**FACULTY
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**BACHELOR THESIS**

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# On the fastest path in the pedestrian flow problem

Department of Numerical Mathematics

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Prague, 21st July 2017 Jiří Zeman

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Abstract: The work treats a macroscopic pedestrian flow model. It shows the link of two possible definitions of the pedestrians' preferred direction of movement, one based on minimization of a functional, the other using the eikonal equation. The eikonal equation is derived in two dimensions, taking into account that the distant endpoint of the fastest path to the exit depends on the location of the pedestrian under consideration. Also, necessary conditions for a piecewise regular curve to be the minimizer of a certain functional in a related two-dimensional variational problem with non-standard Dirichlet boundary condition are formulated.

Keywords: pedestrian flow, eikonal equation, functional minimization

# Contents

# Introduction

Pedestrian motion is studied by contemporary scientists from various fields, including social sciences but also computer science and mathematics. The aim of pedestrian flow models is particularly to capture the phenomena that are present in emergency situations [1]. Analysis of observational data has shown interesting facts, like the one that pedestrians tend to “choose the fastest not the shortest route” ([1], p. 14), but has also revealed some peculiarities of dense crowds in panic situations (bumping into others, faster-is-slower effect, . . . ). Obviously, this can be dangerous during disasters such as the Grenfell Tower fire in London on 14 July 2017.

Today’s models are descriptive but scarcely predictive [2]. Fortunately, we can already benefit from certain achievements in crowd simulation. To give an example, the Esprit arena in Düsseldorf with a capacity of 60 000 people is equipped with a system that monitors visitors, transfers the data to a real-time computer simulation and provides information to decision-makers for evacuation purposes, in case of need.

In this work, we deal with a fluid-based model where the pedestrian at a point $x$ and a time $t$ decides upon their preferred direction of movement according to the overall density distribution at the time $t$. The crowd motion is simulated by the continuity equation and the equations of motion of compressible inviscid flow. Chapter 2 consists of a description of the model.

In literature, a PDE called the eikonal equation is usually added to the continuity equation and the equations of motion. In Chapter 5, this equation is derived on the basis of a variational formalization of the pedestrians’ effort to reach the exit in minimal time. The dependence of the point where the pedestrian completes his/her journey to the exit on the pedestrian’s location at the time $t$ is taken into consideration.

Among all possible escape paths, a pedestrian tries to choose the one along which the travel time is the shortest. The reader may find information about how to express this travel time using a curvilinear integral, in Chapter 3. This concept gives rise to a functional $\tilde{T}$ that has to be minimized to obtain the intended velocity of the modelled pededestrians. Treating the minimization in a generality sufficient for a domain with one exit and $\mathcal{C}^1$-boundary, Chapter 4 extends the result on necessary conditions for a minimum from T. Petrášová [3] to two dimensions, piecewise regular paths and a loose-endpoint boundary condition. This is the novelty of the bachelor thesis.

There are two ways how to determine the preferred direction of movement. The first possibility is to solve the eikonal equation via the Bornemann-Rasch algorithm. A second possibility that has been tested is the numerical minimization of $\tilde{T}$ through approximation of the shortest path by Dijkstra’s algorithm.

Some other examples of models for the movement of pedestrians are mentioned in Chapter 1 to illustrate their diversity.

# Notation

## Euclidean plane

In the Euclidean space $\mathbb{R}^2$, points will be denoted by italic small letters ($x = (x_1, x_2)$, $x_1$, $x_2 \in \mathbb{R}$, $y = (y_1, y_2)$, $y_1$, $y_2 \in \mathbb{R}\ldots$) and vectors by bold italic small letters ($\boldsymbol{a}$, $\boldsymbol{b}$, ...). Subscripts will be added to denote the components of a vector:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1, a_2)^\top$$

or a point: $x = (x_1, x_2)$, $x_1$, $x_2 \in \mathbb{R}$. By default, we shall treat vectors as column vectors, and points as ordered pairs, i. e., written in a single line. Occasionally, the time dimension will be added: $(x, t) = (x_1, x_2, t) \in \mathbb{R}^3$.

Let us also introduce the dot product $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b}^\top \boldsymbol{a} = a_1 b_1 + a_2 b_2$ , $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$, and the Euclidean norm $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + a_2^2}$.

If a set $A \subset \mathbb{R}^2$, let us denote by $\partial A$ its boundary and by $\overline{A} = A \cup \partial A$ its closure.

## Test functions

By $\mathcal{C}_0^\infty([0,1])$, we mean the vector space of all infinitely differentiable functions $\boldsymbol{\psi} : [0,1] \to \mathbb{R}^2$ with a compact support contained in $(0,1)$, which implies that $\boldsymbol{\psi}(0) = \boldsymbol{\psi}(1) = \mathbf{0}$ for every $\boldsymbol{\psi} \in \mathcal{C}_0^\infty([0,1])$.

## Dependence on density

Certain physical quantities $\boldsymbol{q}$ (scalar or vector ones) depend on the position $x$ and time $t$ through density: $\boldsymbol{q}(x,t) = \boldsymbol{q}(\rho(x,t))$. Being aware of the fact that the functions $\boldsymbol{q}$ and $\boldsymbol{q} \circ \rho$ are mathematically different, we keep the same letter $\boldsymbol{q}$ to denote both of them when no confusion can arise, as it is done frequently in physics. To be more specific, it is the case of $\boldsymbol{\mu}$, $V$, $\phi$ and $\mathbf{f}$ in the model.

# 1. Variety of methods for the description of pedestrian motion

There exist two major approaches to pedestrian modelling. *Macroscopic* models treat the crowd as a whole and often derive from physical laws of fluid flow, whereas *microscopic* models regard individual pedestrians separately. The development of microscopic models has started later than that of the macroscopic ones.

A brief comparison of the two families of models, both of which are in the spotlight of contemporary researchers, can be found in [1].

## 1.1 Microscopic models

In contrast to the macroscopic models, these reflect better the diversity of people's walking habits. For instance, some people are easily distracted by shop window displays while others rush resolutely in a fixed direction. Let us take a look at some examples of microscopic models.

The *behavioural force model* stems from social psychology and has been conceived by Helbing and Molnár [4]. Dirk Helbing is a professor of sociology at ETH Zürich who habilitated in physics and has made important contributions to the field of pedestrian simulation. Certain ideas of the behavioural force model have been borrowed by later researchers to incorporate psychological phenomena into microscopic as well as macroscopic models. The macroscopic model in Chapter 2 uses an acceleration term $\mathbf{f}$ from the behavioural force model.

In *cellular automata models*, the plane is divided into a regular rectangular grid with $0.5 \times 0.5\,\mathrm{m}$ cells. Each individual occupies one whole cell, therefore it cannot be accessed by another person when it is already taken. At every discrete time step, the positions of pedestrians in the lattice are recalculated following a set of basic rules. There have also been experiments with a finer grid than $0.5 \times 0.5\,\mathrm{m}$ to make the movement more realistic. (In such a model, each pedestrian spreads over several cells.) Dadová [5] states that the advantages of cellular automata (CA) models lie in easily-programmable obstacle avoidance and collision detection. On the other hand, the discrete nature of CA makes it problematic to model high density situations or smooth changes of velocity .

Pedestrian flow simulations also attract attention of experts in *artificial intelligence* (AI). For example, Schelhorn et al. [6] explain the idea of an *agent-based model*. An agent is described as "an identifiable unit of computer program code which is autonomous and goal-directed". It needs not be spatially located or aware, as the purpose is to figure out the resultant of individual behaviours rather than to model every aspect of reality. Commonly, a model involves tens but no more than thousands of agents. The authors propose their own agent-based pedestrian model. In it, all the pedestrians are created referring to socio-economic data (income, gender, . . . ) and are attributed with behavioural characteristics such as speed or visual range. Then the pedestrians are released in an urban district at parking areas or bus stops according to a Poisson distribution and the movement can start.

## 1.2 Macroscopic models

Macroscopic models can be computationally more efficient than microscopic ones [1]. They often treat the crowd as if it were a sort of fluid and truly, there is "evidence to suggest that crowds act like fluid" at high densities [1, p. 13]. Helbing et al. [7] list some "striking analogies" of the motion of pedestrian crowds and that of fluids or granular flows. However, the oldest models that were only based on principles from physics provably did not reflect enough of the psychological behaviour of pedestrians. For example, in a room filled with smoke, panicking people resort to a herd-like behaviour and gather around an already blocked exit. A fluid-based model would distribute the people among all available exits, though [1]. That is why the more recent models attempt to include e. g. some macroscopic version of the social forces affecting pedestrians.

Piccoli and Tosin's model [2], utilizing *discrete time-evolving measures* instead of PDE, is macroscopic in the way that it does not capture the exact positions of each pedestrian but allows us to measure the crowding (i. e. an estimate of the number of people) for every (Borel) subset of the region in question. In more detail, the authors denote by $\Omega$ the walking area and at every time $n \geq 0$, they introduce a positive Radon measure $\mu_n$ on $\mathcal{B}(\Omega)$[1]. For all $E \in \mathcal{B}(\Omega)$, $\mu_n(E)$ approximates the amount of pedestrians contained in $E$ at time $n$. By virtue of a *motion mapping* $\gamma_n : \Omega \to \Omega$, they obtain a recursive relation between $\mu_{n+1}$ and $\mu_n$. The governing equations are then discretized to yield a numerical scheme.

There are also macroscopic models which use *queuing theory* or *transition matrices*.

[1]The symbol $\mathcal{B}(\Omega)$ denotes the $\sigma$-algebra of all Borel subsets of $\Omega$

# 2. Pedestrian flow model

## 2.1 Formulation of the problem

In 2010, Jiang et al.[8] extended to two dimensions a 1D vehicular traffic model by Payne[9] and Whitham[10] to use it in crowd dynamics. Its description here adopts notation from Felcman[11] and Twarogowska[12].

