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3-Coloring Graphs on Torus

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Abstract: The theory of Dvořák et al. [2015a] shows that a 4-critical triangle-free graph embedded in the torus has only a bounded number of faces of length greater than 4 and that the size of these faces is also bounded. We study the natural reduction in such embedded graphs—identification of opposite vertices in 4-faces. We give a computer-assisted argument showing that there are exactly four 4-critical triangle-free *irreducible* toroidal graphs in which this reduction cannot be applied without creating a triangle. Using this result we demonstrate several properties that are necessary for every triangle-free graph embedded in the torus to be 4-critical. Most importantly we demonstrate that every such graph has at most four 5-faces, or a 6-face and two 5-faces, or a 7-face and a 5-face, in addition to at least seven 4-faces.

Keywords: coloring graph torus

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Introduction

The subject of coloring graphs on surfaces goes back to the work of Heawood [1890], who proved that any graph G drawn on surface Σ is t-colorable for any t satisfying $t \geq H(\Sigma) := \lfloor (7 + \sqrt{24\gamma_\Sigma + 1})/2 \rfloor$ unless Σ is the sphere. The symbol γ_Σ denotes the Euler genus of Σ defined as $\gamma = 2g$ when $\Sigma = S_g$ (orientable surface of genus g), and $\gamma = k$ when $\Sigma = N_k$ (non-orientable surface with k crosscaps). Incidentally, the assertion holds for the sphere as well, as stated by the Four-Color Theorem (Appel and Haken [1977], Appel et al. [1977], Robertson et al. [1997]).

The bound given by Heawood's formula is tight. As proven by Ringel and Youngs [1968], the bound is best possible for all surfaces except the Klein bottle, for which the correct bound is 6.

While Heawood's formula gives a tight bound on the possible values of chromatic number of graphs on almost all surfaces, values close to the bound are achieved by only relatively few graphs. An improvement of Heawoods's formula in this sense was brought by Dirac [1952] and Albertson and Hutchinsonn [1979] who showed that the only graphs with chromatic number exactly $H(\Sigma)$ contain a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

Further improvements are possible for large enough graphs. A graph G is *k-critical* if its chromatic number is exactly k and every proper subgraph of G has a chromatic number at most $k-1$. It follows from Euler's formula that if Σ is a fixed surface and a graph G drawn on Σ has sufficiently many vertices, then G has a vertex of degree at most six. Consequently, for every $k \geq 8$ a graph drawn on Σ is not k-critical except for finitely many exceptions with bounded number of vertices. The same argument can be extended to $k = 7$.

From the work of Thomassen [1997] we have an extension of the same argument even to $k = 6$. Thus it is shown that for every surface there are only finitely many 6-critical graphs that can be drawn on Σ. An immediate consequence from a computational point of view is that for every $k \geq 6$, fixed surface Σ and a graph G drawn on Σ it is possible to efficiently test whether G is $(k-1)$-colorable by testing the presence of all possible k-critical subgraphs of G in linear time. Such algorithm can be constructed if an explicit full list of k-critical graphs on Σ is provided. The lists of 6-critical graphs are explicitly known for the projective plane (Albertson and Hutchinson [1979]), the torus (Thomassen [1994a]) and the Klein bottle (Kawarabayashi et al. [2008], Chenette et al. [2012]).

Since the problem of testing 2-colorability is polynomial-time solvable, and the problem of testing 3-colorability for planar graphs is NP-complete (Garey and Johnson [1979]), the only remaining non-trivial case is 4-colorability. It is an open problem whether there is a polynomial time algorithm for testing 4-colorability of graphs on a fixed surface Σ other than the sphere. However a characterization similar to the one described for (≥ 5)-colorability above does not exist, as shown by an elegant construction of Fisk [1978].

Let us consider the analogous problem for embedded graphs of larger girth. Chromatic number of graphs of girth at least five is characterized by a deep theorem of Thomassen [2003] who showed that for every $k \geq 4$ and every surface Σ there are only finitely many k-critical graphs of girth at least five that can be drawn on Σ. Thus testing $(k-1)$-colorability of graphs with girth at least five again reduces to deciding the presence of finitely many obstructions for any $k \geq 4$. There actually turn out to be no 4-critical graphs of girth at least five on the projective plane and the torus (Thomassen [1994b]) and on the Klein bottle (Thomas and Walls [2004]).

Theorem 1 (Thomassen [2003]). *If G is a graph drawn on torus such that all contractible cycles have length at least five, then G is 3-colorable.*

The presence of cycles of length four complicates matters. Similarly as in the general case, if G is a triangle-free graph drawn on a fixed surface Σ and G has sufficiently many vertices, then G has a vertex of degree at most four. Analogously to the general case we get that G is not 6-critical and the result can be strengthened to show that G is not even 5-critical. Thus testing k-colorability of triangle-free graphs on a fixed surface Σ is a linear-time solvable for any $k \geq 4$. However for $k = 3$ the situation is more complicated. The well-known theorem of Grötzsch [1959] shows that every triangle-free planar graph is 3-colorable. This theorem, while fully characterizing 3-colorability of planar graphs, also motivates the study of the problem on other surfaces.

Theorem 2 (Grötzsch [1959]). *Every planar triangle-free graph is 3-colorable.*

Unfortunately, Grötzsch's theorem cannot be extended to any surface other than the sphere. For instance, the graphs obtained from an odd cycle of length five or more by applying Micielski's construction (Bondy and Murty [1976]) provide an infinite class of 4-critical graphs embeddable in any surface other than the sphere. This of course means that 3-colorability of triangle-free graphs on a fixed surface Σ cannot be decided by testing the presence of finitely many obstructions (however a more sophisticated algorithm already exists as we will see below).

The only non-planar surface for which the 3-colorability problem for triangle-free graphs is fully characterized is the projective plane. Building on earlier work of Youngs [1996], Gimbel and Thomassen [1997] obtained an elegant characterization stating that a triangle-free graph drawn in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane. Less is known regarding surfaces of higher genus.

For a graph G embedded on a surface, let $S(G)$ denote the multiset of lengths of (≥ 5)-faces of G (thus, the aforementioned result of Gimbel and Thomassen [1997] implies that $S(G) = \emptyset$ for every 4-critical projective-planar triangle-free graph G). Dvořák et al. [2015a] proved that for any surface Σ, there exists a constant c_Σ such that every 4-critical triangle-free graph G embedded in Σ *without non-contractible 4-cycles* satisfies $\sum S(G) \leq c_\Sigma$; i.e., G has only a bounded number of faces of length greater than 4 and these faces have bounded size. Such a bound does not hold in general if non-contractible 4-cycles are allowed (but it does hold for toroidal graphs, as we will see below). A more detailed treatment of 4-critical triangle-free graphs with non-contractible 4-cycles was given by Dvořák and Lidický [2015]. Dvořák et al. [2015b] proved that for any surface Σ, a triangle-free graph embedded in Σ with large edgewidth is 3-colorable unless Σ is non-orientable and the graph contains a quadrangulation with an odd orienting cycle. They also designed a linear-time algorithm to test 3-colorability of embedded triangle-free graphs (Dvořák et al. [2016]).

The goal of this thesis is to contribute to filling in the remaining gap, in particular to provide a partial characterization of 3-colorability of triangle-free graphs embedded in the torus. Král' and Thomas [2008] studied the special case of 4-critical triangle-free graphs embedded in the torus without odd-length faces, and showed that there is only one such graph (depicted as I_4 in Figure 5.2). On the other hand, the theory of Dvořák et al. [2015a] can be used to show that if G is a 4-critical triangle-free graph embedded in the torus (even possibly with non-contractible 4-cycles), then $\sum S(G) \leq 500$. We strengthen this bound substantially, providing in particular the exact list of multisets that can be realized as $S(G)$ for some such graph G.

Thomassen [2003] proved that every graph embedded in the torus without contractible (≤ 4)-cycles (but possibly with non-contractible triangles or 4-cycles) is 3-colorable. Consequently, every 4-critical triangle-free graph drawn in the torus has a 4-face. As a first step we show the following much stronger claim.

Lemma 11. *If G is a 4-critical triangle-free graph drawn in the torus, then every vertex of G is incident with a 4-face.*

All these results emphasize the importance of 4-faces in the considered problem. In particular the previously mentioned results together imply that the structure of every 4-critical triangle-free graph drawn on torus can be deconstructed into a set of 4-faces and a constant-limited number of additional edges.

A natural way of dealing with 4-faces is to unify opposite vertices of one of them, effectively removing it from the graph. This operation is of particular interest as while reducing size of the graph it never produces graphs with lower chromatic number. In particular, if we use such an operation on a 4-critical graph and limit it to produce only triangle-free results, it always produces further graphs with chromatic number at least 4 from which we can obtain smaller 4-critical triangle-free graphs. We formalize this process as a *reduction*.

The main focus of our work is a study of graphs that cannot undergo any further reduction. We say that a 4-critical triangle-free graph G embedded in the torus is *irreducible* if each two opposite vertices incident with a 4-face of G are joined by a path of length three, so that their identification creates a triangle. Since all 4-critical graphs eventually reduce into these irreducible graphs, we may use them to study properties of the whole class. A number of properties of 4-critical triangle-free toroidal graphs can be proven by induction using the (inverse) reduction operation in the inductive step. The importance of irreducible graphs stems from the fact that they form the base case of such inductive arguments.

We provide a way to enumerate all the irreducible graphs. Let B denote the graph obtained from K_4 by subdividing edges of its perfect matching twice, with its unique embedding in the torus that has a 4-face. By considering a 4-face in any irreducible graph G and the two paths of length three certifying its irreducibility, we conclude that G contains B as a subgraph. Based on Lemma 11 we can then build up any irreducible graph from B by repeatedly adding 4-cycles that bound faces, and by adding paths of length three between their opposite vertices to guarantee irreducibility.

It is not a priori clear that the number of irreducible graphs is finite. However, we design and implement an algorithm that carries out the enumeration process (and avoids repeated enumeration of isomorphic graphs as well as eliminates graphs containing substructures incompatible with 4-criticality of their supergraph to increase efficiency). Running this algorithm produces four irreducible graphs (depicted in Figure 5.2) and its termination shows that there are no other irreducible graphs.

The explicit knowledge of irreducible graphs enables us to prove a number of properties of 4-critical triangle-free toroidal graphs. All irreducible graphs have representativity at least two and they have at least seven 4-faces. We argue that both of these parameters are non-increasing in respect to reduction operation which shows that both of these bounds also hold for every 4-critical triangle-free toroidal graph.

In the last part, we present a partial analysis of a process inverse to reduction. In doing so, we prove the following theorem. Recall that $S(G)$ denotes the multiset of lengths of (≥ 5)-faces of G.

Theorem 36. *Every 4-critical triangle-free graph G drawn on torus satisfies one of the following properties:*

1. $S(G) = \{7, 5\}$
2. $S(G) = \{6, 5, 5\}$
3. $S(G) = \{5, 5, 5, 5\}$
4. $S(G) = \{5, 5\}$
5. $S(G) = \emptyset$ *and* $G = I_4$ *(see figure 5.2)*

In further work we intend to analyze this reverse process in a more detailed way to provide the complete characterization of 4-critical toroidal triangle-free graphs.

1. Preliminaries

A graph G consists of finite set $V(G)$ of vertices and a finite set $E(G)$ of undirected edges. The graphs we generally consider have no parallel edges and no loops, although some of our constructions will require considering multigraphs with loops. If for an edge e and vertices u and v we have $e = uv$, we say that e connects u and v and that u and v are neighboring vertices.

A *walk* is a sequence of not necessarily distinct vertices $p_0, p_1, ..., p_k$ connected by edges $p_0p_1, p_1p_2, ..., p_{k-1}p_k$. A *path* is a walk with all vertices distinct. We may further specify that a walk is a k-walk resp. k-path (often denoted as P_k) to specify its length in the number of edges. A *closed walk* is a sequence of vertices $p_1, p_2, ..., p_k$ connected by edges $p_1p_2, p_2p_3, ..., p_{k-1}p_k, p_kp_1$. A *cycle* is a closed walk with all vertices pairwise distinct. We may further specify that a closed walk is a closed k-walk resp. that cycle is a k-cycle to specify its length in the number of edges.

