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Evolutionary differential equations in unbounded domains

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Title: Evolutionary differential equations in unbounded domains

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Abstract: We study asymptotic properties of evolution partial differential equations posed in unbounded spatial domain in the context of locally uniform spaces. This context allows the use of non-integrable data and carries an inherent non-compactness and non-separability. We establish the existence of a locally compact attractor for non-local parabolic equation and weakly damped semilinear wave equation and provide an upper bound on the Kolmogorov's ε-entropy of these attractors and the attractor of strongly damped wave equation in the subcritical case using the method of trajectories. Finally we also investigate infinite dimensional exponential attractors of nonlinear reaction-diffusion equation in its natural energy setting.

Keywords: evolution partial differential equation, unbounded domain, attractor, entropy estimate, asymptotic analysis.

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Introduction

There are at least two reasons why one might consider solving differential equation in a spatially unbounded domain: one could hope to gain access to additional symmetries or the Fourier transform or one might be interested in the dynamics resulting from the unboundedness of the spatial domain. In this thesis we adopt the latter approach.

Differential equations in unbounded domains have several specifics. The classical Lebesgue spaces do not contain constants nor other potentially interesting functions, so one should consider the space of initial data. Historically, weighted spaces have been used, see e.g. Abergel [1990]. Roughly in the last 20 years the analysis has been carried out in so-called locally uniform spaces, see e.g. Mielke and Schneider [1995], Feireisl [1996], Zelik [2001b] and Zelik [2001a], which on the other hand are neither separable nor reflexive. One should also think about suitable generalizations of the notions of attractor since we cannot in general expect these to be compact or have finite fractal dimension owing to the unboundedness of the domain.

The unifying theme of this thesis is the asymptotic analysis of dissipative evolution differential equations posed in unbounded domains. We establish the existence of locally compact attractors for several equations and obtain upper bounds on their Kolmogorov's ε-entropy. We also study the infinite dimensional exponential attractors and supply a sufficient and necessary condition for the existence of such an attractor.

This thesis consist of two published paper and two preprints:

[I] D. Pražák and J. Slavík. Attractors and entropy bounds for a nonlinear RDEs with distributed delay in unbounded domains. *Discrete Contin. Dyn. Syst. Ser. B*, 21(4):1259–1277, 2016. ISSN 1531-3492. doi: 10.3934/dcdsb.2016.21.1259.

[II] J. Slavík. A sufficient and necessary condition for infinite dimensional exponential attractor in locally uniform spaces. In preparation.

[III] M. Michálek, D. Pražák, and J. Slavík. Semilinear damped wave equation in locally uniform spaces. *Commun. Pure Appl. Anal.*, 16(5):1673–1695, 2017. ISSN 1534-0392. doi:10.3934/cpaa.2017080.

[IV] J. Slavík. Kolmogorov's ε-entropy of the attractor of the strongly damped wave equation in locally uniform spaces. In preparation.

The thesis is organised as follows: in Section 1 the reader may find basic definitions of function spaces used in the above papers. Section 2 recalls some of the objects of interest of asymptotic analysis of evolution equations in unbounded domains. Section 3 contains a review of results obtained in the papers [I–IV]. Finally Sections 4–7 consist of the papers and preprints themselves.

1. Function spaces

In this section we review definitions and basic properties of weighted spaces, locally uniform spaces and their parabolic variant.

According to Arrieta et al. [2004] the locally uniform spaces have been first introduced in Kato [1975] who studied hyperbolic equations in the unbounded domain $\mathbb{R}^d$, although he considered that only the coefficients of the equations are locally uniform. However, locally uniform spaces are the more natural choice for a phase space in the study of differential equations in unbounded domains mainly for two reasons: first, the classical Lebesgue spaces over $\mathbb{R}^d$ are not nested, i.e. $L^p(\mathbb{R}^d) \subsetneq L^q(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d) \subsetneq L^p(\mathbb{R}^d)$ for $1 \le p, q \le \infty$, $p \neq q$, and secondly, many solutions relevant to physical or biological applications and natural mathematical techniques, such as travelling waves and constants, are not in $L^p(\mathbb{R}^d)$ for any $1 \le p < \infty$.

In the theory of partial differential equations in unbounded domains the locally uniform spaces are used as the space of initial data. The weighted spaces serve mostly as an analytic tool, for example to obtain apriori estimates or describe the continuity of the solution semigroup. The parabolic variant of the locally uniform spaces plays the role of classical Bochner spaces which are essential for the method of trajectories.

1.1 Weighted spaces

A bounded measurable function $\phi : \mathbb{R}^d \to (0, \infty)$ is called a *weight function of growth rate* $\mu \ge 0$ if

$$C_\phi^{-1} e^{-\mu|x-y|} \le \phi(x)/\phi(y) \le C_\phi e^{\mu|x-y|}, \qquad \forall x, y \in \mathbb{R}^d, \tag{1.1}$$

for some $C_\phi > 0$ and

$$|\nabla \phi(x)| \le \tilde{C}_\phi \varepsilon \phi(x), \qquad \text{for a.a. } x \in \mathbb{R}^d, \tag{1.2}$$

and for some $\tilde{C}_\phi > 0$. For $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we define

$$\phi_{\bar{x},\varepsilon}(x) = e^{-\varepsilon|x-\bar{x}|}.$$

One can easily show that $\phi_{\bar{x},\varepsilon}$ is a weight function of growth ε.

For $\varepsilon > 0$, $\bar{x} \in \mathbb{R}^d$ and $p \in [1, \infty)$ we define the *weighted Lebesgue space* $L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by

$$L^p_{\bar{x},\varepsilon}(\mathbb{R}^d) = \{u \in L^p_{\text{loc}}(\mathbb{R}^d); \|u\|^p_{L^p_{\bar{x},\varepsilon}} = \int_{\mathbb{R}^d} |u(x)|^p \phi_{\bar{x},\varepsilon}(x)\, dx < \infty\}.$$

For $p = 2$ we write $\|\cdot\|_{\bar{x},\varepsilon}$ instead of $\|\cdot\|_{L^2_{\bar{x},\varepsilon}}$. The corresponding Sobolev spaces are defined in a straightforward manner. The weighted spaces $L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ are clearly separable and for $1 < p < \infty$ they are reflexive.

Concerning the embeddings of weighted spaces, first observe that the space $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)$ cannot be embedded into $L^q_{\bar{x},\varepsilon}(\mathbb{R}^d)$ for any $q > p$. However, we are able to overcome this limitation once we allow different growth rates.

Assume that $k, l \in \mathbb{N}_0$ and $p, q \in [1, \infty)$ are such that $k \geq l$, $q \geq p$ and $W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{l,q}(\mathbb{R}^d)$, then for $\tilde{\varepsilon} = \varepsilon q/p$ we have the continuous embedding $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$. If the embedding $W^{k,p}(B(0,1)) \hookrightarrow\hookrightarrow W^{l,q}(B(0,1))$ is compact, then for $\tilde{\varepsilon} > \varepsilon q/p$ the embedding $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow\hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$ is also compact.

1.2 Locally uniform spaces

For a weight function ϕ and $p \in [1, \infty)$ we define the locally uniform space $L^p_{b,\phi}(\mathbb{R}^d)$ by

$$L^p_{b,\phi}(\mathbb{R}^d) = \{u \in L^p_{\text{loc}}(\mathbb{R}^d); \|u\|^p_{L^p_{b,\phi}} = \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \int_{B(\bar{x},1)} |u(x)|^p \, dx\}.$$

The corresponding Sobolev spaces are again defined in a straightforward manner. One can also easily show that the norm $\|\cdot\|_{L^p_{b,\phi}}$ is equivalent to the norm

$$\|u\|^p_{L^p_{b,\phi}} \approx \sup_{k \in \mathbb{Z}^d} \phi(k) \int_{C^1_k} |u(x)|^p \, dx, \qquad u \in L^p_{\text{loc}}(\mathbb{R}^d), \tag{1.3}$$

where C^R_x denotes the cube in $\mathbb{R}^d$ of side $R > 0$ centred at $x \in \mathbb{R}^d$. For $p = 2$ we usually write $\|\cdot\|_{b,\phi}$ instead of $\|\cdot\|_{L^2_{b,\phi}}$. Moreover, if $\phi \equiv 1$ we omit the dependence on ϕ and write for example $L^2_b(\mathbb{R}^d)$ instead of $L^2_{b,1}(\mathbb{R}^d)$.

The locally uniform spaces are neither separable nor reflexive. Being in $L^1_{\text{loc}}(\mathbb{R}^d)$, the locally uniform functions are clearly distributions on $\mathbb{R}^d$ and one can also show that locally uniform functions are actually tempered distributions. To the best of our knowledge there are no results studying the locally uniform spaces using the Fourier transform.

The locally uniform spaces have the pleasant property that whenever $k, l \in \mathbb{N}_0$ and $p, q \in [1, \infty)$ are such that the embedding $W^{k,p}(B(0,1)) \hookrightarrow W^{l,p}((B(0,1))$ holds, then the embedding $W^{k,p}_b(\mathbb{R}^d) \hookrightarrow W^{l,q}_b(\mathbb{R}^d)$ also holds. However, none of these embeddings are compact.

The weighted spaces and the locally uniform spaces are connected through the following equivalence of norms. The proof can be found e.g. in Grasselli et al. [2010]. Let $k \in \mathbb{N}_0$, $p \in [1, \infty)$, $\varepsilon > 0$. Let ϕ be a weight function of growth rate $0 \leq \mu < \varepsilon$ and $u \in W^{k,p}_{\text{loc}}(\mathbb{R}^d)$. Then $u \in W^{k,p}_{b,\phi}(\mathbb{R}^d)$ if and only if $u \in W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)$ for every $\bar{x} \in \mathbb{R}^d$ and

$$\sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x})^{1/p} \|u\|_{W^{k,p}_{\bar{x},\varepsilon}} < \infty. \tag{1.4}$$

Moreover, the left-hand side of (1.4) defines a norm equivalent to the $W^{k,p}_{b,\phi}(\mathbb{R}^d)$-norm.

We can also easily show that on $L^2_b(\mathbb{R}^d)$-bounded sets the $L^2_{\text{loc}}(\mathbb{R}^d)$-topology is equivalent to the weighted topology.

Lemma 1.1. *Let $\mathcal{B} \subseteq L^2_b(\mathbb{R}^d)$ and $u_n, u \in \mathcal{B}$, then for every $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$*

$$u_n \to u \text{ in } L^2_{\bar{x},\varepsilon}(\mathbb{R}^d) \Leftrightarrow u_n \to u \text{ in } L^2_{\text{loc}}(\mathbb{R}^d).$$

For $\mathcal{O} \subseteq \mathbb{R}^d$ we denote $\mathbb{I}(\mathcal{O}) = \{k \in \mathbb{Z}^d; C_k^1 \cap \mathcal{O} \neq \emptyset\}$ and define the seminorm $W_{b,\phi}^{k,p}(\mathcal{O})$ by

$$\|u\|_{W_{b,\phi}^{k,p}(\mathcal{O})}^p = \sup_{l \in \mathbb{I}(\mathcal{O})} \phi(l) \|u\|_{W^{k,p}(C_l^1)}^p. \tag{1.5}$$

1.3 Parabolic locally uniform spaces

Let $\ell > 0$ be fixed and denote $Q_\ell = (0, \ell) \times \mathbb{R}^d$. For a weight function ϕ we define the *parabolic locally uniform space* $L_{b,\phi}^2(0, \ell; L^2(\mathbb{R}^d))$ by

$$L_{b,\phi}^2(0, \ell; L^2(\mathbb{R}^d)) = \{u : Q_\ell \to \mathbb{R}; \|u\|_{L_{b,\phi}^2(0,\ell;L^2)}^2 = \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \|u\|_{L^2(0,\ell;L^2(C_{\bar{x}}^1))}^2 < \infty\}.$$

It is easy to see

$$L^2(0, \ell; L_{b,\phi}^2(\mathbb{R}^d)) \subsetneq L_{b,\phi}^2(0, \ell; L^2(\mathbb{R}^d)) \subsetneq L_{\mathrm{loc}}^2(Q_\ell).$$

We also define the space $L_{b,\phi}^2(0, \ell; W^{1,2}(\mathbb{R}^d))$ by

$$L_{b,\phi}^2(0, \ell; W^{1,2}(\mathbb{R}^d)) = \{u : Q_\ell \to \mathbb{R}; \\ \|u\|_{L_{b,\phi}^2(0,\ell;W^{1,2})}^2 = \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \|u\|_{L^2(0,\ell;W^{1,2}(C_{\bar{x}}^1))}^2 < \infty\}$$

and finally the space $L_{b,\phi}^2(0, \ell; W^{-1,2}(\mathbb{R}^d))$ by

$$L_{b,\phi}^2(0, \ell; W^{-1,2}(\mathbb{R}^d)) = \{u : Q_\ell \to \mathbb{R}; \\ \|u\|_{L_{b,\phi}^2(0,\ell;W^{-1,2})}^2 = \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \|u\|_{L^2(0,\ell;W^{-1,2}(C_{\bar{x}}^1))}^2 < \infty\}.$$

Similarly as for locally uniform space in (1.3) one may take the supremum over $k \in \mathbb{Z}^d$ instead of $\bar{x} \in \mathbb{R}^d$ and show that there is an equivalent norm relying on the weight functions $\phi_{\bar{x},\varepsilon}$. More precisely for a weight function ϕ of growth $\mu \geq 0$ and $\varepsilon > \mu$, by Theorem 2.4 in Grasselli et al. [2010] the space $L_{b,\phi}^2(0, \ell; L^2(\mathbb{R}^d))$ admits an equivalent norm

$$\|u\|_{L_{b,\phi}^2(0,\ell;L^2)}^2 \approx \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(t, x)|^2 \phi_{\bar{x},\varepsilon}(x) \, dx \, dt.$$

Similar equivalent norms can be found for the spaces $L_{b,\phi}^2(0, \ell; W^{1,2}(\mathbb{R}^d))$ and $L_{b,\phi}^2(0, \ell; W^{-1,2}(\mathbb{R}^d))$.

For $\mathcal{O} \subseteq \mathbb{R}^d$ we may also define the $L_{b,\phi}^2(0, \ell; L^2(\mathcal{O}))$-seminorm by

$$\|u\|_{L_{b,\phi}^2(0,\ell;L^2(\mathcal{O}))}^2 = \sup_{k \in \mathbb{I}(\mathcal{O})} \phi(k) \|u\|_{L^2(0,\ell;L^2(C_k^1))}^2 \tag{1.6}$$

with obvious extensions to $L_{b,\phi}^2(0, \ell; W^{1,2}(\mathcal{O}))$ and $L_{b,\phi}^2(0, \ell; W^{-1,2}(\mathcal{O}))$.

2. Attractors in locally uniform spaces

In this section we briefly review the basic objects of interest in the asymptotic analysis, namely the global attractor and the exponential attractor, and discuss the issues arising in the context of locally uniform spaces.

The asymptotic properties of autonomous evolutionary equations are studied using dynamical systems. In the rest of this section let (X, d) be a complete metric space and let $S(t) : X \to X$ for every $t \geq 0$. The pair $(X, S(t))$ is called a *dynamical system* if $S(t)$ is a semigroup, i.e. $S(0)$ is an identity on X, $S(t+s) = S(t)S(s) = S(s)S(t)$ for every s, $t \geq 0$, and the mapping $(t, x) \to S(t)x$ is continuous. We will often call the set X the *phase space* of the dynamical system $(X, S(t))$.

However, in the setting of locally uniform spaces we do not usually have the continuity in the phase space, in fact the solutions are in general not even strongly measurable. On the other hand the solution semigroup is usually continuous in the weighted spaces.

Recall that the (non-symmetric) distance between sets A, $B \subseteq X$ is defined by

$$\mathrm{dist}_X(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

2.1 Bi-space attractor

A set $\mathcal{A} \subseteq X$ is called the *global attractor* of the dynamical system $(X, S(t))$ if $\mathcal{A}$ is compact, invariant with respect to $S(t)$, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and $\mathcal{A}$ attracts bounded subsets of X, more precisely for $B \subseteq X$ bounded one has

$$\lim_{t \to \infty} \mathrm{dist}_X(S(t)B, \mathcal{A}) = 0.$$

Owing to the unboundedness of the spatial domain we cannot expect the attractor of evolution equations in locally uniform spaces to be compact in the locally uniform topology. This leads to the definition of the so-called bi-space attractor.

Definition. *Let (X, d) be a complete metric space and $(X, S(t))$ a dynamical system. Let τ be a topology on X weaker than the topology generated by the metric d. A set $\mathcal{A} \subseteq X$ is called a* $((X, d), (X, \tau))$-attractor *of the dynamical system $(X, S(t))$ if*

1. *$\mathcal{A}$ is bounded in (X, d) and compact in (X, τ),*
2. *$\mathcal{A}$ is invariant w.r.t. $S(t)$, in other words $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$,*
3. *$\mathcal{A}$ attracts bounded subsets of (X, d) w.r.t. to the topology τ, i.e. for every $B \subseteq X$ bounded in (X, d) and for every $\mathcal{O} \in \tau$ such that $\mathcal{A} \subseteq \mathcal{O}$ there exists $T_B > 0$ such that $S(t)B \subseteq \mathcal{O}$ for all $t \geq T_B$.*

In the context of differential equations in unbounded domains the space X is usually a (Sobolev) locally uniform space, for now denoted by Φ_b, and the topology τ is the respective local topology Φ_{loc}. We then speak of the (Φ_b, Φ_{loc})-*attractor* or the *locally compact attractor*. Also recall that by Lemma 1.1 the local topology Φ_{loc} on Φ_b-bounded sets is equivalent to the weighted topology, which further stresses out the importance of weighted spaces as a useful analytical tool.

It is well-known that the existence of the global attractor is equivalent to the asymptotic compactness, i.e. the relative compactness of the set

$$\{S(t_n)x_n; n \in \mathbb{N}\} \qquad \text{for every } t_n \to \infty \text{ and } \{x_n\}_{n=1}^{\infty} \subseteq X \text{ bounded,}$$

and dissipativity, i.e. the existence of a bounded (absorbing) set $\mathcal{B}$, which is a set such that for every bounded set $B \subseteq X$ one has $S(t)B \subseteq \mathcal{B}$ for $t \geq t_B = t_B(B)$. The existence criteria for the bi-space attractors are analogous, cf. Section 2 in Babin and Vishik [1992].

2.2 Kolmogorov's ε-entropy

The complexity of global attractors is often described by the *fractal dimension*. Let $A \subseteq X$ be precompact, then we define the fractal dimension of A by

$$\dim_f(A) = \limsup_{\varepsilon \to 0} \frac{\ln N_\varepsilon(A, X)}{\ln \frac{1}{\varepsilon}},$$

where $N_\varepsilon(A, X)$ is the smallest number of balls of diameter ε that cover A in X. Owing to the noncompacntess of locally compact attractors in the locally uniform topology, we see that the fractal dimension will not be of much use considering problems posed in locally uniform spaces over unbounded domains.

The idea is to study the complexity of the locally compact attractor only locally. To this end we define the *Kolmorov's ε-entropy* $H_\varepsilon(A, X)$ by

$$H_\varepsilon(A, X) = \ln N_\varepsilon(A, X).$$

The finiteness of the fractal dimension of the global attractor is then replaced by a particular bound on Kologorov's ε-entropy of the locally compact attractor $\mathcal{A}$ of the form

$$H_\varepsilon\left(\mathcal{A}|_{C^R_{\bar{x}}}, \Phi_b(C^R_{\bar{x}})\right) \leq C\left(R + L \ln \frac{\varepsilon_0}{\varepsilon}\right)^d \ln \frac{\varepsilon_0}{\varepsilon} \tag{2.1}$$

holding for all $\bar{x} \in \mathbb{R}^d$, $R \geq 1$, $\varepsilon \in (0, \varepsilon_0)$ with constants C, L, $\varepsilon_0 > 0$ independent of $\bar{x}$, R and ε. This type of bound has been shown to be optimal for the reaction-diffusion equation and the wave equation, cf. Zelik [2001b] and Zelik [2001a].

2.3 Infinite dimensional exponential attractor

The global (or bi-space) attractor describes the limit asymptotic behaviour of the dynamical system. On the other hand the global and bi-space attractor may attract the solutions arbitrarily slowly and in general it is not possible to express the rate of attraction in terms of the data of the problem. The lack of information on the rate of attraction is solved by the concept of the exponential attractor.

Recall that a set $\mathcal{E} \subseteq X$ is called the *exponential attractor* of the dynamical system $(X, S(t))$ if $\mathcal{E}$ is compact, positively invariant, i.e. $S(t)\mathcal{E} \subseteq \mathcal{E}$ for all $t \geq 0$, $\dim_f(\mathcal{E}) < \infty$ and the set $\mathcal{E}$ exponentially attracts bounded subsets of X, more precisely if there exist a monotone increasing function $Q : [0, \infty) \to (0, \infty)$ and $\gamma > 0$ such that for every $B \subseteq X$ bounded

$$\operatorname{dist}_X(S(t)B, \mathcal{E}) \leq Q\left(\|B\|_X\right) e^{-\gamma t}, \qquad t \geq 0.$$

A necessary and sufficient condition for the existence of a discrete exponential attractor has been given in Pražák [2003] by the means of uniformly good covering of the images of the absorbing set.

Theorem (Theorem 2.1, Pražák [2003])**.** *Let $S : X \to X$ be Lipschitz. Then a discrete dynamica system (X, S^n) admits a discrete exponential attractor if and only if there exist constants a, $b > 0$, $\eta \in (0, 1)$ and $K > 1$ such that*

$$N_{a\eta^n}(S^n\mathcal{B}, X) \leq bK^n \qquad \textit{for all } n \in \mathbb{N},$$

where $\mathcal{B}$ is the absorbing set of the discrete dynamical system (X, S^n).

An analogous concept to the exponential attractor in the context of evolution equations posed in locally uniform spaces over unbounded domains is the infinite dimensional exponential attractor introduced in Efendiev et al. [2004] for the reaction-diffusion equation. Compared to the classical exponential attractor, one requires that the infinite dimensional exponential attractor is only locally compact and that an entropy bound similar to (2.1) holds instead of finite fractal dimension. The infinite dimensional exponential attractor as defined in Efendiev et al. [2004] also has the advantage of attraction in the locally uniform norm rather than only locally.

We will refrain from defining the infinite dimensional attractor in an abstract setting in this section and rather refer the reader to Section 3.2, where we present a rather obvious, nevertheless abstract definition of the infinite dimensional exponential attractor together with a sufficient and necessary condition on its existence. We also discuss the applicability of the abstract model on other equations.

3. Review of the results

In this section we review the results of the papers [I], [II], [III] and [IV]. The obtained results often rely in some way on the method of ℓ-trajectories described in detail in Málek and Pražák [2002]. The method aims to study the asymptotic properties of the solution semigroup through a different semigroup defined on the space of *trajectories of solutions* over a finite[1] time interval. This often allows us to obtain results that can be directly obtained in a higher regularity phase space even in a lower regularity setting.

We remark that in the rest of this section we assume that all the absorbing sets are positively invariant. This can be done without any limitations by a standard procedure. Also for every function space defined in the following by taking supremum over $\bar{x} \in \mathbb{R}^d$ or $k \in \mathbb{Z}^d$, we automatically define respective seminorms over $\mathcal{O} \subseteq \mathbb{R}^d$ similarly as in (1.5) and (1.6).

3.1 Paper [I]

In [I] we have studied the nonlocal differential equation

$$u_t - \operatorname{div} a(\nabla u) + du = F(u^t), \tag{3.1}$$

where $a : \mathbb{R}^N \to \mathbb{R}^N$ represents a nonlinear diffusion, $d, r > 0$, $u^t \equiv u(t+\theta)$ for $\theta \in [-r, 0]$ and

$$F(u^t) = \int_{-r}^{0} \left(\int_{\mathbb{R}^N} b(u(t+\theta, y)) f(x-y) e^{-|x-y|}\, dy \right) \xi(\theta, u(t), u^t)\, d\theta,$$

where b, f and ξ are nonlinear functions specified later. The equation (3.1) is supplemented with the initial data

$$u(0) = u_0 \in L^2_b(\mathbb{R}^N), \qquad u|_{(-r,0)} = \psi \in L^2_b(-r, 0; L^2(\mathbb{R}^N)).$$

We assume that the nonlinear diffusion $a \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfies

$$a(0) = 0, \qquad (a(\zeta) - a(\eta)) \cdot (\zeta - \eta) \geq \kappa |\zeta - \eta|^2, \qquad \forall \zeta, \eta \in \mathbb{R}^N, \tag{3.2}$$

$$|a(\zeta) - a(\eta)| \leq \kappa\gamma |\zeta - \eta|, \qquad \forall \zeta, \eta \in \mathbb{R}^N, \tag{3.3}$$

$$\zeta \to a(\zeta) \cdot \zeta \text{ is a convex function on } \mathbb{R}^N, \tag{3.4}$$

for some $\kappa > 0$, $\gamma \geq 1$. The function $b : \mathbb{R} \to \mathbb{R}$ is assumed to be bounded and Lipschitz continuous and the function $f : \mathbb{R}^N - \mathbb{R}^N \to \mathbb{R}$ to be bounded. Let us denote the natural phase space of the equation (3.1) by

$$H = L^2_b(\mathbb{R}^N) \times L^2_b(-r, 0; L^2(\mathbb{R}^N)).$$

The function $\xi : (-r, 0) \times H \to \mathbb{R}$ governing the distributed delay satisfies the following two conditions:

[1] The method is sometimes called "the method of short trajectories" to distinguish it from methods working with whole trajectories over $(-\infty, \infty)$ or $[0, \infty)$. However, when applied to a wave equation (see for example [III]), these "short" trajectories have to be sufficiently long.

1. For every $M > 0$ there exists $L = L(M) > 0$ such that for every $\bar{x} \in \mathbb{R}^N$ and $(v^i, \psi^i) \in H$, $i = 1, 2$, satisfying

$$\|v^i\|_{\bar{x},\varepsilon}^2 + \int_{-r}^{0} \|\psi^i(\theta)\|_{\bar{x},\varepsilon}^2 \, d\theta \le M, \qquad i = 1, 2,$$

   the estimate

$$\int_{-r}^{0} |\xi(\theta, v^1, \psi^1) - \xi(\theta, v^2, \psi^2)| \, d\theta \\ \le L \left(\|v^1 - v^2\|_{\bar{x},\varepsilon}^2 + \int_{-r}^{0} \|\psi^1(\theta) - \psi^2(\theta)\|_{\bar{x},\varepsilon}^2 \, d\theta \right)^{1/2}$$

   holds for every $\varepsilon > 0$, $\bar{x} \in \mathbb{R}^N$.

2. The function $\xi(\cdot, v, \psi)$ is $L^2(-r, 0)$-integrable in the first variable uniformly w.r.t. $(v, \psi) \in H$, in other words

$$\|\xi(\cdot, v, \psi)\|_{L^2(-r,0)} \le C_\xi$$

   for some $C_\xi > 0$ and all $(v, \psi) \in H$.

The weak solution u satisfies the equation (3.1) in distributions over $(0, \infty) \times \mathbb{R}^N$ and has the regularity

$$u \in C([0, T], L^2_{\bar{x},\varepsilon}(\mathbb{R}^N)) \cap L^2(-r, 0; L^2_{\bar{x},\varepsilon}(\mathbb{R}^N)) \cap L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^N))$$

for all $\varepsilon > 0$ sufficiently small, all $\bar{x} \in \mathbb{R}^N$ and every weight function ϕ of growth rate smaller than ε. However, the solution as a function $u : [0, T] \to L^2_{b,\phi}(\mathbb{R}^d)$ is not in general strongly measurable.

In the paper we show that the equation (3.1) is well-posed and generates a dissipative semigroup $S(t)$. We also establish the existence of a locally compact, more precisely a $(L^2_b(-r, 0; L^2(\mathbb{R}^N)), L^2_{\text{loc}}((-r, 0) \times \mathbb{R}^N))$-attractor $\mathcal{A}$ of the dynamical system $(X, S(t))$, where

$$X = \{u \in C([-r, 0], L^2_{\bar{x},\varepsilon}(\mathbb{R}^N)); u \text{ is a weak solution of (3.1)}\}$$

is the reduced phase space equipped with the $L^2(-r, 0; L^2_{\bar{x},\varepsilon}(\mathbb{R}^d))$-topology for arbitrary $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ sufficiently small. We also show that the usual entropy estimate

$$H_\varepsilon \left(\mathcal{A}|_{B(\bar{x},R)}, L^2_b(-r, 0; B(\bar{x}, R)) \right) \le C \left(R + \tilde{L} \ln \frac{\varepsilon_0}{\varepsilon} \right)^N \ln \frac{\varepsilon_0}{\varepsilon} \tag{3.5}$$

holds for some C, $\tilde{L}$, $\varepsilon_0 > 0$ and all $R \ge 1$, $\bar{x} \in \mathbb{R}^N$ and $\varepsilon \in (0, \varepsilon_0)$. We have used the method of Grasselli et al. [2010] adapted to the delayed equation (3.1).

Let us discuss the method in more detail. As usual we restrict ourselves to the dynamics on the absorbing set, in this case to the trajectories starting in the absorbing set. We define the space of trajectories $\mathcal{B}_\ell$ by

$$\mathcal{B}_\ell = \{\chi \in C([-r, \ell], L^2_{\bar{x},\varepsilon}(\mathbb{R}^N)); \chi \text{ solves (3.1) in } [0, \ell] \text{ and } (\chi(0), \chi|_{(-r,0)}) \in \mathcal{B}\},$$

where is $\varepsilon > 0$ sufficiently small, $\ell > 0$ and $\bar{x} \in \mathbb{R}^N$ are fixed and $\mathcal{B} \subseteq H$ is an absorbing set of the semigroup $S(t)$. We equip the space $\mathcal{B}_\ell$ with the topology $L^2(-r, \ell; L^2_{\bar{x},\varepsilon}(\mathbb{R}^N))$. By a variant of Lemma 1.1[2] the topology on $\mathcal{B}_\ell$ is equivalent to the local topology $L^2_{\mathrm{loc}}((-r, 0) \times \mathbb{R}^N)$. The trajectory solution semigroup $L(t) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ and the end-point mapping $e : \mathcal{B}_\ell \to \mathcal{B}$ are then defined by

$$[L(t)\chi](s) = u(t + s),\ s \in [-r, \ell], \qquad e(\chi) = [L(r + \ell)\chi]|_{[-r,0]}, \qquad \chi \in \mathcal{B}_\ell,$$

where u is the unique solution of (3.1) from the definition of $\mathcal{B}_\ell$.

The mapping $e : L^2_{b,\phi}(-r, \ell; L^2(\mathbb{R}^N)) \to L^2_{b,\phi}(-r, 0; L^2(\mathbb{R}^N))$ is Lipschitz continuous and the mapping $L = L(r + \ell)$ has a smoothing property, more precisely, denoting

$$\|\chi\|_{\mathcal{W}_{b,\phi}} = \|\chi\|_{L^2_{b,\phi}(-r,\ell;W^{1,2})} + \|\partial_t\chi\|_{L^2_{b,\phi}(-r,\ell;W^{-1,2})},$$

that the mapping $L : L^2_{b,\phi}(-r, \ell; L^2(\mathbb{R}^N)) \to \mathcal{W}_{b,\phi}$ is Lipschitz continuous for a weight function ϕ of sufficiently small growth.

We prove that the dynamical system $(\mathcal{B}_\ell, L(t))$ has a global attractor $\mathcal{A}_\ell$ and that $\mathcal{A} = e(\mathcal{A}_\ell)$ is the $(L^2_b(-r, 0; L^2(\mathbb{R}^N)), L^2_{\mathrm{loc}}((-r, 0) \times \mathbb{R}^N))$-attractor of the dynamical system $(X, S(t))$. By the Lipschitz continuity of the end-point mapping e it suffices to establish an upper bound of the Kolmogorov's ε-entropy of the trajectory attractor $\mathcal{A}_\ell$ similar to (3.5). Such a bound can be obtained using the smoothing property of the trajectory semigroup L and the following explicit version of the Aubin-Lions lemma similar to the one in Zelik [2001b].

Lemma (Lemma 2.6, Grasselli et al. [2010])**.** *Let φ be a weight function and let $\mathcal{O} \subseteq \mathbb{R}^N$ satisfy*

$$\#(\mathcal{O}) \le C_0 \operatorname{vol}(\mathcal{O}). \tag{3.6}$$

Let $R > 0$ and $\theta \in (0, 1)$. Then there exists $C_1 > 0$ such that

$$H_{\theta R}\left(LB_R(\chi; \mathcal{W})|_{\mathcal{O}}, L^2_{b,\phi}(-r, \ell; L^2(\mathcal{O})\right) \le C_1 \operatorname{vol}(\mathcal{O}).$$

The constant C_1 depends on C_0, ℓ, θ and the constants μ, C_ϕ, $\tilde{C}_\phi$ in (1.1), (1.2) *but is independent of χ, R, r, $\mathcal{O}$ and the particular form of the function ϕ as long as* (1.1),(1.2) *and* (3.6) *are satisfied.*

We will see similar instances of this lemma in the following papers as they are crucial in obtaining the entropy estimates.

3.2 Paper [II]

In Efendiev et al. [2004] the authors show that solution semigroup of the reaction-diffusion equation

$$u_t - a\Delta u + (\mathbf{L}, \nabla)u + f(u) + \lambda_0 u = g$$

[2]In the paper [I] we use Lemma 2.9. However, there is an error in the paper as we in fact require a slightly different version. Instead of the requirement that $\mathcal{B} \subseteq L^\infty(-r, \ell; L^2_b(\mathbb{R}^d))$, which is not satisfied by the solutions, we should require that

$$\sup_{\bar{x}\in\mathbb{R}^d} \sup_{t\in(-r,\ell)} \|u(t)\|_{\bar{x},\varepsilon} \le C_{\mathcal{B}}, \qquad \text{uniformly for } u \in \mathcal{B}.$$

posed in the space $W_b^{2,p}(\mathbb{R}^d)$ with $p > \max\{2, d/2\}$, where $a \in \mathbb{R}^{d\times d}$ is a constant diffusion matrix with positive symmetric part, $\mathbf{L}$ is a suitable vector field in $\mathbb{R}^d$, $f : \mathbb{R} \to \mathbb{R}$ is a suitable reaction function and $g \in L_b^p(\mathbb{R}^d)$ is an external force, admits an infinite dimensional exponential attractor. The infinite exponential exponential attractor $\mathcal{E} \subseteq W_b^{2,p}(\mathbb{R}^d)$ is by definition bounded in $W_b^{2,p}(\mathbb{R}^d)$, compact in $W_{\text{loc}}^{2,p}(\mathbb{R}^d)$, positively invariant under $S(t)$, exponentially attracts bounded sets in $W_b^{2,p}(\mathbb{R}^d)$ in the locally uniform topology $W_b^{2,p}(\mathbb{R}^d)$ and satisfies the entropy estimate

$$H_\varepsilon\left(\mathcal{E}|_{C^R_{\bar{x}}}, W_b^{2,p}(C^R_{\bar{x}}))\right) \le C\left(R + \tilde{L}\ln\frac{\varepsilon_0}{\varepsilon}\right)^d \ln\frac{\varepsilon_0}{\varepsilon}$$

for some C, $\tilde{L}$, $\varepsilon_0 > 0$ and all $\bar{x} \in \mathbb{R}^d$, $R > 0$ and $\varepsilon \in (0, \varepsilon_0)$. The procedure relies on the embedding $W_b^{2,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Two questions naturally occur: firstly, whether the existence of an infinite dimensional exponential attractor can be established working in the less regular phase space $L_b^2(\mathbb{R}^d)$, and secondly whether one can find a necessary and sufficient condition for the existence of the infinite dimensional exponential attractor similar to the one in Pražák [2003] and apply this criterion directly to different problems.

Using an abstract model of a locally uniform space described below we show in [II] that there is indeed a necessary and sufficient condition for the existence of an infinite dimensional exponential attractor that can be applied to the nonlinear reaction-diffusion equation

$$u_t - \operatorname{div} a(\nabla u) + f(u) + h(\cdot, \nabla u) = g \tag{3.7}$$

posed in the phase space $L_b^2(\mathbb{R}^d)$ with $d \le 3$, where the nonlinear functions $a \in C(\mathbb{R}^d, \mathbb{R}^d)$, $f \in C(\mathbb{R}, \mathbb{R})$, $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are specified below and $g \in L_b^2(\mathbb{R}^d)$ is an external force. However, a direct application of the criterion to problems such as the semilinear damped wave equation is not possible.

The nonlinear diffusion a is assumed to satisfy (3.2–3.4). The nonlinear reaction f satisfies $f(0) = 0$ and

$$\begin{aligned}
|f(r) - f(s)| \le C_1 (1 + |r| + |s|)^{p-2} |r - s|, &\qquad \text{for all } r, s \in \mathbb{R},\\
(f(r) - f(s))(r - s) \ge -C_2|r - s|^2, &\qquad \text{for all } r, s \in \mathbb{R},\\
C_3|r|^p - C_4 \le f(r)r \le C_5(|r|^p + 1), &\qquad \text{for all } r \in \mathbb{R},
\end{aligned}$$

for some $p \in (2, \infty)$ and $C_i > 0$. Finally the nonlinear drift $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is such that

$$\begin{gathered}
\text{the function } \xi \to h(x, \xi) \text{ is globally Lipschitz for a.e. } x \in \mathbb{R}^d,\\
\text{the function } x \to h(x, \xi) \text{ is measurable and essentially bounded for all } \xi \in \mathbb{R}^d.
\end{gathered}$$

We assume the following abstract model of a locally uniform space. Let X_k be closed subsets of some Banach space $\tilde{X}_k$ for every $k \in \mathbb{Z}^d$ and let X_b be an abstract locally uniform space defined by

$$X_b = \prod_{k\in\mathbb{Z}^d} X_k \quad \text{equipped with the norm} \quad \|x\|_{X_b} = \sup_{k\in\mathbb{Z}^d}\|x\|_{X_k}.$$

For $K \subseteq \mathbb{Z}^d$ we will denote

$$X_b(K) = \prod_{k\in K} X_k \quad \text{with the corresponding seminorm} \quad \|x\|_{X_b(K)} = \sup_{k\in K}\|x\|_{X_k}.$$

We emphasize that the above definition implicitly assumes that for $x_k \in X_k$ there exists $x \in X_b$ such that $x|_{\{k\}} = x_k$ for every $k \in \mathbb{Z}^d$, which allows for splicing of local elements x_k, $k \in \mathbb{Z}^d$, into one global element $x \in X_b$. This requirement prevents the criterion stated below from direct application to the wave equation posed in the space $W_b^{1,2}(\mathbb{R}^d) \times L_b^2(\mathbb{R}^d)$, where one cannot join the locally $W^{1,2}$-functions into a $W_b^{1,2}$-function without leaving the absorbing set, or to the trajectory spaces, where we cannot expect that solutions on (spatially) bounded domains with different initial data can be joined together into another solution.

We define the local topology X_{loc} by

$$x_n \to x \text{ in } X_{\text{loc}} \quad \Leftrightarrow \quad x_n|_K \to x|_K \text{ in } X_b(K) \text{ for every } K \subseteq \mathbb{Z}^d \text{ finite.}$$

With a slight abuse of notation we denote the cubes of side $R > 0$ in $\mathbb{Z}^d$ centred in $k \in \mathbb{Z}^d$ by

$$C_k^R = \{j \in \mathbb{Z}^d;\ \max_{i=1,\dots,d} |j_i - k_i| \le R/2\}.$$

Let $S : X_b \to X_b$ be an operator. We define the discrete infinite dimensional exponential attractor in the spirit of Efendiev et al. [2004].

