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MASTER THESIS

Jan Bok

Structural Aspects of Graph Homomorphisms

Computer Science Institute of Charles University

Supervisor of the master thesis: prof. RNDr. Jaroslav Nešetřil, DrSc.
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Zásady pro vypracování:

Student se bude zabývat strukturálními otázkami teorie grafových homomoprfismů, mimo jiné homomorfismů grafů do množiny celých čísel.

Seznam odborné literatury:

Hell, Pavol, and Jaroslav Nešetril. "Graphs and homomorphisms, volume 28 of Oxford Lecture Series in Mathematics and its Applications." (2004).
Godsil, Chris, and Gordon F. Royle. Algebraic graph theory. Vol. 207. Springer Science & Business Media, 2013.
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Title: Structural Aspects of Graph Homomorphisms

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Keywords: graph theory, homomorphisms, lattice path enumeration, random walks, Lipschitz mappings

Abstract:

This thesis is about graph-indexed random walks, Lipschitz mappings and graph homomorphisms. It discusses connections between these notions, surveys the existing results, and shows new results.

Graph homomorphism is an adjacency-preserving mapping between two graphs. Our main objects of study are graph homomorphisms to an infinite path. We are interested in two parameters: maximum range and average range. The average range of a graph is the expected size of the image of a uniformly picked random homomorphism to an infinite path. We obtain formulas for several graph classes and investigate main conjectures on this parameter. For maximum range parameter we show a general formula and an algorithm to compute it for general graphs. Besides that, we study the problem of extending a prescribed partial graph homomorphism to a full graph homomorphism. We show that this problem is polynomial in some cases.

Název: Structural Aspects of Graph Homomorphisms

Autor: Jan Bok

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Supervisor: prof. RNDr. Jaroslav Nešetřil, DrSc., Informatický ústav Univerzity Karlovy

Klíčová slova: teorie grafů, homomorfismy grafů, enumerace cest v mřížce, náhodné procházky, Lipschitzovská zobrazení

Abstrakt:

Tato diplomová práce se zabývá náhodnými procházkami, Lipschitzovskými zobrazeními a grafovými homomorfismy. Diskutujeme propojení těchto pojmů, dáváme přehled dosavadních výsledků a ukazujeme nové výsledky.

Grafový homomorfismus je zobrazení mezi dvěma grafy zachovávající sousednost. Hlavním předmětem zkoumání jsou pro nás homomorfismy grafů do nekonečných cest. Konkrétně nás zajímají dva parametry: maximální rozsah a průměrný rozsah. Průměrný rozsah grafu je očekávaná velikost obrazu uniformně a náhodně zvoleného homomorfismu do nekonečné cesty. Ukazujeme, jak odvodit vztahy pro výpočet průměrného rozsahu na různých třídách grafů a zabýváme se hlavními hypotézami, které se týkají tohoto parametru. Pro maximální rozsah ukazujeme přesný vztah a způsob výpočtu na obecném grafu. Kromě toho studujeme problém rozšiřování částečného homomorfismu, kde ukážeme jeho polynomialitu pro některé případy.

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Contents

Chapter 1

Introduction

This thesis is about graph homomorphisms, Lipschitz mappings, random walks and about their interplay.

We are mostly interested in random homomorphisms into an infinite path graph and some other closely related graphs. Roughly speaking, our main focus is on answering the question of how big is the homomorphic image of a uniformly at random chosen homomorphism to an infinite path.

Our secondary goal is to understand how can we effectively decide if it is possible to extend some partial mapping to a full homomorphism to an infinite path (or to related graphs).

The area covered in this thesis falls into algebraic graph theory, the branch of mathematics studying combinatorial structures known as graphs by using the theory of both abstract and linear algebra. More specifically, this thesis fits into the study of graph homomorphisms. For more background on algebraic graph theory in general, we invite reader to check [21]; for graph homomorphisms, see [24].

We will cover essential preliminaries in later chapter; this thesis should be self-contained and expects only the basic knowledge beforehand. The main aim of this chapter is to informally introduce our work in broad context, leaving the needed formalities and mathematical formulations to the next chapter.

1.1 Prolog

Many processes in real life, for example stock movement or motion of particles, are so complex that it is impossible for us to observe, record and evaluate all the effects that influence their behavior and future state. In these cases it is often convenient to somehow approximate the reality by more simple models.

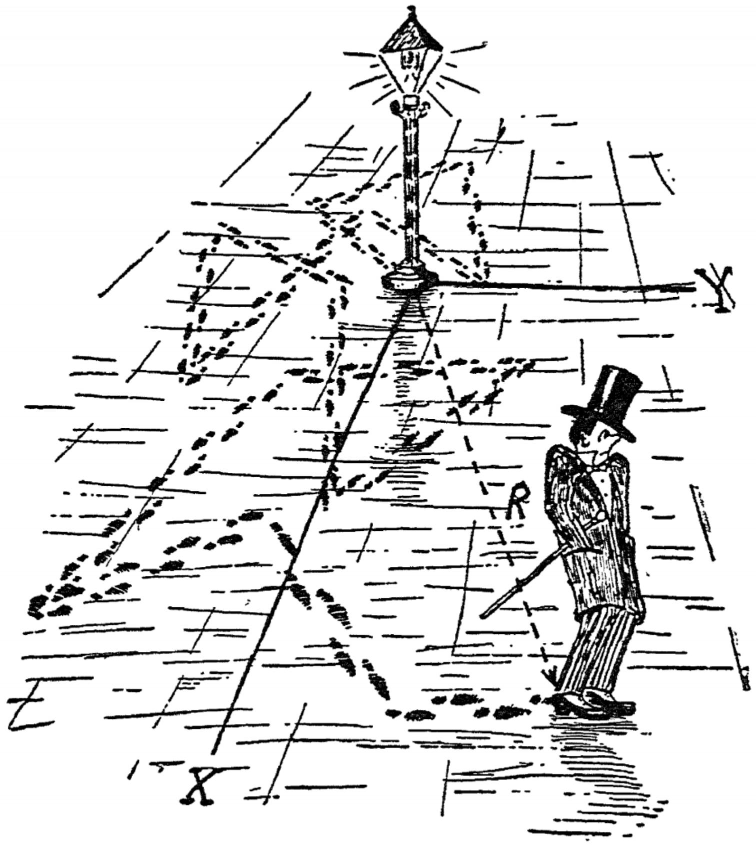

Figure 1.1: Drunkard's walk [20, Chapter VIII].

For example, let us talk about a path traced by a molecule in some fluid. The number of variables affecting the trajectory of such path is often gigantic and contemporary technology is incapable of simulating this process. However, to observer, the movement of such molecule seems to be completely random. That is so called *Brownian motion*. See Figure 1.2 for an illustration. This phenomenon of randomness allows us to analyze these motions by stochastic methods. Note that Brownian motion is the process continuous in time.

Informally, random walk is a discretization of Brownian motion. We have some underlying discrete space and a set of possible steps. In time t, one of the possible steps is taken uniformly at random. The sequence of n consecutive steps is then a random walk.

Consider the classical example, explained e.g. in [59]. A drunken man is standing in the middle of a city square. Every minute he picks a random direction (north, south, east or west) and walks one meter forward. This is also a random walk, this time on a discrete space isomorphic to $\mathbb{Z}^2$.

If we would simplify this example by restricting the drunkard to take only west or east direction in every step, we would get a random walk on $\mathbb{Z}$. See Figure 1.3 for a visualization of 100 steps of eight random walks on $\mathbb{Z}$.