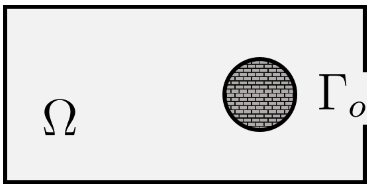


Figure 2.1: A walking facility with a round obstacle and an exit.

The room or area where the pedestrians move finds its mathematical counterpart in an open bounded simply connected set $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ has the form $\partial\Omega = \overline{\Gamma}_w \cup \overline{\Gamma}_o$, where $\Gamma_w$, $\Gamma_o$ are disjoint and expressible as a union of a finite number of connected sets open in $\partial\Omega$. The part $\Gamma_o$, which constitutes the outflow boundary (exits of the room), is assumed to have a non-zero one-dimensional Hausdorff measure. Whereas $\Gamma_w$ represents the walls.

We are interested in the pedestrian flow during a time interval $(0, T)$, $T > 0$. For brevity, let $Q_T$ denote the Cartesian product $\Omega \times (0, T)$.

### 2.1.1 Physical quantities appearing in the model

If the crowd is sufficiently large, it can be treated as continuum and described by similar quantities as in fluid dynamics:

- $\rho : Q_T \longrightarrow (0, \infty)$ is the density of pedestrians (number of individuals per $m^2$).
- $\boldsymbol{v} : Q_T \longrightarrow \mathbb{R}^2$ with components $v_i$, $i = 1, 2$, denotes the average velocity of the crowd at a specified point in $m \cdot s^{-1}$.
- $p : Q_T \longrightarrow (0, \infty)$, the pressure with physical dimension $ped \cdot s^{-2}$, can be understood as a result of interactions among pedestrians. We assume that it depends on density through the power law for isentropic gases

$$p = p(\rho) = p_0 \rho^\gamma, \quad p_0 > 0, \quad \gamma > 1. \tag{2.1}$$

  The notation means $p(x, t) = p(\rho(x, t))$, $(x, t) \in Q_T$. Physically relevant values are $p_0 \in [0.0005, 10]$ and $\gamma \in [2, 5]$.

The pedestrian flow equations are then derived from the physical conservation laws of mass and momentum.

### 2.1.2 Volume and surface forces

Analogously to fluid dynamics, two kinds of forces affecting the pedestrians are considered.

First, surface forces, which act upon the border of any subdomain $\omega \subset \Omega$ and are modelled by the Cauchy stress tensor $\mathcal{T}$. They represent pedestrian reactions to the surrounding people. According to [8], there are several reasons for neglecting viscous forces, thus the flow is regarded as inviscid and the stress tensor takes the form $\mathcal{T} = -p\mathbb{I}$.

Second, volume forces, which are expressed by a formula by Helbing [4]:

$$\mathbf{f} = \frac{1}{\tau}\left(V\boldsymbol{\mu} - \boldsymbol{v}\right), \quad (x,t) \in Q_T. \tag{2.2}$$

In his work, Helbing introduces several so-called *social forces* that appear in pedestrian movement. The above, given by (2.2), describes the adaptation of the current velocity $\boldsymbol{v}$ (at a specified point) to the desired velocity $V\boldsymbol{\mu}$. Here $\boldsymbol{\mu}$ is a unit vector and $V$ its magnitude.

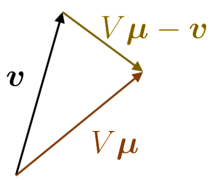


Figure 2.2: The intended velocity $V\boldsymbol{\mu}$, the actual velocity $\boldsymbol{v}$ and the direction in which $\mathbf{f}$ is oriented.

The explanation of $\mathbf{f}$, which has the physical dimension of acceleration, lies in the microscopic description. A free pedestrian at a time $t$ and a point $x$ in an empty room would move with the speed $V(x,t)$ in the preferred direction $\boldsymbol{\mu}(x,t)$ (e. g. straight towards the exit). Surrounded by other individuals, he/she is forced to avoid collisions so the actual velocity is $\boldsymbol{v}(x,t)$ instead (see figure 2.2). This frustrates the unfortunate pedestrian, but as a human being endowed with free will, he/she tries to change it. Such tendency to adjust the instant velocity to the intended one is expressed in the model via subjecting the pedestrian to an artificial acceleration $\mathbf{f}$ oriented in the direction of $(V\boldsymbol{\mu} - \boldsymbol{v})$ but diminished by a relaxation time $\tau$ in seconds. If $\tau$ is chosen small, then $\mathbf{f}$ causes a quick adjustment of velocity and vice versa.

*Remark* 1. The letter $\mathbf{f}$ typically denotes the density of volume forces in fluid dynamics and the notation remains here. But the physical dimension of the quantity, if defined by (2.2), is $m \cdot s^{-2}$, so Jiang et al. [8] derive its form as if it were a volume force but call it, more carefully, a "relaxation term" later.

On macroscopic scale, we suppose $V$ and $\boldsymbol{\mu}$ to be density-dependent. For the speed $V$, Jiang et al.([8], p. 4629) propose an exponential relation based on earlier studies:

$$V = v_{max} e^{-\alpha\left(\frac{\rho}{\rho_{max}}\right)^2}, \quad \alpha > 0. \tag{2.3}$$

If $\rho$ reaches a congestion density $\rho_{max}$, the motion is hardly possible. Realistic values of $\rho_{max}$, used in simulations range from 7 to 9. The free flow speed $v_{max}$ is typically set to a constant value between 1 and 7 $m \cdot s^{-1}$. As to the last parameter in (2.3), the authors of the model choose $\alpha = 7.5$.

### 2.1.3 Preferred direction $\boldsymbol{\mu}$ of movement of the pedestrians

We define $\boldsymbol{\mu}$ to be a unit tangent vector to the fastest path to the exit in the scalar field $V$ at a time $t$. Hence $\boldsymbol{\mu}(x,t)$, $x \in \Omega$, $t \in (0,T)$, depends on the whole speed field $V(\cdot,t)$ and therefore also on the density distribution in $\Omega$ for a given $t \in (0,T)$. The determination of $\boldsymbol{\mu}$ is based on functional minimization (the fastest path is the minimum of a certain functional).

Another way how to find $\boldsymbol{\mu}$ is to solve a non-linear partial differential equation, called the *eikonal equation*. Both approaches will be explained subsequently. Namely, we shall deal with the variational approach in Chapter 3 and with the eikonal equation in Chapter 5.

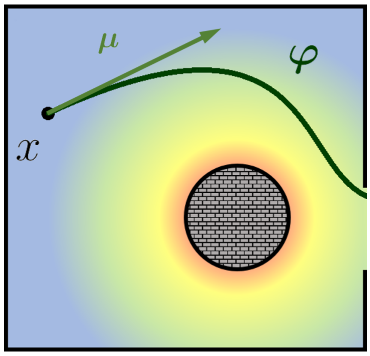

Figure 2.3: The definition of $\boldsymbol{\mu}$.

### 2.1.4 Pedestrian flow equations

Under the aforementioned hypotheses, the conservation of mass and momentum yield the system of pedestrian flow equations (PFEs):

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 \quad \text{in}\, Q_T \tag{2.4}$$

$$\frac{\partial \rho v_i}{\partial t} + \operatorname{div}(\rho v_i \boldsymbol{v}) = \rho \mathrm{f}_i - \frac{\partial p}{\partial x_i} \quad \text{in}\, Q_T, \tag{2.5}$$

$i = 1, 2$. The pressure $p$ is calculated from (2.1) and $\mathrm{f}_i$ from (2.2).

Using the vector notation and an auxiliary variable $\boldsymbol{w} = (w_1, w_2, w_3)^\top = (\rho, \rho v_1, \rho v_2)^\top$ as in [11], we can rewrite (2.4)–(2.5) as

$$\frac{\partial}{\partial t}\underbrace{\begin{pmatrix}\rho\\ \rho v_1\\ \rho v_2\end{pmatrix}}_{\boldsymbol{w}} + \frac{\partial}{\partial x_1}\underbrace{\begin{pmatrix}\rho v_1\\ \rho v_1^2 + p\\ \rho v_1 v_2\end{pmatrix}}_{\boldsymbol{f}_1(\boldsymbol{w})} + \frac{\partial}{\partial x_2}\underbrace{\begin{pmatrix}\rho v_2\\ \rho v_1 v_2\\ \rho v_2^2 + p\end{pmatrix}}_{\boldsymbol{f}_2(\boldsymbol{w})} = \rho \begin{pmatrix}0\\ \mathrm{f}_1\\ \mathrm{f}_2\end{pmatrix}. \tag{2.6}$$

The quantities $\mathrm{f}_1$, $\mathrm{f}_2$ are dependent on $\boldsymbol{v}$ and $\rho$ (see (2.2)), even if it is not indicated explicitly in the equations. Equipped with suitable initial and boundary conditions, the system is solved numerically.

# 3. Fastest path as a minimum of a functional

## 3.1 Fastest path in the field $V$

The fastest path is considered as a curve in $\mathbb{R}^2$. Let us introduce some basic definitions.