Definition. *A set $\mathcal{E} \subseteq X_b$ is called a* discrete infinite dimensional exponential attractor *of the discrete dynamical system (X_b, S) if*

1. *$\mathcal{E}$ is bounded in X_b and compact in X_{loc},*
2. *$\mathcal{E}$ is positively invariant under S, i.e. $S\mathcal{E} \subseteq \mathcal{E}$,*
3. *$\mathcal{E}$ exponentially attracts bounded sets in X_b, i.e. there exist $\gamma > 0$ and a monotone increasing function $Q : \mathbb{R}_0^+ \to \mathbb{R}^+$ such that for every $B \subseteq X_b$ bounded and $n \in \mathbb{N}$ one has*

$$\operatorname{dist}_{X_b}(S^n B, \mathcal{E}) \le Q(\|B\|_{X_b}) e^{-\gamma n},$$

4. *there exist ε_0, C, $L > 0$ such that for every $k \in \mathbb{Z}^d$, $R \ge 1$ and $\varepsilon \in (0, \varepsilon_0)$ the estimate*

$$H_\varepsilon \left(\mathcal{E}|_{C_k^R}, X_b(C_k^R) \right) \le C \left(\# C_k^{R + L \ln \varepsilon_0/\varepsilon} \right) \ln \frac{\varepsilon_0}{\varepsilon}$$

holds with constants C, L and ε_0 independent of k, R and ε.

In Theorem 3.1 in [II] we give the abstract necessary and sufficient condition on the existence of an exponential attractor similar to an analogous criterion for exponential attractors from Pražák [2003], cf. Section 2.3. The proof is similar to the original proof in Efendiev et al. [2004].

Theorem. *Let $S : X_b \to X_b$ be Lipschitz continuous and let $\mathcal{B} \subseteq X_b$ be an absorbing set of the discrete dynamical system (X_b, S), i.e. for every bounded $B \subseteq X_b$ there exists $N = N(B)$ such that for every $n \ge N$ one has $S^n(B) \subseteq \mathcal{B}$. Let there exist α, $L > 0$ such that*

$$\|S^n u - S^n v\|_{X_k} \le L^n \sup_{l \in \mathbb{Z}^d} e^{-\alpha |k-l|} \|u - v\|_{X_l} \tag{3.8}$$

for every $k \in \mathbb{Z}^d$, $n \in \mathbb{N}$ and u, $v \in \mathcal{B}$. Then the dynamical system (X_b, S) has a discrete infinite dimensional exponential attractor if and only if for some ε_0, $L' > 0$ and $\theta \in (0,1)$ and every $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ we have

$$H_{\varepsilon_0\theta^{-n}}\left((S^n\mathcal{B})|_{C_k^n}, X_b(C_k^n)\right) \le C(\#C^{n+L'n})n \tag{3.9}$$

with the constant L' independent of $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$.

The assumption (3.8) quantifies the decay of the effect distant cubes have on the cube C_k^1. It is easy to show that for example a finite speed of propagation implies the estimate (3.8). We note that estimates of the form (3.8), for which we use the term "exponentially finite speed of propagation", arise naturally in the equations in locally uniform spaces.

We then use the above criterion to show that the solution semigroup $S(t) : L_b^2(\mathbb{R}^d) \to L_b^2(\mathbb{R}^d)$ of the nonlinear reaction-diffusion equation (3.7) admits an infinite dimensional exponential attractor in the sense of the above definition with $X_b = L_b^2(\mathbb{R}^d)$ and $X_{\text{loc}} = L_{\text{loc}}^2(\mathbb{R}^d)$. The crucial entropy bound (3.9) is obtained by studying the properties of the trajectory semigroup $L(t)$. For details on the proof, see Theorem 4.4, [II].

3.3 Paper [III]

In [III] we have studied the semilinear wave equation with weak nonlinear damping

$$u_{tt} + g(u_t) - \Delta u + \alpha u + f(u) = h(t), \tag{3.10}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are nonlinear functions, $\alpha > 0$ and $h \in L_b^2(0, \infty; L^2(\mathbb{R}^d))$. The equation is coupled with initial data

$$u(0) = u_0 \in W_b^{1,2}(\mathbb{R}^d), \qquad u_t(0) = u_1 \in L_b^2(\mathbb{R}^d).$$

The nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ is assumed to satisfy $f' \ge -\beta$,

$$\forall r \in \mathbb{R} : |f'(r)| \le \gamma_1(|r|^{p-1} + 1), \qquad \text{and} \qquad \liminf_{|r|\to\infty} f(r)/r > 0$$

for some β, $\gamma_1 > 0$. The nonlinear damping $g \in C^1(\mathbb{R}, \mathbb{R})$ is such that $g(0) = 0$, $g' \ge \gamma_5 > 0$ and

$$\gamma_2|r|^{\mu+1} - \gamma_3 \le g(r)r \le \gamma_4(|r|^{\mu+1} + 1), \qquad \forall r \in \mathbb{R},$$

with $\gamma_i > 0$. We consider the following set of parameters:

$$p \in \left(0, \frac{d}{d-2}\right] \quad \text{for } d > 2, \qquad p \in (0, \infty) \quad \text{for } d = 2, \quad \mu \in [1, \infty), \tag{3.11}$$

Let us denote

$$\Phi_b = W_b^{1,2}(\mathbb{R}^d) \times L_b^2(\mathbb{R}^d), \qquad \Phi_{\text{loc}} = W_{\text{loc}}^{1,2}(\mathbb{R}^d) \times L_{\text{loc}}^2(\mathbb{R}^d). \tag{3.12}$$

A weak solution u is defined to satisfy the equation in distributions over $(0, \infty) \times \mathbb{R}^d$ and has the regularity

$$(u, u_t) \in C([0, \infty), W_{\bar{x},\varepsilon}^{1,2}(\mathbb{R}^d) \times L_{\bar{x},\varepsilon}^2(\mathbb{R}^d)), \qquad u_t \in L^{\mu+1}(0, \infty; L_{\bar{x},\varepsilon}^{\mu+1}(\mathbb{R}^d)),$$

for every $\varepsilon > 0$ and $\bar{x} \in \mathbb{R}^d$. Again the solution $(u, u_t) : [0, T] \to \Phi_b$ is in general not measurable.

We prove that the equation (3.10) is well-posed in the full spectrum of parameters (3.11) and, under additional assumptions, generates a dissipative semigroup $S(t)$. More precisely we show the semigroup $S(t)$ has a $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor if either $\mu = 1$, or $\mu \in (1, (d+2)/(d-2))$ and

$$-g(r)s \leq \kappa f(s)s + C(g(r)r + 1) \qquad \forall r, s \in \mathbb{R}$$

for some $\kappa \in (0, 1)$ and $C > 0$. Even though this assumption allows the use of polynomials such as

$$g(r) = r|r|^{\mu-1}, \quad f(s) = s|s|^{p-1} - as, \quad \text{with } \mu \in [1, 3),\ p \in [\mu, 3),$$

for $a > 0$ small and $d = 3$, it is far from optimal. Under yet another additional assumptions, namely $\mu \in [1, 7/3)$ and

$$C(1 + |r|)^{\mu-1} \leq g'(r) \leq C'(1 + |r|)^{\mu-1}, \qquad r \in \mathbb{R},$$

we establish the usual entropy estimate

$$H_\varepsilon\left(\mathcal{A}|_{B(\bar{x},R)}, \Phi_b(B(\bar{x}, R))\right) \leq C\left(R + \tilde{L}\ln\frac{\varepsilon_0}{\varepsilon}\right)^d \ln\frac{\varepsilon_0}{\varepsilon} \tag{3.13}$$

holding for all $\bar{x} \in \mathbb{R}^d$, $R \geq 1$ and $\varepsilon \in (0, \varepsilon_0)$ with constants C, $\tilde{L}$, $\varepsilon_0 > 0$. A similar result for linear damping has been obtained in Zelik [2001a]. We also remark that the well-posedness of the equation (3.10) for the full spectrum of the parameters (3.11) and the existence of a $(\Phi_b, \Phi_{\mathrm{loc}})$-attractor for linearly bounded damping g, i.e. with $\mu = 1$, was shown in the diploma thesis of M. Michálek, one of the authors of [III].

The well-posedness has been shown by a suitable approximation. The asymptotic results were again obtained using the method of trajectories. Compared to the nonlocal equation in [I] or the reaction-diffusion equation in [II], the trajectory semigroup does not possess a smoothing property. However, it is possible to show that the trajectory semigroup has a squeezing property using the finite speed of propagation. To this end we define a cone-version of locally uniform spaces in the following way:

Let $\ell > 1$ and $v > 1$ be fixed and let ϕ be a weight function. For $k \in \mathbb{Z}^d$ denote

$$Z_k(t) = B(k, v(2\ell - t)), t \in (0, 2\ell), \qquad \widetilde{Z}_k(t) = B(k, v(3\ell - t)), t \in (0, 3\ell).$$

We define the cone-version of locally uniform spaces by

$$\mathcal{E}^{\ell,v}_{b,\phi} = \{(\chi, \chi_t); \chi : (0, \ell)\times\mathbb{R}^3 \to \mathbb{R}, \|\chi\|^2_{\mathcal{E}^{\ell,v}_{b,\phi}} = \sup_{k\in\mathbb{Z}^d} \phi(x_k) \int_0^\ell \int_{Z_k(t)} E[\chi]dx\, dt < \infty\},$$

where E is the energy functional

$$E[u] = \frac{1}{2}\left(|u_t|^2 + |\nabla u|^2 + \alpha|u|^2\right). \tag{3.14}$$

The space of trajectories is then defined by

$$\mathcal{B}_\ell = \{(\chi, \chi_t) \in \mathcal{E}^{\ell,v}_{b,\phi}; \chi \text{ solves the equation (3.10) in } [0,\ell] \text{ with } (\chi(0), \chi_t(0)) \in \mathcal{B}\},$$

where $\mathcal{B}$ again denotes the absorbing set of the solution semigroup $S(t)$.

Similarly as before we define the operators $e : \mathcal{B}_\ell \to \mathcal{B}$ and $L(t) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ by

$$\begin{aligned} e((\chi, \chi_t)) &= (\chi(\ell), \chi_t(\ell)), \\ [L(t)(\chi, \chi_t)](s) &= S(t+s)(\chi(0), \chi_t(0)), \quad s \in (0, \ell), \end{aligned} \tag{3.15}$$

and show that $e : \mathcal{E}^{\ell,v}_{b,\phi} \to \Phi_b$ and $L = L(\ell) : \mathcal{E}^{\ell,v}_{b,\phi} \to \mathcal{E}^{\ell,v}_{b,\phi}$ are Lipschitz continuous. Using the finite speed of propagation we then prove that the trajectory semigroup L has a *locally uniform squeezing property*, more precisely that for every weight function ϕ and every $\theta > 0$ there exist $\ell > 1$, $v > 1$, $\kappa > 0$ and $\mathcal{N} \subseteq \mathbb{Z}^d \cap B(0, 3v\ell)$ such that for every $k \in \mathbb{Z}^d$, χ_1, $\chi_2 \in \mathcal{B}_\ell$ and their respective solutions u_1, u_2 we have

$$\begin{aligned} \phi(k) \int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} E[w]\, dx\, dt &\le \theta \sum_{j \in \mathcal{N}(k)} \phi(j) \int_0^\ell \int_{Z_j(t)} E[w]\, dx\, dt \\ &+ \kappa \left(\phi(k) \int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} |w|^2\, dx\, dt + \sum_{j \in \mathcal{N}(k)} \phi(j) \int_0^\ell \int_{Z_j(t)} |w|^2\, dx\, dt \right), \end{aligned}$$

where $w = u_1 - u_2$ and

$$\mathcal{N}(k) = \{j \in \mathbb{Z}^d; j = i + k \text{ for some } i \in \mathcal{N}\}.$$

Using the locally uniform squeezing property we prove the following covering lemma which is then used to establish the entropy estimate (3.13) together with the Lipschitz continuity of the end-point mapping e.

Lemma (Lemma 6.1, [III])**.** *Let $\mathcal{O} \subseteq \mathbb{R}^3$ be bounded and satisfy* (3.6)*. Let $\varepsilon > 0$, $\delta \in (0,1)$ and $(x_0, x_1) \in \mathcal{B}$. Also let ϕ be a weight function. Then there exist $\ell, v > 1$ such that*

$$H_{\delta\varepsilon}\left((LB)\big|_{\mathcal{O}}, \mathcal{E}^{\ell,v}_{b,\phi}(\mathcal{O}) \right) \le C_1 \operatorname{vol}(\mathcal{O}),$$

where $B = B_\varepsilon((\chi_0, \chi_1); \mathcal{E}^{\ell,v}_{b,\phi}) \cap \mathcal{B}_\ell$ is a ball centered around the ℓ-trajectory (χ_0, χ_1) starting from (x_0, x_1). The constant C_1 depends only on C_0, ℓ and δ and is independent of (x_0, x_1), ε and $\mathcal{O}$ as long as (3.6) *is satisfied.*

3.4 Paper [IV]

In [IV] we have studied the strongly damped wave equation

$$u_{tt} + \beta u_t - \alpha \Delta u_t - \Delta u + f(u) = g, \tag{3.16}$$

where α, $\beta > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function specified later, coupled with the initial conditions

$$u(0) = u_0 \in W^{1,2}_b(\mathbb{R}^d), \qquad u_t(0) = u_1 \in L^2_b(\mathbb{R}^d).$$

For simplicity we choose $\alpha = \beta = 1$. We assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ and that there exist $C > 0$ and $0 \le q \le 4/(d-2)$ such that

$$|f(r) - f(s)| \le C|r-s|\left(1 + |r|^q + |s|^q\right), \qquad \forall r, s \in \mathbb{R}.$$

Moreover, let there exist $k \ge 1$ and $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0]$ we can find $C_\mu, C_0 \in \mathbb{R}$ such that

$$\begin{aligned} kF(s) + \mu s^2 - C_\mu \le sf(s), \qquad \forall s \in \mathbb{R},\\ -C_0 \le F(s), \qquad \forall s \in \mathbb{R}, \end{aligned}$$

where $F(s) = \int_0^s f(r)\,dr$. The nonlinearity is *critical* if $q = 4/(d-2)$ and *subcritical* if $q < 4/(d-2)$. We will use the notation (3.12) and define

$$W_{b,\phi} = W^{1,2}_{b,\phi}(\mathbb{R}^d) \times W^{1,2}_{b,\phi}(\mathbb{R}^d), \qquad W_{\text{loc}} = W^{1,2}_{\text{loc}}(\mathbb{R}^d) \times W^{1,2}_{\text{loc}}(\mathbb{R}^d).$$

In Yang and Sun [2009] the authors show that the equation (3.16) is well-posed in the space Φ_b and generates a dissipative semigroup $S(t)$. Further, the authors establish the existence of an invariant bounded closed set $\mathcal{A} \subseteq W^{2,2}_b(\mathbb{R}^d) \times W^{1,2}_b(\mathbb{R}^d)$ compact in W_{loc}, which attracts bounded sets of Φ_b in the W_{loc}-topology. We show that in the subcritical case the locally compact attractor $\mathcal{A}$ satisfies the entropy estimate

$$H_\varepsilon\left(\mathcal{A}|_{B(\bar{x},R)}, W_b(B(\bar{x},R))\right) \le C\left(R + \tilde{L}\ln\frac{\varepsilon_0}{\varepsilon}\right)^d \ln\frac{\varepsilon_0}{\varepsilon} \tag{3.17}$$

holding for every $R \ge 1$, $\varepsilon \in (0, \varepsilon_0)$ and $\bar{x} \in \mathbb{R}^d$ with the constants C, ε_0, $\tilde{L} > 0$ independent of $\bar{x}$, ε, R.

The method is similar to the one used in [III] – we rely on some form of the squeezing property. However, the strongly damped wave equation does not have a finite speed of propagation, so the particular form of the squeezing property and the covering lemma have to be slightly different. Again we define the trajectory space by

$$\mathcal{B}_\ell = \{(\chi, \chi_t); \chi \in L^2_{\text{loc}}((0,\ell) \times \mathbb{R}^d) \text{ solves (3.16) on } (0,\ell) \text{ with } (\chi(0), \chi_t(0)) \in \mathcal{B}\},$$

where $\mathcal{B}$ is the absorbing set of the semigroup $S(t)$, and define the end-point mapping $e : \mathcal{B}_\ell \to \mathcal{B}$ and the trajectory semigroup $L(t)$ by (3.15). Denoting

$$\begin{aligned} \Phi^\ell_{b,\phi} &= L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)) \times L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d)),\\ W^\ell_{b,\phi} &= L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)) \times L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)), \end{aligned}$$

we show that $e : \Phi^\ell_{b,\phi} \to W_{b,\phi}$ and $L = L(\ell) : \Phi^\ell_{b,\phi} \to W^\ell_{b,\phi}$ are again Lipschitz continuous. We then establish that the operator L has a *parabolic squeezing property*, more precisely that for a weight function ϕ of sufficiently small growth there exists $\varepsilon > 0$ such that for every $\gamma > 0$ we may find ℓ, κ, $R > 0$ so that for every χ_1, $\chi_2 \in \mathcal{B}_\ell$ and their respective solutions u_1 and u_2 we have

$$\begin{aligned} \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_\ell^{2\ell}\int_{\mathbb{R}^d} \left(E[w] + |\nabla w_t|^2\right)\phi_{\bar{x},\varepsilon}\,dx\,dt &\le \gamma \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell\int_{\mathbb{R}^d} E[w]\phi_{\bar{x},\varepsilon}\,dx\,dt\\ &+ \kappa\left(\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell\int_{B(\bar{x},R)} |w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell\int_{B(\bar{x},R)} |w_t|^2\,dx\,dt\right)\\ &+ \kappa\left(\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_\ell^{2\ell}\int_{B(\bar{x},R)} |w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_\ell^{2\ell}\int_{B(\bar{x},R)} |w_t|^2\,dx\,dt\right), \end{aligned}$$

where $w = u_1 - u_2$ and E is the energy functional from (3.14) with $\alpha = 1$.

The desired entropy estimate (3.17) is then obtained again by the Lipschitz continuity of the end-point mapping e and the following version of the covering lemma.

Lemma. *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be bounded satisfying* (3.6)*. Let $\varepsilon > 0$ and $\theta \in (0,1)$. Let $(u_0, u_1) \in \mathcal{B}$ and let $(\chi_0, (\chi_0)_t)$ be the trajectory starting from (u_0, u_1). Let ϕ be a weight function such that the operator L has the parabolic squeezing property for ϕ and denote $B = B_\varepsilon((\chi_0, (\chi_0)_t); \Phi^\ell_{b,\phi}) \cap \mathcal{X}$. Then there exist C_1, $\ell > 0$ such that*

$$H_{\theta\varepsilon}\left((LB)|_{\mathcal{O}}, W^\ell_{b,\phi}(\mathcal{O})\right) \leq C_1 \operatorname{vol}(\mathcal{O}),$$

where the constant C_1 depends only on C_0 and θ and is independent of (u_0, u_1), ε, ϕ and $\mathcal{O}$ as long as (3.6) *holds and the constants in* (1.1) *and* (1.2) *remain the same.*

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4. Paper [I]

Attractors and entropy bounds for a nonlinear RDEs with distributed delay in unbounded domains

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Attractors and entropy bounds for a nonlinear RDEs with distributed delay in unbounded domains*

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Abstract

A nonlinear reaction-diffusion problem, with a general, both spatially and delay distributed reaction term is considered in an unbounded domain R^N. The existence of a unique weak solutions is proved. A locally compact attractor together with entropy bounds is also established.

1 Introduction

We are interested the equation of the form

$$\partial_t u - \operatorname{div} a\,(\nabla u) + du = \int_{-r}^{0} \left(\int_{\Omega} b(u(t+\theta, y)) f(x-y) e^{-|x-y|}\, dy \right) \xi(\theta, u(t), u^t)\, dt \qquad (1.1)$$

where $d > 0$ and $u^t(\theta) \equiv u(t+\theta)$ for $\theta \in [-r, 0]$. The problem is posed in the unbounded spatial domain $x \in \Omega = \mathbb{R}^N$.

The equation can be seen as an abstract prototype of a nonlinear reaction diffusion system, which combines three nontrivial mathematical features: (i) nonlinear diffusion term $-\operatorname{div} a(\nabla u)$, (ii) temporally and spatially distributed delay terms and (iii) the setting of unbounded domains. We will begin by discussing the difficulties related to these three issues, together with a selection of recent references.

Let us start with the last point (iii). It can be said that the dynamics in unbounded domains has attracted a growing attention of the PDE community during the last decade. The problem obviously has an inherent non-compactness or even non-separability. This calls for a careful rethinking of the proper choice for the functional setting, so that the results on the global attractor and its finite-dimensionality, which are generic in *bounded domain* setting, can find a proper generalized expression. A natural choice seem to be some space of *uniformly* locally integrable functions, see [4], [20], [2], [1]. In such a setting, the existence of locally compact attractor admitting natural entropy estimates is the expected result; see also [10] and [3].

Concerning the point (ii), we would remark that presence of temporally and spatially non-local reaction lower order terms arise naturally in describing both living and non-living nature. We can mention the birth-death dynamics of maturing population or the spread of infection on the one hand, and the phenomena of yield or creep occurring in viscoelastic materials, or nonlocal interactions in phase transitions, on the other hand. The available mathematical techniques and results depend essentially on the complexity of non-local terms. In the case of

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linear delay of convolution type, linear techniques (theory of C_0-semigroups, linear stability results) can be used [7, 11, 5].

For a more general non-linear problems, perturbation and topological methods provide sufficient conditions for the existence of robust nontrivial structures like travelling waves, see [6], [18]. In the case of a bounded spatial domain, the existence of a global compact attractor was shown for the equation (1.1) in [14] with a linear diffusion $a(\nabla u) = \nabla u$; cf. also [13]. The existence of a global attractor for a similar linear equation in unbounded domain with $b(y, u(t + \theta, y))$ instead of $b(u(t + \theta, y))$ was proved in [9]. However, the authors in [9] study the equation in classical Sobolev spaces with $\xi \equiv 1$ and certain restrictions on d and r have to be met to obtain the existence of a compact attractor. Another similar linear equation with fixed delay and $N = 1$ was studied in [19] in the setting of bounded uniformly continuous functions. The existence of generalized attractors for delayed systems in unbounded domains was recently established in [17] and [16].

In the following we analyze the equation (1.1) in locally uniform spaces L^2_b in the spirit of [20]. The main advantage of this setting, as compared to standard Lebesgue or weighted Lebesgue spaces, is the possibility to capture arbitrary spatial complexity of the dynamics, including (spatially) periodic patterns. The spatial uniformity of L^2_b spaces, however, makes them similar to L^∞ spaces and thus not a good choice as a target spaces for the underlying dynamical system. For example, one cannot in general expect that the solution will be continuous with values in L^2_b. Several auxiliary weaker spaces are thus necessary to be introduced in the course of the analysis. Here in particular, following [8], we introduce a sort of parabolic version of uniformly local spaces $L^2_{b,1}(0;T;L^2)$ (see Section 2 below for definitions). The smoothing property of the dynamics can be easily proved in this parabolic setting, very much in the spirit of the so-called method of ℓ-trajectories. This leads to the existence and entropy estimates of the global (locally uniform) attractor $\mathcal{A}$. No higher order regularity estimates and in particular, no restrictions on d or r other than $r, d > 0$ are needed. This low-cost (in terms of regularity) approach also enables us to work with a more general assumption on the diffusion term (i): a general non-linear elliptic diffusion is possible, further generalizing the results common in the existing literature, where most often a *linear* dissipation (e.g. the Laplace operator) is considered.

The paper is organized as follows: the locally uniform spaces, corresponding duals and also their parabolic variants are briefly reviewed in Section 2. Existence and uniqueness of the weak solution are proven in Section 3. Locally compact attractor and its entropy estimates are established in Sections 4 and 5.

2 Function spaces and notation

Here we review the locally uniform spaces, following [20], [8].

Definition. *Let $\bar{x} \in \mathbb{R}^N$ and $\varepsilon > 0$. The* weighted Lebesgue space $L^2_{\bar{x},\varepsilon}(\Omega)$ *is defined by the norm*

$$\|u\|^2_{L^2_{\bar{x},\varepsilon}(\Omega)} \equiv \|u\|^2_{\bar{x},\varepsilon} = \int_\Omega |u(x)|^2 e^{-\varepsilon|x-\bar{x}|}\, dx. \tag{2.1}$$

Similarly the spaces $W^{1,2}_{\bar{x},\varepsilon}(\Omega)$ and $W^{-1,2}_{\bar{x},\varepsilon}(\Omega)$ are defined by the norms

$$\|u\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)} = \int_\Omega \left(|\nabla u(x)|^2 + |u(x)|^2\right) e^{-\varepsilon|x-\bar{x}|}\, dx, \tag{2.2}$$

$$\|u\|^2_{W^{-1,2}_{\bar{x},\varepsilon}(\Omega)} = \sup_v \int_\Omega u(x)v(x) e^{-\varepsilon|x-\bar{x}|}\, dx, \tag{2.3}$$

where the last supremum is taken over $v \in W^{1,2}_{\bar{x},\varepsilon}(\Omega)$ with unit norm.

We use the notation

$$(u, v)_{\varepsilon,\bar{x}} \equiv \int_\Omega u(x)v(x) e^{-\varepsilon|x-\bar{x}|}\, dx. \tag{2.4}$$

Similarly we can define the weighted Lebesgue spaces for general $p \in [1, \infty]$ and the same actually holds for all the spaces defined in the rest of this section.

Definition. *The space of* locally uniform L^2 functions *is defined by*

$$L^2_b(\Omega) = \{u \in L^2_{loc}(\Omega);\ \sup_{x_0 \in \Omega} \|u\|_{L^2(B(x_0,1))} < \infty\}. \tag{2.5}$$

Here $B(x_0,r)$ stands for an r-ball centered in x_0. Let x_k, $k \in \mathbb{N}$, enumerate the points with half-integer coordinates, i.e. $(\mathbb{Z}/2)^N$, and let $C_k = C(x_k)$, $k \in \mathbb{N}$, be the unit cubes, centered in x_k. Then clearly the space $L^2_b(\Omega)$ has an equivalent norm

$$\|u\|_{L^2_b(\Omega)} \equiv \|u\|_b = \sup_{k\in\mathbb{N}} \|u\|_{L^2(C_k)}. \tag{2.6}$$

Definition. *Let $\mu \geq 0$. An* admissible weight function of growth rate μ *is a measurable bounded function $\phi : \mathbb{R}^N \to (0,\infty)$ satisfying the inequalities*

$$C^{-1}e^{-\mu|x-y|} \leq \phi(x)/\phi(y) \leq Ce^{\mu|x-y|}, \qquad |\nabla\phi(x)| \leq |\phi(x)| \tag{2.7}$$

for some $C \geq 1$ and every $x, y \in \mathbb{R}^N$.

A typical example of an admissible weight function is the exponential $\phi(x) = e^{-q|x-\bar{x}|}$ with $\bar{x} \in \mathbb{R}^N$ and $q \in [0,1]$. Trivially, $\phi(x) \equiv 1$ is and admissible weight function of growth rate $\mu = 0$. In fact we could define the locally uniform space in a more general manner and arrive at similar relations between weighted Lebesgue spaces and (weighted) locally uniform spaces. For more information see e.g. [2], Section 4.

Definition. *Let ϕ be an admissible weight function. We define the space of* weighted locally uniform L^2 functions *$L^2_{b,\phi}(\Omega)$ by*

$$L^2_{b,\phi}(\Omega) = \{u \in L^2_{loc}(\Omega); \sup_{x_0\in\Omega} \phi(x_0)^{1/2}\|u\|_{L^2(B(x_0,1))} < \infty\}. \tag{2.8}$$

For $\phi \equiv 1$, we simply write $L^2_b(\Omega)$.

Similarly as in the non-weighted case one may observe that the space $L^2_{b,\phi}(\Omega)$ has an equivalent norm

$$\|u\|_{L^2_{b,\phi}(\Omega)} \equiv \|u\|_{b,\phi} = \sup_{k\in\mathbb{N}} \phi(x_k)^{1/2}\|u\|_{L^2(C_k)}. \tag{2.9}$$

Theorem 2.1 ([8], Theorem 2.1)**.** *Let ϕ be an admissible weight function. The space $L^2_{b,\phi}(\Omega)$ admits an equivalent norm*

$$\|u\|^2_{b,\phi} = \sup_{\bar{x}\in\Omega} \phi(\bar{x})^{1/2} \int_\Omega |u(x)|^2 e^{-\varepsilon|x-\bar{x}|}\, dx \tag{2.10}$$

for every $\varepsilon > 0$ and $u \in L^2_b(\Omega)$.

Following the notation of [8], we define the $L^2_{b,\phi}$ seminorms corresponding to a subdomain $\mathcal{O} \subseteq \Omega$. For $\mathcal{O} \subseteq \Omega$ we define

$$\mathbb{I}(\mathcal{O}) = \{k \in \mathbb{N}; C_k \cap \mathcal{O}\}, \tag{2.11}$$

$$\|u\|_{L^2_{b,\phi}(\mathcal{O})} = \sup_{k\in\mathbb{I}(\mathcal{O})} \phi^{1/2}(x_k)\|u\|_{L^2(C_k)}. \tag{2.12}$$

We will need to use so called parabolic uniformly bounded spaces introduced in [8].

Definition. *Let ϕ be an admissible weight function and $\varepsilon > 0$. We define the* parabolic locally uniform spaces *by their respective norms*

$$\|u\|_{L^2_{b,\phi}(-r,\ell;L^2(\Omega))} = \sup_{k\in\mathbb{N}} \phi(x_k)^{1/2}\|u\|_{L^2(-r,\ell;L^2(C_k))}, \tag{2.13}$$

$$\|u\|_{L^2_{b,\phi}(-r,\ell;W^{1,2}(\Omega))} = \sup_{k\in\mathbb{N}} \phi(x_k)^{1/2}\|u\|_{L^2(-r,\ell;W^{1,2}(C_k))}, \tag{2.14}$$

$$\|u\|_{L^2_{b,\phi}(-r,\ell;W^{-1,2}(\Omega))} = \sup_{k\in\mathbb{N}} \phi(x_k)^{1/2}\|u\|_{L^2(-r,\ell;W^{-1,2}(C_k))}. \tag{2.15}$$

Once again, the symbol ϕ is dropped if $\phi = 1$.

A simple variant of Theorem 2.4 from [8] implies that for ϕ of growth rate μ strictly smaller than $\varepsilon > 0$, the parabolic locally uniform spaces admit equivalent norms

$$\|u\|^2_{L^2_{b,\phi}(-r,\ell;L^2(\Omega))} \approx \sup_{\bar{x}\in\Omega} \phi(\bar{x}) \int_{(-r,\ell)\times\Omega} |u(t,x)|^2 e^{-\varepsilon|x-\bar{x}|}\, dx\, dt, \tag{2.16}$$

$$\|u\|^2_{L^2_{b,\phi}(-r,\ell;W^{1,2}(\Omega))} \approx \sup_{\bar{x}\in\Omega} \phi(\bar{x}) \int_{(-r,\ell)\times\Omega} \left(|u(t,x)|^2 + |\nabla u(t,x)|^2\right) e^{-\varepsilon|x-\bar{x}|}\, dx\, dt, \tag{2.17}$$

$$\|u\|^2_{L^2_{b,\phi}(-r,\ell;W^{-1,2}(\Omega))} \approx \sup_{v} \sup_{\bar{x}\in\Omega} \phi(\bar{x}) \int_{(-r,\ell)\times\Omega} u(t,x)v(t,x) e^{-\varepsilon|x-\bar{x}|}\, dx\, dt, \tag{2.18}$$

where the first supremum in the last equivalence is taken over $v \in L^2_{b,\phi}(-r,\ell;W^{1,2}(\Omega))$ with unit norm.

The parabolic uniformly bounded spaces and the Bochner spaces constructed over locally uniform spaces are related in the following way:

$$L^2(-r,\ell;L^2_{b,\phi}(\Omega)) \subsetneq L^2_{b,\phi}(-r,\ell;L^2(\Omega)) \subsetneq L^2_{loc}([-r,\ell]\times\Omega). \tag{2.19}$$

Recall that for $\varepsilon > 0$, a metric space M and a precompact set $K \subseteq M$ the Kolmogorov ε-entropy is defined by

$$H_\varepsilon(K,M) = \log N_\varepsilon(K,M),$$

where $N_\varepsilon(K,M)$ is the smallest number of balls of radius ε that cover the set K in M.

Lemma 2.2 ([8], Lemma 2.6)**.** *Let ϕ be an admissible weight function. Let $\mathcal{O} \subseteq \Omega$ satisfy*

$$\#\mathbb{I}(\mathcal{O}) \le c_1 \operatorname{vol}(\mathcal{O}). \tag{2.20}$$

Denote $Q = [-r,\ell]\times\Omega$ and define

$$\|\chi\|_{W_{b,\phi}(Q)} = \|\chi\|_{L^2_{b,\phi}(-r,\ell;W^{1,2}(\Omega))} + \|\partial_t\chi\|_{L^2_{b,\phi}(-r,\ell;W^{-1,2}(\Omega))}. \tag{2.21}$$

Let $r > 0$ and $\theta \in (0,1)$. Then there exists $c_0 > 0$ such that

$$H_{\theta r}\left(B_r(\chi;W_{b,\phi}(Q)), L^2_{b,\phi}(-r,\ell;L^2(\mathcal{O})\right) \le c_0 \operatorname{vol}(\mathcal{O}), \tag{2.22}$$

where $B_r(x_0;X)$ denotes a ball in the space X with radius r centered at x_0. The constant c_0 depends on c_1, ℓ, θ and μ, C in (2.7), but does not depend on χ, r and the particular form of the weight function ϕ.

Observe that a ball in $\mathbb{R}^N$ satisfies (2.20) with c_1 independent of the radius $r \ge 1$.

We conclude this section with four auxiliary lemmata. The proofs are elementary and therefore omitted. Lemma 2.6 is the standard L^p-estimate for the convolution.

Lemma 2.3. *Let $\varepsilon > 0, \bar{x} \in \Omega$ and let $\mathcal{B} \subseteq L^2_b(\Omega)$ be bounded. Then for every $\delta > 0$ there exists $R > 0$ such that*

$$\int_{\Omega\setminus B(0,R)} |u(x)|^2 e^{-\varepsilon|x-\bar{x}|}\,dx < \delta$$

for every $u \in \mathcal{B}$.

Lemma 2.4. *Let $\mathcal{B} \subseteq L^2_b(\Omega)$ be bounded and let $u_n, u \in \mathcal{B}$. Then*

$$u_n \to u \text{ in } L^2_{\bar{x},\varepsilon}(\Omega) \Leftrightarrow u_n \to u \text{ in } L^2_{loc}(\Omega) \tag{2.23}$$

for every $\bar{x} \in \Omega, \varepsilon > 0$.

Lemma 2.5. *Let $\mathcal{B} \subseteq L^\infty(-r,\ell;L^2_b(\Omega))$ be bounded and let $u_n, u \in \mathcal{B}$. Then*

$$u_n \to u \text{ in } L^2(-r,\ell;L^2_{\bar{x},\varepsilon}(\Omega)) \Leftrightarrow u_n \to u \text{ in } L^2_{loc}((-r,\ell)\times\Omega)$$

for every $\bar{x} \in \Omega, \varepsilon > 0$.

Lemma 2.6. *Let $p \in [1,\infty)$, $\bar{x} \in \mathbb{R}^N$, $u \in L^p_{\bar{x},\varepsilon}(\mathbb{R}^N)$ and let G be a function such that $G_{\varepsilon/p} \in L^1(\mathbb{R}^N)$, where*

$$G_{\varepsilon/p}(y) = G(y)e^{\varepsilon/p\,|y|}.$$

Then the estimate

$$\|u * G\|_{L^p_{\bar{x},\varepsilon}(\mathbb{R}^N)} \le \|u\|_{L^p_{\bar{x},\varepsilon}(\mathbb{R}^N)}\|G_{\varepsilon/p}\|_{L^1(\mathbb{R}^N)} \tag{2.24}$$

holds true.

3 Well-posedness

We impose the following assumptions on the nonlinearities: Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function satisfying

$$a(0) = 0, \qquad (a(\zeta) - a(\eta)) \cdot |\zeta - \eta| \geq \kappa|\zeta - \eta|, \qquad \forall \zeta, \eta \in \mathbb{R}^N, \tag{3.1}$$

$$|a(\zeta) - a(\eta)| \leq \kappa c|\zeta - \eta|, \qquad \forall \zeta, \eta \in \mathbb{R}^N, \tag{3.2}$$

$$\zeta \to a(\zeta) \cdot \zeta \text{ is a convex function on } \mathbb{R}^N, \tag{3.3}$$

for some $\kappa > 0, c \geq 1$.

Let $b : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz, i.e.

$$b(0) = 0, |b(r)| \leq C_b \text{ for every } r \in \mathbb{R}, \tag{3.4}$$

$$|b(r) - b(s)| \leq C_b|r - s| \text{ for every } r, s \in \mathbb{R}. \tag{3.5}$$

Let $f : (\Omega - \Omega) \to \mathbb{R}$ be bounded, i.e.

$$|f(x - y)| \leq C_f \text{ for every } x, y \in \Omega. \tag{3.6}$$

Finally, concerning the form of the distributed delay, we impose the following conditions:

(i) For every $M > 0$ and $0 < \varepsilon < 1$ there exists $L = L(M, \varepsilon) > 0$ such that for every $\bar{x} \in \Omega$ and $(v^i, \psi^i) \in H$ satisfying

$$\|v^i\|^2_{\bar{x},\varepsilon} + \int_{-r}^0 \|\psi^i(\theta)\|^2_{\bar{x},\varepsilon}\, ds \leq M^2, \qquad i = 1, 2,$$

the following holds:

$$\int_{-r}^0 |\xi(\theta, v^1, \psi^1) - \xi(\theta, v^2, \psi^2)| d\theta \leq L\left(\|v^1 - v^2\|^2_{\bar{x},\varepsilon} + \int_{-r}^0 \|\psi^1(\theta) - \psi^2(\theta)\|^2_{\bar{x},\varepsilon}\, d\theta\right)^{1/2}. \tag{3.7}$$

(ii) There exists $C_\xi > 0$ such that for every $(v, \psi) \in H$ we have

$$\|\xi(\cdot, v, \psi)\|_{L^2(-r,0)} \leq C_\xi. \tag{3.8}$$

The space of initial conditions is defined as

$$H \equiv L^2_b(\Omega) \times L^2_b(-r, 0; L^2(\Omega)). \tag{3.9}$$

Function $u : [0, T] \times \Omega \to \mathbb{R}$ will be called (weak) solution, if for every $\bar{x} \in \Omega$ and $0 < \varepsilon < 1$

$$u \in C([0, T]; L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2(-r, 0; L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\Omega)),$$

$$\partial_t u \in L^2(0, T; W^{-1,2}_{\bar{x},\varepsilon}(\Omega)), \tag{3.10}$$

and u satisfies the variational formulation (here and in what follows, $F(u^t)$ denotes the right-hand side of (1.1)):

$$-\int_0^T (u(t), \partial_t\psi(t))\, dt + \int_0^T (a(\nabla u(t)), \nabla\psi(t))\, dt + d\int_0^T (u(t), \psi(t))\, dt = \int_0^T \left(F(u^t), \psi(t)\right) dt \tag{3.11}$$

for every $\psi \in \mathcal{D}((0, T) \times \Omega)$ and the initial conditions

$$u(0) = u_0 \in L^2_b(\Omega), \qquad u|_{(-r,0)} = \varphi \in L^2_b(-r, 0; L^2(\Omega)) \tag{3.12}$$

hold true. For arbitrary $\varepsilon > 0$ and $\bar{x} \in \Omega$ we may use a standard density argument and arrive to the duality with respect to $L^2_{\bar{x},\varepsilon}(\Omega)$:

$$(u(T), \psi(T))_{\varepsilon,\bar{x}} - \int_0^T (u(t), \partial_t \psi(t))_{\varepsilon,\bar{x}}\, dt + \int_0^T \left(a\left(\nabla u(t)\right), \nabla \psi(t) - \varepsilon \frac{x-\bar{x}}{|x-\bar{x}|}\psi(t) \right)_{\varepsilon,\bar{x}} dt$$
$$+ d \int_0^T (u(t), \psi(t))_{\varepsilon,\bar{x}}\, dt = \int_0^T \left(F(u^t), \psi(t)\right)_{\varepsilon,\bar{x}} dt + (u(0), \psi(0))_{\varepsilon,\bar{x}} \quad (3.13)$$

for any $\psi \in L^2(0,T; W^{1,2}_{\bar{x},\varepsilon}(\Omega)) \cap W^{1,2}(0,T; L^2_{\bar{x},\varepsilon}(\Omega))$. Indeed, one can replace ψ in (3.11) by $\psi \chi_n e^{-\varepsilon|x-\bar{x}|}$, where χ_n is some sequence of cut-off functions such that $\chi_n \to 1$, $\nabla\chi_n \to 0$ and $|\chi_n| + |\nabla \chi_n| \le c$ a.e. It is clear that (3.13) in turn implies (3.11).