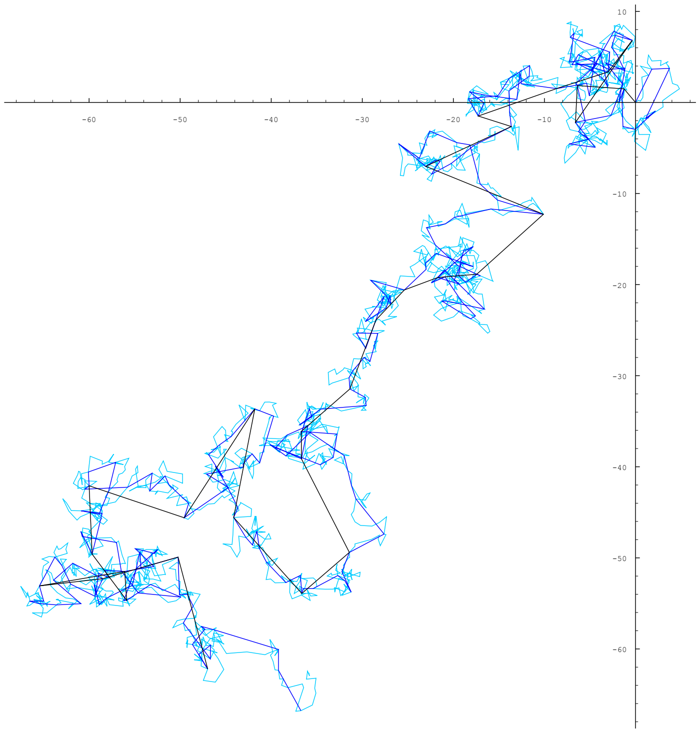

Figure 1.2: An example of Brownian motion [58].

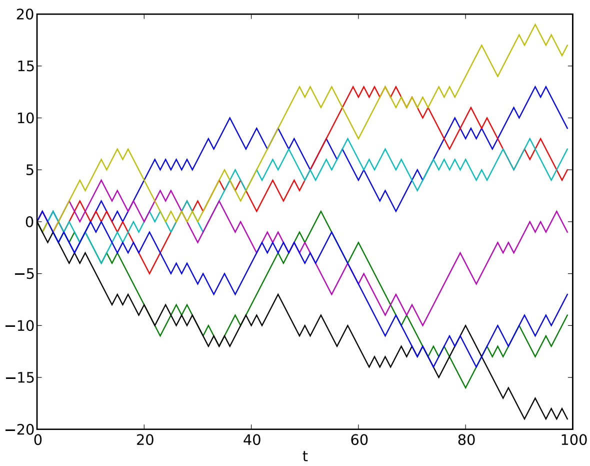

Figure 1.3: An example of eight one-dimensional random walks with 100 steps. The x-axis is time, the y-axis is the position on integers [59].

In this thesis, we are interested in a special kind of *graph homomorphisms* – homomorphisms of graphs into integers. We postpone the formal definitions to Section 2.3. Informally, we will assign integers to vertices of graph in such a way that every two adjacent vertices get assigned integers which are in distance at most M. See Figure 1.4 for an initial example. One can quickly check that the numbers assigned to vertices satisfy this condition for $M = 1$.

This definition, maybe not apparently on the first sight, entails many notions and concepts. We will show that this type of mappings – so called *M-Lipschitz mappings of graphs* – generalize the concept of random walk on $\mathbb{Z}$ we just saw. We will further show a connection with certain problems in statistical physics. We will explore motivations and connections more thoroughly in Chapter 2.

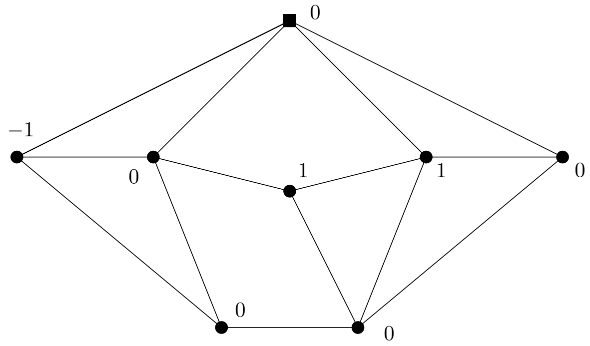

Figure 1.4: An example of a homomorphism of the Their graph [45] into integers. The images of the endpoints of every edge are in distance at most one.

1.2 Structure this thesis

The structure of this thesis is as follows:

- Chapter 2 is about preliminaries. It introduces the main definitions, shows their relations, motivation and some basic properties as well.
- Chapter 3 is a short chapter considering the problem of the maximum range (the size of the homomorphic image) among all M-Lipschitz mappings of a given graph.
- Chapter 4 is devoted to the parameter of the average range and it is the main chapter of this thesis containing our main results. Namely, we prove precise formulas for this parameter for the class of complete graphs, complete bipartite graphs, paths and cycles. We will also survey the most important conjectures regarding this parameter.
- In Chapter 5, we introduce a new algorithmic problem regarding the extension of some special-type prescribed homomorphism. We give a context to other existing extension problems and we show that some cases of this new problem can be solved in polynomial time.
- Chapter 6 concludes the thesis with a brief summary of our contribution and a list of open problems.

Figure 1.5 shows the structure of essential interdependence between chapters:

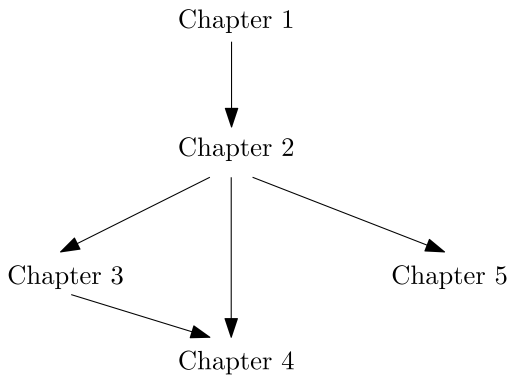

Figure 1.5: The graph of the structure of this thesis.

Chapter 2

Preliminaries

This chapter introduces a basic terminology and presents needed background on the theory of graph homomorphisms. Besides that, basic definitions regarding Lipschitz mappings and their basic properties are discussed. We also show more detailed information on motivations.

2.1 Basic graph terminology and notation

Contrary to the majority of works in graph theory, we will need a more general definition of graph. This section is similar to the introduction of the book by Hell and Nešetřil [24].

Our main definition is the definition of *digraph*. Digraph G is a pair of vertex set $V = V(G)$ and edge set $E = E(G)$ which is a binary relation on V. In our case, V can be infinite. We call digraph *symmetric, reflexive*, or *irreflexive* if its edge set has the corresponding property.

We define *loop* as an edge $(v, v) \in E, v \in V$. Therefore, reflexive digraphs are those with loops in each vertex. We call symmetric digraphs as *undirected graphs* and undirected graphs without loops, i.e. irreflexive undirected graphs, as *simple graphs*. We denote all n-vertex simple graphs by $\mathcal{G}_n$. See Figure 2.1 for inclusion diagram of subclasses of digraphs.

For graph $G = (V, E)$, the cardinality of V is called the *order* of G and the cardinality of E is called the *size* of G. As for other notation, we refer to Diestel's monograph [14].

To avoid cumbersome notation, we will often write uv for undirected edge.

Note. As for the notation specific to Lipschitz mappings of graphs, we decided to introduce our own notation since it was hard to decide which of the many existing notations we should use. We hope that the result is more clear, logical and unified notation than those in the existing literature.

