{Y'_X}$ is defined by (3.4), is the optimal range partner of X for the Riesz potential I_γ.

Moreover, the assumption (3.9) is necessary in the sense that should there exist any rearrangement-invariant Banach function space Z such that $I_\gamma : X \to Z$ is bounded, then (3.9) is satisfied.

Proof. The proof is similar to that of Theorem 2.13.

The fact that Y_X is indeed a rearrangement-invariant Banach function space follows from Lemma 3.10.

We note that the Riesz potential is well-defined on X by Proposition 3.13.

Considering the optimality of Y_X, we just use Lemma 3.6 instead of Lemma 2.8.

Considering the necessity of (3.9), we use the fact (see the proof of Lemma 3.10) that $t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t) \le \frac{n-\gamma}{n} T_\gamma \chi_E^{**}(t)$ for every $t > 0$ in order to prove that $t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t) \in X'(0,\infty)$. Combining Lemma 3.7 and the fact that $X'(0,\infty)$ is rearrangement invariant with Proposition 3.9, we obtain that $X \hookrightarrow \Lambda$.

❑

Remark 3.15. The moreover part of the preceding theorem says that the condition (3.9) is not restrictive because if the given space X is too large, then there is no rearrangement-invariant target space for the Riesz potential.

As with the corresponding theorem for the Hilbert transform (Theorem 2.13), the nature of this theorem is fairly abstract. This means that concrete examples are of great interest. The following point-wise estimate on the operator T_γ will prove useful soon.

Proposition 3.16. *Let $\gamma \in (0,n)$ and $f \in \mathfrak{M}_0$. Then*

$$f^{**}(t) \le \frac{n-\gamma}{n} t^{-\frac{\gamma}{n}} T_\gamma f^{**}(t)$$

for each $t > 0$.

Proof. Clearly

$$T_\gamma f^{**}(t) = \int_t^\infty f^{**}(s) s^{\frac{\gamma}{n}-1}\, ds = \int_t^\infty \frac{1}{s^{2-\frac{\gamma}{n}}} \int_0^s f^*(u)\, du\, ds$$
$$\geq \int_0^t f^*(u)\, du \int_t^\infty \frac{1}{s^{2-\frac{\gamma}{n}}}\, ds = \frac{n}{n-\gamma} t^{\frac{\gamma}{n}} f^{**}(t).$$

❑

Examples 3.17. Assume that $\gamma \in (0, n)$.

1. Set $X = L^{p,q;\mathbb{A}}$ where $p \in (1, \frac{n}{\gamma})$ and $q \in [1, \infty]$. Then X satisfies (3.9) and Y_X is equivalent to $L^{\frac{np}{n-\gamma p},q;\mathbb{A}}$, where $\frac{np}{n-\gamma p} \in (\frac{n}{n-\gamma}, \infty)$.

2. Set $X = L^{\frac{n}{\gamma},q;\mathbb{A}}$ where $q \in (1, \infty)$. Then X satisfies (3.9) if and only if $\alpha_\infty > \frac{1}{q'}$. If $\alpha_\infty > \frac{1}{q'}$, then Y_X is equivalent to

$$\begin{cases} L^{\infty,q;\mathbb{A}-1} & \text{if } \alpha_0 < \frac{1}{q'}, \\ L^{\infty,q;(-\frac{1}{q},\alpha_\infty-1),(-1,0)} & \text{if } \alpha_0 = \frac{1}{q'}, \\ \{f \in \mathfrak{M}_0; \|f\|_{Y_X} = \|f\|_\infty + \|t^{-\frac{1}{q}} \ell^{\alpha_\infty - 1}(t) f^*(t)\|_{q,(1,\infty)} < \infty\} & \text{if } \alpha_0 > \frac{1}{q'}. \end{cases}$$

Moreover, there is no rearrangement-invariant target space if $\alpha_\infty \leq \frac{1}{q'}$.

3. Set $X = L^{\frac{n}{\gamma},1;\mathbb{A}}$. Then X satisfies (3.9) if and only if $\alpha_\infty \geq 0$. If $\alpha_\infty \geq 0$, then Y_X is equivalent to

$$\begin{cases} L^{\infty,1;\mathbb{A}-1} & \text{if } \alpha_0 < 0, \alpha_\infty > 0, \\ \{f \in \mathfrak{M}_0; \|f\|_{Y_X} = \|t^{-1}\ell^{\alpha_0-1}(t) f^*(t)\|_{1,(0,1)} < \infty\} & \text{if } \alpha_0 < 0, \alpha_\infty = 0, \\ L^{\infty,1;(-1,\alpha_\infty-1),(-1,0),(-1,0)} & \text{if } \alpha_0 = 0, \alpha_\infty > 0, \\ L^\infty & \text{if } \alpha_0 = 0, \alpha_\infty = 0, \\ \{f \in \mathfrak{M}_0; \|f\|_{Y_X} = \|f\|_\infty + \|t^{-1}\ell^{\alpha_\infty-1}(t) f^*(t)\|_{1,(1,\infty)} < \infty\} & \text{if } \alpha_0 > 0, \alpha_\infty > 0, \\ L^\infty & \text{if } \alpha_0 > 0, \alpha_\infty = 0. \end{cases}$$

Moreover, there is no rearrangement-invariant target space if $\alpha_\infty < 0$.