Theorem 3.1. *Let* (3.4) - (3.6) *hold and let* $\xi : [-r,0] \times H \to \mathbb{R}$ *satisfy conditions (i–ii). Then for every* $T > 0$ *and* $(u_0, \varphi) \in H$ *there exists a unique* u *solution to* (1.1).

Proof. The proof is a variant of the original proof for the linear case in a bounded domain (see [14], Theorem 1). We need to handle the limit of nonlinear diffusion term (cf. [8], Theorem 3.2); otherwise, standard techniques for unbounded domains are used ([20]).

We approximate the problem (3.11) by a sequence of problems solvable on bounded domain and then pass to the limit. Let $\Omega_n = B_n(0) \subseteq \mathbb{R}^N$ and let $\psi_n \in C^\infty(\Omega, [0,1])$ satisfy $\psi_n \equiv 1$ on $\bar{\Omega}_{n-1}$, $\operatorname{supp}\psi_n \subseteq \Omega_n$, and define $u_{0,n} = u_0 \psi_n$ and $\varphi_n(\theta) = \varphi(\theta)\psi_n$ for $\theta \in [-r, 0]$. Using Theorem 2.1 and the Lebesgue dominated convergence theorem we immediately obtain

$$u_{0,n} \to u_0 \text{ in } L^2_{\bar{x},\varepsilon}(\Omega), \qquad \varphi_n \to \varphi \text{ in } L^2(-r, 0; L^2_{\bar{x},\varepsilon}(\Omega)) \quad (3.14)$$

for every $0 < \varepsilon < 1$ and $\bar{x} \in \Omega$. Next we define the operator

$$A_n : W^{1,2}_0(\Omega_n) \to W^{-1,2}(\Omega_n), \qquad \langle A_n v, z\rangle = \int_{\Omega_n} a(\nabla v(x)) \cdot \nabla z(x) dx$$

and the approximate problem

$$\partial_t u_n + A_n u + d u_n = \int_{-r}^0 \left(\int_{\Omega_n} b(u_n(t+\theta, y)) f(x-y) e^{-|x-y|} dy \right) \xi(\theta, u_n(t), u_n^t) d\theta. \quad (3.15)$$

with the initial condition

$$u_n(0) = u_{0,n}, \qquad u_n|_{(-r,0)} = \varphi_n. \quad (3.16)$$

A nonlinear variant of Theorem 1 from [14] implies that the equation (3.15) with the initial equation (3.16) has a solution

$$u_n \in C([0,T], L^2(\Omega_n)) \cap L^2(-r, T; L^2(\Omega_n)) \cap L^2(0,T; W^{1,2}_0(\Omega_n)).$$

First we aim to show that

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T, L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2(0,T; W^{1,2}_{\bar{x},\varepsilon}(\Omega)) \quad (3.17)$$

for some $u \in L^\infty(0,T, L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2(0,T; W^{1,2}_{\bar{x},\varepsilon}(\Omega))$. Let us extend u_n by zero outside of Ω_n (note that then $u_n \in L^\infty(0,T; L^2_b(\Omega)) \cap L^2(0,T; W^{1,2}_b(\Omega))$ and thus $\xi(\theta, u_n(t), u_n^t)$ makes good sense) and test (3.15) by $u_n(t,x)e^{-\varepsilon|x-\bar{x}|}$ to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n(t,x)|^2 e^{-\varepsilon|x-\bar{x}|} dx$$
$$+ \int_\Omega a\left(\nabla u_n(t,x)\right) \cdot \left(\nabla u_n(t,x) - \varepsilon \frac{x-\bar{x}}{|x-\bar{x}|} u_n(t,x) \right) e^{-\varepsilon|x-\bar{x}|} dx$$
$$+ d \int_\Omega |u_n(t,x)|^2 e^{-\varepsilon|x-\bar{x}|} dx$$
$$= \int_\Omega \left[\int_{-r}^0 \left(\int_\Omega b(u_n(t+\theta, y) f(x-y) e^{-|x-y|} dy \right) \xi(\theta, u_n(t), u_n^t) d\theta \right] u_n(t,x) e^{-\varepsilon|x-\bar{x}|} dx. \quad (3.18)$$

Observe that the integration in the previous equation is over Ω instead of Ω_n. This is possible since $u_n \equiv 0$ outside of Ω_n and $b(0) = 0, a(0) = 0$. Using the boundedness of the functions b and f, from (ii) it follows that

$$\langle F(v^t), z\rangle_{\bar{x},\varepsilon} \leq C\|z\|_{\bar{x},\varepsilon}.$$

The previous estimate, (3.18), (3.1), (3.2) and Young's inequality immediately give

$$\frac{d}{dt}\|u_n(t)\|^2_{\bar{x},\varepsilon} + \sigma\left(\|\nabla u_n(t)\|^2_{\bar{x},\varepsilon} + \|u_n(t)\|^2_{\bar{x},\varepsilon}\right) \leq C_1\|u_n(t)\|^2_{\bar{x},\varepsilon} + C_2 \tag{3.19}$$

for some $\sigma > 0$. Gronwall's inequality applied to

$$\chi_n(t) = \|u_n(t)\|^2_{\bar{x},\varepsilon} + \sigma\left(\int_0^t \|\nabla u_n(s)\|^2_{\bar{x},\varepsilon} ds + \int_0^t \|u_n(s)\|^2_{\bar{x},\varepsilon} ds\right) + C_2$$

gives

$$\chi_n(t) \leq (\|u_n(0)\|^2_{\bar{x},\varepsilon} + C_2)e^{C_1 t},$$

in other words

$$\|u_n(t)\|^2_{\bar{x},\varepsilon} + \int_0^t \|\nabla u_n(s)\|^2_{\bar{x},\varepsilon} + \int_0^t \|u_n(s)\|^2_{\bar{x},\varepsilon} ds \leq (\|u_n(0)\|^2_{\bar{x},\varepsilon} + C_2)e^{C_1 t} - C_2. \tag{3.20}$$

From (3.14) it follows that the sequence $\{u_n(t)\}_{n=1}^{\infty}$ is bounded in $L^\infty(0,T; L^2_{\bar{x},\varepsilon}(\Omega))$, $L^2(0,T; W^{1,2}_{\bar{x},\varepsilon}(\Omega))$ and using a standard argument we finally have (3.17). Observe that if we take the supremum of (3.20) over $\bar{x} \in \Omega$, using (2.10) and (2.17) we obtain

$$u \in L^\infty(0,T; L^2_b(\Omega)) \cap L^2_{b,1}(0,T; W^{1,2}(\Omega)). \tag{3.21}$$

Following a similar argument we can show

$$\partial_t u_n \rightharpoonup \partial_t u \text{ in } L^2(0,T; W^{-1,2}_{\bar{x},\varepsilon}(\Omega)) \tag{3.22}$$

and therefore we immediately have

$$u \in C([0,T], L^2_{\bar{x},\varepsilon}(\Omega)). \tag{3.23}$$

Also note that since the norms $L^2_{\bar{x},\varepsilon}(\Omega)$ are equivalent for different $\bar{x} \in \Omega$, the function u is independent of $\bar{x}$. Next we show

$$u_n \to u \text{ in } L^2(0,T; L^2_{\bar{x},\varepsilon}(\Omega)), \tag{3.24}$$

The first step is to establish the convergence

$$u_n \to u \text{ in } L^2(0,T; L^2(\Omega_m)) \tag{3.25}$$

for $m \in \mathbb{N}$ fixed. We proceed similarly as in the proof of Theorem 3.1 from [8]. The weak convergence (3.17) implies

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T, L^2(\Omega_m)) \cap L^2(0,T; W^{1,2}(\Omega_m)). \tag{3.26}$$

Then we choose $n > m$ and test the equation (3.15) with $v \in L^2(0,T; W^{1,2}_0(\Omega_m))$ extended by zero outside of Ω_m to get

$$\partial_t u_n \rightharpoonup \partial_t u \text{ in } L^2(0,T; W^{-1,2}(\Omega_m)). \tag{3.27}$$

The desired convergence (3.25) then follows from (3.26), (3.27) and the Aubin-Lions lemma.

Using the continuity of u_n and u and the estimate (3.21) we can find a set $\mathcal{B} \subseteq L^2_b(\Omega)$ such that

$$u_n(t) - u(t) \in \mathcal{B} \text{ for a.e. } t \in [0,T]$$

and $\mathcal{B}$ is bounded in $L^2_b(\Omega)$. Choose $\delta > 0$ and use Lemma 2.3 to find $n_0 \in \mathbb{N}$ such that

$$\int_{\Omega\setminus\Omega_{n_0}} |u_n(t,x) - u(t,x)|^2 e^{-\varepsilon_0|x-\bar{x}|}\, dx < \delta$$

for $n \geq n_0$. Since this estimate is uniform in $n \geq n_0$ and $t \in [0, T]$, from the strong convergence in Ω_{n_0}(3.25) we thus obtain the desired convergence (3.24). We emphasize the convergence (3.24) holds for all $\bar{x} \in \mathbb{R}^N$ and $0 < \varepsilon < 1$.

Now we will prove $F(u_n^t) \to F(u^t)$ in $L^2(0, T; W^{-1,2}_{\bar{x},\varepsilon}(\Omega))$. Let $v \in L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\Omega))$ be fixed. We split the estimate into two separate integrals:

$$
\begin{aligned}
&\int_0^T \langle \tilde{F}(u_n^t) - \tilde{F}(u^t), v(t)\rangle_{\bar{x},\varepsilon}\, dt \\
&\quad = \int_0^T \left(\int_\Omega \left[\int_{-r}^0 \left(\int_\Omega b(u_n(t+\theta, y)) f(x-y) e^{-|x-y|} dy \right) \xi(\theta, u_n(t), u_n^t) d\theta \right.\right. \\
&\quad\quad \left.\left. - \int_{-r}^0 \left(\int_\Omega b(u(t+\theta, y)) f(x-y) e^{-|x-y|} dy \right) \xi(\theta, u(t), u^t) d\theta \right] v(t,x) e^{-\varepsilon|x-\bar{x}|} dx \right) dt \\
&\quad = \int_0^T \left(\int_\Omega \left[\int_{-r}^0 \left(\int_\Omega (b(u_n(t+\theta, y)) - b(u(t+\theta, y)))\, f(x-y) e^{-|x-y|} dy \right) \right.\right. \\
&\quad\quad \left.\left. \cdot \xi(\theta, u_n(t), u_n^t) d\theta \right] v(t,x) e^{-\varepsilon|x-\bar{x}|} dx \right) dt \\
&\quad\quad + \int_0^T \left(\int_\Omega \left[\int_{-r}^0 \left(\int_\Omega b(u(t+\theta, y)) f(x-y) e^{-|x-y|} dy \right) \right.\right. \\
&\quad\quad \left.\left. \cdot \left(\xi(\theta, u_n(t), u^t) - \xi(\theta, u(t), u^t)\right) d\theta \right] v(t,x) e^{-\varepsilon|x-\bar{x}|} dx \right) dt \equiv I_1 + I_2.
\end{aligned}
$$

The integral I_1 can be estimated using the boundedness of f, Fubini's theorem, the Lipschitz continuity of b, Hölder's inequality and Lemma 2.6 with $G(y) = e^{-|x-y|}$ in the following way (for convenience we denote $\omega(t, x) \equiv u_n(t, x) - u(t, x)$):

$$
\begin{aligned}
I_1 &\leq C_3 \int_0^T \int_{-r}^0 \left[\int_\Omega \left(\int_\Omega |\omega(t+\theta, y)| e^{-|x-y|}\, dy \right) v(t,x) e^{-\varepsilon|x-\bar{x}|}\, dx \right] \xi(\theta, u_n(t), u_n^t)\, d\theta\, dt \\
&\leq C_3 \int_0^T \|v(t)\|_{\bar{x},\varepsilon} \int_{-r}^0 \left[\int_\Omega \left(\int_\Omega |\omega(t+\theta, y)| e^{-|x-y|}\, dy \right)^2 e^{-\varepsilon|x-\bar{x}|}\, dx \right]^{1/2} \\
&\quad \cdot \xi(\theta, u_n(t), u_n^t)\, d\theta\, dt \\
&\leq C_4 \int_0^T \|v(t)\|_{\bar{x},\varepsilon} \left[\int_{-0}^r \left(\int_\Omega |\omega(t+\theta, x)|^2 e^{-\varepsilon|x-\bar{x}|}\, dx \right)^{1/2} \xi(\theta, u_n(t), u_n^t)\, d\theta \right] dt \\
&\leq C_4 \int_0^T \|v(t)\|_{\bar{x},\varepsilon} \left[\int_{-r}^0 \left(\int_\Omega |\omega(t+\theta, x)|^2 e^{-\varepsilon|x-\bar{x}|}\, dx \right) d\theta \right]^{1/2} \\
&\quad \cdot \left[\int_{-r}^0 |\xi(\theta, u_n(t), u_n^t)|^2\, d\theta \right]^{1/2} dt \\
&\leq C_5 \left(\int_0^T \|v(t)\|^2_{\bar{x},\varepsilon}\, dt \right)^{1/2} \left(\int_0^T \int_{-r}^0 \|\omega(t+\theta)\|^2_{\bar{x},\varepsilon}\, d\theta\, dt \right)^{1/2} \\
&\leq C_6 \left(\|\varphi_n - \varphi\|_{L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))} + \|u_n - u\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))} \right) \|v\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))}.
\end{aligned}
\tag{3.28}
$$

The integral I_2 can be treated in a similar manner. Since b and f are bounded, the condition (3.7) implies

$$
\begin{aligned}
I_2 &\leq C_7 \int_0^T \left(\int_\Omega \left[\int_{-r}^0 |\xi(\theta, u_n(t), u_n^t) - \xi(\theta, u(t), u^t)|\, d\theta \right] v(t,x) e^{-\varepsilon|x-\bar{x}|}\, dx \right) dt \\
&\leq C_8 \int_0^T \left(\|u_n(t) - u(t)\|^2_{\bar{x},\varepsilon} + \int_{-r}^0 \|u_n(t+\theta) - u(t+\theta)\|^2_{\bar{x},\varepsilon}\, d\theta \right)^{1/2} \|v(t)\|_{L^1_{\bar{x},\varepsilon}(\Omega)}\, dt \\
&\leq C_9 \left(\|\varphi_n - \varphi\|_{L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))} + \|u_n - u\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))} \right) \|v\|_{L^2(0,T;L^2(0,T;L^1_{\bar{x},\varepsilon}(\Omega))}.
\end{aligned}
\tag{3.29}
$$

Using the definition of the $L^2(0, T; W^{-1,2}_{\bar{x},\varepsilon}(\Omega))$-norm and the continuous inclusion

$L^2_{\bar{x},\varepsilon}(\Omega) \hookrightarrow L^1_{\bar{x},\varepsilon}(\Omega)$, the estimates (3.28) and (3.29) imply

$$\|F(u_n^t) - F(u^t)\|_{L^2(0,T;W^{-1,2}_{\bar{x},\varepsilon}(\Omega))} \le C_{10}\left(\|\varphi_n - \varphi\|_{L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))} + \|u_n - u\|_{L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))}\right). \tag{3.30}$$

Also note that the argument leading to (3.28) and (3.29) shows the Lipschitz continuity of F

$$\|F(v^t) - F(w^t)\|_{W^{-1,2}_{\bar{x},\varepsilon}(\Omega)} \le C\|v - w\|_{L^2(t-r,t;L^2_{\bar{x},\varepsilon}(\Omega))} \tag{3.31}$$

for $v, w \in L^2(t-r,t;L^2_{\bar{x},\varepsilon}(\Omega))$.

To finish the proof of existence, we need to deal with the nonlinear diffusion. The process is standard. We follow [8, Theorem 3.2]. First we observe that the convergence (3.17) and the assumption (3.2) implies

$$a(\nabla u_n) \rightharpoonup \alpha \text{ in } L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega)) \tag{3.32}$$

for fixed $\bar{x} \in \Omega$. Note that at this stage α might depend on the choice of $\bar{x}$. Test the equation (3.15) by a fixed $v \in L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))$ and take the limit with respect to n to obtain

$$\int_\Omega \frac{\partial}{\partial t}u(t,x)v(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx + \int_\Omega \alpha(t,x)\cdot\left(\nabla v(t,x) - \varepsilon\frac{x-\bar{x}}{|x-\bar{x}|}v(t,x)\right)e^{-\varepsilon|x-\bar{x}|}\,dx + d\int_\Omega u(t,x)v(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx = \int_\Omega F(u^t)(t,x)v(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx. \tag{3.33}$$

Let us go back to (3.18), integrate over $(0,T)$ and take the limit superior with respect to $n \to \infty$ to get

$$\begin{aligned}\limsup_{n\to\infty}\int_0^T\int_\Omega a(\nabla u_n(t,x))\cdot\nabla u_n(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx \\ \le -\frac{1}{2}\liminf_{n\to\infty}\int_\Omega |u_n(T,x)|^2 e^{-\varepsilon|x-\bar{x}|}\,dx + \frac{1}{2}\limsup_{n\to\infty}\int_\Omega |u_n(0,x)|^2 e^{-\varepsilon|x-\bar{x}|}\,dx \\ - \liminf_{n\to\infty} d\int_0^T\int_\Omega |u_n(t,x)|^2 e^{-\varepsilon|x-\bar{x}|}\,dx\,dt \\ + \limsup_{n\to\infty}\int_0^T\int_\Omega F(u_n^t)(t,x)u_n(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx\,dt \\ + \limsup_{n\to\infty}\varepsilon\int_0^T\int_\Omega a(\nabla u_n(t,x))\frac{x-\bar{x}}{|x-\bar{x}|}u_n(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx\,dt.\end{aligned} \tag{3.34}$$

Using (3.14), (3.17) and the lower semicontinuity of norms we immediately see that the first three terms have a well defined limit. Also using (3.24) and (3.32) we have

$$\lim_{n\to\infty}\varepsilon\int_0^T\int_\Omega a(\nabla u_n(t,x))\frac{x-\bar{x}}{|x-\bar{x}|}u_n(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx\,dt = \varepsilon\int_0^T\int_\Omega \alpha(t,x)\frac{x-\bar{x}}{|x-\bar{x}|}u(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx\,dt \tag{3.35}$$

Finally, the strong convergence (3.24) and Lipschitz continuity of F (3.31) imply

$$\begin{aligned}\left(F(u_n^t), u_n\right)_{\bar{x},\varepsilon} &= \left(F(u_n^t), u_n - u\right)_{\bar{x},\varepsilon} + \left(F(u_n^t) - F(u^t), u\right)_{\bar{x},\varepsilon} + \left(F(u^t), u\right)_{\bar{x},\varepsilon} \\ &\to \left(F(u^t), u\right)_{\bar{x},\varepsilon}.\end{aligned}$$

Now we are ready to compare (3.34) and (3.33) (with $v = u$) and integrated over time and use arrive to

$$\limsup_{n\to\infty}\int_0^T\int_\Omega a(\nabla u_n(t,x))\cdot\nabla u_n(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx\,dt \le \int_0^T\int_\Omega \alpha(t,x)\nabla u(t,x)e^{-\varepsilon|x-\bar{x}|}\,dx\,dt. \tag{3.36}$$

Since a induces a maximal monotone operator on $L^2(0,T;L^2_{\bar{x},\varepsilon}(\Omega))$, a standard argument leads us to the equality

$$\alpha(t,x) = a\left(\nabla u(t,x)\right) \qquad \left(e^{-\varepsilon|x-\bar{x}|}\,dx\right)\text{-a.e. in } \Omega \text{ and a.e. in } (0,T). \tag{3.37}$$

Clearly the equality (3.37) holds also a.e. with respect to the standard Lebesgue measure in $(0,T)\times\Omega$ and therefore α is independent of the choice of $\bar{x}$. Therefore we may use (3.37) to substitute in (3.33), which finishes the proof of existence.

The proof of the uniqueness is analogous to the proof for the bounded domain. Let u and v be two solutions with the respective initial conditions $(u_0,\varphi),(v_0,\psi)\in H$ and denote $w(t)=u(t)-v(t)$. Test the equations for u and v by w. Subtracting these and using similar argument as in the derivation of the inequalities (3.28) and (3.29) we obtain

$$\frac{d}{dt}\|w(t)\|^2_{\bar{x},\varepsilon}+\sigma\|w(t)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)} \le C_{10}\|w(t)\|^2_{\bar{x},\varepsilon} + C_{11}\int_{-r}^0 \|w(t+\theta)\|^2_{\bar{x},\varepsilon}\,d\theta \tag{3.38}$$

$$\le C_{12}\left(\|w(t)\|^2_{\bar{x},\varepsilon}+\sigma\int_0^t \|w(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\,ds\right)+C_{13}\int_{-r}^0\|w(\theta)\|^2_{\bar{x},\varepsilon}\,d\theta \tag{3.39}$$

for some $\sigma>0$. The Gronwall's lemma applied to the function

$$Y(t)=\|w(t)\|^2_{\bar{x},\varepsilon}+\sigma\int_0^t\|w(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\,ds$$

implies

$$Y(t)\le\left(Y(0)+\int_{-r}^0\|w(\theta)\|^2_{\bar{x},\varepsilon}\,d\theta\right)C(t). \tag{3.40}$$

The estimate can be rewritten in the form

$$\sup_{t\in[0,T]}\|u(t)-v(t)\|^2_{\bar{x},\varepsilon}+\sigma\int_0^t\|u(s)-v(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\,ds$$

$$\le C(T)\left(\|u_0-v_0\|^2_{\bar{x},\varepsilon}+\int_{-r}^0\|\varphi(\theta)-\psi(\theta)\|^2_{\bar{x},\varepsilon}\,d\theta\right), \tag{3.41}$$

which gives the uniqueness of the solutions. □

Definition. *We define the* solution operator $S(t):H\to H$ *by*

$$S(t)(u_0,\varphi)=\left(u(t),u^t\right),$$

where $u(t)$ is the solution from Theorem 3.1.

As is usual in locally uniform spaces, we cannot generally expect the solution $u(t)$ be continuous in the space $L^2_b(\Omega)$. Continuity in $L^2_b(\Omega)$ can be achieved for more regular initial data; cf. [19]. In the following we will compensate for the lack of additional regularity by working in weighted Lebesgue spaces and by using the method of ℓ-trajectories.

Corollary 3.2. *The operator*

$$S(t):L^2_{\bar{x},\varepsilon}(\Omega)\times L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))\to L^2_{\bar{x},\varepsilon}(\Omega)\times L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))$$

is Lipschitz continuous on H uniformly with respect to $t\in[0,T]$ for every $\bar{x}\in\Omega$ and $0<\varepsilon<1$. The solution operator $S(t)$ paired with H as its phase space form a dynamical system.

Proof. The continuity in time follows from the continuity of the solution and the locally uniform Lipschitz continuity from (3.41). □

4 Locally compact attractor

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold and let ϕ be an admissible weight function with growth rate smaller than 1. Then for $t \geq r$ we have the estimate*

$$\|u(t)\|^2_{L^2_{b,\phi}(\Omega)} + C_1 \|u\|_{L^2_{b,\phi}(t-r,t;W^{1,2}(\Omega))} \leq \|u_0\|_{L^2_{b,\phi}(\Omega)} e^{-\sigma(t-r)} + C_2, \tag{4.1}$$

where $\sigma, C_1, C_2 > 0$ are dependent of the data of the equation and independent of the initial data u_0, φ.

Proof. The proof follows the proof of Theorem 3.2 in [8]. Let u be the solution of (3.11), (3.12). Then using the Cauchy-Schwartz and Young inequalities and (3.1) we have and for $\varepsilon > 0$ small enough we obtain

$$\frac{d}{dt}\|u(t)\|^2_{\bar{x},\varepsilon} + \sigma\left(\|\nabla u(t)\|^2_{\bar{x},\varepsilon} + \|u(t)\|^2_{\bar{x},\varepsilon}\right) \leq C_1 \tag{4.2}$$

for some $C_1, \sigma > 0$. The Gronwall lemma implies

$$\|u(t)\|^2_{\bar{x},\varepsilon} \leq \|u(0)\|^2_{\bar{x},\varepsilon} e^{-\sigma t} + C_2 t e^{-\sigma t} \leq \|u(0)\|^2_{\bar{x},\varepsilon} e^{-\sigma t} + C_3$$

and integrating (4.2) from $t-r$ to t we obtain

$$\|u(t)\|^2_{\bar{x},\varepsilon} + \sigma \int_{t-r}^{t} \left(\|\nabla u(s)\|^2_{\bar{x},\varepsilon} + \|u(s)\|^2_{\bar{x},\varepsilon}\right) ds \leq \|u_0\|^2_{\bar{x},\varepsilon} e^{-\sigma(t-r)} + C_4.$$

Finally we multiply the previous estimate by $\phi(\bar{x})$ and take the supremum over $\bar{x} \in \Omega$. From the definition of parabolic locally uniform spaces we obtain (4.1). □

Corollary 4.2. *The operator*

$$S(t) : L^2_{\bar{x},\varepsilon}(\Omega) \times L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega)) \to L^2_{\bar{x},\varepsilon}(\Omega) \times L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))$$

admits a positively invariant bounded absorbing set $W \subseteq H$. Moreover, the absorbing set W absorbs not only bounded subsets of H, but also the sets of the form $B \times L^2_b(-r,0;L^2(\Omega))$, where $B \subseteq L^2_b(\Omega)$ is bounded.

Proof. The existence of the absorbing set W follows immediately from the previous theorem. Then we find $t_0 > 0$ such that $S(t)\tilde{W} \subseteq \tilde{W}$ for every $t \geq t_0$ and set

$$W = \bigcup_{t \geq t_0} S(t)\tilde{W}.$$

The fact that the set W absorbs even the sets of the form $B \times L^2_b(-r,0;L^2(\Omega))$ for $B \subseteq L^2_b(\Omega)$ bounded follows immediately from the form of the estimate (4.1). □

Since all the solutions from Theorem 3.1 are continuous for $t \geq 0$ and we are interested in the asymptotic dynamics, from now on we may assume that $S(t) : X \to X$, where

$$X = \{u \in C([-r,0], L^2_{\bar{x},\varepsilon}(\Omega)); u \text{ is a solution from Theorem 3.1}\} \tag{4.3}$$

is equipped with $L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))$ topology for fixed $\bar{x} \in \Omega$ and $0 < \varepsilon < 1$. Corollary 4.2 implies that the dynamical system $(X, S(t))$ admits a bounded absorbing set

$$\mathcal{B} \subseteq L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2_{b,1}(-r,0;L^2(\Omega)).$$

Moreover, the continuity of the solutions allows us to assume

$$\mathcal{B} \subseteq C([-r,0];L^2_{\bar{x},\varepsilon}(\Omega)) \cap L^2_{b,1}(-r,0;L^2(\Omega)).$$

Clearly all the interesting dynamics will take place in the absorbing set $\mathcal{B}$. Now we may define the space of short trajectories similarly as in [12].

Definition. *The space of short trajectories is*

$$\mathcal{X} = \{\chi \in C([-r,\ell], L^2_{\bar{x},\varepsilon}(\Omega)); \\ \chi \text{ is a solution from Theorem 3.1 in } [0,\ell] \text{ with } \chi|_{[-r,0]} \in \mathcal{B}\}, \quad (4.4)$$

together with the $L^2(-r,\ell; L^2_{\bar{x},\varepsilon}(\Omega))$ *topology. The evolution operator* $L(t) : \mathcal{X} \to \mathcal{X}$ *is defined by*

$$(L(t)\chi)(s) = u(t+s), \qquad s \in [-r,\ell],$$

where u *is the solution from Theorem 3.1 satisfying* $u|_{(-r,0)} = \chi$.

Finally, the operator $e : \mathcal{X} \to X$ *is defined by*

$$e(\chi) = (L(r+\ell)\chi)|_{[-r,0]}. \quad (4.5)$$

We note that $\mathcal{X}$ in general is not complete with respect to its metric; we will see however in Lemma 4.5 that $L(t)$ is asymptotically compact on $\mathcal{X}$.

Theorem 4.3. *The translation operator* $L(t) : \mathcal{X} \to \mathcal{X}$ *is Lipschitz continuous uniformly with respect to* $t \in [0,T]$*. Moreover, the pair* $(\mathcal{X}, L(t))$ *forms a dynamical system.*

Proof. Let $\chi_1, \chi_2 \in \mathcal{X}$ and let u_1 and u_2 be the respective solutions from Theorem 3.1. Define $w(t) = u_1(t) - u_2(t)$. We start from the estimate (3.38), use the condition (3.2) and integrate over $\tau \in [s,t]$, where $s \in (0,\ell), t \in (\ell, \ell+T)$, to obtain

$$\|w(t)\|^2_{\bar{x},\varepsilon} + \sigma \int_\ell^t \|w(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\, d\tau \le \|w(s)\|^2_{\bar{x},\varepsilon} + C_3 \int_{s-r}^t \|w(\tau)\|^2_{\bar{x},\varepsilon}\, d\tau.$$

Then we integrate over $s \in (0,\ell)$ and get

$$\|w(t)\|^2_{\bar{x},\varepsilon} + \sigma \int_\ell^t \|w(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\, d\tau \le C_4 \left(\int_{-r}^\ell \|w(\tau)\|^2_{\bar{x},\varepsilon}\, d\tau + \int_\ell^t \|w(\tau)\|^2_{\bar{x},\varepsilon}\, d\tau \right).$$

Again we apply the Gronwall's lemma to the function

$$Y(s) = \|w(s)\|^2_{\bar{x},\varepsilon} + \int_\ell^s \|w(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\, d\tau$$

and obtain the estimate

$$\sup_{t\in[0,T]} \|w(t)\|^2_{\bar{x},\varepsilon} + \int_\ell^{\ell+t} \|w(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\, d\tau \le C(T) \int_{-r}^\ell \|w(\tau)\|^2_{\bar{x},\varepsilon}\, d\tau, \quad (4.6)$$

in other words

$$\sup_{t\in[0,T]} \|L_\varepsilon \chi_1 - L_\varepsilon \chi_2\|^2_{\mathcal{X}_\varepsilon} \le C(T) \|\chi_1 - \chi_2\|^2_{\mathcal{X}_\varepsilon}.$$

It remains to prove that $(\mathcal{X}, L(t))$ is a dynamical system. The continuity of the operator $L(t)$ in time follows from the continuity of solutions and the continuity of $L(t)$ in $\mathcal{X}$ follows from the previous part of the proof. □

Definition. *We define the space* $\mathcal{W}$ *as the space* $\mathcal{X}$ *with the norm*

$$\|\chi\|^2_{\mathcal{W}} = \int_{-r}^\ell \|\chi(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\, ds + \int_{-r}^\ell \|\partial_t \chi(s)\|^2_{W^{-1,2}_{\bar{x},\varepsilon}(\Omega)}\, ds.$$

Lemma 4.4. *The mapping* $L(r+\ell) : \mathcal{X} \to \mathcal{W}$ *is Lipschitz continuous.*

Proof. Using the notation of the proof of Theorem 4.3, choosing $t = r + \ell$ in the estimate (4.6) immediately gives

$$\int_\ell^{2\ell+r} \|w(s)\|^2_{W^{1,2}_{\bar{x},\varepsilon}(\Omega)}\, ds \le C \int_{-r}^\ell \|w(s)\|^2_{\bar{x},\varepsilon}\, ds. \quad (4.7)$$

From the equation we have

$$\int_{\ell}^{2\ell+r} \langle \partial_t w(s), \varphi(s)\rangle_{\bar{x},\varepsilon}\, ds = -\int_{\ell}^{2\ell+r} \left(a(\nabla u) - a(\nabla v), \nabla\varphi(s) - \varepsilon \frac{x-\bar{x}}{|x-\bar{x}|}\varphi(s) \right)_{\bar{x},\varepsilon} ds$$
$$- d\int_{\ell}^{2\ell+r} (w(s),\varphi(s))_{\bar{x},\varepsilon}\, ds + \int_{\ell}^{2\ell+r} \langle F(u^s) - F(v^s), \varphi(s)\rangle_{\bar{x},\varepsilon}\, ds. \quad (4.8)$$

Assuming that the $L^2(\ell, 2\ell + r; W^{1,2}_{\bar{x},\varepsilon}(\Omega))$-norm of φ is one, we can use (3.2) to obtain the estimates

$$\int_{\ell}^{2\ell+r} \left(a(\nabla u) - a(\nabla v), \nabla\varphi(s) - \varepsilon \frac{x-\bar{x}}{|x-\bar{x}|}\varphi(s) \right)_{\bar{x},\varepsilon} ds \le C\|w\|_{L^2(\ell,2\ell+r;W^{1,2}_{\bar{x},\varepsilon}(\Omega))}, \quad (4.9)$$

$$\int_{\ell}^{2\ell+r} (w(s),\varphi(s))_{\bar{x},\varepsilon}\, ds \le C\|w\|_{L^2(\ell,2\ell+r;W^{1,2}_{\bar{x},\varepsilon}(\Omega))}, \quad (4.10)$$

and similarly as in the proof of Theorem 3.1 we obtain

$$\int_{\ell}^{2\ell+r} \langle F(u^s) - F(v^s), \varphi(s)\rangle_{\bar{x},\varepsilon}\, ds \le C\|w\|_{L^2(\ell,2\ell+r;L^2_{\bar{x},\varepsilon}(\Omega))}, \quad (4.11)$$

which gives

$$\|\partial_t L(r+\ell)\chi_1 - L(r+\ell)\chi_2\|_{L^2(-r,\ell;W^{-1,2}_{\bar{x},\varepsilon}(\Omega))}$$
$$\le C\left(\|L(r+\ell)\chi_1 - L(r+\ell)\chi_2\|_{L^2(-r,\ell;W^{1,2}_{\bar{x},\varepsilon}(\Omega))} + \|\chi_1-\chi_2\|_{\mathcal{X}} \right). \quad (4.12)$$

Combining (4.8) - (4.12) gives the desired continuity

$$\|L(r+\ell)\chi_1 - L(r+\ell)\chi_2\|_{\mathcal{W}} \le C\|\chi_1-\chi_2\|_{\mathcal{X}}.$$

□

The previous lemma also implies that the mapping $e : \mathcal{X} \to X$ is Lipschitz continuous, since we have the inequality

$$\|e(\chi)\|_X \le \|\chi\|_{\mathcal{W}} \le C\|\chi\|_{\mathcal{X}}. \quad (4.13)$$

Lemma 4.5. *The dynamical system $(\mathcal{X}, L(t))$ is asymptotically compact.*

Proof. Let $\chi_n \in \mathcal{X}$ be a bounded sequence and $t_n \to \infty$. We aim to show

$$L(t_n)\chi_n \to \chi \text{ in } L^2(-r,\ell,L^2_{\bar{x},\varepsilon}(\Omega)) \quad (4.14)$$

where $\chi \in \mathcal{X}$, up to a subsequence.

From Theorem 4.1 and Lemma 4.4 we see that $L(t_n)\chi_n$ is bounded in the norms $L^2(-r,\ell;W^{1,2}(B))$, $W^{1,2}(-r,\ell;W^{-1,2}(B))$, where $B \subseteq \Omega$ is an arbitrary compact set. The Aubins-Lions lemma implies that

$$L(t_n)\chi_n \to \chi \text{ in } L^2(-r,\ell;L^2(B))$$

and therefore $L(t_n)\chi_n \to \chi$ in $L^2_{loc}((-r,\ell)\times\Omega)$. Since the sequence is also bounded in $L^\infty(-r,\ell;L^2_b(\Omega))$, Lemma 2.5 immediately gives us the strong convergence (4.14).

Theorem 4.1 and Lemma 4.4 also imply

$$L(t_n)\chi_n \rightharpoonup \chi \text{ in } L^2(-r,\ell;W^{1,2}_{\bar{x},\varepsilon}(\Omega)), \qquad \partial_t L(t_n)\chi_n \rightharpoonup \partial_t\chi \text{ in } L^2(-r,\ell;W^{-1,2}_{\bar{x},\varepsilon}(\Omega)), \quad (4.15)$$

which together with the strong convergence (4.14), (3.2) and the Lipschitz continuity of F (3.31) justifies taking a limit in the equation in a similar manner as in the proof of existence (see Theorem 3.1) and thus $\chi \in \mathcal{X}$. □

Theorem 4.6. *The dynamical system $(X, S(t))$ has a $\left(L^2_b(-r,0;L^2(\Omega)), L^2_{loc}((-r,0)\times\Omega)\right)$ attractor.*

Proof. First we observe that the dynamical system $(\mathcal{X}, L)$ has a global attractor $\mathcal{A}_\ell$. By Theorem 4.1 it has a bounded absorbing set; the asymptotic compactness was proved in Lemma 4.5 and we apply a standard result (see e.g. [15, Theorem 23.12]). Define

$$\mathcal{A} = e(\mathcal{A}_\ell). \tag{4.16}$$

It remains to check whether $\mathcal{A}$ is the desired attractor.