### 3.1.1 Preliminary definitions

**Definition 1.** *We call a finite sequence of points $\{\sigma_k\}_{k=0}^n$, $\sigma_k \in \mathbb{R}$, $n \in \mathbb{N}$ a* partition *of the interval $[a, b]$ if*

$$a = \sigma_0 < \sigma_1 < \cdots < \sigma_n = b.$$

*By the* norm *of the partition $P = \{\sigma_k\}_{k=0}^n$, we mean the number*

$$\nu(P) = \max\{\sigma_{k+1} - \sigma_k; k = 0, 1, \dots, n-1\}.$$

**Definition 2.** *A* curve *is defined as a continuous mapping $\boldsymbol{\varphi}$ of a compact interval $\emptyset \neq [a, b] \subset \mathbb{R}$ into $\mathbb{R}^2$. It means that $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^\top$ and the functions $\varphi_i : [a, b] \to \mathbb{R}$, $i = 1, 2$, are continuous. The set*

$$\langle \boldsymbol{\varphi} \rangle = \{\boldsymbol{\varphi}(\sigma) \in \mathbb{R}^2; \sigma \in [a, b]\}$$

*is called the* geometric image *of the curve $\boldsymbol{\varphi}$.*

*We say that a curve $\boldsymbol{\varphi}$ is* simple *if $\boldsymbol{\varphi}$ is injective. Further, we say that a curve $\boldsymbol{\varphi}$ is* regular *if $\boldsymbol{\varphi} \in \mathcal{C}^1([a, b], \mathbb{R}^2)$ (one-sided derivatives are considered at the endpoints 0, 1 of the interval) and $\boldsymbol{\varphi}'(\sigma) \neq \mathbf{0}$ for each $\sigma \in [a, b]$.*

*A curve $\boldsymbol{\varphi}$ is said to be* piecewise regular *if there exists a partition $\{\sigma_k\}_{k=0}^n$ of the interval $[a, b]$ such that $\boldsymbol{\varphi}\big|_{[\sigma_k, \sigma_{k+1}]}$ is a regular curve for $k = 0, 1, \dots, n-1$.*

*If there exists a partition $\{\sigma_k\}_{k=0}^n$ of the interval $[a, b]$ such that $\boldsymbol{\varphi}\big|_{[\sigma_k, \sigma_{k+1}]}$ is an affine map for $k = 0, 1, \dots, k-1$, i. e.,*

$$\boldsymbol{\varphi}\Big|_{[\sigma_k, \sigma_{k+1}]}(\sigma) = \frac{\sigma_{k+1} - \sigma}{\sigma_{k+1} - \sigma_k} u_k + \frac{\sigma - \sigma_k}{\sigma_{k+1} - \sigma_k} u_{k+1}, \quad u_k \in \mathbb{R}^2,$$

*then $\boldsymbol{\varphi}$ is called a* polygonal path*.*

*Remark* 2. For our purposes, it will be usually convenient to restrict ourselves to curves with the domain $[a, b] = [0, 1]$.

**Definition 3** (Riemann integral)**.** *Consider a non-empty compact interval $[a, b]$ and a real-valued function $f$ defined on $[a, b]$. We say that a number $I \in \mathbb{R}$ is the* Riemann integral of $f$ on $[a, b]$ *if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there is a $\delta \in \mathbb{R}$, $\delta > 0$, with the following property:*

*Given a partition $P = \{\sigma_k\}_{k=0}^n$ of the interval $[a, b]$ with norm $\nu(P) < \delta$ and $\tilde{\sigma}_k \in [\sigma_{k-1}, \sigma_k]$, $k = 1, \ldots, n$, it follows that the inequality*

$$\left| \sum_{k=1}^{n} f(\tilde{\sigma}_k)(\sigma_k - \sigma_{k-1}) - I \right| < \varepsilon \text{ holds.}$$

*If such an $I$ exists, we denote it by*

$$I =: \int_a^b f(\sigma) d\sigma$$

.

**Definition 4** (length of a curve)**.** *Let $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^2$ be a piecewise regular curve. By the* length $\ell$ of the curve $\boldsymbol{\varphi}$ *we mean the value*

$$\ell(\boldsymbol{\varphi}) = \int_a^b |\boldsymbol{\varphi}'(\sigma)| d\sigma.$$

*Remark* 3. The length of a piecewise regular curve is well-defined because the integral $\int_a^b |\boldsymbol{\varphi}'(\sigma)| d\sigma$ converges, To see it, note that the integrand is a composition of bounded functions that are continuous almost everywhere in an interval of finite Lebesgue measure (if $\sigma_0, \sigma_1, \ldots, \sigma_N$ are the points where $\boldsymbol{\varphi}'$ vanishes or does not exist, then for $n = 0, 1, \ldots, N-1$, the derivatives $\boldsymbol{\varphi}'\big|_{[\sigma_n, \sigma_{n+1}]}$ are continuous thus bounded as well). We applied Lebesgue's criterion for Riemann integrability (Proposition 1).

*Remark* 4. The length of $\boldsymbol{\varphi}$ is equal to the one-dimensional Hausdorff measure of its geometric image.

**Proposition 1** (criterion of Riemann integrability)**.** *A bounded function $g$ on a compact interval $K$ is Riemann-integrable if and only if $g$ is continuous almost everywhere in $K$ in the sense of the Lebesgue measure.*

### 3.1.2 Time to move along a path

Let the time $t \in (0, T)$ be fixed and the speed $V(x, t) = V(\rho(x, t))$ given at every point $x \in \Omega$. We recall that $V$ satisfies (2.3) and acts through (2.2) in the model. Yet we may proceed in greater generality, only supposing that $V \circ \rho$ is positive, continuously differentiable in $\Omega \times (0, T)$ and admitting a positive continuous extension to $\overline{\Omega} \times [0, T]$.[1]

For abbreviation, let $\Omega_a$ stand for the union $\Omega \cup \Gamma_o$, which constitutes the part of $\overline{\Omega}$ that is accessible to pedestrians.

By a path connecting two points $x, x^0 \in \Omega_a$ we mean a piecewise regular curve $\boldsymbol{\varphi}$ such that $\boldsymbol{\varphi}(0) = x$ and $\boldsymbol{\varphi}(1) = x^0$. Sometimes we shall use a more detailed notation $\boldsymbol{\varphi}(\sigma; x)$ for the curve. It states that $\boldsymbol{\varphi}$ is parametrized by the parameter $\sigma \in [0, 1]$ and $x = \boldsymbol{\varphi}(0; x)$.

[1] Which means that there is a continuous function $\tilde{V} : \overline{\Omega} \times [0, T] \to \mathbb{R}$ such that $\tilde{V}\big|_{\Omega \times (0,T)} = V$. Hereafter $V(x, t)$, $x \in \partial\Omega$, will be understood as the value of this continuous extension $\tilde{V}$.

We shall discuss this question: what is the time $\tilde{T}[\boldsymbol{\varphi}]$ needed to move from a point $x \in \Omega_a$ to another point $x^0 \in \Omega_a$ along a piecewise regular curve $\boldsymbol{\varphi}(\sigma; x)$ connecting the points? To get the idea, let us begin with $V = const$. Then

$$\text{time} = \frac{\text{distance}}{\text{speed}}, \text{ which translates into}$$
$$\tilde{T}[\boldsymbol{\varphi}] = \frac{\ell(\boldsymbol{\varphi})}{V},$$

where $\ell(\boldsymbol{\varphi})$ is from Definition 4. Now let $V$ be generally non-constant. Then for every partition $P = \{\sigma_k\}_{k=0}^n$ of the interval $[0,1]$ we can approximate

$$\tilde{T}[\boldsymbol{\varphi}] \approx \sum_{k=1}^{n} \frac{s_k}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)} \tag{3.1}$$

where $s_k = \int_{\sigma_{k-1}}^{\sigma_k} |\boldsymbol{\varphi}'(\sigma)| d\sigma$ are the lengths of $\boldsymbol{\varphi}\Big|_{[\sigma_{k-1}, \sigma_k]}$, $k \in \{1, 2, \dots, n\}$, and each $\tilde{\sigma}_k$ is a point in the interval $(\sigma_{k-1}, \sigma_k)$. Without loss of generality, it will be assumed that the partition $P$ contains all the points where $\boldsymbol{\varphi}'$ equals zero or does not exist.

It appears natural to define $\tilde{T}[\boldsymbol{\varphi}]$ as the value which the sum in (3.1) approaches as $n$ tends to infinity. The symbol $\int_{\boldsymbol{\varphi}} \frac{ds}{V}$ will be used to denote this limit, as precised in the next

**Definition 5.** *Let $\boldsymbol{\varphi} : [0,1] \to \Omega_a$ be a piecewise regular parametric curve. We say that*

$$\tilde{T}[\boldsymbol{\varphi}] = \lim_{\nu(P_n) \to 0} \sum_{k=1}^{n} \frac{s_k}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}$$

*if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $P_n = \{\sigma_k\}_{k=0}^n$ is a partition*[2] *of the interval $[0,1]$ with norm $\nu(P_n) < \delta$ and $\tilde{\sigma}_k \in (\sigma_{k-1}, \sigma_k)$, it follows that*

$$\left| \sum_{k=1}^{n} \frac{s_k}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)} - \tilde{T}[\boldsymbol{\varphi}] \right| < \varepsilon.$$

*If $\lim_{\nu(P_n) \to 0} \sum_{k=1}^{n} \frac{s_k}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}$ exists and is finite, we denote it by $\int_{\boldsymbol{\varphi}} \frac{ds}{V}$. The value $\int_{\boldsymbol{\varphi}} \frac{ds}{V}$ will be referred to as the* curvilinear integral *of the function $\frac{1}{V}$ along the curve $\boldsymbol{\varphi}$.*

The concept of the *time to pass along a curve* $\boldsymbol{\varphi}$ in the speed field $V$ has been formalized using an *ad hoc* definition of the curvilinear integral, based on (3.1). However, $\tilde{T}[\boldsymbol{\varphi}]$ can be calculated as a Riemann integral, namely the Riemann integral of the function $f(\sigma) = \frac{|\boldsymbol{\varphi}'(\sigma)|}{V(\boldsymbol{\varphi}(\sigma), t)}$.

**Lemma 2.** *Assume that the curve $\boldsymbol{\varphi}$ is piecewise regular and $V$ continuous and positive on $\overline{\Omega} \times [0, T]$. Then*

$$\int_{\boldsymbol{\varphi}} \frac{ds}{V} = \int_0^1 \frac{|\boldsymbol{\varphi}'(\sigma)|}{V(\boldsymbol{\varphi}(\sigma), t)} d\sigma, \quad t \in (0, T) \text{ fixed},$$

*where the integral on the right-hand side is a Riemann one.*

[2] The notation $P_n$ means that $P_n$ consists of $n + 1$ points.