We always consider graphs associated with a specific drawing on torus. We explicitly acknowledge this by stating that G is *drawn on torus*. The drawing δ of a graph G on torus T is defined as a pair of functions $\delta : V(G) \to T$ associating distinct vertices of G with distinct points in T, and $\delta : E(G) \to 2^T$ associating distinct edges of G with pairwise disjoint open arcs in T such that for every $uv \in E(G)$ we have $\overline{\delta(uv)} = \delta(uv) \cup \delta(u) \cup \delta(v)$ where bar denotes a set closure and for every $w \in V(G) : \delta(uv) \cap \delta(w) = \emptyset$.

The arcwise connected components of the surface minus the drawing are called *faces*. A face is *2-cell*, if it is homeomorphic to an open disk. The boundary of a 2-cell face is an image of a closed walk called the *facial walk*. A 2-cell face is a *k-face* if its facial walk has length k. A drawing is a *2-cell* drawing if all of its faces are 2-cell and thus have a single facial walk. Generally we will only consider 2-cell drawings of graphs.

A cycle C is *contractible* if its drawing can be continuously transformed within the surface into a single point and is *non-contractible* otherwise. The *interior* of a contractible cycle C is the component of torus minus the drawing of C homeomorphic to an open disk, the other component is the *exterior*. A cycle C is *facial* if it is a facial walk for some face. A cycle C is *separating* if it is contractible and it is not facial, equivalently there are elements of G drawn in the interior of C, as well as in the exterior of C.

We consider two graphs drawn on a surface isomorphic if and only if their drawings can be transformed into one another by a homeomorphism.

A 2-cell embedding of a graph can be described purely combinatorially by specifying the cyclic ordering of edges incident with each vertex. This ordering corresponds to a clockwise order of edges in the drawing as they appear around given vertex. Two graphs are then isomorphic, if and only if the underlying graph structures are isomorphic and the edge orderings are the same or reversed.

For all constructions, we consider the properties of drawing to be hereditary to all shared elements. Thus for a graph G drawn on torus, a drawing of each subgraph of G is well defined. Similarly constructions such as insertion of a new edge into specific face or a contraction of an edge inherit drawings of all shared elements from the original graph. For 2-cell drawings we always define the drawing of the new elements in the only (up to isomorphism) natural way so that the resulting graph is drawn on torus. We consider these drawing manipulations trivial and omit explicit technical descriptions, although we will always keep in mind validity and uniqueness of all such constructions and will perform graph modifications with respect to a drawing.

A cornerstone of our study will be the following theorem. Let C be a proper subgraph of G. We say that G is C-critical for k-coloring, for some constant k, if for every proper subgraph $H \subset G$ such that $C \subseteq H$, there exists a k-coloring of C that extends to a k-coloring of H, but not to a k-coloring of G.

Theorem 3 (Gimbel and Thomassen [1997])**.** *Let G be a connected triangle-free plane graph with the outer face bounded by a cycle C of length at most 6. Then G is C-critical if and only if C is a 6-cycle, all internal faces of G have length exactly four and G contains no separating 4-cycle.*

This theorem enables us to characterize local properties of 4-critical graphs drawn on torus. Let Λ be a subset of the torus homeomorphic to the open disk. We say that a closed walk C *bounds the disk* Λ if Λ is a face of the subgraph of G consisting of vertices and edges of C and C is the facial walk of this face.

Let C be a closed walk bounding a disk Λ. If C is a contractible cycle, then Λ is exactly its interior. Similarly, the facial walk of a 2-cell face f is a closed walk that bounds the disc f. As another example, let C be a contractible 6-cycle $abcdef$ and Δ its interior. Let cf be an edge drawn in the exterior of C. By contraction of cf we obtain a closed walk of length six, $abcdec$, bounding a disk Λ which naturally corresponds to a deformation of Δ. On the other hand if the edge cf is drawn in the interior of C, then the contraction of cf splits Δ into two 3-faces none of which is bound by the closed walk $abcdec$ and so we obtain no closed walk of length 6 bounding a disk as the disk Λ no longer exists.

The closed walk C bounding the disk Λ is *separating* if it is not a facial walk; equivalently, this is the case if there are elements of G drawn in Λ, as well as elements of G that are not elements of C and are drawn outside of the closure of Λ. A *chord* of a closed walk C bounding a disk Λ is an edge connecting two vertices of C that is drawn inside Λ. This is a natural extension of standard definition of chord for (contractible) cycles.

Lemma 4. *Let G be a graph drawn on torus. Let C be a closed walk in G bounding disk Λ, and let H be the subgraph of G drawn in the closure of Λ. If G is 4-critical and Λ is not its face, then H is C-critical for 3-coloring.*

Proof. Let G' denote the subgraph of G drawn in the closure of the complement of Λ. Note that C is the intersection of G' and H (Λ is an open disk).

Consider a proper 3-coloring ϕ of C which can be extended to G'. Then ϕ cannot be extended to H because $G = G' \cup H$ and G is not 3-colorable. However, for every edge $e \in E(H) \setminus E(C)$ there exists a proper 3-coloring of $G - e$. This coloring defines a proper 3-coloring of C which extends to $H - e$ but not to H. Therefore, H is C-critical. □

An intuitive interpretation of Lemma 4 is that 4-critical graphs can be locally thought of as C-critical graphs for 3-coloring.

Finally, we summarize some of the previously mentioned key properties in the following useful lemma.

Corollary 5. *Let G be a 4-critical graph drawn on torus with no contractible triangles. Then G satisfies all of the following:*

- *The graph G has a 4-face.*
- *The graph G has no separating closed walk of length 4 or 5 bounding a disk.*
- *Let Λ be a disk bound by a separating walk of length 6 in G. Then all faces contained in Λ are 4-faces.*

Proof. Joint consequence of Theorem 1, Theorem 3 and Lemma 4. □

2. Deflation

The natural way of dealing with 4-faces in general graphs is to unify their opposite vertices; this effectively removes a 4-face from the graph while keeping some key properties related to chromatic number unchanged. In this section we will formally define such operation and explore some of its structural properties related to triangles and 4-faces.

Let G be a graph drawn on torus with a 4-face f. Consider the following operation. Let $v_1v_2v_3v_4$ denote the cycle bounding f. We identify vertices v_1 and v_3 into a single vertex v' and remove one of the two parallel edges $v'v_2$ and one of the two parallel edges $v'v_4$. We consider v' to be a new vertex.

Let H be the graph obtained by the described operation. We say H is a *deflation* of G. More precisely we may say that H is an *f-deflation* of G. Note that while we generally consider graphs without parallel edges and loops, H may contain them (e.g. if v_1v_3 was an edge in G).

Alternatively we may understand this operation as a contraction of an imaginary v_1v_3 edge placed so that it forms a chord of f (in other words, the imaginary v_1v_3 edge is drawn inside f). In this view we may obtain the new drawing by a natural continuous transformation of the imaginary edge into a single point which becomes the drawing of v'.

Note that exchanging vertices v_1 and v_3 or v_2 and v_4 would not change H. On the other hand, every 4-face can be deflated in two directions, as the role of pairs v_1, v_3 and v_2, v_4 is different. Exchanging the pairs results in two generally different deflations.

Since it is often necessary to denote many elements of a given graph when discussing properties of deflations, we will use the following notation. We say that deflation of G is of form $(v_1v_2v_3v_4) \to (v_2v'v_4)$ if $v_1v_2v_3v_4$ is the facial walk of the deflated 4-face in G and v' is the unification of v_1 and v_3; and therefore the path $v_2v'v_4$ corresponds to the deflated 4-face in the drawing of H.

Consider the path $v_2v'v_4$ in H. Since we consider H being drawn on torus, we can distinguish edges that were incident with v_1 from edges that were incident with v_3 before deflation (up to exchanging v_1 and v_3), depending on from which side they attach to the path $v_2v'v_4$ (this naturally extends even to a potential loop on v'). We observe that there exists only one way to inflate such a path back into a 4-face. Because of symmetry of deflation, we do not need to distinguish the endpoints of the path $v_2v'v_4$.

Lemma 6. *Let G be a graph with no proper 3-coloring. Let G' be a deflation of G. Then G' has no proper 3-coloring, and hence has a 4-critical subgraph.*

Proof. For contradiction, let G' have a proper 3-coloring ψ. Let v_1 and v_3 be the vertices of G unified into vertex v' by deflation.

We observe that ψ is a proper coloring of $G - \{v_1, v_3\}$. We extend ψ by setting $\psi(v_1) = \psi(v_3) := \psi(v')$; ψ is now a proper 3-coloring of G. □

2.1 Substructure inheritance

We define natural correspondence between substructures of a graph and its deflation, which we use to work more easily with deflations. The following definition is merely a formal tool to acknowledge that some substructures in a graph are preserved in its deflation although they may not be strictly speaking identical.

Let G be a graph and G' its deflation of form $(v_1v_2v_3v_4) \to (v_2v'v_4)$. Let H be a subgraph of G containing at most one of v_1 and v_3. We say that H *appears* in G' as $\bar{H}$ if $\bar{H} \subseteq G'$ and $\bar{H}$ is obtained from H by substitution of v_1 or v_3 by v'. Symmetrically for any subgraph H' of G' we say that H' *appears* in G as $\bar{H}'$ if $\bar{H}' \subseteq G$ and $\bar{H}'$ appears in G' as H'.

Similarly, we say that a walk C from G *appears* in G' as a walk $\bar{C}$, if after substitution of all occurrences of v_1 and v_3 in C we obtain $\bar{C}$ which is a valid walk in G'. Notice that we allow C to pass through both v_1 and v_3, possibly several times. Symmetrically a walk C' in G' *appears* in G as a walk $\bar{C}'$ if there exists a walk $\bar{C}'$ in G such that $\bar{C}'$ appears in G' as C'.

Generally, all substructures (subgraphs and walks) of G that are not incident with v_1 or v_3 appear in G' and similarly all substructures of G' not incident with v' appear in G. In other cases the deflation may transform the substructures in various ways depending on the exact way these substructures coincide with the aforementioned vertices. Exact description of some of these cases is the main focus of this section.

We also define a correspondence of faces as a natural extension of previous definition. We say a face f of G *appears* in G' if its bounding walk appears in G' and bounds a face.

Lemma 7. *Let G be a graph and let f be its 4-face. Let G' be the f-deflation of G of form $(v_1v_2v_3v_4) \to (v_2v'v_4)$. Let C' be a closed walk in G' bounding a disk. If v' has at most one instance in C' and C' does not appear in G as a closed walk, then there exists a closed walk D in G, $D = v_1c_1c_2...c_nv_3v_i$, where $i \in \{2, 4\}$, $\{c_1, c_n\} \cap \{v_2, v_4\} = \emptyset$ and $C' = c_1c_2...c_nv'$. Thus the length of D is greater by exactly two than the length of C'. Furthermore, D can be chosen so that it bounds a disk Λ in the drawing of G and $f \subseteq \Lambda$.*

Proof. As C' does not appear in G, we immediately get that $v' \in V(C')$. Let a denote vertex preceding v' on C' and b denote the vertex following v' on C'.

If $a \in \{v_2, v_4\}$, then $av_1, av_3 \in E(G)$ and either $bv_1 \in E(G)$ or $bv_3 \in E(G)$, thus C' would appear in G as its structure has not changed. This would contradict the assumptions. Hence $a \notin \{v_2, v_4\}$. Symmetrically $b \notin \{v_2, v_4\}$.

If $av_1, bv_1 \in E(G)$ or $av_3, bv_3 \in E(G)$ then C' appears in G up to substitution of v' by v_1 or v_3. Otherwise either $av_1, bv_3 \in E(G)$ or $av_3, bv_1 \in E(G)$. As we may switch the role of a and b, we assume the former without loss of generality. We may take C' and define the closed walk C in G by replacing the subwalk $av'b$ with $av_1v_2v_3b$. Similarly, we define $\bar{C}$ by replacing $av'b$ with $av_1v_4v_3$.