Observe that

$$S(t)e(\chi) = e(L(t)\chi),$$

therefore $\mathcal{A}$ is invariant under $S(t)$. The compactness in $L^2_{loc}((-r,0)\times\Omega)$ follows from the compactness of $\mathcal{A}_\ell$, the continuity of $e : \mathcal{X} \to X$ and Lemma 2.5. To show that $\mathcal{A}$ attracts bounded sets of X, we observe that

$$S(r+\ell)B = e(B_\ell)$$

for $B \subseteq X$, where

$$B_\ell = \{\chi \in C([-r,\ell]; L^2_{\bar{x},\varepsilon}(\Omega)); \chi \text{ is a solution from Theorem 3.1 with the initial condition } (\varphi(0),\varphi) \text{ for } \varphi \in B\}.$$

By Theorem 4.1, the set B_ℓ is bounded in $\mathcal{X}$ for B bounded in X. Then we have the estimate

$$\begin{aligned}\operatorname{dist}_X(S(t+r+\ell)B,\mathcal{A}) &= \operatorname{dist}_X(S(t)S(r+\ell)B,\mathcal{A}) \le C_1 \operatorname{dist}_X(S(t)e(B_\ell),\mathcal{A})\\ &= C_1 \operatorname{dist}_X(E(L(t)B_\ell,\mathcal{A}) \le C_2 \operatorname{dist}_{\mathcal{X}}(L(t)B_\ell,\mathcal{A}_\ell),\end{aligned}$$

where the last estimate uses the Lipschitz continuity of e. □

5 Entropy estimates

We estimate the entropy of the attractor constructed in Theorem 4.6 using the general method presented in [20], that has been adapted to the setting of ℓ-trajectories and parabolic uniform spaces in [8]. Actually, the rest of the proof follows the latter article quite closely.

We need some preliminary results. First we formulate the Lipschitz continuity of the operators $L(t), e$ and the smoothing property in the context of parabolic uniformly bounded spaces.

Corollary 5.1. *Let ψ be an admissible weight function of growth rate smaller than 1. Then*

1. *$L(t) : L^2_{b,\psi}(-r,\ell;L^2(\Omega)) \to L^2_{b,\psi}(-r,\ell;L^2(\Omega))$ is Lipschitz continuous uniformly with respect to $t \in [0,T]$,*
2. *$e : L^2_{b,\psi}(-r,\ell;L^2(\Omega)) \to L^2_{b,\psi}(-r,0;L^2(\Omega))$ is Lipschitz continuous,*
3. *the mapping $L(\ell+r) : L^2_{b,\psi}(-r,\ell;L^2(\Omega)) \to W_{b,\psi}(Q)$, where $Q = [-r,\ell]\times\Omega$ and $W_{b,\psi}(Q)$ is defined in (2.21), is Lipschitz.*

The Lipschitz constants only depend on C and μ in (2.7) and not on the particular form of the weight function ψ.

Proof. Multiply (4.6) by $\psi(\bar{x})$ and take supremum over $\bar{x} \in \Omega$ to obtain

$$\sup_{\bar{x}\in\Omega}\int_{(t-r,t+\ell)\times\Omega} |w(s,x)|^2 e^{-\varepsilon|x-\bar{x}|}\,dx\,ds \le C(T)\sup_{\bar{x}\in\Omega}\psi(\bar{x})\int_{(-r,\ell)\times\Omega} |w(s,x)|^2 e^{-\varepsilon|x-\bar{x}|}\,dx\,ds.$$

The first assertion then follows from the equivalence of the norms (2.16).

The remaining assertions can be proved in a similar manner from a variant of (4.13), the equivalence of norms (2.16)-(2.18) and from Lemma 4.4. □

We will also need an auxiliary admissible weight function (cf. [20]). Let $x_0 \in \Omega$ and $R \ge 1$. Then we define

$$\psi(x_0,R) \equiv \psi(x_0,R)(x) = \begin{cases} 1 & \text{if } |x-x_0| \le R+\sqrt{d},\\ e^{(R+\sqrt{d}-|x-x_0|)/2} & \text{otherwise.}\end{cases} \tag{5.1}$$

Observe that $\psi(x_0, R)$ satisfies (2.7) with $\mu = 1/2$ and some $C > 0$ independent of $x_0 \in \mathbb{R}^N$ and $R \geq 1$.

We also define

$$\Omega_{x_0,R} = \Omega \cap B(x_0, R) \subseteq \mathbb{R}^N.$$

Lemma 5.2 ([8], Lemma 5.4)**.** *Let $\lambda_0 > 0$. Then there exists $c_1 > 0$ such that for every $x_0 \in \Omega$, $R \geq 1$, $\lambda \in (0, \lambda_0)$ and $\chi_1, \chi_2 \in \mathcal{X}$ we have*

$$\|\chi_1 - \chi_2\|^2_{L^2_{b,\psi(x_0,R)}(-r,\ell;L^2(\Omega))} \leq \max\{\|\chi_1 - \chi_2\|_{L^2_{b,\psi(x_0,R)}(-r,\ell;L^2(\Omega_{x_0,R(\lambda)}))}, \lambda\},$$

where

$$R(\lambda) = R + c_1 \left(1 + \log \frac{1}{\lambda}\right).$$

We are now ready to prove the entropy estimate.

Theorem 5.3. *Let $x_0 \in \Omega$ and $R > 0$. Then there exist $c_0, c_1, \lambda_0 > 0$ such that for every $x_0 \in \Omega, R \geq 1$ and $\lambda \in (0, \lambda_0)$ the entropy estimate*

$$H_\lambda \left(\mathcal{A}, L^2_b(-r, 0; L^2(\Omega_{x_0,R}))\right) \leq c_0 \left(R + c_1 \log \frac{1}{\lambda}\right)^N \log \frac{1}{\lambda} \tag{5.2}$$

holds.

Proof. The proof follows the proof of Theorem 5.1 in [8] almost word by word. First we observe that it suffices to prove a similar estimate for the global attractor of $(\mathcal{A}_\ell, \mathcal{X})$, namely

$$H_\lambda \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) \leq c_0 \left(R + c_1 \log \frac{1}{\lambda}\right)^N \log \frac{1}{\lambda}. \tag{5.3}$$

The estimate (5.2) then follows immediately using the relation (4.16), the Lipschitz continuity of e induced estimate

$$H_\lambda \left(\mathcal{A}, L^2_{b,\psi(x_0,R)}(-r, 0; L^2(\Omega))\right) \leq H_{\lambda/\kappa} \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right),$$

where $\kappa > 0$ is the Lipschitz constant of the mapping e from Corollary 5.1, and the obvious estimate

$$H_\lambda \left(\mathcal{A}, L^2_b(-r, 0; L^2(\Omega_{x_0,R}))\right) \leq H_\lambda \left(\mathcal{A}, L^2_{b,\psi(x_0,R)}(-r, 0; L^2(\Omega))\right).$$

The proof of the estimate (5.3) relies on the recurrent estimate

$$\begin{aligned} &H_{\alpha/2} \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) \\ &\qquad \leq H_\alpha \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) + c_0 \left(R + c_1 \log \frac{1}{\alpha}\right)^N. \end{aligned} \tag{5.4}$$

Then we may choose $\lambda_0 > 0$ large enough so that

$$H_{\lambda_0} \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) = 0$$

and for $\lambda \in (0, \lambda_0)$ we find $k \in \mathbb{N}$ such that $2^{-k}\lambda_0 \leq \lambda \leq 2^{-k+1}$. The estimate (5.3) then follows from the recurrent estimate (5.4) and the inequality $k \leq c \log 1/\lambda$ holding for all λ sufficiently small:

$$\begin{aligned} &H_\lambda \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) \\ &\quad \leq H_{2^{-k}\lambda_0} \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) - H_{\lambda_0} \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) \\ &\quad \leq \sum_{i=1}^{k} \Big\{ H_{2^{-i}\lambda_0} \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) \\ &\qquad - H_{2^{-i+1}\lambda_0} \left(\mathcal{A}_\ell, L^2_{b,\psi(x_0,R)}(-r, \ell; L^2(\Omega))\right) \Big\} \\ &\quad \leq \sum_{i=1}^{k} c_0 \left(R + c_1 \log \frac{2^{i-1}}{\lambda_0}\right)^N \leq c_0 \left(R + c_1 \log \frac{1}{\lambda}\right)^N \log \frac{1}{\lambda}. \end{aligned}$$

It remains to prove the recurrent estimate (5.4). Recall that the short trajectory attractor $\mathcal{A}_\ell$ is compact in $L^2(-r,0;L^2_{\bar{x},\varepsilon}(\Omega))$, it is also compact in $L^2_{b,\psi(x_0,R)}(-r,\ell;L^2(\Omega))$, therefore for every $\alpha>0$ we may find $m\in\mathbb{N}$ such that

$$\mathcal{A}_\ell \subseteq \bigcup_{i=1}^{m} B_\alpha(\chi_i; L^2_{b,\psi(x_0,R)}(-r,\ell;L^2(\Omega))).$$

Using Corollary 5.1 and the invariance of $\mathcal{A}_\ell$ we have

$$\mathcal{A}_\ell \subseteq \bigcup_{i=1}^{m} B_{\kappa\alpha}(\tilde{\chi}_i; W_{b,\psi(x_0,R)}(Q))$$

for some $\kappa>0$. Lemma 2.2 now implies that

$$H_{\alpha/2}\left(B_{\kappa\alpha}(\tilde{\chi}_i; W_{b,\psi(x_0,R)}(Q)), L^2_{b,\psi(x_0,R)}(-r,\ell;\Omega_{x_0,R(\alpha/2)})\right) \le \tilde{c}_0\left(R+\tilde{c}_1(1+\log\frac{2}{\alpha})\right)^N \le c_0\left(R+c_1\log\frac{1}{\alpha})\right)^N. \quad (5.5)$$

From Lemma 5.2 it follows that an $\alpha/2$-covering in $L^2_{b,\psi(x_0,R)}(-r,\ell;L^2(\Omega_{x_0,R}))$ is also an $\alpha/2$-covering in $L^2_{b,\psi(x_0,R)}(-r,\ell;L^2(\Omega))$, which finishes the proof. □

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5. Paper [II]

A sufficient and necessary condition for an infinite dimensional exponential attractor in locally uniform spaces

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(preprint)

A sufficient and necessary condition for an infinite dimensional exponential attractor in locally uniform spaces

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Abstract

An abstract criterion containing the sufficient and necessary condition for the existence of a discrete infinite dimensional exponential attractor in locally uniform spaces is shown and is applied to a nonlinear reaction-diffusion equation posed in the space $L^2_b(\mathbb{R}^d)$ with $d \leq 3$.

1 Introduction

Exponential attractors are one of the objects of interest in asymptotic analysis of differential equations. Compared to global attractors of finite fractal dimension, exponential attractors provide additional control on the rate of attraction of bounded sets while still retaining finite dimensionality. Exponential attractors are thoroughly studied in the literature and the existence of the exponential attractor has been established for many partial differential equations, see for example [6], [8], [9], [5] and the references therein.

The natural extension of the exponential attractor to problems posed in unbounded domains, more precisely in the context of locally uniform spaces, is the infinite dimensional exponential attractor introduced in [11]. Owing to the unboundedness of the spatial domain, the asymptotic properties of attractors can often be described only locally and in general we cannot expect the attractors to be globally compact. This is resolved by working with local topology and using Kolmogorov's ε-entropy instead of fractal dimension. In [11] the authors studied the reaction-diffusion equation

$$u_t - a\Delta u + (\mathbf{L}, \nabla)u + f(u) + \lambda_0 u = g$$

posed in the space $W^{2,p}_b(\mathbb{R}^d)$ with $p > \max\{2, d/2\}$, where $a \in \mathbb{R}^{d\times d}$ is a constant diffusion matrix with positive symmetric part, $\mathbf{L} \in (W^{1,\infty}_b(\mathbb{R}^d))^d$ is a vector field with sufficiently small divergence, $f : \mathbb{R} \to \mathbb{R}$ is a suitable reaction function, $\lambda_0 > 0$ and $g \in L^p_b(\mathbb{R}^d)$ is an external force. The requirements on p assure the embedding $W^{2,p}_b(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and allow the authors to investigate the necessary properties in $L^\infty(\mathbb{R}^d)$ while using the fact that the solution semigroup can be extended to the space $L^2_b(\mathbb{R}^d)$. A natural question immediately occurs – whether the same result can be obtained without the high regularity of the phase space.

In this paper we establish a sufficient and necessary condition of the existence of an infinite dimensional exponential attractor in an abstract setting suitable for the reaction-diffusion equation

$$u_t - \operatorname{div} a(\nabla u) + f(u) + h(\cdot, \nabla u) = g$$

posed in its natural energy space $L^2_b(\mathbb{R}^d)$, where a, f, h and g are specified in Section 4. The sufficient and necessary condition is similar to the one obtained in [15] for exponential attractors. We also briefly comment on the fact that this abstract setting is not directly extensible to the case of for example wave equations.

The attractors of the reaction-diffusion equation in unbounded domains posed in weighted spaces have been studied for example in [1], [4] and [3]. The attractors of the reaction-diffusion equation in locally uniform spaces have been thoroughly examined in [16], where lower and upper bounds on the Kolmogorov's ε-entropy of the attractor have been established. Other results include the study of spatio-temporal chaos in [17].

Apart from the above mentioned paper [11] the exponential attarctors of evolution equations in locally uniform spaces have been also studied in [10], where the authors study the infinite dimensional exponential attractor of a reaction-diffusion equation with external force vanishing in infinity, and in [7], where the existence of an infinite dimensional exponential attractor has been established for a fourth order nonlinear parabolic equation in the space $W^{4,2}_b(\mathbb{R}^3)$, a space also embedded into $L^\infty(\mathbb{R}^3)$.

The paper is organized as follows: Section 2 briefly reviews the locally uniform spaces and their properties. In Section 3 we establish the sufficient and necessary condition for the existence of an infinite dimensional exponential attractor in an abstract setting. Finally in Section 4 we apply the criterion from Section 3 to a nonlinear reaction-diffusion equation.

2 Function spaces

A measurable bounded function $\phi : \mathbb{R}^d \to (0, \infty)$ is called an *admissible weight function of growth rate* $\nu \geq 0$ if

$$c_\phi^{-1} e^{-\nu|x-y|} \leq \phi(x)/\phi(y) \leq c_\phi e^{\nu|x-y|} \tag{2.1}$$

for some $c_\phi > 0$ and all $x, y \in \mathbb{R}^d$ and further satisfies

$$|\nabla\phi(x)| \leq c\nu\phi(x) \tag{2.2}$$

for almost all $x \in \mathbb{R}^d$ and some $c > 0$. For $\varepsilon > 0$ and $\bar{x} \in \mathbb{R}^d$ we denote

$$\phi_{\bar{x},\varepsilon}(x) = e^{-\varepsilon|x-\bar{x}|}.$$

Clearly $\phi_{\bar{x},\varepsilon}$ is an admissible weight function of growth rate ε. The particular choice of the weight function $\phi_{\bar{x},\varepsilon}$ does not play any role in the following considerations as long as certain decay properties are met (for more information see [2, Section 4]).

For $p \in [1, \infty)$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we define the *weighted Lebesgue space* $L^p_{\varepsilon,\bar{x}}(\mathbb{R}^d)$ by

$$L^p_{\bar{x},\varepsilon}(\mathbb{R}^d) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d); \|u\|^p_{L^p_{\bar{x},\varepsilon}} = \int_{\mathbb{R}^d} |u(x)|^p \phi_{\bar{x},\varepsilon}(x)\, dx < \infty\}.$$

The weighted Sobolev spaces are defined in a straightforward manner. We also denote the dual to $W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by $W^{-1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d) = \left(W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d)\right)^*$.

Let C^R_x denote a closed cube of side $R > 0$ in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$, i.e.

$$C^R_x = \prod_{i=1}^d [x_i - R/2, x_i + R/2], \qquad x \in \mathbb{R}^d, \quad R > 0.$$

For an admissible weight function ϕ and $p \in [1, \infty)$ we define the *weighted locally uniform space* $L^p_{b,\phi}(\mathbb{R}^d)$ by

$$L^p_{b,\phi}(\mathbb{R}^d) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d); \|u\|^p_{L^p_{b,\phi}} = \sup_{k\in\mathbb{Z}^d} \phi(k)\|u\|^p_{L^p(C^1_k)} < \infty\}.$$

If $\phi \equiv 1$ we omit the subscript and write $L^p_b(\mathbb{R}^d)$ instead of $L^p_{b,1}(\mathbb{R}^d)$. The Sobolev variant of the weighted locally uniform spaces is again defined in an obvious manner.

The weighted Lebesgue spaces and the locally uniform spaces are connected through the equivalence of the locally uniform norm. The following result is standard and the proof may be found e.g. in [13, Theorem 2.1].

Theorem 2.1. *Let $\varepsilon > 0$, $1 \le p < \infty$, $k \in \mathbb{N}_0$ and let ϕ be an admissible weight function of growth rate $\mu < \varepsilon$. Then $u \in W^{k,p}_{b,\phi}(\mathbb{R}^d)$ if and only if*

$$\|u\|^p_{\tilde{W}^{k,p}_{b,\phi}} = \sup_{l\in\mathbb{Z}^d} \phi(l)\|u\|^p_{W^{k,p}_{l,\varepsilon}} < \infty.$$

Moreover, the norm $\|\cdot\|_{\tilde{W}^{k,p}_{b,\phi}}$ is equivalent to the norm $\|\cdot\|_{W^{k,p}_{b,\phi}}$ with constants dependent on ε, $\mu - \varepsilon$ and c_ϕ, where c_ϕ is the constant from (2.1).

For $\mathcal{O} \subseteq \mathbb{R}^d$ we denote

$$\mathbb{I}(\mathcal{O}) = \{k \in \mathbb{Z}^d; C^1_k \cap \mathcal{O} \neq \emptyset\}$$

and define the seminorms

$$\|u\|^p_{L^p_{b,\phi}(\mathcal{O})} = \sup_{k\in\mathbb{I}(\mathcal{O})} \phi(k)\|u\|^p_{L^p(C^1_k)} \tag{2.3}$$

with an obvious extension to the Sobolev spaces.

Denote $Q_\ell = (0, \ell) \times \mathbb{R}^d$. Let ϕ be an admissible weight function and $p \in [1, \infty)$. Then we define the *parabolic locally uniform space* $L^p_{b,\phi}(0, \ell; L^p(\mathbb{R}^d))$ by

$$L^p_{b,\phi}(0,\ell; L^p(\mathbb{R}^d)) = \{u \in L^p_{\mathrm{loc}}(Q_\ell); \\ \|u\|^p_{L^p_{b,\phi}(0,\ell;L^p)} = \sup_{k\in\mathbb{Z}^d} \phi(k)\|u\|^p_{L^p(0,\ell;L^p(C^1_k))} < \infty\}.$$

For $p = 2$ the Sobolev variant $L^2_{b,\phi}(0, \ell; W^{1,2}(\mathbb{R}^d))$ is given by

$$L^2_{b,\phi}(0, \ell; W^{1,2}(\mathbb{R}^d)) = \{u \in L^2_{\mathrm{loc}}(Q_\ell);\\ \|u\|^2_{L^2_{b,\phi}(0,\ell;W^{1,2})} = \sup_{k\in\mathbb{Z}^d} \phi(k)\|u\|^2_{L^2(0,\ell;W^{1,2}(C^1_k))} < \infty\}.$$

We also define

$$L^2_{b,\phi}(0, \ell; W^{-1,2}) = \{u : Q_\ell \to \mathbb{R};\\ \|u\|^2_{L^2_{b,\phi}(0,\ell;W^{-1,2})} = \sup_{k\in\mathbb{Z}^d} \phi(k)\|u\|^2_{L^2(0,\ell;W^{-1,2}(C^1_k))} < \infty\}.$$

Similarly as for the weighted spaces one can show that parabolic locally uniform spaces admit an equivalent norm utilizing the weighted spaces.

Theorem 2.2 ([13, Theorem 2.4])**.** *Let $\varepsilon > 0$, $p \in [1, \infty)$ and let ϕ be an admissible weight function of growth $0 \le \nu < \varepsilon$. Then the spaces $L^p_{b,\phi}(0, \ell; L^p(\mathbb{R}^d))$, $L^2_{b,\phi}(0, \ell; W^{1,2}(\mathbb{R}^d))$ and $L^2_{b,\phi}(0, \ell; W^{-1,2}(\mathbb{R}^d))$ admit equivalent norms*

$$\|u\|^p_{L^p_{b,\phi}(0,\ell;L^p)} \approx \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(t, x)|^p e^{-\varepsilon|x-\bar{x}|}\, dx\, dt,$$

$$\|u\|^2_{L^2_{b,\phi}(0,\ell;W^{1,2})} \approx \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} \left(|u(t, x)|^2 + |\nabla u(t, x)|^2\right) e^{-\varepsilon|x-\bar{x}|}\, dx\, dt,$$

$$\|u\|^2_{L^2_{b,\phi}(0,\ell;W^{-1,2})} \approx \sup_{v} \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} u(t, x)v(t, x)e^{-\varepsilon|x-\bar{x}|}\, dx\, dt,$$

where the first supremum in the last equivalence is taken over all functions $v \in L^2_{b,\phi}(0, \ell; W^{1,2}(\mathbb{R}^d))$ with less that unit norm, i.e.

$$\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} \left(|v(t, x)|^2 + |\nabla v(t, x)|^2\right) e^{-\varepsilon|x-\bar{x}|}\, dx \le 1.$$

Moreover, the constants of equivalence depend only on ε, $\nu - \varepsilon$ and c_ϕ in the notation of (2.1)*.*

Similarly as in (2.3) for $\mathcal{O} \subseteq \mathbb{R}^d$ we may define the respective seminorms

$$\|u\|^p_{L^p_{b,\phi}(0,\ell;L^p(\mathcal{O}))} = \sup_{k\in\mathbb{I}(\mathcal{O})} \phi(k)\|u\|^p_{L^p(0,\ell;L^p(C^1_k))}$$

with a straightforward extension to the spaces $L^2_{b,\phi}(0, \ell; W^{1,2}(\mathcal{O}))$ and $L^2_{b,\phi}(0, \ell; W^{-1,2}(\mathcal{O}))$.

3 Abstract criterion

Let X_k be closed subsets of some Banach space $\tilde{X}_k$ for every $k \in \mathbb{Z}^d$ and let X_b be the abstract locally uniform space defined by

$$X_b = \prod_{k\in\mathbb{Z}^d} X_k \quad \text{equipped with the norm} \quad \|x\|_{X_b} = \sup_{k\in\mathbb{Z}^d} \|x\|_{X_k}.$$

For $K \subseteq \mathbb{Z}^d$ we will denote

$$X_b(K) = \prod_{k \in K} X_k \quad \text{with the corresponding seminorm} \quad \|x\|_{X_b(K)} = \sup_{k \in K} \|x\|_{X_k}.$$

We emphasize that the above definition implicitly assumes that for given sequence $x_k \in X_k$ there exists $x \in X_b$ such that $x|_{\{k\}} = x_k$ for every $k \in \mathbb{Z}^d$, which allows for splicing of local elements x_k, $k \in \mathbb{Z}^d$, into one global element $x \in X_b$.

We define the local topology X_{loc} by

$$x_n \to x \text{ in } X_{\text{loc}} \quad \Leftrightarrow \quad x_n|_K \to x|_K \text{ in } X_b(K) \text{ for every } K \subseteq \mathbb{Z}^d \text{ finite.}$$

With a slight abuse of notation, we denote the cubes in $\mathbb{Z}^d$ of side R centred at $k \in \mathbb{Z}^d$ by

$$C_k^R = \{j \in \mathbb{Z}^d; \max_{i=1,\ldots,d} |j_i - k_i| \le R/2\}.$$

In the rest of this section let $S : X_b \to X_b$ be an operator. Recall that for a metric space M and a precompact set $K \subseteq M$ we define the *Kolmogorov's ε-entropy* by

$$H_\varepsilon(K, M) = \ln N_\varepsilon(K, M),$$

where $N_\varepsilon(K, M)$ denotes the smallest number of ε-balls in M with centres in K that cover the set K. Also for $B \subseteq X_b$ we denote

$$\|B\|_{X_b} = \sup_{b \in B} \|b\|_{X_b}.$$

We define the discrete infinite dimensional exponential attractor in the spirit of [11].

Definition. *A set $\mathcal{E} \subseteq X_b$ is called a* discrete infinite dimensional exponential attractor *of the discrete dynamical system (X_b, S) if*

1. *$\mathcal{E}$ is bounded in X_b and compact in X_{loc},*
2. *$\mathcal{E}$ is positively invariant under S, i.e. $S\mathcal{E} \subseteq \mathcal{E}$,*
3. *$\mathcal{E}$ exponentially attracts bounded sets in X_b, i.e. there exist $\gamma > 0$ and a monotone increasing function $Q : \mathbb{R}_0^+ \to \mathbb{R}^+$ such that for every $B \subseteq X_b$ bounded and every $n \in \mathbb{N}$ one has*
$$\operatorname{dist}_{X_b}(S^n B, \mathcal{E}) \le Q(\|B\|_{X_b}) e^{-\gamma n},$$
4. *there exist ε_0, c, $L > 0$ such that for every $k \in \mathbb{Z}^d$, $R \ge 1$ and $\varepsilon > 0$ the estimate*
$$H_\varepsilon\left(\mathcal{E}|_{C_k^R}, X_b(C_k^R)\right) \le c\left(\#C_k^{R + L \ln \varepsilon_0/\varepsilon}\right) \ln \frac{\varepsilon_0}{\varepsilon} \tag{3.1}$$
holds with the constants c, L, ε_0 independent of k, R and ε.

Theorem 3.1. *Let $S : X_b \to X_b$ be Lipschitz continuous and let $\mathcal{B} \subseteq X_b$ be a bounded absorbing set, i.e. for every bounded $B \subseteq X_b$ there exists $N = N(B)$ such that for every $n \ge N$ one has $S^n(B) \subseteq \mathcal{B}$. Let there exist $L > 0$ such that*

$$\|S^n u - S^n v\|_{X_k} \le L^n \sup_{l \in \mathbb{Z}^d} e^{-\alpha|k-l|} \|u - v\|_{X_l} \tag{3.2}$$

for every $k \in \mathbb{Z}^d$, $n \in \mathbb{N}$ and $u, v \in \mathcal{B}$. Then the dynamical system (X_b, S) has a discrete infinite dimensional exponential attractor if and only if for some ε_0, $L' > 0$ and $\theta \in (0,1)$ and every $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ we have

$$H_{\varepsilon_0\theta^{-n}}\left((S^n\mathcal{B})|_{C_k^n}, X_b(C_k^n)\right) \le c(\#C_k^{n+L'n})n \tag{3.3}$$

with constant c, ε_0, L' independent of $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$.

The estimate (3.2) corresponds to "exponentially finite" speed of propagation of information. As we will see in the following section, estimates of this type are natural for the solutions of dissipative evolution equations posed in locally uniform spaces even in the case when the solution of a similar problem posed in a bounded domain does not have a finite speed of propagation.

Proof. The structure of the locally uniform spaces is sufficiently similar to $L^\infty(\mathbb{R}^d)$ that the proof of the theorem may closely follow the proof of [11, Theorem 4.3] where a similar claim is shown in the space $L^\infty(\mathbb{R}^d)$. We will briefly go through the main ideas of the proof and comment on those parts which explicitly use the properties of the $L^\infty(\mathbb{R}^d)$-norm.

Step 1 - The condition (3.3) is necessary. This part is standard. Assume that $\mathcal{E}$ is a discrete infinite dimensional exponential attractor. Putting $\varepsilon = Q(\|\mathcal{B}\|_{X_b})e^{-\gamma n}$ in (3.1) gives

$$\begin{aligned} H_{Q(\|\mathcal{B}\|_{X_b})e^{-\gamma n}}\left(\mathcal{E}|_{C_k^n}, X_b(C_k^n)\right) &\le c\left(\#C_k^{n+L\ln\frac{\varepsilon_0 e^{\gamma n}}{Q(\|\mathcal{B}\|_{X_b})}}\right)\ln\left(\frac{\varepsilon_0 e^{\gamma n}}{Q(\|\mathcal{B}\|_{X_b})}\right) \\ &\le c'(\#C_k^{n+L'n})n. \end{aligned}$$

The set $\mathcal{E}$ attracts bounded sets in the topology X_b and thus

$$\operatorname{dist}_{X_b(C_k^n)}((S^n\mathcal{B})|_{C_k^n}, \mathcal{E}|_{C_k^n}) \le \operatorname{dist}_{X_b}(S^n\mathcal{B}, \mathcal{E}) \le Q(\|\mathcal{B}\|_{X_b})e^{-\gamma n}.$$

From this we readily observe

$$H_{2Q(\|\mathcal{B}\|_{X_b})e^{-\gamma n}}\left((S^n\mathcal{B})|_{C_k^n}, X_b(C_k^n)\right) \le C'(\#C_k^{n+L'n})n,$$

which is of the same form as (3.3) with $\theta = e^{-\gamma}$ and $\varepsilon_0 = 2Q(\|\mathcal{B}\|_{X_b})$.

Step 2 - Construction of the discrete infinite dimensional exponential attractor. Let $n \in \mathbb{N}$ be fixed and let $\varepsilon_0 > 0$ be such that $\mathcal{B} \subseteq B_{\varepsilon_0}(0, X_b)$. Let $\mathbb{M}^n \subseteq \mathbb{Z}^d$ be such that

$$\mathbb{Z}^d = \bigcup_{k\in\mathbb{M}^n}^{\infty} C_k^n, \qquad \text{and} \qquad C_k^n \cap C_l^n = \emptyset \quad \text{for } k, l \in \mathbb{M}^n,\ k \ne l. \tag{3.4}$$

For $k \in \mathbb{M}^n$ let

$$\mathbb{V}_k^n = \{x \in X_b(C_k^n); x \text{ is a center of a ball of the covering verifying (3.3) with } \varepsilon = 2^{-n}\varepsilon_0\}.$$

Clearly

$$\ln(\#\mathbb{V}_k^n) \le c\left(\#C_k^{L''n}\right)n. \tag{3.5}$$

Define

$$\widetilde{\mathbb{V}}^n = \{x \in X_b; x|_{C_k^n} \in \mathbb{V}_k^n \text{ for all } k \in \mathbb{M}^n\}.$$

We emphasize that this is precisely the moment where we need the splicing of local elements into elements in X_b. As usual we throw out the elements of $\widetilde{\mathbb{V}}^n$ that are not sufficiently close to the elements of $S^n\mathcal{B}$ in the locally uniform norm and define

$$\mathbb{V}^n = \{x \in \widetilde{\mathbb{V}}^n; \mathrm{dist}_{X_b}(x, S^n\mathcal{B}) \le \varepsilon_0 2^{-n}\}.$$

For the sets $\mathbb{V}^n$ we have the estimate

$$\mathrm{dist}^{\mathrm{sym}}_{X_b}(\mathbb{V}^n, S^n\mathcal{B}) \le \varepsilon_0 2^{-n}, \tag{3.6}$$

where $\mathrm{dist}^{\mathrm{sym}}_E(A, B)$ denotes the (symmetric) Hausdorff distance of sets A and B in E. Also for $m \in \mathbb{N}$ we have

$$\mathrm{dist}_{X_b}(\mathbb{V}^{n+m}, \mathbb{V}^n) \le \varepsilon_0 2^{1-n}. \tag{3.7}$$

and the entropy estimate

$$H_\varepsilon\left(\mathbb{V}^n|_{C_k^R}, X_b(C_k^R)\right) \le c\left(\#C_k^{R+L''n}\right) n \tag{3.8}$$

for $k \in \mathbb{Z}^d$, $R \ge 1$ and $\varepsilon > 0$. The proof is an abstract version of [11, Lemma 4.4]. The original proof carries over almost word for word with the following difference: in the proof of (3.8) we require that for $K_1, K_2 \subseteq \mathbb{Z}^d$, $K = K_1 \cup K_2$ and $B \subseteq X_b(K)$ relatively compact we have

$$H_\varepsilon\left(B, X_b(K)\right) \le H_\varepsilon\left(B|_{K_1}, X_b(K_1)\right) + H_\varepsilon\left(B|_{K_2}, X_b(K_2)\right). \tag{3.9}$$

Indeed, if we have $N_i \in \mathbb{N}$ such that

$$H_\varepsilon\left(B|_{K_i}, X_b(K_i)\right) \le \ln N_i \qquad \text{for } i = 1, 2,$$

then by the definition of $X_b(K)$, the number of required ε-balls to cover B in $X_b(K)$ is then N_1N_2, which verifies (3.9).

Returning to (3.8), let $m = \lceil R/n \rceil + 1$ and let $K \subseteq \mathbb{M}^n$, where $\mathbb{M}^n$ is the covering from (3.4), be the smallest finite subset such that

$$C_k^R \subseteq C_k^{mn} \subseteq \bigcup_{l \in K} C_l^n.$$

Observe that $\#K \le m^d$. Using (3.9) and (3.5) we estimate

$$\begin{aligned} H_\varepsilon\left(\mathbb{V}^n|_{C_k^R}, X_b(C_k^R)\right) &\le \sum_{l\in K} H_\varepsilon\left(\mathbb{V}^n|_{C_l^n}, X_b(C_l^n)\right) = \sum_{l \in K} \ln(\#\mathbb{V}^n|_{C_l^n}) \\ &\le c \sum_{l \in K}\left(\#C_l^{n+L'n}\right) n \le c(\#K)\left(\#C_k^{n+L'n}\right) n \\ &\le c'\left(\#C_k^{R+L''n}\right) n. \end{aligned}$$

Following a standard argument we set

$$\tilde{\mathcal{E}}_d = \bigcup_{l=0}^{\infty}\bigcup_{n=1}^{\infty} S^l\mathbb{V}^n, \qquad \mathcal{E}_d = \mathrm{cl}_{X_{\mathrm{loc}}} \tilde{\mathcal{E}}_d,$$

where $\mathrm{cl}_Y A$ denoted the closure of the set A in the topology of the space Y.

It can be easily seen that the set $\mathcal{E}_d$ is positively invariant and exponentially attracts bounded sets of X_b using the estimate (3.6). It remains to show that $\mathcal{E}_d$ satisfies the entropy estimate (3.1), which in turn leads to the local compactness of the closed complete set $\mathcal{E}_d$.

Step 3 - entropy estimate and compactness. Let us now prove (3.1). To this end we define the auxiliary set

$$\mathbb{V}^\infty = \bigcup_{n\in\mathbb{N}} \mathbb{V}^n.$$

Using exactly the same argument as in [11, Lemma 4.5] relying on the "exponentially finite" speed of propagation (3.2) and (3.6) we can show that

$$\mathrm{dist}_{X_b}(S^m\mathbb{V}^n, \mathbb{V}^\infty) \le \varepsilon_0 2^{2-\alpha(n+m)} \tag{3.10}$$

for some $\alpha > 0$ and that the entropy bound

$$H_\varepsilon\left(S^m\mathbb{V}^n|_{C_k^R}, X_b(C_k^R)\right) \le c\left(\#C_k^{R+L'(n+m+\ln\varepsilon_0/\varepsilon)}\right) n \tag{3.11}$$

holds for every $\varepsilon > 0$, $R > 0$ and $k \in \mathbb{Z}^d$. A standard argument relying on the properties of the distance and (3.7) and (3.10) leads to

$$\mathrm{dist}_{X_b}\left(\tilde{\mathcal{E}}_d, \bigcup_{l+n\le\kappa\ln\frac{\varepsilon_0}{\varepsilon}} S^l\mathbb{V}^n\right) \le \frac{\varepsilon}{2}$$

holding for some $\kappa > 0$ independent of $\varepsilon > 0$ sufficiently small. This justifies the estimate

$$N_\varepsilon\left(\mathcal{E}_d|_{C_k^R}, X_b(C_k^R)\right) \le \sum_{n+m\le\kappa\ln(\varepsilon_0/\varepsilon)} N_{\varepsilon/2}\left(S^m\mathbb{V}^n|_{C_k^R}, X_b(C_k^R)\right)$$

and employing (3.11), we finally arrive at the desired entropy bound (3.1).

The compactness in the topology $X_b(K)$ for $K \subseteq \mathbb{Z}^d$ then immediately follows from the entropy estimate (3.1) and the closedness of $\mathcal{E}_d$. □

From the proof of Theorem 3.1 we see that the assumption on splicing is essential. Let for a moment $X_b = W_b^{1,2}(\mathbb{R}^d)$ and let $x_k \in W^{1,2}(C_k^1)$ be such that

$$\sup_{k\in\mathbb{Z}^d} \|x_k\|_{W^{1,2}(C_k^1)} < \infty.$$

Then there obviously exists $x \in L_b^2(\mathbb{R}^d)$ such that $x|_{C_k^1} = x_k$ for every $k \in \mathbb{Z}^d$. However, in general $x \notin W_b^{1,2}(\mathbb{R}^d)$. It is also worth noting that this cannot be solved by mollification since then we might easily escape the respective absorbing set. In light of these findings it seems that the notion of an infinite dimensional exponential attractor as defined in [11] might be too demanding in a more general setting.

4 Nonlinear reaction-diffusion equation

We will study the equation

$$u_t - \operatorname{div} a(\nabla u) + f(u) + h(\cdot, \nabla u) = g \qquad \text{in } (0,\infty) \times \mathbb{R}^d, \tag{4.1}$$

where a represents a nonlinear diffusion, f is a reaction function, h is a convective term and $g \in L^2_b(\mathbb{R}^d)$ is an external force. The equation (4.1) is supplied with initial data

$$u(0) = u_0 \in L^2_b(\mathbb{R}^d).$$

In this section we assume that $\varepsilon = 1$ in the definition of $\phi_{\bar{x},\varepsilon}$ and we will often omit the symbol ε.

We assume that the function $a \in C(\mathbb{R}^d, \mathbb{R}^d)$ satisfies $a(0) = 0$ and

$$\begin{gathered}(a(\xi) - a(\eta)) \cdot (\xi - \eta) \geq \kappa |\xi - \eta|^2 \quad \text{for all } \xi, \eta \in \mathbb{R}^d,\\ |a(\xi) - a(\eta)| \leq c\kappa |\xi - \eta|, \qquad \text{for all } \xi, \eta \in \mathbb{R}^d,\\ \text{the function } \xi \to a(\xi) \cdot \xi \text{ is convex on } \mathbb{R}^d,\end{gathered}$$

with suitable constants $\kappa > 0$ and $c \geq 1$. The function $f \in C(\mathbb{R}, \mathbb{R})$ should be such that $f(0) = 0$ and

$$\begin{gathered}|f(r) - f(s)| \leq c\,(1 + |r| + |s|)^{p-2}\,|r - s|, \qquad \text{for all } r, s \in \mathbb{R},\\ (f(r) - f(s))\,(r - s) \geq -c|r - s|^2, \qquad \text{for all } r, s \in \mathbb{R},\\ c|r|^p - c' \leq f(r)r \leq c''\,(|r|^p + 1)\,, \qquad \text{for all } r \in \mathbb{R},\end{gathered}$$

for some $p > (2, \infty)$, c, c', $c'' > 0$. Finally the function $h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is such that

$$\begin{gathered}\xi \to h(x, \xi) \text{ is globally Lipschitz for a.e. } x \in \mathbb{R}^d,\\ x \to h(x, \xi) \text{ is measurable and essentially bounded for every } \xi \in \mathbb{R}^d.\end{gathered}$$

The weak solution is defined in the sense of distributions on $(0, \infty) \times \mathbb{R}^d$ and has the regularity

$$\begin{gathered}u \in C([0, T], L^2_{\bar{x}}(\mathbb{R}^d)) \cap L^2(0, T; W^{1,2}_{\bar{x}}(\mathbb{R}^d)) \cap L^p(0, T; L^p_{\bar{x}}(\mathbb{R}^d)),\\ u_t \in L^2(0, T; W^{-1,2}_{\bar{x}}(\mathbb{R}^d)) + L^{p'}(0, T; L^{p'}_{\bar{x}}(\mathbb{R}^d)),\end{gathered} \tag{4.2}$$

for every $\bar{x} \in \mathbb{R}^d$ and $T > 0$. Moreover, for an admissible weight function ϕ of growth rate $\mu \in [0, 1)$ and for a.e. $t \geq 0$ one has

$$u(t) \in L^2_{b,\phi}(\mathbb{R}^d), \qquad u \in L^2_{b,\phi}(t, t+1; W^{1,2}(\mathbb{R}^d)) \cap L^p_{b,\phi}(t, t+1; L^p(\mathbb{R}^d)).$$

The existence and uniqueness of weak solution of the problem (4.1) is shown in [13, Theorem 3.1] together with the existence of a closed positively invariant absorbing set $\widetilde{\mathcal{B}}$. We also have the following regularity result.

