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DÔKAZY BEZESPORNOSTI ARITMETIKY
CUT ELIMINATION AND CONSISTENCY PROOFS

Abstract of doctoral thesis

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Contents

Part 1

What is the height of Gentzen's reduction trees?

In his consistency proof of 1935 [4], Gentzen constructs reduction procedures for sequents that are derivable in Peano arithmetic, whereas contradictory sequents have no reduction procedures. Reduction procedures generate reduction trees which we interpret as cut-free infinitary derivations. A cut elimination theorem is used to build reduction trees; Gentzen calls it Hilfssatz in this particular proof. Hilfssatz is interesting because of two reasons: (1) The cut elimination strategy applied there eliminates always an uppermost cut, regardless of its complexity. (2) The proof of Hilfssatz makes use of transfinite induction on the height of reduction trees that have been constructed so far. In this part, we analyse Gentzen's proof and particularly the cut elimination strategy of Hilfssatz to quantify the transfinite induction implicitly applied in the consistency proof. We determine an upper bound for the heights of reduction trees that belong to sequents which are derivable in Peano arithmetic. Namely, the heights of these reduction trees are less than $\Phi_\omega(0)$ where Φ_ω is the ω-th Veblen function. The question stays open what the lower bound is, but this seems to be quite difficult. If Gentzen had applied Tait's cut elimination strategy, which reduces the cut-rank, the heights of reduction trees for sequents derivable in PA would be bounded by ε_0.

The second half deals with the question whether the transfinite induction mentioned above is the only tool in the consistency proof that exceeds PA. We shall show that this is the case by formalizing the proof in $I\Pi_3$ plus transfinite induction on the height of reduction trees for sequents derivable in PA. The transfinite induction uses at most Δ_3 induction formulas. One can view it as an improvement of Gentzen's original formulation where he implicitly uses induction formulas that contain the notion of well-foundedness, which is a second order property.

1.1 Gentzen's proof and his cut elimination strategy in Hilfssatz

Now, we explain Gentzen's method of the proof in [4]. First, he defines reduction rules that generate reduction trees for sequents that are valid in the standard model of PA. The leaves of reduction trees are sequents in *endform* whose validity is easy to see. Reduction rules decompose a sequent S into, let us say, simpler sequents. If we consider these simpler sequents, we can get countably many of them, to be premises and the sequent S to be a conclusion, we can interpret reduction rules as deduction (inference) rules. Then, reduction trees are understood as deduction trees, cut-free infinitary derivations. We will stick to this terminology, even if Gentzen does not use the notion of deduction trees. The first person who interpreted Gentzen's reduction rules in this way was probably Schütte [10].

Second, Gentzen takes a sequent calculus for PA and shows that every sequent derivable in PA has a deduction tree. He does it by induction on the complexity of the derivation in PA. Deduction trees for initial sequents of PA have a finite height. Then, he assumes that there already exist deduction trees for premises and he shows how to obtain a deduction tree for the conclusion. To find deduction trees for the conclusions of the rules of negation and the induction rule, he uses Hilfssatz. Gentzen's formulation of Hilfssatz is that one has a deduction tree for $\Gamma, \Delta \to C$ when $\Gamma \to D$ and $D, \Delta \to C$ have deduction trees ([4], p. 108). The proof of Hilfssatz proceeds by induction on the number of logical operations in the cut formula and an embedded transfinite induction on the height of the deduction tree for the second cut premise. The elimination strategy eliminates always an uppermost cut.

Assume that we have cut ϑ. The cut formula is $A\&B$ and both premises of the cut have deduction trees, cut-free infinitary derivations. The rules of $\&R$ and $\&L_2$ are Gentzen's reduction rules. Then, the elimination transforms ϑ into ϑ_1 and ϑ_2:

$$\dfrac{\&R\,\dfrac{\overset{\vdots}{\Gamma\to A}\qquad \overset{\vdots}{\Gamma\to B}}{\Gamma\to A\&B}\qquad \dfrac{\overset{\vdots}{A\&B,A,\Delta\to 0=1}}{A\&B,\Delta\to 0=1}\,\&L_2}{\Gamma,\Delta\to 0=1}\,\vartheta \quad\rightsquigarrow\quad \dfrac{\overset{\vdots}{\Gamma\to A}\qquad \dfrac{\overset{\vdots}{\Gamma\to A\&B}\qquad \overset{\vdots}{A\&B,A,\Delta\to 0=1}}{\Gamma,A,\Delta\to 0=1}\,\vartheta_1}{\Gamma,\Gamma,\Delta\to 0=1}\,\vartheta_2$$

Cut ϑ_1 is removed since the deduction tree for the second cut premise of ϑ_1 has a smaller height in comparison to ϑ. Cut ϑ_2 is removed since cut formula A contains less logical operations than $A\&B$. Contraction is admissible in the calculus and does not change the height of the derivation.