Consider the alternative definition of deflation operation using the imaginary chord v_1v_3 of the face f. Let D' be a closed walk obtained from C' by replacing the subwalk $av'b$ with av_1v_3b using the imaginary edge. Consider the transition from the drawing of G to the drawing of G' by continuous contraction of the drawing of v_1v_3 into a single point which is the drawing of v'. We may clearly see that since C' bounds a disk in G', D' also bounds a disk in G. Let Δ denote the disk bounded by D'. Since D' passes through the middle of f, the disk Δ contains either the imaginary triangle $v_1v_3v_2$ or the imaginary triangle $v_1v_3v_4$. From the symmetry let us assume the former case, otherwise we may switch the roles of v_2 and v_4. We define the walk D by bypassing the edge v_1v_3 via vertex v_4 in the walk D'. Notice that D bounds a disk which is exactly Δ extended by the imaginary triangle $v_1v_3v_4$ and thus contains the whole 4-face f. □

For a graph G and its deflation G', if a closed walk D bounding a disk in G and a closed walk C' in G' correspond as in Lemma 7, then we say C' is a *deflation* of D and D is an *inflation* of C'.

We immediately apply the rather technical Lemma 7 to characterize behavior of triangles and 4-faces in a more convenient way.

Lemma 8. *Let G be a 4-critical graph without contractible triangles drawn on torus. Then deflations of G do not contain contractible triangles. Furthermore, if G is triangle-free, then deflations of G do not contain loops.*

Proof. Let f be a 4-face of G and let G' be the f-deflation of G that is of form $(v_1v_2v_3v_4) \to (v_2v'v_4)$. We will prove the first part by contradiction. Let G' contain a contractible triangle T, $T = t_1t_2t_3$. Since G does not contain any contractible triangles, T does not appear in G. Without loss of generality $t_1 = v'$. From Lemma 7 we have a closed walk D of length five in G bounding a disk Λ which contains f. Since D has length five, the 4-face f cannot be the whole disk Λ, and thus D is separating. This contradicts Corollary 5.

To prove the second part, simply note that loop can only be a result of an edge such that its end vertices are unified. Such an edge connects v_1 and v_3, which is a contradiction with G being triangle-free as $v_1v_2v_3$ and $v_1v_4v_3$ would form triangles. □

Lemma 9. *Let G be a graph drawn on torus without contractible triangles and f one of its 4-faces. Let G' be an f-deflation of G and let C' be a facial 4-cycle in G' such that C' does not appear in G. Then there exists a closed walk C in G of length six, C is an inflation of C', C bounds a disk Λ, all faces of G contained in Λ are 4-faces, there are at least three such faces and one of the vertices from the boundary of f is drawn in the interior of Λ.*

Proof. Let G' be of form $(v_1v_2v_3v_4) \to (v_2v'v_4)$. Since C' does not appear in G, $v' \in V(C')$. We may apply Lemma 7 from which we have a closed walk D of length six bounding a disk Λ which contains f. From the length of D, the 4-face f clearly cannot be the whole disk Λ and so D is separating. From Corollary 5 we have that all faces contained in Λ are 4-faces.

From Lemma 7 we know that at least one of the vertices v_2 and v_4 is element of D. Since we also know that $f \subsetneq \Lambda$, both are drawn in the closure of Λ. If either v_2 or v_4 is not drawn on the boundary of Λ, then the quadrangulation of Λ contains a vertex and since D is of length six, the quadrangulation must consist of at least three 4-faces. Let us assume for contradiction that both v_2 and v_4 are drawn on the boundary of Λ. From Lemma 7 we get that D has a subwalk $av_1v_iv_3b$ where $i \in \{2, 4\}$ and $\{a, b\} \cap \{v_2, v_4\} = \emptyset$. Without loss of generality let $i = 2$ and $D = av_1v_2v_3bv_4$. Since $f \subsetneq \Lambda$, the edges v_1v_4 and v_3v_4 must be drawn in the closure of Λ and therefore inside Λ as they are not on the boundary. We conclude that G contains contractible triangles av_1v_4 and bv_3v_4 embedded in the closure of disk Λ, which is a contradiction. $\square$

3. Reduction

In this section we build on the previously defined deflation operation and focus our attention to 4-critical triangle-free graphs drawn on torus. Deflations of such graphs need not be 4-critical or triangle-free. We will deal with this via encapsulation of the deflation into a more complex and limited operation with immediate useful applications.

Let G be a triangle-free 4-critical graph drawn on torus. Graph H is a *reduction* of G if H is a 4-critical subgraph of a triangle-free deflation of G.

Note that according to Lemma 6, if G is 4-critical, then deflations of G have no proper 3-coloring and hence contain 4-critical subgraphs. Therefore, a reduction of G exists if and only if a triangle-free deflation of G exists. Unlike deflations, reductions never contain parallel edges as graphs with parallel edges are not 4-critical.

A triangle-free 4-critical graph G drawn on torus is *reducible* if G has a reduction; equivalently if G has a 4-face f such that at least one of the two possible f-deflations of G is triangle-free. A triangle-free 4-critical graph G drawn on torus is *irreducible* if G has no reduction; equivalently if deflation of every 4-face in G (in either direction) creates a triangle. Note that according to Lemma 8 such a triangle is always non-contractible.

We only define terms reducible and irreducible for the class of triangle-free 4-critical graphs. Therefore, when we state that a graph G is (ir)reducible, we implicitly state that it is also triangle-free and 4-critical.

It is clear that the number of vertices and edges strictly decreases with every iteration of the reduction operation. We show that the same is true for the number of 4-faces. For a graph G drawn on torus, let $c(G)$ denote the number of its 4-faces.

Lemma 10. *Let G be a 4-critical graph drawn on torus, with no contractible triangles. Let H be a 4-critical subgraph of a deflation of G. Then $c(H) \leq c(G) - 1$. Also, if H contains a 4-face that does not appear in G, then $c(H) \leq c(G) - 2$.*

Proof. Let G' denote a deflation of G of form $(v_1v_2v_3v_4) \rightarrow (v_2v'v_4)$ such that H is a subgraph of G' and let f denote the deflated 4-face.

Suppose H has some 4-faces that are not 4-faces of G and let h be one of them. Since H is a subgraph of G', the boundary of h is a closed walk C' of length four in G'. If C appeared as a closed walk bounding a disk in G, it would bound a 4-face according to Corollary 5. This would contradict the choice of h. Hence we use Lemma 9 on the boundary of h. There exists a closed walk C of length six in G such that C' is a deflation of C. Furthermore C bounds an open disk Λ consisting of a quadrangulation with at least three 4-faces and the drawing of either v_2 or v_4 in its interior.

First suppose there is only one such 4-face h. Then none of the three 4-faces from the associated quadrangulation is present in H as they would be nested inside the 4-face h. Therefore $c(G) \geq c(H) + 2$.

It holds that if v_2 is drawn in the interior of Λ in G, then it is also drawn in the interior of C' in G'; here we use the fact that v_2 is not unified with v_4 during deflation. From symmetry the same holds for v_4. From this we know that each new 4-face in H contains an interior point corresponding to either the drawing of v_2 or v_4 in G'. Since no two faces share an interior point, there are at most two new 4-faces h_1 and h_2.

By an analogous argument, if there was a 4-face g in G other than f that is a member of both quadrangulations (associated with h_1 and h_2), then the interior of g is transformed via deflation into subset of both h_1 and h_2. This is again a contradiction and we get that $\Lambda_1 \cap \Lambda_2 = f$ for Λ_i denoting the disk associated with h_i.

In G we have a total of at least five 4-faces forming the two quadrangulations that are not present on H. In H we have exactly two 4-faces that do not appear in G. Together we get $c(G) \geq c(H) + 3$.

Finally, let us consider the case that H has no 4-faces that do not appear in G. Then it trivially follows that $c(G) > c(H)$, as f is deflated in G'. Together with the previously proven inequalities we get $c(H) \leq c(G) - 1$ for every H, and $c(H) \leq c(G) - 2$ whenever H has a 4-face that does not appear in G. □

The assumptions of Lemma 10 are satisfied whenever G is a reducible graph drawn on torus and H is a reduction of G. As an immediate corollary, any such reduction H has strictly fewer 4-faces than G. The more general formulation of Lemma 10 allows us to obtain a similar result under a less restrictive relation between H and G which will be useful later on.

The following application of the reduction operation is a key improvement of the Theorem 1 (and of Corollary 5).

Lemma 11. *If G is a 4-critical graph drawn on torus such that every triangle is non-contractible, then every vertex of G is incident with a 4-face.*

Proof. For contradiction, let us take a counterexample G with minimum number of vertices. Let v be a vertex of G that is not incident with any 4-face.

As G is 4-critical, $G - v$ has a 3-coloring ψ. According to Corollary 5, G has a 4-face. Let $v_1v_2v_3v_4$ denote a boundary of one such 4-face. Without loss of generality, we assume that $\psi(v_1) = \psi(v_3)$.

Let H denote a 4-critical subgraph of the deflation of G of form $(v_1v_2v_3v_4) \to (v_2v'v_4)$. Since ψ gives a proper 3-coloring of $H - v$ and H is not 3-colorable, it follows that $v \in V(H)$.

We will now show that H is a counterexample with fewer vertices, which will be a contradiction. Firstly, according to Lemma 8 all triangles in G' are non-contractible. We claim that v is not incident with any 4-face in H. For contradiction let C be a cycle that bounds a 4-face in H such that $v \in V(C)$. Clearly C is contractible, and so if C appeared as a 4-cycle in G, it would bound a 4-face according to Corollary 5. However this does not happen since v is not incident with any 4-face in G. Applying Lemma 9 to C however shows that C inflates into a 6-cycle which bounds a quadrangulation in G. Thus all vertices of C are incident with a 4-face in G. This is in contradiction with G being a minimal counterexample. □

As the last direct application we derive a key description of local properties of irreducible graphs related to their 4-faces. This description of irreducible graphs will serve as a basis for their enumeration later on.

Lemma 12. *Let G be an irreducible graph drawn on torus. Let f be one of its 4-faces, and $v_1v_2v_3v_4$ a cycle bounding f. Then the following properties hold:*

- *There exist paths P_1 and P_2 in G such that v_1, v_3 are the ends of P_1, v_2, v_4 are the ends of P_2 and both P_1 and P_2 have length exactly 3.*
- *Paths P_1 and P_2 are edge-disjoint from $v_1v_2v_3v_4$.*
- *Paths P_1 and P_2 are disjoint.*
- *Path P_1 together with edges v_1v_2 and v_2v_3 forms a non-contractible 5-cycle in G. Similarly, P_2 together with edges v_2v_1 and v_1v_4 forms a non-contractible 5-cycle.*

Proof. Since G is irreducible, any deflation of f of form $(v_1v_2v_3v_4) \to (v_2v'v_4)$ produces a triangle. Let us denote one such triangle $t_1t_2t_3$, without loss of generality $t_1 = v'$.

For the first property, without loss of generality v_1 is a neighbor of t_2 and we put $P_1 = v_1t_2t_3v_3$. Symmetrically we get existence of P_2 considering the f-deflation in the other direction.

Let us prove the second property via contradiction. Let P_1 share an edge e with the cycle bounding f. Since P_1 is a proper path, e is either the first or the last edge of P_1. Without loss of generality, let e be the first edge of P_1 and $e = v_1v_2$. If we now consider the edge v_2v_3 and the last two edges of P_1, we get a triangle. Symmetrically we also reach contradiction for P_2.

The third part will also be proven by contradiction. Let $v \in V(P_1) \cap V(P_2)$. Then v is not an endpoint of either path according to the previous property. Without loss of generality we can assume P_1 and P_2 are oriented so that v is the second vertex in both of them. Note that any endpoint of P_1 is a neighbor of any endpoint of P_2. Thus v_1vv_2 is a triangle, which is a contradiction with G being triangle-free.

