Charles University in Prague
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MASTER THESIS

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Lattices and Codes

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I declare that I wrote the thesis by myself and listed all used sources. I agree with lending of this thesis.

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Abstract: This thesis studies triangular configurations, binary matroids, and integer lattices generated by the codewords of a binary code. We study the following hypothesis: the lattice generated by the codewords of a binary code has a basis consisting only of the codewords. We prove the hypothesis for the matroids with the good ear decomposition. We study the operation of edge contraction in the triangular configurations. Especially in cycles and acyclic triangular configurations. For an arbitrary graph we find a triangular configuration with the skeleton containing this graph as a minor. For every binary matroid we construct a triangular configuration such that the matroid is a minor of the configuration. We prove that between the cycle spaces of the matroid and the configuration exists a bijection. The bijection maps the circuits of the matroid to the circuits of the configuration.
Keywords: code, matroid, triangular configuration, integer lattice, minor.

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Abstrakt: Diplomová práce zkoumá trojúhelníkové konfigurace, binární matroidy a mřížky generované binárními kódy. Zkoumáme domněnku, zdali mřížka generovaná kódovými slovy binárního kódu má bázi skládající se pouze z kódových slov. Domněnku dokážeme pro matroidy s dobrou uchovou dekompozicí. Prostudujeme operaci hranové kontrakce trojúhelníkové konfigurace. Hlavně dopad kontrakce na cyklus a acyklickou trojúhelníkovou konfiguraci. Dokážeme, že pro každý graf existuje trojúhelníková konfigurace s kostrou která obsahuje tento graf jako minor. Pro každý binární matroid sestrojíme trojúhelníkovou konfiguraci, která obsahuje tento matroid jako minor. Navíc dokážeme, že mezi prostory cyklů konfigurace a matroidu existuje bijekce. Tato bijekce zobrazuje kružnice matroidu na kružnice konfigurace.
Klíčová slova: kód, matroid, trojúhelníkova konfigurace, mřízka, minor.

Contents

1 Introduction

In this thesis we study triangular configurations, binary matroids, binary codes, and lattices generated by the cycles of a binary matroid.

In Section 2 we define matroids and graphs. We present some basic facts about matroids.

In Section 3 we define the ear extension and the ear decomposition of a binary matroid. Then we prove the existence of an ear decomposition for every connected matroid and that the cycle space of the matroid has a basis consisting of the "ears" vectors.

In Section 4 we define codes, and integer lattices and describe their basic properties.

In Section 5 we introduce the hypothesis about the basis of the lattice generated by a binary code. The hypothesis is that the lattice generated by the codewords of a binary code has a basis consisting only of the codewords. In Subsection 5.3 we present a sufficient condition for constructing a cycle lattice basis of an ear extension of M by extending a cycle lattice basis of the matroid M.

In Section 6 we define the triangular configuration and its geometric representation.

In Section 7 we study the edge contraction of a triangular configuration. We describe the properties of the cycles that remain a cycle after edge contraction.

In Section 8 for every graph we show how to find a triangular configuration with the skeleton that has this graph as a minor.

In Section 9 we construct a cycle lattice basis for triangular configurations. This basis is constructed from a cycle lattice basis of an edge contraction.

In Section 10 for every binary matroid we construct a triangular configuration such that the matroid is a minor of the configuration. We prove that between the cycle spaces of the matroid and the configuration exists a bijection. The bijection maps the circuits of the matroid to the circuits of the configuration. We show a relationship between the weight polynomials of the configuration and the matroid.

2 Preliminaries

In this section we define basic concepts. We assume that the reader is familiar with the linear algebra.

2.1 Matroids

We use standard definitions, which can be found in Oxley [4].

Let M be a set. The **incidence vector** of a set A a subset of M is a vector χ^A of $\{0,1\}^{|M|}$ such that $\chi^A_e := 1$ if and only if $e \in M$.

A **matroid** M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Elements of $\mathcal{I}$ are called **independent sets**. For particular matroid M, we denote the sets E and $\mathcal{I}$ by $E(M)$ and $\mathcal{I}(M)$, respectively. The maximal independent sets are called **bases**. The collection of bases of M is denoted by $\mathcal{B}$ or $\mathcal{B}(M)$.

Any subset of E that is not in $\mathcal{I}$ is called **dependent**. A minimal dependent set is called **circuit**. Let T be a maximal independent set of M. For $e \in E \setminus T$, let C_e denote the **fundamental circuit** of e with respect to T; that is, C_e is the unique circuit such that $e \in C_e \subseteq T \cup \{e\}$.

The collection of circuits of M is denoted by $\mathcal{C}$ or $\mathcal{C}(M)$. An element e of M that is a circuit is called a **loop**. Moreover, elements f, g of M are said **parallel**, if $\{f, g\}$ is a circuit.

A **parallel class** is a maximal subset X of E such that every two distinct elements of X are parallel and no element of X is a loop. A parallel class that has only one element is called **trivial**. A matroid M is called **simple**, if it has neither loops nor non-trivial parallel classes.

Let M be a matroid with a collection of bases $\mathcal{B}$. Then the **dual** of M, denoted by M^*, is the matroid with the collection of bases $\mathcal{B}^* := \{E(M) - B : B \in \mathcal{B}\}$. We omit a proof that the dual is a matroid. The independent sets, circuits, bases in the dual matroid are called **coindependent** sets, **cocircuits**, **cobases**, respectively.

In this text we will frequently work with circuits rather than independent sets. The next lemma shows that a matroid can be defined in terms of circuits.

Lemma 2.1. *Let $\mathcal{C}$ be a collection of subsets of a set E. Then, $\mathcal{C}$ is the collection of circuits of a matroid on E if and only if $\mathcal{C}$ satisfies the following conditions:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(C3) *If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.*

Matroids have a lot of equivalent definitions. Another definitions can be found in Oxley [4].

Let M be a matroid $(E, \mathcal{I})$. Let T be a subset of E. Then the **deletion** of T from M or the **restriction** of M to $E \setminus T$ is the pair $(E \setminus T, \{I \subseteq E \setminus T : I \in \mathcal{I}\})$. It is denoted by $M \setminus T$ or $M \mid (E \setminus T)$, respectively. For the deletion holds that $\mathcal{C}(M \setminus T) = \{C \subseteq E \setminus T : C \in \mathcal{C}(M)\}$.

Let M/T, the **contraction** of T from M, be given by $M/T = (M^* \setminus T)^*$. For the contraction holds that $\mathcal{C}(M/T) = \{C \setminus T : C \in \mathcal{C}(M)\}$.

A matroid N that is obtained from a matroid M by a sequence of deletions and contractions is called a **minor** of M.

Two matroids M_1 and M_2 are **isomorphic**, written $M_1 \cong M_2$, if there is a bijection ψ from $E(M_1)$ to $E(M_2)$ such that for all $X \subseteq E(M_1)$ $\psi(X)$ is independent in M_2 if and only if X is independent in M_1.

A **graph** G consists of a nonempty set $V(G)$ of vertices and a multiset $E(G)$ of edges each of which consists of an unordered pair of (possibly identical) vertices.

The **degree** of a vertex v is the number of edges incident with v, each loop counting as two edges.

A graph H is a **subgraph** of a graph G if $V(H)$ and $E(H)$ are subsets of $V(G)$ and $E(G)$, respectively.

A graph is a **cycle** if every vertex has an even degree. A nonempty cycle that does not contain any other cycle is called **circuit**.

Lemma 2.2. *Let E be the set of edges of a graph G. Let $\mathcal{C}$ be the collection of circuits of G. Then $\mathcal{C}$ is the set of circuits of a matroid on E.*

The matroid from the lemma above is called **cycle matroid** of a graph G. A matroid that is isomorphic to the cycle matroid of a graph is called **graphic**.

In the next lemma we introduce the **vector matroid**.

Lemma 2.3. *Let E be the set of labels of columns of an $m \times n$ matrix A over a field F. Let $\mathcal{I}$ be the set of subsets X of E for which the multiset of columns labeled by X is linearly independent. Then $(E, \mathcal{I})$ is a matroid.*

If a matroid M is isomorphic to a vector matroid of a matrix D over a field F, then M is said to be **representable over** F; D is called **representation** for M over F. A matroid is said to be **representable**, if it is representable over some field F. A matroid that is representable over two element field $\mathbb{Z}_2$ is called **binary**.

If X and Y are sets then their **symmetric difference**, $X \triangle Y$, is the set $(X \cup Y) \setminus (X \cap Y)$. One can easily checks that the operation of symmetric difference is both commutative and associative.

The **cycle** of a binary matroid is a symmetric difference of any set of circuits. We abbreviate notions, and the collection of cycles of a binary matroid M denote by $\mathcal{C}(M)$. Obviously, $\mathcal{C}(M)$ is closed under taking symmetric difference.

The **circuit (cycle) space** and the **cocircuit (cocycle) space** of a binary matroid M are the vector spaces over $\mathbb{Z}_2$ that are generated by the incidence vectors of the cycles and cocycles, respectively, of M.

Lemma 2.4. *Let M be a binary matroid. Let A be a representation of M. Then*

(i) *if C is a cycle of M and C^* is a cocycle of M then $|C \cap C^*|$ is even;*

(ii) *a vector x belongs to the circuit space of M if and only if $Ax = 0$;*

(iii) *a vector x belongs to the cocircuit space of M if and only if x is a linear combination of the rows of the matrix A;*

(iv) *a cycle C is a circuit if and only if C is minimal (that is, it does not contain any other cycle) and nonempty.*

In this text we will work only with binary matroids.

3 Ear Decomposition of Connected Matroids

In this section we introduce the ear decomposition. The decomposition is a generalization of the ear decomposition of 2-vertex connected graph. A similar decomposition that use both operations of contraction and deletion is defined in Oxley [4]. The decomposition that use only deletions is already known but we give our definition and proofs.

A matroid M is **connected** if and only if for every pair of distinct elements of $E(M)$ there is a circuit containing both.

Let V_1 and V_2 be vector spaces then the set of all sums $v_1 + v_2$ of vectors $v_1 \in V_1$ and $v_2 \in V_2$ is called **sum** of vector spaces V_1 and V_2, denoted by $V_1 + V_2$. We abbreviate this notion, and if C_1 and C_2 are collections of sets, then the set of all symmetric differences $c_1 \triangle c_2$ of sets $c_1 \in C_1$ and $c_2 \in C_2$ is called **sum** of collections of sets, denoted by $C_1 + C_2$.

A matroid M is an **ear extension** of a matroid N, if the following conditions are satisfied. N is obtained from M by deleting a nonempty subset T of $E(M)$. There is a circuit C of M such that T is a proper subset of C. The set T is a coparallel class of M. There is no matroid M' such that M' is an ear extension of N and M is an ear extension of M'. The circuit C and the set T are called **ear circuit** and **ear**, respectively, of M.

Lemma 3.1. *Let M be an ear extension of N. Let N be a connected matroid. Then M is a connected matroid.*

Proof. From the definition of the ear extension; $N = M \setminus T$. Let C be a circuit of M containing T. Let u, v be elements of $E(M)$.

If u, v belong to $E(N)$ then there is, by assumptions that N is connected, a circuit containing both.