4. Set $X = L^{\frac{n}{\gamma},\infty;\mathbb{A}}$. Then X satisfies (3.9) if and only if $\alpha_\infty > 1$. If $\alpha_\infty > 1$, then Y_X is equivalent to

$$\begin{cases} L^{\infty,\infty;\mathbb{A}-1} & \text{if } \alpha_0 < 1, \\ L^{\infty,\infty;(0,\alpha_\infty-1),(-1,0)} & \text{if } \alpha_0 = 1, \\ L^{\infty,\infty,(0,\alpha_\infty-1)} & \text{if } \alpha_0 > 1. \end{cases}$$

Moreover, there is no rearrangement-invariant target space if $\alpha_\infty \leq 1$.

5. Set $X = L^{1,1;\mathbb{A}}$. Then X is equivalent to a rearrangement-invariant Banach function space if and only if $\alpha_0 \geq 0$ and $\alpha_\infty \leq 0$. If $\alpha_0 \geq 0$ and $\alpha_\infty \leq 0$, then X satisfies (3.9) and we can use Theorem 3.14 in order to obtain the optimal range partner of X for the Riesz potential I_γ, but we do not know an explicit description of Y_X.

Proof. We note that X is equivalent to a rearrangement-invariant Banach function space in the first four examples by Theorem 1.59.

We shall prove the second and the fourth example and also the first one (for $q \in (1, \infty]$) in the course of one computation. The third example and the first one for $q = 1$ can be done in a similar way.

Assume that $p \in (1, \infty)$ and $q \in (1, \infty]$. We need to check when the condition (3.9) is satisfied, that is by Lemma 3.9 and Lemma 3.7, when

$$\int_0^\infty t^{\frac{q'}{p'}-1} \ell^{-\mathbb{A}q'}(t)(t+1)^{\frac{\gamma-n}{n}q'}\, dt < \infty.$$

It is easy to see that this integral is finite if and only if either

$$p \in (1, \frac{n}{\gamma})$$

or

$$p = \frac{n}{\gamma} \text{ and } \alpha_\infty > \frac{1}{q'}.$$

This, in particular, proves the moreover part of the examples by virtue of the moreover part of Theorem 3.14. Henceforth, we assume one of the two conditions. By Theorem 1.62, the associate space of X is equivalent to $L^{p',q';-\mathbb{A}}$. Using Theorem 1.64, we compute that

$$\begin{aligned} \|T_\gamma g^{**}\|_{p',q';-\mathbb{A}} &= \|t^{\frac{1}{p'}-\frac{1}{q'}} \ell^{-A}(t) \int_t^\infty g^{**}(s) s^{\frac{\gamma-n}{n}}\, ds\|_{q'} \\ &\lesssim \|t^{\frac{1}{p'}+\frac{1}{q}} \ell^{-\mathbb{A}}(t) g^{**}(t) t^{\frac{\gamma}{n}-1}\|_{q'} = \|t^{\frac{1}{p'}+\frac{\gamma}{n}-\frac{1}{q'}} \ell^{-\mathbb{A}}(t) g^{**}(t)\|_{q'} \\ &= \|g\|_{(r',q';-\mathbb{A})}, \end{aligned}$$

where $\frac{1}{p'} + \frac{\gamma}{n} = \frac{1}{r'}$, that is, $r' = \frac{np}{(n+\gamma)p-n}$.

The opposite inequality

$$\|g\|_{(r',q';-\mathbb{A})} \lesssim \|T_\gamma g^{**}\|_{p',q';-\mathbb{A}}$$

follows immediately from Proposition 3.16.

If $p \in (1, \frac{n}{\gamma})$, then $r' \in (1, \frac{n}{\gamma})$. By Theorem 1.60, $L^{(r',q';-\mathbb{A})}$ is equivalent to $L^{r',q';-\mathbb{A}}$. Hence Y_X is equivalent to $L^{r,q;\mathbb{A}}$, where $r = \frac{np}{n-\gamma p} \in (\frac{n}{n-\gamma}, \infty)$, by Theorem 1.62. This is nothing else than the first example for $q \in (1, \infty]$.