*Proof:* Let us consider the curve $\boldsymbol{\varphi}$ to be regular. For a piecewise regular curve, the proof would be analogous, but somewhat more technical. We shall denote by $I$ the Riemann integral:

$$I := \int_0^1 \frac{|\boldsymbol{\varphi}'(\sigma)|}{V(\boldsymbol{\varphi}(\sigma), t)} d\sigma.$$

By the Lebesgue criterion (Proposition 1), the integral $I$ is convergent. Let $\varepsilon > 0$ be given. By assumption,

1. Find an $M$ such that $|\frac{1}{V(x,t)}| \leq M$ for all $(x,t) \in \overline{\Omega} \times [0,T]$.

2. Take a $\delta_1 > 0$ satisfying for every pair $\sigma^*$, $\sigma'$ in $[0,1]$, $|\sigma^* - \sigma'| < \delta_1$, the inequality $|\boldsymbol{\varphi}'(\sigma^*) - \boldsymbol{\varphi}'(\sigma')| < \frac{\varepsilon}{2M}$ (uniform continuity of the continuous function $\boldsymbol{\varphi}'$ on the compact interval $[0,1]$).

3. Thirdly, consider a $\delta_2 > 0$ to have $|\sum_{k=1}^{n} \frac{|\boldsymbol{\varphi}'(\tilde{\sigma}_k)|}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k),t)}(\sigma_k - \sigma_{k-1}) - I| < \frac{\varepsilon}{2}$ for every partition with norm less than $\delta_2$ and $\tilde{\sigma}$ interior points of the partition intervals. Here we used the definition of the Riemann integral – we know that $I$ converges.

By the mean value theorem for integrals

$$\sum_{k=1}^{n} \frac{s_k}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)} = \sum_{k=1}^{n} \frac{|\boldsymbol{\varphi}'(\sigma_k^*)|}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}(\sigma_k - \sigma_{k-1}), \quad \sigma_k^* \in (\sigma_{k-1}, \sigma_k). \tag{3.2}$$

Setting $\delta := \min\{\delta_1, \delta_2\}$, we get, for any partition $P = \{\sigma_k\}_{k=0}^{n}$ with norm $\nu(P) < \delta$,

$$\begin{aligned}
&\left|\sum_{k=1}^{n} \frac{|\boldsymbol{\varphi}'(\sigma_k^*)|}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}(\sigma_k - \sigma_{k-1}) - I\right| = \left|\sum_{k=1}^{n} \frac{|\boldsymbol{\varphi}'(\sigma_k^*)| - |\boldsymbol{\varphi}'(\tilde{\sigma}_k)|}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}(\sigma_k - \sigma_{k-1}) + \right. \\
&\left. + \sum_{k=1}^{n} \frac{|\boldsymbol{\varphi}'(\tilde{\sigma}_k)|}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}(\sigma_k - \sigma_{k-1}) - I\right| \leq \sum_{k=1}^{n} \frac{\Big||\boldsymbol{\varphi}'(\sigma_k^*)| - |\boldsymbol{\varphi}'(\tilde{\sigma}_k)|\Big|}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}(\sigma_k - \sigma_{k-1}) + \\
&+ \left|\sum_{k=1}^{n} \frac{|\boldsymbol{\varphi}'(\tilde{\sigma}_k)|}{V(\boldsymbol{\varphi}(\tilde{\sigma}_k), t)}(\sigma_k - \sigma_{k-1}) - I\right| \leq \sum_{k=1}^{n} \frac{\varepsilon}{2M} M(\sigma_k - \sigma_{k-1}) + \frac{\varepsilon}{2} \leq \\
&\leq \frac{\varepsilon}{2}(\sigma_n - \sigma_0) + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where the reverse triangle inequality $\Big||\boldsymbol{a}| - |\boldsymbol{b}|\Big| \leq |\boldsymbol{a} - \boldsymbol{b}|$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$, came in handy, too.

❑

Altogether, we have

$$\begin{aligned}
\tilde{T}[\boldsymbol{\varphi}] &\stackrel{\text{def}}{=} \int_{\boldsymbol{\varphi}} \frac{ds}{V} = && \text{(physical motivation)} \\
&= \int_0^1 \frac{|\boldsymbol{\varphi}'(\sigma; x)|}{V(\boldsymbol{\varphi}(\sigma; x), t)} d\sigma, && \text{(lemma 2)}
\end{aligned} \tag{3.3}$$

if $\boldsymbol{\varphi}$ is given. The relation (3.3) can be used in a practical calculation.

Let us set

$$L(\sigma;\varphi_1,\varphi_2,\varphi_1',\varphi_2') := \frac{\sqrt{(\varphi_1')^2+(\varphi_2')^2}}{V(\varphi_1,\varphi_2,t)}, \qquad \sigma\in[0,1],\ (\varphi_1,\varphi_2)\in\Omega_a,\ (\varphi_1',\varphi_2')\in\mathbb{R}^2. \tag{3.4}$$

In the definition (3.4), we regard $\sigma$, $\varphi_1$, $\varphi_2$, $\varphi_1'$ and $\varphi_2'$ as symbols for independent variables of the function $L$. To shorten notation, the form $L(\sigma;\boldsymbol{\varphi},\boldsymbol{\varphi}')$ will also be used. Then we may substitute the values and the derivative of a piecewise regular curve $\boldsymbol{\varphi}:[0,1]\to\Omega_a$ into $L$. In (3.3), this leads to

$$\tilde{T}[\boldsymbol{\varphi}] = \int_0^1 L(\sigma;\boldsymbol{\varphi}(\sigma),\boldsymbol{\varphi}'(\sigma))d\sigma. \tag{3.5}$$

If $V$ is positive and of class $\mathcal{C}^1$ (as stated in the beginning of this Sec. 3.1.2), it is clear that $L\in\mathcal{C}^1((0,1)\times\Omega_a^2\times\mathbb{R}^2)$.

### 3.1.3 Fastest path (shortest time)

For a chosen $t\in[0,T]$, the pedestrian at a point $x$ intends to follow a path $\boldsymbol{\varphi}$ with the property that the time $\tilde{T}[\boldsymbol{\varphi}]$ is minimal:

$$\tilde{T}[\boldsymbol{\varphi}] = \min_{\substack{\boldsymbol{\eta}(\cdot;x):\,[0,1]\to\Omega_a\\ \boldsymbol{\eta}(1;x)\in\Gamma_o}} \tilde{T}[\boldsymbol{\eta}]. \tag{3.6}$$

Note that if $\boldsymbol{\varphi}(\sigma;x)$ is a minimizing curve as above, the endpoint $x^0=\boldsymbol{\varphi}(1;x)$, $x^0\in\Gamma_o$, *depends* on $x$, so in fact, $x^0=x^0(x)$. This can easily be imagined if one considers the situation as in figure 2.3. Apparently, there are places in the room from which it is more advantageous to go round the obstacle from its left side and others from which a bypass from the right would promise a faster escape.

Further, since $\tilde{T}[\boldsymbol{\varphi}]$ is given by the curvilinear integral of the function $\frac{1}{V}$ which is time-dependent, the minimizer $\boldsymbol{\varphi}$ also depends on $t$, which is not explicitly considered in its definition.

## 3.2 Preferred direction $\boldsymbol{\mu}$ of motion

As it has already been stated in Section 2.1.3, $\boldsymbol{\mu}$ at a point $x$ and a time $t$ will be a unit tangent vector to the path $\boldsymbol{\varphi}(\sigma;x)$ starting at $x$ which minimizes the time needed to reach the outflow boundary $\Gamma_o$ at $t$. That is,

$$\boldsymbol{\mu}(x,t) = \frac{\boldsymbol{\varphi}'(0;x)}{|\boldsymbol{\varphi}'(0;x)|}, \qquad \text{where}\quad \boldsymbol{\varphi} = \operatorname*{argmin}_{\substack{\boldsymbol{\eta}(\sigma;x)\\ \boldsymbol{\eta}(1;x)\in\Gamma_o}} \tilde{T}[\boldsymbol{\eta}]. \tag{3.7}$$

We write simply $\boldsymbol{\varphi}'(\sigma;x)$ while we actually mean $\frac{\partial\boldsymbol{\varphi}}{\partial\sigma}(\sigma;x)$.

# 4. Necessary conditions for the minimizer $\boldsymbol{\varphi}$

The purpose of this chapter is to generalize the ideas from Section 3.2.1 in the bachelor thesis [3]. We shall see that even as a vector-valued function, $\boldsymbol{\varphi}$ must satisfy the Euler-Lagrange equations. The required differentiability class for $\boldsymbol{\varphi}$ will be weakened to “piecewise regular” instead of $\mathcal{C}^2$.

In addition, we deal with a variational problem with non-standard Dirichlet boundary condition which means that the endpoint $x^0$ is allowed to move freely in $\Gamma_o$ (Sec. 4.2).

A very good general reference for this chapter is [13].