If u, v belong to T then there is also a circuit containing both, as T is a subset of some circuit.

So, suppose that $u \in E(N)$ and $v \in T = E(M) \setminus E(N)$. Let w be an element of $E(N)$ such that $w \in C$. As N is connected, there is a circuit C' containing w, u. Then $C \triangle C'$ is a circuit, as $C \cap C' \neq \emptyset$. And this circuit contains elements u, v.

Therefore M is connected. □

Proposition 3.1. *Let M be a connected matroid and suppose that $|E(M)| \geq 2$. Then there is a sequence $M_0, \ldots, M_n$ of connected matroids such that M_0 contains just one circuit. The matroid M_i is an ear extension of M_{i-1}, for $i = 1, \ldots, n$. Moreover $M = M_n$.*

Proof. As the matroid M has two distinct elements and is connected, then it contains at least one circuit C'. Set $M_0 := M \mid C'$. Let $M_0, \ldots, M_n$ be a maximal desired sequence. For a contradiction suppose that $M_n \neq M$.

As M is connected, there is a circuit containing both elements of $E(M) \setminus E(M_n)$ and $E(M_n)$. The collection of these circuits denote by $\mathcal{D}$. Let C be a circuit from $\mathcal{D}$ such that for every $C' \in \mathcal{D}$; $(C' \setminus E(M_n)) \neq (C \setminus E(M_n))$ holds $(C' \setminus E(M_n)) \nsubseteq (C \setminus E(M_n))$. Let T be the set $C \setminus E(M_n)$ and let M_{n+1} be the matroid on the ground set $E(M_n) \cup T$ and with the collection of cycles $\mathcal{C}(M_n) + \{0, C\}$. Obviously; $M_{n+1} \setminus T = M_n$, and the set T is a proper subset of C.

For a circuit $C' \in \mathcal{C}(M) \setminus \mathcal{C}(M_n)$; $C' \subseteq E(M_{n+1})$ holds $(C \triangle C') \subseteq E(M_n)$. Thus $\mathcal{C}(M_{n+1}) = \mathcal{C}(M_n) + \{0, C\} = \{C' \subseteq E(M_n) : C' \in \mathcal{C}(M_n)\} \cup \{C' \triangle C \subseteq E(M_n) : C' \in \mathcal{C}(M_{n+1})\} = \{C' \subseteq E(M_{n+1}) : C' \in \mathcal{C}(M)\} = \mathcal{C}(M \mid E(M_{n+1}))$. Therefore $M \mid E(M_{n+1}) = M_{n+1}$, and M_{n+1} is a minor of M.

Let D be a circuit of M_{n+1} containing an element of T. Obviously, D is a symmetric difference of C and some circuit of M_n. As the set T and every circuit of M_n are disjoint, then D contains the entire set T. Therefore all elements of T are pairwise coparallel. Let u be an element of T. Let v, w be elements of $E(M_n)$. Let D be a circuit containing u, v. Let D' be a circuit of M_n containing v, w. Then the circuit $D \triangle D'$ does not contain the element v and contains the element u. So, the set T is a coparallel class of M_{n+1}. As there is no proper subset T' of T such that $M_{n+1} \setminus T'$ is an ear extension of M_n, then M_{n+1} is an ear extension of M_n.

This is a contradiction with the maximality of the sequence. Therefore $M_n = M$. $\square$

The sequence in the lemma above is called **ear decomposition** of a matroid M. For a connected matroid M with at least two elements. We define the **ear basis** in this way. Let C_0 denote a circuit of M_0. Let C_i denote the ear circuit of M_i. Then the set $\{\chi^{C_0}, \chi^{C_1}, \ldots, \chi^{C_n}\}$ is called ear basis of $\mathcal{C}(M)$.

Proposition 3.2. *Let M be a connected binary matroid such that $|E(M)| \geq 2$. Let β be an ear basis of M. Then the set β is a basis of the circuit space $\mathcal{C}(M)$.*

Proof. We apply induction on the dimension of the circuit space. If dim $\mathcal{C}(M) = 1$ then the matroid M has one circuit C. Hence, the ear decomposition of M is M_0. Therefore $\beta = \{\chi^C\}$. This is a basis of $\mathcal{C}(M)$.

Suppose that dim $\mathcal{C}(M) > 1$. The matroid M has an ear decomposition $M_0, \ldots, M_{n-1}, M_n$. By the induction assumptions, the ear basis β' of the matroid M_{n-1} is a basis of the circuit space of M_{n-1}. Let C' be a cycle of $\mathcal{C}(M) \setminus \mathcal{C}(M_{n-1})$. Let C_n be the ear circuit of M_n. Let T be the ear of M_n. As T is a coparallel class of M, then $\chi_i^{C'} = \chi_i^{C_n}$ for $i \in T$. Thus, the vector $\chi^{C'} + \chi^{C_n}$ belongs to $\mathcal{C}(M_{n-1})$. Therefore, the vector $\chi^{C'}$ is a linear combination of χ^{C_n} and the vectors of the set β'. Hence, the set $\beta = \{\chi^{C_0}, \ldots, \chi^{C_n}\}$ is a basis of the circuit space $\mathcal{C}(M)$. $\square$

4 Lattices and Codes

4.1 Codes

An **alphabet** is a set of **symbols** $\Sigma = \{s_0, \dots, s_m\}$. Let Σ^n be a set of n-tuples of symbols. Elements of Σ^n are called **words**. A **code** is a subset W of Σ^n. Elements of W are called **codewords**.

If W forms a vector space over a field $\mathbb{F}$, then W is called **linear code**. A **binary linear code** is a linear code over two elements field. The **weight** of a codeword x is the number of nonzero coordinates, denoted by $w(x)$.

4.2 Lattices

4.2.1 Definitions

A **lattice** in $\mathbb{R}^d$ is the set

$$\mathbb{Z}(X) := \{\sum_{x \in X} \lambda_x x | \lambda_x \in \mathbb{Z}\, \forall x \in X\} \tag{4.1}$$

where X is a set of real vectors of $\mathbb{R}^d$. If $\mathbb{Z}(X)$ is a full dimensional lattice, then the **dual lattice** of $\mathbb{Z}(X)$ is the set

$$(\mathbb{Z}(X))^* := \{x \in \mathbb{R}^d | xy \in \mathbb{Z}\, \forall y \in \mathbb{Z}(X)\}. \tag{4.2}$$

The following well known relation taken from Fleiner *et al.* [1] is between a lattice and its dual lattice.

Proposition 4.1. *Let $\mathbb{Z}(X)$ be a full dimensional lattice in $\mathbb{R}^d$. Let $(\mathbb{Z}(X))^*$ be the dual lattice. Let N be an integer. Then*

$$N\mathbb{Z}^d \subseteq \mathbb{Z}(X) \Leftrightarrow (\mathbb{Z}(X))^* \subseteq \frac{1}{N}\mathbb{Z}^d \tag{4.3}$$

.

Proof. At first, we observe that

$$\{x \in \mathbb{R}^d | xy \in \mathbb{Z}\, \forall y \in N\mathbb{Z}^d\} = \frac{1}{N}\mathbb{Z}^d. \tag{4.4}$$

" $\Rightarrow$ "

As $N\mathbb{Z}^d \subseteq \mathbb{Z}(X)$ then

$$(\mathbb{Z}(X))^* = \{x | xy \in \mathbb{Z}\, \forall y \in \mathbb{Z}(X)\} \subseteq \{x | xy \in \mathbb{Z}\, \forall y \in N\mathbb{Z}^d\} = \frac{1}{N}\mathbb{Z}^d. \tag{4.5}$$

" $\Leftarrow$ "

We suppose that $(\mathbb{Z}(X))^* \subseteq \frac{1}{N}\mathbb{Z}^d$. Thus

$$(\mathbb{Z}(X))^* = \{x|xy \in \mathbb{Z}\,\forall y \in \mathbb{Z}(X)\} \subseteq \{x|xy \in \mathbb{Z}\,\forall y \in N\mathbb{Z}^d\} = \frac{1}{N}\mathbb{Z}^d. \quad (4.6)$$

Therefore $N\mathbb{Z}^d \subseteq \mathbb{Z}(X)$.

□

4.2.2 Bases of Lattices

In this section we define the basis of a lattice and show that every rational lattice admits a basis. The following definitions and proofs in this section are taken from Schrijver [5].

A **basis** of lattice $\mathbb{Z}(M)$ is a linear independent subset B of $\mathbb{R}^d$ such that $\mathbb{Z}(B) = \mathbb{Z}(M)$. A matrix of full row rank is said to be in **Hermite normal form** if it has the form $\begin{bmatrix} B & 0 \end{bmatrix}$, where B is a nonsingular, lower triangular, nonnegative matrix, in which each row has a unique maximum entry, which is located on the main diagonal of B.

The following operations on a matrix are called **elementary (unimodular) column operations**:

- **(i)** exchanging two columns;
- **(ii)** multiplying a column by -1;
- **(iii)** adding an integral multiple of one column to another column.

Theorem 4.1. *Each rational matrix of full row rank can be brought into Hermite normal form by a series of elementary column operations.*

Proof. Let A be a rational matrix of full row rank. Without loss of generality, A is integral. Suppose we have transformed A, by elementary column operations, to the form $\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$ where B is lower triangular and with positive diagonal. Now with elementary column operations we can modify D so that its first row $(\delta_{11}, \dots, \delta_{1k})$ is nonnegative, and so that the sum $\delta_{11} + \cdots + \delta_{1k}$ is as small as possible. We may assume that $\delta_{11} \geq \delta_{12} \geq \cdots \geq \delta_{1k}$. Then $\delta_{11}\, 0$, as A has full row rank. Moreover, if $\delta_{12}\, 0$, by subtracting the second column of D from the first column of D, the first row will have smaller sum, contradicting our assumption. Hence $\delta_{12} = \cdots = \delta_{1k} = 0$, and we have obtained a larger lower triangular matrix.

By repeating this procedure, the matrix A finally will be transformed into $\begin{bmatrix} B & 0 \end{bmatrix}$ with $B = (\beta_{ij}$ lower triangular with positive diagonal. Next do the following:

for $i = 2, \dots, n$ (:= order of B), do the following: for $j = 1, \dots, i-1$, add an integer multiple of the ith column of B to the jth column of B so that (i, j)th entry of B will be nonnegative and less than β_{ii}.

It is easy to see that after these elementary column operations the matrix is in Hermite normal form. □

Corollary 4.1. *Every lattice generated by rationals vectors $a_1, \dots, a_m$ has a basis.*

Proof. We may assume that $a_1, \dots, a_m$ span all space. (Otherwise we could apply a linear transformation to a lower dimensional space.) Let A be the matrix with columns $a_1, \dots, a_m$ (so A has full row rank). Let $\begin{bmatrix} B & 0 \end{bmatrix}$ be the Hermite normal form of A. Then the columns of B are linearly independent vectors generating the same lattice as $a_1, \dots, a_m$. □

4.3 Notations

We fix some notations.