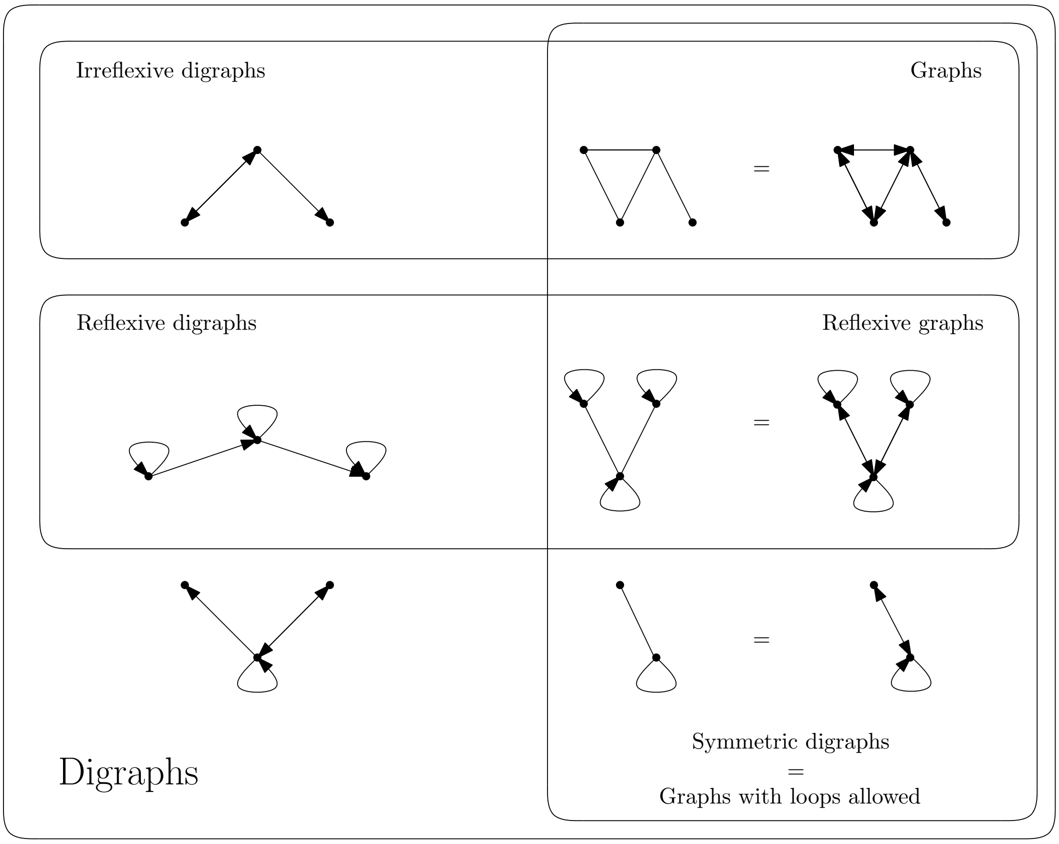

Figure 2.1: Main subclasses of digraphs. [24]

2.2 Graph homomorphisms

We define a *graph homomorphism* between digraphs G and H as a mapping $f : V(G) \to V(H)$ such that for every edge $uv \in E(G)$, $f(u)f(v) \in E(H)$. That means that graph homomorphism is an adjacency-preserving mapping between the vertex sets of two digraphs. The set $I := \{w \in V(H) \mid \exists v \in V(G) : f(v) = w\}$ for a graph homomorphism f is called the *homomorphic image* of f.

A function f is a *vertex-surjective* homomorphism if it is surjective, the same goes for *vertex-injective* homomorphisms. By the property of preserving adjacency, we get for each graph homomorphism f a mapping $f^* : E(G) \to E(H)$ by setting $f^*(uv) := f(u)f(v)$. We are then able to define *edge-surjective* and *edge-injective* homomorphisms analogously. Bijective variants can be defined in the same way as well.

By *graph endomorphism*, we mean a graph homomorphism of G to itself. By *graph isomorphism*, we mean a graph homomorphism which is bijective (both edge-bijective and vertex-bijective).

For a comprehensive and more complete source on graph homomorphisms, the reader is invited to see [24]. A quick introduction is given in [21] as well.

2.3 Main definition: Lipschitz mapping of a graph

Definition 2.1. For $M \in \mathbb{N}$, an *M-Lipschitz mapping* of a connected graph $G = (V, E)$ with root $v_0 \in V$ is a mapping $f : V \to \mathbb{Z}$ such that $f(v_0) = 0$ and for every edge $(u, v) \in E$ it holds that $|f(u) - f(v)| \leq M$. The set of all M-Lipschitz mappings of a graph G is denoted by $\mathcal{L}_M(G)$.

By the term *Lipschitz mappings of graph* we mean the union of sets of M-Lipschitz mappings for every $M \in N$.

The importance of having rooted graphs is the following. We want to have finitely many Lipschitz mappings for a fixed graph G. Mappings with $f(v_0) \neq 0$ are just linear shifts of some mapping with $f(v_0) = 0$. Formally, consider a mapping f' with $f'(v_0) = a$. Then we can define a linear transformation T_a as $T_a(f) \longmapsto f - a$. Applying T_a to f' yields a Lipschitz mapping of G with v_0 as its root.

We note that we are interested in connected graphs only. Components without the root would also allow infinitely many new 1-Lipschitz mappings. Consider this example.

Example. Let H denote the graph with the vertex set $\{a, b, c\}$ and the edge set $\{bc\}$. Let us root H in a. Assuming that we would extend Definition 2.1 to disconnected graphs, all the 1-Lipschitz mappings of H would form the following, clearly infinite, set:

$$\Big\{\{(a, 0), (b, x), (c, y)\} \,|\, |x - y| \leq 1, x, y \in \mathbb{Z}\Big\}.$$

In literature, we will often meet a slightly different definition of 1-Lipschitz mappings. In it the restriction $|f(u) - f(v)| \leq 1$, for all $uv \in E$, is removed and instead, the restriction $|f(u) - f(v)| = 1$, for all $uv \in E$, is added. In [38] authors call these mappings *strong Lipschitz mappings*. We generalize this in the following definition.

Definition 2.2. For $M \in \mathbb{N}$, a *strong M-Lipschitz mapping* of a connected graph $G = (V, E)$ with root $v_0 \in V$ is a mapping $f : V \to \mathbb{Z}$ such that $f(v_0) = 0$ and for every edge $(u, v) \in E$ it holds that $|f(u) - f(v)| = M$. The set of all M-Lipschitz mappings of a graph G is denoted by $\mathcal{L}_{\pm M}(G)$.

Note that strong M-Lipschitz mappings are a special case of M-Lipschitz mappings of graph. Also, B-Lipschitz mappings are a superset of A-Lipschitz mapping whenever $B \geq A$. See Figure 2.2 for the Hasse diagram of various types of Lipschitz mappings.

Analogously, by the term *strong Lipschitz mappings of graph* we mean the union of sets of strong M-Lipschitz mappings for every $M \in N$.

The following two lemmata shed some light on the definition of strong Lipschitz mappings.