Theorem 4.1 ([13, Theorem 3.8])**.** *Let $d \leq 3$. For every $q \in (1, \infty)$ there exists a nonnegative function $Q : \mathbb{R}^+ \times \mathbb{R}^+_0 \to \mathbb{R}^+$ increasingly monotone in each of its arguments such that*

$$\|u(t)\|_{L^q_b} \leq Q(\tau^{-1}, \|u_0\|_{L^2_b}), \qquad t \geq \tau > 0, \quad u_0 \in L^2_b(\mathbb{R}^d),$$

where $u(t)$ is a weak solution of (4.1) *satisfying $u(0) = u_0$.*

From now on let $d \le 3$. Theorem 4.1 allows us to find a closed positively invariant absorbing set $\mathcal{B} = S(\tau)\widetilde{\mathcal{B}}$, where $\tau > 0$ is fixed, such that for every admissible weight function ϕ of growth rate $\mu \in [0,1)$ and $u_0 \in \mathcal{B}$ we have

$$\|u_0\|^2_{L^2_{b,\phi}} + \|u_0\|^p_{L^p_b} + \|u\|^2_{L^2_{b,\phi}(0,t;W^{1,2})} \le c_{t,p}, \tag{4.3}$$

where u is a solution of (4.1) with $u(0) = u_0$.

Let $S(t) : L^2_b(\mathbb{R}^d) \to L^2_b(\mathbb{R}^d)$ denote the solution semigroup of the equation (4.1).

Again we define the infinite dimensional exponential attractor as in [11].

Definition. *A set $\mathcal{E} \subseteq L^2_b(\mathbb{R}^d)$ is called an* infinite dimensional exponential attractor *of the dynamical system $(L^2_b(\mathbb{R}^d), S(t))$ if*

1. *$\mathcal{E}$ is bounded in $L^2_b(\mathbb{R}^d)$ and compact in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$,*
2. *$\mathcal{E}$ is positively invariant w.r.t. $S(t)$, i.e. $S(t)\mathcal{E} \subseteq \mathcal{E}$ for every $t \ge 0$,*
3. *the set $\mathcal{E}$ exponentially attracts bounded sets of $L^2_b(\mathbb{R}^d)$ in the locally uniform topology $L^2_b(\mathbb{R}^d)$, i.e. there exist $\gamma > 0$ and an increasing function $Q : \mathbb{R}^+_0 \to \mathbb{R}^+$ such that for every $B \subseteq L^2_b(\mathbb{R}^d)$ bounded and $t \ge 0$*

$$\mathrm{dist}_{L^2_b}(S(t)B, \mathcal{E}) \le Q(\|B\|_{L^2_b})e^{-\gamma t},$$

4. *there exist ε_0, c, $L > 0$ such that for $R \ge 1$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon \in (0, \varepsilon_0)$ we have the entropy estimate*

$$H_\varepsilon\left(\mathcal{E}|_{C^R_{\bar{x}}}, L^2_b(C^R_{\bar{x}})\right) \le c\left(R + L\ln\frac{\varepsilon_0}{\varepsilon}\right)^d \ln\frac{\varepsilon_0}{\varepsilon}$$

with constants c, L, ε_0 independent of ε, $\bar{x}$ and R.

Before we show that the dynamical system $(L^2_b(\mathbb{R}^d), S(t))$ has an infinite dimensional exponential attractor, we need to define the trajectory semigroup and review several of its properties. Let us define the trajectory space by

$$\mathcal{B}_\ell = \{\chi : [0,\ell] \times \mathbb{R}^d \to \mathbb{R}; \chi \text{ is a weak solution of (4.1) with } \chi(0) \in \mathcal{B}\}.$$

The trajectory semigroup $L(t) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ and the end-point mapping $e : \mathcal{B}_\ell \to \mathcal{B}$ are defined by

$$[L(t)\chi](s) = S(t+s)\chi(0), \quad s \in [0,\ell], \qquad e(\chi) = \chi(\ell).$$

We summarize the properties of S, L and e in the following lemma. Let us denote

$$\|\chi\|_{W_{b,\phi}} = \|\chi\|_{L^2_{b,\phi}(0,\ell;W^{1,2})} + \|\partial_t\chi\|_{L^2_{b,\phi}(0,\ell;W^{-1,2})}.$$

Lemma 4.2. *Let $\ell > 0$ and let ϕ be an admissible weight function of growth $\mu \in [0,1)$. The following holds:*

1. *the operators $S(t) : L^2_{b,\phi}(\mathbb{R}^d) \to L^2_{b,\phi}(\mathbb{R}^d)$ and $L(t) : L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d)) \to L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d))$ are uniformly Lipschitz continuous w.r.t. $t \in [0,T]$,*

2. *there exist c, $\alpha > 0$ such that*

$$\|S(n\ell)u - S(n\ell)v\|_{L^2_b(C^1_k)} \le c^n \sup_{l\in\mathbb{Z}^d} e^{-\alpha|l-k|}\|u-v\|_{L^2_b(C^1_l)} \tag{4.4}$$

for every $n \in \mathbb{N}$, $k \in \mathbb{Z}^d$ and every u, $v \in \mathcal{B}$,

3. *the mapping $e : L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d) \to L^2_{b,\phi}(\mathbb{R}^d)$ is Lipschitz continuous,*

4. *the set $\mathcal{B}_\ell$ is positively invariant under $L(t)$, i.e. $L(t)\mathcal{B}_\ell \subseteq \mathcal{B}_\ell$ for every $t \ge 0$,*

5. (smoothing property) *there exist $K > 0$ such that for every χ_1, $\chi_2 \in \mathcal{B}_\ell$ one has*

$$\|L(\ell)\chi_1 - L(\ell)\chi_2\|_{W_{b,\phi}} \le K\|\chi_1-\chi_2\|_{L^2_{b,\phi}(0,\ell;L^2)}. \tag{4.5}$$

Moreover, the Lipschitz constants of L, S and e and the constant K in (4.5) *are independent of the particular form of ϕ as long as the constants μ and c_ϕ from* (2.1) *remain the same.*

Proof. The set $\mathcal{B}_\ell$ is positively invariant immediately from the definitions. The uniform Lispchitz continuity of $S(t)$ and $L(t)$ has been shown in [13, Theorem 5.2] by the means of the estimate

$$\int_{\mathbb{R}^d} |w(t,x)|^2 e^{-|x-\bar{x}|}\,dx \le c\int_{\mathbb{R}^d} |w(s,x)|^2 e^{-|x-\bar{x}|}\,dx, \qquad 0 \le s \le t \le T',$$

where $w = u - v$ and $T' \ge 0$ is sufficiently large, and Theorems 2.1 and 2.2, respectively. Let us now show (4.4). Clearly

$$c\int_{C_{\bar{x}}} |w(t,x)|^2\,dx \le \int_{\mathbb{R}^d} |w(t,x)|^2 e^{-|x-\bar{x}|}\,dx$$

for some $c > 0$ independent of $\bar{x} \in \mathbb{R}^d$. On the other hand by taking $\bar{x} = k \in \mathbb{Z}^d$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} |w(s,x)|^2 e^{-|x-k|}\,dx &= \sum_{l\in\mathbb{Z}^d}\int_{C^1_l} |w(s,x)|^2 e^{-|x-k|}\,dx \\ &\le \sum_{l\in\mathbb{Z}^d}\int_{C^1_l} |w(s,x)|^2 e^{-|l-k|+|l-x|}\,dx \\ &\le c\sup_{l\in\mathbb{Z}^d}\left(e^{-|l-k|/2}\int_{C^1_l} |w(s,x)|^2\,dx\right)\left(\sum_{l\in\mathbb{Z}^d} e^{-|l-k|/2}\right)\end{aligned}$$

and the estimate (4.4) for $n = 1$ immediately follows by setting $t = \ell$, $s = 0$, since the last sum converges. For $n > 1$ the result follows from the Lipschitz continuity of $S(\ell)$ and the case $n = 1$. The Lipschitz continuity of e follows in a similar way. The smoothing property (4.5) has been shown in [13, Theorem 5.3]. □

For $\bar{x} \in \mathbb{R}^d$ and $R \ge 1$ we define the auxiliary weight function $\psi(\bar{x}, R)$ by

$$\psi(\bar{x},R)(x) = \begin{cases} 1, & |x-\bar{x}| \le R+\sqrt{d}, \\ \exp\left((R+\sqrt{d}-|x-\bar{x}|)/2\right), & \text{otherwise.}\end{cases}$$

Clearly $\psi(\bar{x}, R)$ is an admissible weight function of growth $1/2$ for every $\bar{x} \in \mathbb{R}^d$ and $R \geq 1$. Also for $B \subseteq L^2_b(0,\ell;L^2(\mathbb{R}^d))$, $\varepsilon > 0$, $\bar{x} \in \mathbb{R}^d$ and $R \geq 1$ one has

$$H_\varepsilon\left(B|_{C^R_{\bar{x}}}, L^2_b(0,\ell;L^2_b(C^R_{\bar{x}})\right) \leq H_\varepsilon\left(B, L^2_{b,\psi(\bar{x},R)}(0,\ell;L^2(R^d))\right). \tag{4.6}$$

Also it is easy to show that, using the notation of (2.1), $c_{\psi(\bar{x},R)} = 1$ for all $\bar{x} \in \mathbb{R}^d$ and $R \geq 1$.

Lemma 4.3 ([13, Lemma 5.4])**.** *Let $\nu_0 > 0$ be such that*

$$\mathcal{B}_\ell \subseteq B_{\nu_0}(\chi_0; L^2_b(0,\ell;L^2(\mathbb{R}^d))$$

for some $\chi_0 \in \mathcal{B}_\ell$. Then for every $\bar{x} \in \mathbb{R}^d$, $R \geq 1$, $\nu \in (0, \nu_0)$ and χ_1, $\chi_2 \in \mathcal{B}_\ell$ we have

$$\|\chi_1 - \chi_2\|_{L^2_{b,\psi(\bar{x},R)}(0,\ell;L^2)} \leq \max\{\nu, \|\chi_1 - \chi_2\|_{L^2_{b,\psi(\bar{x},R)}(0,\ell;L^2(C_k^{n+\ln(\nu_0/\nu)}))}\}.$$

We will also need the following explicit version of Aubin-Lions lemma.

Lemma 4.4 ([13, Lemma 2.6])**.** *Let $\mathcal{O} \subseteq \mathbb{R}^d$ satisfy*

$$\#\mathbb{I}(\mathcal{O}) \leq c_1 \operatorname{vol}(\mathcal{O}) \tag{4.7}$$

and let $r > 0$ and $\theta \in (0, 1)$ be given. Then

$$H_{\theta r}\left(B_r(\chi, W_{b,\phi}(\mathbb{R}^d))|_{\mathcal{O}}, L^2_{b,\phi}(0,\ell;L^2(\mathcal{O}))\right) \leq c_0 \operatorname{vol}(\mathcal{O})$$

with the constant c_0 depending on c_1, r and ℓ and independent of χ, r, $\mathcal{O}$ and ϕ as long as (4.7) *holds and the constants in* (2.1) *and* (2.2) *remain the same.*

We are now ready to prove the main theorem.

Theorem 4.5. *The equation* (4.1) *possesses an infinite dimensional exponential attractor.*

Proof. The proof combines the abstract criterion from Theorem 3.1 with the technique of [13, Theorem 5.1].

Let $\ell > 0$ be arbitrary and let $\varepsilon_0 > 0$ be such that $\mathcal{B} \subseteq B_{\varepsilon_0}(u, L^2_b(\mathbb{R}^d))$ for some $u \in L^2_b(\mathbb{R}^d)$. First we show that the discrete dynamical system $(L^2_b(\mathbb{R}^d), S(2n\ell))$ has a discrete infinite dimensional exponential attractor. The "exponentially finite" speed of propagation and the Lipschitz continuity of $S(2\ell)$ have been shown in Lemma 4.2, therefore by Theorem 3.1 it remains to establish the entropy estimate (3.3), i.e.

$$H_{\varepsilon_0 2^{-n}}\left(S(2n\ell)\mathcal{B}|_{C^n_k}, L^2_b(C^n_k)\right) \leq c(n + R'n)^d n \tag{4.8}$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ and some $R' > 0$.

Let $\nu_0 > 0$ and $\chi_0 \in \mathcal{B}_\ell$ be such that

$$\mathcal{B}_\ell \subseteq B_{\nu_0}(\chi_0; L^2_b(0, 2\ell; L^2(\mathbb{R}^d))) \qquad \text{and} \qquad \nu_0 > \frac{\varepsilon_0}{2c_e K},$$

where c_e is the Lipschitz constant of e and K is the constant from the smoothing property, cf. Lemma 4.2. For simplicity we assume that the constant K in the smoothing property (4.5) is the same for $L(\ell)$ and $L(2\ell)$ and $K > 1/2$. Since $S(2\ell n\mathcal{B}) = e(L(2\ell(n-1)+\ell)\mathcal{B}_\ell)$, using (4.6) and the Lipschitz continuity of e we observe

$$H_{\varepsilon_0 2^{-n}}\left(S(2\ell n\mathcal{B})|_{C_k^n}, L_b^2(C_k^n)\right) \leq H_{\varepsilon_0 2^{-n}}\left(S(2\ell n)\mathcal{B}, L^2_{b,\psi(k,n)}(\mathbb{R}^d)\right)$$
$$\leq H_{\varepsilon_0 2^{-n}/c_e}\left(L(2\ell(n-1)+\ell)\mathcal{B}_\ell, L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))\right).$$

Therefore it suffices to establish the entropy bound

$$H_{\varepsilon_0 2^{-n}/c_e}\left(L(2\ell(n-1)+\ell)\mathcal{B}_\ell, L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))\right) \leq c(n+R'n)^d n \quad (4.9)$$

for some $R' > 0$ and all $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$.

Denote $\rho = \nu_0 c_e/\varepsilon_0$ and let $n \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ be fixed. By smoothing property (4.5) we have

$$L(\ell)\mathcal{B}_\ell \subseteq B_{K\nu_0}(L(\ell)\chi_0; W_{b,\psi(k,n)}(\mathbb{R}^d)).$$

Using Lemma 4.4 with $\theta = \varepsilon_0/2c_eK\nu_0 < 1$ we get

$$H_{\varepsilon_0/2c_e}\left(L(\ell)\mathcal{B}_\ell|_{C_k^{n+\ln(\rho 2)}}, L^2_{b,\psi(k,n)}(0,\ell;L^2(C_k^{n+\ln(\rho 2)}))\right) \leq c(n+\ln(\rho 2))^d$$

and, since by Lemma 4.3 every $\varepsilon_0 2/c_e$-covering in $L^2_{b,\psi(k,n)}(0,\ell;L^2(C_k^{n+\ln(\rho 2)}))$ is also an $\varepsilon_0 2/c_e$-covering in the space $L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))$, we obtain

$$H_{\varepsilon_0/2c_e}\left(L(\ell)\mathcal{B}_\ell, L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))\right) \leq c(n+\ln(\rho 2))^d.$$

If $n = 1$ we are done, otherwise we proceed by finite induction. Assume that for $m < n$ the entropy bound

$$H_{\varepsilon_0 2^{-m}/c_e}\left(L(2\ell(m-1)+\ell)\mathcal{B}_\ell, L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))\right) \leq c(n+\ln(\rho 2^m))^d m$$

holds, that is

$$L(2\ell(m-1)+\ell)\mathcal{B}_\ell \subseteq \bigcup_{j=1}^{N_m} B_{\varepsilon_0 2^{-m}/c_e}(\chi_j^m; L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d)))$$

holds for some $\chi_j^m \in L(2\ell(m-1)+\ell)\mathcal{B}_\ell$ and $N_m \in \mathbb{N}$ such that $\ln N_m \leq c(n+\ln(\rho 2^m))^d m$. Applying $L(2\ell)$ to both sides of the above inclusion and using the smoothing property (4.5) for $L(2\ell)$ we obtain

$$L(2\ell m+\ell)\mathcal{B}_\ell \subseteq \bigcup_{j=1}^{N_m} B_{K\varepsilon_0 2^{-m}/c_e}(L(2\ell)\chi_j^m; W_{b,\psi(k,n)}(\mathbb{R}^d)).$$

Invoking Lemma 4.4 with $\theta = 1/2K$ we get

$$H_{\varepsilon_0 2^{-(m+1)}/c_e}\left(B_{K\varepsilon_0 2^{-m}/c_e}(L(2\ell)\chi_j^m; W_{b,\psi(k,n)}(\mathbb{R}^d))|_{C_k^{n+\ln(\rho 2^{m+1})}},\right.$$
$$\left.L^2_{b,\psi(k,n)}(0,\ell;L^2(C_k^{n+\ln(\rho 2^{m+1})}))\right) \leq c(n+\ln(\rho 2^{m+1}))^d$$

uniformly for $1 \le j \le N_m$, which again by Lemma 4.3 leads to

$$H_{\varepsilon_0 2^{-(m+1)}/c_e}\big(B_{K\varepsilon_0 2^{-m}/c_e}(L(2\ell)\chi_j^m; W_{b,\psi(k,n)}(\mathbb{R}^d)),\\ L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))\big) \le c(n+\ln(\rho 2^{m+1}))^d$$

uniformly for $1 \le j \le N_m$. Now we arrive to

$$\begin{aligned} H_{\varepsilon_0 2^{-(m+1)}/c_e} &\Big(L(2\ell m+\ell)\mathcal{B}_\ell, L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))\Big)\\ &\le H_{\varepsilon_0 2^{-m}/c_e}\Big(L(2\ell(m-1)+\ell)\mathcal{B}_\ell, L^2_{b,\psi(k,n)}(0,\ell;L^2(\mathbb{R}^d))\Big)\\ &\quad + c(n+\ln(\rho 2^{m+1}))^d\\ &\le c(n+\ln(\rho 2^{m+1}))^d(m+1), \end{aligned}$$

which finishes the proof of the entropy bound (4.9) and therefore also of the desired entropy bound (4.8) with $R' = \ln(\rho 2)$. By Theorem 3.1 the discrete dynamical system $(L^2_b(\mathbb{R}^d), S(2\ell n))$ has a discrete infinite dimensional exponential attractor $\mathcal{E}_d$.

The extension to the continuous time is standard and follows the argument of e.g. [12, Theorem 2.27]. We set

$$\mathcal{E} = \bigcup_{0\le t\le 2\ell} S(t)\mathcal{E}_d.$$

Then $\mathcal{E}$ is clearly bounded in $L^2_b(\mathbb{R}^d)$ and positively invariant w.r.t. $S(t)$. The exponential attraction follows from the exponential attraction of the discrete infinite dimensional exponential attractor $\mathcal{E}_d$ and the uniform Lipschitz continuity of $S(t)$ on finite time intervals. To obtain the entropy estimate, we use the additional regularity of the absorbing set $\mathcal{B}$ (4.3) and the regularity of the solution (4.2) to get

$$\|S(t_1)u - S(t_2)v\|_{L^2_{\bar x}} \le c\|u-v\|_{L^2_{\bar x}} + c|t_1-t_2|^{1/2}, \qquad u,v\in\mathcal{B}, \quad t_1,t_2\in[0,2\ell],$$

where the constant $c>0$ is independent of $\bar x \in \mathbb{R}^d$. Similarly as in Lemma 4.2 we get

$$\|S(t_1)u - S(t_2)v\|_{L^2_b(C^R_{\bar x})} \le \max\{\varepsilon, \|u-v\|_{L^2_b(C^{R+R''\ln(\varepsilon_0/\varepsilon)}_{\bar x})}\}\\ + c|t_1-t_2|^{1/2}, \qquad u,v\in\mathcal{B}, \quad t_1,t_2\in[0,2\ell],$$

for all $\bar x \in \mathbb{R}^d$ and $R \ge 1$ and some $R''>0$ independent of $\bar x$ and R. The desired entropy bound then follows form a standard argument. The compactness of $\mathcal{E}$ is then a direct corollary of finite entropy of $\mathcal{E}$. □

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6. Paper [III]

Semilinear damped wave equation in locally uniform spaces

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Semilinear damped wave equation in locally uniform spaces*

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Abstract

We study a damped wave equation with a nonlinear damping in the locally uniform spaces and prove well-posedness and existence of a locally compact attractor. An upper bound on the Kolmogorov's ε-entropy is also established using the method of trajectories.

1 Introduction

We study the semilinear damped wave equation

$$u_{tt} + g(u_t) - \Delta u + \alpha u + f(u) = h, \qquad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

where f and g are nonlinear continuous functions described in more detail in Section 3, with initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \qquad x \in \mathbb{R}^d.$$

We focus on proving the well-posedness of the problem in the context of locally uniform spaces, the existence of a locally compact attractor and mainly on establishing an upper bound on the Kolmogorov's ε-entropy. We use the method of trajectories introduced in [15], which has been previously used in a similar context for showing the finite dimensionality of the global attractor of (1.1) in bounded domains in [17]. However, the approach applied to the bounded domain problem cannot be used directly due to a different nature of embeddings in weighted spaces and requires a slightly different technique. To this end, we introduce a variant of locally uniform spaces which seems suitable for equations with a finite speed of propagation. Also as usual in locally uniform spaces, the problem has an inherent non-compactness and non-separability. In

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order to obtain the dissipation of energy, we formulate additional assumptions that allow for the nonlinearities in the equation to be superlinear. One could expect that a suitable control of dispersion could yield dissipative estimates under still weaker growth restrictions on the nonlinearities.

This equation has been intensely studied in the setting of bounded domains. The existence of a global attractor with supercritical nonlinearities has been shown in [8] and for critical nonlinearities in [19] with less restrictive conditions on the damping. The finite dimensionality has been discussed in [17] and has been achieved even for critical nonlinearities in [4] and [12]. In the case of linear damping, the results include the existence of a global attractor on manifolds [11], asymptotic regularity of solutions of perturbed damped wave equation with nonlinearity of arbitrary growth [21] and recently in [10] the existence and smoothness of a global attractor for the equation with critical nonlinearity in $d = 3$.

In the context of locally uniform spaces, a linearly damped wave equation has been studied in [7] and [22]. In [22] the author has also established an upper bound on the Kolmogorov's ε-entropy of the locally compact attractor. Some results, including well-posedness and the existence of a locally compact attractor, have also been shown for a strongly damped wave equation in [20] and recently for a wave equation with fractional damping in [18]. To the best of our knowledge, nonlinear damping in this setting has not yet been studied.

The paper is organized as follows: In Section 2 we review the locally uniform spaces. In Section 3, the well-posedness of the equation (1.1) is established. In Section 4, we discuss additional assumptions that lead to dissipative estimates. In Section 5, we introduce the trajectory setting and prove a local variant of squeezing property, which is used in Section 6 to establish the existence of the locally compact attractor and an upper bound on its Kolmogorov's ε-entropy.

2 Function spaces

In this section we review the basic facts about weighted Sobolev spaces and locally uniform spaces. These spaces and their relations have been studied in [23] and [2].

By an *admissible weight function of growth rate* $\nu \le 0$ we understand $\phi : \mathbb{R}^d \to (0, +\infty)$ measurable and bounded satisfying

$$C_\phi^{-1} e^{-\nu|x-y|} \le \phi(x)/\phi(y) \le C_\phi e^{\nu|x-y|} \tag{2.1}$$

for some $C_\phi \ge 1$ and every $x, y \in \mathbb{R}^d$.

For $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we define the *weight function* $\phi_{\bar{x},\varepsilon}$ *with center in* $\bar{x}$ *and decay rate* ε by

$$\phi_{\bar{x},\varepsilon}(x) = e^{-\varepsilon|x-\bar{x}|}. \tag{2.2}$$

Then clearly for every multiindex α there exists $C_\alpha > 0$ such that

$$|D^\alpha \phi_{\bar{x},\varepsilon}| \le C_\alpha \varepsilon^{|\alpha|} \phi_{\bar{x},\varepsilon}. \tag{2.3}$$

We emphasize that, thanks to [2, Proposition 4.1], the particular choice of the weight function (2.2) does not play any role in the definition of the locally uniform spaces below as long as certain decay properties are met. Also note that by the above definition, $\phi_{\bar{x},\varepsilon}$ is an admissible weight function with growth ε.

For $p \in [1,\infty)$ we define the *weighted Lebesgue space* $L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by

$$L^p_{\bar{x},\varepsilon}(\mathbb{R}^d) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d); \|u\|_{L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |u|^p \phi_{\bar{x},\varepsilon}\, dx\right)^{1/p} < \infty\}.$$

In the special case $p = 2$ we use the notation

$$\|u\|_{\bar{x},\varepsilon} \equiv \|u\|_{L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)}.$$

We denote the scalar product on $L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by $(\cdot,\cdot)_{\bar{x},\varepsilon}$. The scalar product on $L^2(\mathbb{R}^d)$ will be denoted by $(\cdot,\cdot)$. We will sometimes omit the space domain $\mathbb{R}^d$ in the notation of function spaces.

Clearly the embedding $L^p_{\bar{x},\varepsilon_1}(\mathbb{R}^d) \hookrightarrow L^p_{\bar{x},\varepsilon_2}(\mathbb{R}^d)$ holds for $\varepsilon_1 \leq \varepsilon_2$.

The weighted Sobolev spaces are defined in an obvious manner and allow the continuous embedding

$$W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$$

with $k \geq l$ and $q \geq p$ such that $W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{l,q}(\mathbb{R}^d)$ and $\tilde{\varepsilon} = \varepsilon q/p$. We stress out that the embedding

$$W^{1,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow L^q_{\bar{x},\varepsilon}(\mathbb{R}^d)$$

does not hold for any $q > p$.

The weighted spaces also allow certain compact embeddings. More precisely, let $k \geq l$ and $q \geq p$ be such that $W^{k,p}(B(0,1)) \hookrightarrow\hookrightarrow W^{l,q}(B(0,1))$. Then we have the compact embedding

$$W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow\hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$$

with $\tilde{\varepsilon} > \varepsilon p/q$, which gives that for example the embedding

$$\{u \in L^\infty(0,T_0; W^{1,2}_{\bar{x},\varepsilon}), u_t \in L^\infty(0,T_0; L^2_{\bar{x},\tilde{\varepsilon}})\} \hookrightarrow\hookrightarrow L^m(0,T_0; L^s_{\bar{x},\tilde{\varepsilon}}), \qquad (2.4)$$

where $1 < m < \infty$ and $1 \leq s < 2d/(d-2)$, is compact and continuous for $s = 2d/(d-2)$.

Let C_k denote a closed unit cube in $\mathbb{R}^d$ centered at $x_k \in (\mathbb{Z}/2)^d$, i.e.

$$C_k = \prod_{i=1}^{d} [x_{k,i} - 1/2, x_{k,i} + 1/2], \qquad k \in \mathbb{N}.$$

The weighted *locally uniform Lebesgue space* $L^p_{b,\phi}$ for $p \in [1,\infty)$ and an admissible weight function ϕ is defined by

$$L^p_{b,\phi}(\mathbb{R}^d) = \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^d); \|u\|_{L^p_{b,\phi}(\mathbb{R}^d)} = \sup_{k\in\mathbb{N}} \phi^{1/p}(x_k)\|u\|_{L^p(C_k)} < \infty\}.$$

If $\phi \equiv 1$, we omit the subscript and write for example L^2_b instead of $L^2_{b,1}$. For $p = 2$ we use a simplified notation

$$\|u\|_{b,\phi} \equiv \|u\|_{L^2_{b,\phi}(\mathbb{R}^d)}.$$

The spaces $L^p_{b,\phi}(\mathbb{R}^d)$ are neither separable nor reflexive.

Locally uniform Sobolev spaces are again constructed in a straightforward manner. The standard embeddings holding on bounded domains also hold for locally uniform spaces, namely $W^{1,2}_b(\mathbb{R}^d) \hookrightarrow L^{2d/(d-2)}_b(\mathbb{R}^d)$. However, none of these embeddings are compact.

The weighted Lebesgue spaces and the locally uniform spaces are connected through the equivalence of the locally uniform norm. The following lemmata are standard and their proofs can be found e.g. in [2, 9] and [23].

Lemma 2.1. *Let $\varepsilon > 0$, $1 \le p < \infty$, $k \in \mathbb{N}_0$ and let ϕ be an admissible weight function with growth $\nu < \varepsilon$. Then $u \in W^{k,p}_{b,\phi}(\mathbb{R}^d)$ if and only if*

$$\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})\|u\|^p_{W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)} < \infty.$$

Moreover, the norm

$$\|u\|^p_{\tilde{W}^{k,p}_{b,\phi}(\mathbb{R}^d)} := \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})\|u\|^p_{W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)}$$

is equivalent to the original $W^{k,p}_{b,\phi}(\mathbb{R}^d)$norm.

Lemma 2.2. *Let $p \in [1,\infty)$, $\varepsilon > 0$ and $\bar{x} \in \mathbb{R}^d$. Then there exist constants C_1 and C_2 such that*

$$C_1 \int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}(x)|u(x)|^p\,dx \le \int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}(x) \int_{B(x,1)} |u(y)|^p\,dy\,dx \le C_2 \int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}(x)|u(x)|^p\,dx.$$

Finally we define so-called *parabolic locally uniform space* $L^p_b(0,T;L^p(\mathbb{R}^d))$ by

$$L^p_b(0,T;L^p) = \{u : (0,T)\times\mathbb{R}^d \to \mathbb{R};\ \|u\|_{L^p_b(0,T;L^p)} := \sup_{k\in\mathbb{N}} \|u\|_{L^p(0,T;L^p(C_k))} < \infty\}.$$

These spaces and their weighted variants have been studied in [9].

3 Well-posedness for locally uniform data

In this section we prove the existence and uniqueness of weak solutions of (1.1) for infinite energy data. We will make use of the following energy spaces which arise in the case of (1.1) in unbounded domains

$$\Phi_{\bar{x},\varepsilon} = W^{1,2}_{\bar{x},\varepsilon} \times L^2_{\bar{x},\varepsilon}, \qquad \Phi_b = W^{1,2}_b \times L^2_b, \qquad \Phi_{\mathrm{loc}} = W^{1,2}_{\mathrm{loc}} \times L^2_{\mathrm{loc}}.$$

We consider Φ_b as the phase space for the asymptotic analysis. However, it is well known that the locally uniform spaces are not separable, hence there are problems with attaining the initial conditions and approximating less regular data. There are at least two ways how to overcome this inconvenience. The first one is to consider Sobolev spaces with the weight functions like $\phi_{\bar{x},\varepsilon}$ with better properties. The second way is to use a phase space which is defined as closure of smooth functions in $\|\cdot\|_{\Phi_b}$ (such approach was considered e.g. in [16]). Both settings combined with the finite speed of propagation property of wave equations lead to the uniqueness and existence result. We have chosen the second approach.

Let us denote $L^p_{\mathrm{loc}}(I; W^{k,p}_{\mathrm{loc}}(\Omega))$ the set of measurable functions u on $I \times \Omega$ such that for any compact $J \subseteq I$ and $K \subseteq \Omega$ we have $u \in L^p(J; W^{k,p}(K))$. Particularly, $u \in L^\infty_{\mathrm{loc}}(0,\infty; X)$ means that u is strongly (Bochner) measurable on $(0,\infty)$ and $u \in L^\infty_{\mathrm{loc}}(0,T;X)$ for any $T > 0$.

We impose the following requirements on the nonlinearities of studied equation:

(F1) $f \in C^1(\mathbb{R})$,

(F2) $\forall r \in \mathbb{R} : |f'(r)| \le \gamma_1(|r|^{p-1} + 1)$,

(F3) $f' \ge -\beta$,

(F4) $\liminf_{|r|\to\infty} f(r)/r > 0$,

(G1) $g \in C^1(\mathbb{R})$, $g(0) = 0$,
$g' \ge \gamma_5 > 0$ for some $\gamma_5 > 0$,

(G2) for every $r \in \mathbb{R}$:

$$\gamma_2|r|^{\mu+1} - \gamma_3 \le g(r)r \le \gamma_4(|r|^{\mu+1} + 1),$$

where $\gamma_j, \beta > 0$. In what follows, we consider the following set of parameters:

$$p \in \left(0, \frac{2^*}{2}\right] \quad \text{for } d > 2, \qquad p \in (0,\infty) \quad \text{for } d = 2, \quad \mu \in [1,\infty), \tag{3.1}$$

where $2^* = 2d/(d-2)$.

Also from (G1) we observe

$$\gamma_5|u-v|^2 \le (g(u) - g(v))\,(u-v) \tag{3.2}$$

holds for every $u, v \in \mathbb{R}$.

We use the notation

$$\begin{aligned} E[u](t) &= \frac{1}{2}\left(|\partial_t u(t)|^2 + |\nabla u(t)|^2 + \alpha|u(t)|^2\right), \\ F[u](t) &= E[u](t,x) + F(u(t)), \end{aligned}$$

where $F(r) = \int_0^r f(s)\,ds$. Also note that by (F3) we have

$$F(r) = \int_0^r f(s) + \beta s\,ds - \frac{\beta}{2}r^2 \le f(r)(r) + \frac{\beta}{2}r^2 \tag{3.3}$$

and from (F4) we obtain $f(r)r \ge \eta r^2 - C_\eta$ for $\eta > 0$ sufficiently small. Combining this with (3.3) gives

$$f(r)r = \kappa f(r) + (1-\kappa)f(r)r \ge \kappa F(r) - \left(\frac{\beta\kappa}{2} + \eta\,(1-\kappa)\right)r^2 - C_\eta(1-\kappa) \tag{3.4}$$

for $\kappa \in [0,1]$, $\eta > 0$.

Definition. *Let* u_0, $u_1 \in \Phi_b$ *and* $h \in L^2_b(0,\infty; L^2)$. *We call* $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ *a* weak solution *of* (1.1) *if for every* $\varepsilon > 0$ *and* $\bar{x} \in \mathbb{R}^d$ *we have*

$$\begin{aligned} (u, u_t) \in C([0,\infty), \Phi_{\bar{x},\varepsilon}), \quad & u_t \in L^{\mu+1}_{loc}(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}(\mathbb{R}^d)), \\ u(0) = u_0, \quad u_t(0) = u_1, \qquad & \|u\|^2_{W^{1,2}_b(\mathbb{R}^d)} + \|u_t\|^2_{L^2_b(\mathbb{R}^d)} \in L^\infty_{loc}((0,\infty)) \end{aligned} \tag{3.5}$$

and the equality

$$\begin{aligned} -\int_0^\infty (u_t(t), \psi_t(t,\cdot))\, dt + \int_0^\infty (\nabla u(t), \nabla\psi(t,\cdot))\, dt + \int_0^\infty (\alpha u(t), \psi(t,\cdot))\, dt \\ + \int_0^\infty (g(u_t(t)), \psi(t,\cdot))\, dt + \int_0^\infty (f(u(t)), \psi(t,\cdot))\, dt = \int_0^\infty (h(t), \psi(t,\cdot))\, dt \end{aligned} \tag{3.6}$$

holds for every test function $\psi \in \mathcal{D}((0,\infty) \times \mathbb{R}^d)$ *(or equivalently for every Lipschitz compactly supported function* ψ*).*

The identity (3.6) has an equivalent version closely connected to the energy space $\Phi_{\varepsilon,\bar{x}}$. By using $\psi\phi_{\varepsilon,\bar{x}}$ with $\psi \in \mathcal{D}((0,\infty) \times \mathbb{R}^d)$ as a test function in (3.6), we obtain

$$\begin{aligned} -\int_0^T \Big(u_t(t), \partial_t\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(g(u_t(t)), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \\ + \int_0^T \left(\Big(\nabla u(t), \nabla\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} + \Big(\alpha u(t), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon}\right) dt \\ + \int_0^T \Big(f(u(t)), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(\nabla u(t), \psi(t,\cdot)\nabla\phi_{\bar{x},\varepsilon}\Big)\, dt \\ = \int_0^T \Big(h(t), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \end{aligned} \tag{3.7}$$

By a standard density argument, the identity (3.7) holds also for any function $\psi\theta$ where $\theta = \theta(t)$ is a smooth function compactly supported in $(0,\infty)$ and

$$\psi \in L^\infty_{\mathrm{loc}}(0,\infty; W^{1,2}_{\bar{x},\varepsilon}(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,\infty; L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)) \cap L^{\mu+1}_{\mathrm{loc}}(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}(\mathbb{R}^d)).$$

Moreover, if

$$\psi \in C([0,\infty); W^{1,2}_{\bar{x},\varepsilon}(\Omega)) \cap L^{\mu+1}_{\mathrm{loc}}(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}(\mathbb{R}^d)) \tag{3.8}$$

then

$$\begin{aligned} -\int_0^T \Big(u_t(t), \partial_t\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \Big(u_t(T), \psi(T,\cdot)\Big)_{\bar{x},\varepsilon} - \Big(u_t(0), \psi(0,\cdot)\Big)_{\bar{x},\varepsilon} \\ + \int_0^T \Big(\nabla u(t), \nabla\psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(g(u_t(t)), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \\ + \int_0^T \Big(\alpha u(t), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt + \int_0^T \Big(f(u(t)), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt \\ = \int_0^T \Big(h(t), \psi(t,\cdot)\Big)_{\bar{x},\varepsilon} dt - \int_0^T \Big(\nabla u(t), \psi(t,\cdot)\nabla\phi_{\bar{x},\varepsilon}\Big)\, dt. \end{aligned} \tag{3.9}$$

is satisfied for every $T \in (0,\infty)$. To this end, we test (3.7) by $\psi\theta_n$ with

$$\theta_n'(t) = n\eta\left(n\left(t - \frac{1}{n}\right)\right)\chi_{\left(0,\frac{2}{n}\right)} - n\eta\left(n\left(t - T + \frac{1}{n}\right)\right)\chi_{\left(T-\frac{2}{n},T\right)}, \quad \theta_n(0) = 0$$

where η is the standard non-negative mollifier compactly supported in $(-1,1)$ and χ_I denotes the characteristic function of $I \subset \mathbb{R}$. Observe that $\theta' \rightharpoonup^* \delta_0 - \delta_T$ in the space of Radon measures on $[0,T]$ and $\theta \to 1$ in $L^s([0,T])$ for all $s \in [1,\infty)$. Hence, we conclude (3.9) by letting $n \to \infty$ and using the continuity of ψ with respect to time. The weak formulation is therefore equivalent to (3.9) with test functions (3.8).

The lack of regularity of u_t with respect to the space variables prevents us from using it as a test function in (3.9). On the other hand, one can test the weak formulation (3.9) by the time difference

$$D_\tau u(t,x) = \frac{u(t+\tau,x) - u(t-\tau,x)}{2\tau}$$

where we take $u(s,x) = u(0,x)$ for $s < 0$. Indeed, as $u \in AC([0,\infty),L^2_{\bar{x},\varepsilon})$ we have for a fixed $t \in (0,\infty)$

$$\left\|\frac{u(t+\tau,x) - u(t,x)}{\tau}\right\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}} = \left\|\frac{1}{\tau}\left(\int_t^{t+\tau} u_t(s)\,ds\right)\right\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}} \leq \frac{1}{\tau}\int_t^{t+\tau} \|u_t(s)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}}\,ds,$$

thus $D_\tau u \in L^{\mu+1}_{\text{loc}}\left(0,\infty; L^{\mu+1}_{\bar{x},\varepsilon}\right)$. In the rest of the paper, with an obvious abuse of terminology, we will use the phrase "testing by u_t" instead of taking the time differences as test functions and sending $\tau \to 0^+$. For more details see e.g. [13].