To prove the last property we first notice that the previous properties imply that the paths together with the specified edges are proper cycles. It is only necessary to show that the 5-cycles are non-contractible. Let us consider a triangle $t_1t_2t_3$ created by deflation of G. According to Lemma 8 it is a non-contractible triangle. Cycle $v_1v_2v_3t_2t_3$ is therefore also non-contractible. From symmetry we get the same property for the other 5-cycle. □

4. Bound on the number of 4-faces

In this section we use the previous results to give an explicit combinatorial lower bound on the number of 4-faces of all triangle-free graphs drawn on torus that are not 3-colorable. We do this by lower-bounding the number of 4-faces in all irreducible graphs. As we previously proved in Lemma 10, the number of 4-faces strictly decreases with each iteration of reduction. Thus the number of 4-faces of all 4-critical triangle-free graphs drawn on torus (and thus of all their triangle-free supergraphs) is lower-bounded by some irreducible graph with the minimal number of 4-faces. We define an operation that further simplifies irreducible graphs and use it to show that any irreducible graph requires at least four 4-faces to allow existence of an obstruction to 3-colorability. This bound will be improved later on as a consequence of the main result based on a computer enumeration.

Let us consider a slightly different kind of reduction. Let G be a 4-critical graph drawn on torus with no contractible triangles. Let G' denote a loop-free deflation of G. A graph H is a *strong reduction* of G if H is a 4-critical subgraph of G'. Note that this is a valid definition since Lemma 6 gives existence of such a 4-critical subgraph.

Both reduction and strong reduction of G are 4-critical subgraphs of deflations of G. The difference is in the conditions we require. For the (simple) reduction we require G to be triangle-free and the deflation to be free of (non-contractible) triangles. In the process of strong reduction, we allow non-contractible triangles, we only require G to be free of contractible triangles and the intermediate deflation to be free of loops.

Recall that Lemma 8 states that deflation does not create contractible triangles and that there are no loops in deflations of a triangle-free graph. In particular every reduction is also strong reduction, while irreducible graphs have further strong reductions.

Analogously with the terminology regarding the (simple) reduction operation, we say that graph G is *strongly reducible* if it has a strong reduction, and *strongly irreducible* otherwise. We also say that 4-face of G is *strongly reducible* if it can be deflated so that no loops occur, and *strongly irreducible* otherwise.

We only define terms strongly reducible and strongly irreducible for 4-critical graphs, thus if we state that graph is strongly (ir)reducible, we implicitly state that it is also 4-critical. Unlike with the simple (ir)reducibility, we do not imply the graph to be triangle-free as we in fact allow non-contractible triangles.

Lemma 13. *Graph K_4 is the only strongly irreducible graph drawn on torus with no contractible triangles.*

Proof. Let G be a strongly irreducible graph with no contractible triangles drawn on torus. According to Corollary 5 there exists a 4-face f in G. Let $v_1v_2v_3v_4$ be the facial walk of f. Since G is irreducible, identifying v_1 and v_3 produces a loop. Consequently $v_1v_3 \in E(G)$ and similarly $v_2v_4 \in E(G)$. From this we deduce that vertices v_1, v_2, v_3, v_4 induce a clique of size 4 (which we denote K_4) in G.

The graph K_4 is 4-critical, it can be drawn on torus without contractible

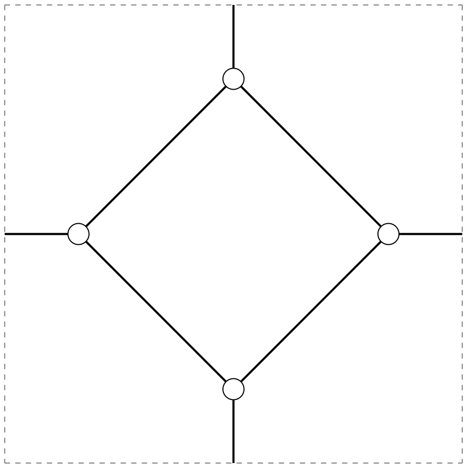


Figure 4.1: K_4 drawn on torus without contractible triangles

triangles (see Figure 4.1) and is thus strongly irreducible. On the other hand, if $K_4 \subsetneq G$, then G is not 4-critical. Therefore $G = K_4$. $\square$

Note that Lemma 13 can be interpreted as the following simplification of the characterization of strong irreducibility: any 4-critical graph drawn on torus is strongly irreducible if and only if at least one of its 4-faces is strongly irreducible.

Corollary 14. *Let G be a triangle-free 4-critical graph drawn on torus. Then G contains at least two 4-faces.*

Proof. Recall Lemma 10 which was used to show that number of 4-faces decreases with every reduction. Notice that assumptions of this lemma also hold for strong reduction. According to Lemma 13, any triangle-free graph G is strongly reducible and repeated iteration of strong reduction eventually transforms it into K_4. Since K_4 drawn on torus without contractible triangles has exactly one 4-face, we immediately get that G has at least two 4-faces. $\square$

4.1 Junctions

Let G be a graph drawn on torus and let H be a result of iterated application of strong reduction on G. Vertex v is a *G-junction* in H if $v \in V(H)$ and $v \notin V(G)$. Note that any such vertex v is a result of unification of two or more vertices of the original graph G.

We show that G-junctions relate to existing triangles in the (iterated) strong reductions of G. In fact we show that all triangles cluster around these G-junctions in a particular way. By tracking the G-junctions we may avoid clustering triangles in a way that is compatible with the existence of K_4 as a subgraph.

As we demonstrated before, if G is a 4-critical graph then iteration of strong reduction (in an arbitrary way) necessarily produces K_4. This obstruction to 3-colorabilty is therefore both sufficient and necessary condition. We exploit this formulation to show that if G has only a few 4-faces we may in fact design a sequence of strong reductions of G such that K_4 is never reached, thus proving that G cannot in fact be 4-critical.

Lemma 15. *Let J be an irreducible graph drawn on torus, let G be an iterated strong reduction of J and let H be a strong reduction of G. Then H contains exactly one J-junction w such that w is not present in G and all triangles in H not appearing in G are incident with w. Furthermore, all triangles in H are incident with a J-junction.*

Proof. Let G' be a deflation of G such that $H \subseteq G'$, and let w be the new vertex in G'. Since both G and H are 4-critical and $H - w \subset G$, we get $w \in V(H)$ and w is a J-junction. On the other hand, from the definition of J-junction, it is clear that one deflation can only produce a single new J-junction. It is a simple observation that any new triangle in H is a result of unification of two vertices from G. Thus for every such triangle T in H we get $w \in V(T)$.

To prove the second part, note that every newly formed triangle is incident with a J-junction when it first appears during iterated strong reduction. Consider for contradiction a situation where in one steps of the iteration one of the triangles loses incidence with one of its J-junctions. One possibility is that the J-junction was unified with a different vertex by deflation. However then the triangle becomes incident with a new J-junction and the property still holds. The only other possibility is that the J-junction is removed from the graph when taking a 4-critical subgraph of a deflation. That however also destroys the triangle. □

Lemma 16. *Let G be an irreducible graph drawn on torus. Let H be a result of iterated strong reduction of G. Let f be a strongly irreducible 4-face in H and C its bounding cycle. Then C is incident with two G-junctions. Furthermore if f appears as a 4-face in G, then C is incident with two G-junctions neighboring in C.*

Proof. According to Lemma 13, C is a subgraph of K_4 and therefore each triplet of its vertices forms a triangle in H.

Since G was originally triangle-free, all of these triangles were formed by strong reductions and are incident with G-junctions according to Lemma 15. Let j_1 be a G-junction in C. Vertices $V(C)\setminus\{j_1\}$ form a triangle which is not incident with j_1. Therefore, there exists a G-junction $j_2 \in V(C)$ such that $j_1 \neq j_2$.

Suppose f is a face of G. Let us assume for contradiction that $C = j_1aj_2b$ where neither a nor b is a G-junction in H, thus $a, b \in V(G)$. We already know that aj_1b is a triangle in H but not in G. As $a, b \in V(G)$ we also have $ab \in E(G)$. From G being triangle-free we have that a and b do not share any neighbors in G. This is however contradiction with f being a 4-face in G. □

The previous two lemmas characterize conditions necessary for a 4-face to be strongly irreducible, in other words for K_4 to arise as a result of strong reduction. To guarantee avoiding K_4 we need to use Lemma 16 in its stronger form. To do that we put forward the following specific observation.

Lemma 17. *Let G be an irreducible graph on torus with at most three 4-faces. Let H be a product of iterated strong reduction of G such that H is strongly irreducible. Let f denote a 4-face of H. Then f is also a 4-face in G.*

Proof. Let us assume for contradiction that f is a 4-face of H, but it is not a 4-face in G. Recall Lemma 10 stating that if a new 4-face is produced by deflation then the total number of 4-faces is reduced by at least two. Since H has at least one 4-face, this means at most one strong reduction took place transforming G into H. Lemma 15 then shows that there is at most one G-junction in H. This is in contradiction with Lemma 16 as an irreducible 4-face f must be incident with at least two G-junctions. □

Lemma 18. *Let G be a triangle-free graph drawn on torus without separating 4-cycles and with at most three 4-faces. Then G is 3-colorable.*

Proof. Suppose G is not 3-colorable. Without loss of generality we can assume that G is 4-critical. We will show that for every such graph there exists a way to perform (iterated) strong reduction so that K_4 is never reached. If G was not 3-colorable, this would contradict Lemma 13.

Suppose iterated strong reduction reaches K_4. This happens in at most two steps according to Lemma 10. On the other hand, since every iteration produces at most one junction, Lemma 16 shows that at least two iterations are needed. And Lemma 17 shows that only 4-faces from G are present in the whole process. In particular we can assume that each step removes exactly one 4-face and creates no new 4-faces.

Let x, y, z denote 4-faces of G and X, Y, Z their respective bounding cycles. Let us now consider several cases.

Case 1: Let x and y be two 4-faces such that x and y share three or more vertices. Since G has no loops, parallel edges or triangles, no two vertices of X can be identical or connected by any edge that is not part of X. The bounding cycle Y cannot share all edges with X as they are not identical, thus Y has a vertex v that is not a vertex of X and is connected to two vertices $a, b \in V(X)$. If ab was an edge then abv would form a triangle, so a and b are opposite in X. Clearly we have a vertex c of X such that edges ac and bc are shared between X and Y. The degree of c is two, which contradicts G being 4-critical.

Case 2: Let x and y be two 4-faces such that x and y share an edge. Let $X = abcd$ and $Y = cdef$. Since case 1 does not apply, no vertices a, b, c, d, e, f can be identical in G. Let G' be the y-deflation of G such that vertices f and d are unified into a junction v and let H be a 4-critical subgraph of G'. If $bv \in E(H)$, then either bd or $bf \in E(G)$ and either bcd or bcf forms a triangle in G. Therefore, $bv \notin E(H)$ and x can be deflated so that b and v are unified into junction v', which is the only junction in the resulting graph with a single 4-face. This contradicts Lemma 16.

Case 3: Let x and y share two vertices a and c but case 2 does not occur. Let P_X be a shortest path in X connecting a and c; and let P_Y denote a shortest path connecting a and c in Y. As X and Y are edge-disjoint, P_X and P_Y are also edge disjoint. Since G contains no triangles, the union $P_X \cup P_Y$ is a closed walk of length at least four. As P_X and P_Y have each length at most two, both must be of length two and therefore a and c are opposite in both faces x and y. Let $X = abcd$ and $Y = ceaf$. None of the vertices a, b, c, d, e, f can be identical in G. Let G' be the y-deflation of G such that a and c are unified into a G-junction v and let H be a 4-critical subgraph of G'. Then x becomes $vbvd$ in G', and since H does not contain parallel edges, x is not a 4-face in H. This is a contradiction with previous observation that each strong reduction removes exactly one 4-face.

Case 4: Let $X = abcd$. Assume that $a, c \notin V(Y \cup Z)$. Let G' be the x-deflation of G that a and c are unified into a G-junction v and let H be a 4-critical subgraph of G'. The vertex v is not incident with any remaining 4-face in H. This means that in the following strong reduction v cannot become incident with any present 4-face and according to observation from the beginning of this proof, no new 4-face is created. As at most one additional junction can be created, we get a contradiction with Lemma 16.