- The **cycle basis** is a basis of cycle space over $GF(2)$ of some binary matroid.
- The **ear basis** is a basis of cycle space over $GF(2)$ of some binary matroid obtained from an ear decomposition.
- The **lattice basis** is a basis of lattice.
- The **cycle lattice basis** is a basis of lattice generated by cycle space of some binary matroid consisting only of elements of cycle space (cycles).

5 Lattices of Binary Matroids

5.1 Definitions and the Introduction to the Problem

Let M be a binary matroid, then the **cycle lattice** of M is the set

$$\mathbb{Z}(M) := \{ \sum_{C \in \mathcal{C}(M)} \lambda_C \chi^C \mid \lambda_C \in \mathbb{Z} \, \forall C \in \mathcal{C}(M)\}. \tag{5.1}$$

We study the following hypothesis taken from Fleiner *et al.* [1].

Hypothesis 5.1. *Let M be a binary code. Let $\mathbb{Z}(M)$ be a lattice generated by the code M. Then $\mathbb{Z}(M)$ has a basis consisting only of codewords.*

No code is known for which the hypothesis fails. The best known results are the following two theorems taken from Fleiner *et al.* [1].

Theorem 5.1. *Let M be a binary matroid with no F_7^* minor. Then the lattice $\mathbb{Z}(M)$ has a basis consisting only of circuits.*

The matroids with no F_7^* minor contain the class of regular matroids, which extends the graphic and cographic matroids.

Theorem 5.2. *Let M be a binary matroid on E with no F_7^*. Let M' be an one-element extension of M. Then every cycle lattice basis B_M of $\mathbb{Z}(M)$ can be extended to a cycle lattice basis B of $\mathbb{Z}(M')$.*

These matroids contain the class of graft matroids (that is, one-element extensions of graphic matroids).

Our goal is to consider the hypothesis for a geometrically defined class of binary codes. Especially the class of codes generated by the cycles of a triangular configuration.

We will work only with connected matroids. In disconnected matroid does not exists a cycle that contains two elements of two different components of connectivity. Therefore the lattice generated by a component of connectivity does not contains any cycle of the others components of connectivity.

5.2 Basic Facts

Fleiner *et al.* [1] showed propositions 5.1 and 5.2.

Proposition 5.1. *Let M be a binary matroid, then the following holds obviously for all $x \in \mathbb{Z}(M)$.*

(i) $\sum_{e \in D} x_e$ *is even for all cocircuits $D \in \mathcal{C}^*(M)$,*

(ii) $x_f = x_g$ *if* f *and* g *are coparallel in* M*,*

(iii) $x_e = 0$ *if* e *is coloop of* M*.*

A matroid M has the **lattice of circuits property** if the conditions (i)-(iii) characterize the lattice $\mathbb{Z}(M)$.

Hypothesis 5.1 is open even for matroids with the lattice of circuits property.

Let M be a binary matroid, then the collection of all parallel classes of M is denoted by $P(M)$. From the lemma above follows that the dimension of $\mathbb{Z}(M)$ is equal to the number of coparallel classes of M; dim $\mathbb{Z}(M) = |P(M^*)|$.

Proposition 5.2. *A matroid* M *has the lattice of circuits property if and only if* $2\chi^P$ *belongs to* $\mathbb{Z}(M)$ *for every coparallel class* P *of* M*.*

If we want to prove that a matroid does not have the lattice of circuits property, it suffices to find a vector of $(\mathbb{Z}(M))^*$ not in $\frac{1}{2}\mathbb{Z}$.

Let M be a cosimple binary matroid. Let T be a maximal independent subset of $E(M)$. Let C_e $(e \in T')$ be the corresponding fundamental circuits. Let W be the matrix whose rows are the incidence vectors of the sets $C_e \cap C_f$ (for $e, f \in T'$). Lovász and Seress [3] have shown that

Proposition 5.3. *Let* M *be a cosimple binary matroid.* M *has the lattice of circuits property if and only if the matrix* W *has full column rank over* $GF(2)$*.*

The following proposition is taken from Fleiner *et al.* [1].

Proposition 5.4. *Let* M *be a cosimple binary matroid. If we could find a set* I *of pairs* (e, f) $(e \neq f \in T')$ *for which the submatrix* W_I *with rows* C_e $(e \in T')$ *and* $C_e \cap C_f$ $((e, f) \in I)$ *has its determinant equal to* 1*, then the set* $\{C_e(e \in T'), C_e \triangle C_f((e, f) \in I)\}$ *would be a cycle basis of* $\mathbb{Z}(M)\}$*.*

For $r \geq 2$, the **projective space** $\mathcal{P}_r$ is the binary matroid represented by the $r \times (2^r - 1)$ matrix whose columns are all nonzero $0, 1$-vectors of length r. Lovász and Seress [3] have shown that

Theorem 5.3. *Let* M *be a cosimple binary matroid with no* $\mathcal{P}_{r+1}^*$ *minor, then* $2^{r-1}\mathbb{Z}^E \subseteq \mathbb{Z}(M)$*.*

Fleiner *et al.* [1] showed that $\mathbb{Z}(\mathcal{P}_r^*)$ has obviously a cycle basis, since the nonempty cycles of $\mathcal{P}_r^*$ are linearly independent over $\mathbb{R}$. Fleiner *et al.* [1] showed that $\mathbb{Z}(\mathcal{P}_r)$ has a basis consisting only of cycles of $\mathcal{P}_r$.

5.3 New Results

In this section we give some new results. At first we prove a technical lemma about coparallel classes of an ear extension of a matroid.

Lemma 5.1. *Let N be a connected binary matroid with the collection of coparallel classes $P(N^*)$. Let M be an ear extension of N. Let C be the ear circuit. Let T be the ear of the extension. Then $P(M^*) = \{T\} \cup \{P \cap C, P \setminus C \mid P \in P(N^*)\} \setminus \{\emptyset\}$.*

Proof. From the definition of the ear extension, we know that the set T is a coparallel class of M. Let P' be an arbitrary element of $P(M^*) \setminus \{T\}$. The set P' is equal to $P \cap C$ or $P \setminus C$ where $P \in P(N^*)$.

Suppose that $|P'| \geq 2$. Let u, v be two distinct elements of P'. Let C' be a circuit of $\mathcal{C}(N)$. As P' is a subset of some coparallel class of N, $|C' \cap \{u, v\}|$ is even. Let C' be a circuit of $\mathcal{C}(M) \setminus \mathcal{C}(N)$. Then the circuit C' is equal to $C \triangle D$ where $D \in \mathcal{C}(N)$. If $P' = P \cap C$, $|C \cap \{u, v\}|$ is equal to 2. If $P' = P \setminus C$, $|C \cap \{u, v\}|$ is equal to 0. As $|D \cap \{u, v\}|$ is even, $|(C \triangle D) \cap \{u, v\}|$ is even. Hence, every two elements of P' are coparallel.

Let u be an element of the set P'. Let v be an element of $E(M) \setminus (P' \cup T)$. If $v \in P$ then $|C \cap \{u, v\}| = 1$, where C is the ear circuit. If $v \notin P$ then u, v do not belong to the same coparallel class of N. Hence, there is a circuit D of $\mathcal{C}(N)$ such that $|D \cap \{u, v\}| = 1$. Therefore the set P' is a coparallel class of M. □

Let N be a connected binary matroid. Let M be an ear extension of N. Let C be an ear circuit of the extension. Then denote by $I(N, M)$ the set $\{P' \in P(M^*) | P' \subset C, P' \subsetneq P \in P(N^*)\}$. The elements of the set $I(N, M)$ are the new coparallel classes of M contained in the ear circuit C. The cardinality of $I(N, M)$ is equal to $|P(M^*)| - |P(N^*)| - 1$.

The following basis construction is a generalization of the basis construction of the lattice of a graph introduced in Loebl and Matamala [2].

Theorem 5.4. *Let N be a connected binary matroid with the lattice of circuits property and let M be an ear extension of N. Let m denote the cardinality of $I(N, M) = \{P'_1, \ldots, P'_m\}$. If $\mathbb{Z}(N)$ has a cycle basis $B = \{\beta_1, \ldots, \beta_d\}$ such that $\beta_i \supseteq P'_i$ and $\beta_i \cap P'_{i+1} = \emptyset, \ldots, \beta_i \cap P'_m = \emptyset$ for $i = 1, \ldots, m$. Then the lattice of the matroid M has circuits property and has a cycle lattice basis. This basis contains the basis B.*

Proof. We show that the set

$$B' := B \cup \{\beta'_i \mid \beta'_i = \beta_i \triangle C, i = 1, \ldots, m\} \cup \{C\} \tag{5.2}$$

of cycles of M is a basis of $\mathbb{Z}(M)$ and generates the vectors $2\chi^{P'}$ for all coparallel classes P' of $P(M^*)$.

Let P' be a coparallel class of M. If $P' \in P(N^*)$, then the vector $2\chi^{P'}$ is generated by the set B since N has the lattice of circuits property.

If $P' = P'_1$, then

$$2\chi^{P'_1} = \chi^{\beta_1} + \chi^{C} - \chi^{\beta'_1} + \sum_{P \in P(N^*)} \lambda_P 2\chi^{P} \tag{5.3}$$

since $\beta'_1 = \beta_1 \triangle C$ and $P'_1 \subseteq \beta_1 \cap C$ and $\beta_1 \cap P'_2 = \emptyset, \ldots, \beta_1 \cap P'_m = \emptyset$.

For $P'_i \in I(N, M) \setminus \{P'_1\}$ we have

$$2\chi^{P'_i} = \chi^{\beta_i} + \chi^{C} - \chi^{\beta'_i} + \sum_{j \in \{1,\ldots,i-1\}} \lambda_{P'_j} 2\chi^{P'_j} + \sum_{P \in P(N^*)} \lambda_P 2\chi^{P}. \tag{5.4}$$

If P' is the ear T of the extension, then

$$2\chi^{T} = 2\chi^{C} + \sum_{j \in \{1,\ldots,m\}} \lambda_{P'_j} 2\chi^{P'_j} + \sum_{P \in P(N^*)} \lambda_P 2\chi^{P}. \tag{5.5}$$

If $P' \in P(M^*) \setminus (I(N, M) \cup T)$, then $P = P_1 \setminus P_2$ where $P_1 \in P(N^*)$ and $P_2 \in I(N, M)$. Therefore

$$2\chi^{P} = 2\chi^{P_1} - 2\chi^{P_2}. \tag{5.6}$$

As $P(M^*) \subseteq P(N^*) \cup I(N, M) \cup \{T\}$, the matroid M has the lattice of circuits property.

Finally, we prove that B' generates all cycles of M over $\mathbb{R}$. Let C' be a cycle of M. If $C' \in \mathcal{C}(N)$ then C' is generated by the set B.

Suppose that $C' \notin C(N)$. From the definition of the ear extension, we can express C' as $C' = C \triangle D$ where $D \in \mathcal{C}(N)$. Thus

$$\chi^{C'} = \chi^{D} + \chi^{C} + \sum_{P'_i \in I(N,M)\}} \lambda_{P'_i} 2\chi^{P'_i}. \tag{5.7}$$

The set B' has the right cardinality, because $|B'| = |B| + |I(N, M)| + 1 = |P(N^*)| + |P(M^*)| - |P(N^*)| - 1 + 1 = P(M^*) = \dim \mathbb{Z}(M)$.