Although Gentzen's Hilfssatz proves that the cut elimination works in his infinitary calculus, the proof mentions no bounds on heights of cut-free derivations and exactly these bounds are necessary to quantify the transfinite induction implicitly used in the consistency proof. We applied Veblen hierarchy, which is excellently explained in ([11], pp. 73–84), to estimate the heights of cut-free infinitary derivations, deduction trees, constructed in Hilfssatz.

1.2 An upper bound on the height of deduction trees for sequents that are derivable in PA

In Theorem 1, we proved an upper bound on heights of deduction trees that are constructed in Hilfssatz. To obtain an accurate answer to the question stated in the title of this part, we would need to prove the lower bound on the heights of these deduction trees, too. Unfortunately, this seems to be difficult.

The numbers of the theorems and the definitions correspond to the theorems and definitions in the thesis.

Theorem 1. *Assume that sequents $\Gamma \to D$ and $D, \Delta \to C$ have deduction trees T_1 and T_2 with heights α_1 and α_2, respectively, and $|D| = n$. Then, sequent $\Gamma, \Delta \to C$ has a deduction tree whose height is at most $\Phi_{n-1}(\alpha_1+\alpha_2)$, where $\Phi_{-1} = Id$.*

The proof of Theorem 1 applies induction on the number of the logical operations in cut formula D and induction on the height of T_2. Function Φ_{n-1} is the $(n-1)$-th Veblen function.

So far, we have investigated Gentzen's infinitary calculus. Now, we move on to PA. All mathematical initial sequents of PA have deduction trees whose heights are finite. Furthermore, PA includes logical initial sequents of the form of $D \to D$. Gentzen defines an algorithm that constructs deduction trees of finite height for them. Then, he continues by induction on the complexity of the derivation in PA and shows how to construct a deduction tree for the conclusion of a rule when its premises have deduction trees. We took his construction and, with the help of Theorem 1, we estimated the height of the deduction trees for sequents that are derivable in PA:

Theorem 2. *Following Gentzen's procedure, we can construct for every sequent that is derivable in PA a deduction tree whose height is less than $\Phi_\omega(0)$.*

1.3 Formalization of Gentzen's proof of 1935

We showed that Gentzen's consistency proof of 1935 can be formalized in $I\Pi_3$ plus transfinite induction up to $\Phi_\omega(0)$. The induction formulas of the transfinite induction are at most Δ_3.

To formalize the proof, we need some finite representation of deduction trees. The only interesting case is when the last derivation rule in the chosen tree T is an ω-rule, the rule that has countably many premises. Assume that the last inference rule in T is of the form:

$$\frac{\Gamma\to F(0) \qquad \Gamma\to F(\bar{1}) \qquad \Gamma\to F(\bar{2}) \ \ldots}{\Gamma\to\forall x F(x)}\ \forall R$$

The code of T has the following form:

$$\langle\ \ulcorner\alpha\urcorner, \ulcorner\Gamma \to \forall x F(x)\urcorner, \ulcorner\forall R\urcorner, \ulcorner\varphi_1(n,y)\urcorner\ \rangle$$

where α is an upper bound on the height of T, $\forall R$ is the derivation rule by which the endsequent $\Gamma \to \forall x F(x)$ is derived and $\varphi_1(n,y)$ is a Σ_1-formula that holds true when y is a code of a deduction tree for the premise $\Gamma \to F(\bar{n})$ of the rule of $\forall R$. This is a variant of the idea suggested by Schwichtenberg in ([12], p. 886). He uses codes of primitive recursive functions to enumerate codes of deduction trees for premises of the ω-rules. If the last inference rule in T has finitely many premises, we enumerate the codes for their deduction trees explicitly.

With the help of our finite representation, we defined the following formulas:

- `DedTreeAxiom`(x,z,y) that holds true when x is a code of a logical initial sequent $D \to D$, z is a code of a list of decisions that were made during the decomposition of the succedent formula D and y is a code of a deduction tree for $D \to D$.

- `Wk`(x,z,y) that holds true when x is a code of a deduction tree with the endsequent $\Gamma \to C$, z is a code of a multiset Δ and y is a code of a deduction tree with the endsequent $\Gamma, \Delta \to C$.