Let us now assume that none of the cases 1–4 occurs. From symmetry we assume this to be true even when we permute the labels of the three 4-faces. We will show that G is not triangle-free which is a contradiction. Since cases 2 and 3 do not occur, we get that $|V(X \cap Y)| \leq 1, |V(Y \cap Z)| \leq 1, |V(X \cap Z)| \leq 1$. Also, since case 4 does not occur, we get that $|V(X \cap (Y \cup Z))| \geq 2$ and by symmetry also $|V(Y \cap (X \cup Z))| \geq 2$ and $|V(Z \cap (X \cup Y))| \geq 2$. Note that if $a \in V(X \cap Y \cap Z)$ then the first three inequalities do not allow any more shared vertices and the remaining three inequalities are necessarily violated. Thus we have $|V(X \cap Y \cap Z)| = 0$ and $|V(X \cap Y)| = |V(X \cap Z)| = |V(Y \cap Z)| = 1$.

Let a, b and c be vertices such that $a \in V(X \cap Y)$, $b \in V(Y \cap Z)$ and $c \in V(X \cap Z)$. Note that $a \neq b \neq c \neq a$ from the inequalities above. Since case 4 does not occur, a and c are not opposite in X, and therefore are neighbors. Similarly a and b are neighbors in Y and b and c are neighbors in Z. We get that abc is a triangle in G which is a contradiction. $\square$

5. Irreducible graph enumeration

In this section, we describe a theoretical basis for enumeration of all irreducible graphs drawn on torus via a computer program. We do this in several steps. First we provide a process that can enumerate all irreducible graphs, but is potentially infinite. Then we introduce extra steps that significantly limit the extent and complexity of created graphs. Finally we introduce a number of optimizations for practical implementation of the process. Using this process, we show that there are only four irreducible graphs, which we present in the section 5.5. Based on the explicit knowledge of these graphs, we derive new properties of all 4-critical triangle-free graphs drawn on torus.

Recall the description of 4-faces of irreducible graphs from Lemma 12. Let f be a 4-face of an irreducible graph G and let F be its bounding cycle. Each pair a, b of opposite vertices of F is *linked* if it is connected via a 3-path P such that $P \cap F = \{a, b\}$ and P together with a path connecting a and b in F forms a non-contractible 5-cycle. The pair a, b is *unlinked* otherwise. A *link* is any such 3-path and an *ab-link* is a link with end vertices a and b. A face f is *linked* if both of the opposite pairs of vertices of its bounding cycle are linked and their links do not intersect. Otherwise, f is *unlinked*. Similarly, graph G is *linked* if all of its 4-faces are linked, and *unlinked* otherwise.

Let B denote the unique (up to isomorphism) inclusionwise-minimal graph drawn on torus with a linked 4-face. We call this graph the *base graph* and denote it as B. Equivalently, we may define the base graph as the graph obtained from K_4 by subdividing edges of its perfect matching twice, with its unique embedding in the torus that has a 4-face. Notice that base graph has exactly two 2-cell faces, a 4-face and a 16-face. Thus, all faces of connected supergraphs of B are always 2-cell faces.

Since all irreducible graphs drawn on torus are linked and contain at least one (linked) 4-face, we have $B \subset G$ for any irreducible graph G drawn on torus. This simple observation serves as a basis for our enumeration.

Recall Lemma 11 which can be narrowed into a statement that every vertex of any irreducible graph drawn on torus is incident with a 4-face. This key property serves as a driving force of our enumeration via the following structural argument.

Lemma 19. *Let G be an irreducible graph drawn on torus. Let H be a subgraph of G and f a 2-cell face of H such that f is not a face of G. Then in the drawing of G either the interior of f contains a chord or G has a 4-face g and an edge e such that $g \subsetneq f$ and e has one end contained in the boundary of f and the other end contained in the boundary of g.*

Proof. Since f is not a face of G, either f contains a chord or at least one vertex of G. Suppose the former does not hold. Since G is connected, there exists a vertex $v \in V(G) \setminus V(H)$ such that v is connected to the boundary of f via an edge e. According to Lemma 11 v is incident with a 4-face which we can choose to play the role of g. □

5.1 Expansion

The previous lemma provides us with a crude process to generate all irreducible graphs. We describe this basic ingredient of enumeration and prove its correctness.

Let H be a graph drawn on torus such that all faces of H are 2-cell. Let c be a face in H of length at least six. Perform one of the following operations:

1. Insert a new 4-cycle into the interior of c so that it shares some vertices (and possibly some edges) with the boundary of c.

2. Insert a new 4-cycle F and an edge e into the interior of c so that F and the boundary of c share no vertices and e is incident with both of them.

3. Insert a chord into c.

If H' is the result of applying one of the previous operations, then H' is an *expansion* of H, also H' is an *expansion of face* c. Note that expansion of a graph is always a strict supergraph.

Lemma 20. *Let G be an irreducible graph drawn on torus. Let H_1 be a connected graph drawn on torus such that $B \subseteq H_1 \subseteq G$. Then there exists a sequence of connected graphs $H_1, H_2, ..., H_k$ such that $H_k = G$ and for every value of i from 1 to $k-1$ a graph H_{i+1} is an expansion of H_i.*

Proof. Consider a graph H_i such that $H_i \subsetneq G$. Recall that all faces of H_1 are 2-cell faces as H_1 is a connected supergraph of B. We will show that applying one of the described expansion operations produces a graph H' such that $H_i \subset H' \subseteq G$.

Let c be a 2-cell face of H that does not correspond to a face in G and let C be its facial walk. From Corollary 5 we know that the length of C is at least six.

If c contains no vertices of the drawing of G, then it contains a chord and we can apply operation 3. On the other hand, if a vertex of G is drawn inside c, we use Lemma 19 on c to show that it contains a 4-face g of G. If the boundary of g shares vertices with C, then we apply operation 1. Otherwise, we get from Lemma 19 that there exists a choice of g and an edge e in G such that e and g are constructed applying via 2. $\square$

To better understand what structures are possible to construct via the expansion operation 1, we specifically describe the possible options in the following lemma. As a simplification, we also allow insertion of a chord, which technically falls under the expansion operation 3.

Lemma 21. *Let G be an irreducible graph drawn on torus. Let H be a subgraph of G and C a facial walk of one of the 2-cell faces c of H such that c is not a face in G. If G has a 4-face $f \subsetneq c$ and the boundary of f shares at least one vertex with C, then H can be expanded into a subgraph of G by inserting elements into c in one of the following ways (see Figure 5.1 for illustration):*

1. *Insert a new 4-cycle sharing exactly one edge with C (Figure 5.1(a)).*

2. *Insert a new 4-cycle sharing exactly two edges with C and these edges are consecutive on C (Figure 5.1(b)).*

3. *Insert a new 4-cycle sharing exactly one vertex with C (Figure 5.1(c)).*
4. *Insert a new 4-cycle sharing exactly two of its opposite vertices with C (Figure 5.1(d)).*
5. *Insert a new chord into C (Figure 5.1(e)).*

Proof. Let F be the boundary of f.

Case 1: Suppose that one of the edges of F is a chord of C. Then it suffices to insert a chord applying option 5. Note that applying the option 5 may not construct the 4-face c but it still gives a valid expansion of H.

Case 2: Suppose $|V(F \cap C)| = 4$. Then all edges of F are either also edges of C or chords. Supposing case 1 does not occur, we reach a contradiction.

Case 3: Suppose $|V(F \cap C)| = 3$ and suppose case 1 does not occur. Then two consecutive edges of F are also consecutive edges of C and the option 2 applies.

Case 4: Suppose $|V(F \cap C)| = 2$ and suppose case 1 does not occur. If the shared vertices are consecutive in F, then the edge connecting them is an edge of C and the option 1 applies. Otherwise the vertices are opposite in F and the option 4 applies.

Case 5: Suppose $|V(F \cap C)| = 1$. Then the option 3 clearly applies. □

5.2 Linkage

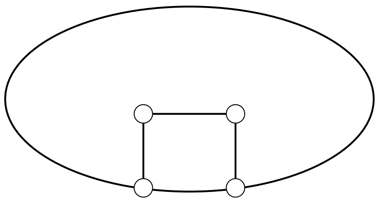
While the iterated expansion can construct any irreducible graph, it does not utilize the previously described property that all 4-faces of an irreducible graph are linked, as implied by Lemma 12. Since the expansion alone generates any irreducible graph, it would naturally create all the necessary links without any explicit construction. However the expansion alone is not finite. As an example, consider a face f of length six. The iterated expansion can iteratively construct an arbitrarily complex quadrangulation inside f, for example by arranging new 4-faces in concentric layers, each layer forming a new separating 6-cycle embedded inside the previous one. If however we force all 4-faces to be linked, then all the nested 4-faces have a very limited distance from the boundary of f. Though we do not formally prove that such limitations make enumeration finite, it is implied by the fact that our practical implementation takes only a finite time to finish.

Let H be a graph drawn on torus such that all faces of H are 2-cell faces and H is not linked. Consider the following operation:

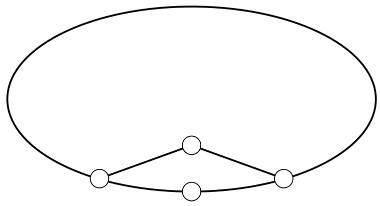
Let f be a 4-face unlinked in H and a, b a pair of unlinked opposite vertices on its boundary. Let P be an path $axyb$ for some vertices x, y such that $H \cup P$ is a triangle-free graph drawn on torus with linked pair a, b. Then $H \cup P$ is a *linkage* of H and P is a *link*.

While vertices a and b are vertices of H, vertices x or y may be new vertices. The resulting linkage of an unlinked graph is always its strict supergraph.

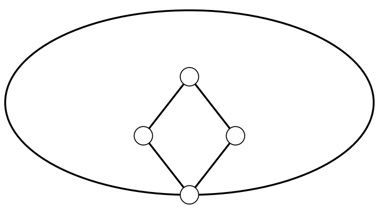
We intend to use iterated linkage operation to construct linked supergraphs of a given graph. It is important to realize that iteration is necessary. It may seem that defining linkage of H as H together with links for all unlinked vertex pairs would produce a linked supergraph of H in a single step. However such operation may produce new 4-faces and thus give a result with new unlinked 4-faces. For



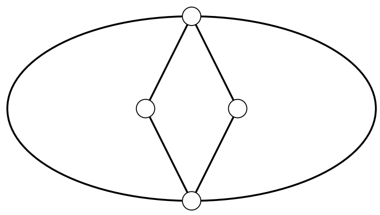
(a) One common edge



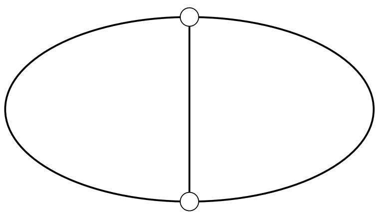
(b) Two common edges



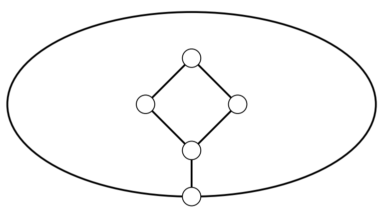
(c) One common vertex



(d) Two common vertices



(e) Chord



(f) No common elements

Figure 5.1: Expansion operation 1, as described by Lemma 21, and expansion operation 2

this reason we adopt weaker and less technical definition which constructs one link at a time and achieves the desired properties through iteration.

Recall that we defined a chord of a closed walk C bounding a disk Λ as an edge connecting two vertices of C that is drawn inside Λ. Let f be a 2-cell face of a graph and let F be its facial walk. A P_k*-chord* of f is a chord of its facial walk subdivided into a path of length k (a path with k edges).

To better understand the way a link may be constructed, we offer the following description.

Lemma 22. *Let G be an unlinked triangle-free graph drawn on torus such that all faces of H are 2-cell faces and let H be a linkage of G. Then H can be constructed from G by extending G in one of the following ways:*

1. *Add at most three and at least one chord into faces of G.*
2. *Add a P_2-chord and at most one additional chord into faces of G.*
3. *Add a P_3-chord into one of the faces of G.*

Proof. Let P denote the link in H. Let us consider several cases:

Case 1: Suppose $|V(G) \cap V(P)| = 4$. Then clearly all edges of P are either also edges of G or chords of faces of G and the option 1 applies as $G \subsetneq G \cup P$.