Therefore, the set B' is a cycle lattice basis of $\mathbb{Z}(M)$. $\square$

In the following paragraphs we discuss whether Theorem 5.4 may provide a proof of Hypothesis 5.1 restricted to the matroids with the lattice of circuits property.

Let N be the ear extension of the graphic matroid M in figure 5.1 with the following cycle space $\mathcal{C}(N) + \{e_1, e_2, e_3, t\}$. Then $I(N, M) = \{e_1, e_2, e_3\}$.

A basis vector containing e_1 have to cover e_2 or e_3. Therefore M does not have a basis required by Theorem 5.4. Thus, there exists an ear extension of a matroid with the lattice of circuits property which does not have the basis required by Theorem 5.4. Therefore Theorem 5.4 does not provide a proof of the hypothesis.

Moreover, the matroid M does not have the lattice of circuits property, by Proposition 5.5.

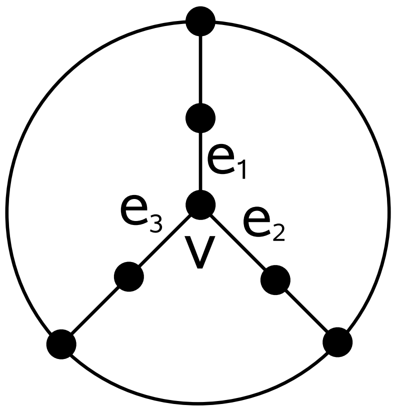

Figure 5.1: A matroid with no good basis.

Next, we demonstrate that an ear decomposition of a matroid with the lattice of circuits property may contain a matroid that does not have the lattice of circuits property. For instance the matroid S_8 taken from Fleiner *et al.* [1]. The matroid S_8 is an ear extension of a graph G in figure 5.2. S_8 has the following cycle space $\mathcal{C}(S_8) = \mathcal{C}(G) + \{e_1, e_2, e_3, e_4, t\}$. S_8 has the lattice of circuit property. As S_8 contains as a minor the dual fano matroid F_7^*, there exists an ear decomposition which contains the dual fano matroid.

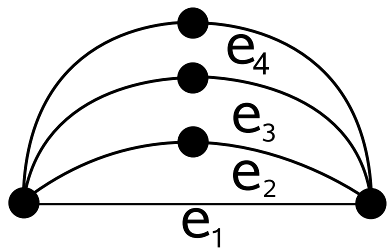

Figure 5.2: A graph G of graft S_8.

The following proposition is taken from Fleiner *et al.* [1].

Proposition 5.5. *Let M be a binary matroid. Let N be an ear extension of M. Let $I(N, M)$ be equal to $\{P_1, \dots, P_k\}$. Let P_i' be an element of $P^*(M)$*

such that $P'_i \supset P_i$. If there exists elements $e_1 \in P_1, \dots, e_k \in P_k, f_1 \in P'_1 \setminus P_1, \dots, f_k \in P'_k \setminus P_k$ ($k \geq 3$) such that the set $\{e_1, \dots, e_k\}$ is a cocircuit of M, then N does not have the lattice of circuits property.

Proof. We can suppose without loss of generality that the matroids M and N are cosimple. We show that N does not have the lattice of circuits property by constructing a vector $x \in \frac{1}{4}\mathbb{Z}^{E(N)} \setminus \frac{1}{2}\mathbb{Z}^{E(N)}$ belonging to the dual lattice $(\mathbb{Z}(N))^*$. For this, set $x(e_i) = x(f_i) := \frac{1}{4}(i = 1, \dots, k)$, $x(t) := 0, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}$ if k is congruent to $0, 1, 2, 3$, respectively, and $x(e) := 0$ for all remaining elements $e \in E(N)$. □

Corollary 5.1. *Let N be a connected graphic matroid with the lattice of circuits property. Let M be a graphic matroid that is an ear extension of N. If $\mathbb{Z}(N)$ has a cycle basis B, then $\mathbb{Z}(M)$ has a cycle basis B'. The basis B' contains the basis B. Moreover M has the lattice of circuits property.*

Proof. Let G, G' be graphs such that $M(G) \cong N$ and $M(G') \cong M$. Let C be an ear circuit of the ear extension M and T be the ear. Then T is a path. Let t_1, t_2 be end vertices of the path. Let $P_1, P_2 \in P(M^*)$ be coparallel classes such that $t_1 \in e \in P_1$ and $t_2 \in f \in P_2$ and $C \cap P_1 \neq \emptyset$ and $C \cap P_2 \neq \emptyset$.

If both end vertices have degree greater than 2 in G, then the set $I(N, M)$ is empty.

If $P_1 = P_2$ and at least one end vertex has degree 2 in G then $I(N, M) = \{P_1 \cap C\}$.

Suppose that $P_1 \neq P_2$. If one end vertex t_i of T has degree 2 in G, then $I(N, M) = \{C \cap P_i\}$.

If both end vertices of T have degree 2 in G, then $I(N, M) = \{C \cap P_1, C \cap P_2\}$. In that case there is a circuit D of G such that $D \cap P_1 \neq \emptyset$ and $D \cap P_2 = \emptyset$, since P_1 and P_2 are distinct coparallel classes.

Hence, we can use Theorem 5.4 and prove the corollary. □

By using the corollary above, we give a new proof of the theorem that the lattice of a graphic matroid has a basis consisting only of cycles of the matroid.

Theorem 5.5. *Let M be a connected graphic matroid. Then the lattice $\mathbb{Z}(M)$ has a basis consisting only of cycles of M.*

Proof. Let $M_0, \dots, M_n$ be an ear decomposition of M. Obviously M_0 has the lattice of circuits property and $\mathbb{Z}(M_0)$ has a basis consisting only of cycles. By using corollary 5.1 repeatedly to the decomposition, we obtain that $\mathbb{Z}(M) = \mathbb{Z}(M_n)$ has a basis consisting only of cycles. □

The next theorem about basis extension is a reformulation of the theorem taken from Fleiner *et al.* [1].

Theorem 5.6. *Let N be a binary matroid. Let M be an ear extension of N. If $\mathbb{Z}(N)$ has a cycle lattice basis $B = \{\beta_1, \dots, \beta_d\}$ and $|P(M^*)| = |I(N, M)|$, then $\mathbb{Z}(M)$ has the following cycle lattice basis $B' = B \cup \{D \triangle C | D \in B\} \cup \{C\}$.*

Proof. As B' has the right cardinality, it suffice to verify that it generates all cycles of M. For this, let E be a cycle of N; then

$$\chi^E = \sum_{\beta \in B} \lambda_\beta \chi^\beta, \tag{5.8}$$

where the λ'_Bs are integers. Therefore,

$$\chi^{(E \triangle C)} = \sum_{\beta \in B} \lambda_\beta \chi^{(\beta \triangle C)} + \Big(1 - \sum_{\beta \in B} \lambda_\beta\Big) \chi^C \tag{5.9}$$

belongs to $\mathbb{Z}(B')$. □

Remark 5.1. The comparison of Theorems 5.4 and 5.6. Let M be a binary matroid. Let N be an ear extension of M. The theorems construct a lattice basis of the matroid N by extending a basis of the matroid M.

The main difference is that Theorem 5.4 requires on the lattice of the matroid N circuits property and the "good" cycle lattice basis. Whereas Theorem's 5.6 assumptions are that the number of coparallel classes of M is equal to $2|P(N^*)| + 1$.

The matroid in Figure 5.1 and the ear extension defined above do not satisfy the assumptions of theorem 5.4 and satisfy the assumptions of Theorem 5.6. The matroid with the lattice of circuits property in Figure 5.2 and the matroid S_8 do not satisfy assumptions of both Theorems 5.4 and 5.6.

6 Triangular Configurations

6.1 Definitions

A **triangular configuration** is a triple $\Delta = (V, E, T)$ consisting of a finite set V of points, a finite set E of edges satisfying $E \subseteq \binom{V}{2}$, and a finite set T of triangles satisfying $T \subseteq \binom{V}{3}$ and for every $t \in T$ holds $\binom{t}{2} \subseteq E$.

A **geometric representation** of a triangular configuration (V, E, T) in $\mathbb{R}^d$ is an injective mapping $f : V \mapsto \mathbb{R}^d$ satisfying

1. for every $t \in T$ holds the set $f(t) := \{f(x)|x \in t\}$ is affinely independent,
2. for every $e, e' \in E$ such that $e \neq e'$ holds $conv(f(e)) \cap conv(f(e'))$ is equal to $\emptyset$ or $f(v)$ for some vertex $v \in V$,
3. for every $t, t' \in T$ such that $t \neq t'$ holds $conv(f(t)) \cap conv(f(t'))$ is equal to $\emptyset$; or $f(v)$ for some vertex $v \in V$; or $conv(f(e))$ for some edge $e \in E$.

Let Δ be a triangular configuration, we denote by

- $V(\Delta)$ the set of vertices of a triangulation;
- $E(\Delta)$ the set of edges of a triangulation;
- $T(\Delta)$ the set of triangles of a triangulation.

Let v_1, v_2, v_3 be vertices of Δ. Let e be an edge of Δ. Let t be a triangle of Δ. The edge e can be written as $\{v_1, v_2\}$ or v_1v_2 where v_1 and v_2 are the vertices of the edge. The triangle t can be written as $\{v_1, v_2, v_3\}$ or $v_1v_2v_3$, or efg or $\{e, f, g\}$ where e, f, g are the edges of the triangle.

If we admit that T is a multiset in the definition of the triangular configuration, we say that the triple Δ is a **multitriangular configuration**.

A triangular configuration S is a **subconfiguration** of a triangular configuration R, if $V(S)$; $E(S)$; and $T(S)$ are subsets of $V(R)$; $V(R)$; and $T(R)$, respectively. We say that R contains S.

Let Δ be a triangular configuration. Then the pair $(V(\Delta), E(\Delta))$ forms a graph. This graph is called **skeleton** and is denoted by $G(\Delta)$.

The **edge degree** $d_E(e)$ of an edge e of Δ is the number of triangles containing the edge.

$$d_E(e) := |\{t : t \in T(\Delta), e \subset t\}| \tag{6.1}$$

We define the **incidence matrix** of a triangular configuration $A = (A_{et})$ in this way. The rows are indexed by edges and the columns by triangles. We set

$$a_{et} := \begin{cases} 1 & \text{if the edge } e \text{ belongs to the triangle } t;\ e \subset t, \\ 0 & \text{otherwise.} \end{cases}$$

A triangular configuration is a **cycle** if every edge has an even degree.

If R and S are triangular configurations then their **symmetric difference**, denoted by $R \triangle S$, is the triangular configuration $(V(R) \cup V(S), E(R) \cup E(S), T(R) \triangle T(S))$.

The **Euler characteristic** χ of a triangular configuration Δ is defined according to the formula

$$\chi = |V(\Delta)| - |E(\Delta)| + |T(\Delta)|. \tag{6.2}$$

6.2 Triangular Matroid

In this section we prove that the collection of the cycles of a triangular configuration forms a cycle space of some binary matroid.

