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Matematicko-fyzikální fakulta

# DIPLOMOVÁ PRÁCE

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Weighted inequalities for Hardy-type operators and their application in the Interpolation Theory

Katedra matematické analýzy

Vedoucí diplomové práce: Doc. RNDr. Luboš Pick, DSc.

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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

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## Abstract

**Název práce:** Váhové nerovnosti pro operátory Hardyova typu a jejich aplikace v teorii interpolací
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**Abstrakt:** Studujeme reálné interpolační prostory $(X_0, X_1)_{\varrho,q}$, kde $\varrho$ je obecný funkční parametr (nikoli nutně mocninná váha). Použitím diskretizační metody diskretizujeme normu v $(X_0, X_1)_{\varrho,q}$. Výsledná norma je dána pomocí odpovídající kvazikonkávní funkce $h$ a její dikretizační posloupnosti, prostor s touto normou značíme $(X_0, X_1)^*_{h,q}$. Podáme přímý důkaz věty V. I. Ovchinnikova a A. S. Titenkovova, která charakterizuje prostor $(L_{p_0}, L_{p_1})^*_{h,q}$ v jazyce nerostoucího přerovnání. Dále najdeme vztah mezi dilatačními indexy kvazikonkávní funkce $h$ a její diskretizační posloupností. Pokud jsou dilatační indexy funkce $h$ nelimitní, prostor $(L_{p_0}, L_{p_1})_{h,q}$ splývá s nějakým klasickým Lorentzovým prostorem $\Lambda^q(\varphi)$. V případě limitního dilatačního indexu ukážeme, že prostor $(L_{p_0}, L_{p_1})^*_{h,q}$ může být reprezentovaný jako extrapolační prostor.
**Klíčová slova:** diskretizační posloupnost, dilatační indexy, reálné interpolační prostory, extrapolační prostory.

**Title:** Weighted inequalities for Hardy-type operators and their application in the Interpolation Theory
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**Abstract:** We study real interpolation spaces $(X_0, X_1)_{\varrho,q}$, where $\varrho$ is a parameter function, not necessarily a power weight. Using a discretization method we "discretize" the norm in $(X_0, X_1)_{\varrho,q}$. The resulting norm is given by the corresponding quasiconcave function $h$ and its discretizing sequence, we denote the space endowed with this norm by $(X_0, X_1)^*_{h,q}$. We give a direct proof of a theorem due to V. I. Ovchinnikov and A. S. Titenkov, which characterizes the space $(L_{p_0}, L_{p_1})^*_{h,q}$ in terms of the non-increasing rearrangement. Further, we find a relation between the dilation indices of a quasiconcave function $h$ and its discretizing sequence. In the case when the dilation indices of $h$ are not limiting, the space $(L_{p_0}, L_{p_1})_{h,q}$ coincides with some classical Lorentz space $\Lambda^q(\varphi)$. If the dilation indices are limiting, then we characterize the space $(L_{p_0}, L_{p_1})^*_{h,q}$ as an extrapolation space.
**Keywords:** discretizing sequence, dilation indices, real interpolation spaces, extrapolation spaces.

# Preface

The real interpolation spaces $(X_0, X_1)_{\varrho,q}$, where $\varrho$ is a parameter function, are a generalization of interpolation spaces by Lions and Peetre, where only a function of the form $\varrho(t) = t^\theta$, $\theta \in [0,1]$, is admitted. In particular, the norm in this space is given by

$$\|f\|_{(X_0,X_1)_{\varrho,q}} = \left(\int_0^\infty \left(\frac{K(f,t;X_0,X_1)}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q}.$$

The space $(X_0, X_1)_{\varrho,q}$ was in detail studied in [9], [6] and [12]. In these papers, the equivalence theorem, the duality of this space and also the reiteration theorem are given. In this thesis we introduce a different approach to this space.

In Section 1, we establish the notation and the definitions of some commonly used function spaces such as $L_p$, $L_{p,q}$, $L_{p,q}(\log L)_\gamma$.

Section 2 contains a discretization method introduced in [5] and [4]. We define a quasiconcave function, its discretizing sequence and a fundamental function, then we bring in a discretization theorem. We use this powerful method in sections 5–7 to obtain our results.

In Section 3 we discuss the connection of the dilation indices of a quasiconcave function and its discretizing sequence. It turns out that if the dilation indices are not limiting, then, roughly speaking, the discretizing sequence behaves like a geometric one.

Section 4 presents some embeddings between classical Lorentz spaces, which are a counterpart of Hölder's inequality and we will use them as a very helpful technical tool.

Section 5 contains the real interpolation method, that is the definition of a compatible couple, the Peetre K-functional and the interpolation spaces $(X_0, X_1)_{\varrho,q}$ and $(X_0, X_1)^*_{h,q}$. The space $(X_0, X_1)^*_{h,q}$ is the discrete version of $(X_0, X_1)_{\varrho,q}$ and the norm is given by

$$\|f\|_{(X_0,X_1)^*_{h,q}} = \left(\sum_{k\in\mathbb{Z}} \left(\frac{K(f,t_k;X_0,X_1)}{h(t_k)}\right)^q\right)^{1/q},$$

where $h$ is a quasiconcave function and $\{t_k\}$ is a discretizing sequence for $h$. This definition was firstly used by S. Jansson in [8]. He showed the equivalence and duality theorems for this space. We use the discretization method to make the connection between $(X_0, X_1)_{\varrho,q}$ and $(X_0, X_1)^*_{h,q}$. However, if the dilation indices of $h$ are not limiting, then these two spaces coincide. Finally, we show the basic properties of these spaces.

We consider the interpolation space $(L_{p_0}, L_{p_1})_{\varrho,q}$ and using the discretization method we characterize this space in terms of the non-increasing rearrangement. Further, we give the direct proof of the theorem, which was obtained by V. I. Ovchinnikov and A. S. Titenkov in [11]. At the end of Section 6, we identify $(X_0, X_1)_{\varrho,q}$ as a classical Lorentz space $\Lambda^q(\varphi)$, under the assumption that $\varrho$ is a quasiconcave function with non-limiting dilation indices.

In Section 7, we describe the space $(X_0, X_1)^*_{h,q}$, where $h$ is a quasiconcave function with limiting dilation indices, as an extrapolation space. As an application of this result, we obtain a description of the grand Lebesgue space in the form of an interpolation space.

# 1 Introduction

This section is intended for the introduction of some basic definitions and notation, which we will use in the sequel.

Let $(R, \mu)$ be a totally $\sigma$–finite measure space with a non-atomic measure $\mu$, and let $\mathcal{M}(R, \mu)$ be the set of all extended complex-valued $\mu$-measurable functions on $R$. For $f \in \mathcal{M}(R, \mu)$, let $f_*(t) = \mu(\{x \in R : |f(x)| > t\})$, $t > 0$ be the *distribution function* of $f$. The *non-increasing rearrangement* of $f$ is defined by

$$f^*(t) = \inf\{s > 0 : f_*(s) \leq t\}, \quad t \in [0, \infty).$$

We further denote

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds, \quad t \in [0, \infty).$$

Let $1 \leq p \leq \infty$, then the *Lebesgue space* $L_p = L_p(R) = L_p(\mu) = L_p(R, \mu)$ is the set of all $p$-integrable functions in $\mathcal{M}(R, \mu)$. The norm is defined by

$$\|f\|_{L_p} = \left( \int_R |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad 1 \leq p < \infty,$$
$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in R} |f(x)|, \quad p = \infty.$$

As usual, functions which differ only on a set of measure zero are identified.

If moreover $1 \leq q \leq \infty$, then the *Lorentz space* $L_{p,q}$ is defined as follows. We have $f \in L_{p,q}$, if and only if

$$\|f\|_{L_{p,q}} = \left( \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty, \quad 1 \leq q < \infty,$$
$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t) < \infty, \quad q = \infty.$$

Assume in addition that $\mu(R) = 1$ and $\gamma \in \mathbb{R}$. Then the *Lorentz-Zygmund space* $L_{p,q}(\log L)_\gamma$ consists of all $\mu$-measurable functions $f$ on $R$ for which

$$\|f\|_{L_{p,q}(\log L)_\gamma} = \left( \int_0^1 (t^{1/p}(1 - \log t)^\gamma f^*(t))^q \frac{dt}{t} \right)^{1/q}, \quad 1 \leq q < \infty,$$
$$\|f\|_{L_{p,\infty}(\log L)_\gamma} = \sup_{0<t<1} t^{1/p}(1 - \log t)^\gamma f^*(t), \quad q = \infty,$$

is finite.

If $L \subseteq \mathbb{Z}$ then the space $l_q^L$ consists of all sequences $a = \{a_k\}_{k \in \mathbb{Z}}$ for which is the pseudonorm

$$\|a\|_{l_q^L} = \left( \sum_{k \in L} |a_k|^q \right)^{1/q}, \quad 1 \leq q < \infty,$$
$$\|a\|_{l_\infty^L} = \sup_{k \in L} |a_k|, \quad q = \infty,$$

finite. In the case when $L = \mathbb{Z}$, we write $l_q^{\mathbb{Z}} = l_q$.

Let $1 \leq p, q \leq \infty$ and let $L_k \subseteq \mathbb{Z}$ for $k \in \mathbb{Z}$. Then $l_q(l_p^{L_k})$ is the space of sequences generated by pseudonorm

$$\|a\|_{l_q(l_p^{L_k})} = \left\| \|a\|_{l_p^{L_k}} \right\|_{l_q}.$$

If X and Y are two (pseudo)normed linear spaces, then the space $X \oplus Y$ is generated by (pseudo)norm $\|a\|_{X \oplus Y} = \|a\|_X + \|a\|_Y$.

If we write $A \lesssim B$, it will mean that there exists some positive constant $C$ independent of appropriate quantities such that $A \leq CB$. If simultaneously $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. We say that two functions $f$, $g$ are *equivalent* on $(0, \infty)$ if there exists a positive constant $C$ such that

$$C^{-1} f(t) \leq g(t) \leq C f(t) \quad \text{for all } t \in (0, \infty).$$

## 2 The Discretization Method

The discretization method was firstly introduced in [5]. Then it was further studied e.g. in [4], which we will mainly follow.

**Definition 2.1.** Let $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of positive real numbers. We say that $\{a_k\}$ is *strongly increasing* or *strongly decreasing* and write $\{a_k\}\upuparrows$ or $\{a_k\}\downdownarrows$ when

$$\inf_{k\in\mathbb{Z}}\frac{a_{k+1}}{a_k}>1 \quad \text{or} \quad \sup_{k\in\mathbb{Z}}\frac{a_{k+1}}{a_k}<1,$$

respectively.

For the proof of the first lemma, see [5, Proposition 2.1].

**Lemma 2.2.** *Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{\sigma_k\}_{k\in\mathbb{Z}}$ and $\{\tau_k\}_{k\in\mathbb{Z}}$ be sequences of nonnegative numbers. Let $p\in(0,\infty)$.*

(i) *If $\sigma_k\upuparrows$, then*

$$\sum_{k\in\mathbb{Z}}\left(\sum_{m=k}^{\infty}a_m\right)^p\sigma_k\approx\sum_{m\in\mathbb{Z}}a_m^p\sigma_m.$$

(ii) *If $\tau_k\downdownarrows$, then*

$$\sum_{k\in\mathbb{Z}}\left(\sum_{m=-\infty}^{k}a_m\right)^p\tau_k\approx\sum_{m\in\mathbb{Z}}a_m^p\tau_m.$$

The following lemma can be proved by a similar argument as Lemma 2.2.

**Lemma 2.3.** *Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{\sigma_k\}_{k\in\mathbb{Z}}$ and $\{\tau_k\}_{k\in\mathbb{Z}}$ be sequences of nonnegative numbers. Let $p\in(0,\infty)$.*

(i) *If $\sigma_k\upuparrows$, then*

$$\sup_{k\in\mathbb{Z}}\left(\sum_{m\geq k}a_m\right)^p\sigma_k\approx\sup_{m\in\mathbb{Z}}a_m^p\sigma_m.$$

(ii) *If $\tau_k\downdownarrows$, then*

$$\sup_{k\in\mathbb{Z}}\left(\sum_{m\leq k}a_m\right)^p\tau_k\approx\sup_{m\in\mathbb{Z}}a_m^p\tau_m.$$

**Definition 2.4.** Let $\varphi$ be a continuous strictly increasing function on $[0,\infty)$ such that $\varphi(0)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$. Then we say that $\varphi$ is *admissible.*

Given such $\varphi$, we say that a function $h$ is *$\varphi$-quasiconcave* if $h$ is equivalent to a non-decreasing function on $[0,\infty)$ and $h/\varphi$ is equivalent to a non-increasing function on $(0,\infty)$. If moreover

$$\lim_{t\to 0+} h(t) = \lim_{t\to\infty}\frac{1}{h(t)} = \lim_{t\to\infty}\frac{h(t)}{\varphi(t)} = \lim_{t\to 0+}\frac{\varphi(t)}{h(t)} = 0, \tag{1}$$

then $h$ is *non-degenerate.* The set of all non-degenerate $\varphi$-quasiconcave functions will be denoted by $\Omega_\varphi$.

We say that $h$ is *quasiconcave* when $h$ is $\varphi$-quasiconcave with $\varphi(t)=t$. If $h$ is moreover non-degenerate, we write $h\in\Omega_{0,1}$.

*Remark* 2.5. It will be useful to note that

$$h\in\Omega_\varphi \Longleftrightarrow \frac{\varphi}{h}\in\Omega_\varphi.$$

**Definition 2.6.** Assume that $\varphi$ is admissible and $h\in\Omega_\varphi$. We say that $\{\mu_k\}_{k\in\mathbb{Z}}$ is a *discretizing sequence for $h$ with respect to $\varphi$* if

(i) $\mu_0=1$ and $\varphi(\mu_k)\upuparrows$,

(ii) $h(\mu_k)\upuparrows$ and $\frac{h(\mu_k)}{\varphi(\mu_k)}\downdownarrows$,

(iii) there is a decomposition $\mathbb{Z}=\mathbb{Z}_1\cup\mathbb{Z}_2$ such that $\mathbb{Z}_1\cap\mathbb{Z}_2=\emptyset$ and for every $t\in[\mu_k,\mu_{k+1}]$

$$\begin{aligned} h(\mu_k) &\approx h(t) \quad \text{if} \quad k\in\mathbb{Z}_1\,,\\ \frac{h(\mu_k)}{\varphi(\mu_k)} &\approx \frac{h(t)}{\varphi(t)} \quad \text{if} \quad k\in\mathbb{Z}_2\,, \end{aligned}$$

where the constants of equivalence are independent of $k\in\mathbb{Z}$.

We say that $\{\mu_k\}_{k\in\mathbb{Z}}$ is a *discretizing sequence for $h$* when $\{\mu_k\}_{k\in\mathbb{Z}}$ is a discretizing sequence for $h$ with respect to $\varphi(t)=t$.

The following lemma and its proof can be found in [4, Lemma 2.7], but because of its importance for us, the proof is brought in.

**Lemma 2.7.** *Let $\varphi$ be an admissible function and assume that $h\in\Omega_\varphi$. Then for arbitrary $a>4$ we can define the sequence $\{\mu_k\}$ by $\mu_0=1$ and*

$$\mu_{k+1}=\inf\left\{t:\min\left\{\frac{h(t)}{h(\mu_k)},\frac{h(\mu_k)\,\varphi(t)}{\varphi(\mu_k)\,h(t)}\right\}=a\right\} \qquad \textit{when } k\geq 0,$$

$$\mu_{k-1} = \inf\left\{t : \min\left\{\frac{h(\mu_k)}{h(t)}, \frac{h(t)\,\varphi(\mu_k)}{\varphi(t)\,h(\mu_k)}\right\} = a\right\} \qquad \textit{when } k \le 0.$$

*Then $\{\mu_k\}$ is a discretizing sequence for $h$ with respect to $\varphi$.*

*Proof.* We should verify the properties (i), (ii) and (iii) from Definition 2.6. Set

$$\mathbb{Z}_1 = \{k \in \mathbb{Z} : ah(\mu_k) = h(\mu_{k+1})\}, \quad \mathbb{Z}_2 = \mathbb{Z} \setminus \mathbb{Z}_1 .$$

Then

$$\frac{h(\mu_k)}{\varphi(\mu_k)} = a\frac{h(\mu_{k+1})}{\varphi(\mu_{k+1})} \text{ for } k \in \mathbb{Z}_2 .$$

Since $h \in \Omega_\varphi$, it follows from the definition of the sequence $\{\mu_k\}$ that $\mu_{k+1} > \mu_k$ for every $k \in \mathbb{Z}$. Hence, we get for every $k \in \mathbb{Z}$,

$$\frac{h(\mu_k)}{\varphi(\mu_k)} \ge a\frac{h(\mu_{k+1})}{\varphi(\mu_{k+1})} \ge a^2\frac{h(\mu_k)}{\varphi(\mu_{k+1})}$$

and therefore, for $k \in \mathbb{Z}$,

$$\frac{\varphi(\mu_{k+1})}{\varphi(\mu_k)} \ge a^2 > 1. \tag{2}$$

This shows that $\varphi(\mu_k) \upuparrows$. Moreover, by the definition of $\{\mu_k\}$, we have for $k \in \mathbb{Z}$,

$$\frac{h(\mu_{k+1})}{h(\mu_k)} \ge a > 1, \quad \text{and} \quad \frac{h(\mu_{k+1})\,\varphi(\mu_k)}{\varphi(\mu_{k+1})\,h(\mu_k)} \le \frac{1}{a} < 1, \tag{3}$$

so $h(\mu_k) \upuparrows$ and $\frac{h(\mu_k)}{\varphi(\mu_k)} \downdownarrows$.