Theorem 3.1. *Let μ and p be as in* (3.1)*, i.e. $\mu \in [1,\infty)$ and $p \in (0, 2^*/2]$ for $d > 2$ or $p \in (0,\infty)$ for $d = 2$. Then for every $(u_0,u_1) \in \Phi_b$ and $h \in L^2_b(0,\infty;L^2)$ there exists a unique weak solution of* (1.1) *which satisfies the energy equality*

$$\begin{aligned}&\int_{\mathbb{R}^d} F[u](t_2)\phi_{\bar{x},\varepsilon}\,dx - \int_{\mathbb{R}^d} F[u](t_1)\phi_{\bar{x},\varepsilon}\,dx \\ &+ \int_{t_1}^{t_2}\Big(g(u_t),u_t\Big)_{\bar{x},\varepsilon}dt + \int_{t_1}^{t_2}\Big(\nabla u, u_t\nabla\phi_{\bar{x},\varepsilon}\Big)\,dt = \int_{t_1}^{t_2}\Big(h(t),u_t\Big)_{\bar{x},\varepsilon}dt \quad (3.10)\end{aligned}$$

for every $0 \leq t_1 < t_2 < \infty$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$.

We remark that both existence and uniqueness of solutions can be shown even in the so-called super-critical case, particularly when $\mu \in [1,\infty)$ and

$$p \in \left(\frac{2^*}{2}, 2^* - 1\right), \; p \leq \frac{2^*\mu}{\mu+1}, \quad f \in C^2(\mathbb{R}), \quad |f''(r)| \leq \gamma_1(|r|^{p-2}+1).$$

The existence part remains as is below. Uniqueness follows by combination of the approach presented in [3] for bounded domains with the localisation technique developed in [1, Section 7].

As usual in the context of locally uniform spaces, one cannot expect the strong time continuity of solutions in the phase space Φ_b. Taking $d = 1$, one can check that

$$u(t,x) = e^{-t/2}\theta(x-t)$$

with

$$\theta(x) = \int_0^x \sum_{n=1}^{\infty} (-1)^{n+1} n \chi_{n,n+\frac{1}{n^2}}(y)\, dy$$

is a weak solution of

$$u_{tt} - u_{xx} + u_t + \frac{1}{4}u = 0,$$
$$(u(0,x),u_t(0,x)) = \left(\theta(x), -\frac{\theta}{2}(x) - \theta'(x)\right) \in \Phi_b.$$

However,

$$\|u(t_1) - u(t_2)\|_{W_b^{1,2}} \geq \frac{1}{2e},$$

holds for all $t_1,\ t_2 \in (0,\delta)$, $t_1 \neq t_2$, provided $\delta > 0$ is small enough. Hence, $u\colon [0,\delta) \to W_b^{1,2}$ is not continuous as the range of u is not separable. Moreover, the function u is not strongly (Bochner) measurable.

Proof of Theorem 3.1. Assume that $\varepsilon > 0$ and $\bar{x}$ are given. It is sufficient to show existence of solutions on $(0,T)$ for fixed $T \in (0,\infty)$ independent on the initial data together with time continuity, particularly $(u,u_t) \in C([0,T];\Phi_{\bar{x},\varepsilon})$. The existence of global solutions then follows from a continuation argument.

Step 1 - approximations and solutions on bounded domains. We approximate the nonlinear term f by Lipschitz functions and the initial data by compactly supported data. Let $\{f^k\}_k$ be a sequence of functions such that for every $k \in \mathbb{N}$ the function $f^k \in C^1(\mathbb{R})$ is globally Lipschitz, satisfies (F1)-(F4), $f^k \to f$ pointwise, $f^k(t) = f(t)$ for $t \in (-k,k)$ and $|f^k| \leq |f|$.

For $k \in \mathbb{N}$, we define function

$$\phi^k(x) = \begin{cases} 1 & \text{for } x \in B(0,k), \\ k+1-|x| & \text{for } x \in B(0,k+1)\setminus B(0,k), \\ 0 & \text{for } x \in B(0,k+1)^c. \end{cases}$$

Let $u_0^k = \eta_k * (u_0\phi^k)$, $u_1^k = \eta_k * (u_1\phi^k)$, $h^k = \eta_k * (h\phi^k)$ where $\eta_k = k^d\eta(k|x|)$ and η is the standard mollifier. We get

$$(u_0^k,u_1^k) \to (u_0,u_1) \quad \text{in } \Phi_{\bar{x},\varepsilon}, \qquad \|(u_0^k,u_1^k)\|_{\Phi_{\bar{x},\varepsilon}} \leq \|(u_0,u_1)\|_{\Phi_{\bar{x},\varepsilon}} \tag{3.11}$$

$$h^k \to h \quad \text{in } L^2((0,T);L^2_{\bar{x},\varepsilon}), \qquad \|h^k\|_{L^2_{\bar{x},\varepsilon}} \leq \|h\|_{L^2_{\bar{x},\varepsilon}} \tag{3.12}$$

as a direct consequence of approximation by mollifiers and decay of $\phi_{\bar{x},\varepsilon}$.

Existence and uniqueness of strong solutions on bounded domains is a well known result (see e.g. [14]). The finite speed of propagation holds as the source term f^k is Lipschitz. Hence, for every $k \in \mathbb{N}$ we can construct

$$u^k \in W^{1,2}_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R}^d)) \cap L^2_{\mathrm{loc}}([0,\infty);W^{2,2}(\mathbb{R}^d)),\ u_t \in L^{\mu+1}_{\mathrm{loc}}([0,\infty);L^{\mu+1}(\mathbb{R}^d))$$

which is a global strong solution (the equation (3.13) is satisfied almost everywhere in $(0,T) \times \mathbb{R}^d$) of

$$u_{tt}^k + g(u_t^k) - \Delta u^k + \alpha u^k + f^k(u) = h^k, \qquad t > 0, \quad x \in \mathbb{R}^d \tag{3.13}$$

satisfying the initial conditions

$$u^k(0,x) = u_0^k(x), \quad u_t^k(0,x) = u_1^k(x), \qquad x \in \mathbb{R}^d.$$

Moreover, $u^k(t,\cdot)$ is compactly supported for any $t \in [0,\infty)$ and

$$(u^k,u_t^k) \in C([0,T]; W^{1,2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \hookrightarrow C([0,T]; \Phi_{\bar{x},\varepsilon}).$$

Step 2 - uniform estimates in weighted Lebesgue spaces. Let us multiply both sides of (3.13) by $u_t^k \phi_{\bar{x},\varepsilon}$ and integrate the resulting equality w.r.t. x over $\mathbb{R}^d$. We get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \left(E[u^k] + F^k(u^k)\right) \phi_{\bar{x},\varepsilon}\, dx + \int_{\mathbb{R}^d} g(u_t^k) u_t^k \phi_{\bar{x},\varepsilon}\, dx \\ = \int_{\mathbb{R}^d} h^k u_t^k \phi_{\bar{x},\varepsilon}\, dx - \int_{\mathbb{R}^d} \nabla u^k u_t^k \nabla \phi_{\bar{x},\varepsilon}\, dx \\ \leq \int_{\mathbb{R}^d} E[u^k] \phi_{\bar{x},\varepsilon}\, dx + \frac{1}{2} \int_{\mathbb{R}^d} |h^k|^2 \phi_{\bar{x},\varepsilon}\, dx, \end{aligned}$$

where F^k is the primitive function of f^k such that $F^k(0) = 0$. From Gronwall's lemma and condition (G3), we obtain

$$\begin{aligned} \|u^k(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u_t^k(\tau)\|^2_{L^2_{\bar{x},\varepsilon}} + \gamma_2 \int_0^\tau \|u_t^k(t)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}}\, dt \\ \leq e^\tau \Bigg(\|u^k(0)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u_t^k(0)\|^2_{L^2_{\bar{x},\varepsilon}} + \|F^k(u^k(0))\|_{L^1_{\bar{x},\varepsilon}} \\ + \int_0^\tau \frac{1}{2} \|h^k(t)\|^2_{L^2_{\bar{x},\varepsilon}}\, dt + \tau \int_{\mathbb{R}^d} \gamma_3 \phi_{\bar{x},\varepsilon}\, dx \Bigg) \end{aligned} \tag{3.14}$$

for arbitrary $\tau \in (0,T)$. Therefore,

$$\sup_{\tau \in (0,T)} \left(\|u^k(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u_t^k(\tau)\|^2_{L^2_{\bar{x},\varepsilon}} \right) + \int_0^T \|u_t^k(t)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}}\, dt \leq C$$

for some $C > 0$ depending only on u_0, u_1, h and T. Applying the basic weak compactness arguments and (2.4), there is a subsequence of $\{u^k\}_{n\in\mathbb{N}}$ (not relabelled) and measurable functions u, $\bar{f}$, $\bar{g}$ such that

$$\begin{aligned} (u^k,u_t^k) &\rightharpoonup^* (u,u_t) && \text{in } L^\infty((0,T); \Phi_{\bar{x},\varepsilon}), \\ u_t^k &\rightharpoonup u_t && \text{in } L^{\mu+1}((0,T); L^{\mu+1}_{\bar{x},\varepsilon}), \\ u^k &\rightharpoonup^* u && \text{in } L^\infty \left((0,T); L^{2^*}_{\bar{x},\frac{2^*}{p}\varepsilon}\right), \\ f^k(u^k) &\rightharpoonup^* \bar{f} && \text{in } L^\infty \left((0,T); L^{\frac{2^*}{p}}_{\bar{x},\frac{2^*}{p}\varepsilon}\right), && (3.15) \\ g(u_t^k) &\rightharpoonup \bar{g} && \text{in } L^{(\mu+1)/\mu} \left((0,T); L^{(\mu+1)/\mu}_{\bar{x},\varepsilon}\right), && (3.16) \\ u^k &\to u && \text{almost everywhere in } (0,T) \times \mathbb{R}^d. && (3.17) \end{aligned}$$

Using supremum over $\bar{x} \in \mathbb{R}^d$ on both sides of (3.14) gives us (cf. Lemma 2.1)

$$\sup_{t \in (0,T)} \left(\|u^k(t)\|^2_{W^{1,2}_b} + \|u^k_t(t)\|^2_{L^2_b} \right) + \sup_{\bar{x} \in \mathbb{R}^d} \int_0^T \|u^k_t(t)\|^{\mu+1}_{L^{\mu+1}_{\bar{x},\varepsilon}} \, dt \le C \tag{3.18}$$

where $C > 0$ depends only on $\|(u_0,u_1)\|_{\Phi_b}$, $\|h\|_{L^2_b(0,\infty;L^2)}$ and $T > 0$. Thus, using (3.17), (3.15) and assumptions on f^k together with the embedding

$$W^{1,2}_b \hookrightarrow L^{2^*}_b \hookrightarrow L^{2^*}_{\bar{x},\varepsilon},$$

we have $f^k(u^k)$ uniformly bounded in $L^\infty\left((0,T); L^{\frac{2^*}{p}}_{\bar{x},\varepsilon}\right)$, therefore

$$f^k(u^k) \to f(u) \quad \text{in } L^r\left((0,T); L^q_{\bar{x},\varepsilon}\right) \tag{3.19}$$

for any $q \in \left[1, \frac{2^*}{p}\right)$ and $r \in [1,\infty)$, hence $\bar{f} = f(u)$.

Step 3 - stability in $C([0,T]; \Phi_{\bar{x},\varepsilon})$ and existence. Let us subtract the equation for u^l from the equation for u^k, multiply the difference by $(u^k_t - u^l_t)\phi_{\bar{x},\varepsilon}$ and integrate over $\mathbb{R}^d$ with respect to x. Using the monotonicity of g and standard estimates, we obtain

$$\begin{aligned}
&\|u^k(\tau) - u^l(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u^k_t(\tau) - u^l_t(\tau)\|^2_{L^2_{\bar{x},\varepsilon}} \\
&\quad \le \|u^k(0) - u^l(0)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u^k_t(0) - u^l_t(0)\|^2_{L^2_{\bar{x},\varepsilon}} + \|h^k - h^l\|^2_{L^2\left((0,\tau); L^2_{\bar{x},\varepsilon}\right)} \\
&\qquad + \|f^k(u^k) - f^l(u^l)\|_{L^{(\mu+1)/\mu}\left((0,\tau); L^{(\mu+1)/\mu}_{\bar{x},\varepsilon}\right)} \|u^k_t - u^l_t\|_{L^{\mu+1}\left((0,\tau); L^{\mu+1}_{\bar{x},\varepsilon}\right)} \\
&\qquad + C \int_0^\tau \left(\|u^k(t) - u^l(t)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u^k_t(t) - u^l_t(t)\|^2_{L^2_{\bar{x},\varepsilon}} \right) dt.
\end{aligned}$$

for every $\tau \in (0,T)$. From Gronwall's lemma, we infer that

$$\begin{aligned}
&\sup_{\tau \in (0,T)} \|u^k(\tau) - u^l(\tau)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u^k_t(\tau) - u^l_t(\tau)\|^2_{L^2_{\bar{x},\varepsilon}} \\
&\le C(T) \left(\|u^k(0) - u^l(0)\|^2_{W^{1,2}_{\bar{x},\varepsilon}} + \|u^k_t(0) - u^l_t(0)\|^2_{L^2_{\bar{x},\varepsilon}} + \|h^k - h^l\|_{L^2(0,\tau; L^2_{\bar{x},\varepsilon})} \right) \\
&\quad + C(T) \|f^k(u^k) - f^l(u^l)\|_{L^{(\mu+1)/\mu}\left((0,\tau); L^{(\mu+1)/\mu}_{\bar{x},\varepsilon}\right)} \|u^k_t - u^l_t\|_{L^{\mu+1}\left((0,\tau); L^{\mu+1}_{\bar{x},\varepsilon}\right)}.
\end{aligned} \tag{3.20}$$

Observe that (3.20) together with (3.11), (3.12), (3.16) and (3.19) gives

$$(u^k, u^k_t) \to (u, u_t) \quad \text{in } C([0,T]; \Phi_{\bar{x},\varepsilon}). \tag{3.21}$$

Finally, we conclude that $u^k_t \to u_t$ almost everywhere in $(0,T) \times \mathbb{R}^d$ which in combination with (3.16) implies that $\bar{g} = g(u_t)$. Summing up the results on convergence given above and noting that u^k satisfies (3.6), we can pass to the limit in (3.9).

The energy equality (3.10) holds for u^k and using the convergence results above, in particular (3.21), it follows that it holds also for u. The relation (3.5) follows from Gronwall's lemma and taking supremum over $\bar{x} \in \mathbb{R}^d$ (see also estimates leading to (3.18)).

Step 4 - uniqueness of weak solutions. Let us test the weak formulation for u^1 and u^2 by $(u^1-u^2)_t$ keeping in mind that actually we test by $D_\tau[u^1-u^2]$ and send $\tau \to 0$. Subtracting both equalities. we obtain the energy equality in the following form:

$$\begin{aligned}\int_{\mathbb{R}^d} E(u^1(\tau)-u^2(\tau))\phi_{\bar{x},\varepsilon}\,dx + \int_0^\tau\int_{\mathbb{R}^d} \left(f(u^1)-f(u^2)\right)(u^1_t-u^2_t)\phi_{\bar{x},\varepsilon}\,dx\,dt \\ + \int_0^\tau\int_{\mathbb{R}^d} \left(g(u^1_t)-g(u^2_t)\right)(u^1_t-u^2_t)\phi_{\bar{x},\varepsilon}\,dx\,dt \\ = \int_0^\tau\int_{\mathbb{R}^d} \left(\nabla u^1-\nabla u^2\right)(u^1_t-u^2_t)\nabla\phi_{\bar{x},\varepsilon}\,dx\,dt\end{aligned}$$

for any $\tau \in (0,T)$; hence, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} E(u^1(\tau)-u^2(\tau))\phi_{\bar{x},\varepsilon}\,dx \\ &\le 2\int_0^\tau\int_{\mathbb{R}^d} E(u^1(t)-u^2(t))\phi_{\bar{x},\varepsilon}\,dx\,dt + \frac{1}{2}\int_0^\tau\int_{\mathbb{R}^d} |f(u^1)-f(u^2)|^2\phi_{\bar{x},\varepsilon}\,dx\,dt\end{aligned}$$

according to the monotonicity of g. Using assumptions (F2) and (3.5), we get

$$\int_{\mathbb{R}^d} E(u^1(\tau)-u^2(\tau))\phi_{\bar{x},\varepsilon}\,dx \le C\int_0^\tau\int_{\mathbb{R}^d} E(u^1(t)-u^2(t))\phi_{\bar{x},\varepsilon}\,dx\,dt. \tag{3.22}$$

Indeed, the inner integral containing f can be estimated using Lemma 2.2 as follows:

$$\begin{aligned}&\int_{\mathbb{R}^d} \left|f(u^1(t,x))-f(u^2(t,x))\right|^2\phi_{\bar{x},\varepsilon}\,dx \\ &\quad\le C_1\int_{\mathbb{R}^d}\phi_{\bar{x},\varepsilon}(x)\left\|f(u^1(t,\cdot))-f(u^2(t,\cdot))\right\|^2_{L^2(B(x,1))}\,dx \\ &\quad\le C_2\int_{\mathbb{R}^d}\phi_{\bar{x},\varepsilon}(x)\left\|u^1(t,\cdot)-u^2(t,\cdot)\right\|^2_{L^{2^*}(B(x,1))}\,dx \\ &\quad\le C_3\int_{\mathbb{R}^d}\phi_{\bar{x},\varepsilon}(x)\left\|u^1(t,\cdot)-u^2(t,\cdot)\right\|^2_{W^{1,2}(B(x,1))}\,dx \\ &\quad\le C_4\int_{\mathbb{R}^d} E(u^1(t)-u^2(t))\phi_{\bar{x},\varepsilon}\,dx,\end{aligned}$$

where $C_2 = C_2\left(\|u^1\|_{L^\infty((0,T);W^{1,2}_b)},\|u^2\|_{L^\infty((0,T);W^{1,2}_b)}\right)$.

Hence, $u^1(t) = u^2(t) \in \Phi_{\bar{x},\varepsilon}$ almost everywhere in $[0,T]$ as a consequence of Gronwall's lemma. □

Theorem 3.2. *The solution operator $S(T) : \Phi_b \to \Phi_b$ defined by*

$$S(T)(u_0,u_1) = (u(T),u_t(T))$$

where $(u(T),u_t(T))$ is the weak solution of (1.1) with $(u(0),u_t(0)) = (u_0,u_1)$, is locally Lipschitz. Moreover, if $\mathcal{B} \subseteq \Phi_b$ is bounded, then $S(T) : (\mathcal{B},\|\cdot\|_{\Phi_{\bar{x},\varepsilon}}) \to (\Phi_b,\|\cdot\|_{\Phi_{\bar{x},\varepsilon}})$ is Lipschitz.

Proof. Assume that (u_0,u_1), $(v_0,v_1) \in \mathcal{B}$. Following the same line as in the proof of uniqueness, we obtain (3.22). Standard application of Gronwall's lemma gives

$$\|(u-v)(T)\|_{\Phi_{\bar{x},\varepsilon}} \leq C(T,\mathcal{B})\|(u-v)(0)\|_{\Phi_{\bar{x},\varepsilon}}. \tag{3.23}$$

Finally, applying supremum over $\bar{x} \in \mathbb{R}^d$ on both sides of (3.23), from Lemma 2.1 we infer

$$\|(u-v)(T)\|_{\Phi_b} \leq \tilde{C}(T,\mathcal{B})\|(u-v)(0)\|_{\Phi_b}. \qquad \square$$

4 Dissipation of energy

In contrast to the bounded domain case, the energy of the solutions does not necessarily decrease over time. This may be attributed to the last element in (3.7) and the absence of the embeddings between the weighted spaces of the same weight. Thus, it seems that an additional assumption has to be made in order to show any dissipation of energy. As we will see below, we can either have linearly bounded g and possibly superlinear function f, or we can ensure the dissipation by connecting the growths of the functions f and g. By a *dissipation assumption*, we understand one of the following:

(D1) $\mu = 1$,

(D2) $\mu \in (1,(d+2)/(d-2))$ and there exists $\kappa \in (0,1)$ and $C > 0$ such that

$$-g(r)s \leq \kappa f(s)s + C\left(g(r)r + 1\right) \qquad \forall r, s \in \mathbb{R}.$$

The assumption (D1), i.e. linearly bounded damping, is well studied in the case of the bounded domain. The assumption (D2) is a variant of an assumption from [5] and allows for example the use of the functions

$$g(r) = r|r|^{\mu-1}, \quad f(s) = |s|^{p-1}s - as, \qquad \text{where } \mu \in [1,3) \text{ and } p \in [\mu,3) \tag{4.1}$$

with $d = 3$ and $0 < a < \alpha$. However, we remark that the assumption (D2) implicitly gives restriction on the admissible growth exponent μ, for instance in the example (4.1) we observe that $\mu \leq 3$, though formally we allow $\mu < 5$. The assumption (D2) seems to be unnatural since it prohibits the following simple choice of nonlinearities $f(s) = 0$, $g(r) = r|r|$. In the future, we hope that a less restrictive condition than (D2) will be obtained.

We emphasize that the upper entropy bound established the last section does not depend on the particular choice of the dissipation condition.

Lemma 4.1. *Let either of the conditions (D1), (D2) hold. Then there exist* $\varepsilon, \zeta > 0, C_0, C_1 > 0$ *such that for every weak solution* $(u(t), u'(t))$ *with initial condition* $(u_0, u_1) \in \Phi_b$ *the estimate*

$$\int_{\mathbb{R}^d} F[u](T)\phi_{\bar{x},\varepsilon}\,dx \leq C_1 e^{-\zeta T}\int_{\mathbb{R}^d} F[u](0)\phi_{\bar{x},\varepsilon}\,dx + C_0 \tag{4.2}$$

holds for all $T > 0$.

Proof. Let $T > 0$ and $t_1, t_2 \in [0, T], t_1 < t_2$. We test the equation by $u_t + \delta u$, where $\delta > 0$ will be determined later. We obtain the equality

$$\begin{aligned}
&\int_{\mathbb{R}^d} F[u](t_2)\phi_{\bar{x},\varepsilon}\,dx + \delta\Big(u_t(t_2), u(t_2)\Big)_{\bar{x},\varepsilon} - \int_{\mathbb{R}^d} F[u](t_1)\phi_{\bar{x},\varepsilon}\,dx \\
&\quad - \delta\Big(u_t(t_1), u(t_1)\Big)_{\bar{x},\varepsilon} + \int_{t_1}^{t_2} \Big(g(u_t(t)), u_t(t)\Big)_{\bar{x},\varepsilon}\,dt - \delta \int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\,dt \\
&\qquad + \delta \int_{t_1}^{t_2} \Big(f(u(t)), u(t)\Big)_{\bar{x},\varepsilon}\,dt + \delta \int_{t_1}^{t_2} \|\nabla u(t)\|^2_{\bar{x},\varepsilon} + \alpha\|u(t)\|^2_{\bar{x},\varepsilon}\,dt \\
&\qquad = \int_{t_1}^{t_2} \Big(h(t), u_t(t) + \delta u(t)\Big)_{\bar{x},\varepsilon}\,dt - \delta \int_{t_1}^{t_2} \Big(g(u_t(t)), u(t)\Big)_{\bar{x},\varepsilon}\,dt \\
&\qquad\qquad - \int_{t_1}^{t_2} \Big(\nabla u(t), (u_t(t) + \delta u(t))\nabla\phi_{\bar{x},\varepsilon}\Big)\,dt. \qquad (4.3)
\end{aligned}$$

For $\delta_1 \in (0, 1)$ and $\eta > 0$, we use (3.4) to get

$$\begin{aligned}
\int_{t_1}^{t_2} \Big(f(u(t)), u(t)\Big)_{\bar{x},\varepsilon}\,dt &\geq \delta_1 \int_{t_1}^{t_2} \int_{\mathbb{R}^d} F(u(t))\phi_{\bar{x},\varepsilon}\,dx\,dt \\
&\quad - \left(\frac{\delta_1\beta}{2} + \eta\,(1 - \delta_1)\right) \int_{t_1}^{t_2} \|u(t)\|^2_{\bar{x},\varepsilon}\,dt - C_\eta(1 - \delta_1)C_\varepsilon(t_2 - t_1) \qquad (4.4)
\end{aligned}$$

for some $C_\eta > 0$. Also for $\delta_2 > 0$ we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big(h, u_t(t) + \delta u(t)\Big)_{\bar{x},\varepsilon}\,dt \\
&\qquad \leq \frac{1}{2\delta_2} \int_{t_1}^{t_2} \|h(t)\|^2_{\bar{x},\varepsilon}\,dt + \delta_2 \int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon} + \delta^2\|u(t)\|^2_{\bar{x},\varepsilon}\,dt. \qquad (4.5)
\end{aligned}$$

Other elementary estimates and (2.3) give

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big(\nabla u(t), (u_t(t) + \delta u(t))\nabla\phi_{\bar{x},\varepsilon}\Big)\,dt \\
&\qquad \leq C\varepsilon \int_{t_1}^{t_2} \|\nabla u(t)\|^2_{\bar{x},\varepsilon} + \|u_t(t)\|^2_{\bar{x},\varepsilon} + \delta^2\|u(t)\|^2_{\bar{x},\varepsilon}\,dt \qquad (4.6)
\end{aligned}$$

and

$$-\delta\delta_3 \int_{t_1}^{t_2} \Big(u_t(t), u(t)\Big)_{\bar{x},\varepsilon}\,dt \geq -\frac{\delta\delta_3}{2}\left(\int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\,dt + \int_{t_1}^{t_2} \|u\|^2_{\bar{x},\varepsilon}\,dt\right). \qquad (4.7)$$

Assume that (D1) holds. Then

$$\begin{aligned}
&-\delta \int_{t_1}^{t_2} \Big(g(u_t(t)), u(t)\Big)_{\bar{x},\varepsilon}\,dt \\
&\qquad \leq \frac{\gamma_4^2\delta}{\alpha} \int_{t_1}^{t_2} \|u_t(t)\|^2_{\bar{x},\varepsilon}\,dt + \frac{\delta\alpha}{2} \int_{t_1}^{t_2} \|u(t)\|^2_{\bar{x},\varepsilon}\,dt + \frac{\gamma_4^2\delta}{\alpha}C_\varepsilon(t_2 - t_1). \qquad (4.8)
\end{aligned}$$

The assertion the follows by inserting the estimates (G2), (4.4–4.8) and (3.2) into (4.3) and finishing the argument by choosing the constants $\delta_1, \delta, \kappa, \delta_2, \varepsilon, \delta_3$ (possibly in this order) sufficiently small and by Gronwall's lemma applied to

$$\zeta(t) = \int_{\mathbb{R}^d} F[u](t)\phi_{\bar{x},\varepsilon}\,dx + \delta\Big(u_t(t), u(t)\Big)_{\bar{x},\varepsilon}.$$

Under the assumption (D2), we have

$$\begin{aligned} -\delta \int_{t_1}^{t_2} &\Big(g(u_t(t)), u(t)\Big)_{\bar{x},\varepsilon} dt \\ &\le \delta\left(\kappa \int_{t_1}^{t_2} \Big(f(u(t)), u(t)\Big)_{\bar{x},\varepsilon} dt + \int_{t_1}^{t_2} \Big(g(u_t(t)), u_t(t))\Big)_{\bar{x},\varepsilon} dt + CC_\varepsilon\right). \end{aligned}$$

The conclusion is then reached similarly as in the case (D1) using (3.2) multiplied by $1-\delta$. □

Theorem 4.2. *Let the assumptions of Lemma 4.1 hold. Then there exists a closed positively invariant absorbing set $\mathcal{B} \subseteq \Phi_b$ bounded in Φ_b.*

Proof. Let $\varepsilon, \zeta, C_0, C_1 > 0$ be as in Lemma 4.1. Using the standard embedding $W_b^{1,2}(\mathbb{R}^d) \hookrightarrow L_b^{p+1}(\mathbb{R}^d)$ and the equivalence of weighted and locally uniform norms in Lemma 2.1 we have

$$\int_{\mathbb{R}^d} F_1(u_0)\phi_{\bar{x},\varepsilon}\,dx \le C\|u_0\|^{p+1}_{W_b^{1,2}(\mathbb{R}^d)} + CC_\varepsilon.$$

Inserting into (4.2) we obtain

$$\int_{\mathbb{R}^d} F[u](T)\phi_{\bar{x},\varepsilon}\,dx \le e^{-\zeta T} Q\left(\|u_0\|_{W_b^{1,2}(\mathbb{R}^d)}, \|u_1\|_{L_b^2(\mathbb{R}^d)}\right) + C$$

which leads to

$$\sup_{\bar{x}\in\mathbb{R}^d} \int_{\mathbb{R}^d} E[u](T)\phi_{\bar{x},\varepsilon}\,dx \le e^{-\zeta T} Q\left(\|u_0\|_{W_b^{1,2}(\mathbb{R}^d)}, \|u_1\|_{L_b^2(\mathbb{R}^d)}\right) + \tilde{C}$$

Set $\tilde{\mathcal{B}} = B(0,\tilde{C}) \subseteq \Phi_b$ and find $t_0 > 0$ such that $S(t)\tilde{\mathcal{B}} \subseteq \tilde{\mathcal{B}}$. We define

$$\mathcal{B} = \overline{\bigcup_{t\ge t_0} S(t)\tilde{\mathcal{B}}}^{\Phi_{\mathrm{loc}}}$$

and observe that $\mathcal{B}$ is positively invariant, cf. Theorem 3.2. □

5 Locally uniform squeezing property

In this section we introduce the trajectory setting and prove that the solution operator in the space of trajectories satisfies a local variant of the so-called squeezing property (cf. [6]), which will in turn lead to the asymptotic compactness and an upper bound on Kolmogorov's ε-entropy. To achieve this, we require additional assumptions on μ and the damping nonlinearity g. We note that one can obtain the asymptotic compactness required for the existence of a locally compact attractor also without these additional assumptions by means of a standard decomposition argument.

From now on, let $h \equiv 0$ and for simplicity we assume $d = 3$. In addition, we require

$$\begin{gathered}\mu \in [1, 7/3), \qquad p \in [0, 3), \\ C(1 + |r|)^{\mu-1} \le g'(r) \le C(1 + |r|)^{\mu-1} \qquad \forall r \in \mathbb{R}.\end{gathered}$$

These assumptions and the properties of f lead to the estimates

$$\begin{aligned}|f(r) - f(s)| &\le C(1 + (|r| + |s|)^{p-1})|r - s|, \\ (g(r) - g(s))(r - s) &\ge C \int_0^1 (1 + |tr + (1 - t)s|^{\mu-1})|r - s|^2 \, dt, \\ |g(r) - g(s)| &\le C \int_0^1 (1 + |tr + (1 - t)s|^{\mu-1})|r - s| \, dt, \\ |g(r) - g(s)| &\le (1 + (|r| + |s|)^{\mu-1})|r - s|.\end{aligned} \tag{5.1}$$

Let $\ell > 1$ and $v > 1$ be fixed and let ϕ be an admissible weight function. We define the space of trajectories by

$$\mathcal{E}^{\ell,v}_{b,\phi} = \{(\chi, \chi_t); \chi : Q_\ell \to \mathbb{R}, \|u\|^2_{\mathcal{E}^{\ell,v}_{b,\phi}} = \sup_{k \in \mathbb{N}} \phi(x_k) \int_0^\ell \int_{Z_k(t)} \mathcal{E}[u] dx dt < \infty\},$$

$$\begin{aligned}\mathcal{B}_\ell = \{&(\chi, \chi_t) \in \mathcal{E}^{\ell,v}_{b,\phi}; \\ &\chi \text{ is a weak solution to the equation (1.1) in } [0, \ell] \text{ with } (\chi(0), \chi_t(0)) \in \mathcal{B}\}.\end{aligned}$$

where we denote $Q_\ell = (0, \ell) \times \mathbb{R}^3$ and

$$\begin{aligned}Z_k(t) &= B(x_k, v(2\ell - t)),\ t \in (0, 2\ell), \quad K(x_k) = \{(t, x) \in Q_\ell : x \in Z_k(t)\}, \\ \widetilde{Z}_k(t) &= B(x_k, v(3\ell - t)),\ t \in (0, 3\ell), \quad \widetilde{K}(x_k) = \{(t, x) \in Q_{2\ell} : x \in \widetilde{Z}_k(t)\}.\end{aligned}$$

Note that the half-cone $\{(t, x) \in \widetilde{K}(0); 0 < t < \ell\}$ can be covered by a finite number of cones $K(x_j)$, $j \in \mathcal{N}$, $x_j \in B(0, 3v\ell)$. We emphasize that the size of $\mathcal{N}$ is independent of ℓ.

We define the operators $e : \mathcal{B}_\ell \to \Phi_b$ and $L(t) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ by

$$\begin{aligned}e((\chi, \chi_t)) &= (\chi(\ell), \chi_t(\ell)), \\ [L(t)(\chi, \chi_t)](s) &= S(t + s)(\chi(0), \chi_t(0)), \qquad s \in (0, \ell).\end{aligned}$$

Let $\mathcal{O} \subseteq \mathbb{R}^3$ and let ϕ be an admissible weight function. We define

$$\begin{aligned}\|u\|^2_{\Phi_{b,\phi}(\mathcal{O})} &= \sup_{k \in \mathbb{I}(\mathcal{O})} \phi(x_k) \int_{C_k} |u|^2 + |\nabla u|^2 + |u_t|^2 \, dx, \\ \|u\|^2_{\mathcal{E}^{\ell,v}_{b,\phi}(\mathcal{O})} &= \sup_{k \in \mathbb{I}(\mathcal{O})} \phi(x_k) \int_0^\ell \int_{Z_k(t)} E[u] \, dx \, dt,\end{aligned}$$

where

$$\mathbb{I}(\mathcal{O}) = \{k \in \mathbb{N}; C_k \cap \mathcal{O} \neq \emptyset\}.$$

Again, if $\phi \equiv 1$ we write $\Phi_b(\mathcal{O})$ instead of $\Phi_{b,1}(\mathcal{O})$.

Lemma 5.1. *Let $\ell, v > 1$ and ϕ be arbitrary weight function. The following holds:*

1. *$L = L(\ell) : \mathcal{B}_\ell \to \mathcal{B}_\ell$ is Lipschitz continuous,*
2. *$e : \mathcal{B}_\ell \to \Phi_b$ is Lipschitz continuous,*
3. *$\mathcal{B}_\ell$ is positively invariant under $L(t)$, i.e. $L(t)\mathcal{B}_\ell \subseteq \mathcal{B}_\ell$ for every $t \geq 0$.*

Proof. The proof follows from the finite speed of propagation and is similar to [17, Lemma 2.1].

Assume for u is a sufficiently smooth solution of (1.1). Then using a standard result on differentiation, integration by parts and the equation (1.1) we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\widetilde{Z}_k(t)} E[u](t)\,dx &= \int_{\widetilde{Z}_k(t)} \partial_t E[u](t)\,dx - v\int_{\partial\widetilde{Z}_k(t)} E[u](t)\,dS_x \\ &= \int_{\widetilde{Z}_k(t)} u_t\left(u_{tt} - \Delta u + au\right)\,dx + \int_{\partial\widetilde{Z}_k(t)} u_t \nabla u\cdot n - vE[u]\,dS_x \\ &= \int_{\widetilde{Z}_k(t)} u_t\left(g(u_t) + f(u)\right)\,dx + \int_{\partial\widetilde{Z}_k(t)} u_t \nabla u\cdot n - vE[u]\,dS_x, \end{aligned} \tag{5.2}$$

where n denotes the outward normal to $\widetilde{Z}_k(t)$ in the space domain $\mathbb{R}^3$.

Let $\chi_1, \chi_2 \in \mathcal{B}_\ell$ and let u^1, u^2 be the respective weak solutions. Set $w = u^1 - u^2$ and let $0 < t_1 < t_2 < 2\ell$. Integrating the identity (5.2) over $t \in (t_1, t_2)$, approximating by more regular data and by mollification, we get to the equation

$$\begin{aligned}&\int_{\widetilde{Z}_k(t_2)} E[w](t_2)\,dx - \int_{\widetilde{Z}_k(t_1)} E[w](t)\,dx + \int_{t_1}^{t_2}\int_{\widetilde{Z}_k(t)} \left(g(u_t^1) - g(u_t^2)\right) w_t\,dx\,dt \\ &= -\int_{t_1}^{t_2}\int_{\widetilde{Z}_k(t)} \left(f(u^1) - f(u^2)\right) w_t\,dx\,dt + \int_{t_1}^{t_2}\int_{\partial\widetilde{Z}_k(t)} w_t \nabla w\cdot n - vE[w]\,dS_x\,dt,\end{aligned}$$

where n is denotes the outward normal to $\partial\widetilde{Z}_k(t)$. Since $v > 1$, the boundary integral is non-positive and using (G1) and a similar estimate on the first element on the right-hand side of the previous equation as in the proof of uniqueness, we arrive to

$$\int_{\widetilde{Z}_k(t_2)} E[w](t_2)\,dx \leq \int_{\widetilde{Z}_k(t_1)} E[w](t)\,dx + C\int_{t_1}^{t_2}\int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt.$$

Invoking Gronwall's lemma we get

$$\int_{\widetilde{Z}_k(t)} E[w](t)\,dx \leq \left(1 + C(t-s)e^{C(t-s)}\right)\int_{\widetilde{Z}_k(s)} E[w](s)\,dx$$

for $0 < s < t < 2\ell$. Integrating over $s \in (0, \ell)$ and $t \in (\ell, 2\ell)$ leads to

$$\int_{\ell}^{2\ell}\int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt \leq C\sum_{j\in\mathcal{N}}\int_0^{\ell}\int_{x_k + Z_j(t)} E[w]\,dx\,dt.$$

We multiply the equation by $\phi(x_k)$ and use the property (2.1) to get

$$\phi(x_k)\int_{\ell}^{2\ell}\int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt \leq C\#\mathcal{N}\max_{j\in\mathcal{N}}\phi(x_j)\int_0^{\ell}\int_{x_k+Z_j(t)} E[w]\,dx\,dt. \tag{5.3}$$

The Lipschitz continuity of L in $\mathcal{E}^{\ell,v}_{b,\phi}$ follows by taking supremum over $k \in \mathbb{N}$ and estimating the maximum on the right-hand side by the supremum over $j \in \mathbb{N}$. The Lipschitz continuity of e can be obtained in a similar manner.