- `Ct`(x,z,y) that holds true when x is a code of a deduction tree with the endsequent $\Gamma, A, A \to C$, z is a code of a formula A and y is a code of a deduction tree with the endsequent $\Gamma, A \to C$.

- `MultiCt`(x,z,y) that holds true when x is a code of a deduction tree with the endsequent $\Gamma, \Delta, \Delta \to C$, z is a code of a multiset Δ and y is a code of a deduction tree with the endsequent $\Gamma, \Delta \to C$.

- $\texttt{Elim}(x_1, x_2, z, y)$ that holds true when x_1 is a code of a deduction tree with the endsequent $\Gamma \to D$, x_2 is a code of a deduction tree with the endsequent $D, \Delta \to C$, z is a code of the cut formula D and y is a code of a deduction tree with the endsequent $\Gamma, \Delta \to C$.

- $\texttt{DedTree}(x, y)$ that holds true when x is a code of a derivation in PA and y is a code of a deduction tree for the endsequent of x.

We proved in $\mathrm{I}\Pi_3$ plus TI up to $\Phi_\omega(0)$ that these formulas are total and correct. A total formula gives us some y for all possible input values x, z or x_1, x_2, z. If the input is corrupt, the formula holds true for an arbitrary y. A formula is correct when it yields a code of a proper deduction tree y for a proper input. What we mean by a proper deduction tree is defined by predicate $DT(x)$ that expresses that x is a code of a proper deduction tree. Roughly speaking, x is a code of a proper deduction tree when it fulfills some local conditions and all its subtrees are also proper deduction trees. We constructed the predicate $DT(x)$ using the partial truth predicates and the Fixed-point theorem.

At a metalevel, formula $DT(x)$ contains the information that deduction trees are well-founded; recall that deduction trees are in fact cut-free infinitary derivations. Since every node is assigned an ordinal number that represents the height of the particular subtree and these numbers decrease towards the leaves, we know that x such that $DT(x)$ is well-founded. Nevertheless, this cannot be proved in PA because the proof theoretic ordinal of PA is ε_0 and our upper bound on heights of deduction trees is $\Phi_\omega(0)$. Numbers below $\Phi_\omega(0)$ can be compared in $\mathrm{I}\Sigma_1$, but $\mathrm{I}\Sigma_1$ does not prove that they do not build infinite decreasing sequences.

The main reason for not requiring that formula $DT(x)$ speaks about well-foundedness explicitly is that well-foundedness is a second order property and a consistency proof of PA is trivial when second order properties are allowed. Formula $DT(x)$ appears in the induction formulas that we use to prove the totality and the correctness of the formulas above and we want to use only arithmetic induction formulas in the proof. We do not mind applying transfinite induction on the height of deduction trees, but the induction formulas are always arithmetic and of a bounded complexity.

Eventually, the assertion that gives us the consistency of PA is that for every sequent S derivable in PA we can construct T such that S is the endsequent of T and $DT(\overline{T})$. The point is that sequent $\to 0 = 1$ has no deduction tree even if we allowed deduction trees with infinite branches. This reasoning is an important improvement of Gentzen's original proof in [4] in which he implicitly uses transfinite induction on the height of deduction trees with induction formulas that explicitly speak about well-foundedness.

Part 2

Comparison between Tait's and Gentzen's cut elimination strategy in classical propositional logic

The most problematic part of Gentzen's consistency proof of 1935 is Hilfssatz, the cut elimination theorem, that eliminates uppermost cuts regardless of the complexity. The analysis of the cut elimination strategy of Hilfssatz, which is described in the previous part, showed that Gentzen implicitly applied transfinite induction up to α, $\varepsilon_0 \leq \alpha \leq \Phi_\omega(0)$, in the consistency proof. It must be stressed that Gentzen himself does not speak about any transfinite induction in connection with this proof. We know that if he had applied Tait's cut elimination strategy, the one that decreases the cut-rank of the derivation, he would obtain transfinite induction up to ε_0.

In this part, we deal with the question to what extent cut-free derivations differ when they are produced by distinct cut elimination strategies, particularly we are interested in Gentzen's strategy and Tait's strategy. We show that both strategies yield the same cut-free derivations in classical propositional logic. Hence, not only are the heights of cut-free derivations the same but also their structures.

Our proof applies an elimination algorithm of a single cut inspired by the method of Buss ([3], pp. 37–40) that makes global changes to the derivations. This algorithm is deterministic. A cut elimination strategy is a list of properties that a cut must have to be eliminated in a particular state. We will use only strategies that are nondeterministic in the sense that any cut with suitable properties can be chosen for elimination. We define a strategy, which we call *general cut elimination strategy*, that includes both investigated strategies, the one of Gentzen and the one of Tait. We prove that general cut elimination strategy has the weak Church-Rosser property in classical propositional logic. It can be seen that it also has the strong normalization property. The weak Church-Rosser property and strong normalization yield the Church-Rosser property that ensures that normal forms, in our case cut-free propositional derivations, are given unambiguously.