**Definition 6.** *Let $X$ be a normed linear space. A mapping $F : X \to \mathbb{R}$ is called a* functional.

**Definition 7** ($\delta$-neighbourhood)**.** *The* $\delta$-neighbourhood *of a piecewise regular curve $\boldsymbol{\varphi} : [a, b]\mathbb{R}^2$ is the set $N^\delta(\boldsymbol{\varphi}) = \{(\sigma, x);\ \sigma \in [a, b],\ |x_i - \varphi_i(\sigma)| < \delta,\ i = 1, 2\}$.*

**Definition 8** (local minimum)**.** *Let $F$ be a functional on the space $\mathcal{C}([a, b])$ equipped with the norm $\|\boldsymbol{\varphi}\| = \max_{[a,b]} |\varphi_1(\sigma)| + \max_{[a,b]} |\varphi_2(\sigma)|$. We say that $\boldsymbol{\varphi} \in \check{\mathcal{C}}^1([a, b])$ is a* point of local minimum *of the functional $F$ if $F(\boldsymbol{\eta}) \geq F(\boldsymbol{\varphi})$ for all $\boldsymbol{\eta}$ that belong to a $\delta$-neighbourhood of $\boldsymbol{\varphi}$ for a certain $\delta > 0$.*

## 4.1 Euler-Lagrange equation in two dimensions

Suppose that, for a given $x \in \Omega$, the piecewise regular curve $\boldsymbol{\varphi}(\cdot, x) : [0, 1] \to \Omega_a = \Omega \cup \Gamma_o$ is a point of local minimum of the functional $\tilde{T}$ (given by Definition 5) with respect to all piecewise regular curves $\boldsymbol{\eta}$ with the properties below:

$$\begin{aligned} \tilde{T}[\boldsymbol{\eta}] \geq \tilde{T}[\boldsymbol{\varphi}], \quad \text{for all } \boldsymbol{\eta}(\cdot; x) : [0, 1] \to \Omega_a, \\ N^\delta(\boldsymbol{\varphi}) \ni \boldsymbol{\eta}(\cdot; x) \quad \text{for some } \delta > 0, \quad \boldsymbol{\eta}(0; x) = x, \quad \boldsymbol{\eta}(1; x) = x^0 \in \Gamma_o, \end{aligned} \tag{4.1}$$

where we keep the point $x^0$ fixed, for now.

Let us consider an $0 < \varepsilon < 1$ and express $\boldsymbol{\eta}$ in the form $\boldsymbol{\eta} = \boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}$, $\boldsymbol{\psi} : [0, 1] \to \mathbb{R}^2$ being a piecewise regular curve. In view of the conditions

$$\begin{aligned} \boldsymbol{\eta}(0; x) = x = x + \varepsilon\boldsymbol{\psi}(0) \\ \boldsymbol{\eta}(1; x) = x^0 = x^0 + \varepsilon\boldsymbol{\psi}(1), \end{aligned}$$

we require $\boldsymbol{\psi}$ to have the properties $\boldsymbol{\psi}(0) = \boldsymbol{\psi}(1) = \mathbf{0}$. Since $\boldsymbol{\psi}$ will be used as a *test function* in a certain integral identity, we shall call it so.

By (4.1), a function $h$ of one variable, defined by the next equation, assumes a minimum for $\varepsilon = 0$:

$$h(\varepsilon) := \tilde{T}[\boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}] - \tilde{T}[\boldsymbol{\varphi}] \geq 0, \quad h(0) = 0.$$

Therefore if $h'(0)$ exists, it can only be zero:

$$h'(0) = \left( \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \tilde{T}[\boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}] \right) \Bigg|_{\varepsilon=0} = 0. \tag{4.2}$$

The value in the equation is also known as the first variation or the Gâteaux differential of $\tilde{T}$ at the point $\boldsymbol{\varphi}$ in the direction $\boldsymbol{\psi}$. For abbreviation, we shall borrow the following convention from matrix calculus:

$$\frac{\partial L}{\partial \boldsymbol{\varphi}} = \left(\frac{\partial L}{\partial \varphi_1}, \frac{\partial L}{\partial \varphi_2}\right)^\top, \frac{\partial L}{\partial \boldsymbol{\varphi}'} = \left(\frac{\partial L}{\partial \varphi_1'}, \frac{\partial L}{\partial \varphi_2'}\right)^\top$$

and omit the arguments of $\boldsymbol{\varphi}(\sigma; x)$, $\boldsymbol{\psi}(\sigma)$ etc. in the following lines.

Let us compute the first variation of $\tilde{T}$, recalling (3.5):

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{T}[\boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}]\right)\Bigg|_{\varepsilon=0} &= \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_0^1 L(\sigma, \boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}, \boldsymbol{\varphi}' + \varepsilon\boldsymbol{\psi}')d\sigma\right)\Bigg|_{\varepsilon=0} = \\ &= \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(\sigma, \boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}, \boldsymbol{\varphi}' + \varepsilon\boldsymbol{\psi}')d\sigma\right)\Bigg|_{\varepsilon=0}.\end{aligned} \tag{4.3}$$

The reasons why interchanging the order of differentiation and integration was possible are given in Section 4.3. Successively applying the chain rule and integration by parts, we get

$$\begin{aligned}0 = \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(\sigma, \boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}, \boldsymbol{\varphi}' + \varepsilon\boldsymbol{\psi}')d\sigma\right)\Bigg|_{\varepsilon=0} &= \int_0^1 \frac{\partial L}{\partial \boldsymbol{\varphi}} \cdot \boldsymbol{\psi} + \frac{\partial L}{\partial \boldsymbol{\varphi}'} \cdot \boldsymbol{\psi}' d\sigma = \\ &= \int_0^1 \left(\frac{\partial L}{\partial \boldsymbol{\varphi}} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \boldsymbol{\varphi}'}\right)\right) \cdot \boldsymbol{\psi} d\sigma + \left[\frac{\partial L}{\partial \boldsymbol{\varphi}'} \cdot \boldsymbol{\psi}\right]_{\sigma=0}^1.\end{aligned} \tag{4.4}$$

(The previous steps were correct despite the possible discontinuities of $\boldsymbol{\varphi}'$ and $\boldsymbol{\psi}'$ as we could decompose the integration interval, perform the operations on each partial integral and then add up the integrals again.) Due to the boundary conditions $\boldsymbol{\psi}(0) = \boldsymbol{\psi}(1) = \mathbf{0}$, the term $[\frac{\partial L}{\partial \boldsymbol{\varphi}'} \cdot \boldsymbol{\psi}]_{\sigma=0}^1$ vanishes. Hence we get from (4.2) the necessary condition for $\boldsymbol{\varphi}$ to be a minimizer:

$$\int_0^1 \left(\frac{\partial L}{\partial \boldsymbol{\varphi}} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \boldsymbol{\varphi}'}\right)\right) \cdot \boldsymbol{\psi} d\sigma = 0, \tag{4.5}$$

with $\boldsymbol{\psi}$ defined in the beginning of this section. In particular, the last equality must be valid for all $\boldsymbol{\psi} \in \mathcal{C}_0^\infty([0,1])$. Consequently, and writing the functions' arguments in full, as in (3.4), we obtain

$$\begin{aligned}\frac{\partial L}{\partial \varphi_i}\big(\sigma; \varphi_1(\sigma), \varphi_2(\sigma), \varphi_1'(\sigma), \varphi_2'(\sigma)\big) - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \varphi_i'}\big(\sigma; \varphi_1(\sigma), \varphi_2(\sigma), \varphi_1'(\sigma), \varphi_2'(\sigma)\big)\right) = \\ = 0, \quad i = 1, 2\end{aligned} \tag{4.6}$$

for all $\sigma$ in $[0,1]$ where $\boldsymbol{\varphi}'$ is continuous. Thus we have established that if $\boldsymbol{\varphi}$ is a point of local minimum of $\tilde{T}$, it inevitably satisfies the differential equation (4.6). The next theorem summarizes this result.

**Theorem 3.** *Let the functional* $\tilde{T}$ *be given by* (3.5), $L$ *by* (3.4) *and* $L \in \mathcal{C}^1((0,1) \times \Omega_a^2 \times \mathbb{R}^2)$. *Let* $\boldsymbol{\varphi}$ *be a piecewise regular point of local minimum of* $\tilde{T}$ *with respect to all piecewise regular curves* $\boldsymbol{\eta}$ *satisfying* (4.1). *Then for all* $\sigma$ *in* $[0,1]$ *where* $\boldsymbol{\varphi}'$ *is continuous,*

$$\frac{\partial L}{\partial \boldsymbol{\varphi}} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \boldsymbol{\varphi}'}\right) = 0. \tag{4.7}$$

*The equation above is called the* Euler-Lagrange equation.

Since we have sought a minimizer that is only piecewise regular, there exists another kind of necessary conditions for $L$ and $\boldsymbol{\varphi}$ that must hold. They are called the *Weierstrass-Erdmann corner conditions*. Details are given in [13].

## 4.2 Loosening the endpoint

Logically, a pedestrian tries to reach the exit and does not care about the particular point at the outflow boundary that he/she passes through. Thus we shall only require $\boldsymbol{\eta}(0;x) \in \Gamma_o$ while keeping the other boundary condition fixed: $\boldsymbol{\eta}(0;x) = x$.

$$\boldsymbol{\eta}(0;x) = x, \quad \boldsymbol{\eta}(1;x) \in \Gamma_o. \tag{4.8}$$

In this section, let us assume that there is only one exit in the room (mathematically, that $\Gamma_o$ is connected) and that there exists a simple curve $\boldsymbol{\gamma}$ of class $\mathcal{C}^1$, $\boldsymbol{\gamma} : [0,1] \to \partial\Omega$, such that $\langle \boldsymbol{\gamma} \rangle = \overline{\Gamma}_o$.

The ideas used here are quite similar to those in the previous section. Let $x \in \Omega$ be fixed. We assume that the piecewise regular curve $\boldsymbol{\varphi}(\cdot, x) : [0,1] \to \Omega_a$ is a point of local minimum of the functional $\tilde{T}$ (see Definition 5) with respect to all piecewise regular curves $\boldsymbol{\eta}$ within the set $\mathcal{U}_{ad}$ of all admissible curves:

$$\begin{gathered}\tilde{T}[\boldsymbol{\varphi}] \leq \tilde{T}[\boldsymbol{\eta}] \\ \forall \boldsymbol{\eta} \in \mathcal{U}_{ad} := \\ = \{\boldsymbol{\eta}(\cdot;x) : [0,1] \to \Omega_a \text{ piecewise regular; } \boldsymbol{\eta}(0;x) = x;\ \boldsymbol{\eta}(1;x) \in \Gamma_o\} \\ \boldsymbol{\eta}(\cdot;x) \in N^\delta(\boldsymbol{\varphi}) \quad \text{for some } \delta > 0.\end{gathered} \tag{4.9}$$

The minimizing curve $\boldsymbol{\varphi}(\cdot;x)$ joins $x$ to a point $x^0 = x^0(x) \in \Gamma_o$ and we have $\boldsymbol{\varphi}(1;x) = \boldsymbol{\gamma}(\xi_0)$ for some $\xi_0 \in (0,1)$. Let us consider a family of curves $\boldsymbol{\eta}_\xi$ in the form

$$\boldsymbol{\eta}_\xi(\sigma) = \boldsymbol{\varphi}(\sigma) + \sigma(\boldsymbol{\gamma}(\xi) - \boldsymbol{\varphi}(1)), \quad \sigma \in [0,1],\ \xi \in (0,1).$$

Here we omitted $x$ in the argument of $\boldsymbol{\varphi}$, as $x$ is fixed. It is readily seen that $\boldsymbol{\eta}_\xi \in \mathcal{U}_{ad}$ for all $\xi \in (0,1)$ and that $\boldsymbol{\eta}_{\xi_0} = \boldsymbol{\varphi}$.