The positive invariance of $\mathcal{B}_\ell$ follows immediately from the definitions. $\square$

Definition. *We say that the mapping $L : \mathcal{B}_\ell \to \mathcal{B}_\ell$ has a* locally uniform squeezing property *(LUSP) for an admissible weight function ϕ if for every $\theta > 0$ there exists $\ell > 1$, $v > 1$, $\kappa > 0$ and $\mathcal{N} \subseteq \mathbb{N}$ such that $x_j \in B(0, 3v\ell) \subseteq \mathbb{R}^d$ for every $j \in \mathcal{N}$ and for every $k \in \mathbb{N}$ and $\chi_1, \chi_2 \in \mathcal{B}_\ell$ and the respective solutions u_1, u_2 we have*

$$\phi(x_k) \int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt \le \theta \sum_{j\in\mathcal{N}(k)} \phi(x_j) \int_0^\ell \int_{Z_j(t)} E[w]\,dx\,dt$$
$$+ \kappa \left(\phi(x_k) \int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} |w|^2\,dx\,dt + \sum_{j\in\mathcal{N}(k)} \phi(x_j) \int_0^\ell \int_{Z_j(t)} |w|^2\,dx\,dt \right), \quad (5.4)$$

where

$$\mathcal{N}(k) = \{j \in \mathbb{N}; x_j = x_i + x_k \text{ for some } i \in \mathcal{N}\}.$$

The above definition contains a slight abuse of terminology as one has to first choose $\theta > 0$ and only then find suitable ℓ and v to get the squeezing property of $L = L(\ell) : \mathcal{B}_\ell \to \mathcal{B}_\ell$. However, this will not be of any concern later on as $\theta > 0$ will be chosen only once.

Lemma 5.2. *The operator $L = L(\ell)$ has (LUSP) for every admissible weight function.*

Proof. The proof in similar to [17, Lemma 3.1]. Let us restrict ourselves to the case $\mu \in (1, 7/3)$ and $p \in (1, 3)$ since the remaining cases are similar or easier.

Let $\tau \in (0, \ell)$, $\chi_1, \chi_2 \in \mathcal{B}_\ell$ with the respective solutions u^1, u^2 and denote $w = u^1 - u^2$. Similarly as in the proof of Lemma 5.1 we get

$$\begin{aligned}\int_{\widetilde{Z}_k(2\ell)} E[w](2\ell)\,dx\,dt &+ \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} (g(u^1_t) - g(u^2_t))w_t\,dx\,dt \\ &+ \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} (f(u^1) - f(u^2))w_t\,dx\,dt \qquad (5.5)\\ = \int_{\widetilde{Z}_k(\tau)} E[w](\tau)\,dx &+ \int_\tau^{2\ell} \int_{\partial\widetilde{Z}_k(t)} w_t \nabla w \cdot n - vE[w]\,dS_x\,dt,\end{aligned}$$

$$\begin{aligned}\int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} |\nabla w|^2 + \alpha|w|^2\,dx\,dt &+ \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} (f(u^1) - f(u^2))w\,dx\,dt \\ + \int_{\widetilde{Z}_k(2\ell)} ww_t\,dx\,dt &= \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} |w_t|^2 - (g(u^1) - g(u^2))w\,dx\,dt \qquad (5.6)\\ &+ \int_{\widetilde{Z}_k(\tau)} ww_t\,dx + \int_\tau^{2\ell} \int_{\partial\widetilde{Z}_k(t)} w\nabla w \cdot n - vw_t w\,dS_x\,dt\end{aligned}$$

Using the estimates (5.1) in (5.5) we have

$$\int_{\widetilde{Z}_k(2\ell)} E[w](2\ell) - \int_{\widetilde{Z}_k(\tau)} E[w](\tau)\,dx + C_1 \int_\tau^{2\ell} \|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt \\ + C_1 \int_\tau^{2\ell} \mathcal{J}(t)\,dt \le \int_\tau^{2\ell} \int_{\partial \widetilde{Z}_k(t)} w_t \nabla w \cdot n - vE[w]\,dS_x\,dt \\ + C \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} (1 + (|u^1| + |u^2|)^{p-1} |w||w_t|\,dx\,dt, \quad (5.7)$$

where

$$\mathcal{J}(t) = \int_{\widetilde{Z}_k(t)} \int_0^1 (1 + |su_t^1 + (1-s)u_t^2|^{\mu-1})|w_t|^2\,ds\,dx.$$

We estimate the first element on the right-hand side using the dissipation of energy by

$$C \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} (1 + (|u^1| + |u^2|)^{p-1} |w||w_t|\,dx\,dt \\ \le C \int_\tau^{2\ell} \|1 + |u^1| + |u^2|\|^{p-1}_{L^{(p-1)r_1}(\widetilde{Z}_k(t))} \|w_t\|_{L^2(\widetilde{Z}_k(t))} \|w\|_{L^{r_2}(\widetilde{Z}_k(t))}\,dt \\ \le \int_\tau^{2\ell} C_1/2 \|w_t\|^2_{L^2(\widetilde{Z}_k(t))} + C\|w\|^2_{L^{r_2}(\widetilde{Z}_k(t))}\,dt, \quad (5.8)$$

where we put $r_1 = 6/(p-1)$ and $1/r_1 + 1/r_2 = 1/2$, therefore $r_2 \in (2,6)$. Combining (5.7) and (5.8) we arrive to

$$\int_{\widetilde{Z}_k(2\ell)} E[w](2\ell)\,dx - \int_{\widetilde{Z}_k(\tau)} E[w](\tau)\,dx + \frac{C_1}{2} \int_\tau^{2\ell} \|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt \\ + C_1 \int_\tau^{2\ell} \mathcal{J}(t)\,dt \le C \int_\tau^{2\ell} \|w\|^2_{L^{r_2}(\widetilde{Z}_k(t))} dt \\ + \int_\tau^{2\ell} \int_{\partial \widetilde{Z}_k(t)} w_t \nabla w \cdot n - vE[w]\,dS_x\,dt. \quad (5.9)$$

Returning to (5.6), by the estimates (5.1) we have

$$\int_\tau^{2\ell} \|\nabla w\|^2_{L^2(\widetilde{Z}_k(t))} + \alpha \|w\|^2_{L^2(\widetilde{Z}_k(t))}\,dt \\ \le \int_\tau^{2\ell} \|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt + C \int_{\widetilde{Z}_k(2\ell)} E[w](2\ell)\,dx + C \int_{\widetilde{Z}_k(\tau)} E[w](\tau)\,dx \\ + C \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} \int_0^1 (1 + |su_t^1 + (1-s)u_t^2|^{\mu})|w_t||w|\phi_{\bar{x},\varepsilon}\,ds\,dx\,dt \\ + C \int_\tau^{2\ell} \int_{\widetilde{Z}_k(t)} (1 + (|u^1| + |u^2|)^{p-1})|w|^2 \phi_{\bar{x},\varepsilon}\,dx\,dt \\ + \int_\tau^{2\ell} \int_{\partial \widetilde{Z}_k(t)} w \nabla w \cdot n - v w_t w\,dS_x\,dt. \quad (5.10)$$

Similarly as in (5.8) we estimate the fourth element on the right-hand side of (5.10) as

$$C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)}\int_0^1 (1+|su_t^1+(1-s)u_t^2|^{\mu-1})|w_t||w|\phi_{\bar{x},\varepsilon}\,ds\,dx\,dt \\ \leq C\int_\tau^{2\ell}\mathcal{J}\,dt + C\int_\tau^{2\ell}\|w\|^2_{L^{2s_2}(\widetilde{Z}_k(t))}\,dt, \quad (5.11)$$

where we use the dissipation of energy and set $s_1 = 2/(\mu-1)$ and $1/s_1+1/s_2 = 1$, therefore $2s_2 \in (2,6)$. Similarly the fifth element (5.10) is estimated by

$$C\int_\tau^{2\ell}\int_{\widetilde{Z}_k(t)}(1+(|u^1|+|u^2|)^{p-1})|w|^2\,dx\,dt \leq C\int_\tau^{2\ell}\|w\|^2_{L^{2z_2}(\widetilde{Z}_k(t))}\,dt, \quad (5.12)$$

where we again used the dissipation estimate and set $z_1 = 6/(p-1)$ and $1/z_1 + 1/z_2 = 1$, therefore $2z_2 \in (2,3)$. Set $s = \max(2s_2, 2z_2)$. Combining the estimates (5.10–5.12) we obtain

$$\int_\tau^{2\ell}\|\nabla w\|^2_{L^2(\widetilde{Z}_k(t))} + \alpha\|w\|^2_{L^2(\widetilde{Z}_k(t))}\,dt \\ \leq \int_\tau^{2\ell}\|w_t\|^2_{L^2(\widetilde{Z}_k(t))}\,dt + C\left(\int_{\widetilde{Z}_k(2\ell)}E[w](2\ell)\,dx + \int_{\widetilde{Z}_k(\tau)}E[w](\tau)\,dx\right) \\ + C\int_\tau^{2\ell}\mathcal{J}\,dt + C\int_\tau^{2\ell}\|w\|^2_{L^s(\widetilde{Z}_k(t))}\,dt + \int_\tau^{2\ell}\int_{\partial\widetilde{Z}_k(t)} w\nabla w\cdot n - vw_t w\,dS_x\,dt. \quad (5.13)$$

Define $r = \max(s, r_2)$. Multiply (5.13) by $\delta > 0$, add it to (5.9) and choose $v \geq (1+\delta)/(1-\delta)$ and $\delta > 0$ small enough to get

$$\zeta\int_\ell^{2\ell}\int_{\widetilde{Z}_k(t)}E[w](t)\,dx\,dt \leq C\int_0^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt + 2\int_{\widetilde{Z}_k(\tau)}E[w](\tau)\,dx$$

for some $\zeta > 0$ and integrate by τ from 0 to ℓ to obtain

$$\zeta\ell\int_\ell^{2\ell}\int_{\widetilde{Z}_k(t)}E[w](t)\,dx\,dt \\ \leq C\ell\int_0^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt + 2\int_0^\ell\int_{\widetilde{Z}_k(t)}E[w](t)\phi_{\bar{x},\varepsilon}\,dx\,dt. \quad (5.14)$$

Now split the integral

$$\int_0^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt = \int_0^\ell\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt + \int_\ell^{2\ell}\|w\|^2_{L^r(\widetilde{Z}_k(t))}\,dt$$

and divide the equation (5.14) by $\zeta\ell$. Next we employ Ehrling's lemma, namely

$$\|w\|^2_{L^r(\Omega)} \leq \gamma\|w\|^2_{W^{1,2}(\Omega)} + C\|w\|^2_{L^2(\Omega)}$$

for $\Omega = B(x,R) \subseteq \mathbb{R}^d$ with $x \in \mathbb{R}^d$, $R > 0$, $\gamma > 0$ arbitrary and $C = C(\gamma, R)$, on the arguments of the split integrals. Indeed, this is possible since the diameters of the domains in question, i.e. $\widetilde{Z}_k(t)$ for $t \in (0, 2\ell)$, are bounded. Combining these estimates with (5.14) we obtain

$$\left(1 - \frac{C\gamma}{\zeta}\right) \int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt \le \left(\frac{2}{\zeta\ell} + \frac{C\gamma}{\zeta}\right) \sum_{j \in \mathcal{N}(k)} \int_0^\ell \int_{Z_j(t)} E[w]\,dx\,dt + \frac{C}{\zeta}\left(\int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} |w|^2\,dx\,dt + \sum_{j \in \mathcal{N}(k)} \int_0^\ell \int_{Z_j(t)} |w|^2\,dx\,dt\right), \quad (5.15)$$

where $\mathcal{N}(k) \subseteq \mathbb{N}$ is a finite set of size N such that the union of cones $K(x_j)$ over $j \in \mathcal{N}$ covers the cone $\widetilde{K}(x_k)$. Now let $\tilde{\theta} > 0$ be such that $\tilde{\theta} C_\phi \exp(\nu 3 v \ell) < \theta$, where $\nu > 0$ is the growth of the admissible function ϕ. By choosing ℓ sufficiently large and γ sufficiently small we get

$$\int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} E[w]\,dx\,dt \le \tilde{\theta} \sum_{j \in \mathcal{N}(k)} \int_0^\ell \int_{Z_j(t)} E[w]\,dx\,dt + C\left(\int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} |w|^2\,dx\,dt + \sum_{j \in \mathcal{N}(k)} \int_0^\ell \int_{Z_j(t)} |w|^2\,dx\,dt\right). \quad (5.16)$$

It remains to insert the weight function with sufficiently small growth which is easily done by multiplying (5.16) by $\psi(x_k)$, invoking (2.1) and using the restriction on $\tilde{\theta}$. □

The critical case $p = 3$ contains essential difficulties and would be an interesting problem for the consideration in future.

6 Locally compact attractor and entropy estimate

Let M be a metric space and $K \subseteq M$ be relatively compact. Let $N_\varepsilon(K, M)$ denote the smallest number of balls of radii ε that cover K in M. We define the *Kolmogorov's ε-entropy* by

$$H_\varepsilon(K, M) = \ln N_\varepsilon(K, M).$$

A number of typical examples of upper and lower bounds on the Kolmogorov's ε-entropy in various situations can be found e.g. in [22].

The following lemma is crucial for the estimate of Kolmogorov's ε-entropy and considerably simplifies the proof of asymptotic compactness. We note that an estimate of this kind may be used to establish an infinite dimensional exponential attractor. We postpone this issue to a subsequent paper together with an abstract criterion and applications to other equations.

Lemma 6.1. *Let $\mathcal{O} \subseteq \mathbb{R}^3$ be bounded and satisfy*

$$\#\mathbb{I}(\mathcal{O}) \le C_0 \operatorname{vol}(\mathcal{O}). \quad (6.1)$$

Let $\varepsilon > 0$, $\delta \in (0,1)$ and $(x_0, x_1) \in \mathcal{B}$. Also let ϕ be an admissible weight function. Then there exist $\ell, v > 1$ such that

$$H_{\delta\varepsilon}\left((LB)\big|_{\mathcal{O}}, \mathcal{E}^{\ell,v}_{b,\phi}(\mathcal{O})\right) \leq C_1 \operatorname{vol}(\mathcal{O}), \tag{6.2}$$

where $B = B_\varepsilon((\chi_0,\chi_1); \mathcal{E}^{\ell,v}_{b,\phi}) \cap \mathcal{B}_\ell$ is a ball centered around the ℓ-trajectory (χ_0, χ_1) starting from (x_0, x_1). The constant C_1 depends only on C_0, ℓ and δ and is independent of (x_0, x_1), ε and $\mathcal{O}$ as long as (6.1) is satisfied.

Proof. The proof adapts the techniques from [17, Lemma 4.1] and [9, Lemma 2.6], the main difference being working with hyperbolic trajectories space instead of parabolic ones.

Without loss of generality, assume that $0 \in \mathcal{N}$. First find $\ell, v > 1$ such that (5.4) holds for $\theta > 0$ satisfying $4\theta\#\mathcal{N} < \delta^2$ and fix $\lambda > 0$ for which

$$4\theta\#\mathcal{N} + \kappa\lambda^2(\#\mathcal{N} + 1) < \delta^2.$$

Let $k \in \mathbb{I}(\mathcal{O})$. Define

$$\begin{aligned} P(\chi, \chi_t) &= \left(\phi(x_j)\chi|_{K(x_j)}\right)_{j\in\mathcal{N}} \cup \left(\phi(x_k)L\chi|_{K(x_k)}\right), \qquad (\chi, \chi_t) \in \mathcal{B}_\ell, \\ X &= \{P(\chi, \chi_t);\ (\chi, \chi_t) \in B\}. \end{aligned}$$

We equip the space X with the norm

$$\|y\|_X^2 = \max\left\{\max_{j\in\mathcal{N}(k)}\left\{\int_0^\ell \int_{Z_j(t)} |y_j|^2\, dx\, dt\right\}, \int_0^\ell \int_{Z_k(t)} |z|^2\, dx\, dt\right\},$$

where $y = (y_j; j \in \mathcal{N}(k)) \cup (z) \in X$. Since $\mathcal{B}_\ell$ (and thus B) is uniformly bounded on every cone $K(x_i)$, $i \in \mathbb{N}$, by Aubin-Lions lemma there exists $N \in \mathbb{N}$ and $(\chi^i, \chi^i_t) \in B$, $i = 1, \ldots, N$, such that

$$X \subseteq \bigcup_{i=1}^N B_{\lambda\varepsilon}\left(P(\chi^i, \chi^i_t); X\right).$$

It is important to note that N is independent of k and ε, which follows from the estimate

$$\|y\|_X^2 \leq C\varepsilon^2$$

holding uniformly for $\varepsilon > 0$ and $k \in \mathbb{I}(\mathcal{O})$ with C depending only on the Lipschitz constant of L.

Choose $(\chi, \chi_t) \in B$. Then $P(\chi, \chi_t) \in B_{\varepsilon\lambda}(P(\chi^i, \chi^i_t); X)$ for some $1 \leq i \leq N$. Let u and u^i be the respective solution for χ and χ^i and let $w = u - u^i$. Using (LUSP) we may estimate

$$\begin{aligned} &\phi(x_k)\int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} E[w]\, dx\, dt \\ &\qquad \leq \theta \sum_{j\in\mathcal{N}(k)} \int_0^\ell \int_{Z_j(t)} E[w]\, dx\, dt + \kappa \sum_{j\in\mathcal{N}(k)} \phi(x_j)\int_0^\ell \int_{Z_j(t)} |w|^2\, dx\, dt \\ &\qquad\quad + \kappa\phi(x_k)\int_\ell^{2\ell} \int_{\widetilde{Z}_k(t)} |w|^2\, dx\, dt \\ &\qquad \leq 4\theta\varepsilon^2\#\mathcal{N} + \kappa\varepsilon^2\lambda^2(\#\mathcal{N} + 1) < \delta^2\varepsilon^2, \end{aligned}$$

therefore we have

$$H_{\delta\varepsilon}\left((LB)|_{C_k}, \mathcal{E}^{\ell,v}_{b,\phi}(C_k)\right) \le \ln N$$

uniformly for every $k \in \mathbb{I}(\mathcal{O})$.

The final estimate follows directly from (6.1) since for covering in $\mathcal{E}^{\ell,v}_{b,\phi}(\mathcal{O})$, one needs to consider the product of all the coverings in $\mathcal{E}^{\ell,v}_{b,\phi}(C_k)$, $k \in \mathbb{I}(\mathcal{O})$. □

Proposition 6.2. *The dynamical system $(S(t), \Phi_b)$ is asymptotically compact in the local topology Φ_{loc}.*

Proof. Let $\{x_n\} \subseteq \Phi_b$ be bounded, let $t_n \to \infty$ and let $K \subseteq \mathbb{R}^d$ be compact. Without loss of generality we may assume $x_n \in \mathcal{B}$. Find $\ell, v > 1$ such that (6.2) holds for $\phi \equiv 1$ and $\theta = 1/2$. Let $B \subseteq \mathbb{R}^d$ be a sufficiently large ball such that $K \subseteq B$ and $\mathcal{N}(k) \subseteq B$ for every $k \in \mathbb{I}(K)$.

Passing to a subsequence we may find $\chi_n \in \mathcal{B}_\ell$ such that $S(t_n)x_n = e(L^n\chi_n)$. Using (6.2) we are able to recurrently find a Cauchy subsequence $\{L^n\chi_n\}$ in $\mathcal{E}^{\ell,v}_{b,\phi}(B)$. The proof will be finished once we show that the sequence $e(L\chi_n)$ is Cauchy in $\Phi_b(K)$ and this follows from (5.3) by taking supremum over $k \in \mathbb{I}(K)$. □

Using the dissipation of energy and the local asymptotic compactness we are able to show the existence of a locally compact attractor. The proof of the following theorem follows exactly as in [7] or [22] and will be omitted here.

Theorem 6.3. *There exists a unique set $\mathcal{A} \subseteq \Phi_b$ invariant under $S(t)$ and compact in Φ_{loc} such that $\mathcal{A}$ attracts sets bounded in Φ_b in the local topology Φ_{loc}, i.e. for every $B \subseteq \Phi_b$ bounded*

$$\lim_{t\to\infty} \operatorname{dist}_{\Phi_{loc}}(S(t)B, \mathcal{A}) = 0.$$

We denote

$$\mathcal{A}_\ell = \{\chi \in \mathcal{E}^{\ell,v}_{b,\phi}; \chi \text{ solves the equation in } [0,\ell] \text{ with } (\chi(0), \chi_t(0)) \in \mathcal{A}\}.$$

It is clear that $e(\mathcal{A}_\ell) = \mathcal{A}$ and $L(\mathcal{A}_\ell) = \mathcal{A}_\ell$.

Before we proceed to the entropy estimate, we define an auxiliary weight function in the spirit of [22]. Let $x_0 \in \mathbb{R}^d$, $R > 0$ and $\nu > 0$ be fixed. We define

$$\psi(x_0,R) = \psi(x_0,R)(x) = \begin{cases} 1, & |x - x_0| \le R + \sqrt{d}, \\ e^{\nu\left(R+\sqrt{d}-|x-x_0|\right)}, & \text{otherwise.} \end{cases}$$

Clearly $\psi(x_0,R)$ is an admissible weight function with growth ν and one has

$$H_\varepsilon\left(\mathcal{A}, \Phi_b(B(x_0,R))\right) \le H_\varepsilon\left(\mathcal{A}, \Phi_{b,\psi(x_0,R)}(\mathbb{R}^d)\right). \tag{6.3}$$

The statement of the following lemma is formally the same as in [9]. However, we should keep in mind that we are working with a different trajectories norm, even if the proof of the lemma runs exactly in the same way as in the original proof.

Lemma 6.4 ([9, Lemma 5.4])**.** *For every $\varepsilon_0 > 0$ there exist $C > 0$ such that for every $x_0 \in \mathbb{R}^3$, $R \geq 1$, $\varepsilon \in (0, \varepsilon_0)$ and $\chi_1, \chi_2 \in \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}$ it holds that*

$$\|\chi_1 - \chi_2\|_{\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}} \leq \max\left\{\|\chi_1 - \chi_2\|_{\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0,R_\varepsilon))}, \varepsilon\right\},$$

where

$$R(\varepsilon) = R + C\left(1 + \ln\frac{1}{\varepsilon}\right).$$

Theorem 6.5. *There exists $C_0, C_1, \varepsilon_0 > 0$ such that for every $x_0 \in \mathbb{R}^3$, $R \geq 1$ and $\varepsilon \in (0, \varepsilon_0)$ one has the bound*

$$H_\varepsilon\left(\mathcal{A}|_{B(x_0,R)}, \Phi_b\left(B\left(x_0, R\right)\right)\right) \leq C_0\left(R + C_1 \ln\frac{1}{\varepsilon}\right)^3 \ln\frac{1}{\varepsilon}.$$

Proof. The proof uses a similar technique to [9, Theorem 5.1] and is standard. Let $\ell, v > 1$ and let $\psi(x_0, R)$ have sufficiently small growth such that Lemma 6.1 holds with $\delta = 1/2$ and for $\psi(x_0, R)$. By (6.3), the Lipschitz continuity of e shown in Lemma 5.1 and the fact that $C_k \subseteq B_k(\ell)$ allows us to estimate

$$H_\varepsilon\left(\mathcal{A}, \Phi_b\left(B\left(x_0, R\right)\right)\right) \leq H_\varepsilon\left(\mathcal{A}, \Phi_{b,\psi(x_0,R)}\right) \leq H_{\varepsilon/\operatorname{Lip}(e)}\left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right).$$

We find $\varepsilon_0 > 0$ and $\chi \in \mathcal{A}_\ell$ such that $\mathcal{A}_\ell \subseteq B_{\varepsilon_0}(\chi; \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)})$, in other words

$$H_{\varepsilon_0}\left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right) = 0.$$

The proof will be finished once we establish the bound

$$H_{\varepsilon_0 2^{-k}}\left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right) \leq kC_0\left(R + C(1 + \ln\frac{2^k}{\varepsilon_0})\right)^3 \tag{6.4}$$

since then for $\varepsilon \in (0, \varepsilon_0)$ we find $k \in \mathbb{N}$ such that $2^{-k}\varepsilon_0 \leq \varepsilon < 2^{-k+1}\varepsilon_0$ and the desired entropy bound follows from $k < C \ln 1/\varepsilon$ holding for ε sufficiently small.

To prove the recurrent estimate (6.4) we use induction. Let first $k = 1$. Then from Lemma 6.1 we have

$$H_{\varepsilon_0/2}\left(\mathcal{A}_\ell|_{B(x_0,R(\varepsilon_0/2))}, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0, R(\varepsilon_0/2)))\right) \leq C_0\left(R + C(1 + \ln\frac{2^k}{\varepsilon_0})\right)^3.$$

By Lemma 6.4 the $\varepsilon_0/2$-covering in the space $\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0, R(\varepsilon_0/2)))$ is also a $\varepsilon_0/2$-covering in $\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}$.

Now let the bound (6.4) hold for $k > 1$, i.e.

$$\mathcal{A}_\ell \subseteq \bigcup_{i=1}^{N} B_{\varepsilon_0 2^{-k}}\left(\chi_i; \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}\right) \tag{6.5}$$

for some $\chi_i \in \mathcal{A}_\ell$. Apply the mapping L to (6.5) to get

$$L(\mathcal{A}_\ell) = \mathcal{A}_\ell \subseteq \bigcup_{i=1}^{N} B_{\mathrm{Lip}(L)\varepsilon_0 2^{-k}} \left(L\chi_i; \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)} \right), \tag{6.6}$$

where we used the invariance of $\mathcal{A}_\ell$ under L from Lemma 5.1. By Lemma 6.1 each of the balls on the right-hand side of (6.6) can be covered by balls with radii $\varepsilon_0/2^{-(k+1)}$in $\mathcal{E}^{\ell,v}_{b,\psi(x_0,R)}(B(x_0, R(\varepsilon_0 2^{-k+1})))$ so that

$$\begin{aligned}
&H_{\varepsilon_0 2^{-(k+1)}} \left(\mathcal{A}_\ell|_{B(x_0,R_0(\varepsilon_0/2^{-(k+1)}))}, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)} \left(B \left(x_0, R_0(\varepsilon_0/2^{-(k+1)}) \right) \right) \right) \\
&\quad \le H_{\varepsilon_0 2^{-k}} \left(\mathcal{A}_\ell, \mathcal{E}^{\ell,v}_{b,\psi(x_0,R)} \right) + C \left(R_0 + C(1 + \ln \frac{2^{k+1}}{\varepsilon_0}) \right)^3 \\
&\quad \le (k+1)C \left(R_0 + C(1 + \ln \frac{2^{k+1}}{\varepsilon_0}) \right)^3 .
\end{aligned}$$

The proof is finished by another use of Lemma 6.4 as in the step $k = 1$. □

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7. Paper [IV]

Kolmogorov's ε-entropy of the attractor of the strongly damped wave equation in locally uniform spaces

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(preprint)

Kolmogorov's ε-entropy of the attractor of the strongly damped wave equation in locally uniform spaces

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Abstract

We establish an upper bound on the Kolmogorov's entropy of the locally compact attractor for strongly damped wave equation posed in locally uniform spaces in subcritical case using the method of trajectories.

1 Introduction

We are interested in the asymptotic properties of the strongly damped wave equation

$$u_{tt} + \beta u_t - \alpha \Delta u_t - \Delta u + f(u) = g, \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function specified later and α, $\beta > 0$, supplemented by the initial datum

$$u(0) = u_0 \in W^{1,2}_b(\mathbb{R}^d), \qquad u_t(0) = u_1 \in L^2_b(\mathbb{R}^d).$$

The strongly damped wave equation has a number of relevant physical applications, see e.g. [5].

The asymptotic properties of the equation (1.1) posed in a bounded domain has been thoroughly studied in the literature. Let us only briefly mention some of the results. In [2] the authors established the existence of a global attactor for the critical case. The existence of an exponential attractor for the subcritical, resp. critical case, has been show in [10], resp. [14]. The existence of a global attractor for critical and supercritical exponents has been also shown for a variant of the strongly damped wave equation with memory in [5]. The finite dimensionality of the attractor has been shown in [6]. The situation in supercritical case is studied in detail in [8].

In unbounded domains the results are more scarce. In [1] and [4] the authors study the equation (1.1) posed in the classical space $W^{1,2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and show the existence of a connected universal attractor in the subcritical and critical case. In the context of locally uniform spaces, the wave equation with weak linear damping, i.e. with $\alpha = 0$, has been studied in detail in [15]. The strongly damped wave equation has been studied in [3], where the well-posedness of the equation in a subspace of locally uniform space $\dot{W}_b^{2,p}(\mathbb{R}^d) \times \dot{L}_b^p(\mathbb{R}^d)$, $p > d/2$, $p \geq 2$, of functions continuous w.r.t. spatial translations in the locally uniform norm and the existence of a locally compact attractor has been shown for the critical case. In [13] the authors generalized these results to the space of locally uniform functions $W_b^{1,2}(\mathbb{R}^d) \times L_b^2(\mathbb{R}^d)$ and obtained a result on the asymptotic regularity of the solutions. The results of [13] will be reviewed in more detail at the end of this section as they serve as a starting point of our investigation. In [12] the author studies a variant of the strongly damped wave equation with fractional damping and shows the existence of a locally compact attractor in the critical case together with space-time regularity of the solutions.

The aim of this paper is to establish an upper bound on the Kolmogorov's ε-entropy of the attractor of the equation (1.1) in the subcritical case. To this end we use the method of trajectories and a technique similar to the ones used for a wave equation with nonlinear damping in [11] for bounded domains, resp. in [9] for unbounded domains. In [9] the result is achieved using a local squeezing property obtained from finite speed of propagation. However, with strong damping the equation (1.1) no longer has a finite speed of propagation and the argument must be slightly adapted. Compared to the equation studied in [12], the strongly damped wave equation does not posses a smoothing property.

Let ϕ be an admissible weight function, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$. We denote

$$\Phi_{\bar{x},\varepsilon} = W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d) \times L^2_{\bar{x},\varepsilon}(\mathbb{R}^d), \quad W_{\bar{x},\varepsilon} = W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d) \times W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d),$$
$$\Phi_{b,\phi} = W^{1,2}_{b,\phi}(\mathbb{R}^d) \times L^2_{b,\phi}(\mathbb{R}^d), \quad W_{b,\phi} = W^{1,2}_{b,\phi}(\mathbb{R}^d) \times W^{1,2}_{b,\phi}(\mathbb{R}^d),$$
$$W_{\mathrm{loc}} = W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d) \times W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d),$$

with the convention that we omit the subscript ϕ if $\phi \equiv 1$ and write for example Φ_b instead of $\Phi_{b,1}$. For definitions of admissible weight functions and weighted and locally uniform spaces see Section 2.

For simplicity let us choose $\alpha = \beta = 1$. The nonlinear term $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

- (*growth condition*) there exist $C > 0$ and $0 \leq q \leq 4/(d-2)$ such that

  $$|f(r) - f(s)| \leq C|r - s| \left(1 + |r|^q + |s|^q\right), \qquad \forall r, s \in \mathbb{R}.$$

  The nonlinearity is *critical* if $q = 4/(d-2)$ and *subcritical* if $q < 4/(d-2)$.

- (*dissipation condition*) there exist $k \geq 1$ and $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0]$ there exist $C_\mu, C_0 \in \mathbb{R}$ such that

  $$kF(s) + \mu s^2 - C_\mu \leq sf(s), \qquad \forall s \in \mathbb{R},$$
  $$-C_0 \leq F(s), \qquad \forall s \in \mathbb{R},$$

  where $F(s) = \int_0^s f(r)\, dr$.

These conditions are the same as in [3] and [13].

The *weak solution* of (1.1) is defined in the sense of distributions on $(0, \infty) \times \mathbb{R}^d$ and has the regularity

$$(u, u_t) \in C([0, T]; \Phi_{\bar{x},\varepsilon}), \qquad \|u\|^2_{W_b^{1,2}} + \|u_t\|^2_{L_b^2} \in L^\infty((0, T)),$$

for every $T > 0$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$. Using a standard density argument it can be shown that a weak solution u satisfies

$$\Big(u_t(T), \varphi(T)\Big)_{\bar{x},\varepsilon} - \Big(u_t(0), \varphi(0)\Big)_{\bar{x},\varepsilon} - \int_0^T \Big(u_t(t), \varphi_t(t)\Big)_{\bar{x},\varepsilon}\, dt$$
$$+ \int_0^T \Big(u_t(t), \varphi(t)\Big)_{\bar{x},\varepsilon}\, dt + \int_0^T \Big(\nabla u_t(t), \nabla\varphi(t)\Big)_{\bar{x},\varepsilon}\, dt$$
$$+ \int_0^T \Big(\nabla u(t), \nabla\varphi(t)\Big)_{\bar{x},\varepsilon}\, dt + \int_0^T \Big(f(u(t)), \varphi(t)\Big)_{\bar{x},\varepsilon}\, dt$$
$$+ \int_0^T \Big(\nabla u_t(t), \varphi\nabla\phi_{\bar{x},\varepsilon}\Big)\, dt + \int_0^T \Big(\nabla u(t), \varphi\nabla\phi_{\bar{x},\varepsilon}\Big)\, dt = \int_0^T \Big(g, \varphi(t)\Big)_{\bar{x},\varepsilon}\, dt$$

for every $T > 0$, $\bar{x} \in \mathbb{R}^d$, $\varepsilon > 0$ and every function

$$\varphi \in L^2(0, T; W^{1,2}_{\bar{x},\varepsilon}(\mathbb{R}^d)) \cap W^{1,2}(0, T; L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)).$$

The existence and uniqueness of weak solutions has been shown in [13, Section 3] using semigroup theory in the subspace of more regular initial data continuous with respect to spatial to translations. We also have the following dissipative estimates: there exist t_0, $C > 0$ such that for every $t > t_0$ we have

$$\|u\|_{W_b^{1,2}} + \|u_t\|_{W_b^{1,2}} + \|u_{tt}\|_{L_b^2} \leq C. \tag{1.2}$$

For proofs see [13, Section 4]. Let us denote the absorbing set by $\mathcal{B}$ and assume that $\mathcal{B}$ is closed and positively invariant.

In [13], the authors also show the existence of a locally compact attractor in the critical case, namely the existence an invariant set $\mathcal{A} \subseteq \Phi_b$ bounded and closed in $W_b^{2,2}(\mathbb{R}^d) \times W_b^{1,2}(\mathbb{R}^d)$ and compact in W_{loc}, which attracts the bounded sets of Φ_b in the W_{loc}-norm, and the asymptotic regularity, namely the existence of a closed and bounded set $\mathcal{B}_1 \subseteq W_b^{2,2}(\mathbb{R}^d) \times W_b^{1,2}(\mathbb{R}^d)$, a constant $\nu > 0$, and a positive monotonically increasing function $Q(\cdot)$ such that for every bounded $B \subseteq \Phi_b$ we have

$$\mathrm{dist}_{\Phi_b}\left(S(t)B, \mathcal{B}_1\right) \leq Q(\|B\|_{\Phi_b})e^{-\nu t} \qquad \forall t > 0.$$

For proofs see [13, Theorem 1.1 and 1.2]. It is worth noting that the technique presented in this paper do not rely on the asymptotic regularity of the attractor.

This paper is organized as follows: in Section 2 we review the basic definitions of function spaces used in the rest of the paper. In Section 3 we define the trajectory spaces and the trajectory semigroup and show that the trajectory semigroup has a parabolic squeezing property which is then used in Section 4 to establish an upper estimate on the locally compact attractor of the equation (1.1).

2 Function spaces

A function $\phi : \mathbb{R}^d \to (0, \infty)$ is called an *admissible weight function* of growth $\mu \geq 0$ if

$$C_\phi^{-1} e^{-\mu|x-y|} \leq \phi(x)/\phi(y) \leq C_\phi e^{\mu|x-y|} \tag{2.1}$$

for some $C_\phi \geq 1$ and all $x, y \in \mathbb{R}^d$ and

$$|\nabla\phi(x)| \leq \tilde{C}_\phi \mu \phi(x) \tag{2.2}$$

for almost all $x \in \mathbb{R}^d$ and some $\tilde{C}_\phi > 0$. For $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we denote

$$\phi_{\bar{x},\varepsilon}(x) = \exp(-\varepsilon|x-y|).$$

Clearly $\phi_{\bar{x},\varepsilon}$ is an admissible weight function of growth ε.

For $p \in [1,\infty)$, $\bar{x} \in \mathbb{R}^d$ and $\varepsilon > 0$ we define the *weighted Lebesgue space* $L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by

$$L^p_{\bar{x},\varepsilon}(\mathbb{R}^d) = \{u \in L^p_{\text{loc}}(\mathbb{R}^d); \|u\|^p_{L^p_{\bar{x},\varepsilon}} = \int_{\mathbb{R}^d} |u(x)|^p \phi_{\bar{x},\varepsilon}(x)\ dx < \infty\}.$$

In the case $p = 2$ we use the notation $\|\cdot\|_{L^2_{\bar{x},\varepsilon}} \equiv \|\cdot\|_{\bar{x},\varepsilon}$ and denote the scalar product in $L^2_{\bar{x},\varepsilon}(\mathbb{R}^d)$ by $(\cdot,\cdot)_{\bar{x},\varepsilon}$. The weighted Sobolev spaces are defined in an obvious manner.

Concerning the embeddings of weighted spaces, first observe that the space $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)$ cannot be embedded into $L^q_{\bar{x},\varepsilon}(\mathbb{R}^d)$ for any $q > p$. However, this limitation no longer stands once we allow different growth rates. Assume that $k, l \in \mathbb{N}_0$ and $p, q \in [1,\infty)$ are such that $k \geq l$, $q \geq p$ and $W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{l,q}(\mathbb{R}^d)$, then for $\tilde{\varepsilon} = \varepsilon q/p$ we have the continuous embedding $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$. If the embedding $W^{k,p}(B(0,1)) \hookrightarrow\hookrightarrow W^{l,q}(B(0,1))$ is compact, then for $\tilde{\varepsilon} > \varepsilon q/p$ the embedding $W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d) \hookrightarrow\hookrightarrow W^{l,q}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d)$ is also compact.

Let ϕ be an admissible weight function and $p \in [1,\infty)$. We define the *weighted locally uniform space* $L^p_{b,\phi}(\mathbb{R}^d)$ by

$$L^p_{b,\phi}(\mathbb{R}^d) = \{u \in L^p_{\text{loc}}(\mathbb{R}^d); \sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})^{1/p}\|u\|_{L^p(C^1_{\bar{x}})} < \infty\},$$

where C^R_x denotes the cube in $\mathbb{R}^d$ of side $R > 0$ and centred at $x \in \mathbb{R}^d$. We equip the space with a norm equivalent to $\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})^{1/p}\|u\|_{L^p(C^1_{\bar{x}})}$ defined by

$$\|u\|_{L^p_b} = \sup_{k\in\mathbb{Z}^d} \phi(k)^{1/p}\|u\|_{L^p(C^1_k)}. \tag{2.3}$$

Also one can see that if we take any bounded neighbourhood of $\bar{x}$ in (2.3) instead of C^1_k, we again obtain an equivalent norm.

The weighted spaces and locally uniform spaces are connected through the following equivalence of norms. For proof see e.g. [7, Theorem 2.1].

Theorem 2.1. *Let $k \in \mathbb{N}_0$, $p \in [1,\infty)$ and $\varepsilon > 0$. Let ϕ be a weight function of growth rate $0 \leq \mu < \varepsilon$ and $u \in W^{k,p}_{\text{loc}}(\mathbb{R}^d)$. Then $u \in W^{k,p}_{b,\phi}(\mathbb{R}^d)$ if and only if $u \in W^{k,p}_{\bar{x},\varepsilon}(\mathbb{R}^d)$ for every $\bar{x} \in \mathbb{R}^d$ and*

$$\sup_{\bar{x}\in\mathbb{R}^d} \phi(\bar{x})^{1/p}\|u\|_{W^{k,p}_{\bar{x},\varepsilon}} < \infty. \tag{2.4}$$

Moreover, the left-hand side of (2.4) defines a norm equivalent to the $W^{k,p}_{b,\phi}(\mathbb{R}^d)$-norm.