Finally, $h \in \Omega_\varphi$, whence for $t \in [\mu_k, \mu_{k+1}]$,

$$h(\mu_k) \le h(t) \le h(\mu_{k+1}) = ah(\mu_k), \qquad \text{when } k \in \mathbb{Z}_1;$$

$$\frac{1}{a}\frac{h(\mu_k)}{\varphi(\mu_k)} = \frac{h(\mu_{k+1})}{\varphi(\mu_{k+1})} \le \frac{h(t)}{\varphi(t)} \le \frac{h(\mu_k)}{\varphi(\mu_k)}, \qquad \text{when } k \in \mathbb{Z}_2,$$

which shows (iii). □

**Definition 2.8.** Let $\varphi$ be an admissible function and suppose that $\nu$ is a non-negative Borel measure on $[0,\infty)$. We say that the function $\psi$ defined as

$$\psi(t) = \int_{[0,t]} d\nu(s) + \varphi(t)\int_{[t,\infty)} \frac{d\nu(s)}{\varphi(s)}, \quad t \in (0,\infty), \tag{4}$$

is the *fundamental function of the measure* $\nu$ with respect to $\varphi$.

The measure $\nu$ is *non-degenerate* if the following conditions are satisfied for every $t \in (0,\infty)$:

$$\int_{[0,t]} d\nu(s) + \varphi(t)\int_{[t,\infty)} \frac{d\nu(s)}{\varphi(s)} < \infty, \quad \int_{[0,1]} \frac{d\nu(s)}{\varphi(s)} = \int_{[1,\infty)} d\nu(s) = \infty.$$

*Remark* 2.9. If $\nu$ is a non-negative non-degenerate Borel measure on $[0,\infty)$ and if $\psi$ is the fundamental function of $\nu$ with respect to some admissible function $\varphi$ then $\psi$ is $\varphi$-quasiconcave and moreover $\psi \in \Omega_\varphi$. This follows from the another expression of $\psi$:

$$\psi(t) \approx \int_{[0,\infty)} \frac{\varphi(s)}{\varphi(t)+\varphi(s)} d\nu(s), \qquad t \in (0,\infty),$$

and from standard limiting theorems. For details see [4, Remark 2.10 (i)].

*Remark* 2.10. Conversely, if $\varphi$ is an admissible function and $h \in \Omega_\varphi$, then there exists a representation measure $\nu$, see [4, Lemma 2.8].

The next theorem is proved in [4, Corollary 2.13].

**Theorem 2.11.** *Let $q \in (0,\infty)$, let $u$ be an admissible function, let $w$ be $u$-quasiconcave function, let $\nu$ be a non-negative non-degenerate Borel measure on $[0,\infty)$ and let $\psi$ be the fundamental function of $\nu$ with respect to $u^q$. Let $\{t_k\}$ be a discretizing sequence for $\psi$ with respect to $u^q$. Then*

$$\left(\int_{[0,\infty)} \left(\frac{w(t)}{u(t)}\right)^q d\nu(t)\right)^{1/q} \approx \left(\sum_{k\in\mathbb{Z}} \left(\frac{w(t_k)}{u(t_k)}\right)^q \psi(t_k)\right)^{1/q}.$$

# 3 Dilation Indices

In this section we will study the dilation indices of a quasiconcave function, say $h$, and their connection to a discretizing sequence for $h$.

**Definition 3.1.** Let $h$ be a quasiconcave function, then we denote

$$\overline{h}(t) = \sup_{s>0} \frac{h(ts)}{h(s)}, \quad \underline{h}(t) = \inf_{s>0} \frac{h(ts)}{h(s)}, \qquad \text{for } t > 0,$$

and we define the *dilation indices* of $h$ by

$$\alpha_h = \sup_{0<t<1} \frac{\log \overline{h}(t)}{\log t}, \qquad \beta_h = \inf_{t>1} \frac{\log \overline{h}(t)}{\log t}.$$

**Example 3.2.** Let $h(t) = t^\theta$, where $0 < \theta < 1$. Then $\overline{h}(t) = t^\theta$ and so $\alpha_h = \beta_h = \theta$.

**Proposition 3.3.** *Let $h$ be a quasiconcave function, then*

(i) $\overline{h}$ *and* $\underline{h}$ *are non-decreasing,* $\underline{h} \leq \overline{h}$, *and* $\overline{h}(1) = \underline{h}(1) = 1$;

(ii) $\overline{h}(st) \leq \overline{h}(s)\overline{h}(t), \quad s,t > 0$, *in other words,* $\overline{h}$ *is submultiplicative;*

(iii) $\underline{h}(st) \geq \underline{h}(s)\underline{h}(t), \quad s,t > 0$;

(iv) $0 < \underline{h}(s)h(t) \leq h(st) \leq \overline{h}(s)h(t), \quad s,t > 0$;

(v) $\overline{h}(t) \leq \max(1,t), \quad \underline{h}(t) \geq \min(t,1), \quad t > 0$;

(vi) $\overline{h}(t)\,\underline{h}(1/t) = 1$.

*Proof.* The statements in (i) and (iv) are quite trivial.

Let $t, \tau > 0$ then

$$\overline{h}(t\tau) = \sup_{s>0} \frac{h(t\tau s)}{h(s)} \frac{h(\tau s)}{h(\tau s)} \leq \sup_{s>0} \frac{h(t\tau s)}{h(\tau s)} \sup_{s>0} \frac{h(\tau s)}{h(s)} = \overline{h}(t)\overline{h}(\tau),$$

which proves the submultiplicativity of $\overline{h}$. The assertion in (iii) is analogous.

From (i) it is clear that $\overline{h}(t) \leq 1$ for $t \in (0,1)$, and $\underline{h}(t) \geq 1$ for $t \geq 1$. Since $h$ is quasiconcave and hence $h(t)/t$ is non-increasing, we have for $t \geq 1$

$$\overline{h}(t) = \sup_{s>0} \frac{h(ts)}{ts} \frac{ts}{h(s)} \leq \sup_{s>0} \frac{h(s)}{s} \frac{ts}{h(s)} = t,$$

and for $t \in (0,1)$

$$\underline{h}(t) = \inf_{s>0} \frac{h(ts)}{ts} \frac{ts}{h(s)} \geq \inf_{s>0} \frac{h(s)}{s} \frac{ts}{h(s)} = t,$$

so (v) is proved.

If we take $t > 0$, then

$$\overline{h}(t) = \sup_{s>0} \frac{h(ts)}{h(s)} = \sup_{s>0} \frac{h(s)}{h(s/t)} = \frac{1}{\inf_{s>0} \frac{h(s/t)}{h(s)}} = \frac{1}{\underline{h}(1/t)},$$

which is (vi). □

**Proposition 3.4.** *The dilation indices $\alpha = \alpha_h$ and $\beta = \beta_h$ of the quasiconcave function $h$ are given by the limits*

$$\alpha = \lim_{t\to\infty} \frac{\log \underline{h}(t)}{\log t}, \qquad \beta = \lim_{t\to\infty} \frac{\log \overline{h}(t)}{\log t}, \tag{5}$$

*and they satisfy $0 \le \alpha \le \beta \le 1$.*

*Proof.* By Proposition 3.3 (vi), we get

$$\alpha = \sup_{0<t<1} \frac{\log \overline{h}(t)}{\log t} = \sup_{0<t<1} \frac{-\log \underline{h}(1/t)}{-\log 1/t} = \sup_{t>1} \frac{\log \underline{h}(t)}{\log t}.$$

Let $\varepsilon > 0$, then there exists $t > 1$ such that $\alpha - \varepsilon < \log \underline{h}(t)/\log t \le \alpha$. Choose a positive integer $N$ satisfying

$$\alpha - \varepsilon < \frac{N}{N+1} \frac{\log \underline{h}(t)}{\log t} \le \alpha.$$

Then for every $s \ge t^N$, we find an integer $n \ge N$ such that $t^n \le s < t^{n+1}$. Using Proposition 3.3 (iii), we have

$$\frac{\log \underline{h}(s)}{\log s} \ge \frac{\log \underline{h}(t^n)}{\log t^{n+1}} \ge \frac{\log \underline{h}^n(t)}{\log t^{n+1}} = \frac{n}{n+1} \frac{\log \underline{h}(t)}{\log t} \ge \frac{N}{N+1} \frac{\log \underline{h}(t)}{\log t} > \alpha - \varepsilon.$$

This proves that $\lim_{s\to\infty} \log \underline{h}(s)/\log s = \alpha$, which is the first identity in (5). The second one could be proved in the similar way and it is even simpler.

The inequalities $0 \le \alpha \le \beta \le 1$ follow from (5) and Proposition 3.3 (i), (v). □

*Remark* 3.5. Let $\alpha = \alpha_h$ and $\beta = \beta_h$ be the dilation indices of a quasiconcave function $h$, and let $\varepsilon > 0$. We define functions $f_\varepsilon$ and $g_\varepsilon$ by

$$f_\varepsilon(t) = \begin{cases} t^{\beta+\varepsilon}, \ 0 < t < 1; \\ t^{\alpha-\varepsilon}, \ t \ge 1; \end{cases} \quad \text{and} \quad g_\varepsilon(t) = \begin{cases} t^{\alpha-\varepsilon}, \ 0 < t < 1; \\ t^{\beta+\varepsilon}, \ t \ge 1. \end{cases}$$

If $0 < \alpha \le \beta < 1$, then $f_\varepsilon(t) \lesssim h(t) \lesssim g_\varepsilon(t)$ for $t > 0$.

**Theorem 3.6.** *Let $h \in \Omega_{0,1}$ and assume that $\{t_k\}$ is a discretizing sequence for $h$. Let $\alpha_h$ and $\beta_h$ be the dilation indices of $h$. Then*

$$\sup_{k\in\mathbb{Z}_1} \frac{t_{k+1}}{t_k} < \infty \Leftrightarrow \alpha_h > 0, \qquad \sup_{k\in\mathbb{Z}_2} \frac{t_{k+1}}{t_k} < \infty \Leftrightarrow \beta_h < 1.$$

*Proof.* Let us assume that $\frac{t_{k+1}}{t_k} \le b < \infty$ for all $k$ in $\mathbb{Z}_1$. It is clear from (2) that $a^2 \le b$. If we take such $k$ that $t_{k+1} \le b\,t_k$, then by (3) we have

$$h(b\,t_k) \ge h(t_{k+1}) \ge ah(t_k).$$

On the other hand, the inequality $b\,t_k < t_{k+1}$ implies that $k$ is in the set $\mathbb{Z}_2$, and so

$$h(b\,t_k) = \frac{h(b\,t_k)}{b\,t_k} b\,t_k \ge \frac{h(t_{k+1})}{t_{k+1}} b\,t_k = \frac{b}{a}\frac{h(t_k)}{t_k} t_k \ge ah(t_k).$$

Altogether we get $h(b\,t_k) \ge ah(t_k)$ for each $k$ in $\mathbb{Z}$.

Now, our aim is to prove the following inequality:

$$\underline{h}(b^2) = \inf_{s>0} \frac{h(b^2 s)}{h(s)} \ge a. \tag{6}$$

Let us assume that $s$ lies in some $[t_k, t_{k+1})$. Then we distinguish several possibilities. If $\frac{t_{k+1}}{b} \le s < t_{k+1}$, then

$$\frac{h(b^2 s)}{h(s)} \ge \frac{h(b\,t_{k+1})}{h(t_{k+1})} \ge \frac{ah(t_{k+1})}{h(t_{k+1})} \ge a.$$

Otherwise $t_k < \frac{t_{k+1}}{b}$, and so $k \in \mathbb{Z}_2$. At first, we take such $s$ that $\frac{t_{k+1}}{b^2} \le s < \frac{t_{k+1}}{b}$, then

$$\frac{h(b^2 s)}{h(s)} \ge \frac{h(t_{k+1})}{h(\frac{t_{k+1}}{b})} = \frac{h(t_{k+1})}{\frac{t_{k+1}}{b}} \frac{\frac{t_{k+1}}{b}}{h(\frac{t_{k+1}}{b})} \ge b\frac{h(t_{k+1})}{t_{k+1}}\frac{t_k}{h(t_k)} = \frac{b}{a} \ge a.$$

Secondly, let $t_k \le s < \frac{t_{k+1}}{b^2}$, then

$$\frac{h(b^2 s)}{h(s)} = b^2 \frac{h(b^2 s)}{b^2 s}\frac{s}{h(s)} \ge b^2 \frac{h(t_{k+1})}{t_{k+1}}\frac{t_k}{h(t_k)} = \frac{b^2}{a} \ge a^3 \ge a.$$

Taking the infimum over all positive $s$ we obtain (6).

Let $(b^2)^{n-1} < t \le (b^2)^n$, where $n$ is a natural number. Also by Proposition 3.3 (iii), we have

$$\frac{\log \underline{h}(t)}{\log t} \ge \frac{(n-1)\log \underline{h}(b^2)}{n \log b^2} \ge \frac{n-1}{n}\frac{\log a}{\log b^2} \longrightarrow \frac{\log a}{\log b^2}, \qquad n \to \infty.$$

Proposition 3.4 gives us that $\alpha_h = \frac{\log a}{2\log b} > 0$, as we wanted to prove.

Now, let us assume that $\sup_{k\in\mathbb{Z}_1}\{t_{k+1}/t_k\} = \infty$, then for each $n$ in $\mathbb{N}$ there exists $k(n)$ in $\mathbb{Z}_1$ such that $a^{n+1}t_{k(n)} \leq t_{k(n)+1}$. If we take $a^n < t \leq a^{n+1}$, then

$$\underline{h}(t) = \inf_{s>0}\frac{h(ts)}{h(s)} \leq \frac{h(a^{n+1}t_{k(n)})}{h(t_{k(n)})} \leq \frac{h(t_{k(n)+1})}{h(t_{k(n)})} = a,$$

and therefore

$$\frac{\log \underline{h}(t)}{\log t} \leq \frac{\log a}{\log a^n} = \frac{1}{n} \longrightarrow 0, \quad n \to \infty.$$

And again Proposition 3.4 yields $\alpha_h = 0$. This concludes the proof of the first equivalence.

We now turn our attention to the second equivalence. Assume that $t_{k+1}/t_k \leq b < \infty$ for all $k$ in $\mathbb{Z}_2$. We will first look at such $k$ for which $t_{k+1} \leq b\,t_k$. For these, we have also by (3) that

$$\frac{h(b\,t_k)}{h(t_k)} = b\frac{h(b\,t_k)}{b\,t_k}\frac{t_k}{h(t_k)} \leq b\frac{h(t_{k+1})}{t_{k+1}}\frac{t_k}{h(t_k)} \leq \frac{b}{a}.$$

If we take $k$ for which $b\,t_k < t_{k+1}$, then this $k$ has to belong to the set $\mathbb{Z}_1$, and we obtain

$$\frac{h(b\,t_k)}{h(t_k)} \leq \frac{h(t_{k+1})}{h(t_k)} = a \leq \frac{b}{a}.$$

The conclusion is that for every $k$ in $\mathbb{Z}$ holds $\frac{h(b\,t_k)}{h(t_k)} \leq \frac{b}{a}$.

We are going to prove the following inequality

$$\overline{h}(b^2) = \sup_{s>0}\frac{h(b^2 s)}{h(s)} \leq \frac{b^2}{a}. \tag{7}$$

Let $s$ be a positive real number, then we can find the interval $[t_k, t_{k+1})$ in which $s$ lies. Furthermore, if $\frac{t_{k+1}}{b} \leq s < t_{k+1}$, then

$$\frac{h(b^2 s)}{h(s)} = b^2\frac{h(b^2 s)}{b^2 s}\frac{s}{h(s)} \leq b^2\frac{h(b\,t_{k+1})}{b\,t_{k+1}}\frac{t_{k+1}}{h(t_{k+1})} = b\frac{h(b\,t_{k+1})}{h(t_{k+1})} \leq \frac{b^2}{a}.$$

Otherwise $t_k < \frac{t_{k+1}}{b}$, which implies that $k$ lies in $\mathbb{Z}_1$. At first, we assume that $\frac{t_{k+1}}{b^2} \leq s < \frac{t_{k+1}}{b}$, then

$$\frac{h(b^2 s)}{h(s)} = b^2\frac{h(b^2 s)}{b^2 s}\frac{s}{h(s)} \leq b^2\frac{h(t_{k+1})}{t_{k+1}}\frac{\frac{t_{k+1}}{b}}{h(\frac{t_{k+1}}{b})} \leq b\frac{h(t_{k+1})}{h(t_k)} = b\,a = \frac{b\,a^2}{a} \leq \frac{b^2}{a}.$$

Finally, if $t_k \leq s < \frac{t_{k+1}}{b^2}$, then

$$\frac{h(b^2 s)}{h(s)} \leq \frac{h(t_{k+1})}{h(t_k)} = a \leq \frac{b}{a} < \frac{b^2}{a},$$

and by taking the supremum over all positive $s$ we obtain (7). Let $(b^2)^n < t \leq (b^2)^{n+1}$, where $n$ is a natural number. Since $\overline{h}$ is submultiplicative, we have

$$\frac{\log \overline{h}(t)}{\log t} \leq \frac{(n+1)\log \overline{h}(b^2)}{n \log b^2} \leq \frac{n+1}{n} \frac{\log \frac{b^2}{a}}{\log b^2} \longrightarrow \frac{\log \frac{b^2}{a}}{\log b^2}, \quad n \to \infty.$$

By Proposition 3.4 we get the desired inequality $\beta_h = \frac{\log \frac{b^2}{a}}{\log b^2} < 1$, so one direction of the second equivalence is proved.