Lemma 6.1. *Let R and S be cycles. Then $R \triangle S$ is a cycle.*

Proof. We show that every edge of $R \triangle S$ has an even degree. Let e be an arbitrary edge. Let T_R and T_S be subsets of $T(R)$ and $T(R)$, respectively, containing the edge e. Then the degree of the edge e is equal to $|T_R| + |T_S| - 2|T_R \cap T_S|$. Thus, the degree is even. Therefore $R \triangle S$ is a cycle. □

Denote by $\mathcal{C}(\Delta)$ the collection of the cycles contained in a triangular configuration Δ. From the lemma above we know that $\mathcal{C}(\Delta)$ is closed under taking symmetric difference.

Lemma 6.2. *Let Δ be a triangular configuration. Let A be the incidence matrix of Δ. Let C be a subconfiguration of Δ. Then $A\chi^{T(C)} = 0$ if and only if C is a cycle.*

Proof. C is a cycle. $\Leftrightarrow$ For every edge $e_i \in E(\Delta)$ indexing row a_{i*} holds $|\{t | e_i \subset t, t \in T(C)\}|$ is even. $\Leftrightarrow$ $A\chi^{T(C)} = 0$. □

From the lemma above follows that the incidence vectors of the cycles contained in a triangular configuration forms a circuit space of a binary matroid. This matroid is called **triangular matroid**, and is denoted by $M(\Delta)$; where Δ is the configuration. The incidence matrix is a representation of the matroid.

Let Δ be a triangular configuration. Let T' be a subset of $T(\Delta)$. Then the **deletion** of T' from Δ is the triple $(V(\Delta), E(\Delta), T(\Delta) \setminus T')$.

Proposition 6.1. *Let Δ be a triangular configuration. Let T' be a subset of $T(\Delta)$. Then $M(\Delta \setminus T') = M(\Delta) \setminus T'$.*

Proof. Follows directly from the definition. □

Proposition 6.2. *Let C be a circuit. Then $|T(C)| \geq 4$.*

Proof. As $T(C)$ is nonempty, there is a triangle t. Since every edge has an even degree, every edge of t is incident with at least one triangle different from t. These triangles are pairwise different, since three point determine an unique triangle. Hence, we have found four triangles. □

Proposition 6.3. *Every triangular matroid is simple.*

Proof. Follows directly from the definition. □

Remark 6.1. Triangular matroid with a geometric representation is a simplicial complex.

Corollary 6.1. *Let C be a circuit then $|E(C)| \geq 6$ and $|V(C)| \geq 4$.*

Proof. Follows directly from Proposition 6.2. □

Proposition 6.4. *Let Δ be a triangular configuration. Let V^* be the set of isolated vertices (vertices not contained in any edge) of Δ. Let E^* be the set of edges of Δ with degree 0. Then $M(\Delta) \cong M((V(\Delta) \setminus V^*, E(\Delta) \setminus E^*, T(\Delta)))$.*

Proof. We consider the incidence matrices of Δ and $(V(\Delta) \setminus V^*, E(\Delta) \setminus E^*, T(\Delta))$. As isolated vertices are not contained in any edge, removing it from triangular configurations does not affect incidence matrices. An edge with degree 0 corresponds with a zero row vector. If we remove such edge, then we delete a zero row vector. Such modification of a representation matrix does not affect a matroid represented by this matrix. Thus $M(\Delta) \cong M((V \setminus V^*, E \setminus E^*, T))$. □

Let Δ be a triangular configuration. Let v be a vertex of Δ. We define recursively the set Δ_v.

$$\Delta_v := \{t\}; t \in T(\Delta), v \subset t \tag{6.3}$$

$$\Delta_v := \{t \in T(\Delta) | v \subset t, \exists t' \in \Delta_v : t' \cap t \in E(\Delta)\} \tag{6.4}$$

For some vertex v may exists more distinct sets $\Delta_{v1}, \Delta_{v2}, \ldots, \Delta_{vn}$. We denote the collection of all sets Δ_v by Δ^v.

The set Δ^v of the vertex v depicted in Figure 6.1 contains three sets $\Delta_{v1}, \Delta_{v2}, \Delta_{v2}$. The set $\Delta^{v'}$ of the vertex v' contains only one set $\Delta_{v'}$. A

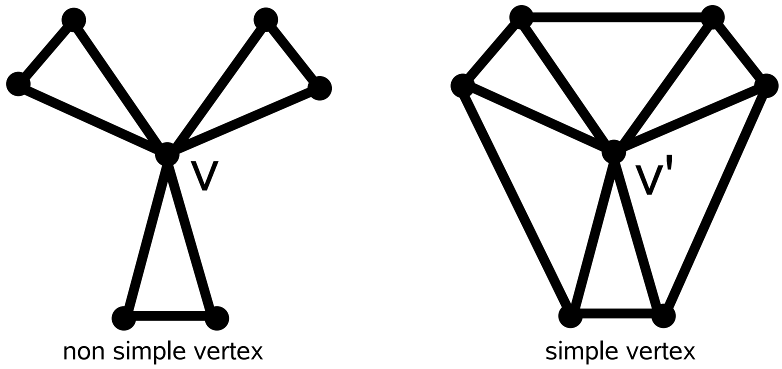

Figure 6.1: Two vertices with distinct collections of Δ_v sets

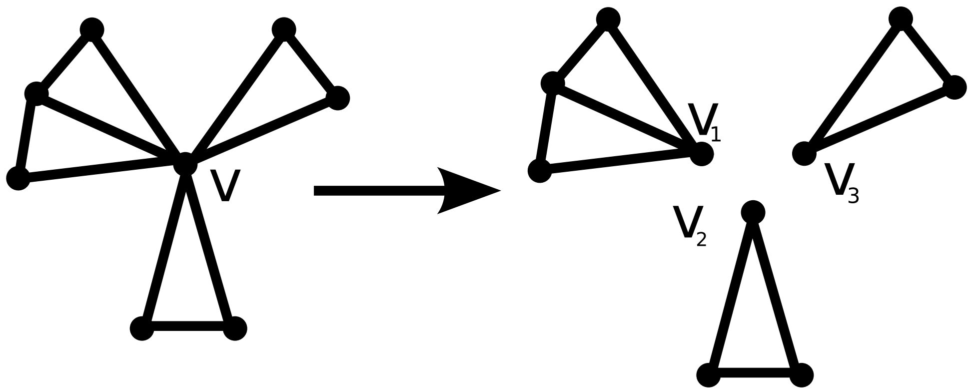

Figure 6.2: Vertex simplification

vertex v is called **simple vertex**, if the cardinality of Δ^v is equal to one. Let v be a vertex with the set $\Delta^v = \{\Delta_{v1}, \dots, \Delta_{vn}\}$. Then the **simplification** of a vertex v is a substitution of the vertex v by vertices $v_1, \dots, v_n$ and each triangle vxy belonging to the set Δ_{vi} is replaced by the triangle v_ixy.

Proposition 6.5. *Let Δ be a triangular configuration. Let Δ' be a triangular configuration obtained from Δ by simplification of all vertices. Then $M(\Delta) \cong M(\Delta')$.*

Proof. Let v be an arbitrary vertex of Δ. Let t, t' be triangles of Δ such that $v \subset t, v \subset t'$. Let t_s, t'_s be the triangles t, t'; respectively; after simplification of the vertex v.

If $t \cap t' \in E(\Delta)$, then these triangles belongs to the same set Δ_v. Hence $t_s \cap t'_s \in E(\Delta')$.

If $t \cap t' \notin E(\Delta)$, then $t \cap t' \in V(\Delta)$. In case that the vertex v is splitted in simplification, then $t_s \cap t'_s = \emptyset$. Hence $t_s \cap t'_s \notin E(\Delta')$. In the other case that the vertex v is not splitted, then $t_s \cap t'_s \in V(\Delta')$. Thus $t_s \cap t'_s \notin E(\Delta')$.

Thus, the operation of simplification preserve triangle incidence. Therefore, both matroids has identical representation matrices. Hence, they are isomorphic. □

7 Edge Contraction of Triangular Configurations

7.1 Definitions

Let $\Delta = (V, E, T)$ be a triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. The **edge contraction** is the triangular configuration $\Delta/_E e = \Delta' = (V', E', T')$ with the vertex set

$$V' := (V \setminus \{v_1, v_2\}) \cup \{v_e\},$$

the edge set

$$E' := \{vw \in E | \{v, w\} \cap \{v_1, v_2\} = \emptyset\} \cup \{v_e w | v_1 w \in E \setminus \{e\} \vee v_2 w \in E \setminus \{e\}\},$$

and the triangle set

$$\begin{aligned} T' := \{uvw \in T | \{u, v, w\} \cap \{v_1, v_2\} = \emptyset\} \\ \cup \{v_e vw | v_1 vw \in T \vee v_2 vw \in T; v \neq v_1, v \neq v_2, w \neq v_1, w \neq v_2\}. \end{aligned}$$

From the definition, triangular configurations are closed under taking edge

Edge contraction v_e

e

Figure 7.1: Edge contraction

contractions. Unfortunately, we do not know whether edge contraction has a geometric representation.

For a skeleton of a triangular configuration holds $G(\Delta/_E e) = G(\Delta)/e$.

In this section we survey the effect of the edge contraction to a triangular matroid. This effect depends only on the triangles that are incident with the contracted edge.

Let Δ be a triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. We say that the edge e is **deleting**, if Δ contains the triangles xyv_1 and xyv_2 where $x, y \in V(\Delta)$.

The set $\{e, f, g\}$ of edges of Δ is said to be the **empty triangle**, if $e \cap f \neq \emptyset, f \cap g \neq \emptyset, g \cap e \neq \emptyset$ and Δ does not contains the triangle $\{e, f, g\}$.

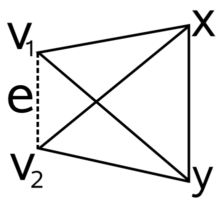

Figure 7.2: Deleting edge

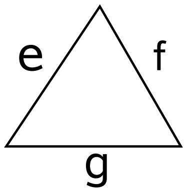

Figure 7.3: Empty triangle

Let Δ be a triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. Define

$$\begin{aligned} D_e &:= \{t | e \subset t, t \in T(\Delta)\}, \\ D'_e &:= \{\{t_1, t_2\} | v_1 \in t_1, v_2 \in t_2, t_1 \cap t_2 \in E(\Delta), t_1, t_2 \in T(\Delta)\}, \\ D'_{1e} &:= \{t_1 | \{t_1, t_2\} \in D'_e\}, \\ D'_{2e} &:= \{t_2 | \{t_1, t_2\} \in D'_e\}. \end{aligned} \tag{7.1}$$

For an edge contraction of Δ holds $T(\Delta/_E e) = T(\Delta) \setminus (D_e \cup D'_{2e})$.

7.2 Cycles and Acyclic Sets

Proposition 7.1. *A contraction along an edge $e = \{v_1, v_2\}$ of a cycle C is a cycle if and only if C does not contains a triangle (xyv_1) or (xyv_2); $x, y \in V(C)$ (that is, the edge e is not deleting).*

Proof. Let $C' = (V', E', T')$ be an edge contraction of a cycle $C = (V, E, T)$ along the edge $e = \{v_1, v_2\}$.