For $\mathcal{O} \subseteq \mathbb{R}^d$ denote

$$\mathbb{I}(\mathcal{O}) = \{k \in \mathbb{Z}^d; C_k^1 \cap \mathcal{O} \neq \emptyset\}.$$

We define the $W^{k,p}_{b,\phi}(\mathcal{O})$-seminorm by

$$\|u\|_{W^{k,p}_{b,\phi}(\mathcal{O})} = \sup_{l \in \mathbb{I}(\mathcal{O})} \phi(l)^{1/p} \|u\|_{W^{k,p}(C_l^1)}. \tag{2.5}$$

We will need the following estimate.

Lemma 2.2 ([15, Proposition 1.2]). *For $1 \leq p < \infty$ and $\varepsilon > 0$ fixed there exist C_1, $C_2 > 0$ such that for $\bar{x} \in \mathbb{R}^d$ and $u \in L^p_{\bar{x},\varepsilon}(\mathbb{R}^d)$ with we have*

$$C_1 \int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}(x)|u(x)|^p \, dx \leq \int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}(x) \left(\int_{B(x,1)} |u(y)|^p \, dy \right) dx \leq C_2 \int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}(x)|u(x)|^p \, dx.$$

Let $\ell > 0$ and let ϕ be an admissible weight function. We define the parabolic locally uniform space $L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d))$ by

$$L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d)) = \{u : (0,\ell) \times \mathbb{R}^d \to \mathbb{R}; \|u\|^2_{L^2_{b,\phi}(0,\ell;L^2)} = \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \|u\|^2_{L^2(0,\ell;L^2(C^1_{\bar{x}}))} < \infty\},$$

and the space $L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d))$ by

$$L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)) = \{u : (0,\ell) \times \mathbb{R}^d \to \mathbb{R}; \|u\|^2_{L^2_{b,\phi}(0,\ell;W^{1,2})} = \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \|u\|^2_{L^2(0,\ell;W^{1,2}(C^1_{\bar{x}}))} < \infty\}.$$

Similarly as for the locally uniform spaces one can show that there exists an equivalent norm on the parabolic locally uniform spaces using the weighted norm.

Lemma 2.3 ([7, Theorem 2.4]). *Let $\varepsilon > 0$ be fixed and let ϕ be an admissible weight function of growth rate $\mu \in [0, \varepsilon)$. Then*

$$\|u\|^2_{L^2_{b,\phi}(0,\ell;L^2)} \approx \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(x,t)|^2 \phi_{\bar{x},\varepsilon}(x) \, dx \, dt,$$

$$\|u\|^2_{L^2_{b,\phi}(0,\ell;W^{1,2})} \approx \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} \left(|u(x,t)|^2 + |\nabla u(x,t)|^2\right) \phi_{\bar{x},\varepsilon}(x) \, dx \, dt.$$

In particular the previous lemma implies that for an admissible weight function ϕ of growth rate $\mu \in [0, \min\{\varepsilon_1, \varepsilon_2\})$ for some $\varepsilon_1, \varepsilon_2 > 0$ one has

$$\sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(x)|^2 \phi_{\bar{x},\varepsilon_2}(x) \, dx \, dt \approx \|u\|^2_{L^2_{b,\phi}(0,\ell;L^2)} \approx \sup_{\bar{x} \in \mathbb{R}^d} \phi(\bar{x}) \int_0^\ell \int_{\mathbb{R}^d} |u(x)|^2 \phi_{\bar{x},\varepsilon_1}(x) \, dx \, dt$$

and similarly in the case of $L^2_{b,\phi}(0,\ell;W^{1,2})$.

For $\mathcal{O}\subseteq\mathbb{R}^d$ we can define the $L^2_{b,\phi}(0,\ell;L^2(\mathcal{O}))$ and $L^2_{b,\phi}(0,\ell;W^{1,2}(\mathcal{O}))$ seminorms similarly as in (2.5).

Lemma 2.4 (Ehrling's lemma)**.** *Let p, $q\geq 1$ and ε, $\tilde\varepsilon>0$ be such that the embedding $W^{1,p}_{\bar x,\varepsilon}(\mathbb{R}^d)\hookrightarrow\hookrightarrow L^q_{\bar x,\tilde\varepsilon}(\mathbb{R}^d)$ holds. Then for every $\theta>0$ and $1\leq\alpha<q$ there exist C, $R>0$ such that for every $u:(0,\ell)\times\mathbb{R}^d\to\mathbb{R}$ one has*

$$\int_0^\ell \|u(t)\|^\alpha_{L^q_{\bar x,\tilde\varepsilon}}\,dt\leq\theta\int_0^\ell\|u(t)\|^\alpha_{W^{1,p}_{\bar x,\varepsilon}}\,dt+C\int_0^\ell\int_{B(\bar x,R)}|u(t,x)|^\alpha\,dxdt. \tag{2.6}$$

We remark that for an admissible weight function ϕ one can get a parabolic locally uniform version of Ehrling's lemma by multiplying (2.6) by $\phi(\bar x)$ and applying supremum over $\bar x\in\mathbb{R}^d$.

Proof. The proof is standard. Observe that it suffices to show the stationary case

$$\|u(t)\|^\alpha_{L^q_{\bar x,\tilde\varepsilon}}\leq\theta\|u(t)\|^\alpha_{W^{1,p}_{\bar x,\varepsilon}}+C\int_{B(\bar x,R)}|u(t,x)|^\alpha\,dx\qquad\text{for a.a. } t\in(0,\ell),$$

since the desired result follows by integration over $t\in(0,\ell)$. For contradiction, assume that there exist a sequence u_n such that, after renormalization,

$$1=\|u_n\|^\alpha_{L^q_{\bar x,\tilde\varepsilon}}>\theta\|u_n\|^\alpha_{W^{1,p}_{\bar x,\varepsilon}}+n\int_{B(\bar x,n)}|u_n(x)|^\alpha\,dx.$$

Clearly the sequence u_n is bounded in $W^{1,p}_{\bar x,\varepsilon}(\mathbb{R}^d)$ and therefore by the compactness of the embedding we have $u_n\to u$ in $L^q_{\bar x,\tilde\varepsilon}(\mathbb{R}^d)$ for some $u\in L^q_{\bar x,\tilde\varepsilon}(\mathbb{R}^d)$ with unit norm, in particular $u\neq 0$. Taking the limit for $n\to\infty$ leads to $u_n\to 0$ in $L^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$. However, since the convergence in $L^q_{\bar x,\tilde\varepsilon}(\mathbb{R}^d)$ implies the convergence in $L^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$, we arrive to a contradiction. Indeed, let $B\subseteq\mathbb{R}^d$ be bounded with nonempty interior and let $C>0$ be such that $C\phi_{\bar x,\tilde\varepsilon}\geq 1$ in B. Then by Hölder's inequality we have

$$\begin{aligned}\int_B|u_n(x)-u(x)|^\alpha\,dx&\leq C\int_B|u_n(x)-u(x)|^\alpha\phi_{\bar x,\tilde\varepsilon}(x)\,dx\\&\leq C\left(\int_B|u_n(x)-u(x)|^q\phi_{\bar x,\tilde\varepsilon}(x)\,dx\right)^{\frac{\alpha}{q}}\left(\int_B\phi_{\bar x,\tilde\varepsilon}(x)\,dx\right)^{\frac{q-\alpha}{q}}\\&\leq C'\left(\int_{\mathbb{R}^d}|u_n(x)-u(x)|^q\phi_{\bar x,\tilde\varepsilon}\,dx\right)^{\frac{q}{\alpha}}=C'\|u_n-u\|^\alpha_{L^q_{\bar x,\tilde\varepsilon}}.\end{aligned}$$

□

3 Squeezing property

We define the energy functional by

$$E[u](t,x)=\frac{1}{2}\left(|u_t(t,x)|^2+|u(t,x)|^2+|\nabla u(t,x)|^2\right).$$

Let us define the space of trajectories

$\mathcal{X} = \{(\chi, \chi_t); \chi \in L^2_{\mathrm{loc}}((0,\ell)\times\mathbb{R}^d) \text{ solves (1.1) on } (0,\ell) \text{ with } (\chi(0),\chi_t(0)) \in \mathcal{B}\}.$

Let $\ell > 0$ be fixed. The trajectory semigroup $L(t) : \mathcal{X} \to \mathcal{X}$ and the end-point operator $e : \mathcal{X} \to \Phi_b$ are given by

$$(L(t)(\chi,\chi_t))(s) = (S(t)\chi(s), \partial_t S(t)\chi), \ s \in (0,\ell), \qquad e(\chi) = (\chi(\ell), \chi_t(\ell)).$$

Let us also denote $L \equiv L(\ell)$. For an admissible weight function ϕ we also define

$$\begin{aligned}\Phi^\ell_{b,\phi} &= L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)) \times L^2_{b,\phi}(0,\ell; L^2(\mathbb{R}^d)),\\ W^\ell_{b,\phi} &= L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d)) \times L^2_{b,\phi}(0,\ell; W^{1,2}(\mathbb{R}^d))\end{aligned}$$

and define respective seminorms similarly as in (2.5) for the parabolic spaces.

Lemma 3.1. *There exists $\mu_0 > 0$ such that for all admissible weight functions of growth $\mu \in [0,\mu_0)$ and all $\ell > 0$ the operators $L : \Phi^\ell_{b,\phi} \to W^\ell_{b,\phi}$ and $e : \Phi^\ell_{b,\phi} \to W_{b,\phi}$ are Lipschitz continuous on $\mathcal{X}$.*

We remark that for the asymptotic analysis in the next section we will use a weaker version of Lemma 3.1, more precisely the Lipschitz continuities $L : W^\ell_{b,\phi} \to W^\ell_{b,\phi}$ and $e : W^\ell_{b,\phi} \to W_{b,\phi}$, both of which follow from the proof by adding $\|\nabla w_t(s)\|^2_{\bar{x},\varepsilon}$ to the right-hand side of (3.1). A similar remark also applies to Lemma 4.1.

Proof. Let χ_1, $\chi_2 \in \mathcal{X}$, let u_1 and u_2 be the respective solutions and denote $w = u_1 - u_2$. By Lemma [13, Lemma 9.2] the semigroup $S(t) : \Phi_{\bar{x},\varepsilon} \to W_{\bar{x},\varepsilon}$ is Lipschitz continuous on $\mathcal{B}$ uniformly w.r.t. $t \in [0,T]$, i.e.

$$\begin{aligned}&\|w(t)\|^2_{\bar{x},\varepsilon} + \|\nabla w(t)\|^2_{\bar{x},\varepsilon} + \|w_t(t)\|^2_{\bar{x},\varepsilon} + \|\nabla w_t(t)\|^2_{\bar{x},\varepsilon}\\ &\qquad \le C_{t,s}\left(\|w(s)\|^2_{\bar{x},\varepsilon} + \|\nabla w(s)\|^2_{\bar{x},\varepsilon} + \|w_t(s)\|^2_{\bar{x},\varepsilon}\right) \qquad (3.1)\end{aligned}$$

for $0 < s < t$ and $\varepsilon > 0$ sufficiently small. The Lipschitz continuity of L then follows by integration over $s \in (0,\ell)$, $t \in (\ell, 2\ell)$, multiplication by $\phi(\bar{x})$, applying supremum over $\bar{x} \in \mathbb{R}^d$ to both sides of the estimate and using the equivalence of norms from Lemma 2.3. The Lipschitz continuity of e follows in a similar manner. ☐

Definition. *The mapping $L : \mathcal{X} \to \mathcal{X}$ has a* parabolic squeezing property *for an admissible weight function ϕ if there exists $\varepsilon > 0$ such that for every $\gamma > 0$ we may find ℓ, κ, $R > 0$ so that for every χ_1, $\chi_2 \in \mathcal{X}$ and their respective solutions u_1 and u_2 we have*

$$\begin{aligned}&\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{\mathbb{R}^d}\left(E[w] + |\nabla w_t|^2\right)\phi_{\bar{x},\varepsilon}\,dx\,dt\\ &\le \gamma \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{\mathbb{R}^d}E[w]\phi_{\bar{x},\varepsilon}\,dx\,dt\\ &\quad + \kappa\left(\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{B(\bar{x},R)}|w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{B(\bar{x},R)}|w_t|^2\,dx\,dt\right)\\ &\quad + \kappa\left(\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w_t|^2\,dx\,dt\right),\end{aligned} \qquad (3.2)$$

where $w = u_1 - u_2$.

Lemma 3.2. *Let the nonlinear term* f *be subcritical, i.e. let* $0 \le q < 4/(d-2)$. *Then for every admissible function* ϕ *of sufficiently small growth the operator* L *has the parabolic squeezing property.*

Proof. The proof is similar to [11, Lemma 3.1]. Let χ_1, $\chi_2 \in \mathcal{X}$ and let u_1, u_2 be the respective solutions. Let $0 < \tau < \ell$ and denote $w = u_1 - u_2$. We test both the equations for u_1 and u_2 by $w_t + w/2$ to get

$$\begin{aligned}\frac{1}{2}\left(\|w_t(2\ell) + \frac{1}{2}w(2\ell)\|^2_{\bar{x},\varepsilon} + \frac{1}{8}\|w(2\ell)\|^2_{\bar{x},\varepsilon} + \frac{3}{4}\|\nabla w(2\ell)\|^2_{\bar{x},\varepsilon}\right) + \frac{1}{2}\int_\tau^{2\ell} \|w_t\|^2_{\bar{x},\varepsilon}\,dt \\ + \int_\tau^{2\ell} \|\nabla w_t\|^2_{\bar{x},\varepsilon} + \frac{1}{2}\|\nabla w\|^2_{\bar{x},\varepsilon}\,dt + \int_\tau^{2\ell} \left(f(u_1) - f(u_2), w_t + \frac{1}{2}w\right)_{\bar{x},\varepsilon} dt \\ + \int_\tau^{2\ell} \left(\nabla w_t, (w_t + \frac{1}{2}w)\nabla\phi_{\bar{x},\varepsilon}\right) + \left(\nabla w, (w_t + \frac{1}{2}w)\nabla\phi_{\bar{x},\varepsilon}\right) dt \\ = \frac{1}{2}\left(\|w_t(\tau) + \frac{1}{2}w(\tau)\|^2_{\bar{x},\varepsilon} + \frac{1}{8}\|w(\tau)\|^2_{\bar{x},\varepsilon} + \frac{3}{4}\|\nabla w(\tau)\|^2_{\bar{x},\varepsilon}\right). \quad (3.3)\end{aligned}$$

Using Hölder's inequality and the growth estimates on the nonlinearity f we obtain

$$\begin{aligned}I &\equiv \left|\int_{\mathbb{R}^d} (f(u_1) - f(u_2))w_t\phi_{\bar{x},\varepsilon}\,dx\right| \\ &\le \left(\int_{\mathbb{R}^d} |f(u_1) - f(u_2)|^{p_1}\phi_{\bar{x},\varepsilon}\,dx\right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^d} |w_t|^{p_2}\phi_{\bar{x},\varepsilon}\,dx\right)^{\frac{1}{p_2}} \\ &\le C_1 \left(\int_{\mathbb{R}^d} |w|^{p_1}\left(1 + |u_1|^q + |u_2|^q\right)^{p_1} \phi_{\bar{x},\varepsilon}\,dx\right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^d} |w_t|^{p_2}\phi_{\bar{x},\varepsilon}\,dx\right)^{\frac{1}{p_2}}.\end{aligned}$$

For Ehrling's lemma we will need the compact embedding $W^{1,2}(B(0,1)) \hookrightarrow\hookrightarrow L^{p_2}(B(0,1))$ and thus we require

$$1 \le p_2 < \frac{2d}{d-2} \qquad \text{and} \qquad p_1 > \frac{2d}{d+2}.$$

By Lemma 2.2 we get

$$\begin{aligned}I \le C_2 &\left(\int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}\left(\int_{B(x,1)} |w|^{p_1}\left(1 + |u_1|^q + |u_2|^q\right)^{p_1}\,dy\right) dx\right)^{\frac{1}{p_1}} \\ &\cdot \left(\int_{\mathbb{R}^d} |w_t|^{p_2}\phi_{\bar{x},\varepsilon}\,dx\right)^{\frac{1}{p_2}}\end{aligned}$$

with $C_2 > 0$ dependent of ε. Employing Hölder's inequality again we get

$$\begin{aligned}I \le C_3 &\left(\int_{\mathbb{R}^d} |w_t|^{p_2}\phi_{\bar{x},\varepsilon}\,dx\right)^{\frac{1}{p_2}} \left(\int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon}\left(\int_{B(x,1)} |w|^{p_1 r_1}\,dy\right)^{\frac{1}{r_1}}\right. \\ &\left.\cdot \left(\int_{B(x,1)} 1 + |u_1|^{p_1 r_2 q} + |u_2|^{p_1 r_2 q}\,dy\right)^{\frac{1}{r_2}} dx\right)^{\frac{1}{p_1}}\end{aligned}$$

To have the embeddings $W^{1,2}(B(x,1)) \hookrightarrow L^{p_1 r_1}(B(x,1))$ and $W^{1,2}(B(x,1)) \hookrightarrow L^{p_1 r_2 q}(B(x,1))$ we require

$$1 \le p_1 r_1 \le \frac{2d}{d-2} \qquad \text{and} \qquad 1 \le p_1 r_2 q \le \frac{2d}{d-2}. \tag{3.4}$$

For $\theta = p_1 - (d+2)/(d-2)$ this leads to the choice of r_1 to satisfy

$$1 \le r_1 \le \left(1 - \frac{\theta(d+2)}{2d + \theta(d+2)}\right)\frac{d+2}{d-2} \qquad \text{and thus} \qquad r_2 > \frac{d+2}{4 + \theta\frac{d^2-4}{2d}}. \tag{3.5}$$

It is easy to check that (3.4) is still holds for r_2 sufficiently close to the bound in (3.5) and θ sufficiently small. Returning to the estimate of the integral I we again use Hölder's inequality, the embeddings above and the dissipation estimates (1.2) to get

$$I \le C_4 \left(\int_{\mathbb{R}^d} \phi_{\bar{x},\varepsilon} \left(\int_{B(x,1)} |w|^2 + |\nabla w|^2 \, dy\right)^{\frac{p_1 s_1}{2}} dx\right)^{\frac{1}{p_1 s_1}} \cdot \left(\int_{\mathbb{R}^d} |w_t|^{p_2} \phi_{\bar{x},\varepsilon} \, dx\right)^{\frac{1}{p_2}}.$$

We choose s_1 in such a way that

$$\frac{p_1 s_1}{2} = 1, \qquad \text{in another words} \qquad s_1 = \frac{d+2}{d + \theta\frac{d+2}{d}}.$$

Clearly $s_1 > 1$ for θ sufficiently small. Using a standard embedding and the estimate from Lemma 2.2 we get

$$I \le C_5 \left(\int_{\mathbb{R}^d} \left(|w|^2 + |\nabla w|^2\right) \phi_{\bar{x},\varepsilon} \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |w_t|^{p_2} \phi_{\bar{x},\varepsilon} \, dx\right)^{\frac{1}{p_2}}.$$

Finally we use Young's inequality to obtain

$$I \le \eta \int_{\mathbb{R}^d} \left(|w|^2 + |\nabla w|^2\right) \phi_{\bar{x},\varepsilon} \, dx + C_6 \left(\int_{\mathbb{R}^d} |w_t|^{p_2} \phi_{\bar{x},\varepsilon} \, dx\right)^{\frac{2}{p_2}}$$

for $\eta > 0$ determined later. Similarly we have

$$\left|\int_{\mathbb{R}^d} (f(u_1) - f(u_2)) w \phi_{\bar{x},\varepsilon} \, dx\right| \le \eta \left(\|w\|^2_{\bar{x},\varepsilon} + \|\nabla w\|^2_{\bar{x},\varepsilon}\right) + C\|w\|^2_{L^{p_2}_{\bar{x},\varepsilon}}.$$

We can also estimate

$$\left(\nabla w, w_t \nabla \phi_{\bar{x},\varepsilon}\right) \le C\varepsilon \left(\|\nabla w\|^2_{L^2_{\bar{x},\varepsilon}} + \|w_t\|^2_{L^2_{\bar{x},\varepsilon}}\right),$$
$$\left(\nabla w_t, w_t(t) \nabla \phi_{\bar{x},\varepsilon}\right) \le C\varepsilon \left(\nu\|\nabla w_t\|^2_{L^2_{\bar{x},\varepsilon}} + C_\nu \|w_t\|^2_{L^2_{\bar{x},\varepsilon}}\right),$$

with $\nu > 0$ and $C_\nu > 0$. Putting the previous estimates into (3.3) and choosing ε, η, ν sufficiently small we get

$$
\begin{aligned}
C\left(\|w_t(2\ell) + \frac{1}{2}w(2\ell)\|^2_{\bar{x},\varepsilon} + \|w(2\ell)\|^2_{\bar{x},\varepsilon} + \|\nabla w(2\ell)\|^2_{\bar{x},\varepsilon}\right) \\
+ \zeta \int_\ell^{2\ell} \|w_t\|^2_{\bar{x},\varepsilon} + \|\nabla w_t\|^2_{\bar{x},\varepsilon} + \|\nabla w\|^2_{\bar{x},\varepsilon} + \|w\|^2_{\bar{x},\varepsilon}\, dt \\
\le \int_{\mathbb{R}^d} E[w](\tau)\phi_{\bar{x},\varepsilon}\, dx + C\int_0^{2\ell} \|w\|^2_{\bar{x},\varepsilon} + \|w_t\|^2_{L^{p2}_{\bar{x},\varepsilon}} + \|w\|^2_{L^{p2}_{\bar{x},\varepsilon}}\, dt
\end{aligned}
$$

for some $\zeta > 0$. We note that from now on the value of ε will not change. We integrate over $\tau \in (0,\ell)$ to get

$$
\begin{aligned}
\zeta\ell \int_\ell^{2\ell}\int_{\mathbb{R}^d} \left(E[w] + |\nabla w_t|^2\right)\phi_{\bar{x},\varepsilon}\, dx\, dt \\
\le \int_0^\ell \int_{\mathbb{R}^d} E[w]\phi_{\bar{x},\varepsilon}\, dx\, dt + C\ell \int_0^{2\ell} \|w\|^2_{\bar{x},\varepsilon} + \|w\|^2_{L^{p2}_{\bar{x},\varepsilon}} + \|w_t\|^2_{L^{p2}_{\bar{x},\varepsilon}}\, dt.
\end{aligned}
$$

Applying the weighted version of Ehrling's lemma (Lemma 2.4) to the functions $w(t)$ and $w_t(t)$ both on the time intervals $(0,\ell)$ and $(\ell, 2\ell)$ we get

$$
\begin{aligned}
&\zeta\ell \int_\ell^{2\ell}\int_{\mathbb{R}^d} \left(E[w] + |\nabla w_t|^2\right)\phi_{\bar{x},\varepsilon}\, dx\, dt \\
&\le \int_0^\ell \int_{\mathbb{R}^d} E[w]\phi_{\bar{x},\varepsilon}\, dx\, dt + C\ell\int_0^{2\ell}\|w\|^2_{\bar{x},\varepsilon}\, dt + C\ell\theta\Bigg(\int_0^\ell \|w\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\, dt \\
&\quad + \int_0^\ell \|w_t\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\, dt + \int_\ell^{2\ell} \|w\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\, dt + \int_\ell^{2\ell}\|w_t\|^2_{W^{1,2}_{\bar{x},\tilde{\varepsilon}}}\, dt\Bigg) \\
&\quad + C\ell\left(\int_0^\ell\int_{B(\bar{x},R)} |w|^2 + |w_t|^2\, dx\, dt + \int_\ell^{2\ell}\int_{B(\bar{x},R)} |w|^2 + |w_t|^2\, dx\, dt\right)
\end{aligned}
$$

for some $R > 0$ fixed, $\theta > 0$ determined later and some $\tilde{\varepsilon} > 0$ such that $W^{1,2}_{\bar{x},\tilde{\varepsilon}}(\mathbb{R}^d) \hookrightarrow\hookrightarrow L^q_{\bar{x},\varepsilon}(\mathbb{R}^d)$, i.e. $2\varepsilon/q > \tilde{\varepsilon}$. If we restrict ourselves to admissible functions ϕ of growth $\mu \in [0, \min\{\varepsilon, \tilde{\varepsilon}\})$, multiply by $\phi(\bar{x})$ and apply supremum over $\bar{x} \in \mathbb{R}^d$, then by Lemma 2.3 and by choosing θ sufficiently small we obtain

$$
\begin{aligned}
&\tilde{\zeta}\ell \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x}) \int_\ell^{2\ell}\int_{\mathbb{R}^d}\left(E[w] + |\nabla w_t|^2\right)\phi_{\bar{x},\varepsilon}\, dx\, dt \\
&\le C\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{\mathbb{R}^d} E[w]\phi_{\bar{x},\varepsilon}\, dx\, dt \\
&\quad + C\ell\left(\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{B(\bar{x},R)}|w|^2\, dx\, dt + \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_0^\ell\int_{B(\bar{x},R)}|w_t|^2\, dx\, dt\right) \\
&\quad + C\ell\left(\sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w|^2\, dx\, dt + \sup_{\bar{x}\in\mathbb{R}^d}\phi(\bar{x})\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w_t|^2\, dx\, dt\right).
\end{aligned}
$$

for some $0 < \tilde{\zeta} < \zeta$. The conclusion follows by dividing by $\tilde{\zeta}\ell$ and choosing ℓ sufficiently large. □

4 Entropy estimate

Let X be a metric space and let $K \subseteq X$ be precompact. We define the *Kolmogorov's ε-entropy* by

$$H_\varepsilon\left(K, X\right) = \ln N_\varepsilon(K, X),$$

where $N_\varepsilon(K, X)$ is the smallest number of ε-balls in X with centres in K that cover the set K.

Lemma 4.1. *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be bounded and let*

$$\mathbb{I}(\mathcal{O}) \leq C_0 \operatorname{vol}(\mathcal{O}) \tag{4.1}$$

for some $C_0 > 0$. Let $\varepsilon > 0$ and $\theta \in (0, 1)$. Let $(u_0, u_1) \in \mathcal{B}$ and let $(\chi_0, (\chi_0)_t)$ be the trajectory starting from (u_0, u_1). Let ϕ be an admissible weight function such that the operator L has the parabolic squeezing property for ϕ and denote $B = B_\varepsilon((\chi_0, (\chi_0)_t); \Phi^\ell_{b,\phi}) \cap \mathcal{X}$. Then there exist C_1, $\ell > 0$ such that

$$H_{\theta\varepsilon}\left((LB)|_{\mathcal{O}}, W^\ell_{b,\phi}(\mathcal{O})\right) \leq C_1 \operatorname{vol}(\mathcal{O}),$$

where the constant C_1 depends only on C_0 and θ and is independent of (u_0, u_1), ε, ϕ and $\mathcal{O}$ as long as (4.1) *holds and the constants in* (2.1) *and* (2.2) *remain the same.*

Proof. The proof combines the technique of [11, Lemma 4.1] and [7, Lemma 2.6] and adapts these to the squeezing property at hand. We will prove the assertion for $\phi \equiv 1$. The general case then follows by the same argument as in [7, Lemma 2.6], namely by showing that

$$\|\chi\|_{L^2_{b,\phi}(0,\ell;W^{1,2}(\mathcal{O}))} \approx \|F\chi\|_{L^2_{b,1}(0,\ell;W^{1,2}(\mathcal{O}))}$$

with $F : \chi \to \phi^{1/2}\chi$.

First fix $0 < \gamma < \theta^2$ and using Lemma 3.2 find κ, $\ell > 0$ such that L has the squeezing property for the weight function ϕ and γ. Let $\delta > 0$ be such that $\gamma + 4\kappa\delta^2 < \theta^2$. For $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$ fixed we denote

$$P_{x_1,x_2,x_3,x_4}\left((\chi, \partial_t\chi)\right) = \left(\chi|_{B(x_1,R)}, \partial_t\chi|_{B(x_2,R)}, L\chi|_{B(x_3,R)}, \partial_t L\chi|_{B(x_4,R)}\right),$$

where $R > 0$ comes from the parabolic squeezing property (3.2). Employing the standard Aubin-Lions lemma and the Lipschitz continuity of L we observe that the set

$$X(x_1, x_2, x_3, x_4) = \{P_{x_1,x_2,x_3,x_4}\left((\chi, \partial_t\chi)\right); (\chi, \partial_t\chi) \in B\}$$

equipped with the product topology

$$\prod_{i=1}^{4} L^2(0, \ell; L^2(B(x_i, R)))$$

can be covered by N balls of diameter $\delta\varepsilon$ with N independent of ε and x_i.

Let now $\chi_1, \chi_2 \in B$, let u_1, u_2 be their respective solutions and set $w = u_1 - u_2$. Then we find $x_i^M \in \mathbb{R}^d$ such that

$$\begin{aligned}\sup_{\bar{x}\in\mathbb{R}^d}\int_0^\ell\int_{B(\bar{x},R)}|w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d}\int_0^\ell\int_{B(\bar{x},R)}|w_t|^2\,dx\,dt \\ + \sup_{\bar{x}\in\mathbb{R}^d}\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w|^2\,dx\,dt + \sup_{\bar{x}\in\mathbb{R}^d}\int_\ell^{2\ell}\int_{B(\bar{x},R)}|w_t|^2\,dx\,dt \\ \le \int_0^\ell\int_{B(x_1^M,R)}|w|^2\,dx\,dt + \int_0^\ell\int_{B(x_2^M,R)}|w_t|^2\,dx\,dt \\ + \int_\ell^{2\ell}\int_{B(x_3^M,R)}|w|^2\,dx\,dt + \int_\ell^{2\ell}\int_{B(x_4^M,R)}|w_t|^2\,dx\,dt + \frac{1}{M}\end{aligned}$$

with $M \in \mathbb{N}$ large enough to have $\gamma\varepsilon^2 + 4\kappa\delta^2\varepsilon^2 + \kappa/M \le \theta^2\varepsilon^2$. By the previous observation we may cover the set $X(x_1^M, x_2^M, x_3^M, x_4^M)$ by $\delta\varepsilon$-balls centered at $P_{x_1^M,x_2^M,x_3^M,x_4^M}\left((\chi^i, \partial_t\chi^i)\right)$ for some $(\chi^i, \partial_t\chi^i) \in B$, $i = 1, \dots, N$. For arbitrary $(\chi, \partial_t\chi) \in B$ we may now find $(\chi^i, \partial_t\chi^i) \in B$ such that

$$\|P_{x^M}((\chi, \partial_t\chi)) - P_{x^M}\left((\chi^i, \partial_t\chi^i)\right)\|_{X(x_1^M,x_2^M,x_3^M,x_4^M)} < \delta\varepsilon.$$

The squeezing property now leads to the estimate

$$\sup_{\bar{x}\in\mathbb{R}^d}\int_\ell^{2\ell}\int_{\mathbb{R}^d}\left(E[w] + |\nabla w_t|^2\right)\,dx\,dt \le \gamma\varepsilon^2 + 4\kappa\delta^2\varepsilon^2 + \frac{\kappa}{M} \le \theta^2\varepsilon^2,$$

which finishes the proof. □

Using the previous lemma one can show that in the subcritical case the dynamical system $(S(t), \Phi_b)$ is asymptotically compact in the space W_{loc} (details can be found in [9, Proposition 6.2]). We remark that in [13] the authors obtain the same result even in the critical case using a suitable decomposition.

We will use the following auxiliary function in the spirit of [15]: let $\bar{x} \in \mathbb{R}^d$, $R > 0$ and $\nu > 0$. Define

$$\psi(\bar{x}, R) = \psi(\bar{x}, R)(x) = \begin{cases} 1, & |x - \bar{x}| \le R + \sqrt{d}, \\ \exp\left(\nu\left(R + \sqrt{d} - |x - \bar{x}|\right)\right), & \text{otherwise.}\end{cases}$$

The function $\psi(\bar{x}, R)$ is clearly an admissible weight function of growth ν with, in the notation of (2.1), $C_{\psi(\bar{x},R)} = 1$ for every $\bar{x} \in \mathbb{R}^d$ and $R > 0$. Also we have

$$H_\varepsilon\left(B|_{B(\bar{x},R)}, W_b(B(\bar{x}, R))\right) \le H_\varepsilon\left(B, W_{b,\psi(\bar{x},R)}\right), \tag{4.2}$$

where $W_b(B(\bar{x}, R))$ is a seminorm defined similarly as in (2.5) and $B \subseteq W_b^\ell$.

Lemma 4.2 ([7, Lemma 5.4])**.** *For every $\varepsilon_0 > 0$ we there exists $R' > 0$ such that for every $\bar{x} \in \mathbb{R}^d$, $R \ge 1$, $\varepsilon \in (0, \varepsilon_0)$ and $\chi_1, \chi_2 \in W_{b,\psi(\bar{x},R)}^\ell$ one has*

$$\|\chi_1 - \chi_2\|_{W_{b,\psi(\bar{x},R)}^\ell} \le \max\left\{\varepsilon, \|\chi_1 - \chi_2\|_{W_{b,\psi(\bar{x},R)}^\ell(B(\bar{x},R+R'\ln(\varepsilon_0/\varepsilon)))}\right\}.$$

Recall that $\mathcal{A} \subseteq W_b^{2,2}(\mathbb{R}^d) \times W_b^{1,2}(\mathbb{R}^d)$ is the locally compact attractor of the set (1.1) defined in Section 1.

Theorem 4.3. *There exist constants C_0, C_1, $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $\bar{x} \in \mathbb{R}^d$ and $R \geq 1$ one has the estimate*

$$H_\varepsilon\left(\mathcal{A}|_{B(\bar{x},R)}, W_b(B(\bar{x},R))\right) \leq C_0\left(R + C_1 \ln\frac{\varepsilon_0}{\varepsilon}\right)^d \ln\frac{\varepsilon_0}{\varepsilon}.$$

Proof. The proof is standard and runs in almost the same way as in [9, Theorem 6.5] and [7, Theorem 5.1] with only minor differences.

Let $\bar{x} \in \mathbb{R}^d$, $R \geq 1$ and let $\psi(\bar{x}, R)$ be of sufficiently small growth such that L has the squeezing property for $\psi(\bar{x}, R)$ and let $\ell > 0$ be such that Lemma 4.1 holds with $\theta = 1/2\,\mathrm{Lip}(L) < 1$, where $\mathrm{Lip}(L)$ denotes the Lipschitz constant of L from Lemma 3.1. The smallness of growth of $\psi(\bar{x}, R)$ can always be achieved by choosing ν small in the definition of $\psi(\bar{x}, R)$. By the Lipschitz continuity of e and the property of the weight function $\psi(\bar{x}, R)$ (4.2) we get

$$H_\varepsilon\left(\mathcal{A}|_{B(\bar{x},R)}, W_b(B(\bar{x},R))\right) \leq H_\varepsilon\left(\mathcal{A}, W_{b,\psi(\bar{x},R)}\right) \leq H_{\varepsilon/\,\mathrm{Lip}(e)}\left(\mathcal{A}_\ell, W^\ell_{b,\psi(\bar{x},R)}\right),$$

where

$$\mathcal{A}_\ell = \{(\chi, \chi_t) \in \Phi^\ell_b; (\chi(0), \chi_t(0) \in \mathcal{A}\}.$$

By the dissipation estimates (1.2) and the invariance of $\mathcal{A}$ we observe that actually $\mathcal{A}_\ell \subseteq W^\ell_b$ and $\mathcal{A}_\ell$ is invariant w.r.t. $L(t)$. Also the dissipation estimates (1.2) imply that for some $\chi \in \mathcal{A}_\ell$ and $\varepsilon_0 > 0$ sufficiently large we have

$$\mathcal{A}_\ell \subseteq B_{\varepsilon_0/\,\mathrm{Lip}(e)}((\chi, \chi_t); W^\ell_{b,\psi(\bar{x},R)}),$$

in other words

$$H_{\varepsilon_0/\,\mathrm{Lip}(e)}\left(\mathcal{A}_\ell, W^\ell_{b,\psi(\bar{x},R)}\right) = 0.$$

The key proof of the proof is to show that for $k \in \mathbb{N} \cup \{0\}$ one has

$$H_{\varepsilon_0 2^{-k}/\,\mathrm{Lip}(e)}\left(\mathcal{A}_\ell, W^\ell_{b,\psi(\bar{x},R)}\right) \leq C\left(R + C' \ln 2^k\right)^d k \tag{4.3}$$

for some $C' > 0$. Indeed, once we have established (4.3) for given $\varepsilon \in (0, \varepsilon_0)$ we may find $k \in \mathbb{N}$ such that $2^{-k}\varepsilon_0 \leq \varepsilon < 2^{-k+1}\varepsilon_0$ and the desired entropy bound follows.

The estimate (4.3) clearly holds for $k = 0$. Assume that (4.3) holds for $k \geq 0$, i.e.

$$\mathcal{A}_\ell \subseteq \bigcup_{i=1}^{N_k} B_{\varepsilon_0 2^{-k}/\,\mathrm{Lip}(e)}\left((\chi^i, \chi^i_t); W^\ell_{b,\psi(\bar{x},R)}\right) \tag{4.4}$$

for some $N_k \in \mathbb{N}$ such that $\ln N_k \leq C(R + C' \ln 2^k)^d k$ and $(\chi^i, \chi^i_t) \in \mathcal{A}_\ell$ for $1 \leq i \leq N_k$. Applying L to (4.4) and recalling the invariance of $\mathcal{A}_\ell$ under L and the Lipschitz continuity of L, we get

$$\mathcal{A}_\ell = L(\mathcal{A}_\ell) \subseteq \bigcup_{i=1}^{N} B_{\mathrm{Lip}(L)\varepsilon_0 2^{-k}/\,\mathrm{Lip}(e)}\left((L\chi^i, \partial_t L\chi^i); W^\ell_{b,\psi(\bar{x},R)}\right) \tag{4.5}$$

By Lemma 4.1 with $\theta = 1/2\,\mathrm{Lip}(L)$ each of the balls on the right-hand side of (4.5) localized to the spatial domain $B(\bar{x}, R + R' \ln 2^{k+1})$ can be covered by

$\varepsilon_0 2^{-(k+1)}$-balls in the space $W^\ell_{b,\psi(\bar{x},R)}$ in such a way that

$$\begin{aligned} H_{\varepsilon_0 2^{-(k+1)}/\operatorname{Lip}(e)} &\left(\mathcal{A}_\ell|_{B(\bar{x},R+R'\ln 2^{k+1})}, W^\ell_{b,\psi(\bar{x},R)}(B(\bar{x},R+R'\ln 2^{k+1}))\right) \\ &\le H_{\varepsilon_0 2^{-k}/\operatorname{Lip}(e)}\left(\mathcal{A}_\ell, W^\ell_{\bar{x},\psi(\bar{x},\varepsilon)}\right) + C\left(R+R'\ln 2^{k+1}\right)^d \\ &\le C\left(R+R'\ln 2^{k+1}\right)^d(k+1). \end{aligned}$$

The proof is finished since by Lemma 4.2 every $\varepsilon_0 2^{-(k+1)}/\operatorname{Lip}(e)$-covering in the space $W^\ell_{b,\psi(\bar{x},R)}(B(\bar{x},R(\varepsilon_0 2^{-(k+1)})))$ is also an $\varepsilon_0 2^{-(k+1)}/\operatorname{Lip}(e)$-covering in $W^\ell_{b,\psi(\bar{x},R)}$. □

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