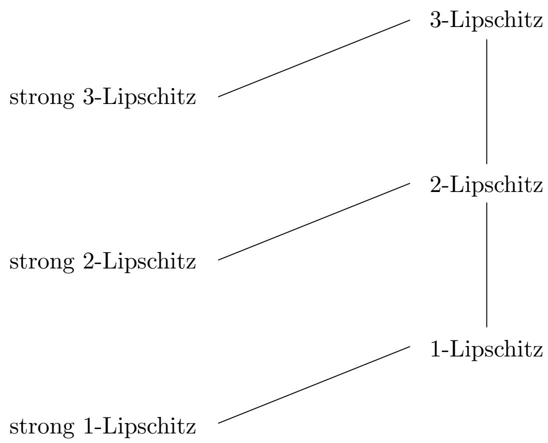

Figure 2.2: The Hasse diagram of different types of Lipschitz mappings of graphs.

Lemma 2.1. *A graph has a strong M-Lipschitz mapping if and only if it is bipartite.*

Proof. First of all, observe the fact that in strong M-Lipschitz mapping, all vertices are mapped to some number of the form $k \cdot M$, where $k \in \mathbb{Z}$.

Consider a non-bipartite graph G and a strong M-Lipschitz mapping f of G. There is a well-known characterization of bipartite graphs:

A graph is bipartite if and only if it does not contain
a cycle of odd length as a subgraph.

Therefore, G contains some odd cycle C with edges $v_1v_2, \ldots, v_lv_1$. Let us denote

$$e_i := v_{(i+1 \bmod l)} - v_{(i \bmod l)}, \forall i \in \{1, \ldots, n\}.$$

We see that $e_i \in \{+M, -M\}$. Moreover, $\sum_{i=0}^{n} e_i = 0$ from the definition of e_i. However, this sum has an odd number of summands and thus we get a contradiction. □

Lemma 2.2. *For every graph G there exists a bijection between $\mathcal{L}_{\pm A}(G)$ and $\mathcal{L}_{\pm B}(G)$ for every $A, B \geq 1$.*

Proof. For fixed $A, B \in \mathbb{N}$, let us define the function $F_{A\to B} : \mathcal{L}_{\pm A}(G) \to \mathcal{L}_{\pm B}(G)$:

$$F_{A\to B}(f_A) := B \cdot A^{-1} \cdot f_A.$$

We have to show that this is the desired bijection.

Injectivity follows from the existence of the inverse function. In our case we have the inverse function $F_{B\to A}(f_B) := A \cdot B^{-1} \cdot f_B$. Furthermore, surjectivity is clear since $F_{B\to A}$ is defined for all $f_B \in \mathcal{L}_{\pm B}(G)$. $\square$

2.4 Motivation and different views

Homomorphism to $\mathbb{Z}$ and to an infinite path

M-Lipschitz mappings map graph vertices to integers. There is a natural bijection between M-Lipschitz mappings and graph homomorphisms to a suitable graph associated with $\mathbb{Z}$. Consider a graph Z_M with the vertex set $V(Z_M) = \{v_i | i \in \mathbb{Z}\}$ and and the edge set $E(Z_M) = \{v_i v_j : |i - j| \leq M\}$. Every M-Lipschitz mapping corresponds to a graph homomorphism to Z_M.

We can define a graph $Z_{\pm M}$ analogously for strong mappings.

Note that in the case of strong 1-Lipschitz mappings, we get that they correspond to homomorphisms to an infinite path and in the case of 1-Lipschitz mappings, we get a correspondence to homomorphisms to an infinite path with loops added to each vertex (Figure 2.5). See Figure 2.3 for an example of such homomorphism of C_4.

Note. We emphasize that one can generalize the definition of Lipschitz mapping to map not into $\mathbb{Z}$ but into $\mathbb{Z}^d$ for some natural number d instead. This was studied too, for example in [16]. In Figure 2.4 we have an example of a generalized Lipschitz mapping into $\mathbb{Z}^2$.

Gas models and physical motivation

As is noted by Zhao [62] and Cohen et al. [9], a homomorphism from G to P_3 with loops added to each vertex corresponds to a partial (not necessarily proper) coloring of the vertices of G with red or blue, allowing vertices to be left "uncolored" such that no red vertex is adjacent to a blue vertex. This coloring is known as the *Widom–Rowlinson configuration* [57] and this target graph is denoted as H_{WR}. Observe that Widom-Rowlinson configuration corresponds to some 1-Lipschitz mapping with the size of the homomorphic image at most 3. See Figure 2.6 for an example.

Widom-Rowlinson configurations have a physical interpretation. Consider particles of a gas B (blue vertices) and of a gas R (red vertices). W-R configurations

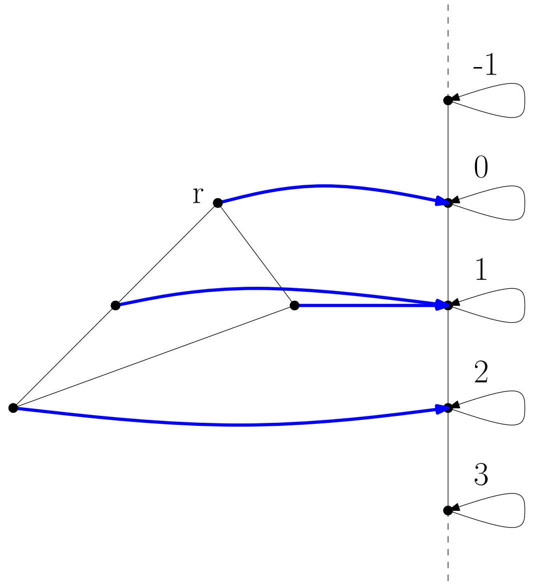

Figure 2.3: A homomorphism of C_4 rooted in r to Z_1 graph. In fact, this homomorphism is a 1-Lipschitz mapping of C_4.

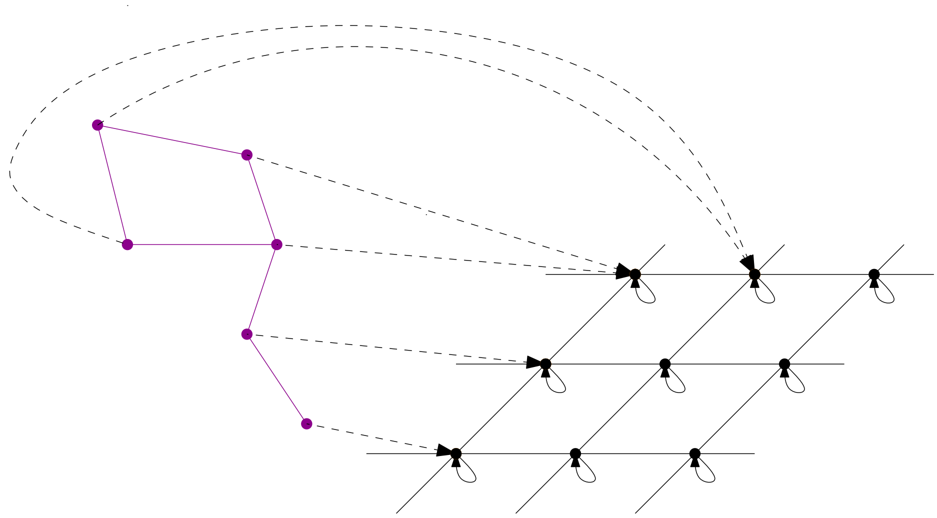

Figure 2.4: A homomorphism of a graph into $\mathbb{Z}^2$ grid.

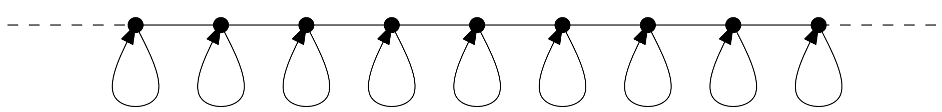

Figure 2.5: An infinite path with loops – Z_1

then model situations in which particles of gases A and B do not interact. This model is sometimes referred to as the *hard-core model*. The name emphasizes the hard restriction that particles of gases cannot be directly adjacent, i.e. their molecules do not interact.