If $p = \frac{n}{\gamma}$, then $r' = 1$. If $q \in (1, \infty)$ (and hence $q' \in (1, \infty)$), we obtain the second example by virtue of Theorem 1.63. If $q = \infty$ (and hence $q' = 1$), we combine Theorem 1.60 with Theorem 1.62 in order to finish the fourth example. □

Remark 3.18. We note that when the case $p = q = 1$ is considered, one encounters similar problems like those mentioned in Remark 2.17.

We stress here the important case of Lebesgue spaces. If $p \in (1, \frac{n}{\gamma})$, then $L^{\frac{np}{n-\gamma p},p}$ is the optimal range partner of L^p for the Riesz potential I_γ. It is important to note here that this improves the known fact that I_γ is of strong type $(p, \frac{np}{n-\gamma p})$ because $p < \frac{np}{n-\gamma p}$. Hence $L^{\frac{np}{n-\gamma p},p}$ is strictly smaller than $L^{\frac{np}{n-\gamma p}}$ (cf. [24, Proposition 8.2.1]).

We conclude this chapter by characterizing the optimal rearrangement-invariant domain space for a given rearrangement-invariant target space.

Theorem 3.19. *Let $\gamma \in (0,n)$ and suppose Y is a rearrangement-invariant Banach function space such that*

$$m_\varphi \hookrightarrow Y, \tag{3.10}$$

where

$$\varphi(t) = (t+1)^{1-\frac{\gamma}{n}}, \quad t > 0.$$

Then $X_Y = X_Y(\varrho_{X_Y}^{(\gamma)})$, where $\varrho_{X_Y}^{(\gamma)}$ is defined by

$$\varrho_{X_Y}^{(\gamma)}(f) = \|T_\gamma f^{**}\|_{Y(0,\infty)}, \quad f \in \mathfrak{M}_0^+,$$

is the optimal domain partner of Y for the Riesz potential I_γ in the class of rearrangement-invariant Banach function spaces.

Moreover, the condition (3.10) *is necessary in the sense that should there exist any rearrangement-invariant Banach function space W such that $I_\gamma : W \to Y$ is bounded, then $m_\varphi \hookrightarrow Y$.*

Proof. By Proposition 1.54 applied to Y', (3.10) is equivalent to the fact that $(t+1)^{\frac{\gamma-n}{n}} \in Y(0,\infty)$. As $Y(0,\infty)$ is rearrangement invariant, we have that $t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t) \in Y(0,\infty)$ by virtue of Lemma 3.7. By Proposition 3.9, this is nothing else than $Y' \hookrightarrow \Lambda$. Hence, X_Y is indeed a rearrangement-invariant Banach function space by Lemma 3.10.

Now, we shall address the question whether the Riesz potential is well-defined on X_Y. Let $f \in X_Y$ and assume that $f \geq 0$ a.e. Then $I_\gamma f$ is well-defined, albeit it might a priori be equal to infinity on a set of positive measure. It follows from the very definition of $\|\cdot\|_{X_Y}$ and Theorem 3.4 that there exists a positive constant C (independent of f) such that

$$\|I_\gamma f\|_Y \leq C\|f\|_{X_Y} < \infty.$$

In particular, not only is $I_\gamma f$ well-defined, but it is also finite a.e. (cf. Theorem 1.5). Therefore, for a general $f \in X_Y$, we write

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f^+(y)}{|x-y|^{n-\gamma}}\, dy - \int_{\mathbb{R}^n} \frac{f^-(y)}{|x-y|^{n-\gamma}}\, dy.$$

Hence the Riesz potential is well-defined (in fact, finite a.e.) on X_Y as the integrals on the right hand side are finite for a.e. $x \in \mathbb{R}^n$.

The optimality of X_Y can be proven in a similar way to the proof of Theorem 2.18. We just use Theorem 3.4 and Theorem 3.5 instead of Theorem 2.5 and Theorem 2.6.