In order to prove the reverse direction, let $\sup_{k \in \mathbb{Z}_2}\{t_{k+1}/t_k\} = \infty$, which means that for every natural number $n$ there exists $k(n)$ in $\mathbb{Z}_2$ such that $a^n t_{k(n)} \leq t_{k(n)+1}$. Let us consider such $t$ that $a^n < t \leq a^{n+1}$, then

$$\begin{aligned} \overline{h}(t) &= \sup_{s>0} \frac{h(ts)}{h(s)} \geq \frac{h(a^n t_{k(n)})}{h(t_{k(n)})} = a^n \frac{h(a^n t_{k(n)})}{a^n t_{k(n)}} \frac{t_{k(n)}}{h(t_{k(n)})} \\ &\geq a^n \frac{h(t_{k(n)+1})}{t_{k(n)+1}} \frac{t_{k(n)}}{h(t_{k(n)})} = a^{n-1}. \end{aligned}$$

Moreover,

$$\frac{\log \overline{h}(t)}{\log t} \geq \frac{(n-1)\log a}{(n+1)\log a} = \frac{n-1}{n+1} \longrightarrow 1, \quad n \to \infty.$$

Using once again Proposition 3.4, we obtain $\beta_h = 1$.

The proof is complete. □

# 4 Embeddings between Classical Lorentz Spaces

In this section we will recall some well-known embeddings between classical Lorentz spaces. The *weight* is a non-negative measurable function on $(0, \infty)$.

**Definition 4.1.** Let $1 \leq p < \infty$ and let $v$ be a weight. We define the *classical Lorentz space* $\Lambda^p(v)$ by

$$\Lambda^p(v) = \left\{ f \in \mathcal{M}(R, \mu) : \|f\|_{\Lambda^p(v)} = \left( \int_0^\infty f^*(t)^p v(t)\, dt \right)^{1/p} < \infty \right\}.$$

**Theorem 4.2** (embedding $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$). *Let $1 < p \leq q < \infty$ and let $v, w$ be weights. Then the inequality*

$$\left( \int_0^\infty f^*(t)^q w(t)\, dt \right)^{1/q} \leq C_1 \left( \int_0^\infty f^*(t)^p v(t)\, dt \right)^{1/p} \tag{8}$$

*holds with some constant $0 < C_1 < \infty$ if and only if there exists constant $0 < C_2 < \infty$ such that*

$$\left( \int_0^t w(s)\, ds \right)^{1/q} \leq C_2 \left( \int_0^t v(s)\, ds \right)^{1/p}, \qquad t > 0.$$

*This constant $C_2$, if it exists, can be used in (8) as $C_1$.*

This result can be found in [13, Remark (i), p. 148].

**Corollary 4.3.** *Let $0 < p \leq q < \infty$, $0 < a < b \leq \infty$ and $\tau \in (0,1)$. Then the following inequality holds:*

$$\left( \int_a^b f^*(t)^q\, dt \right)^{1/q} \leq ((1-\tau)\, a)^{\frac{1}{q}-\frac{1}{p}} \left( \int_{\tau a}^b f^*(t)^p\, dt \right)^{1/p}.$$

*Proof.* If we use Theorem 4.2 with the weights $w(t) = \chi_{(a,b)}(s)$ and $v(s) = \chi_{(\tau a, b)}(s)$, it suffices to show

$$\left( \int_0^t \chi_{(a,b)}(s)\, ds \right)^{1/q} \leq ((1-\tau)\, a)^{\frac{1}{q}-\frac{1}{p}} \left( \int_0^t \chi_{(\tau a,b)}(s)\, ds \right)^{1/p}, \qquad t > 0.$$

Since $\frac{1}{q} - \frac{1}{p} \leq 0$ we have

$$\sup_{a<t<b} \frac{(t-a)^{1/q}}{(t-\tau a)^{1/p}} \leq \sup_{a<t} \frac{(t-\tau a)^{1/q}}{(t-\tau a)^{1/p}} = \sup_{a<t} (t-\tau a)^{\frac{1}{q}-\frac{1}{p}} \leq (a-\tau a)^{\frac{1}{q}-\frac{1}{p}},$$

which gives us the desired inequality. □

# 5 The Real Interpolation Method

We will construct the real interpolation spaces $(X_0, X_1)_{\varrho,q}$ with the function parameter $\varrho$. These spaces are a generalization of the interpolation spaces by Lions and Peetre, and they were studied in [9], [6] and [12]. We take into account the discrete definition, which was introduced by S. Jansson in [8]. We shall make a connection between these two definitions and give some properties of these spaces.

**Definition 5.1.** A pair $(X_0, X_1)$ of Banach spaces $X_0$ and $X_1$ is called a *compatible couple* if there is some Hausdorff topological vector space $H$ in which each of $X_0$ and $X_1$ is continuously embedded.

If $(X_0, X_1)$ is given we denote the *intersection* of $X_0$ and $X_1$ by $X_0 \cap X_1$, and it consists of all elements $h$ in $H$ that are in both $X_0$ and $X_1$. For $h$ in $X_0 \cap X_1$ we define

$$\|h\|_{X_0 \cap X_1} = \max(\|h\|_{X_0}, \|h\|_{X_1}).$$

Further we denote the *sum* of $X_0$ and $X_1$ by $X_0 + X_1$, that is, the set of elements $h$ in $H$ that are representable in the form $h = x_0 + x_1$ for some $x_0$ in $X_0$ and $x_1$ in $X_1$. For each $h$ in $X_0 + X_1$ we define

$$\|h\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : h = x_0 + x_1,\ x_0 \in X_0,\ x_1 \in X_1\}.$$

The next proposition is standard, see for example [1, Theorem III.1.3].

**Proposition 5.2.** *Suppose that $(X_0, X_1)$ is a compatible couple of Banach spaces. Then $X_0 \cap X_1$ and $X_0 + X_1$ are also Banach spaces.*

We will now define the Peetre K-functional.

**Definition 5.3.** Let $(X_0, X_1)$ be a compatible couple of Banach spaces. The *K-functional* is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1} : f = f_0 + f_1\},$$

where the infimum extends over all representations $f = f_0 + f_1$ of $f$ with $f_0 \in X_0$ and $f_1 \in X_1$.

The following lemma is an easy consequence of the definition of the K-functional.

**Lemma 5.4.** *For every $f$ in $X_0 + X_1$, the K-functional $K(f, t; X_0, X_1)$ is a positive, increasing and concave function of $t > 0$, and*

$$t^{-1}K(f, t; X_0, X_1) = K(f, t^{-1}; X_1, X_0). \tag{9}$$

*In particular, $t^{-1}K(f, t; X_0, X_1)$ is decreasing on $(0, \infty)$.*

*Moreover*

$$K(f,t;X_0,X_1) \le \max(1,t/s)K(f,s;X_0,X_1), \quad s,t>0, \tag{10}$$

*and*

$$\min(1,t/s)K(f,s;X_0,X_1) \le K(f,t;X_0,X_1), \quad s,t>0. \tag{11}$$

**Definition 5.5.** The function $\varrho : (0,\infty) \to (0,\infty)$ belongs to the function class $\mathcal{V}_q$, $1 \le q \le \infty$, if $\varrho$ is measurable and satisfies for all positive $t$:

$$\int_0^\infty \left(\frac{\min(s,t)}{\varrho(s)}\right)^q \frac{ds}{s} < \infty, \quad q < \infty,$$

$$\sup_{s>0} \frac{\min(s,t)}{\varrho(s)} < \infty, \quad q = \infty.$$

**Definition 5.6.** Let $(X_0, X_1)$ be a compatible couple of Banach spaces and let $\varrho \in \mathcal{V}_q$. Then we define the functional

$$\|f\|_{(X_0,X_1)_{\varrho,q}} = \begin{cases} \left(\int\limits_0^\infty \left[\frac{K(f,t;X_0,X_1)}{\varrho(t)}\right]^q \frac{dt}{t}\right)^{1/q}, & 1 \le q < \infty, \\ \sup\limits_{t>0} \frac{K(f,t;X_0,X_1)}{\varrho(t)}, & q = \infty, \end{cases}$$

for all measurable functions on $(R,\mu)$ and we denote

$$(X_0,X_1)_{\varrho,q} = \left\{f \in \mathcal{M}(R,\mu) : \|f\|_{(X_0,X_1)_{\varrho,q}} < \infty\right\}.$$

*Remark* 5.7. These spaces are the generalization of well-known interpolation spaces $(X_0,X_1)_{\theta,q}$, i.e. let $\varrho_\theta(t) = t^\theta$, where $0 < \theta < 1$, then

$$(X_0,X_1)_{\theta,q} = (X_0,X_1)_{\varrho_\theta,q}, \qquad 1 \le q \le \infty.$$

We will work with discrete Definition 5.8 in the following sections, but one could say that Definition 5.6 is more natural, so we are going to show that they are in some sense equivalent, see Corollary 5.12.

**Definition 5.8.** Let $(X_0, X_1)$ be a compatible couple of Banach spaces and let $h \in \Omega_{0,1}$. Assume that $\{t_k\}_{k\in\mathbb{Z}}$ is a discretizing sequence for $h$. Then we define

$$\|f\|_{(X_0,X_1)^*_{h,q}} = \begin{cases} \left(\sum\limits_{k\in\mathbb{Z}} \left[\frac{K(f,t_k;X_0,X_1)}{h(t_k)}\right]^q\right)^{1/q}, & 1 \le q < \infty, \\ \sup\limits_{k\in\mathbb{Z}} \frac{K(f,t_k;X_0,X_1)}{h(t_k)}, & q = \infty, \end{cases}$$

for all measurable functions on $(R,\mu)$ and we denote

$$(X_0,X_1)^*_{h,q} = \left\{ f \in \mathcal{M}(R,\mu) : \|f\|_{(X_0,X_1)^*_{h,q}} < \infty \right\}.$$

**Proposition 5.9.** *Let $(X_0,X_1)$ be a given compatible couple of Banach spaces, let $\varrho \in \mathcal{V}_q$ and let $h \in \Omega_{0,1}$. Assume that $\{t_k\}_{k\in\mathbb{Z}}$ is a discretizing sequence for $h$. If $1 \le q \le \infty$, then $(X_0,X_1)_{\varrho,q}$ and $(X_0,X_1)^*_{h,q}$ are Banach spaces.*

*Proof.* Let $\Phi_{\varrho,q}$, respective $\Phi^*_{h,q}$, be the functional defined by

$$\Phi_{\varrho,q}(\varphi(t)) = \left\| \frac{\varphi(t)}{\varrho(t)} \right\|_{L_q(dt/t)}, \quad \text{resp. } \Phi^*_{h,q}(\varphi(t)) = \left\| \frac{\varphi(t_k)}{h(t_k)} \right\|_{l_q},$$

where $\varphi$ is a non-negative function and $\{t_k\}$ is the discretizing sequence for $h$. Since $K(f,t;X_0,X_1)$ is a norm on $X_0+X_1$ and since $\Phi_{\varrho,q}$, resp. $\Phi^*_{h,q}$, has all three properties of a norm, it is easy to see that $(X_0,X_1)_{\varrho,q}$ and $(X_0,X_1)^*_{h,q}$ are normed vector spaces. To establish completeness, suppose that $\{f_n\}_{n=1}^\infty$ is a sequence in $X_0+X_1$ with $\sum_{n=1}^\infty \|f_n\|_{(X_0,X_1)_{\varrho,q}} < \infty$, resp. $\sum_{n=1}^\infty \|f_n\|_{(X_0,X_1)^*_{h,q}} < \infty$. Since $L_q(dt/t)$, resp. $l_q$, is complete, the series

$$\frac{1}{\varrho(t)} \sum_{n=1}^\infty K(f_n,t;X_0,X_1), \quad \text{resp. } \frac{1}{h(t_k)} \sum_{n=1}^\infty K(f_n,t_k;X_0,X_1),$$

converges in $L_q(dt/t)$, resp. $l_q$, and hence it is finite for all $t>0$, resp. $t_k$. Now $X_0+X_1$ is also complete, and $K(f,t;X_0,X_1)$ is an equivalent norm, so this implies that $K(\sum f_n,t;X_0,X_1) \le \sum K(f_n,t;X_0,X_1)$, resp. $K(\sum f_n,t_k;X_0,X_1) \le \sum K(f_n,t_k;X_0,X_1)$. Applying $\Phi_{\varrho,q}$, resp. $\Phi^*_{h,q}$, we obtain

$$\|\sum_{n=1}^\infty f_n\|_{(X_0,X_1)_{\varrho,q}} \le \sum_{n=1}^\infty \|f_n\|_{(X_0,X_1)_{\varrho,q}} < \infty,$$

respectively with the norm of $(X_0,X_1)^*_{h,q}$, and this establishes the completeness of $(X_0,X_1)_{\varrho,q}$, resp. $(X_0,X_1)^*_{h,q}$. □

**Proposition 5.10.** *Let $(X_0,X_1)$ be a given compatible couple and suppose that $h \in \Omega_{0,1}$. Let $\alpha_h$ and $\beta_h$ be the dilation indices of $h$ satisfying $0 < \alpha_h \le \beta_h < 1$. Assume that $\{t_k\}$ is a discretizing sequence for $h$. If $1 \le q < \infty$, then $(X_0,X_1)_{h,q} = (X_0,X_1)^*_{h,q}$ with equivalence of norms.*

*Proof.* Since $0 < \alpha_h \le \beta_h < 1$, then, using Remark 3.5, it is easy to see that $h \in \mathcal{V}_q$. By Definition 2.6 (i) and Theorem 3.6, there are real numbers $a$ and $b$ such that $1 < a < b < \infty$ and $a \le \frac{t_{k+1}}{t_k} \le b$ for all $k$ in $\mathbb{Z}$.

Using also (10), we obtain

$$\begin{aligned}
\|f\|^q_{(X_0,X_1)_{h,q}} &= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} \left(\frac{K(f,t;X_0,X_1)}{h(t)}\right)^q \frac{dt}{t} \\
&\leq \sum_{k\in\mathbb{Z}} \log\frac{t_{k+1}}{t_k} \left(\frac{K(f,t_{k+1};X_0,X_1)}{h(t_k)}\right)^q \\
&\leq \log b \sum_{k\in\mathbb{Z}} \left(\max\left(1,\frac{t_{k+1}}{t_k}\right) \frac{K(f,t_k;X_0,X_1)}{h(t_k)}\right)^q \\
&\leq b^q \log b\, \|f\|^q_{(X_0,X_1)^*_{h,q}}.
\end{aligned}$$

On the other hand, we have also by (11),

$$\begin{aligned}
\|f\|^q_{(X_0,X_1)_{h,q}} &= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} \left(\frac{K(f,t;X_0,X_1)}{h(t)}\right)^q \frac{dt}{t} \\
&\geq \sum_{k\in\mathbb{Z}} \log\frac{t_{k+1}}{t_k} \left(\frac{K(f,t_k;X_0,X_1)}{h(t_{k+1})}\right)^q \\
&\geq \log a \sum_{k\in\mathbb{Z}} \left(\min\left(1,\frac{t_k}{t_{k+1}}\right) \frac{K(f,t_{k+1};X_0,X_1)}{h(t_{k+1})}\right)^q \\
&\geq \left(\frac{1}{b}\right)^q \log a\, \|f\|^q_{(X_0,X_1)^*_{h,q}}.
\end{aligned}$$

Therefore the norms in $(X_0,X_1)_{h,q}$ and $(X_0,X_1)^*_{h,q}$ are equivalent. □

In the following proposition, the case $q=\infty$ is treated.

**Proposition 5.11.** *Let $(X_0,X_1)$ be a compatible couple and suppose that $h\in\Omega_{0,1}$. Let $\{t_k\}$ be a discretizing sequence for $h$. Then $(X_0,X_1)_{h,\infty} = (X_0,X_1)^*_{h,\infty}$ with equivalence of norms.*

*Proof.* We shall prove the following equivalence

$$\sup_{t>0} \frac{K(f,t;X_0,X_1)}{h(t)} \approx \sup_{k\in\mathbb{Z}} \frac{K(f,t_k;X_0,X_1)}{h(t_k)}. \tag{12}$$

Clearly, the righthand side is less or equal to the lefthand side, so it is remaining to show the converse inequality. Since $\{t_k\}$ is the discretizing sequence for $h$, there is a decomposition $\mathbb{Z}=\mathbb{Z}_1\cup\mathbb{Z}_2$ such that for every $t\in[t_k,t_{k+1}]$ and $k\in\mathbb{Z}_1$,

$$\frac{K(f,t;X_0,X_1)}{h(t)} \leq \frac{K(f,t_{k+1};X_0,X_1)}{h(t_k)} \approx \frac{K(f,t_{k+1};X_0,X_1)}{h(t_{k+1})},$$

and for $t \in [t_k, t_{k+1}]$ and $k \in \mathbb{Z}_2$,

$$\frac{K(f,t;X_0,X_1)}{t}\frac{t}{h(t)} \le \frac{K(f,t_k;X_0,X_1)}{t_k}\frac{t_{k+1}}{h(t_{k+1})} \approx \frac{K(f,t_k;X_0,X_1)}{t_k}\frac{t_k}{h(t_k)}.$$

From this the desired converse inequality in (12) follows. □

In the sequel, we will use namely the following general discretization method.