For more on Widom-Rowlinson configurations and on the connection of graph homomorphisms to phase transition models, see [6] and [7] respectively.

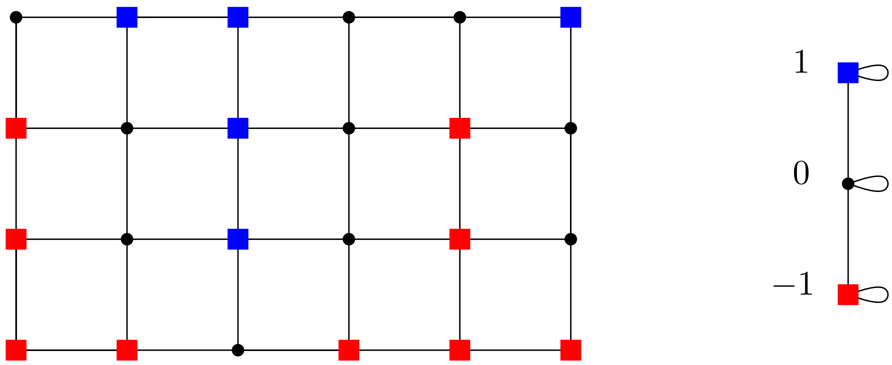

Figure 2.6: An example of Widom–Rowlinson configuration on a grid. On the right is the target graph H_{WR}.

Random walks

We emphasize the importance of this section as it gives a valuable context to this thesis. There is yet another view on Lipschitz mappings of graphs. We encountered this view in the introductory chapter of this thesis.

We will first introduce a formal definition of random walks on $\mathbb{Z}$.

Definition 2.3 (Simple random walk in $\mathbb{Z}$)**.** A *simple random walk (on $\mathbb{Z}$)* of n steps is a sequence $X_1, X_2, \ldots, X_n$ of random variables, each attaining values from the set $\{-1, 0, 1\}$. We further assume that these random variables are uniformly distributed and independent. The *displacement* of the walk after i steps is defined as

$$S_i := \sum_{s=1}^{i} X_s.$$

In this definition, X_s represents the movement on $\mathbb{Z}$ from time $(s-1)$ to s. We assume that the walk starts at time 0 at 0.

We note that every simple random walk X on $\mathbb{Z}$ induces a lattice path s_n defined as $s_0 := (0, 0)$ and $s_i := (i, S_i)$ for $i \in \{1, \ldots, n\}$.

The bijection with 1-Lipschitz mappings of a path graph P_n should be clear now. However, for completeness, we will explain it in more detail. Consider a path graph P_n and a 1-Lipschitz mapping f of G. Denote the vertices of P_n as $v_1, v_2, \ldots, v_n$ consecutively. Every difference $f(v_i) - f(v_{i-1})$, for $i \in \{2, \ldots, n\}$, corresponds to a change (or a step) in the random walk.

We will occasionally use the term *G-indexed random walk* for M-Lipschitz mappings of G. The term is used in such way in other paper as well, e.g. in [29, 19].

This term expresses that 1-Lipschitz mappings are a generalization of simple random walks on $\mathbb{Z}$ with simple random walks being the P_n-indexed random walks.

Finally, see Figure 2.7 for an illustration.

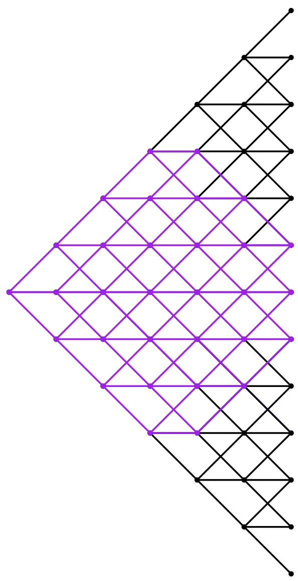

Figure 2.7: A visualization of all C_7-indexed walks and all P_7-indexed walks. Lattice paths (in the sense of Section 2.4) that use only the purple edges correspond to all C_7-indexed walks. Those that use both purple and black edges are the P_7-indexed walks.

For more information regarding random walks, see Lovász's seminal survey [39] or, for more general treatment, Häggström's comprehensive monograph on Markov chains [23], or more up-to-date monograph [37].

2.5 Number of Lipschitz mappings

Although examining the number of Lipschitz mappings of a given graph is not the main problem of this thesis, it is a problem of independent interest. More generally, counting the number of homomorphisms between two graphs is natural and widely studied question. Let us mention the quickly evolving theory of graph limits [40].

We show some basic properties regarding the number of 1-Lipschitz mappings of a given graph.

Theorem 2.1. *For every connected graph $G = (V, E)$ and for every two vertices $a, b \in V$ such that $ab \notin E$, holds that*

$$\mathcal{L}_1(G) \geq \mathcal{L}_1(G \cup \{a, b\}).$$

We will postpone the proof of this theorem to Chapter 3.

Lemma 2.3. *For every tree $T \in \mathcal{G}_n$, the number of M-Lipschitz mappings of T is equal to $(2M+1)^{n-1}$. As a special case, the number of 1-Lipschitz mappings of T is 3^{n-1}.*

Proof. The root r of T is allways mapped to zero. Perform a breadth-first search on T rooted in r. For each visited vertex other than r, only its parent vertex has a value. Thus we can choose any of the $2M+1$ integers at distance at most M from the parent vertex value. We proceed inductively. □

We can state the following corollary on the graphs with extremal numbers of 1-Lipschitz mappings.

Corollary 2.1. *Among connected graphs of order n, trees have the maximum number of 1-Lipschitz mappings and a complete graph K_n has the minimum number of 1-Lipschitz mappings.*

Finally, we remark that for strong Lipschitz mappings on trees, the very similar claim to Lemma 2.3 holds.

Remark 2.1. For every tree $T \in \mathcal{G}_n$, the number of strong M-Lipschitz mappings is equal to 2^{n-1}.

2.6 Maximum and average range

First of all, let us define the *range* of a mapping.

Definition 2.4. The *range* of a Lipschitz mapping f of G is the size of the homomorphic image of f. Formally:

$$\mathrm{r}_G(f) := |\{z \in \mathbb{Z} \mid z = f(v) \text{ for some } v \in V(G)\}| .$$

Remark 2.2. The homomorphic image of a connected subgraph under some 1-Lipschitz mapping is a closed interval in $\mathbb{Z}$. The image is not necessarily closed interval if we would have a graph with more than two components, but as we said, we are interested only in connected graphs.

We define the *average range* of graph G as follows.

Definition 2.5. *(Average range)* The *average range* of graph G over all M-Lipschitz mappings is defined as

$$\bar{\mathrm{r}}_M(G) := \frac{\sum_{f \in \mathcal{L}(G)} \mathrm{r}(f)}{|\mathcal{L}(G)|}.$$

We can view this quantity as the expected size of the homomorphic image of an uniformly picked random M-Lipschitz mapping of G.

Definition 2.6. *(Maximum range)* The *maximum range* of graph G is defined as

$$\mathrm{r}^{\max}(G) = \max_{f \in \mathcal{L}(G)} \mathrm{r}(f).$$

The average range is the main theme of Chapter 4 and we devote Chapter 3 to the maximum range.