Case 2: Suppose $|V(G) \cap V(P)| = 3$. Let $P = axyb$ and let $y \in V(G)$. Then clearly edges ax and xy are contained in the interior of the same face of G as $G \cup P$ is drawn on torus, and thus axy is a P_2-chord of a face in G. Since $y, b \in V(G)$, either $yb \in E(H)$ or yb is a chord in G. Either way, the option 2 applies.

Case 1: Suppose $|V(G) \cap V(P)| = 2$. Let $P = axyb$. Similarly to case 2, all edges ax, xy, yb are contained in the interior of the same face of G and thus P is a P_3-chord and the option 3 applies. □

Finally we prove that a suitable combination of expansion and linkage gives us a correct means of enumeration.

Lemma 23. *Let G be an irreducible graph drawn on torus. Let H_1 be a connected graph drawn on torus such that $B \subseteq H_1 \subseteq G$. Then there exists a sequence of connected graphs $H_1, H_2, ..., H_k$ where $H_k = G$ and for every i from 1 to $k-1$, if H_i is linked then H_{i+1} is expansion of H_i, and H_{i+1} is a linkage of H_i otherwise.*

Proof. We proceed via induction. We can assume that in the i-th step $H_i \subsetneq G$ and we will prove that there exists a graph H_{i+1} such that $H_i \subsetneq H_{i+1} \subseteq G$.

From Lemma 20 we know that if $B \subseteq H_i \subset G$ then there exists a suitable expansion H_{i+1}. This solves the case when H_i is linked.

If H_i is unlinked, consider a pair of unlinked vertices a, b opposite on the boundary of some 4-face in H. Since $H_i \subsetneq G$ and G is linked, there exists an a, b-link P in G. Linkage of H_i using P gives a suitable graph H_{i+1}. □

5.3 Enumeration

Let us focus on the algorithmic side of enumeration. We formalize the scheme from Lemma 23 to provide the following concept algorithm, which enumerates all irreducible graphs drawn on torus, starting from the base graph.

function CONCEPT-ENUMERATION
  CONCEPT-SEARCH(B)
  Output remembered graphs
end function
function CONCEPT-SEARCH(Graph G)
  if G is linked and 4-critical **then** remember G and return **end if**
  if G is linked **then**
    ψ := set of all expansions of G
  else
    ψ := set of all linkages of G
  end if
  for $H \in \psi$ **do** CONCEPT-SEARCH(H) **end for**
end function

The Concept-Search method is a simple DFS-like search through all possible sequences of expansions and linkages, following the construction from Lemma 23. To show correctness, for every irreducible graph H we use Lemma 23 to produce a graph sequence with the first element B and the last element H. For any element H' from such sequence, when the variable G is equal to H' in some level of recursion, then the next element in the sequence appears in the set ψ. Inductively this means that the sequence is explored at some point during the search, if we assume that the program finishes in a finite time. Thus, if the program finishes, all irreducible graphs drawn on torus are present in the output.

To make the actual enumeration effective, we adopt a number of pruning mechanisms. The main idea is that instead of actually following the program above, we will perform a DFS search through the state space of all possible values of G in the Concept-Search method.

5.3.1 Isomorphism based pruning

Since our goal is to search the state-space of the concept-program above, we can cut off any branch where G is a graph already encountered in a different branch, more specifically if a graph H isomorphic to G has already been encountered. Recall that according to our definition of isomorphism, a graph is isomorphic to its mirror image. To apply this in practice, we use a canonical graph encoding which is invariant to renaming vertices, renaming edges and mirror image.

5.3.2 Separation based pruning

Recall Corollary 5 stating that a triangle-free 4-critical graph drawn on torus contains no separating closed walks of length four or five bounding a disk. This property is inherited by any subgraph. In every recursion node all graphs from ψ are supergraphs of G and so we can exclude all branches where G contains any such separating closed walk. In practice we only need to test walks incident with (some of) the new edges maintaining the desired property incrementally.

5.3.3 Systematic search

To enumerate the set ψ in a systematic manner, we use Lemmas 21 and 22. Both of these lemmas describe the expansion and linkage operations in a form of simple constructions that can be systematically explored.

Recall that we require that every expansion and linkage is always triangle-free. This condition can also be ensured effectively by extending schemes from the mentioned lemmas introducing early checks ensuring this property holds for any actually constructed graph. E.g. each new edge must connect vertices with distance at least three.

In order to achieve a systematic exploration of the whole state space, we show that it is possible to switch the order of operations. We show this in the following lemma and then discuss the rather technical applications as strategy-based pruning mechanisms.

Lemma 24. *Let G be an irreducible graph drawn on torus and let H_1 be a connected graph drawn on torus such that $B \subseteq H_1 \subset G$. Let κ denote the set of all sequences from Lemma 23. If H_1 is linked, then for any face c of H_1, either c is a face of G or there exists a sequence from κ such that c is expanded in H_2. If H_1 is unlinked, then for any unlinked 4-face f of H_1 and a, b fixed pair of its unlinked vertices there exists a sequence in κ such that pair a, b is linked in H_2.*

Proof. First consider H_1 linked. Without loss of generality let c be a face of H_1 that is not a face of G. Either the interior of c contains a vertex in the drawing of G, then we use Lemma 19 to find a suitable 4-face in the interior of c, or c contains a chord in the drawing of G. In either case, there exists a graph H_2 such that H_2 is an expansion of H_1 with expanded face c and $B \subseteq H_2 \subseteq G$.

Consider H_1 unlinked. Let f be an unlinked 4-face of H_1 and a, b its fixed unlinked vertex pair. Then clearly G contains an a, b-link P. Let us put $H_2 = H_1 \cup P$. Clearly H_2 is a linkage of H_1 such that $B \subseteq H_2 \subseteq G$.

Finally note that in both cases H_2 satisfies conditions of Lemma 23 which gives us a sequence S with the first element H_2 and the last element G. Concatenation of H_1 and any sequence S $(H_2, ..., G)$ gives a sequence from κ with the desired properties. □

5.3.4 Linking-strategy based pruning

One of the corollaries of Lemma 24 is that if G is unlinked, the order in which we link the unlinked vertex pairs does not matter. It clearly follows that we can fix any strategy of choosing such an unlinked pair and only link this chosen pair in every node of recursion (where G is unlinked).

We propose a strategy to first compute the number of possible links for every unlinked vertex pair and then explore all linkages for the pair with the lowest number of possibilities. Computing the number of possibilities precisely is a non-trivial task, however we can easily obtain an upper bound tight enough for practical purposes, as we will describe later. This strategy reduces the branching factor of the search and may completely eliminate some unlinked graphs from the search while keeping all linked (and therefore all irreducible) graphs. It also allows for an early detection of bad search branches, as whenever there is any unlinked vertex pair with no possible links in graph G (upper bound is zero), then the whole branch may be cut off, as there exists no linked triangle-free supergraph of G, and therefore no irreducible supergraph of G.

To bound the number of links, we use a modified DFS search from one of the vertices with depth limited to three. The DFS is modified so that it may traverse faces via non-existing k-chords. It also does not check which vertices have been visited (as we look for all alternatives) and is forbidden to visit the other two vertices of the 4-face being linked (recall Lemma 12). Such DFS reaches the other vertex repeatedly, at least once for each possible link. We ignore all cases where the other vertex is reached via path of length shorter than three to avoid counting invalid-links.

The one sided error is caused by the fact that DFS does not alter the structure of the graph and may therefore traverse a single short face multiple times although this face cannot in fact obtain multiple chords while also avoiding triangles. In all other cases it is possible to guarantee that the resulting linkage would be triangle-free, if constructed.

5.3.5 Expansion-strategy based pruning

Similarly to the previous pruning mechanism, we want to apply the Lemma 24 to expansions. To do that, we need to introduce a technical modification to the search algorithm. We would like to fix a strategy which always chooses one specific face to be expanded so that we limit branching factor and avoid cases where the same graph is constructed via different order of equivalent operations. Obtaining strategy satisfying the former is much easier than in the previous application, as the number of possible expansions of a face is very well determined by the length of the face (more precisely its square) up to some errors caused by the requirements to avoid triangles and separating cycles, which is negligible for practical purposes.

Lemma 24 shows that we can indeed fix any such strategy as long as the chosen face of G is not a face of the irreducible supergraph we are attempting to reach. The clear issue is that we do not know which faces to expand and which to keep unexpanded. Indeed this often differs for different irreducible graphs we wish to reach through the search. We therefore modify the search when G is linked so that a fixed face is chosen, all possible expansions of this face are explored, and one additional possibility of not-expanding this face is explored. In doing so we face a number of technical difficulties.

The recursion node representing the choice to not-expand a face has the same value of G as its predecessor. Since our strategy of face choice is fixed, it would presumably produce the same decision which leads to a possibly infinite branch of recursion. Therefore, we need to keep the set of faces of G which have previously been decided to be kept not-expanded and our strategy needs to respect these previously made decisions by always choosing a face which is not in this set. Another problem is that the previously described isomorphism-based pruning optimization would cut off branches with the same value of G and different sets of not-expanded faces. We do not wish to modify this optimization to consider two graphs with different sets of not-expanded faces to be non-isomorhic as this would allow the program to produce a wide variety of otherwise isomorphic graphs. Instead, each time we decide to not-expand a face, we exempt this particular recursion node from being cut off by the isomorphism-based pruning.

For this modification to be correct we also need to require that the not-expand possibility is always explored as last (as we will demonstrate later). Furthermore, we should also modify the linkage so that it does not consider links that traverse not-expanded faces via P_k-chords. Such modification of linkage generation clearly only removes duplicities (recall Lemma 19). A simple advantage of this modification is that we avoid further technical issues regarding identification of not-expanded faces after splitting them into multiple faces.

We will now show correctness of the expansion-strategy based pruning, in particular we will show that the isomorphism-based pruning, which is oblivious to the sets of not-expanded faces, does not cause loss of any output. Let A be a search algorithm with all previously described pruning mechanisms, and let B be the same algorithm without the isomorphism-based pruning. Consider τ, the recursion tree of the algorithm B together with ordering of all recursion nodes according to the order of visitation by the algorithm B. Note that the equivalent tree for algorithm A is a subtree of τ and has the same ordering of all shared nodes. For a recursion node a of τ, let us denote $G(a)$ the graph in a and $N(a)$ the set of not-expanded faces of $G(a)$ in a.

Suppose for contradiction H is an irreducible graph not found by the search algorithm A. Let c be the first (according to visitation order) node in τ such that $G(c) = H$ and path leading to c has been cut off in a predecessor node a because of a previous node b where $G(a)$ is isomorphic to $G(b)$. We know that a is not a descendant of b, as then either $G(b) \subsetneq G(a)$ or a is a not-expand possibility of its parent in which case it would be exempt from being cut off. First consider the case where all elements in $N(b)$ are also faces in H. Then the graph H is reachable from b and thus c was not first or H was previously found. Therefore, there exists an element f in $N(b)$ not present in $N(a)$ such that it represents a face which is not a face in H. Let d be the lowest predecessor of b such that $N(d)$ contains no such element. From the choice of d there exists a descendant e of d on the path from d to b in τ that has a not-expanded face f which is not a face in H. Therefore, face chosen by the fixed strategy in d is f. However this means that all the other descendants of d representing all possibilities of expansion of f precede node e, including an expansion which is a subgraph of H. Thus c is not first or H has been found.

5.4 Enumeration Algorithm

We are now ready to put together the full algorithm, applying all the previously described pruning mechanisms.