Utilizing the fact that $\boldsymbol{\varphi}$ is a minimizer,

$$H(\xi) := \tilde{T}[\boldsymbol{\eta}_\xi] - \tilde{T}[\boldsymbol{\varphi}] \geq 0, \quad H(\xi_0) = 0$$

and if $H'(0)$ exists, then

$$H'(\xi_0) = \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{T}[\boldsymbol{\eta}_\xi]\right)\bigg|_{\xi=\xi_0} = 0. \tag{4.10}$$

Let us compute the derivative in 4.10, recalling (3.5):

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{T}[\boldsymbol{\eta}_\xi]\right)\bigg|_{\xi=\xi_0} &= \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^1 L(\sigma, \boldsymbol{\eta}_\xi, \boldsymbol{\eta}'_\xi) d\sigma\right)\bigg|_{\xi=\xi_0} = \\ &= \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\xi} L(\sigma, \boldsymbol{\eta}_\xi, \boldsymbol{\eta}'_\xi) d\sigma\right)\bigg|_{\xi=\xi_0}.\end{aligned} \tag{4.11}$$

One may adapt the reasoning in Section 4.3 to justify the differentiation under the integral sign. We can introduce an auxiliary function

$$\boldsymbol{\Phi}(\sigma, \boldsymbol{\varphi}(\sigma), \boldsymbol{\varphi}'(\sigma)) = \int_0^\sigma \frac{\partial L}{\partial \boldsymbol{\varphi}}(\zeta, \boldsymbol{\varphi}(\zeta), \boldsymbol{\varphi}'(\zeta)) d\zeta. \tag{4.12}$$

This relation, when combined with the fact that $\boldsymbol{\eta}'_{\xi_0} = \boldsymbol{\varphi}'$ (which is independent of $\xi$), guarantees that the expression below equals zero

$$\left( -\boldsymbol{\Phi} \cdot \frac{\partial \boldsymbol{\eta}'_\xi}{\partial \xi} \right) \bigg|_{\xi=\xi_0} = 0. \tag{4.13}$$

and we can add it to (4.11):

$$\begin{aligned} 0 = \left( \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\xi} L(\sigma, \boldsymbol{\eta}_\xi, \boldsymbol{\eta}'_\xi) d\sigma \right) \bigg|_{\xi=\xi_0} &= \int_0^1 \frac{\partial L}{\partial \boldsymbol{\varphi}} \cdot \frac{\partial \boldsymbol{\eta}_\xi}{\partial \xi} + \frac{\partial L}{\partial \boldsymbol{\varphi}'} \cdot \frac{\partial \boldsymbol{\eta}'_\xi}{\partial \xi} d\sigma = \\ &= \int_0^1 \frac{\partial \boldsymbol{\eta}'_\xi}{\partial \xi} \cdot \left( \frac{\partial L}{\partial \boldsymbol{\varphi}'} - \boldsymbol{\Phi} \right) d\sigma + \left[ \boldsymbol{\Phi} \cdot \frac{\partial \boldsymbol{\eta}_\xi}{\partial \xi} \right]_{\sigma=0}^1 . \end{aligned} \tag{4.14}$$

Let us simplify this equation. Since $\boldsymbol{\varphi}$ has to satisfy the Euler-Lagrange equations in integrated form, there exists a constant $\boldsymbol{c} \in \mathbb{R}^2$ such that

$$\frac{\partial L}{\partial \boldsymbol{\varphi}'} - \boldsymbol{\Phi} = \boldsymbol{c}.$$

which ensures that

$$\boldsymbol{\Phi} \Big|_{\sigma=1} = \left( \frac{\partial L}{\partial \boldsymbol{\varphi}'} \right) \bigg|_{\sigma=1} - \boldsymbol{c}. \tag{4.15}$$

From the definition of $\boldsymbol{\eta}_\xi$,

$$\frac{\partial \boldsymbol{\eta}_\xi}{\partial \xi} \bigg|_{\sigma=0} = \mathbf{0}. \tag{4.16}$$

Hence, thanks to (4.15) and (4.16), the equation (4.14) reduces to

$$\left( \frac{\partial L}{\partial \boldsymbol{\varphi}'} \cdot \frac{\partial \boldsymbol{\eta}_\xi}{\partial \xi} \right) \bigg|_{\sigma=1,\, \xi=\xi_0} = 0. \tag{4.17}$$

But $\frac{\partial \boldsymbol{\eta}_\xi}{\partial \xi} = \boldsymbol{\gamma}(\xi)$, establishing a second condition that the minimizer $\boldsymbol{\varphi}$ must satisfy (at the loose endpoint). Sagan [13, page 225] uses the term *transversality condition.*

**Theorem 4** (transversality condition)**.** *Suppose $L \in \mathcal{C}^1((0,1) \times \Omega_a^2 \times \mathbb{R}^2)$. Let the piecewise regular curve $\boldsymbol{\varphi} : [0,1] \to \Omega_a$ be a point of local minimum of the functional $\tilde{T}$ with respect to all piecewise regular curves $\boldsymbol{\eta} \in \mathcal{U}_{ad}$ satisfying* (4.8)*. Assume there is a simple curve $\boldsymbol{\gamma}$ of class $\mathcal{C}^1$, $\boldsymbol{\gamma} : [0,1] \to \partial\Omega$, such that $\langle \gamma \rangle = \overline{\Gamma}_o$. Further assume that $\boldsymbol{\varphi}(1; x) = \boldsymbol{\gamma}(\xi_0)$ for a certain $\xi_0 \in (0,1)$. Then*

$$\frac{\partial L}{\partial \boldsymbol{\varphi}'} \bigg|_{\sigma=1} \cdot \boldsymbol{\gamma}'(\xi_0) = 0.$$

The transversality condition is only applicable in a free-endpoint variational problem.

A natural question that emerges is what the *sufficient* conditions for a local minimum are. Unfortunately, a full discussion would be beyond the scope of this work. In the related theory, the sign of the expression

$$\frac{1}{(\varphi_1')^2 + (\varphi_2')^2}\Big(\frac{\partial^2 L}{\partial\varphi_1\partial\varphi_1'} + \frac{\partial^2 L}{\partial\varphi_2\partial\varphi_2'}\Big)$$

is examined (provided that $L$ is of class $\mathcal{C}^2$ and $|\boldsymbol{\varphi}'| > 0$) and the concept of *embeddability* of $\boldsymbol{\varphi}$ in a *field* is introduced. For details, see [13], Chapter 3.

## 4.3 Appendix: Differentiation under the integral sign

**Definition 9.** *For $a, b \in \mathbb{R}$, $a < b$, we shall write* $\mathcal{L}^1([a,b]) = \{g : [a,b] \to \mathbb{R}$ Lebesgue measurable; $\int_a^b |g(\sigma)| d\sigma < \infty\}$.

**Theorem 5.** [1] *Let $a, b \in \mathbb{R}$, $a < b$, $K = [a,b]$, $J \subset \mathbb{R}$ an open interval and $g : J \times K \to \mathbb{R}$. Furthermore, suppose that*

1. *$g(\theta, \cdot)$ is Lebesgue measurable for every $\theta \in J$,*
2. *there exists $\frac{\partial g}{\partial\theta}(\theta, \sigma) \in \mathbb{R}$ almost everywhere (with respect to the Lebesgue measure) in $K$ and for all $\theta \in J$,*
3. *there exists $G \in \mathcal{L}^1(K)$ such that for all $\theta \in J$, $\left|\frac{\partial g}{\partial\theta}(\theta, \sigma)\right| \le G(\sigma)$ for a. e. $\sigma \in K$,*
4. *there is a $\theta_0 \in J$ for which $g(\theta_0, \cdot) \in \mathcal{L}^1(K)$.*

*Under these assumptions, $g(\theta, \cdot) \in \mathcal{L}^1(K)$ for all $\theta \in J$ and the following equality holds:*

$$\frac{\partial}{\partial\theta}\int_a^b g(\sigma, \theta) d\sigma = \int_a^b \frac{\partial g}{\partial\theta}(\sigma, \theta) d\sigma, \quad \theta \in J.$$

Let us briefly justify that the hypotheses of the above theorem are satisfied in (4.3). Firstly, the derivatives $\varphi_1'$, $\varphi_2'$, $\psi_1'$, $\psi_2'$ are piecewise continuous and bounded. Addition and multiplication of piecewise continuous bounded functions or plugging them into the square root function preserves their piecewise continuity as well as boundedness. Thus the function in question is Lebesgue measurable (1. assertion in Th. 5). As to the next condition (2.), the integrand can be differentiated as follows for all $\sigma$ in $[0,1]$ except a finite number of points where $\boldsymbol{\varphi}'$ or $\boldsymbol{\psi}'$ have jump discontinuities, i. e., almost everywhere:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\sqrt{(\varphi_1'(\sigma) + \varepsilon\psi_1'(\sigma))^2 + (\varphi_2'(\sigma) + \varepsilon\psi_2'(\sigma))^2}}{V(\varphi_1(\sigma), \varphi_2(\sigma), t)} =$$

$$= \frac{(\varphi_1'(\sigma) + \varepsilon\psi_1'(\sigma))\psi_1'(\sigma) + (\varphi_2'(\sigma) + \varepsilon\psi_2'(\sigma))\psi_2'(\sigma)}{\sqrt{(\varphi_1'(\sigma) + \varepsilon\psi_1'(\sigma))^2 + (\varphi_2'(\sigma) + \varepsilon\psi_2'(\sigma))^2} V(\varphi_1(\sigma), \varphi_2(\sigma), t)}.$$

[1] The theorem was specified to better suit our purposes. A more general statement, with a measure space in place of $K$, can be found, e. g., in Malliavin, P. *Integration and Probability*, page 40. Springer, New York, 1995.