Whenever we want to talk about the counterparts of these definitions for strong Lipschitz mappings, we denote it with $\pm$ in subscript. For example, $\bar{\mathrm{r}}_{\pm M}$ is the average range of strong M-Lipschitz mapping of graph.

Whenever we write average range or maximum range without saying which M-Lipschitz mappings we use, it should be clear from the context what M do we mean.

It is worth noting that for maximum range and average range, the choice of root does not matter. That is why we often omit the details of a picking one. For better analysis in proofs we occasionally pick a root in some convenient way.

Lemma 2.4. *For a connected graph G rooted in r_1, (G, r_1), and G rooted in r_2, (G, r_2), the following holds:*

$$\bar{\mathrm{r}}_M((G, r_1)) = \bar{\mathrm{r}}_M((G, r_2)),$$

and

$$\mathrm{r}_M^{\max}((G, r_1)) = \mathrm{r}_M^{\max}((G, r_2)).$$

In other words, maximum range and average range are invariant under the choice of root.

Proof. Let us define the function $R : \mathcal{L}_M((G, r_1)) \to \mathcal{L}_M((G, r_2))$ as

$$R(f) := r_2 - r_1 + f.$$

It suffices to observe that R is a *range-preserving function*, i.e.

$$\mathrm{r}(f) = \mathrm{r}(R(f)).$$

That implies the desired conclusion. $\square$

Analogous lemma for strong M-Lipschitz mappings is true as well.

Height vs. range

In literature, *height* parameter is often studied; for example in [61, 2].

Definition 2.7. The *height* for the graph G is the function $\mathrm{h}_G : \mathcal{L}(G) \to \mathbb{R}$ defined as

$$\mathrm{h}_G(f) := \max_{u \in V(G)} f(u) - \min_{v \in V(G)} f(v).$$

We note that it is important to distinguish between the size of the homomorphic image, i.e. the range, and the height of a Lipschitz mapping. Observe that the following relation holds:

$$\mathrm{h}_G(f) = \mathrm{r}_G(f) - 1. \tag{2.1}$$

Chapter 3

Maximum range

In preliminaries we introduced the notion of *maximum range* (Definition 2.6). We will show how can we algorithmically compute it. Also, we will show the relation of this parameter to other existing ones. This shorter chapter also provides results which will be useful in analyzing average range in Chapter 4.

3.1 Diameter

In this section we observe one important fact giving us an upper bound on the range of a graph. Then we will show that this upper bound is tight. We remind the reader the graph parameter called *diameter*.

Definition 3.1. The *diameter* $\mathrm{diam}(G)$ of a graph G is the maximum distance between any two vertices of G. We define the *distance* of two vertices $u, v \in V(G)$, $d(u, v)$, as the length of the shortest path between these two vertices. The distance of vertex to itself is defined as zero. The distance between two vertices from different components is defined as ∞.

We will first prove an important, yet easy lemma.

Lemma 3.1. *For any connected graph G with* $\mathrm{diam}(G)$ *and every M-Lipschitz mapping f of G, holds that*

$$\mathrm{r}(f) \leq M \cdot (\mathrm{diam}(G) + 1).$$

Proof. The existence of f with $\mathrm{r}(f) > M \cdot (\mathrm{diam}(G)+1)$ would imply the existence of a path subgraph in G with endpoints u, v such that their images would be in distance $|f(u) - f(v)| = M \cdot (\mathrm{diam}(G) + 1)$. However, that would mean that all paths between u and v have to map to some connected subgraph of $\mathbb{Z}$ of size greater than $M \cdot (\mathrm{diam}(G) + 1)$ which is a contradiction with the definition of diameter. $\square$

Now we show that we can always construct a mapping where equality holds and thus we conclude that the diameter and the maximum range are tightly connected.

Theorem 3.1. *For any connected graph G, $\mathrm{r}_M^{\max}(G) = M \cdot (\mathrm{diam}(G) + 1)$.*

Proof. From the definition of the diameter, there must exist vertices u_1 and u_2 such that their distance is equal to $\mathrm{diam}(G)$. Without loss of generality we set $r := u_1$. Now let us define the mapping $f : V(G) \to \mathbb{Z}$ so that for every $v \in V$ we have $f(v) := M \cdot d(r, v)$.

We see that $f(r) = 0$, and $f(u_2) = M \cdot d(r, u_2)$ so the image of the shortest path connecting u_1 and u_2 has the size $\mathrm{diam}(G) + 1$. On the top of that, for every $uv \in E(G)$, $|f(u) - f(v)| \le M$, otherwise we would get a contradiction with the definition of the distance. Thus f is an M-Lipschitz mapping and its range has to be at least $M \cdot (\mathrm{diam}(G) + 1)$. Combining this with Lemma 3.1, we get the claim we wanted to prove. $\square$

3.2 The case of strong Lipschitz mappings

By Lemma 2.1 we showed that strong Lipschitz mappings can exist on bipartite graphs only. We will now extend Theorem 3.1.

Theorem 3.2. *For any bipartite connected graph G, $\mathrm{r}_{\pm M}^{\max}(G) = M \cdot (\mathrm{diam}(G)+1)$.*

Proof. We can take Lipschitz mapping f as in Theorem 3.1. However, we have to check if it is a strong M-Lipschitz mapping.

Suppose that f is not a strong M-Lipschitz mapping. That means that there exist two vertices $a, b \in V(G)$ such that $ab \in E(G)$ and $f(a) = f(b)$. Furthermore, from the definition of f, $d(r, a) = d(r, b)$. Define $l := d(r, a)$.

From the definition of the distance, we get that there exist an (r, a)-path and (r, b)-path, both of length l. Since G is bipartite, parts to which vertices belong have to alternate along the (r, a)-path and along the (r, b)-path as well. Additionally, parts to which a and b belong are determined by the parity of their distance from root. But that means that a and b belong to the same part. Since they are neighbors, we get a contradiction.

Finally, we note that the previous argument works also in the case of r being either the vertex a or b. $\square$

3.3 Application

We will apply our results to prove Theorem 2.1, which was left unproven. We will first need the so called “lemma about cherries”. It is widely known lemma, but we prove it here for completeness.

Lemma 3.2. *A graph G is a disjoint union of complete graphs if and only if it does not contain $K_{1,2}$ as an induced subgraph.*

Proof. The if part is obvious.

For the only if part: There must exist a connected component with two vertices u and v in it such that there is no edge between them. Consider some shortest path between u and v. It must have at least three vertices. Take some three vertices u, x, y that are consecutive on this path. We claim that they induce $K_{1,2}$. If they do not form $K_{1,2}$, the first and the third vertex must be connected. But that contradicts the choice of the shortest path between u and v. □

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. The graph G cannot be a complete graph. Therefore, by Lemma 3.2, induced $K_{1,2}$ exists in G. Let vertices a and b from the statement of Theorem 2.1 be the two non-adjacent vertices of induced $K_{1,2}$.

We see that G has the diameter at least 2, since a and b are in distance 2. Let us root G in a for auxiliary reasons.

By the construction of 1-Lipschitz mapping from Theorem 3.1, there must exist a mapping f with $f(b) = d(a, b) = 2$.

Clearly, $\mathcal{L}_1(G \cup ab) \subseteq \mathcal{L}_1(G)$. However, f cannot be a 1-Lipschitz mapping of $G \cup ab$ rooted in a. That implies

$$|\mathcal{L}_1(G \cup ab)| \leq |\mathcal{L}_1(G)| - 1,$$

and we are done. □

3.4 Algorithmic aspects

Let us consider the following algorithmic problems – M-MaxRange and M-Strong-MaxRange.