For the purposes of the description, we use the following notation. Let f be a face of a graph. We denote $l(f)$ the length of the face f. Let G be a graph, N a set of faces of G and a, b an unlinked pair of opposite vertices of on of the 4-faces of G. Let $link_{G,N}(a, b)$ denote the upper bound on the number of possible links in G such that the links do not traverse any face from the set N.

function ENUMERATION
  SEARCH(B, $\emptyset$)
  Output remembered graphs
end function

```
function SEARCH(Graph G, faceset N)
    if G is linked then
        if G is 4-critical then remember G and return end if
        F := {f' : f' is face of G, f' ∉ N, l(f') ≥ 6}
        if F = ∅ then return end if
        f := argmin{l(f') : f' ∈ F}
        ψ := all expansions of f in G with no separating walk of length 4 or 5
        for H ∈ ψ previously unencountered do SEARCH(H,N) end for
        SEARCH(H,N ∪ {f})
    end if
    if G is unlinked then
        a, b := argmin{link_{G,N}(a', b') : a', b' unlinked pair in G}
        n := link_{G,N}(a, b)
        if n = 0 then return end if
        ψ := all linkages of a, b in G with no separating walk of length 4 or 5
        for H ∈ ψ previously unencountered do SEARCH(H,N) end for
    end if
end function
```

5.5 Results

Theorem 25. *There are exactly four non-isomorphic irreducible triangle-free graphs drawn on torus: I_4, I_5, I_7^a, I_7^b, as depicted in figure 5.2.*

Proof. Proof of this theorem is computer-assisted and was performed by implementation of a search program as described in this chapter.

We provide an example of implementation in attachment 1. □

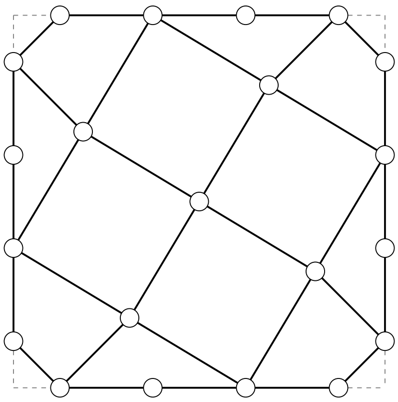
Corollary 26. *Let G be a triangle-free graph drawn on torus without separating walks of length 4 and with at most six 4-faces. Then G is 3-colorable.*

Proof. Suppose G is not 3-colorable. Without loss of generality, we can assume that G is 4-critical. If G is irreducible then Theorem 25 shows G has either thirteen 4-faces (for I_4), eight 4-faces (for I_5) or seven 4-faces (for I_7^a and I_7^b). If G is reducible, then Lemma 10 shows that G has more 4-faces then its reduction and thus more than one of the irreducible graphs. Together we reach a contradiction with the assumed number of 4-faces in G. □

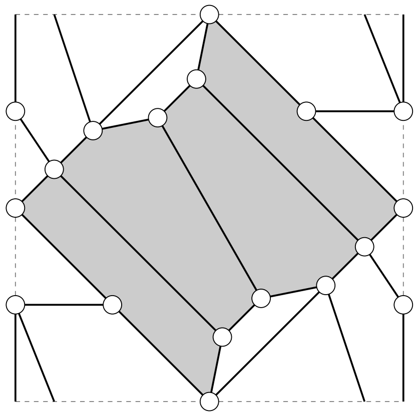
Consider a graph G drawn on torus. The *representativity* of G is the minimum number of intersections of G and l where l is a non-contractible simple closed curve on torus. Let $r(G)$ denote the representativity of G.

Lemma 27 (Mohar and Thomassen [2001])**.** *Let G be a graph and H a minor of G. Then the representativity of H is upper-bounded by the representativity of G.*

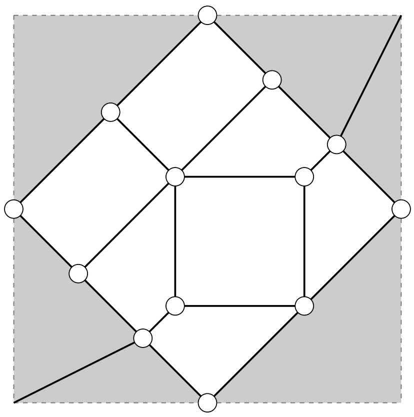
Corollary 28. *Let G be a triangle-free graph drawn on torus such that $r(G) \leq 1$. Then G is 3-colorable.*



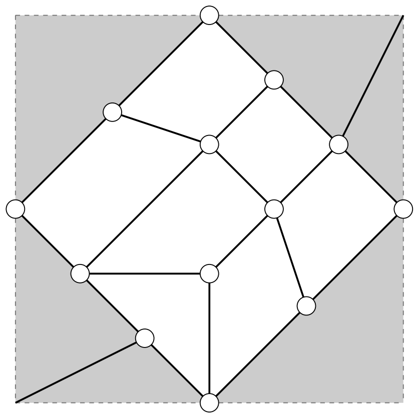
(a) Graph I_4



(b) Graph I_5



(c) Graph I_7^a



(d) Graph I_7^b

Figure 5.2: Irreducible 4-critical graphs drawn on torus

Proof. From Theorem 25 we may observe that all irreducible graphs have representativity at least two.

For contradiction, let G have no proper 3-coloring. Without loss of generality, we can assume that G is 4-critical. Then iterated reduction of G terminates in an irreducible graph. Thus the iterated reduction must have increased the representativity parameter in at least one of the steps.

Consider the case when $r(G) = 0$. Let l be a non-contractible simple closed curve with no intersection with the drawing of G. We cut the torus along l transforming it into a cylinder with no edges going over the boundary. Such drawing is equivalent to a planar drawing of G. Thus G is a planar triangle-free graph. This is in contradiction with Theorem 2. Thus we may assume that the representativity of G is exactly one.

Consider a reducible graph H drawn on torus such that $r(G) = 1$. Let l be a non-contractible simple closed curve intersecting the drawing of H exactly once. Then l is contained (up to the intersection point) within a face h of H, as l without its intersection with H is still a single curve. The existence of such intersection implies that at least one vertex appears twice on the facial walk of the face h.

Let f be a 4-face of H with a facial walk C. No vertex appears twice on C as otherwise G would contain either a loop or a parallel edge. We conclude that the curve l does not pass through f for any choice of f.

Let H' be an f-deflation of H identifying u and v from the boundary of f. Recall the definition of deflation via imaginary edge uv. Consider the graph $\bar{H}$ obtained by adding this edge to H. Clearly $r(\bar{H}) = r(H)$ as the curve l does not pass through f. By contraction of the edge uv in $\bar{H}$ we obtain a supergraph of H', thus H' is a minor of $\bar{H}$. Let H'' be a reduction of H. Since H'' is a subgraph of H' for a suitable H', we get that any reduction of H is a minor of $\bar{H}$. Together with Lemma 27 we get that $r(H'') \leq r(\bar{H}) = r(H)$. This is a contradiction with G reducing to an irreducible graph via iterated reduction. □

Corollary 29. *Let G be a triangle-free 4-critical graph drawn on torus. Then every face of G is bound by a cycle.*

Proof. Let f be a face of G and C its bounding walk. Suppose C is not a cycle, then there is an vertex v that repeats on C. Let us add two imaginary pendant edges e_1 and e_2 into the face f incident with v in such a way that in the clock-wise ordering of edges around v there is at least one edge between e_1 and e_2 and at least one edge between e_2 and e_1.

Consider a simple open curve l contained inside the face f with its ends touching the drawing of v from the same direction as the hypothetical drawings of e_1 and e_2 would. The curve l together with a drawing of v form a closed curve l' which intersects the drawing of G in exactly one point. According to Corollary 28 $r(G) \geq 1$ and so l' is contractible. This means that v is a vertex cut, which is a contradiction with G being 4-critical. □

6. Face lengths in k-critical graphs

In this section we apply the results of irreducible graph enumeration to characterize face lengths in triangle-free 4-critical graphs drawn on torus. As we already know, all such 4-critical graphs eventually reduce to irreducible graphs. Based on the explicit knowledge of all the irreducible graphs we may reverse the process of reduction to generate the whole class of triangle-free 4-critical graphs drawn on torus. We do not attempt to actually construct the 4-critical graphs, we focus on study of possible lengths of their faces.

In the section about deflations, we already implicitly dealt with structural changes during deflation and its inverse. The main issue we solve in this section is the inverse of taking of a 4-critical subgraph of a deflation. To reconstruct the original, and potentially much more complex graph, we return back to Lemma 4. Based on this lemma we know that once we have a suitable subgraph of a 4-critical graph, each of its 2-cell faces f can be filled with a C-critical planar graph, where C is the bounding cycle of the outer face of the planar graph of the same length as the boundary of f. Provided all faces we encounter are 2-cell faces, such operation would construct the original 4-critical graph. Before we formalize this operation, we need theoretical tools to deal with the necessary planar graphs.

A graph G embedded in the plane has one unbounded (outer) face; all other faces of G are *internal*. For a graph G embedded in the plane let $S(G)$ denote the multiset of lengths of internal (≥ 5)-faces of G and for graph G drawn on torus let $S(G)$ denote the multiset of lengths of all (≥ 5)-faces of G.

Let $\mathcal{G}_{g,k}$ denote the set of all plane graphs of girth at least g and with outer face formed by a cycle C of length k that are C-critical. Let $\mathcal{S}_{g,k}$ denote set $\{S(H) : H \in \mathcal{G}_{g,k}\}$.

We use a definition from the work of Dvořák et al. [2015a]. Let S_1 and S_2 be multisets of integers. We say that S_2 is a *one-step refinement* of S_1 if there exists $k \in S_1$ and a set $Z \in \mathcal{S}_{4,k} \cup \mathcal{S}_{4,k+2}$ such that $S_2 = (S_1 \setminus \{k\}) \cup Z$. We say that S_2 is a *refinement* of S_1 if it can be obtained from S_1 by a (possibly empty) sequence of one-step refinements.

Lemma 30 (Dvořák et al. [2015a]). *For every $k \geq 7$, each element of $\mathcal{S}_{4,k}$ other than $\{k-2\}$ is a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$. In particular, if $S \in \mathcal{S}_{4,k}$ then the maximum of S is at most $k-2$, and if the maximum is equal to $k-2$, then $S = \{k-2\}$.*

Theorem 31 (Thomassen [1994b]). *Let G be a planar graph of girth at least 5. Then G is 3-colorable. Furthermore, if G has an outer cycle C of length ≤ 9, then any 3-coloring of $G[V(C)]$ can be extended to a 3-coloring of G, unless C has length 9 and $G - C$ has a vertex joined to three vertices of C, which have three distinct colors.*

Lemma 32. *The following inclusions hold:*

$\mathcal{S}_{4,4} = \mathcal{S}_{4,5} = \emptyset$ *and* $\mathcal{S}_{4,6} \subseteq \{\emptyset\}$

$\mathcal{S}_{4,7} \subseteq \{\{5\}\}$

$\mathcal{S}_{4,8} \subseteq \{\{6\}, \{5,5\}, \emptyset\}$

$\mathcal{S}_{4,9} \subseteq \{\{7\}, \{6, 5\}, \{5, 5, 5\}, \{5\}\}$

Proof. Recall Theorem 3 which implies the first inclusion.

To obtain the remaining inclusions, we will use Lemma 30. First we reformulate the corollary of Theorem 31 as similar inclusions. Note that in the formulation of Theorem 31 a 3-coloring of $G[V(C)]$ refers to any 3-coloring of C respecting the restictions enforced by any chords of C. We consider all 3-colorings of C, and thus any cycle with a chord is also C-critical. We obtain the following inclusions:

$\mathcal{S}_{5,k} = \emptyset$ for $k \leq 7$

$\mathcal{S}_{5,8} \subseteq \{\{5, 5\}\}$

$\mathcal{S}_{5,9} \subseteq \{\{6, 5\}, \{5, 5, 5\}\}$

To derive the superset of $\mathcal{S}_{4,7}$, we take $\{5\}$ together with refinements of $\mathcal{S}_{4,5} \cup \mathcal{S}_{5,7}$. However this union is empty. Therefore, we get no refinements and the inclusion for $\mathcal{S}_{4,7}$ holds.

Similarly, for the superset of $\mathcal{S}_{4,8}$ we take $\{6\}$ together with refinements of $\{5, 5\}$ and $\emptyset$. Notice that a one-step refinement where $k = 5$ has $Z = \{5\}$ and therefore yields the original elements. From this we see that refinement creates no new elements.