From this relation, it is possible to find an integrable majorant (3. condition in Th. 5), using positivity and boundedness of $V$ on $\overline{\Omega}$, boundedness of $\psi_1'$, $\psi_2'$ and that of $\frac{x+y}{\sqrt{x^2+y^2}}$ in a neighbourhood of the origin. If we set $\varepsilon = 0$, we recognize the convergent integral $I$ as in the proof of Lemma 2, which gives the 4. assumption.

# 5. Eikonal equation

We are modelling pedestrians in a panic situation, their motivation is to escape from the area as fast as they can. For this reason, at every point $x$ and time $t$, they aim to minimize their travel time to the exit and to move in a direction $\boldsymbol{\mu}(x,t)$ that would ensure it. Relation (3.7) offers a definition of $\boldsymbol{\mu}$ proceeding directly from physical reality – it uses the path $\boldsymbol{\varphi}(\sigma;x)$ along which the time is shortest possible. However, in [8], p. 5, and other related works in the field, the authors define $\boldsymbol{\mu}$ differently, via the *eikonal equation*.

The procedure is based on the idea that we shall not search for $\boldsymbol{\varphi}$ itself but shall obtain a partial differential equation for a certain scalar function $\phi$ whose normalized gradient will be $-\boldsymbol{\mu}$. Let us examine this approach.

## 5.1 Time to the exit: the functional $\tilde{T}[\boldsymbol{\varphi}]$ and the potential $\phi$

A functional $\tilde{T}$ which associates to a curve the time to move along it in a speed field $V$, has already been defined in Section 3.1.2. Our point of view will be changed now, let us suppress the variational aspect by saying that the minimizing curve $\boldsymbol{\varphi}$ defined by (4.1) is known to us for every $t \in [0,T]$. In return, let us emphasize the dependence on time.

**Definition 10.** *Let $\phi : \Omega_a \times [0,T] \longrightarrow \mathbb{R}$ (with $\Omega_a = \Omega \cup \Gamma_o$ as before) be given by*

$$\phi(x,t) = \int_0^1 \frac{|\boldsymbol{\varphi}'(\sigma;x)|}{V(\boldsymbol{\varphi}(\sigma;x),t)} d\sigma$$

*where $\boldsymbol{\varphi}$ is the minimizing curve from* (4.1) *at $t \in [0,T]$. The function $\phi$ is called the* cost potential *([8]) or simply a* (scalar) potential *in $\Omega_a$ ([12]).*

*Remark* 5. If we look at Lemma (2), we observe that $\phi(x,t) = \tilde{T}[\boldsymbol{\varphi}(\sigma;x,t)]$(where $t$ was explicitly added to the arguments of $\boldsymbol{\varphi}$). Thus we may interpret $\phi(x,t)$, $x \in \Omega_a$, as the shortest time to travel from the point $x$ to some point $x^0 = x^0(x) \in \Gamma_o$ at $t \in [0,T]$.

*Remark* 6. There may be well-founded objections to denoting the "shortest time to the exit" by two distinct symbols, $\tilde{T}[\boldsymbol{\varphi}]$ and $\phi(x,t)$. This occurs since they originate from different backgrounds: $\tilde{T}$ as the functional in the variational problem (4.1) and $\phi$ as the usual unknown in the eikonal equation (5.4).

## 5.2 Derivation of the eikonal equation

We shall work in a frozen time frame with $t \in [0,T]$ fixed, like in Sections 3.1.2, 3.2 and Chapter 4. In the light of (3.5) and Remark 5, the definition of $\phi$ yields

$$\phi(x,t) = \int_0^1 L(\sigma, \boldsymbol{\varphi}(\sigma;x), \boldsymbol{\varphi}'(\sigma;x)) d\sigma.$$

Let us calculate the components of $\nabla\phi$ at a point $x \in \Omega$. To this end, we remind that $\boldsymbol{\varphi}(0; x_1, x_2) = (x_1, x_2)^\top$ (see Section 3.1.2) and compute

$$\frac{\partial \boldsymbol{\varphi}}{\partial x_1}(0; x) = (1, 0)^\top$$
$$\frac{\partial \boldsymbol{\varphi}}{\partial x_2}(0; x) = (0, 1)^\top.$$

Again, it will be correct to interchange differentiation and integration (by Theorem 5), for similar reasons as in (4.3) (those were given in Sec. 4.3). A sufficient condition for the 2. condition from Th. 5 to hold is $\boldsymbol{\varphi}'(\sigma; \cdot) \overset{\text{def}}{=} \frac{\partial \boldsymbol{\varphi}}{\partial \sigma}(\sigma; \cdot) \in \mathcal{C}^1(\Omega)$. This means, besides other things, that the tangent vector $\boldsymbol{\varphi}(0; x)$ to the fastest path should depend continuously on $x$.[1] For the 3. hypothesis of Th. 5 to be true, it is enough to suppose the existence of a neighbourhood of $x$ in which $\frac{\partial^2 \boldsymbol{\varphi}}{\partial x_i \partial \sigma}$, $i = 1, 2$, are bounded.

$$\frac{\partial \phi}{\partial x_i} = \int_0^1 \frac{\partial L}{\partial \boldsymbol{\varphi}} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} + \frac{\partial L}{\partial \boldsymbol{\varphi}'} \cdot \frac{\partial \boldsymbol{\varphi}'}{\partial x_i} d\sigma =$$
$$= \int_0^1 \underbrace{\left( \frac{\partial L}{\partial \boldsymbol{\varphi}} - \frac{\mathrm{d}}{\mathrm{d}\sigma} \left( \frac{\partial L}{\partial \boldsymbol{\varphi}'} \right) \right)}_{=0} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} d\sigma + \left[ \frac{\partial L}{\partial \boldsymbol{\varphi}'} \cdot \frac{\partial \boldsymbol{\varphi}'}{\partial x_i} \right]_{\sigma=0}^1 = - \left. \frac{\partial L}{\partial \varphi_i'} \right|_{\sigma=0}, \qquad (5.1)$$
$$i = 1, 2.$$

Integration by parts, the Euler-Lagrange equation (4.7) and, in the last step, Theorem 4 were used.

To rewrite these partial derivatives of $L$ in terms of $\varphi_1$, $\varphi_2$ and $V$, let us differentiate the definitory relation of $L$ (3.4) with respect to $\varphi_1'$ and $\varphi_2'$.

$$L(\sigma; \varphi_1, \varphi_2, \varphi_1', \varphi_2') = \frac{\sqrt{(\varphi_1')^2 + (\varphi_2')^2}}{V(\varphi_1, \varphi_2, t)},$$

$$\frac{\partial L}{\partial \varphi_1'} = \frac{1}{2} \frac{2\varphi_1'}{V(\boldsymbol{\varphi}, t)|\boldsymbol{\varphi}'|}, \quad \frac{\partial L}{\partial \varphi_2'} = \frac{1}{2} \frac{2\varphi_2'}{V(\boldsymbol{\varphi}, t)|\boldsymbol{\varphi}'|}. \qquad (5.2)$$

Plugging into relation (5.1) then gives:

$$\frac{\partial \phi}{\partial x_i} = - \left. \frac{\varphi_i'}{V\,\underbrace{(\boldsymbol{\varphi}, t)}_{=x}|\boldsymbol{\varphi}'|} \right|_{\sigma=0}, \qquad i = 1, 2.$$

It follows that the gradient of $\phi$ at a point $x$ has the form

$$\nabla \phi(x, t) = - \frac{\boldsymbol{\varphi}'(0; x)}{V(x, t)|\boldsymbol{\varphi}'(0; x)|}. \qquad (5.3)$$

By passing to norm, we get an equation, the solution of which $\phi$ necessarily is.

$$|\nabla \phi(x, t)| = \frac{1}{V(x, t)}, \quad x \in \Omega,\ t \in (0, T) \qquad (5.4)$$

[1] An assumption that seems reasonable in a room with only one way out and no obstacles, but less justifiable in case the pedestrians consider multiple differently located exits.

*Voilà*, (5.4) is called the eikonal equation. In Remark 5, it was pointed out that $\phi(x,t)$ has the meaning of the shortest time to travel from $x$ to any point $x^0(x) \in \Gamma_o$. For this reason, we must impose two supplementary conditions on $\phi$:

$$\phi(x,t) \geq 0, \quad (x,t) \in \Omega_a \times [0,T] \tag{5.5}$$

$$\phi\Big|_{\Gamma_o} = 0. \tag{5.6}$$

The condition of non-negativity (5.5) is important from a physical viewpoint, since if $\phi$ is solution to equation (5.4), then so is $-\phi$. Twarogowska et al. ([12]), do not mention the requirement on $\phi$ to be non-negative, though.

## 5.3 Preferred direction $\boldsymbol{\mu}$ and the potential $\phi$

Combining equation (5.3) with the definition of $\boldsymbol{\mu}$ (3.7), we deduce that the vectors $\boldsymbol{\mu}(x,t)$ and $\nabla\phi(x,t)$, $[x,t] \in \Omega_a \times [0,T]$, have opposite directions. Moreover, we know that $|\boldsymbol{\mu}| = 1$. Hence we get an equality that relates $\phi$ to $\boldsymbol{\mu}$

$$\boldsymbol{\mu} = -\frac{\nabla\phi}{|\nabla\phi|}. \tag{5.7}$$

In [8], the equality obtained is not stated as a corollary but as the definition of $\boldsymbol{\mu}$.