" $\Leftarrow$ "

From the definition of the contraction, the edges whose degree are changed are the new edges and the edges that in C lies in a triangle containing v_1 or v_2.

We show that the new edges created by the contraction have an even degree. Let $\{v, v_e\}$ be a new edge. From the assumption about degrees in C, there are two sets of triangles of an even cardinality $T_1 = \{v_1va \in T | a \in V\}$, $T_2 = \{v_2vb \in T | b \in V\}$. From the definition, the sets T_1 and T_2 become, in the contraction, the sets $T_1' = \{(v_e va)|(v_1 va) \in T_1; v \neq v_2, a \neq v_2\}$ and $T_2' = \{(v_e vb)|(v_2 vb) \in T_2; v \neq v_1, b \neq v_1\}$.

We distinguish the following cases.

- Let T_1 and T_2 be empty. Then the degree of $\{v, v_e\}$ is equal to zero, by the definition.

- Let one of T_1 or T_2 be empty. Let us say T_2. Then the degree of $\{v, v_e\}$ is equal to the degree of $\{v, v_1\}$.

- Let T_1 and T_2 be non empty. If the sets T_1' and T_2' are disjoint, then the degree of the edge $\{v, v_e\}$ is equal to the cardinality of $T_1' \cup T_2'$. $|T_1' \cup T_2'| = |T_1| + |T_1| - 2|T_1 \cap T_2|$. Hence, this cardinality is even.

Suppose that T_1' and T_2' are not disjoint, then in the configuration C are the triangles $(v_1 vx), (v_2 vx)$ and the edge $e = \{v_1, v_2\}$. This is the contradiction, as we suppose that there are not such triangles.

Now we show that the edges that in C lies in a triangle containing v_1 or v_2 have in C' an even degree. Let $\{x, y\}$ be an edge incident with triangles (xyv_1) or (xyv_2). If this edge is incident only with the one triangle, then the degree of this edge in C' is equal to the degree in C, since instead of the triangle (xyv_1) there is (xyv_e). From the assumptions, we know that there is only one triangle (xyv_1) or (xyv_2).

We have proved that all degrees in the contraction of a cycle C are even. Therefore, C' is a cycle.

" $\Rightarrow$ "

Suppose that in C exists triangles (xyv_1) and (xyv_2). The edge $\{x, y\}$ has an even degree in C and is incident with the triangles (xyv_1) and (xyv_2). From the definition, this edge remains in C'. In C' the triangles (xyv_1) and (xyv_2) are deleted. Hence, $\{x, y\}$ is incident with only one new triangle (xyv_e). Thus, the degree of $\{x, y\}$ is odd. Therefore, C' is not a cycle. □

Corollary 7.1. *Let C be a cycle. Let e be an edge of C. If $G(C)$ has no a subgraph K_4 containing the edge e, then the contraction along the edge e is a cycle.*

Proof. Let e be $\{v_1, v_2\}$. We show that C does not contains a triangle (xyv_1) or (xyv_2); $x, y \in V(C)$.

Let C contain triangles (xyv_1) and (xyv_2). Then there is a K_4 with the edges $e = \{v_1, v_2\}, \{v_1, x\}, \{v_1, y\}, \{v_2, x\}, \{v_2, y\}, \{x, y\}$. A contradiction. Now, we use the previous proposition. □

Proposition 7.2. *Let Δ be a triangular configuration that is not a cycle. Let e be an edge of Δ. Then $\Delta/_E e$ is not a cycle if and only if at least one of the following conditions is satisfied*

(i) *there exists an edge f with an odd degree such that $e \cap f = \emptyset$,*

(ii) *there exists an edge f with an odd degree such that $e \cap f \neq \emptyset$ and there does not exists an edge g such that $f \cap g \neq \emptyset \neq g \cap e$,*

(iii) *there exists an edge f such that $e \cap f \neq \emptyset$ and there exists an edge g such that $f \cap g \neq \emptyset \neq g \cap e$; and $d(f)$ and $d(g)$ have different parities.*

Proof. " $\Leftarrow$ "

If the first or the second condition is satisfied by an edge f, then no triangle incident with f is removed or added by the contraction. Therefore this edge has an odd degree in $\Delta/_E e$.

If the third condition is satisfied, then the edges f and g are merged into one edge $\{v_e, x\}$. The degree of the edge is equal to $d(f) + d(g)$, if there is no triangle incident with both f and g in Δ. If there is such triangle, then the degree is equal to $d(f) - 1 + d(g) - 1$. As $d(f)$ and $d(g)$ have different parities, the edge $\{v_e, x\}$ has an odd degree.

" $\Rightarrow$ "

Suppose that $\Delta/_E e$ is a cycle. Obviously, the first and second conditions are not satisfied.

Each edge $\{v_e, x\}$ has an even degree equal to $d(\{v_1, x\}) + d(\{v_2, x\})$ or $d(\{v_1, x\}) + d(\{v_2, x\}) - 2$ depending on if in Δ exists a triangle $v_1 v_2 x$. Thus, $\{v_1, x\}$ and $\{v_2, x\}$ have the same parity. Therefore, the third condition is not satisfied. □

Corollary 7.2. *Let Δ be an acyclic triangular configuration. Let e be an edge. Then $\Delta/_E e$ is acyclic if and only if there is no subconfiguration Δ' of Δ that does not satisfy the following conditions*

(i) *there exists an edge f with an odd degree such that $e \cap f = \emptyset$,*

(ii) *there exists an edge f with an odd degree such that $e \cap f \neq \emptyset$ and there does not exists an edge g such that $f \cap g \neq \emptyset \neq g \cap e$,*

(iii) *there exists an edge f such that $e \cap f \neq \emptyset$ and there exists an edge g such that $f \cap g \neq \emptyset \neq g \cap e$ and $d(f)$ and $d(g)$ have different parities.*

Proof. Directly from Proposition 7.2. □

Let Δ be a triangular configuration. Let e be an edge of Δ. If $\Delta/_E e$ is acyclic, we say that Δ is **e-acyclic**. A triangular configuration is **e-contractable** if every acyclic subconfiguration is e-acyclic.

Let Δ^* be a triangular configuration obtained from Δ by filling all empty triangles incident with e.

Corollary 7.3. *Let Δ be an acyclic triangular configuration. Let e be an edge of Δ. Then $\Delta/_E e$ is acyclic if and only if there does not exists $\Delta' \subseteq \Delta^*$ such that every edge of $E(\Delta') \setminus \{e\}$ has an even degree.*

Proof. " $\Rightarrow$ "

We know that $\Delta/_E e$ is acyclic. For a contradiction suppose that there exists $\Delta' \subseteq \Delta^*$ such that every edge of $E(\Delta') \setminus \{e\}$ has an even degree. Then $\Delta'/_E e$ is a cycle, by Proposition 7.2. $\Delta'/_E e$ is contained in $\Delta/_E e$, since every added triangle of Δ^* is deleted by the contraction. Thus $\Delta/_E e$ is not acyclic. This is a contradiction.

" $\Leftarrow$ "

Let Δ'' be a subconfiguration of Δ. Suppose that Δ'' does not satisfy the conditions (i)–(iii) of Proposition 7.2. Let t be an empty or nonempty triangle of Δ' containing the edge e. Then the edges of t excepting the edge e have the same parities.

Let Δ' be a configuration obtained from Δ'' by deleting (filling) the triangles (empty triangles) that contains the edge e and an edge distinct from e with an odd degree. Then every edge of $E(\Delta') \setminus \{e\}$ has an even degree. Δ' is a subconfiguration of Δ^*. This is the contradiction with our assumptions. Thus, Δ'' satisfy the assumptions of Proposition 7.2.

Hence, $\Delta''/_E e$ is acyclic. Therefore, $\Delta/_E e$ is acyclic. □

7.3 Triangular Matroid

Corollary 7.4. *Let Δ be an triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. If Δ does not contain a triangle xyv_1 or xyv_2 where $x, y \in V(C)$ (that is, $D'_e = \emptyset$), then $\mathcal{C}(\Delta/_E e) \supseteq \mathcal{C}(M(\Delta)/D_e)$.*

Proof. Let C be a cycle of Δ that contains the edge e. Since C does not contain a triangle xyv_1 or xyv_2 where $x, y \in V(C)$, then $C/_E e$ is a circuit with the triangle set $T(C) \setminus D_e$; by Proposition 7.1. Thus, $C/_E e$ belongs to $\mathcal{C}(M(\Delta)/D_e)$ □

Corollary 7.5. *Let Δ be a triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. If Δ contains triangles xyv_1 and xyv_2 where $x, y \in V(C)$, then $\mathcal{C}(\Delta/_E e) \supseteq \mathcal{C}(M(\Delta)/D_e \setminus D'_{1e})$ and $\mathcal{C}(\Delta/_E e) \not\supseteq \{C \setminus (D'_{1e} \cup D_e) | C \in \mathcal{C}(\Delta), \{t_1, t_2\} \subseteq C, \{t_1, t_2\} \in D'_e\}$.*

Proof. Let C be a cycle of Δ that contains the edge e. If C does not contain any element of D'_e, then $C/_E e$ is a cycle; by the previous corollary. Therefore $\mathcal{C}(\Delta/_E e) \supseteq \mathcal{C}(M(\Delta)/D_e \setminus D'_{1e})$.

Let C' be an element of $\{C \setminus (D'_{1e} \cup D_e) | C \in \mathcal{C}(\Delta), \{t_1, t_2\} \subseteq C, \{t_1, t_2\} \in D'_e\}$. Then $C' = T(C) \setminus (D'_{1e} \cup D_e) = T(C/_E e)$ where C is a cycle of Δ , by the definition of the edge contraction. $C/_E e$ is not a cycle, by Proposition 7.1. Thus, $C' \notin \mathcal{C}(\Delta/_E e)$. $\square$

If a triangular configuration is e-contractable, we can exactly express the cycle space of the edge contraction.