Below, we will use the following standard notions. The *principal* formula of a rule is the one that is derived by the rule. The principal formula is derived from *auxiliary* formulas. The formulas of the rule that are not auxiliary nor principal are the *side* formulas of the rule.

Perhaps the most important notion that we use in this part is the notion of a *thread*. To define this, we need the notion of an *ancestor* of a formula in a derivation. Roughly speaking, ancestors of a formula B that is in the lower sequent of an inference rule are formulas from the upper sequent of that rule that are logically connected to B from the lower sequent. Hence, if B is the principal formula of the rule, then the auxiliary formulas are ancestors of B. If B is a side formula of the rule, then the corresponding occurrences of B among the side formulas in the upper sequent are ancestors of B.

Definition 31. *Assume that we have a derivation P in classical propositional logic. We choose an occurrence of formula B in P. A* thread *for this occurrence of formula B are all occurrences of B in P that are ancestors of the chosen occurrence of B.*

The elimination of a single cut is defined in the way that it replaces the thread for the cut formula by threads for its immediate subformulas and these subformulas become then cut formulas of the new cuts. Hence, the cut on $A \vee B$ is replaced by a cut on A and by a cut on B. The cut on $\neg B$ is replaced by a cut on B and the cut on a propositional variable A disappears. We do not use other logical operations since we work in classical propositional logic. We are able to arrange the cut elimination in the way that we always obtain at most two simpler cuts instead of the eliminated cut. Moreover, the new cuts are situated exactly at the position of the original cut. The crucial point is that they are not distributed throughout the whole derivation and we know what is above and below them.

Next, we need to define a cut elimination strategy that covers both investigated strategies, the one of Gentzen and the one of Tait. It is called *general cut elimination strategy*.

Definition 37. Tait's elimination strategy *selects one of the most complex cuts such that there are only cuts of smaller complexity above it if any and this one is then eliminated.*

Definition 38. Gentzen's elimination strategy *selects an uppermost cut such that there are no other cuts above it and this one is then eliminated.*

Definition 39. General cut elimination strategy *selects an arbitrary cut such that there are only cuts of smaller complexity above it if any and this one is then eliminated.*

The cut chosen by the general cut elimination strategy does not have to be one of the most complex cuts in the derivation, but it may be, and it does not have to be an uppermost cut, but it may be. This strategy is considered to be nondeterministic in the same sense as Tait's strategy and Gentzen's strategy.

Since only cuts of smaller complexity than the eliminated one are created and reproduced during the elimination based on the general cut elimination strategy, general cut elimination strategy always terminates. This means that general cut elimination has the strong normalization property.

We focus on the weak Church-Rosser property of general cut elimination. Objects that are going to be rewritten are the whole derivations and the only rewriting rule is the elimination of a single cut. We proved that general cut elimination has the weak Church-Rosser property that says that if we have a derivation and we apply two different elimination steps, we reach two different states that can be both further rewritten so that, in a finite number of steps, the derivations will be the same again. Strong normalization and the weak Church-Rosser property give us that the normal forms (cut-free derivations) are given unambiguously. This means that Tait's and Gentzen's cut elimination strategy, respectively, yield cut-free derivations not only of the same height, but also of the same form.

Theorem 4. *General cut elimination in classical propositional logic has the weak Church-Rosser property.*

To be more precise, the theorem must be considered relative to the calculus and the cut elimination algorithm that we used.

The proof of Theorem 4 consists in the fact that the elimination of a single cut, let us denote the cut by ϑ, changes principal formulas only of such inference rules whose principal formulas belong to the thread for the cut formula. Other rules have only their side formulas changed in the way that threads for other cut formulas are not violated. Thus, from the point of view of the other cuts, the elimination of ϑ did not change any important property of the derivation.

Publications

Horská, A.: Where is the Gödel-Point hiding: Gentzen's consistency proof of 1936 and his representation of constructive ordinals. Springer (2014)

Horská, A.: What Is the Height of Gentzen's Reduction Trees? Submitted on November 8th 2016, Notre Dame Journal of Formal Logic

Horská, A., Punčochář, V.: Sbírka úloh z logiky. The exercise book was written in 2013 as part of the FRVŠ project - Sbírka úloh ke kurzu logiky pro studenty filosofie. It is used in logic courses for philosophers.

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