Problem:	Maximum range problem – M-MaxRange
Input:	A connected graph G.
Question:	What is the maximum range of M-Lipschitz mapping of G, i.e. $\mathrm{r}_M^{\max}(G)$?

Problem:	Strong maximum range problem – M-Strong-MaxRange
Input:	A connected bipartite graph G.
Question:	What is the maximum range of strong M-Lipschitz mapping of G, i.e. $\mathrm{r}_{\pm M}^{\max}(G)$?

Because of Theorem 3.1, we can use the existing algorithms for finding graph diameter and distance in graphs for both of these problems. The following table is a quick survey of them. In it we denote by V and E the order and the size of the input graph, respectively.

Name of algorithm	**Complexity**	**Source**
Floyd-Warshall algorithm	$O(V^3)$	[18]
Johnson's algorithm	$O(V^2 \cdot \log V + VE)$ with Fibonacci heaps for Dijkstra subroutine	[28]
Seidel's algorithm	$O(V^{2.376} \cdot \log V)$	[51]

Table 3.1: Summary of selected algorithms for graph diameter.

Observe that these algorithms are suitable for general graphs. We can achieve a better complexity for some classes. Take for example trees for which we can compute diameter by a linear-time algorithm using one clever depth-first search traversal.

Connection to the surjective homomorphism problem

We saw that 1-Strong-MaxRange is easily solvable by algorithms for graph diameter. However, we would like to show a broader context of this problem by pointing out a connection with problems of finding surjective graph homomorphisms. We need to formulate Surjective Coloring problem first. We note, that by surjective homomorphisms we mean vertex-surjective homomorphism, not the edge-surjective one.

Problem:	Surjective Coloring
Input:	Graphs G and H.
Question:	Does there exist a graph homomorphism of G to H that is surjective?

The graph G is called the *guest graph* and the graph H the *host graph*. If all the guest graphs are from a graph class $\mathcal{G}$ and all the host graphs are from a graph class $\mathcal{H}$, we speak about the Surjective $(\mathcal{G}, \mathcal{H})$-Homomorphism problem.

Golovach et al. [22] proved the following theorem.

Theorem 3.3. *[22, Proposition 1] The* Surjective $(\mathcal{G}, \mathcal{H})$-Homomorphism *problem can be solved in polynomial time in the following two cases:*

1. *$\mathcal{G}$ is the class of complete graphs and $\mathcal{H}$ is the class of all graphs;*
2. *$\mathcal{G}$ is the class of all graphs and $\mathcal{H}$ is the class of paths.*

We get the following corollary for 1-Strong-MaxRange. Observe that we need to binary search for suitable n^* such that a vertex-surjective homomorphism to a host graph H isomorphic to P_{n^*} exists.

Corollary 3.1. *The problem* 1-Strong-MaxRange *can be solved in polynomial time.*

Proof. We have a connected bipartite graph G as the input graph. Let us denote its order n. We need to binary search for suitable n^*. We start with the closed interval $I = [1, n]$ and we choose $n^* := \left\lceil \frac{n}{2} \right\rceil$. Start the instance of Surjective Coloring with G and P_{n^*}. Depending on the result, we will continue to binary search in some corresponding subinterval and set n^* to a new value, i.e. we binary search for the maximum $n^* \in I$ such that G admits a surjective homomorphism to P_{n^*}. At the end of the algorithm, we will output the resulting n^* as the answer. $\square$

For more information on surjective homomorphism problems, we refer to [5, 24].

Chapter 4

Average range

At the first place, we will explain the main problem on the average range, present connections to other similar problems and show some properties of this graph parameter. Then, each of the subsequent sections will deal with a special class of graphs which is widely known. Our focus is to give precise formulas for the average range of some classes of graphs. We have obtained such formulas for complete graphs, compete bipartite graphs, paths, and cycles. These are one of the main results of this chapter and thesis.

4.1 Main conjectures on average range

Fundamental conjectures on the average range of (strong) 1-Lipschitz mappings say that paths P_n are extremal with regard to this parameter on the n-vertex graphs.

The first one is from Benjamini, Häggström and Mossel.

Conjecture 4.1. *[1]* (Benjamini-Häggström-Mossel) *For any connected bipartite graph $G \in \mathcal{G}_n$, $\bar{\mathrm{r}}_{\pm 1}(G) \leq \bar{\mathrm{r}}_{\pm 1}(P_n)$ holds.*

Newer version which generalizes the previous one is the following conjecture by Loebl, Nešetřil and Reed.

Conjecture 4.2. *[38]* (Loebl-Nešetřil-Reed) *For any connected graph $G \in \mathcal{G}_n$, $\bar{\mathrm{r}}_1(G) \leq \bar{\mathrm{r}}_1(P_n)$ holds.*

We will occasionally abbreviate Conjecture 4.1 to BHM Conjecture and Conjecture 4.2 to LNR Conjecture.

In [38] Leobl et al. further proved that LNR Conjecture holds modulo a constant factor.

Theorem 4.1. *[38] For every connected $G \in \mathcal{G}_n$ there exists an absolute constant C such that $\bar{\mathrm{r}}_1(G) \leq C\sqrt{n}$.*

The paper also presents other two interesting and useful theorems. In the following two theorems, the function f denotes an uniformly chosen 1-Lipschitz mapping of G.

Theorem 4.2. *[38] For every connected $G \in \mathcal{G}_n$ there exists an absolute constant C such that for any u and v at distance d in G, the expected value satisfies*

$$E[|f(u) - f(v)|] \leq C\sqrt{d}.$$

Theorem 4.3. *[38] For any pair of vertices u and v in a connected $G \in \mathcal{G}_n$, $E[|f(u) - f(v)|]$ is no more than the expected value obtained when u and v are the endpoints of P_n.*

4.2 Related results and problems

This section presents a couple of results and problems related to the average range parameter.

Counting endomorphisms

Csikvári and Lin showed [12, Theorem 1.8] that in the class of n-vertex trees, the path P_n has the smallest number of endomorphisms and the star S_n the largest number of endomorphisms. The problem is in fact similar to ours and the use of KC-transformation is the important tool to prove the recent result on LNR and BHM Conjecture on trees [61].

Lattice paths and their enumeration and height

As we saw in Section 2.4, analyzing the range of P_n is in fact similar to enumerating a special kind of lattice paths.

Lattice path enumeration is an important area of combinatorics. For a survey on lattice paths, see [35] or specific chapters of [17]. Let us remind you Catalan numbers and their bijection with Dyck paths [55]; paths from $(0, 0)$ to $(n, 0)$ with the possible steps to northeast $(1, 1)$ and to southeast $(1, -1)$ such that we are forbidden to go below the x-axis. We note that for Dyck paths the average height is known (see [13]).

Flat Lipschitz functions

Peled et al. [50] studied *grounded M-Lipschitz functions* on rooted d-regular trees. A *d-regular tree* is a tree with every non-leaf vertex of the degree equalt to d.

Definition 4.1. A *grounded M-Lipschitz mapping* f of a graph $G = (V, E)$ is a function $f : V \to \mathbb{Z}$ such that every leaf vertex is mapped to zero and for every edge $uv \in E$, $|f(u) - f(v)| \leq M$ holds.

Clearly, this definition is slightly different from our definition of M-Lipschitz mapping in two things:

- Peled et al. do not have the restriction on root image.
- We do not have the restriction on the images of leaves.