**Corollary 5.12.** *Let $(X_0, X_1)$ be a compatible couple of Banach spaces and suppose that $q \in [1, \infty)$. Let $\varrho \in \mathcal{V}_q$ and let $\nu$ be the measure defined by $d\nu(s) = \frac{s^{q-1}}{\varrho^q(s)}\,ds$. Let $\psi$ be the fundamental function of the measure $\nu$ with respect to $\varphi(t) = t^q$. Let $h$ be defined by*

$$h(t) = \left(\frac{t^q}{\psi(t)}\right)^{1/q} = \left(t^{-q}\int_0^t \frac{s^{q-1}ds}{\varrho^q(s)} + \int_t^\infty \frac{ds}{s\,\varrho^q(s)}\right)^{-1/q}, \quad t > 0,$$

*then $h \in \Omega_{0,1}$ and $(X_0, X_1)_{\varrho,q} = (X_0, X_1)^*_{h,q}$ with the equivalence of norms.*

*Proof.* By Remark 2.9, we have $\psi \in \Omega_\varphi$, and so $h(t) = \left(\frac{t^q}{\psi(t)}\right)^{1/q}$ is increasing, and $\frac{h(t)}{t} = \left(\frac{1}{\psi(t)}\right)^{1/q}$ is decreasing. It is clear that $h$ also satisfies the non-degeneracy conditions (1), hence $h \in \Omega_{0,1}$. Now, we prove that we can take the same discretizing sequence for $h$ and $\psi$. Let $\{t_k\}$ be a discretizing sequence for $h$, which can be obtained from Lemma 2.7, then by (2) and (3), we have

$$\frac{t_{k+1}}{t_k} \ge a, \quad \frac{h(t_{k+1})}{h(t_k)} \ge a, \quad \frac{h(t_{k+1})}{t_{k+1}}\frac{t_k}{h(t_k)} \le \frac{1}{a}, \quad k \in \mathbb{Z}.$$

Therefore

$$\frac{t_{k+1}^q}{t_k^q} \ge a^q, \quad \frac{\psi(t_{k+1})}{\psi(t_k)} \ge a^q, \quad \frac{\psi(t_{k+1})}{t_{k+1}^q}\frac{t_k^q}{\psi(t_k)} \le \frac{1}{a^q}, \quad k \in \mathbb{Z}.$$

This shows that $t_k^q \uparrow\uparrow$, $\psi(t_k) \uparrow\uparrow$ and $\frac{\psi(t_k)}{t_k^q} \downarrow\downarrow$, hence the properties (i) and (ii) in Definition 2.6 are fulfilled. To establish the condition (iii), we take $k \in \mathbb{Z}_2$, then $\frac{h(t_k)}{t_k} = a\frac{h(t_{k+1})}{t_{k+1}}$ and so

$$a^q\psi(t_k) = \left(a\frac{t_k}{h(t_k)}\right)^q = \left(\frac{t_{k+1}}{h(t_{k+1})}\right)^q = \psi(t_{k+1}).$$

On the other hand, when $k \in \mathbb{Z}_1$, then $ah(t_k) = h(t_{k+1})$ and we have

$$\frac{\psi(t_k)}{t_k^q} = \left(\frac{1}{h(t_k)}\right)^q = \left(\frac{a}{h(t_{k+1})}\right)^q = a^q\frac{\psi(t_{k+1})}{t_{k+1}^q},$$

and the condition (iii) in Definition 2.6 is also verified, hence $\{t_k\}$ is a discretization sequence for $\psi$ with respect to $\varphi$.

We should use Theorem 2.11 with $u(t) = t, w(t) = K(f, t; X_0, X_1), q = q$ and the measure $\nu$ defined by

$$d\nu(t) = \frac{t^{q-1}}{\varrho^q(t)}\, dt.$$

By Lemma 5.4, the K-functional is quasiconcave, the measure $\nu$ is non-degenerate, since $\varrho \in \mathcal{V}_q$, and $\{t_k\}$ is the discretizing sequence for $\psi$ with respect to $\varphi$, so all assumptions in Theorem 2.11 are satisfied. Therefore

$$\begin{aligned}
\|f\|_{(X_0,X_1)_{\varrho,q}} &= \left(\int_0^\infty \left(\frac{K(f,t;X_0,X_1)}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q} \\
&= \left(\int_0^\infty \left(\frac{K(f,t;X_0,X_1)}{t}\right)^q d\nu(t)\right)^{1/q} \\
&\approx \left(\sum_{k\in\mathbb{Z}} \left(\frac{K(f,t_k;X_0,X_1)}{t_k}\right)^q \psi(t_k)\right)^{1/q} \\
&= \left(\sum_{k\in\mathbb{Z}} \left(\frac{K(f,t_k;X_0,X_1)}{h(t_k)}\right)^q\right)^{1/q} = \|f\|_{(X_0,X_1)^*_{h,q}},
\end{aligned}$$

as we wanted to prove. □

*Remark* 5.13. From Proposition 5.11 and Corollary 5.12 we also obtain that the definition of $(X_0, X_1)^*_{h,q}$ does not depend on the choice of the discretizing sequence $\{t_k\}$.

*Remark* 5.14. If we consider the special case $(X_0, X_1)_{\theta,q} = (X_0, X_1)_{\varrho_\theta,q}$, where $\varrho_\theta(t) = t^\theta$, then the fundamental function of the measure $\nu$, which is defined by $d\nu(t) = t^{q(1-\theta)-1}dt$, is $\psi(t) = \frac{1}{q\theta(1-\theta)} t^{q(1-\theta)}$. Corollary 5.12 yields

$$(X_0, X_1)_{\theta,q} = (X_0, X_1)^*_{h,q}, \qquad \text{where } h(t) = t^\theta.$$

**Theorem 5.15** (The interpolation theorem). *Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be two compatible couples. Suppose that $T$ is a bounded linear operator from $X_i$ to $Y_i$ with the norm $M_i$, $i = 0, 1$. Let $\varrho \in \mathcal{V}_q$ and $1 \le q \le \infty$. Then*

$$\|Tf\|_{(Y_0,Y_1)_{\varrho,q}} \le M_0\,\overline{\varrho}(M_1/M_0)\|f\|_{(X_0,X_1)_{\varrho,q}}.$$

*Proof.* Let $f = f_0 + f_1$, where $f_i \in X_i$, $i = 0, 1$, then

$$\|Tf_0\|_{Y_0} + t\|Tf_1\|_{Y_1} \le M_0\|f_0\|_{X_0} + M_1 t\|f_1\|_{X_1} = M_0\left(\|f_0\|_{X_0} + \frac{M_1}{M_0} t\|f_1\|_{X_1}\right).$$

By taking the infimum over all representations $f = f_0 + f_1$ we obtain

$$K(Tf, t; Y_0, Y_1) \le M_0\, K\left(f, \frac{M_1}{M_0}t; X_0, X_1\right).$$

Therefore

$$\|Tf\|_{(Y_0,Y_1)_{\varrho,q}} = \left\| \frac{K(Tf, t; Y_0, Y_1)}{\varrho(t)} \right\|_{L_q(dt/t)} \le M_0 \left\| \frac{K\left(f, \frac{M_1}{M_0}t; X_0, X_1\right)}{\varrho(t)} \right\|_{L_q(dt/t)}.$$

The substitution $s = \frac{M_1}{M_0}t$ and Proposition 3.3 (iv), (vi) yield

$$\begin{aligned} \|Tf\|_{(Y_0,Y_1)_{\varrho,q}} &\le M_0 \left\| \frac{K(f, s; X_0, X_1)}{\varrho(\frac{M_0}{M_1}s)} \right\|_{L_q(ds/s)} \\ &\le M_0 \left\| \frac{K(f, s; X_0, X_1)}{\underline{\varrho}(\frac{M_0}{M_1})\varrho(s)} \right\|_{L_q(ds/s)} = M_0\, \overline{\varrho}\left(\frac{M_1}{M_0}\right) \|f\|_{(X_0,X_1)_{\varrho,q}}, \end{aligned}$$

as we wanted to prove. □

We collect a number of important properties of the interpolation spaces $(X_0, X_1)_{\varrho,q}$ and $(X_0, X_1)^*_{h,q}$.

**Proposition 5.16.** *Let $(X_0, X_1)$ be a given compatible couple of Banach spaces and let $\varrho \in \mathcal{V}_q$, where $1 \le q \le \infty$. Then*

(i) $X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\varrho,q} \hookrightarrow X_0 + X_1$;

(ii) *If* $\varrho \lesssim \omega$, *then* $(X_0, X_1)_{\varrho,q} \hookrightarrow (X_0, X_1)_{\omega,q}$;

(iii) $(X_0, X_1)_{\varrho,q} = (X_1, X_0)_{\omega,q}$, *where* $\omega(t) = t\,\varrho(1/t)$;

(iv) *If $X_0 = X_1$ with equal norms, then $(X_0, X_1)_{\varrho,q} = X_0 = X_1$ with equivalent norms.*

*Proof.* Let $\Phi_{\varrho,q}$ be the functional defined by

$$\Phi_{\varrho,q}(\varphi(t)) = \left(\int_0^\infty \left(\frac{|\varphi(t)|}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q}, \quad 1 \le q < \infty,$$

and with the usual modification when $q = \infty$. From (11) in Lemma 5.4 we have

$$\min(1, t)\, \|f\|_{X_0+X_1} = \min(1, t)\, K(f, 1; X_0, X_1) \le K(f, t; X_0, X_1),$$

and applying $\Phi_{\varrho,q}$, we obtain

$$\Phi_{\varrho,q}(\min(1,t))\, \|f\|_{X_0+X_1} \le \|f\|_{(X_0,X_1)_{\varrho,q}}.$$

Since $\varrho \in \mathcal{V}_q$, we have

$$C = \Phi_{\varrho,q}(\min(1,t)) = \left(\int_0^1 \left(\frac{t}{\varrho(t)}\right)^q \frac{dt}{t} + \int_1^\infty \left(\frac{1}{\varrho(t)}\right)^q \frac{dt}{t}\right)^{1/q} < \infty,$$

and so we obtain $(X_0,X_1)_{\varrho,q} \hookrightarrow X_0 + X_1$.

In order to proof $X_0 \cap X_1 \hookrightarrow (X_0,X_1)_{\varrho,q}$, we write

$$K(f,t;X_0,X_1) \le \min(1,t)\, \|f\|_{X_0 \cap X_1},$$

which is an easy consequence of the definition of the K-functional. Again, we apply $\Phi_{\varrho,q}$ to this inequality to obtain

$$\|f\|_{(X_0,X_1)_{\varrho,q}} \le \Phi_{\varrho,q}(\min(1,t))\, \|f\|_{X_0\cap X_1} = C\, \|f\|_{X_0 \cap X_1}.$$

This completes the proof of (i).

Let $\varrho \le c\omega$, then $\Phi_{\omega,q} \le c\,\Phi_{\varrho,q}$ and so $\|f\|_{(X_0,X_1)_{\omega,q}} \le c\, \|f\|_{(X_0,X_1)_{\varrho,q}}$, which proves (ii).

Let $q < \infty$, then by (9) we obtain

$$\|f\|^q_{(X_0,X_1)_{\varrho,q}} = \int_0^\infty \left(\frac{K(f,t;X_0,X_1)}{\varrho(t)}\right)^q \frac{dt}{t} = \int_0^\infty \left(\frac{t\,K(f,1/t;X_1,X_0)}{\varrho(t)}\right)^q \frac{dt}{t}.$$

Using the substitution $s = 1/t$, we achieve

$$\|f\|^q_{(X_0,X_1)_{\varrho,q}} = \int_0^\infty \left(\frac{K(f,s;X_1,X_0)}{s\,\varrho(1/s)}\right)^q \frac{ds}{s} = \|f\|^q_{(X_0,X_1)_{\omega,q}}.$$

Since the case $q = \infty$ is analogous, (iii) is proved.

Let $X_0 = X_1$ with equal norms, then it is easy to see that

$$K(f,t;X_0,X_1) = \min(1,t)\|f\|_{X_0}.$$

Applying $\Phi_{\varrho,q}$, we obtain

$$\|f\|_{(X_0,X_1)_{\varrho,q}} = \Phi_{\varrho,q}(\min(1,t))\|f\|_{X_0},$$

which shows (iv). □

**Proposition 5.17.** *Let $(X_0,X_1)$ be a given compatible couple of Banach spaces and let $h \in \Omega_{0,1}$. If $1 \le q \le \infty$, then*

(i) $X_0 \cap X_1 \hookrightarrow (X_0, X_1)^*_{h,q} \hookrightarrow X_0 + X_1$;

(ii) $(X_0, X_1)^*_{h,q} = (X_1, X_0)^*_{g,q}$, *where* $g(t) = t\,h(1/t)$;

(iii) *If* $X_0 = X_1$ *with equal norms, then* $(X_0, X_1)^*_{h,q} = X_0 = X_1$ *with equivalent norms.*

*Proof.* Again, we denote

$$\Phi^*_{h,q}(\varphi(t)) = \left( \sum_{k\in\mathbb{Z}} \left( \frac{|\varphi(t_k)|}{h(t_k)} \right)^q \right)^{1/q},$$

where $\{t_k\}$ is the discretizing sequence for $h$. In the case $q = \infty$, we should replace the sum by the supremum.

Directly from the definition of K-functional, we get

$$\min(t,1)\,\|f\|_{X_0+X_1} = \min(t,1)\,K(f,1;X_0,X_1) \le K(f,t;X_0,X_1),$$

and applying $\Phi^*_{h,q}$, we obtain

$$\Phi^*_{h,q}(\min(t,1))\,\|f\|_{X_0+X_1} \le \|f\|_{(X_0,X_1)^*_{h,q}}.$$

Since $\frac{h(t_k)}{t_k} \downdownarrows$ and $h(t_k) \upuparrows$, we have for some $a > 1$ that

$$\frac{t_k}{h(t_k)} \le \frac{1}{a}\frac{t_{k+1}}{h(t_{k+1})}, \quad \text{and} \quad \frac{1}{h(t_{k+1})} \le \frac{1}{a}\frac{1}{h(t_k)}, \quad \text{for } k \in \mathbb{Z}.$$

This implies

$$\begin{aligned} C^q = \Phi^*_{h,q}(\min(t,1))^q &= \sum_{k=-\infty}^{0} \left( \frac{t_k}{h(t_k)} \right)^q + \sum_{k=1}^{\infty} \left( \frac{1}{h(t_k)} \right)^q \\ &\le \sum_{k=-\infty}^{0} \left( \left(\frac{1}{a}\right)^{-k} \frac{1}{h(1)} \right)^q + \sum_{k=1}^{\infty} \left( \left(\frac{1}{a}\right)^{k} \frac{1}{h(1)} \right)^q < \infty, \end{aligned}$$

and so we have $(X_0, X_1)^*_{h,q} \hookrightarrow X_0 + X_1$.

In order to prove $X_0 \cap X_1 \hookrightarrow (X_0, X_1)^*_{h,q}$, we write

$$K(f,t;X_0,X_1) \le \min(t,1)\,\|f\|_{X_0\cap X_1},$$

and again, we apply $\Phi^*_{h,q}$ to this inequality

$$\|f\|_{(X_0,X_1)^*_{h,q}} \le \Phi^*_{h,q}(\min(t,1))\,\|f\|_{X_0\cap X_1} = C\,\|f\|_{X_0\cap X_1}.$$

This completes the proof of (i).

By (9) we have

$$\|f\|_{(X_0,X_1)^*_{h,q}} = \left\| \frac{K(f,t_k;X_0,X_1)}{h(t_k)} \right\|_{l_q} = \left\| \frac{t_k\, K(f,1/t_k;X_1,X_0)}{h(t_k)} \right\|_{l_q}.$$

Since $h$ is quasiconcave, the function $g(t) = \frac{h(1/t)}{1/t}$ is increasing and $\frac{g(t)}{t} = h(1/t)$ is decreasing. It is easy to see that $g$ satisfies also the non-degenerate conditions (1), so $g \in \Omega_{0,1}$. By the substitution $s_k = 1/t_{-k}$ for $k \in \mathbb{Z}$, we obtain

$$\|f\|_{(X_0,X_1)^*_{h,q}} = \left\| \frac{K(f,s_k;X_1,X_0)}{s_k\, h(1/s_k)} \right\|_{l_q} = \left\| \frac{K(f,s_k;X_1,X_0)}{g(s_k)} \right\|_{l_q} = \|f\|_{(X_1,X_0)^*_{g,q}}.$$

To make the last equation clear, we show that $\{s_k\}$ is a discretizing sequence for $g$:

$$s_0 = 1, \quad s_k = \frac{1}{t_{-k}} \upuparrows, \quad g(s_k) = \frac{h(t_{-k})}{t_{-k}} \upuparrows, \quad \frac{g(s_k)}{s_k} = h(t_{-k}) \downdownarrows.$$

For such $k$ that $-k-1$ lies in $\mathbb{Z}_2$, we have

$$a\, g(s_k) = a s_k\, h(1/s_k) = a\frac{h(t_{-k})}{t_{-k}} = \frac{h(t_{-k-1})}{t_{-k-1}} = s_{k+1}\, h(1/s_{k+1}) = g(s_{k+1}),$$

on the other hand, when $-k-1$ is in $\mathbb{Z}_1$:

$$\frac{g(s_k)}{s_k} = h(1/s_k) = h(t_{-k}) = a\, h(t_{-k-1}) = a\, h(1/s_{k+1}) = a\,\frac{g(s_{k+1})}{s_{k+1}}.$$

Since $g$ is quasiconcave, all the three properties of Definition 2.6 are verified and (ii) follows.