**Theorem 6.** *Let $t \in (0,T)$ be fixed and $x \in \Omega$. Let $\boldsymbol{\varphi}(\cdot;x) : [0,1] \to \Omega_a$ be the the fastest path connecting $x$ to a point on the outflow boundary $\Gamma_o$ in the speed field $V(\cdot,t)$. Then the direction $\boldsymbol{\mu}$ defined by* (3.7) *satisfies*

$$\boldsymbol{\mu}(x,t) = -\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|}$$

*where $\phi$ is the solution of the fully non-linear partial differential equation* (5.4) *with boundary condition* (5.6) *and the condition of non-negativity* (5.5).

# Summary: two formulations of the problem

Summarizing Chapters 2, 3 and 5, we see that the Pedestrian flow equations (PFEs) can be formulated equivalently in two ways.

$$
\text{(PFEs I)}\begin{cases}
\text{For given constants } p_0,\ \gamma,\ \tau,\ v_{max},\ \alpha,\ \rho_{max} \\
\text{find } \rho,\ (v_1, v_2)^\top = \boldsymbol{v} \text{ such that} \\
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 \quad \text{in } Q_T, \\
\frac{\partial \rho v_i}{\partial t} + \operatorname{div}(\rho v_i \boldsymbol{v}) = \rho \frac{1}{\tau}(V\mu_i - v_i) - \frac{\partial p}{\partial x_i}, \quad i = 1, 2, \quad \text{in } Q_T, \\
p = p_0 \rho^\gamma, \quad V = v_{max} e^{-\alpha\left(\frac{\rho}{\rho_{max}}\right)^2}, \quad \boldsymbol{\mu}(x,t) = \frac{\boldsymbol{\varphi}'(0;x)}{|\boldsymbol{\varphi}'(0;x)|}, \quad \text{in } Q_T \\
\boldsymbol{\varphi} = \operatorname{argmin}_{\substack{\eta(\sigma;x) \\ \boldsymbol{\eta}(1;x) \in \Gamma_o, \boldsymbol{\eta}\text{piecewise regular}}} \int_{\boldsymbol{\eta}} \frac{ds}{V}, \quad \sigma \in [0,1] \\
\text{so that given initial and boundary conditions are satisfied.}
\end{cases}
$$

$$
\text{(PFEs II)}\begin{cases}
\text{For given constants } p_0,\ \gamma,\ \tau,\ v_{max},\ \alpha,\ \rho_{max} \\
\text{find } \rho,\ (v_1, v_2)^\top = \boldsymbol{v} \text{ such that} \\
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 \quad \text{in } Q_T, \\
\frac{\partial \rho v_i}{\partial t} + \operatorname{div}(\rho v_i \boldsymbol{v}) = \rho \frac{1}{\tau}(V\mu_i - v_i) - \frac{\partial p}{\partial x_i}, \quad i = 1, 2, \quad \text{in } Q_T, \\
p = p_0 \rho^\gamma, \quad V = v_{max} e^{-\alpha\left(\frac{\rho}{\rho_{max}}\right)^2}, \quad \boldsymbol{\mu}(x,t) = -\frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|} \quad \text{in } Q_T \\
|\nabla\phi(x,t)| = \frac{1}{V(x,t)} \quad \text{in } Q_T \\
\phi(x,t) \geq 0, \quad \text{in } Q_T, \quad \phi\Big|_{\Gamma_o} = 0 \\
\text{so that given initial and boundary conditions are satisfied.}
\end{cases}
$$

Clearly, the difference between the systems (PFEs I) and (PFEs II) lies in the way how the intended direction $\boldsymbol{\mu}$ of pedestrians' movement is formalized mathematically in the model.

Solution of the system (PFEs I) uses numerical minimization of the functional $\boldsymbol{\eta} \mapsto \int_{\boldsymbol{\eta}} \frac{ds}{V}$. To do this, Dijkstra's shortest path algorithm is run on a graph induced by the mesh on which the mass and momentum equations are discretized. As an output, it provides a polygonal path which approximates $\boldsymbol{\varphi}$. For details, see [3] or [14].

The model (PFEs II) contains a nonlinear partial differential equation, the *eikonal equation*, which is solved with the Bornemann-Rasch algorithm [11]. How the equation is obtained from the hypothesis of pedestrians' time-to-the-exit-minimizing behaviour, was explained in Sec. 5.2.

The initial and boundary conditions are stated in [3].

# Conclusion

The work was intended as an attempt to show how the eikonal equation:

$$|\nabla\phi| = \frac{1}{V} \tag{5.8}$$

is related to functional minimization:

$$\boldsymbol{\varphi} = \underset{\substack{\boldsymbol{\eta}(\sigma;x) \\ \boldsymbol{\eta}(1;x)\in\Gamma_o}}{\operatorname{argmin}} \tilde{T}[\boldsymbol{\eta}], \tag{5.9}$$

$$\text{where } \tilde{T}[\boldsymbol{\varphi}] = \int_{\boldsymbol{\varphi}} \frac{ds}{V} \tag{5.10}$$

in the problem of modelling pedestrians trying reach the exit of a domain $\Omega \subset \mathbb{R}^2$ in minimal time. In the two relations, $\phi$ is a scalar potential in the area filled with people, $V$ is the magnitude of the desired velocity of pedestrians and the functional $\tilde{T}[\boldsymbol{\eta}]$ gives the time needed to move along a curve $\boldsymbol{\eta}$.

The model contains the continuity equation and the equations of inviscid compressible flow to interrelate the density, velocity and pressure inside the crowd.

In the need of having the model complete, the unit vector field $\boldsymbol{\mu}$ in $\Omega$ has to be defined. This quantity appears in (2.2) and points the preferred direction of movement of the pedestrians.

Both relations (5.8), (5.9) can be used for defining $\boldsymbol{\mu}$.

1. In the first approach, the scalar potential is obtained as a solution of (5.8). After that, $\boldsymbol{\mu}$ may be defined by

$$\boldsymbol{\mu} = -\frac{\nabla\phi}{|\nabla\phi|}.$$

2. In a panic (which is the case we usually study), pedestrians try to escape from the area as fast as they can. The second approach uses this idea more explicitly. Therefore it is convenient to define $\boldsymbol{\mu}$ as a unit tangent vector to a hypothetical path $\boldsymbol{\varphi}$ such that the time needed to pass to the outflow boundary along $\boldsymbol{\varphi}$ is minimal, i. e. $\boldsymbol{\mu} = \frac{\boldsymbol{\varphi}'}{|\boldsymbol{\varphi}'|}$. This leads to a variational problem. T. Petrášová [3] derived the necessary condition for $\boldsymbol{\varphi}$ to be a minimizing curve, on the basis on one-dimensional reasoning, for $\boldsymbol{\varphi}$ of class $\mathcal{C}^2$ and with a fixed-endpoint boundary condition. In this work, the result was generalized to two dimensions, assuming $\boldsymbol{\varphi}$ to be only piecewise regular and considering a free-endpoint boundary condition, which is more natural for the purpose of the model. Namely, the Euler-Lagrange equations and the transversality condition were derived for the case of the pedestrian model. The weakened regularity makes it more complicated to interchange the order of differentiation and integration correctly.

This bachelor thesis also shows an application of the curvilinear integral, which has the meaning of the time needed to move along a path in an environment of variable density which affects the speed $V$ at which the movement is feasible.

# Bibliography

[1] P. Marno. Crowded – macroscopic and microscopic models for pedestrian dynamics. Master's thesis, University of Reading, 2012.

[2] B. Piccoli and A. Tosin. Pedestrian flows in bounded domains with obstacles. *Continuum Mechanics and Thermodynamics*, 21(2):85–107, 2009.

[3] T. Petrášová. Application of the Dijkstra's algorithm in the pedestrian flow problem. bachelor thesis, Charles University in Prague, 2016.

[4] D. Helbing and P. Molnár. Social force model for pedestrian dynamics. *Physical Review E*, pages 4282–4286, 1995.

[5] J. Dadová. *Cellular Automata Approach for Crowd Simulation*. PhD thesis, Comenius University, Bratislava, 2012.

[6] T. Schelhorn, D. O'Sullivan, M. Haklay, and M. Thurstain-Goodwin. STREETS: An agent-based pedestrian model. Presented at the conference *Computers in Urban Planning and Urban Management* (Venice, 1999).

[7] D. Helbing, I. Farkas, P. Molnár, and T. Vicsek. Simulation of pedestrian crowds in normal and evacuation situations. In M. Schreckenberg and S. D. Sharma, editors, *Pedestrian and Evacuation Dynamics*, pages 21–58. Springer, Berlin, 2002.

[8] Y.-Q. Jiang, P. Zhang, S. C. Wong, and R. X. Liu. A higher-order macroscopic model for pedestrian flows. *Physica A: Statistical Mechanics and its Applications*, 389(21):4623–4635, 2010.

[9] H. J. Payne. *Models of Freeway Traffic and Control*. Simulation Councils, Incorporated, 1971.

[10] G. B. Whitham. *Linear and nonlinear waves*. Wiley, New York, 1974.

[11] P. Kubera and J. Felcman. On a numerical flux for the pedestrian flow equations. *JAMSI*, 11(2):79–96, 2015.

[12] M. Twarogowska, P. Goatin, and R. Duvigneau. Numerical study of macroscopic pedestrian flow models. Research Report 8340, INRIA, 2013.

[13] H. Sagan. *Introduction to the Calculus of Variations*. McGraw-Hill, 1969.

[14] J. Felcman and P. Kubera. On the Eikonal Equation in the Pedestrian Flow Problem. 2016. Accepted for publication.

[15] J. Felcman and O. Havle. On a numerical flux for the shallow water equations. *Applied Mathematics and Computation*, 217(11):5160–5170, 2011.