Finally, for superset of $\mathcal{S}_{4,9}$ we consider $\{7\}$ together with all refinements of $\{6, 5\}, \{5, 5, 5\}$ and $\{5\}$. From the previous case we already know that it suffices to consider one-step refinements for $k = 6$, where Z is either $\{6\}, \{5, 5\}$ or $\emptyset$. Clearly we get no new elements and therefore we do not need to consider iteration of one-step refinements. □

6.1 Amplification

Let G be a reducible 4-critical triangle-free graph drawn on torus and H a reduction of G. We would like to design an inverse operation producing G from H. Recall that H is a 4-critical subgraph of a deflation H' of G.

Consider the following process applied to graph H. First we pick an arbitrary vertex $v' \in V(H)$. Then we consider all possible P_3 paths P such that v' is the middle vertex of P and $H \cup P$ is drawn on torus. While P can be subgraph of H, we also allow for paths introducing one or two new vertices and edges as long as these are placed into distinct faces of G. Finally we inflate P in graph $H \cup P$. We denote a set of all results of such process as $\lambda(H)$, a set of *partial amplifications* of H.

Consider a graph $H' \in \lambda(H)$. For some faces f of H' with boundary walk F we replace f with a C-critical triangle-free planar graph for C a cycle of the same length, identifying C with F. We denote set of all possible triangle-free 4-critical results of this process $\Lambda(H)$, a set of *amplifications* of H.

Lemma 33. *Let G be a triangle-free 4-critical graph drawn on torus and let $H \in \lambda(G)$. Then every face of H is bound by a cycle.*

Proof. Let $v_2v'v_4$ denote the path P_3 used to obtain H from G and let $v_1v_2v_3v_4$ be the inflated face f in H. The face f is bound by a cycle, as all if the vertices from its bounding path are distinct. Let us consider any face g of H, distinct from f.

For every choice of g and its facial walk C there exists a corresponding face g' in G bound by a facial walk C'. According to Corollary 29, C' is a cycle. If $C = C'$ then the lemma holds for g. We can therefore assume $C \neq C'$, which can only happen if $v' \in C'$.

First suppose that there were no new vertices added into the interior of g to form the path $v_2v'v_4$, then C is obtained from C' by substitution of v' by either v_1 or v_3 and as none of these two vertices exist in G, no vertices on C repeat and C is a cycle in H.

Now suppose that exactly v_2 was added into the interior of g (the case for v_4 is symmetrical). Then C is obtained from C' by substitution of v' by either a substring $v_1v_2v_3$ or $v_3v_2v_1$. Again, since none of the vertices v_1, v_2, v_3 exist in G and v' is in C' only once, we have that C is a cycle in H.

Together we get that every face in G' is bound by a cycle. □

Lemma 34. *Let G be a reducible graph drawn on torus and H its reduction. Then $G \in \Lambda(H)$.*

Proof. Let G' be a deflation of G such that H is its 4-critical subgraph.

Let v' be the vertex of G' which is a unification of two vertices from G, let us denote these two vertices v_1 and v_3. Clearly $v' \in V(H)$ as otherwise $H \subsetneq G$ which is a contradiction with both H and G being 4-critical. Let P be the P_3 subgraph of G' obtained by deflating a 4-face of G and let us denote its vertices $v_2v'v_4$. First suppose that $v_2, v_4 \notin V(H)$ and both are located in the interior of a single face g of H. Then $H \subsetneq G$, which is a contradiction. Otherwise the process of partial amplification constructs an inflation H' of $H \cup P$ such that $H' \subseteq G$.

Consider $H' \in \lambda(H)$ such that $H' \subseteq G$. Clearly H' is connected and all faces of H' are 2-cell faces, as this holds for H. From the Lemma 33 we also have, that each face of H' is bound by a cycle.

Consider each face f of H' that is not a face of G and let C denote the boundary of f. Let Δ be the closed disk bounded by C. Recall Lemma 4 stating that the subgraph of G drawn in Δ is C-critical. Therefore, for each such face f there exists a C-critical triangle-free planar graph such that it is subgraph of G drawn in Δ. We replace all such faces f with the respective planar graphs, concluding that $G \in \Lambda(H)$. □

We now focus on changes in the multiset of lengths of faces. To describe the effect of amplification, we define an amplification of the integer multiset and show its relevance to the graph amplification in the following lemma.

Let I be a multiset of integers. A multiset A is an *amplification* of I if there exists a partition $\{A_i : i \in I\}$ of A into multisets such that $A_i \in S_{4,i}$ for each $i \in I$. In this definition we understand the elements $i \in I$ in a labeled way, that is each instance of an integer has its own associated multiset A_i.

For an integer multiset I and an integer j we also define a multiset $I + j$ as $\{i + j | i \in I\}$.

Lemma 35. *Let H and G be 4-critical triangle-free graphs drawn on torus such that $G \in \Lambda(H)$. Then there exist integer multisets I, J, J', K, K' such that $S(H) = I \dot{\cup} J \dot{\cup} K$, and $S(G) = I \cup J' \cup K'$, where J' is an amplification of J, K' is an amplification of $K + 2$ and $|K| \leq 2$.*

Proof. Let $H' \in \lambda(H)$ such that $H' \subseteq G$. Then from definition of $\Lambda(H)$ we get the following relation between $S(H')$ and $S(G)$. There exist partitions $I \cup L$ of $S(H')$ and $I \cup L'$ of $S(G)$ where L' is an amplification of L. The multiset L corresponds to the lengths of faces that are replaced with critical triangle-free planar graph during the process of constructing graph amplification G from the partial amplification H'. Notice that all 4-faces are kept as 4-faces.

Let us consider the relation of $S(H)$ and $S(H')$. Let P be the P_3 subgraph such that H' is inflation of P in $H \cup P$. Each face of H such that no elements of P are inserted into it remains a face of H' as inflation does not alter faces other than the inflated 4-face. There are at most two faces with new inserted elements.

First let us consider faces that have an end vertex of P and a corresponding edge inserted into them. Each such face increases its length by two, this follows from the joint consequence of Lemma 33 and 7. We show that the elements of $S(H)$ with increased size in $S(H')$ correspond to faces in $S(H')$ which are replaced by non-trivial planar graphs in the second step of amplification. Consider the vertex v added into the interior of some face f. After inflation, this vertex has degree two. Since all vertices in G have degree at least three, one of the faces incident with v has to be replaced by a non-trivial planar graph. Since v is only incident with a 4-face and the face f, the face f has to be replaced.

The last class of faces are faces with edge forming a chord inserted into them. Let a cycle C bound such a face. This change can be simply understood as a replacement by a C-critical triangle-free-planar graph, as chord gives a trivial C-critical graph. The resulting sub-faces may or may not be replaced by a planar graph in the second part of the amplification. Either way, the cycle C remains intact in G, and therefore its interior must contain a C-critical graph in G. Therefore it is possible to describe the insertion of a chord and potential replacement of resulting faces as a single replacement of the original face.

To finish the proof we need to consider possible transformations of 4-faces, as these are hidden in $S(H)$. Clearly, if a 4-face of H' is not replaced with a non-trivial planar graph, it does not produce any new faces. On the other hand, there is no suitable non-trivial planar graph to replace a 4-face. The only remaining possibility is if a 4-face increases its length to six (due to an end-vertex of P being placed in its interior). Using the same argument as above, we get that such 6-face is replaced with a non-trivial planar graph, which is always a quadrangulation and thus contains only more 4-faces that are not expressed in $S(G)$. Thus value of $S(G)$ is not affected by 4-faces in H.

The multiset I from the formulation of this lemma I corresponds to the lengths of common faces of H and H' that are kept in G. Multiset J corresponds to lengths of faces of H which are replaced with non-trivial planar graphs. These faces come either from the set L, if they are replaced in the second step of amplification, or come from the two faces with extra elements inserted during the first part of the amplification. The multiset K corresponds to lengths of the at most two faces of H which have end-vertices of P inserted into them. □

We are now ready to prove the final result, a description of lengths of faces of all triangle-free 4-critical graphs drawn on torus, up to the number of their 4-faces.

Theorem 36. *Every 4-critical triangle-free graph G drawn on torus satisfies one of the following properties:*

1) $S(G) = \{7, 5\}$
2) $S(G) = \{6, 5, 5\}$
3) $S(G) = \{5, 5, 5, 5\}$
4) $S(G) = \{5, 5\}$
5) $S(G) = \emptyset$ *and* $G = I_4$ *(see Figure 5.2.)*

Proof. For each 4-critical triangle-free graph G drawn on torus we have an irreducible graph H such that H is a result of iterated reduction of G. Then clearly iterated amplification of H constructs G. Also, from Theorem 25 we know that $S(H)$ has one of the values $\emptyset, \{5, 5, 5, 5\}$ or $\{7, 5\}$.

Consider application of Lemma 35 to the multiset $S(F)$ for some graph F. We adopt the same notation of multisets I, J, K. We will understand amplification of F into $F' \in \Lambda(F)$ as a transformation of $S(F)$ into $S(F')$. Notice that amplification replaces each value of $S(F)$ with a multiset of values. The possible values of such multiset are determined only by the partition of elements of $S(F)$ into multisets I, J, K. Note that we do not consider the condition $|K| \leq 2$.

Consider value 5 in $S(F)$. Notice that 5 cannot be an element of J as there is no suitable non-trivial planar graph ($\mathcal{S}_{4,5} = \emptyset$). If 5 is an element of K, then amplification replaces it with one of the elements of $\mathcal{S}_{4,7}$, however there is only one such element, namely $\{5\}$. Together we get that value 5 can only remain as value five and thus never decomposes into any multiset (including empty multiset).

Analogously, consider value 6 in $S(F)$. If 6 is an element of J, it can only be replaced by $\emptyset$. If 6 is an element of K, it can only be replaced by elements $\{6\}$ or $\{5, 5\}$. Together we get that value 6 either remains unchanged, decomposes into two values 5, or decomposes into no values.

Finally, consider value 7 in $S(F)$. If 7 is an element of J, it can only be replaces by $\{5\}$. If 7 is an element of K, it can either remain unchanged or be replaced with elements $\{6, 5\}, \{5, 5, 5\}$ or $\{5\}$.

Now we will apply iterated amplification to the specific possible values of $S(H)$ to obtain all possible values for $S(G)$.

If $S(H) = \{7, 5\}$ then only the value 7 can decompose, namely into into $\{6, 5\}, \{5, 5, 5\}$ or $\{5\}$. Further amplifications can then only decompose value six into two further values 5. Together we get possible values for $S(G)$ to be $\{7, 5\}, \{6, 5, 5\}, \{5, 5, 5, 5\}$ and $\{5, 5\}$.

Similarly, if $S(H) = \{5, 5, 5, 5\}$ then $S(G) = \{5, 5, 5, 5\}$ as values 5 do not decompose.

Finally, if $S(H) = \emptyset$, then clearly also $S(G) = \emptyset$. Furthermore, from the work of Král' and Thomas [2008] we know that the only even-faced triangle-free 4-critical graph drawn on torus is exactly I_4. $\square$

Conclusion

We introduced a framework of studying the class of triangle-free graphs embedded in the torus that are not 3-colorable. Our method is an inductive argument on the subclass of graphs that are 4-critical. The basis of our induction is formed by the *irreducible* graphs and the inductive step is the *reduction* operation, resp. its inverse operation *amplification*.

We demonstrate that each 4-critical triangle-free graph embedded in the torus must satisfy several properties. Particularly, the multiset of lengths of its (≥ 5)-faces must take one of only five possible values. On the other hand we demonstrate that every triangle-free graph embedded in the torus with representativity at most one or with at most six 4-faces is always 3-colorable.

To achieve our results we used a computer-assisted enumeration of the *irreducible* graphs, which completely characterizes all basis of the induction argument. The inductive step, which is mainly based on the analysis of *amplification*, allows further refinement of our result. Our analysis focuses only on the development of certain properties while neglecting the fine structure of individual graphs. Such depth of analysis seems unattainable without further computer-assisted enumeration. In future work, we intent to extend our result in this direction in order to provide a structural description of all possible 4-critical triangle-free graphs embedded in the torus.

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List of Figures

Attachments

1. Implementation of enumeration algorithm from section 5.4