Let $X_0 = X_1$ with equal norms, then we can express the K-functional as follows

$$K(f,t;X_0,X_1) = \min(t,1)\|f\|_{X_0}.$$

Applying $\Phi^*_{h,q}$, we obtain

$$\|f\|_{(X_0,X_1)^*_{h,q}} = \Phi^*_{h,q}(\min(t,1))\|f\|_{X_0},$$

which shows (iii). □

One can notice that in Proposition 5.17 there is missing a counterpart of Proposition 5.16 (ii). The following theorem fills in this gap.

**Theorem 5.18.** *Let $(X_0,X_1)$ be a given compatible couple of Banach spaces and let $g,h \in \Omega_{0,1}$. Suppose that $\{s_l\}$ is the discretizing sequence for $g$. Then $(X_0,X_1)^*_{h,q} \hookrightarrow (X_0,X_1)^*_{g,p}$ if one of the following conditions holds:*

(i) $1 \le q < p < \infty$ *and*

$$\left\| \frac{g(s_l)}{h(s_l)} \right\|_{l_r} < \infty.$$

*where $r$ is given by $1/r = 1/q - 1/p$.*

(ii) $1 \le p \le q < \infty$ *and*

$$\sup_{t>0} \frac{g(t)}{h(t)} < \infty.$$

*Proof.* Since there is no other compatible couple than $(X_0, X_1)$, we can denote $K(f,t;X_0,X_1)$ by $K(f,t)$. Let $\{t_k\}$ be the discretizing sequence for $h$, then we have

$$\begin{aligned}\|f\|^q_{(X_0,X_1)^*_{h,q}} &= \sum_{k\in\mathbb{Z}} \left(\frac{K(f,t_k)}{h(t_k)}\right)^q \\ &= \sum_{l\in\mathbb{Z}_1} \sum_{s_l \le t_k < s_{l+1}} \left(\frac{K(f,t_k)}{h(t_k)}\right)^q + \sum_{l\in\mathbb{Z}_2} \sum_{s_l \le t_k < s_{l+1}} \left(\frac{K(f,t_k)}{h(t_k)}\right)^q = J_1 + J_2,\end{aligned}$$

where $\mathbb{Z}_1$ and $\mathbb{Z}_2$ are defined in Definition 2.6 with discretizing sequence $\{s_l\}$ for the quasiconcave function $g$. By Definition 2.6, $h(t_k) \upuparrows$ and so $\frac{1}{h(t_{k+1})} \le \frac{1}{a}\frac{1}{h(t_k)}$ for $k \in \mathbb{Z}$. Hence, we have for fixed $l \in \mathbb{Z}_1$,

$$\sum_{s_l \le t_k < s_{l+1}} \left(\frac{1}{h(t_k)}\right)^q \le \left(\frac{1}{h(s_l)}\right)^q \left(1 + \left(\frac{1}{a}\right)^q + \left(\frac{1}{a}\right)^{2q} + \cdots\right) = \frac{a^q}{a^q - 1}\left(\frac{1}{h(s_l)}\right)^q.$$

Therefore

$$J_1 \le \frac{a^q}{a^q-1} \sum_{l\in\mathbb{Z}_1} \left(\frac{K(f,s_{l+1})}{h(s_l)}\right)^q = \frac{a^{2q}}{a^q-1} \sum_{l\in\mathbb{Z}_1} \left(\frac{K(f,s_{l+1})}{g(s_{l+1})}\right)^q \left(\frac{g(s_l)}{h(s_l)}\right)^q,$$

where we used the fact that $a\, g(s_l) = g(s_{l+1})$ for $l \in \mathbb{Z}_1$.

If $1 \le q < p < \infty$ and $r$ is given by $1/r = 1/q - 1/p$, we can use the Hölder inequality with exponents $p/q$ and $r/q$ to obtain

$$J_1 \lesssim \left(\sum_{l\in\mathbb{Z}_1} \left(\frac{K(f,s_{l+1})}{g(s_{l+1})}\right)^p\right)^{q/p} \left(\sum_{l\in\mathbb{Z}_1} \left(\frac{g(s_l)}{h(s_l)}\right)^r\right)^{q/r} \le \|f\|^q_{(X_0,X_1)^*_{g,p}} \left\|\frac{g(s_l)}{h(s_l)}\right\|^q_{l_r}.$$

We can estimate $J_2$ in the similar way. By Definition 2.6, $\frac{h(t_k)}{t_k} \downdownarrows$ and so $\frac{t_k}{h(t_k)} \le \frac{1}{a}\frac{t_{k+1}}{h(t_{k+1})}$ and we obtain for any $l \in \mathbb{Z}_2$

$$\sum_{s_l \le t_k < s_{l+1}} \left(\frac{t_k}{h(t_k)}\right)^q \le \left(\frac{s_{l+1}}{h(s_{l+1})}\right)^q \left(1 + \left(\frac{1}{a}\right)^q + \left(\frac{1}{a}\right)^{2q} + \cdots\right).$$

Hence

$$J_2 = \sum_{l\in\mathbb{Z}_2} \sum_{s_l \le t_k < s_{l+1}} \left(\frac{K(f,t_k)}{t_k}\right)^q \left(\frac{t_k}{h(t_k)}\right)^q$$
$$\le \frac{a^q}{a^q-1} \sum_{l\in\mathbb{Z}_2} \left(\frac{K(f,s_l)}{s_l}\right)^q \left(\frac{s_{l+1}}{h(s_{l+1})}\right)^q$$
$$= \frac{a^{2q}}{a^q-1} \sum_{l\in\mathbb{Z}_2} \left(\frac{K(f,s_l)}{g(s_l)}\right)^q \left(\frac{g(s_{l+1})}{h(s_{l+1})}\right)^q,$$

where we used that $\frac{g(s_l)}{s_l} = a\frac{g(s_{l+1})}{s_{l+1}}$ for $l \in \mathbb{Z}_2$. Again we apply the Hölder inequality with exponents $p/q$ and $r/q$ and we obtain

$$J_2 \lesssim \left(\sum_{l\in\mathbb{Z}_2} \left(\frac{K(f,s_l)}{g(s_l)}\right)^p\right)^{q/p} \left(\sum_{l\in\mathbb{Z}_2} \left(\frac{g(s_{l+1})}{h(s_{l+1})}\right)^r\right)^{q/r} \le \|f\|^q_{(X_0,X_1)^*_{g,p}} \left\|\frac{g(s_l)}{h(s_l)}\right\|^q_{l_r}.$$

Altogether, we have for $1 \le q < p < \infty$ that

$$\|f\|_{(X_0,X_1)^*_{h,q}} \lesssim \left\|\frac{g(s_l)}{h(s_l)}\right\|_{l_r} \|f\|_{(X_0,X_1)^*_{g,p}},$$

which proves (i).

If $1 \le p \le q < \infty$, then we should use instead of the Hölder inequality above the following fact: Let $\{a_n\}$ be a non-negative sequence of real numbers and $0 < \alpha \le 1$, then

$$\left(\sum_{n\in\mathbb{Z}} a_n\right)^\alpha \le \sum_{n\in\mathbb{Z}} a_n^\alpha.$$

This is actually true, if we denote $b_k = \frac{a_k}{\sum_{n\in\mathbb{Z}} a_n}$, then each $b_k$ is less or equal to 1, so $b_k \le b_k^\alpha$, and we have

$$1 = \sum_{k\in\mathbb{Z}} b_k \le \sum_{k\in\mathbb{Z}} b_k^\alpha = \frac{1}{\left(\sum_{n\in\mathbb{Z}} a_n\right)^\alpha} \sum_{k\in\mathbb{Z}} a_k^\alpha.$$

This inequality with $\alpha = p/q$ yields

$$\|f\|^q_{(X_0,X_1)^*_{h,q}} = J_1 + J_2 \lesssim \sup_{l\in\mathbb{Z}} \left(\frac{g(s_l)}{h(s_l)}\right)^q \sum_{l\in\mathbb{Z}} \left(\frac{K(f,s_l)}{g(s_l)}\right)^q$$
$$\le \left(\sup_{l\in\mathbb{Z}} \frac{g(s_l)}{h(s_l)}\right)^q \left(\sum_{l\in\mathbb{Z}} \left(\frac{K(f,s_l)}{g(s_l)}\right)^p\right)^{q/p}$$
$$= \left(\sup_{t>0} \frac{g(t)}{h(t)}\right)^q \|f\|^q_{(X_0,X_1)^*_{g,p}},$$

which shows (ii). □

## 6 A Characterization of $(L_{p_0}, L_{p_1})_{\varrho,q}$

In this section we shall characterize the space $(L_{p_0}, L_{p_1})^*_{h,q}$ in terms of the non-increasing rearrangement. We will further identify the space $(L_{p_0}, L_{p_1})_{\varrho,q}$ with some classical Lorentz space $\Lambda^q(\varphi)$.

We now consider $(L_{p_0}, L_{p_1})$, where $1 < p_0 < p_1 < \infty$, the special case of a compatible couple. Then we have the characterization of the K-functional due to T. Holmstedt, see also [1, Theorem V.2.1].

**Theorem 6.1.** *Let $1 < p_0 < p_1 < \infty$ and let $\sigma$ be given by $1/\sigma = 1/p_0 - 1/p_1$. Then*

$$K(f,t;L_{p_0},L_{p_1}) \approx \left(\int_0^{t^\sigma} f^{**}(s)^{p_0} ds\right)^{1/p_0} + t\left(\int_{t^\sigma}^\infty f^{**}(s)^{p_1} ds\right)^{1/p_1},$$

*for all $f$ in $L_{p_0} + L_{p_1}$ and all $t > 0$.*

Now, we are going to show that the double star in Theorem 6.1 can be replaced by the single star.

**Corollary 6.2.** *Let $1 < p_0 < p_1 < \infty$ and let $\sigma$ be given by $1/\sigma = 1/p_0 - 1/p_1$. Then*

$$K(f,t;L_{p_0},L_{p_1}) \approx \left(\int_0^{t^\sigma} f^{*}(s)^{p_0} ds\right)^{1/p_0} + t\left(\int_{t^\sigma}^\infty f^{*}(s)^{p_1} ds\right)^{1/p_1},$$

*Proof.* Clearly by Theorem 6.1 and by $f^* \le f^{**}$, we get one inequality. We turn now our attention to show the converse one. Since $p_0 > 1$, we can use the Hardy inequality and we get

$$\left(\int_0^{t^\sigma} \left(\frac{1}{s}\int_0^s f^*(y)dy\right)^{p_0} ds\right)^{1/p_0} \lesssim \left(\int_0^{t^\sigma} f^*(s)^{p_0} ds\right)^{1/p_0}. \tag{13}$$

But also $p_1 > 1$ and again by the Hardy inequality and the Hölder inequality we obtain

$$\begin{aligned}
&t\left(\int_{t^\sigma}^\infty \left(\frac{1}{s}\int_0^s f^*(y)dy\right)^{p_1} ds\right)^{1/p_1} \\
&= t\left(\int_0^{t^\sigma} f^*(y)dy\right)\left(\int_{t^\sigma}^\infty \frac{1}{s^{p_1}} ds\right)^{1/p_1} + t\left(\int_{t^\sigma}^\infty \left(\frac{1}{s}\int_{t^\sigma}^s f^*(y)dy\right)^{p_1} ds\right)^{1/p_1} \\
&\lesssim t^{1+\frac{\sigma(1-p_1)}{p_1}} \int_0^{t^\sigma} f^*(y)dy + t\left(\int_{t^\sigma}^\infty f^*(s)^{p_1} ds\right)^{1/p_1} \qquad (14)\\
&\le t^{1+\frac{\sigma(1-p_1)}{p_1}} \left(\int_0^{t^\sigma} f^*(y)^{p_0} dy\right)^{1/p_0} t^{\sigma\frac{1}{p_0'}} + t\left(\int_{t^\sigma}^\infty f^*(s)^{p_1} ds\right)^{1/p_1} \\
&= \left(\int_0^{t^\sigma} f^*(y)^{p_0} dy\right)^{1/p_0} + t\left(\int_{t^\sigma}^\infty f^*(s)^{p_1} ds\right)^{1/p_1}.
\end{aligned}$$

Combining (13) and (14) we get the desired converse inequality. $\square$

*Remark* 6.3. When we are working with the norm of $(X_0, X_1)^*_{h,q}$, we will consider that a discretizing sequence $\{t_k\}$ for $h$ is given by Lemma 2.7. Therefore, there exists $a > 4$ such that

$$ah(t_k) = h(t_{k+1}) \text{ for } k \in \mathbb{Z}_1 \quad \text{and} \quad \frac{h(t_k)}{t_k} = a\frac{h(t_{k+1})}{t_{k+1}} \text{ for } k \in \mathbb{Z}_2.$$

**Theorem 6.4.** *Let $1 < p_0 < p_1 < \infty$ and $1 \le q \le \infty$. Let $h \in \Omega_{0,1}$ and assume that $\{t_k\}$ is a discretizing sequence for $h$. Let $\sigma$ be given by $1/\sigma = 1/p_0 - 1/p_1$. Then*

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \approx \left\| \frac{1}{h(t_k)} \left( \int_{t_k^\sigma}^{t_{k+1}^\sigma} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q^{\mathbb{Z}_1}} + \left\| \frac{t_k}{h(t_k)} \left( \int_{t_k^\sigma}^{t_{k+1}^\sigma} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q^{\mathbb{Z}_2}}.$$

*Proof.* By the definition of $(L_{p_0}, L_{p_1})^*_{h,q}$ and Corollary 6.2, we obtain

$$\begin{aligned} \|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} &= \left\| \frac{K(f, t_k; L_{p_0}, L_{p_1})}{h(t_k)} \right\|_{l^q} \\ &\approx \left\| \frac{\left(\int_0^{t_k^\sigma} f^*(s)^{p_0} ds\right)^{1/p_0} + t_k \left(\int_{t_k^\sigma}^\infty f^*(s)^{p_1} ds\right)^{1/p_1}}{h(t_k)} \right\|_{l^q}. \end{aligned}$$

If $a, b \in l^q$, we have $\|a\|_{l^q} + \|b\|_{l^q} \approx \|a+b\|_{l^q}$. In fact, the following inequalities hold: $\|a\|_{l^q} + \|b\|_{l^q} \le 2\|a+b\|_{l^q} \le 2(\|a\|_{l^q} + \|b\|_{l^q})$. It follows that

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \approx \left\| \frac{1}{h(t_k)} \left( \int_0^{t_k^\sigma} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l^q} + \left\| \frac{t_k}{h(t_k)} \left( \int_{t_k^\sigma}^\infty f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l^q}. \tag{15}$$

Let us denote the interval $[t_k^\sigma, t_{k+1}^\sigma)$ as $I_k$. For a while, let $q < \infty$. Since $\frac{1}{h(t_k)} \downdownarrows$ and $\frac{t_k}{h(t_k)} \upuparrows$, then by Lemma 2.2,

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \approx \left\{ \sum_{k\in\mathbb{Z}} \left( \sum_{m=-\infty}^{k} \int_{I_{m-1}} f^*(s)^{p_0} ds \right)^{q/p_0} \left( \frac{1}{h(t_k)} \right)^q \right\}^{1/q}$$

$$+\left\{\sum_{k\in\mathbb{Z}}\left(\sum_{m=k}^{\infty}\int_{I_m}f^*(s)^{p_1}ds\right)^{q/p_1}\left(\frac{t_k}{h(t_k)}\right)^q\right\}^{1/q}$$

$$\approx\left\{\sum_{k\in\mathbb{Z}}\left(\frac{1}{h(t_k)}\left(\int_{I_{k-1}}f^*(s)^{p_0}ds\right)^{1/p_0}\right)^q\right\}^{1/q}$$

$$+\left\{\sum_{k\in\mathbb{Z}}\left(\frac{t_k}{h(t_k)}\left(\int_{I_k}f^*(s)^{p_1}ds\right)^{1/p_1}\right)^q\right\}^{1/q}.$$

In the case $q=\infty$, it works in the similar way, but we should use Lemma 2.3 in place of Lemma 2.2. Altogether, we have

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}}\approx\left\|\frac{1}{h(t_{k+1})}\left(\int_{I_k}f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q}$$

$$+\left\|\frac{t_k}{h(t_k)}\left(\int_{I_k}f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q}.$$

Now, our aim is to prove the following equivalence:

$$\left\|\frac{1}{h(t_{k+1})}\left(\int_{I_k}f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q}+\left\|\frac{t_k}{h(t_k)}\left(\int_{I_k}f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q}$$

$$\approx\left\|\frac{1}{h(t_k)}\left(\int_{I_{k-1}\cup I_k}f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q^{\mathbb{Z}_1}}+\left\|\frac{t_k}{h(t_k)}\left(\int_{I_k}f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q^{\mathbb{Z}_2}}.$$

It is obvious that the following inequality holds:

$$\left\|\frac{t_k}{h(t_k)}\left(\int_{I_k}f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q^{\mathbb{Z}_2}}\le\left\|\frac{t_k}{h(t_k)}\left(\int_{I_k}f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q},$$

and we have also

$$\left\|\frac{1}{h(t_k)}\left(\int_{I_{k-1}}f^*(s)^{p_0}ds+\int_{I_k}f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q^{\mathbb{Z}_1}}$$

$$\le 2^{1/p_0}\left\|\frac{1}{h(t_k)}\left(\int_{I_{k-1}}f^*(s)^{p_0}ds\right)^{1/p_0}+\frac{a}{h(t_{k+1})}\left(\int_{I_k}f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q^{\mathbb{Z}_1}}$$

$$\leq 2^{1/p_0}(1+a)\left\| \frac{1}{h(t_{k+1})}\left(\int_{I_k} f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q},$$

because $ah(t_k) = h(t_{k+1})$ for $k \in \mathbb{Z}_1$, so we obtain one half of the desired equivalence.