Lastly, considering the necessity of (3.10), assume that there exists a rearrangement-invariant Banach function space W such that $I_\gamma : W \to Y$ is bounded (in particular, it means that I_γ is well-defined on W). Combining Theorem 3.5 with the fact that $I_\gamma : W \to Y$ is bounded, we obtain that there exists a positive constant C (independent of f) such that

$$\|T_\gamma f^{**}\|_{Y(0,\infty)} \leq C\|f^*\|_{W(0,\infty)}$$

for each $f \in W$. Now, we choose $f = \chi_E$ where $E \subseteq \mathbb{R}^n$ is such that $|E| = 1$. Combining the preceding inequality with the fact that $t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t) \leq \frac{n-\gamma}{n} T_\gamma \chi_E^{**}(t)$ for each $t > 0$ (see the proof of Lemma 3.10), we arrive at

$$\|t^{\frac{\gamma}{n}-1}\chi_{[1,\infty)}(t)\|_{Y(0,\infty)} \leq C\frac{n-\gamma}{n}\|\chi_{(0,1)}\|_{W(0,\infty)} < \infty,$$

which is nothing else than $m_\varphi \hookrightarrow Y$ arguing as at the beginning of the proof (just in the opposite direction).

❑

Conclusion

Within the previous two chapters, we have seen that the procedure for finding the optimal range space (or optimal domain space) for a given operator involves a couple of steps. Assume that we are given an operator T.

We begin with reducing the problem of the boundedness of T, which is a rather complicated n-dimensional operator, to the boundedness of some simpler one-dimensional operator. As we work in the scope of rearrangement-invariant spaces, the first natural reduction is to reduce the boundedness of T between X and Y to the boundedness of $(Tf)^*$ between $X(0,\infty)$ and $Y(0,\infty)$. Even though we now work in the one-dimensional setting, this is still not satisfying because, even for very simple operators, it is not obvious what $(Tf)^*$ is. As we have seen, a crucial step is to find a one-dimensional operator, say Q, such that $(Tf)^* \approx Qf^*$, which not only reduces the problem to the one-dimensional setting, but also reduces the problem to non-increasing functions. Regrettably, such an operator Q is hard to get and, in fact, it is often impossible, although for the *Hardy-Littlewood maximal operator* M, it is well-known (see e.g. [2, Chapter 3, Theorem 3.8]) that $(Mf)^* \approx f^{**}$. Fortunately, keeping in mind that we work in the scope of rearrangement-invariant spaces, a weaker estimate

$$Qg^* \lesssim (Tf)^* \lesssim Qf^*, \tag{4.1}$$

where g is equimeasurable with f, is enough for us. Such rearrangement inequalities are fortunately known for many classical operators and a large amount of work has already been done on this matter (see e.g. [2, 4, 10]). Once we have a sharp estimate (4.1), we can use the duality argument to find the candidate for the optimal rearrangement-invariant norm. Then one needs to find a reasonable characterization of the situation when the candidate is indeed a rearrangement-invariant norm.

Although it is obvious that none of these steps is automatic for a given operator, we have successfully followed the outlined procedure and characterized the optimal spaces for the Hilbert transform (recall Theorem 2.13, Theorem 2.18 and Theorem 2.19) and for the Riesz potential (see Theorem 3.14 and Theorem 3.19). It is worth noting once more that we have fully characterized the optimal rearrangement-invariant spaces for these operators. The provided necessary and sufficient conditions for the existence of optimal spaces, expressed by means of embeddings between function spaces, are quite easily verifiable in applications. Even though the descriptions of the optimal spaces are rather abstract, which follows from the nature of the construction, we were able to provide several concrete, yet fairly general, examples (recall Examples 2.16, Example 2.21 and Examples 3.17).

As the results, as well as the course of actions which led to them, in Chapter 2 and Chapter 3 are similar, which should come as no surprise since both operators are integral operators of convolution type, one may want to extract the properties of the operators which were really needed and try to formulate more general theorems. That was not, however, the aim of this thesis. The aim of this thesis was to provide applicable theorems with all the necessary details so that it should be quite easy to follow the ideas presented here and adopt them to similar operators. As the amount of the work which we had to done in order to be able to provide concrete examples indicates, there is a non-trivial amount of work concealed behind applications of the theorems even without making them more abstract.

The obtained results lead us, as is usual, to further questions. We could want to look for optimal spaces in narrower class of function spaces. We could, for example, look for optimal spaces in the class of Orlicz spaces. As Orlicz spaces are rearrangement-invariant Banach function spaces, we can, of course, use our results and obtain the optimal rearrangement-invariant function space. There is, however, no guarantee that the obtained optimal space is an Orlicz space itself. In fact, there can be no optimal Orlicz space even though the optimal rearrangement-invariant space exits (see [19]). Another natural question is what the situation is like when we study not integral operators but supremal operators. Such operators have been intensively studied lately, namely the *fractional maximal operator* for which a sharp estimate of (4.1) type is known (see [4]). The supremal nature of supremal operators causes several difficulties which we do not face when dealing with integral operators. Fortunately, a large amount of work on this matter has been done and results like [16, Theorem 3.9] give us an opportunity to look for optimal spaces even for such operators.

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