Corollary 7.6. *Let Δ be an triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. Let Δ be e-contractable. If Δ does not contain a triangle xyv_1 or xyv_2 where $x, y \in V(\Delta)$, then $M(\Delta/_E e) \cong M(\Delta)/D_e$.*

Proof. By Corollary 7.4, we know that $\mathcal{C}(\Delta/_E e) \supseteq \mathcal{C}(M(\Delta)/D_e)$. Suppose that $\mathcal{C}(\Delta/_E e)$ contains a cycle C' such that $T(C') \neq T(C) \setminus D_e$ for every $C \in \mathcal{C}(\Delta)$. As Δ is e-contractable, there is a cycle C'' of Δ such that $C''/_E e = C'$, $T(C'') \setminus D_e = T(C')$. This is a contradiction. Hence, $\mathcal{C}(\Delta/_E e) = \mathcal{C}(M(\Delta)/D_e)$. $\square$

Corollary 7.7. *Let Δ be a triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. Let Δ be e-contractable. If Δ contains triangles xyv_1 and xyv_2 where $x, y \in V(\Delta)$, then $\mathcal{C}(\Delta/_E e) = \{C \setminus D_e | \{t_1, t_2\} \nsubseteq C, \{t_1, t_2\} \in D'_e, C \in \mathcal{C}(\Delta)\}$.*

Proof. By Corollary 7.5, $\mathcal{C}(\Delta/_E e) \supseteq \mathcal{C}(M(\Delta)/D_e \setminus D'_{1e})$ and $\mathcal{C}(\Delta/_E e) \nsupseteq \{C \setminus (D'_{1e} \cup D_e) | C \in \mathcal{C}(\Delta), \{t_1, t_2\} \subseteq C', \{t_1, t_2\} \in D'_e\}$. Suppose that there exists a cycle $C \in \mathcal{C}(\Delta/_E e) \setminus \{C \setminus D_e | \{t_1, t_2\} \nsubseteq C, \{t_1, t_2\} \in D'_e, C \in \mathcal{C}(\Delta)\}$. As Δ is e-contractable, $T(C) = T(C') \setminus D_e$ for $C' \in \mathcal{C}(\Delta)$ such that $\{t_1, t_2\} \nsubseteq C'; \{t_1, t_2\} \in D'_e$. This is a contradiction. Hence, $\mathcal{C}(\Delta/_E e) = \{C \setminus D_e | \{t_1, t_2\} \nsubseteq C, \{t_1, t_2\} \in D'_e, C \in \mathcal{C}(\Delta)\}$. $\square$

7.4 Euler Characteristic

Proposition 7.3. *Let Δ be a triangular configuration with the Euler characteristic χ. Let Δ' be an edge contraction along an edge $\{v_1, v_2\}$. Let Δ be e-contractable. If Δ does not contains a triangle xyv_1 or xyv_2 where $x, y \in V(\Delta)$, then Δ' has the Euler characteristic equal to χ.*

Proof. By Corollary 7.6, $M(\Delta') \cong M(\Delta)/D_e$ where $D_e = \{t | e \subset t, t \in T(\Delta)\}$. The contraction removes the edge $\{v_1, v_2\}$ and one of the vertices v_1, v_2; by the definition. Thus, $|V(\Delta')| = |V(\Delta)| - 1$ and $|E(\Delta')| = |E(\Delta)| - 1$.

The number of triangles of Δ' is equal to $|T(\Delta)| - |D_e|$. One edge distinct of $\{v_1, v_2\}$ is removed with each removed triangle. Hence,

$$\begin{aligned}
\chi &= |V(\Delta')| - |E(\Delta')| + |T(\Delta')| \\
&= (|V(\Delta)| - 1) - (|E(\Delta)| - |D_e| - 1) + (|T(\Delta)| - |D_e|) \qquad (7.2) \\
&= |V(\Delta)| - |E(\Delta)| + |T(\Delta)|.
\end{aligned}$$

□

8 Skeleton of Triangular Configurations

In this section we study the minors of a skeleton of a simple triangular circuit. We show that a skeleton may contain any arbitrary graph as a minor. We give the smallest triangular circuit with a nonplanar skeleton.

Proposition 8.1. *Let G be a graph. Then there exists a simple triangular circuit Δ such that G is a minor of $G(\Delta)$.*

Proof. Before we construct Δ. We define two particular triangular configurations, which serve as basic building blocks. The triangular vertex (sphere), depicted in Figure 8.1, is obtained by a sufficient dense triangulation of a sphere.

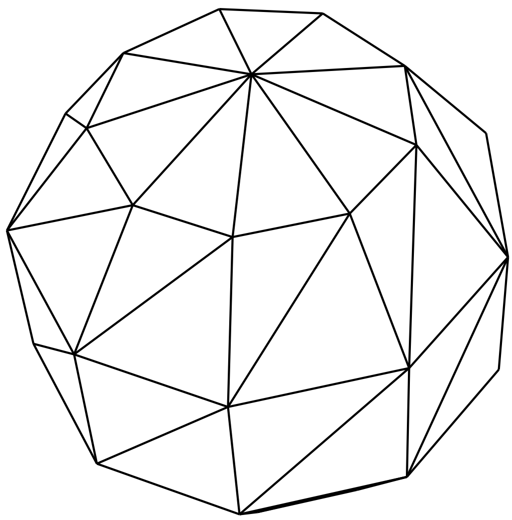

Figure 8.1: A triangular sphere.

The triangular edge or tunnel, depicted in Figure 8.2, is obtained by sticking together a number of basic buildings blocks. Dash triangles in the figure denote empty triangles, the others are regular triangles. Blocks are stuck together at the ending empty triangles depicted by dash lines.

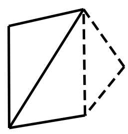

Figure 8.2: A triangular tunnel.

Now, we construct the desired triangular circuit Δ. For each vertex of G we add to Δ a triangular vertex with the number of triangles at least equal to the degree of the vertex. For each edge uv we remove one triangle from the both triangular vertices u and v and we connect these empty triangles by a sufficient large triangular tunnel (edge). Obviously, every edge of Δ

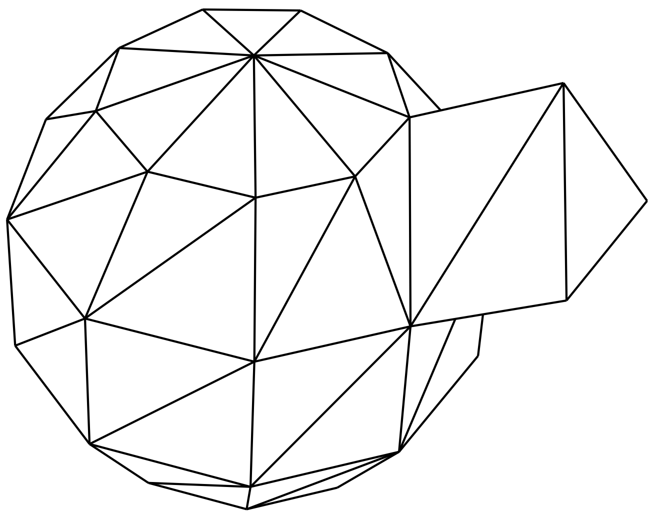

Figure 8.3: An example of a vertex of degree 1.

has degree 2. Therefore, Δ is a simple triangular circuit.

Now, we construct the desired graph G. We take one vertex from every triangular vertex of Δ and put it in the set of vertices of G. Let u, v be vertices of two triangular vertices connected by a tunnel. We take a path between them leading throw the tunnel and contract it to the one edge. We put this edge to the set of edges of G. Obviously, the graph G is a minor of the skeleton of Δ.

Examples of triangular vertices connected by some edges are in Figures 8.3 and 8.4. □

In the next proposition, we give a small triangular circuit that has a nonplanar skeleton.

Proposition 8.2. *There is a triangular circuit C such that $G(C)$ is a nonplanar graph.*

Proof. The desired circuit is depicted in Figure 8.5. The skeleton of the circuit contains $K_{3,3}$ as a subgraph. The circuit contains triangles that are not depicted by the gray color on each picture. One partition class of $K_{3,3}$ is depicted by square nodes, the second by round nodes. □

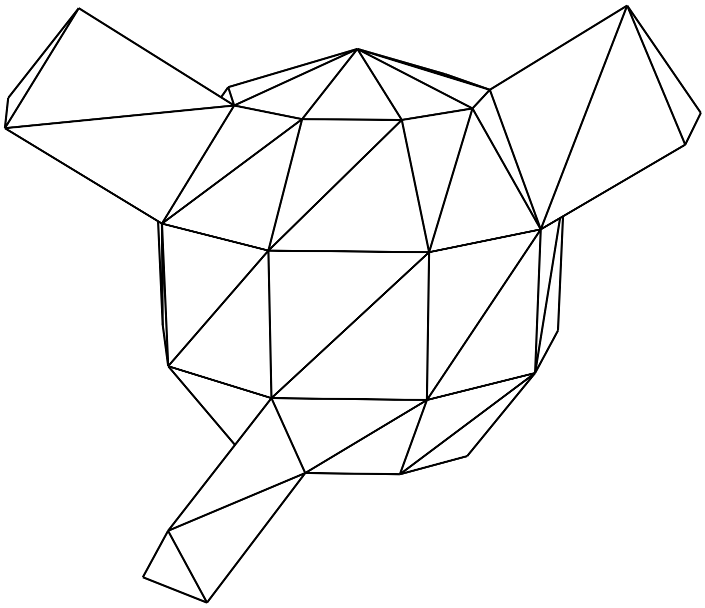

Figure 8.4: An example of a vertex of degree 3.

Figure 8.5: The circuit Δ_1 with the nonplanar skeleton.

9 Lattices of Triangular Configurations

In this section we study the lattice generated by the set of cycles of a triangular configuration.

9.1 Some Interesting Triangular Configurations

We give a particular triangular configuration which does not have the lattice of circuits property and show that its lattice has a basis consisting only of cycles. This triangular configuration contains F_7^* as a minor.

Proposition 9.1. *There exists a triangular configuration which does not have the lattice of circuits property.*

Proof. The desired triangular configuration is depicted in the picture D_4 in Figure 9.2. Every triangular configuration $D_i; i \geq 1$ is an ear extension of D_{i-1}.

Figure 9.1: The circuit Δ_2 contains all triangles that are not depicted by the gray color. The triangles containing a dash line have value 0 and the others have $\frac{1}{4}$.

The ear circuits of these extensions are depicted in detail in Figures 9.1 and 8.5. We assign to the triangles some values. The values of triangles of D_0 are equal to the values of triangles of Δ_2. The values of triangles of D_i; $i = 1, 2, 3$ are the same as the values of triangles of D_{i-1} and the ear circuit. The values of triangles of $T(D_4) \setminus T(D_3)$ are assigned to 0.

Now we observe that D_4 does not have the lattice of circuits property. We construct a cosimple matroid M from $M(D_4)$ by contracting all coparallel classes into one element. The matroid M has well defined assignment of values, since all triangles in one coparallel class of $M(D_4)$ have the same value. This assignment of values of M belongs to the dual lattice of M. Thus, the dual lattice contains the point $x \in \frac{1}{4}\mathbb{Z} \setminus \frac{1}{2}\mathbb{Z}$. Therefore, D_4 does not have the lattice of circuits property.

□

Proposition 9.2. *The lattice of the triangular configuration D_4 has a basis consisting only of cycles.*

Proof. We construct a cycle lattice basis by using Theorems 5.4 and 5.6. Obviously, the triangular configuration D_0 has a cycle lattice basis, since it contains only one circuit. A cycle lattice basis of the ear extension D_i; $i = 1, 2, 3$ is constructed from a cycle lattice basis of D_{i-1} by using Theorem 5.4. A cycle lattice basis of D_4 is constructed from a cycle lattice basis of D_3 by using Theorem 5.6. □

9.2 Local Constructions and Edge Contraction

We give a sufficient condition when it is possible extend a cycle lattice basis of a triangular configuration to the cycle lattice basis of its edge contraction.