Since $\|a\|_{l_q} \leq C(\|a\|_{l_q^{\mathbb{Z}_1}} + \|a\|_{l_q^{\mathbb{Z}_2}})$, $a \in l_q$, for the proof of the second half of the equivalence it suffices to show two estimates

$$\left\| \frac{1}{h(t_{k+1})}\left(\int_{I_k} f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q^{\mathbb{Z}_2}} \lesssim \left\| \frac{t_k}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q^{\mathbb{Z}_2}}, \tag{16}$$

and

$$\left\| \frac{t_k}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q^{\mathbb{Z}_1}} \lesssim \left\| \frac{1}{h(t_k)}\left(\int_{I_{k-1}\cup I_k} f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q^{\mathbb{Z}_1}}. \tag{17}$$

We start with proving (16). Using the Hölder inequality with exponents $\frac{p_1}{p_0}$ and $\frac{p_1}{p_1-p_0}$ we get

$$\begin{aligned}\left(\int_{I_k} f^*(s)^{p_0}\cdot 1ds\right)^{1/p_0} &\leq \left[\left(\int_{I_k} f^*(s)^{p_1}ds\right)^{p_0/p_1} |I_k|^{(p_1-p_0)/p_1}\right]^{1/p_0}\\ &= \left(\int_{I_k} f^*(s)^{p_1}ds\right)^{1/p_1} (t_{k+1}^\sigma - t_k^\sigma)^{1/\sigma} \leq t_{k+1}\left(\int_{I_k} f^*(s)^{p_1}ds\right)^{1/p_1}.\end{aligned}$$

By this and the equality $a\frac{h(t_{k+1})}{t_{k+1}} = \frac{h(t_k)}{t_k}$ for $k \in \mathbb{Z}_2$ we finally obtain

$$\begin{aligned}\left\| \frac{1}{h(t_{k+1})}\left(\int_{I_k} f^*(s)^{p_0}ds\right)^{1/p_0}\right\|_{l_q^{\mathbb{Z}_2}} &\leq \left\| \frac{t_{k+1}}{h(t_{k+1})}\left(\int_{I_k} f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q^{\mathbb{Z}_2}}\\ &= a\left\| \frac{t_k}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_1}ds\right)^{1/p_1}\right\|_{l_q^{\mathbb{Z}_2}},\end{aligned}$$

which is (16).

We now turn our attention to the estimate (17). If we use Corollary 4.3 with the parameter $\tau = a^{-\sigma}$, we get

$$\left(\int_{t_k^\sigma}^{t_{k+1}^\sigma} f^*(s)^{p_1}ds\right)^{1/p_1} \leq ((1-a^{-\sigma})t_k^\sigma)^{\frac{1}{p_1}-\frac{1}{p_0}}\left(\int_{a^{-\sigma}t_k^\sigma}^{t_{k+1}^\sigma} f^*(s)^{p_0}ds\right)^{1/p_0}.$$

And because of that $[a^{-\sigma}t_k^\sigma, t_{k+1}^\sigma) \subseteq [t_{k-1}^\sigma, t_{k+1}^\sigma)$ which follows from (2), and the fact that $\frac{1}{p_1} - \frac{1}{p_0} = -\frac{1}{\sigma}$, we have also

$$t_k \left( \int_{t_k^\sigma}^{t_{k+1}^\sigma} f^*(s)^{p_1} ds \right)^{1/p_1} \leq (1 - a^{-\sigma})^{-1/\sigma} \left( \int_{t_{k-1}^\sigma}^{t_{k+1}^\sigma} f^*(s)^{p_0} ds \right)^{1/p_0},$$

and we consequently obtain the estimate (17).

The last step in the proof is the following equivalence

$$\left\| \frac{1}{h(t_k)} \left( \int_{I_{k-1} \cup I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q^{\mathbb{Z}_1}} + \left\| \frac{t_k}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q^{\mathbb{Z}_2}}$$

$$\approx \left\| \frac{1}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q^{\mathbb{Z}_1}} + \left\| \frac{t_k}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q^{\mathbb{Z}_2}}.$$

Since the lefthand side is greater than the righthand one, this direction of the equivalence is clear. In order to prove the converse one, we will distinguish two cases. When $(k-1) \in \mathbb{Z}_1$, we have $h(t_k) = ah(t_{k-1})$ and so

$$\frac{1}{h(t_k)} \left( \int_{I_{k-1}} f^*(s)^{p_0} ds \right)^{1/p_0} = \frac{1}{a} \frac{1}{h(t_{k-1})} \left( \int_{I_{k-1}} f^*(s)^{p_0} \right)^{1/p_0}.$$

On the other hand, when $(k-1) \in \mathbb{Z}_2$, Hölder's inequality with exponents $\frac{p_1}{p_0}$ and $\frac{p_1}{p_1 - p_0}$ yields

$$\begin{aligned} \frac{1}{h(t_k)} \left( \int_{I_{k-1}} f^*(s)^{p_0} ds \right)^{1/p_0} &\leq \frac{1}{h(t_k)} \left( \left( \int_{I_{k-1}} f^*(s)^{p_1} ds \right)^{p_0/p_1} |I_{k-1}|^{1-p_0/p_1} \right)^{p_0} \\ &\leq \frac{1}{h(t_k)} \left( \int_{I_{k-1}} f^*(s)^{p_1} ds \right)^{1/p_1} (t_k^\sigma)^{1/\sigma} \\ &= \frac{1}{a} \frac{t_{k-1}}{h(t_{k-1})} \left( \int_{I_{k-1}} f^*(s)^{p_1} ds \right)^{1/p_1}. \end{aligned}$$

Altogether, we obtain

$$\left\| \frac{1}{h(t_k)} \left( \int_{I_{k-1} \cup I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q^{\mathbb{Z}_1}}$$

$$\lesssim \left\| \frac{1}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q^{\mathbb{Z}_1}} + \left\| \frac{t_k}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q^{\mathbb{Z}_2}}.$$

This completes the proof. □

We show as a corollary of Theorem 6.4 another characterization of $(L_{p_0}, L_{p_1})^*_{h,q}$, which was also given by V. I. Ovchinnikov and A. S. Titenkov in [11]. However, they considered the subset of the set of all quasiconcave functions, so called *linear step functions.* These are functions on $(0,\infty)$ which are in parts either constant, or homogenously linear. We give a new direct proof of this theorem. In particular, we give an elementary proof which does not make use of the Brudnyi-Krugljak theory.

**Definition 6.5.** Let $M_k = \{j \in \mathbb{Z} : 2^{j/\sigma} \in [t_k, t_{k+1})\}$, then we denote $M_k^i = M_k$ when $k \in \mathbb{Z}_i$ and $M_k^i = \emptyset$ when $k \notin \mathbb{Z}_i$, where $i = 1, 2$. We shall assume that a discretization parameter $a$ for the sequence $\{t_k\}$ is greater than $2^{1/\sigma}$, therefore each set $M_k$ is nonempty.

*Remark* 6.6. There appears the space $l_q\left(l_{p_0}^{M_k^1} \oplus l_{p_1}^{M_k^2}\right)$ in the next theorem. We recall that the norm in this space is given by

$$\|\{a_j\}_{j\in\mathbb{Z}}\|_{l_q\left(l_{p_0}^{M_k^1}\oplus l_{p_1}^{M_k^2}\right)} = \left\{\sum_{k\in\mathbb{Z}}\left(\sum_{j\in M_k^1}|a_j|^{p_0}\right)^{q/p_0} + \left(\sum_{j\in M_k^2}|a_j|^{p_1}\right)^{q/p_1}\right\}^{1/q}.$$

**Theorem 6.7.** *Let $1 < p_0 < p_1 < \infty$ and $1 \le q \le \infty$. Let $h \in \Omega_{0,1}$, and assume that $\{t_k\}$ is a discretizing sequence for $h$. Let $\sigma$ be given by $1/\sigma = 1/p_0 - 1/p_1$. Then*

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \approx \left\|\frac{2^{j/p_0} f^*(2^j)}{h(2^{j/\sigma})}\right\|_{l_q\left(l_{p_0}^{M_k^1}\oplus l_{p_1}^{M_k^2}\right)}.$$

*Proof.* It will be useful to define $\{A_k\}_{k\in\mathbb{Z}}, \{\tilde{A}_k\}_{k\in\mathbb{Z}}, \{B_k\}_{k\in\mathbb{Z}}, \{C_k\}_{k\in\mathbb{Z}}$ and $\{D_k\}_{k\in\mathbb{Z}}$ as follows:

$$A_k = \frac{1}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_0} ds\right)^{1/p_0}, \quad \tilde{A}_k = \frac{1}{h(t_{k+1})}\left(\int_{I_k} f^*(s)^{p_0} ds\right)^{1/p_0},$$

$$B_k = \frac{t_k}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_1} ds\right)^{1/p_1},$$

$$C_k = \frac{1}{h(t_k)}\left(\sum_{j\in M_k} 2^j f^*(2^j)^{p_0}\right)^{1/p_0} \approx \left(\sum_{j\in M_k}\left[\frac{2^{j/p_0}}{h(2^{j/\sigma})} f^*(2^j)\right]^{p_0}\right)^{1/p_0}, \tag{18}$$

$$D_k = \frac{t_k}{h(t_k)}\left(\sum_{j\in M_k} 2^j f^*(2^j)^{p_1}\right)^{1/p_1} \approx \left(\sum_{j\in M_k}\left[\frac{2^{j/p_1+j/\sigma}}{h(2^{j/\sigma})} f^*(2^j)\right]^{p_1}\right)^{1/p_1}, \tag{19}$$

where the equivalence in (18) holds for $k$ in $\mathbb{Z}_1$ and the one in (19) holds for $k$ in $\mathbb{Z}_2$. It is actually true, according to Definition 2.6 (iii), we have for each $k$ in $\mathbb{Z}_1$ that

$$\frac{1}{h(t_k)} \approx \frac{1}{h(2^{j/\sigma})}, \quad j \in M_k,$$

and for $k$ in $\mathbb{Z}_2$ we have

$$\frac{t_k}{h(t_k)} \approx \frac{2^{j/\sigma}}{h(2^{j/\sigma})}, \quad j \in M_k.$$

The proof will be complete when we show that

$$\|A_k\|_{l_q^{\mathbb{Z}_1}} + \|B_k\|_{l_q^{\mathbb{Z}_2}} \lesssim \|C_k\|_{l_q^{\mathbb{Z}_1}} + \|D_k\|_{l_q^{\mathbb{Z}_2}} \lesssim \|\tilde{A}_k\|_{l_q} + \|B_k\|_{l_q}, \tag{20}$$

because we know from the proof of Theorem 6.4 that

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \approx \|\tilde{A}_k\|_{l_q} + \|B_k\|_{l_q} \approx \|A_k\|_{l_q^{\mathbb{Z}_1}} + \|B_k\|_{l_q^{\mathbb{Z}_2}}.$$

Let us start with $k \in \mathbb{Z}_1$. If we denote $j_k = \min M_k$ we have

$$A_k = \frac{1}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_0} ds\right)^{1/p_0} = \frac{1}{h(t_k)}\left(\sum_{j\in\mathbb{Z}} \int_{(2^j,2^{j+1})\cap I_k} f^*(s)^{p_0} ds\right)^{1/p_0}$$

$$\leq \frac{1}{h(t_k)}\left(\sum_{j\in M_k} 2^j f^*(2^j)^{p_0}\right)^{1/p_0} + \frac{1}{h(t_k)}\left(\int_{t_k^\sigma}^{2^{j_k}} f^*(s)^{p_0} ds\right)^{1/p_0} = C_k + I.$$

When $(k-1) \in \mathbb{Z}_1$, the situation is simple:

$$I = \frac{1}{a\, h(t_{k-1})}\left(\int_{t_k^\sigma}^{2^{j_k}} f^*(s)^{p_0} ds\right)^{1/p_0} \leq \frac{(2^{j_k-1} f^*(2^{j_k-1})^{p_0})^{1/p_0}}{a\, h(t_{k-1})} \leq \frac{1}{a} C_{k-1}.$$

If $(k-1) \in \mathbb{Z}_2$, then by the Hölder inequality and the fact that $(2^{j_k} - t_k^\sigma) \leq 2^{j_k-1} < t_k^\sigma$, we obtain

$$I \leq \frac{1}{h(t_k)}\left(\int_{t_k^\sigma}^{2^{j_k}} f^*(s)^{p_1} ds\right)^{1/p_1} (2^{j_k} - t_k^\sigma)^{1/\sigma}$$

$$\leq \frac{a\, t_{k-1}}{h(t_{k-1})}(2^{j_k-1} f^*(2^{j_k-1})^{p_1})^{1/p_1} \leq a\, D_{k-1}.$$

We now assume that $k$ is in the set $\mathbb{Z}_2$ and we estimate

$$B_k = \frac{t_k}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_1} ds\right)^{1/p_1} = \frac{t_k}{h(t_k)}\left(\sum_{j\in\mathbb{Z}} \int_{(2^j,2^{j+1})\cap I_k} f^*(s)^{p_1} ds\right)^{1/p1}$$

$$\leq \frac{t_k}{h(t_k)}\left(\sum_{j\in M_k} 2^j f^*(2^j)^{p_1}\right)^{1/p_1} + \frac{t_k}{h(t_k)}\left(\int_{t_k^\sigma}^{2^{j_k}} f^*(s)^{p_1} ds\right)^{1/p_1} = D_k + II.$$

In the same fashion as above we first assume that $(k-1)$ is in the set $\mathbb{Z}_2$, then

$$II = \frac{a\, t_{k-1}}{h(t_{k-1})}\left(\int_{t_k^\sigma}^{2^{j_k}} f^*(s)^{p_1} ds\right)^{1/p_1} \leq \frac{a\, t_{k-1}}{h(t_{k-1})}\left(2^{j_k-1} f^*(2^{j_k-1})^{p_1}\right)^{1/p_1} \leq a\, D_{k-1}.$$

Secondly, we take such a $k$ that $(k-1)$ lies in $\mathbb{Z}_1$, then using Corollary 4.3 we obtain

$$II \lesssim \frac{1}{h(t_k)}\left(\int_{a^{-\sigma}t_k^\sigma}^{2^{j_k}} f^*(s)^{p_0} ds\right)^{1/p_0} \lesssim \frac{1}{h(t_{k-1})}\left(\int_{t_{k-1}^\sigma}^{2^{j_k}} f^*(s)^{p_0} ds\right)^{1/p_0}$$

$$\leq \frac{1}{h(t_{k-1})}\left(\int_{t_{k-1}^\sigma}^{t_k^\sigma} f^*(s)^{p_0} ds\right)^{1/p_0} + \frac{\left(2^{j_k-1} f^*(2^{j_k-1})^{p_0}\right)^{1/p_0}}{h(t_{k-1})} \lesssim 2C_{k-1} + E_{k-2},$$

where $E_k$ stands for $C_k$ when $k$ lies in $\mathbb{Z}_1$, and for $D_k$ when $k$ lies in $\mathbb{Z}_2$. The last inequality follows from the case above. Altogether, it is

$$\|A_k\|_{l_q^{\mathbb{Z}_1}} + \|B_k\|_{l_q^{\mathbb{Z}_2}} \lesssim \|C_k\|_{l_q^{\mathbb{Z}_1}} + \|D_k\|_{l_q^{\mathbb{Z}_2}},$$

so the first estimate in (20) is proved.

