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Report by Dolors Herbera Espinal of the doctoral thesis: **Tilting theory of commutative rings.**

by RNDr. Michal Hrbek

Tilting and cotilting modules are right now one of the tools that the Representation Theory of finite dimensional algebras has given to Algebra. They were introduced on the late 70's by Brenner and Butler, and its importance was coming from the the partial equivalences induced between certain categories of finitely generated modules. Around ten years later J. Rickard showed that tilting complexes were crucial to understand equivalences between bounded derived categories of modules.

In the last ten years, the development of cluster categories and cluster equivalences motivated a deep revision of the topic. New concepts like cluster tilting module, τ-tilting module and/or silting equivalence were introduced in the setting of Representation Theory to encompass the new ideas.

The development of the theory of tilting and cotilting modules for general rings and outside the world of finitely generated modules has been done in the last 30 years and it has certainly proved to be a rich and challenging world. Much more recently, also silting and cosilting modules/complexes have been introduced and their theory extends and complements the existing one of tilting/cotilting modules.

A strong effort has been made in the direction of classifying all tilting and cotilting classes (of cofinite type) over some classes of rings. This thesis is a crucial contribution to the problem as it gives a full classification of tilting classes over commutative rings parametrizing them in terms of certain subsets of the spectrum of the ring. In order to be able to appreciate the results we need to give some definitions.

Let R be any ring, and let $\mathcal{S}$ be a class of finitely generated modules of projective dimension at most n and having a projective resolution consisting of finitely generated modules. Then

$$\mathcal{S}^{\perp_\infty} = \{M \mid \operatorname{Ext}^i_R(S, M) = 0 \text{ for any } S \in \mathcal{S} \text{ and any } i > 0\}$$

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is an n-tilting class. In 2005, after a brilliant sequence of papers, it was proved that n-tilting classes are of this form, that is to say that all *tilting classes are of finite type*. On the other hand,

$$\mathcal{S}^{\top_\infty} = \{M \mid \operatorname{Tor}_i^R(S, M) = 0 \text{ for any } S \in \mathcal{S} \text{ and any } i > 0\}$$

is a cotilting class, but there are examples showing that not all cotilting classes appear this way (cotilting classes constructed in this way are called of *cofinite type*). Moreover, if $\mathcal{S}_0$ denotes another class of finitely generated modules of projective dimension at most n, having a projective resolution consisting of finitely generated modules, then $\mathcal{S}^{\perp_\infty} = \mathcal{S}_0^{\perp_\infty}$ if and only if $\mathcal{S}^{\top_\infty} = \mathcal{S}_0^{\top_\infty}$ if and only if $\mathcal{S}$ and $\mathcal{S}_0$ have the same resolving closure. Therefore there is a bijective correspondence between n-tilting classes and n-cotilting classes of cofinite type.

From this point of view, over a ring R the classification of tilting classes (or the classification of cotilting classes of cofinite type) should follow as a consequence of a classification of the strongly finitely presented modules of finite projective dimension. However, over most rings, such category is of wild type so it was apparently hopeless to get such classifications in terms of invariants of the ring. The first break through was given by Angeleri, Pospisil, Šťovíček and Trlifaj, solving the case of commutative noetherian rings. They showed the existence of a bijective correspondence between tilting classes and suitable sequences of sets of prime ideals of the ring closed under generalization. This made apparent that one should need a suitable notion of spectrum of a category of modules to go further on the classification. **In this thesis the author overcomes all these difficulties and achieves the classification for general commutative rings.**

The thesis consists of an introduction and three papers. The first two are already published and the third one is still a preprint:

- the first one is published in the prestigious Journal of Algebra, which is the most reputed journal in algebra. It is worth notice that is a paper with only one author.
- The second one is coauthored with L. Angeleri and it is published in *International Mathematics Research Notices* which is a general mathematics journal with hight impact factor: in the JCR2016 list and inside the category of Mathematics is at the position 74 out of a total of 310 (so it lies in Q1).

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- The third one is still a preprint and is coauthored with J. Šťovíček. I am sure that its brilliant and surprising results will be also published in a highly reputed journal (certainly also in Q1!).

A short summary of the three papers:

- The first one solves the problem of classifying 1-tilting classes over commutative rings. Also a construction of its corresponding tilting modules, up to equivalence, is given.

  As an application the author is able to give a new proof of a result by Bazzoni: if T is a 1-tilting module over a commutative ring whose tilting class is closed under direct limits then it must be projective. It has been recently shown that this fails for 1-tilting modules over general rings (even assuming that the ring is two-sided noetherian).

  The ideas developed in this paper are the first step towards the further developments in the rest of the thesis.

- The second paper applies the techniques developed in the first one to classify silting classes. While in the first paper it is shown that there is a bijective correspondence between tilting classes and faithful perfect Gabriel localizations of finite type, here it is show that dropping faithfulness one gets a bijective correspondence with silting classes.

- The third paper closes the problem of the classification of n-tilting modules over a commutative ring, as well as of n-cotilting modules of finite type. A surprising and deep connection with the vanishing of the first n degrees of Koszul (co)homology, of Čech (co)homology and of local (co)homology is also found. In addition, an explicit construction of the cotilting modules of cofinite type is also given.

The long list of new tools that one has to consider to achieve such general classification is quite long: vaguely associated primes, Thomason's subsets of the spectrum, Hochster topologies, grade of a module, contra-adjusted modules,.... I want to stress that the connection with the vanishing of other (co)homology theories discovered here is absolutely surprising, tremendously interesting and opens a whole new world of connections with very classical topics in Algebra.

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With no doubt, the thesis proves the author's ability for creative scientific work and also his ability to master very different concepts and use them in a very deep way.

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