Proposition 9.3. *Let Δ be a triangular configuration. Let $e = \{v_1, v_2\}$ be an edge of Δ. Let B be a cycle basis of $\mathbb{Z}(\Delta)$. Let Δ' be an edge contraction of Δ along the edge e. Let Δ be e-contractable. If Δ does not contain a triangle xyv_1 or xyv_2 where $x, y \in V(\Delta)$, then $\mathbb{Z}(\Delta')$ has a cycle basis.*

Proof. By Corollary 7.6, $M(\Delta') \cong M(\Delta)/D_e$ where $D_e = \{t | e \subset t, t \in T(\Delta)\}$. Thus, $\mathcal{C}(M(\Delta')) = \{C \setminus D_e | C \in \mathcal{C}(M(\Delta))\}$. A consequence of Proposition 7.1 is that $|\mathcal{C}(M(\Delta))| = |\mathcal{C}(M(\Delta'))|$. Hence, the set $B' := \{\beta \setminus D_e | \beta \in B\}$ is a basis of $\mathbb{Z}(\Delta')$. □

$D_0 =$ $D_1 =$

$D_2 =$ $D_3 =$

$D_4 =$

Figure 9.2: The triangular configuration D_4, which does not have the lattice of circuits property and its ear decomposition.

10 Minors of Triangular Configurations

In this section we survey how rich is the class of triangular matroids and their minors. The hypothesis posed by Whittle is that for every binary matroid exists triangular configuration containing this matroid as a minor. We prove the hypothesis. This hypothesis is equivalent with that for every P_r^*; $r \geq 3$ there exists a triangular configuration having P_r^* as a minor. In Section 9 on Figure 9.2 we find a configuration having $P_3^* = F_7^*$ as a minor.

Theorem 10.1. *Let M be a binary matroid. Then there exists a triangular configuration Δ such that $M(\Delta)/S \cong M$, where S is a subset of $E(M)$. Moreover, there exists a bijection between $\mathcal{C}(M)$ and $\mathcal{C}(\Delta)$ mapping circuits to circuits, and $dim\, \mathcal{C}(M) = dim\, \mathcal{C}(\Delta)$.*

Proof. Let n be the cardinality of the ground set of the matroid M. Let r denote the dimension of the cycle space $\mathcal{C}(M)$ a subspace of $GF(2)^n$. Let B be a cycle basis of $\mathcal{C}(M)$. We construct the desired configuration in this way. We put n triangles into a space of sufficient large dimension (Figure 10.1). Denote these triangles as $t_1, \ldots, t_n$.

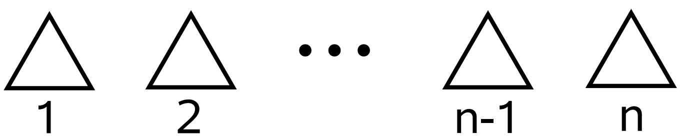

Figure 10.1: Triangles representing the entries of the vectors of $\mathcal{C}(M)$.

For every basis vector $b_i \in B$ we construct the following triangular configuration Δ_{b_i} (Figure 10.2). The configuration Δ_{b_i} is obtained from a sufficiently dense triangular sphere (Figure 8.1, a sphere with the number of triangles greater than n). For every nonzero entry of the vector b_i we remove a triangle from the sphere and add triangular tunnel (Figure 8.2) between the new empty triangle and the triangle t_j where j is a position of a nonzero entry in the vector b_i. Thus, Δ_{b_i} contains t_j if and only if $b_i^j = 1$. We denote the cardinality $|T(\Delta_{b_i})|$ by $w(\Delta_{b_i})$.

The desired triangular configuration Δ is the union of the triangular configurations Δ_{b_i}, $i = 1, \ldots, n$; $\Delta = \bigcup_{i=1}^{d} \Delta_{b_i}$ (Figure 10.3).

It is convenient construct the configurations Δ_{b_i} such that $w(\Delta_{b_i}) - w(b_i) = w(\Delta_{b_j}) - w(b_j)$ where $i, j = 1, \ldots, d$. We denote the number $w(\Delta_{b_i}) - w(b_i)$ by $w(\Delta)$.

The triangular configuration Δ obviously contains all symmetric differences of the triangular configurations Δ_{b_i}, $i = 1, \ldots, d$. For a symmetric

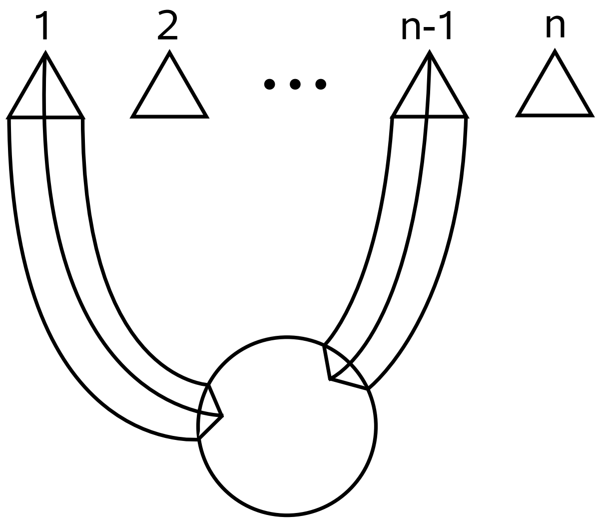

Figure 10.2: A triangular cycle representing a basis vector of $\mathcal{C}(M)$.

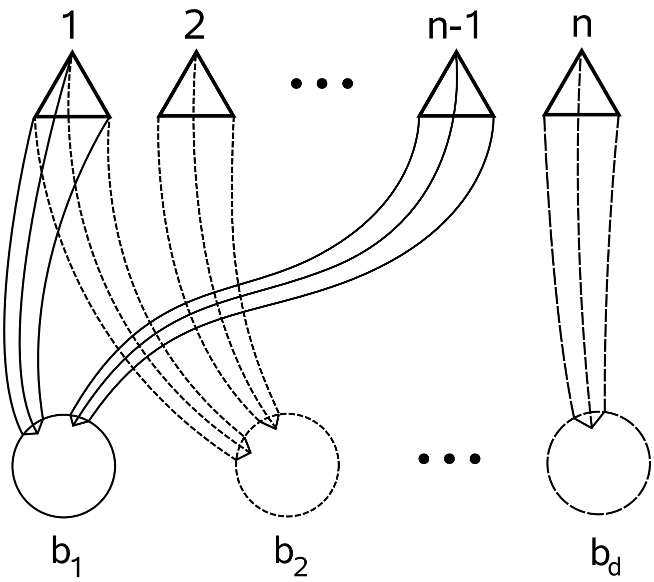

Figure 10.3: A triangular representation of M.

difference of Δ_{b_i} and Δ_{b_j} holds that $\Delta_{b_i} \triangle \Delta_{b_j}$ contains triangle t_k if and only if kth entry of the vector $b_i + b_j$ is equal to 1. Using induction we have that $\triangle_{i\in I}\Delta_{b_i}$ contains a triangle t_k if and only if kth entry of the vector $\sum_{i\in I} b_i$ is equal to 1. Therefore, $\mathcal{C}(M(\Delta)/S) \supseteq \mathcal{C}(M)$ where $S = E(M(\Delta)) \setminus E(M)$.

We define a mapping $f : \mathcal{C}(M) \mapsto \mathcal{C}(\Delta)$ in the following way. Let x be an element of $\mathcal{C}(M)$. The vector x is equal to $\sum_{i\in I} b_i$. We define $f(x)$ as $\triangle_{i\in I}\Delta_{b_i}$. From the paragraph above follows that f is an injective mapping.

Now we prove that $dim\, \mathcal{C}(M) = dim\, \mathcal{C}(\Delta)$. Suppose that there exists a circuit of Δ that is not a symmetric difference of Δ_{b_i}, $i = 1, \ldots, d$. Let C be a such circuit with the minimal possible number of triangles $|T(C)|$. It is obvious that $T(C)$ contains $T(\Delta_{b_i}) \setminus \{t_1, \ldots, t_n\}$ for some $i \in \{1, \ldots, t\}$. For the circuit $C \triangle \Delta_{b_i}$ holds that $|T(C \triangle \Delta_{b_i})| < |T(C)|$, since $T(\Delta_{b_i})$ is sufficiently large. This is a contradiction. Thus, every circuit of Δ is a symmetric difference of Δ_{b_i}, $i = 1, \ldots, d$. Hence, $dim\, \mathcal{C}(M) = dim\, \mathcal{C}(\Delta)$.

Therefore, $\mathcal{C}(M(\Delta)/S) = \mathcal{C}(M)$ and $M(\Delta)/S \cong M$.

As $|\mathcal{C}(M)| = |\mathcal{C}(\Delta)|$, the mapping f is a bijection.

Now we show that f maps circuits to circuits. Let c be a circuit of $\mathcal{C}(M)$. For a contradiction suppose that $f(c)$ is not a circuit. Then there are cycles c_1 and c_2 of $\mathcal{C}(M)$ such that $f(c_1) \cup f(c_2) = f(c)$. By the definition of f, $c = c_1 \cup c_2$. This is a contradiction. Thus, the mapping f maps circuits to circuits. □

The triangular configuration in the theorem above we call **triangular representation** of a binary matroid with respect to the basis B. A triangular configuration such that all $w(\Delta_{b_i}) - w(b_i)$ are the same is called **normal**.

Let M be a binary matroid. Let C be the cycle space of M. Let d be the dimension of C. The **weight polynomial** of the code C is defined according to the formula

$$W(C) := \sum_{c\in C} x^{w(c)}. \tag{10.1}$$

Now, we survey a connection between the weight polynomial of a matroid and the weight polynomial of its triangular representation.

Let B be a basis of C. Let Δ be a normal triangular representation of C. We say that an element c of C has degree of combination i if it is a sum of i basis vectors. We denote the degree of combination of a vector c by $dc(c)$.

For instance a basis vector has degree 1. We define

$$W_i(C) := \sum_{c\in C, dc(c)=i} x^{w(c)}. \tag{10.2}$$

It is obvious that

$$W(C) = \sum_{i=0}^{d} W_i(C). \tag{10.3}$$

Proposition 10.1. *Let M be a binary matroid. Let B be a basis of $\mathcal{C}(M)$. Let Δ be a normal triangular representation of M with respect to B. Then*

$$W_i(\mathcal{C}(\Delta)) = W_i(\mathcal{C}(M))x^{iw(\Delta)}. \tag{10.4}$$

Proof. Let c be a cycle of $\mathcal{C}(M)$ of degree i. The cycle c is equal to $\sum_{j\in J} b_j$ where $|J| = i$. Then there exists a cycle c' of $\mathcal{C}(\Delta)$ equal to $\triangle_{j\in J}\Delta_{b_j}$. The weight of Δ_{b_j} is equal to $w(\Delta) + w(b_j)$. Thus, the weight of the cycle c' is equal to $iw(\Delta) + w(c)$.

Therefore,

$$\begin{aligned} W_i(\mathcal{C}(\Delta)) &= \sum_{c'\in\mathcal{C}(\Delta), dc(c')=i} x^{w(c')} \\ &= \sum_{c\in\mathcal{C}(M), dc(c)=i} x^{w(c)+iw(\Delta)} \\ &= W_i(\mathcal{C}(M))x^{iw(\Delta)}. \end{aligned} \tag{10.5}$$

□

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