For the second one we take $k$ from the set $\mathbb{Z}_1$. Then

$$C_k = \frac{1}{h(t_k)}\left(\sum_{j\in M_k} 2^j f^*(2^j)^{p_0}\right)^{1/p_0} \leq 2^{1/p_0}\frac{1}{h(t_k)}\left(\sum_{j\in M_k}\int_{2^{j-1}}^{2^j} f^*(s)^{p_0} ds\right)^{1/p_0}$$

$$\lesssim \frac{1}{h(t_{k+1})}\left(\int_{I_k} f^*(s)^{p_0} ds\right)^{1/p_0} + \frac{1}{h(t_k)}\left(\int_{2^{j_k-1}}^{t_k^\sigma} f^*(s)^{p_0} ds\right)^{1/p_0} \leq \tilde{A}_k + \tilde{A}_{k-1}.$$

On the other hand, if we take $k$ which belongs to the set $\mathbb{Z}_2$, we have

$$D_k = 2^{1/p_1}\frac{t_k}{h(t_k)}\left(\sum_{j\in M_k} 2^{j-1} f^*(2^j)^{p_1}\right)^{1/p_1} \lesssim \frac{t_k}{h(t_k)}\left(\sum_{j\in M_k}\int_{2^{j-1}}^{2^j} f^*(s)^{p_1} ds\right)^{1/p_1}$$

$$\leq \frac{t_k}{h(t_k)}\left(\int_{I_k} f^*(s)^{p_1} ds\right)^{1/p_1} + \frac{t_k}{h(t_k)}\left(\int_{2^{j_k-1}}^{t_k^\sigma} f^*(s)^{p_1} ds\right)^{1/p_1} = B_k + III,$$

and again we distinguish two possibilities. If $(k-1)$ lies in $\mathbb{Z}_2$, then

$$III \approx \frac{t_{k-1}}{h(t_{k-1})}\left(\int_{2^{j_k-1}}^{t_k^\sigma} f^*(s)^{p_1} ds\right)^{1/p_1} \leq \frac{t_{k-1}}{h(t_{k-1})}\left(\int_{I_{k-1}} f^*(s)^{p_1} ds\right)^{1/p_1} = B_{k-1},$$

and if $(k-1)$ lies in $\mathbb{Z}_1$, then using Corollary 4.3 and $t_{k-1}^\sigma \le a^{-\sigma} t_k^\sigma < \frac{1}{2} t_k^\sigma \le \frac{1}{2} 2^{j_k}$, we obtain

$$III \le \frac{t_k}{h(t_k)} \left( \int_{\frac{1}{2}t_k^\sigma}^{t_k^\sigma} f^*(s)^{p_1} ds \right)^{1/p_1} \lesssim \frac{1}{h(t_k)} \left( \int_{a^{-\sigma}t_k^\sigma}^{t_k^\sigma} f^*(s)^{p_0} ds \right)^{1/p_0} \le \tilde{A}_{k-1}.$$

Thus, finally

$$\|C_k\|_{l_q^{\mathbb{Z}_1}} + \|D_k\|_{l_q^{\mathbb{Z}_2}} \lesssim \|\tilde{A}_k\|_{l_q} + \|B_k\|_{l_q},$$

which is the second estimate in (20). The proof is complete. □

In order to simplify the characterization of $(L_{p_0}, L_{p_1})^*_{h,q}$, we will make restrictions on the dilation indices of $h$.

**Proposition 6.8.** *Let $1 < p_0 < p_1 < \infty$, $1 \le q \le \infty$, let $h \in \Omega_{0,1}$ and let $\{t_k\}$ be the discretizing sequence for $h$. Let $\alpha_h$ and $\beta_h$ be the dilation indices of $h$. Let $\sigma$ be given by $1/\sigma = 1/p_0 - 1/p_1$.*

(i) *If $\alpha_h > 0$, then*

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{t_k}{h(t_k)} \left( \int_{t_k^\sigma}^{\infty} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q}.$$

(ii) *If $\beta_h < 1$, then*

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{1}{h(t_k)} \left( \int_0^{t_k^\sigma} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q}.$$

*Proof.* By Theorem 6.4, we have

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{1}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q^{\mathbb{Z}_1}} + \left\| \frac{t_k}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q^{\mathbb{Z}_2}},$$

where $I_k = [t_k^\sigma, t_{k+1}^\sigma)$.

Let $\alpha_h > 0$, then, by Theorem 3.6, we have $t_k \approx t_{k+1}$ for $k \in \mathbb{Z}_1$. Using the Hölder inequality with exponents $\frac{p_1}{p_0}$ and $\frac{p_1}{p_1 - p_0}$, we obtain

$$\frac{1}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \le \frac{1}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} |I_k|^{\frac{1}{p_0} - \frac{1}{p_1}}.$$

Using the fact that $t_k \approx t_{k+1}$ for $k \in \mathbb{Z}_1$, we have

$$|I_k|^{\frac{1}{p_0}-\frac{1}{p_1}} = (t_{k+1}^\sigma - t_k^\sigma)^{\frac{1}{\sigma}} \le t_{k+1} \lesssim t_k,$$

and therefore

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \lesssim \left\| \frac{t_k}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q}.$$

Now, since $\frac{t_k}{h(t_k)} \uparrow\uparrow$, we shall use Lemma 2.2 when $q < \infty$, or Lemma 2.3 in the case $q = \infty$ to obtain

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \lesssim \left\| \frac{t_k}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q} \approx \left\| \frac{t_k}{h(t_k)} \left( \int_{t_k^\sigma}^{\infty} f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q}.$$

which is one side of the desired equivalence. The converse one is trivial from the proof of Theorem 6.4, more precisely (15), so (i) follows.

On the other hand, if $\beta_h < 1$, then $t_k \approx t_{k+1}$ for $k \in \mathbb{Z}_2$, by Theorem 3.6. This, together with Definition 2.6, implies

$$h(t_k) \approx h(t_{k+1}), \quad k \in \mathbb{Z}.$$

Corollary 4.3 used on $[a^{-\sigma} t_k^\sigma, t_{k+1}^\sigma) \subset [t_{k-1}^\sigma, t_{k+1}^\sigma)$ gives us

$$\begin{aligned} \frac{t_k}{h(t_k)} \left( \int_{I_k} f^*(s)^{p_1} ds \right)^{1/p_1} &\lesssim \frac{t_k}{h(t_k)} \left( \int_{I_{k-1}\cup I_k} f^*(s)^{p_0} ds \right)^{1/p_0} (t_k^\sigma)^{\frac{1}{p_1}-\frac{1}{p_0}} \\ &\lesssim \frac{1}{h(t_k)} \left( \int_{I_{k-1}} f^*(s)^{p_0} ds \right)^{1/p_0} + \frac{1}{h(t_{k+1})} \left( \int_{I_k} f^*(s)^{p_0} ds \right)^{1/p_0}. \end{aligned}$$

It follows that

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \lesssim \left\| \frac{1}{h(t_{k+1})} \left( \int_{I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q}.$$

Since $\frac{1}{h(t_k)} \downarrow\downarrow$, we can again use Lemma 2.2 when $q < \infty$, or Lemma 2.3 when $q = \infty$ to obtain

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \lesssim \left\| \frac{1}{h(t_{k+1})} \left( \int_{I_k} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q} \approx \left\| \frac{1}{h(t_k)} \left( \int_0^{t_k^\sigma} f^*(s)^{p_0} ds \right)^{1/p_0} \right\|_{l_q}.$$

The opposite inequality is clear from (15), the proof of (ii) is complete. □

*Remark* 6.9. In the case $q = \infty$ we have the following formulas:

(i) if $\alpha_h > 0$ then

$$\|f\|_{(L_{p_0}, L_{p_1})_{h,\infty}} \approx \sup_{t>0} \frac{t}{h(t)} \left( \int_{t^\sigma}^{\infty} f^*(s)^{p_1} ds \right)^{1/p_1};$$

(ii) if $\beta_h < 1$ then

$$\|f\|_{(L_{p_0}, L_{p_1})_{h,\infty}} \approx \sup_{t>0} \frac{1}{h(t)} \left( \int_{0}^{t^\sigma} f^*(s)^{p_0} ds \right)^{1/p_0}.$$

This follows from Proposition 5.11, Proposition 6.8 and the fact that $\frac{t_k}{h(t_k)} \approx \frac{t_{k+1}}{h(t_{k+1})}$ when $\alpha_h > 0$ and $\frac{1}{h(t_k)} \approx \frac{1}{h(t_{k+1})}$ when $\beta_h < 1$.

**Proposition 6.10.** *Let $1 < p_0 < p_1 < \infty$, $1 \le q \le \infty$. Let $h \in \Omega_{0,1}$ and assume that $\{t_k\}$ is a discretizing sequence for $h$. Let $\alpha_h$ and $\beta_h$ be the dilation indices of $h$. Let $\sigma$ be given by $1/\sigma = 1/p_0 - 1/p_1$.*

(i) *If $\alpha_h > 0$, then*

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{2^{j/p_0} f^*(2^j)}{h(2^{j/\sigma})} \right\|_{l_q\left(l_{p_1}^{M_k}\right)};$$

(ii) *if $\beta_h < 1$, then*

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{2^{j/p_0} f^*(2^j)}{h(2^{j/\sigma})} \right\|_{l_q\left(l_{p_0}^{M_k}\right)};$$

(iii) *if $0 < \alpha_h \le \beta_h < 1$, then*

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{2^{j/p_0} f^*(2^j)}{h(2^{j/\sigma})} \right\|_{l_q}.$$

*Proof.* From Theorem 6.7 we have

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{2^{j/p_0} f^*(2^j)}{h(2^{j/\sigma})} \right\|_{l_q\left(l_{p_0}^{M_k^1} \oplus l_{p_1}^{M_k^2}\right)}. \tag{21}$$

If $\alpha_h > 0$, then we obtain from Theorem 3.6 that

$$\sup_{k \in \mathbb{Z}_1} \frac{t_{k+1}}{t_k} < \infty,$$

so this implies that the number of elements of $M_k^1$ is uniformly bounded. Since all norms on a finite-dimensional space are equivalent, in fact, it suffices to know that

$$(a_1 + a_2 + \cdots + a_n)^{p_i} \approx (a_1^{p_i} + a_2^{p_i} + \cdots + a_n^{p_i}), \quad i = 0, 1,$$

we can replace $l_{p_0}^{M_k^1}$ in (21) by $l_{p_1}^{M_k^1}$ and we get (i).

Analogously, we can obtain (ii) and (iii). □

**Theorem 6.11.** *Let $1 < p_0 < p_1 < \infty$, $q \in [1, \infty)$. Let $h \in \Omega_{0,1}$ and assume that $\alpha_h$ and $\beta_h$ are the dilation indices of $h$ satisfying $0 < \alpha_h \le \beta_h < 1$. Let $\sigma$ be given by $1/\sigma = 1/p_0 - 1/p_1$ and let*

$$\varphi(t) = \frac{t^{q/p_0}}{t\, h^q(t^{1/\sigma})}, \qquad t > 0.$$

*Then $(L_{p_0}, L_{p_1})_{h,q} = \Lambda^q(\varphi)$ with equivalence of norms.*

*Proof.* Since $0 < \alpha_h \le \beta_h < 1$, we have, by Proposition 5.10, that $(L_{p_0}, L_{p_1})_{h,q} = (L_{p_0}, L_{p_1})^*_{h,q}$ with equivalent norms. From Proposition 6.10 we obtain

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{2^{j/p_0} f^*(2^j)}{h(2^{j/\sigma})} \right\|_{l_q}. \tag{22}$$

We should estimate the norm of $f$ in $\Lambda^q(\varphi)$, that is,

$$\|f\|^q_{\Lambda^q(\varphi)} = \int_0^\infty \left( \frac{t^{1/p_0} f^*(t)}{h(t^{1/\sigma})} \right)^q \frac{dt}{t} = \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left( \frac{t^{1/p_0} f^*(t)}{h(t^{1/\sigma})} \right)^q \frac{dt}{t}.$$

Since $1/p_0 = 1/\sigma + 1/p_1$, we have

$$\frac{t^{1/p_0}}{h(t^{1/\sigma})} = t^{1/p_1} \frac{t^{1/\sigma}}{h(t^{1/\sigma})},$$

so this function is increasing in $t \in (0, \infty)$ by the quasiconcavity of $h$. By this and again by the quasiconcavity of $h$, we obtain

$$\begin{aligned} \|f\|^q_{\Lambda^q(\varphi)} &\le \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left( \frac{(2^{j+1})^{1/p_0} f^*(2^j)}{h((2^{j+1})^{1/\sigma})} \right)^q \frac{dt}{t} \\ &\le 2^{q/p_0} \log 2 \sum_{j \in \mathbb{Z}} \left( \frac{2^{j/p_0} f^*(2^j)}{h(2^{j/\sigma})} \right)^q \approx \|f\|^q_{(L_{p_0}, L_{p_1})^*_{h,q}}, \end{aligned}$$

where we used (22).

Conversely,

$$\|f\|^q_{\Lambda^q(\varphi)} \geq \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \left( \frac{2^{j/p_0} f^*(2^{j+1})}{h(2^{j/\sigma})} \right)^q \frac{dt}{t}$$

$$\geq 2^{-q/p_0} \log 2 \sum_{j\in\mathbb{Z}} \left( \frac{(2^{j+1})^{1/p_0} f^*(2^{j+1})}{h((2^{j+1})^{1/\sigma})} \right)^q \approx \|f\|^q_{(L_{p_0},L_{p_1})^*_{h,q}},$$

again by (22).

Altogether, $(L_{p_0}, L_{p_1})_{h,q} = \Lambda^q(\varphi)$ with equivalence of norms. □

*Remark* 6.12. In same fashion we can prove Theorem 6.11 replacing the condition $0 < \alpha_h \leq \beta_h < 1$ by one of the following conditions:

(i) $\alpha_h > 0$ and $q = p_1$;

(ii) $\beta_h < 1$ and $q = p_0$.

**Example 6.13.** If we consider the special case $\varrho_\theta(t) = t^\theta$, where $0 < \theta < 1$, then we know from Example 3.2 that $\alpha_{\varrho_\theta} = \beta_{\varrho_\theta} = \theta$, and so by Theorem 6.11 we obtain

$$(L_{p_0}, L_{p_1})_{\theta,q} = (L_{p_0}, L_{p_1})_{\varrho_\theta,q} = L_{p_\theta,q}, \quad \text{where } \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

**Example 6.14.** Let $\mu(R) = 1$ and $\varrho_{\theta,\gamma}(t) = t^\theta(1 - \log t)^\gamma$, where $0 < t, \theta < 1$ and $\gamma \in \mathbb{R}$. Then $\alpha_{\varrho_{\theta,\gamma}} = \beta_{\varrho_{\theta,\gamma}} = \theta$ and, by Theorem 6.11, we have

$$(L_{p_0}, L_{p_1})_{\varrho_{\theta,\gamma},q} = L_{p_\theta,q}(\log L)_{-\gamma}, \quad \text{where } \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

# 7 Extrapolation Spaces

Throughout this section we will consider that $\mu(R) = 1$. Then, by Hölder's inequality, we obtain $L_{p_1} \hookrightarrow L_{p_0}$ with the constant of the embedding equal to 1. Therefore, $K(f, t; L_{p_0}, L_{p_1}) = \|f\|_{X_0}$ for $t \geq 1$, and we have

$$\|f\|_{(L_{p_0}, L_{p_1})_{\varrho,q}} \approx \left( \int_0^1 \left( \frac{K(f, t; L_{p_0}, L_{p_1})}{\varrho(t)} \right)^q \frac{dt}{t} \right)^{1/q}.$$

In this setting we shall characterize the space $(L_{p_0}, L_{p_1})^*_{h,q}$ as an extrapolation space.

**Lemma 7.1.** *Let $1 < p_0 < p_1 < \infty$, $1 \leq q \leq \infty$, let $h \in \Omega_{0,1}$ and let $\{t_k\}$ be the discretizing sequence for $h$. Suppose that the dilation indices of $h$ satisfy $0 < \alpha_h \leq \beta_h \leq 1$. Then*

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{1}{\varphi(r_k)} \left( \int_{t_k^\sigma}^1 f^*(s)^{r_k} ds \right)^{1/r_k} \right\|_{l_q},$$

*where $\{r_k\}_{k=-\infty}^{-1}$ is the decreasing sequence such that*

$$p_0 = r_{-1} < r_k < r_{k-1} < p_1, \quad (t_{k+1}^\sigma)^{\frac{1}{p_1} - \frac{1}{r_k}} = c > 1, \quad k = -2, -3, \ldots, \tag{23}$$

*$\sigma$ is given by $1/\sigma = 1/p_0 - 1/p_1$, $\varphi(r_k) = \frac{h(t_k)}{t_k}$ and $I_k = [t_k^\sigma, t_{k+1}^\sigma)$.*

*Proof.* By Proposition 6.8 (i), we have

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{t_k}{h(t_k)} \left( \int_{t_k^\sigma}^1 f^*(s)^{p_1} ds \right)^{1/p_1} \right\|_{l_q}.$$

With the aid of Corollary 4.3 with the parameter $\tau = a^{-\sigma}$, we obtain

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \lesssim \left\| \frac{t_k}{h(t_k)} \; \left((1 - a^{-\sigma}) t_k^\sigma\right)^{\frac{1}{p_1} - \frac{1}{r_{k-1}}} \left( \int_{t_{k-1}^\sigma}^1 f^*(s)^{r_{k-1}} ds \right)^{1/r_{k-1}} \right\|_{l_q}.$$

Clearly, we see from (23) that $r_k \nearrow p_1$ and we have also by (23) that

$$\left((1 - a^{-\sigma}) t_k^\sigma\right)^{\frac{1}{p_1} - \frac{1}{r_{k-1}}} \leq (1 - a^{-\sigma})^{\frac{1}{p_1} - \frac{1}{p_0}} c.$$

Since $\alpha_h > 0$, by Theorem 3.6, we have $t_k \approx t_{k+1}$ for $k \in \mathbb{Z}_1$ and so $\frac{h(t_k)}{t_k} \approx \frac{h(t_{k+1})}{t_{k+1}}$ for all $k \in \mathbb{Z}$ by Definition 2.6 (iii). Using also the Hölder inequality with exponents $\frac{r_k}{r_{k-1}}$ and $\frac{r_k}{r_k - r_{k-1}}$, we have

$$\begin{aligned}
&\left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_{k-1}}(I_{k-1} \cup (t_k^\sigma, 1))} \right\|_{l_q} \\
&\leq \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_{k-1}}(I_{k-1})} \right\|_{l_q} + \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_{k-1}}((t_k^\sigma, 1))} \right\|_{l_q} \\
&\lesssim \left\| \frac{t_{k-1}}{h(t_{k-1})} \|f^*\|_{L_{r_{k-1}}(I_{k-1})} \right\|_{l_q} + \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_k}((t_k^\sigma, 1))} (1 - t_k^\sigma)^{\frac{1}{r_k} - \frac{1}{r_{k-1}}} \right\|_{l_q}.
\end{aligned}$$

Since $\frac{1}{r_k} - \frac{1}{r_{k-1}} > 0$, and $(1 - t_k^\sigma) < 1$, it is $(1 - t_k^\sigma)^{\frac{1}{r_k} - \frac{1}{r_{k-1}}} < 1$, and therefore

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \lesssim \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_k}((t_k^\sigma, 1))} \right\|_{l_q},$$

which is one side of desired equivalence.

In order to get the converse one, we use Hölder's inequality with exponents $\frac{p_1}{r_k}$ and $\frac{p_1}{p_1 - r_k}$ to obtain

$$\left( \int_{t_k^\sigma}^1 f^*(s)^{r_k} ds \right)^{1/r_k} \leq \left( \int_{t_k^\sigma}^1 f^*(s)^{p_1} ds \right)^{1/p_1} (1 - t_k^\sigma)^{\frac{1}{r_k} - \frac{1}{p_1}}.$$

The term $(1 - t_k^\sigma)^{\frac{1}{r_k} - \frac{1}{p_1}}$ can be estimated by 1 and so we obtain

$$\left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_k}((t_k^\sigma, 1))} \right\|_{l_q} \leq \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{p_1}((t_k^\sigma, 1))} \right\|_{l_q} \approx \|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}}.$$

This completes the proof. □

**Theorem 7.2.** *Let* $1 < p_0 < p_1 < \infty$, $1 \leq q \leq \infty$ *and let* $h \in \Omega_{0,1}$ *such that the following conditions hold:*

(i) *the dilation indices of* $h$ *satisfy* $0 < \alpha_h \leq \beta_h \leq 1$;

(ii) *let* $b_\varepsilon$ *be the largest number such that* $\frac{h(t)}{t} t^\varepsilon$ *is increasing on* $(0, b_\varepsilon)$, *then*

$$\inf_{0<\varepsilon<1} (b_\varepsilon)^\varepsilon = b > 0;$$

(iii) *there exists* $a \geq b^{-4}$ *such that*

$$\frac{h(t^2)}{t^2} \leq a\,\frac{h(t)}{t}, \qquad t > 0.$$

*Assume that* $\{t_k\}$ *is the discretizing sequence for* $h$ *with the parameter* $a$ *(see Lemma 2.7). Let* $\{r_k\}_{k=-\infty}^{-1}$ *be the decreasing sequence such that*

$$p_0 = r_{-1} < r_k < p_1, \qquad \text{and} \qquad (t_{k+1}^{\sigma})^{\frac{1}{r_k}-\frac{1}{p_1}} = b^4, \qquad \text{for} \qquad k = -2, -3, \ldots, \tag{24}$$

*where* $\sigma$ *is given by* $1/\sigma = 1/p_0 - 1/p_1$. *Then*

$$\|f\|_{(L_{p_0}, L_{p_1})^*_{h,q}} \approx \left\| \frac{\|f\|_{L_{r_k}}}{\varphi(r_k)} \right\|_{l_q},$$

*where* $\varphi(r_k) = \frac{h(t_k)}{t_k}$.

*Proof.* According to Lemma 7.1, it is sufficient to prove

$$\left\| \frac{\|f\|_{L_{r_k}}}{\varphi(r_k)} \right\|_{l_q} \lesssim \left\| \frac{\|f^*\|_{L_{r_k}((t_k^{\sigma},1))}}{\varphi(r_k)} \right\|_{l_q}.$$

By the Minkowski inequality we obtain

$$\left\| \frac{\|f\|_{L_{r_k}}}{\varphi(r_k)} \right\|_{l_q} \leq \left\| \frac{t_k}{h(t_k)} \sum_{m=-\infty}^{k-1} \|f^*\|_{L_{r_k}(I_m)} \right\|_{l_q} + \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_k}((t_k^{\sigma},1))} \right\|_{l_q} = I + II.$$

To estimate the first term, we apply Hölder's inequality with exponents $\frac{r_m}{r_k}$ and $\frac{r_m}{r_m - r_k}$, and we obtain

$$I \leq \left\| \frac{t_k}{h(t_k)} \sum_{m=-\infty}^{k-1} \|f^*\|_{L_{r_m}(I_m)} \left( \frac{t_{m+1}}{h(t_{m+1})} \right)^2 \left( \frac{h(t_{m+1})}{t_{m+1}} \right)^2 (t_{m+1}^{\sigma})^{\frac{1}{r_k}-\frac{1}{r_m}} \right\|_{l_q}.$$

Now, by (24), we can replace $(t_{m+1}^{\sigma})^{\frac{1}{r_k}-\frac{1}{r_m}}$ by $(t_{m+1}^{\sigma})^{\frac{1}{r_k}-\frac{1}{p_1}}$. Let $1 < q < \infty$. The discrete version of Hölder's inequality with exponents $q$ and $q'$ yields

$$I \lesssim \Bigg\| \frac{t_k}{h(t_k)} \left( \sum_{m=-\infty}^{k-1} \|f^*\|^q_{L_{r_m}(I_m)} \left( \frac{t_{m+1}}{h(t_{m+1})} \right)^{2q} \right)^{1/q}$$
$$\left( \sum_{m=-\infty}^{k-1} \left( \frac{h(t_{m+1})}{t_{m+1}} \right)^{2q'} t_{m+1}^{\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)q'} \right)^{1/q'} \Bigg\|_{l_q}.$$

Since $h$ satisfies (iii), we have

$$\frac{1}{a} \le \frac{h(t_{m+1})}{t_{m+1}} \frac{t_{m+1}^2}{h(t_{m+1}^2)}, \qquad m < 0.$$

Then by (3) and the monotonicity of $\frac{t}{h(t)}$, we get $t_m \le t_{m+1}^2$. This implies

$$\frac{t_m}{t_{m+1}} \le t_{m+1} \le t_{k+1}, \quad m \le k < 0,$$

and by (24) we obtain

$$\left(\frac{t_m}{t_{m+1}}\right)^{\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)} \le (t_{k+1})^{\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)} = b^4 \in (0,1), \quad m \le k < -1. \tag{25}$$

Now, our aim is to prove the following statement:

$$\frac{h(t_k)}{t_k} t_k^{\frac{1}{4}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)} \text{ is increasing on } (0, t_k). \tag{26}$$

It suffices to prove that

$$t_k < t_{k+1} \le b_{\frac{1}{4}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)}.$$

By (24), we have

$$b = (b^4)^{\frac{1}{4}} = t_{k+1}^{\frac{1}{4}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)} \le \left(b_{\frac{1}{4}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)}\right)^{\frac{1}{4}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)},$$

which follows from (ii), so (26) is proved.

Using (25) and (26) we obtain

$$\begin{aligned}
&\left(\sum_{m=-\infty}^{k-1} \left(\frac{h(t_{m+1})}{t_{m+1}}\right)^{2q'} t_{m+1}^{\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)q'}\right)^{1/q'} \\
&= \left(\sum_{m=-\infty}^{k-1} \left(\frac{h(t_{m+1})}{t_{m+1}^{1-\frac{1}{4}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)}}\right)^{2q'} t_{m+1}^{\frac{1}{2}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)q'}\right)^{1/q'} \\
&\le \left(\sum_{m=-\infty}^{k-1} \left(\frac{h(t_k)}{t_k^{1-\frac{1}{4}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)}}\right)^{2q'} t_k^{\frac{1}{2}\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)q'} (b^4)^{\frac{1}{2}q'(k-m+1)}\right)^{1/q'} \\
&= \left(\frac{h(t_k)}{t_k}\right)^2 t_k^{\sigma\left(\frac{1}{r_k}-\frac{1}{p_1}\right)} \left(\sum_{i=0}^{\infty} b^{2q'i}\right)^{1/q'} \lesssim \left(\frac{h(t_k)}{t_k}\right)^2.
\end{aligned}$$

Therefore and also by $\frac{h(t_k)}{t_k} \downdownarrows$, we have

$$
\begin{aligned}
I &\lesssim \left( \sum_{k=-\infty}^{-1} \left( \frac{t_k}{h(t_k)} \right)^q \sum_{m=-\infty}^{k-1} \|f^*\|^q_{L_{r_m}(I_m)} \left( \frac{t_{m+1}}{h(t_{m+1})} \right)^{2q} \left( \frac{h(t_k)}{t_k} \right)^{2q} \right)^{1/q} \\
&= \left( \sum_{m=-\infty}^{-2} \|f^*\|^q_{L_{r_m}(I_m)} \left( \frac{t_{m+1}}{h(t_{m+1})} \right)^{2q} \sum_{k=m+1}^{-1} \left( \frac{h(t_k)}{t_k} \right)^q \right)^{1/q} \\
&\le \left( \sum_{m=-\infty}^{-2} \|f^*\|^q_{L_{r_m}(I_m)} \left( \frac{t_{m+1}}{h(t_{m+1})} \right)^{2q} \left( \frac{h(t_{m+1})}{t_{m+1}} \right)^q \sum_{k=m+1}^{-1} \left( \frac{1}{a} \right)^{q(k-m-1)} \right)^{1/q} \\
&\approx \left( \sum_{m=-\infty}^{-2} \|f^*\|^q_{L_{r_m}(I_m)} \left( \frac{t_{m+1}}{h(t_{m+1})} \right)^q \right)^{1/q} \lesssim \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_k}(I_k)} \right\|_{l_q},
\end{aligned}
$$

where we also used the fact that $\frac{h(t_k)}{t_k} \approx \frac{h(t_{k+1})}{t_{k+1}}$ for all $k \in \mathbb{Z}$. This follows from Theorem 3.6 and Definition 2.6 (iii).

Altogether we obtain

$$
\left\| \frac{t_k}{h(t_k)} \|f\|_{L_{r_k}} \right\|_{l_q} \lesssim \left\| \frac{t_k}{h(t_k)} \|f^*\|_{L_{r_k}((t_k^\sigma, 1))} \right\|_{l_q}.
$$

In the cases $q = 1$ or $q = \infty$, we should replace sums by supremums and everything works in the same way. The proof is complete. □

**Example 7.3.** Let $h(t) = t(\log e/t)^\gamma$, where $\gamma > 0$ and $t \in (0, 1)$. Clearly $h \in \Omega_{0,1}$. Since we want to use Theorem 7.2, we should verify the following conditions:

(i) the dilation indices of $h$ are both equal to 1;

(ii) the function $\frac{h(t)}{t} t^\varepsilon = t^\varepsilon (\log e/t)^\gamma$ is increasing on $(0, e^{1-\frac{\gamma}{\varepsilon}})$, so $b_\varepsilon = e^{1-\frac{\gamma}{\varepsilon}}$. Therefore

$$
b = \inf_{0<\varepsilon<1} (b_\varepsilon)^\varepsilon = \inf_{0<\varepsilon<1} e^{\varepsilon - \gamma} = e^{-\gamma} > 0;
$$

(iii) in our case, we should verify

$$
\begin{aligned}
(\log e/t^2)^\gamma &\le a (\log e/t)^\gamma, \\
\frac{e}{t^2} &\le \left( \frac{e}{t} \right)^{a^{1/\gamma}}, \\
t^{a^{1/\gamma}} &\le e^{a^{1/\gamma} - 1} t^2,
\end{aligned}
$$

which holds when $a \ge 2^\gamma$, so we can take $a = b^{-4} = e^{4\gamma}$.

Theorem 7.2 yields

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,q}} \approx \left\| \frac{\|f\|_{L_{r_k}}}{\varphi(r_k)} \right\|_{l_q}.$$

Now, we shall identify a sequence $\{r_k\}$ and a function $\varphi$. A simple computation gives us that $t_k = e^{1-a^{-k/\gamma}} = e^{1-e^{-4k}}$. Thus,

$$\varphi(r_k) = \frac{h(t_k)}{t_k} = (\log e/t_k)^\gamma = a^k = e^{4k\gamma},$$

and by (24) we finally obtain

$$\frac{1}{r_k} = \frac{1}{p_1} - \frac{4\gamma}{\sigma}\frac{1}{1-e^{-4k}}.$$

We define the grand Lebesgue space in the next definition. This space was introduced in [7].

**Definition 7.4.** Let $1 < p < \infty$, then the *grand Lebesgue space* $L^{p)}$ is determined by the norm

$$\|f\|_{L^{p)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L_{p-\varepsilon}}.$$

**Theorem 7.5.** *Let* $1 < p_0 < p_1 < \infty$ *and* $h(t) = t(1-\log t)^{1/p_1}$. *Then* $L^{p_1)} = (L_{p_0}, L_{p_1})_{h,\infty}$ *with equivalence of norms.*

*Proof.* By Proposition 5.11, we have $(L_{p_0}, L_{p_1})_{h,\infty} = (L_{p_0}, L_{p_1})^*_{h,\infty}$ with equivalent norms. Let $\{t_k\}$ be the discretizing sequence for h with the parameter $b^{-4} = e^{4/p_1}$. We know from Example 7.3 that

$$\|f\|_{(L_{p_0},L_{p_1})^*_{h,\infty}} \approx \sup_{k<0} \frac{\|f\|_{L_{r_k}}}{\varphi(r_k)},$$

where $\varphi(r_k) = \frac{h(t_k)}{t_k}$ and $\{r_k\}$ is defined by (24). If we extend $\varphi$ monotonously on the whole interval $(p_0, p_1)$, we have

$$\sup_{k<0} \frac{\|f\|_{L_{r_k}}}{\varphi(r_k)} \approx \sup_{p_0<s<p_1} \frac{\|f\|_{L_s}}{\varphi(s)}.$$

Indeed, let $s \in [r_k, r_{k+1}]$, then by Hölder's inequality and the fact that $\varphi(r_k) \approx \varphi(r_{k+1})$, we obtain

$$\frac{\|f\|_{L_s}}{\varphi(s)} \leq \frac{\|f\|_{L_{r_{k+1}}}}{\varphi(r_k)} \approx \frac{\|f\|_{L_{r_{k+1}}}}{\varphi(r_{k+1})}.$$

Now, we should express $\varphi(r_k)$ in terms of $r_k$. Using (24), we obtain

$$\begin{aligned}\varphi(r_k) \approx \varphi(r_{k+1}) &= \frac{h(t_{k+1})}{t_{k+1}} = (1-\log t_{k+1})^{1/p_1} \\ &= \left(1-\log e^{-\frac{4}{\sigma}\frac{r_k}{p_1-r_k}}\right)^{1/p_1} = \left(1+\frac{4}{\sigma}\frac{r_k}{p_1-r_k}\right)^{1/p_1} \\ &\approx \left(1+\frac{r_k}{p_1-r_k}\right)^{1/p_1} \approx \left(\frac{1}{p_1-r_k}\right)^{1/p_1}.\end{aligned}$$

Altogether, we get

$$\|f\|_{(L_{p_0},L_{p_1})_{h,\infty}} \approx \sup_{p_0<s<p_1} (p_1-s)^{1/p_1}\|f\|_{L_s} = \sup_{0<\varepsilon<p_1-p_0} \varepsilon^{1/p_1}\|f\|_{L_{p_1-\varepsilon}}.$$

Since $\varepsilon^{\frac{1}{p_1}} \approx \varepsilon^{\frac{1}{p_1-\varepsilon}}$, we eventually obtain

$$\|f\|_{(L_{p_0},L_{p_1})_{h,\infty}} \approx \|f\|_{L^{p_1)}},$$

as we wanted to prove. □

*Remark* 7.6. Combining Theorem 7.5 and Remark 6.9 we obtain

$$\|f\|_{L^{p)}} \approx \sup_{0<t<1} (\log e/t)^{-1/p}\left(\int_{t^p}^1 f^*(s)^p ds\right)^{1/p}.$$

These results were also established by A. Fiorenza and G. E. Karadzhov in [3]. However, they used the theory of the $\Delta^{(p)}$ methods of extrapolation developed in [10].

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