Peled et al. proved that the probability that the uniformly at random chosen grounded M-Lipschitz mapping maps the root of a d-regular tree to a value greater than $M + t$ is doubly-exponentially small in t.

It is shown in another paper by Peled et al. [49] that on expander graphs random M-Lipschitz function takes only $M + 1$ different values on the large part of the graph. The motivation was to investigate some class of highly connected graphs and see if this high connectivity implies smaller fluctuations of a typical random Lipschitz mapping.

For more sources on expanders, we recommend surveys [25, 41].

Long-range connections

Spinka studied in his master thesis [52] and in the subsequent paper with Peled [53] a generalized version of paths and probabilities of obtaining certain ranges for random strong 1-Lipschitz mapping on these generalized paths.

We will need to define what graphs $P_{n,d}$ are.

Definition 4.2. For every $n, d \geq 1$, let the graph $P_{n,d}$ be defined as

$$V(P_{n,d}) := \{0, 1, \ldots, n\},$$
$$E(P_{n,d}) := \{(i, j) : |i - j| \in \{1, 3, \ldots, 2d + 1\}\}.$$

Observe that for $P_{n,0}$ we get a standard path graph P_n. The main result of Spinka's thesis is the following theorem.

Theorem 4.4. *[52] For any positive integers n, d and r, we have*

$$P(\mathrm{r}(f_{n,d}) \geq 3 + r) \leq \binom{n}{r} 2^{-dr}$$

and

$$P(\mathrm{r}(f_{n,d}) < 3) \leq 2^{1-n/2}.$$

Furthermore, if $n2^{-d(n)} \to 0$ as $n \to \infty$, then

$$P(\mathrm{r}(f_{n,d(n)}) = 3) \to 1 \text{ as } n \to \infty.$$

4.3 General properties

In this section we will show two important examples warning us not to use intuition when working with the parameter of average range. By a *cut vertex* of a connected graph G we mean a vertex of G such that its removal disconnects the graph. Observe that by this definition, leaves are not cut vertices. The following examples are taken from [61].

Less cut vertices does not imply lesser average range

See Figure 4.1. Graph T_1 has

$$\bar{\mathrm{r}}_1(T_1) = 23003009/4782969 \approx 4.80936$$

and graph T_2 has

$$\bar{\mathrm{r}}_1(T_2) = 23013183/4782969 \approx 4.81148.$$

However, T_1 has 12 cut vertices in total but T_2 has only 11 cut vertices in total.

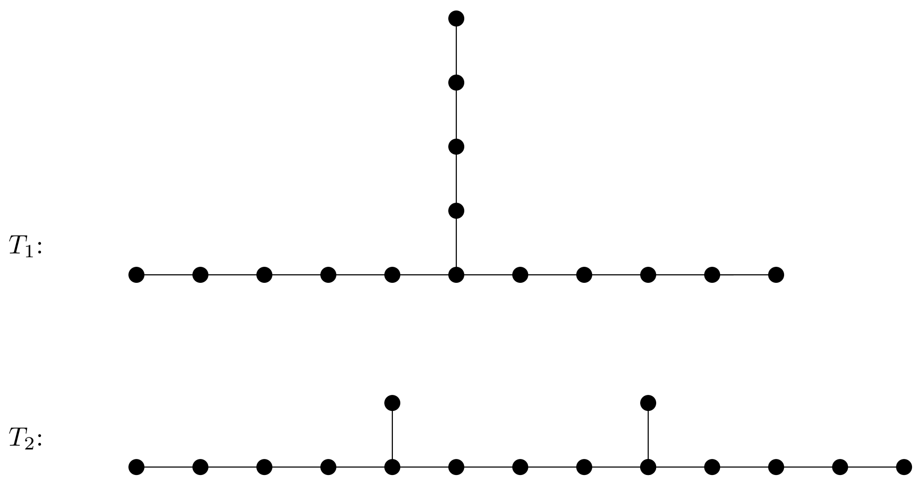

Figure 4.1: An example of two trees. Tree T_1 has more cut vertices than T_2. However, the average range of T_1 is lesser than the average range of T_2.

Adding an edge can increase the average range

See Figure 4.2. Graph G_2 is formed from G_1 by adding edge e. However, G_1 has

$$\bar{\mathrm{r}}_1(G_1) = 31717/9315 \approx 3.40494,$$

and G_2 has

$$\bar{\mathrm{r}}_1(G_2) = 31231/9153 \approx 3.41211.$$

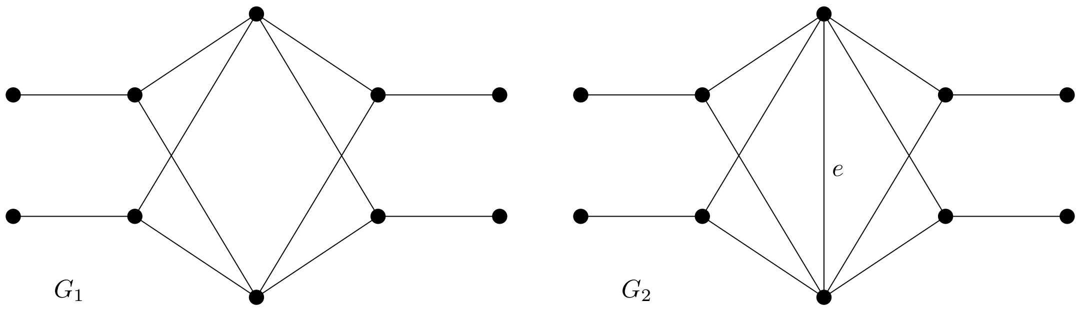

Figure 4.2: An example of two graphs G_1 and G_2. We have $G_2 = G_1 \cup \{e\}$ but $\bar{\mathrm{r}}_1(G_1) \leq \bar{\mathrm{r}}_1(G_2)$.

4.4 Experimental results

With an aid of our own computer program, we performed an experimental checking of the validity of Conjectures 4.1 and 4.2 for small-order graphs.

Corollary 4.1. *Conjecture 4.2 holds for graphs up to* 10 *vertices. Conjecture 4.1 holds for bipartite graphs up to* 13 *vertices.*

For details on our computer program, check Appendix A.

Note. All the following sections will deal with some particular class of graphs.

4.5 Complete graphs

For completeness of the picture we will show the formula for $\bar{\mathrm{r}}_1(K_n)$ and prove that complete graphs are minimal graphs with respect to the average range.

Theorem 4.5. *For a complete graph K_n we have $\bar{\mathrm{r}}_1(K_n) = 2 - (2^n - 1)^{-1}$.*

Proof. Let us count the number of 1-Lipschitz mappings of K_n. We cannot choose the image of the root r but we can do it for other vertices. Namely, we must choose integers from interval $[-1, 1]$ due to the fact that every vertex $v \neq r$ is a neighbor of the root. Furthermore, 1-Lipschitz mapping f with vertices $u, v \in V(K_n)$ such that $f(u) = -1$ and $f(v) = 1$ cannot exist since $uv \in E(K_n)$. Thus, apart from the trivial case of setting image of all vertices to 0, we can choose to map vertices other than r either to -1 and 0, or to 1 and 0 – exclusively. For each of this choice we have $2^{n-1} - 1$ of such 1-Lipschitz mappings and each of them has the range equal to 2. We conclude:

$$\bar{\mathrm{r}}_1(K_n) = \frac{2 \cdot 2 \cdot (2^{n-1} - 1) + 1}{2 \cdot (2^{n-1} - 1) + 1} = 2 - \frac{1}{2^n - 1}.$$

□

Theorem 4.5 implies the limiting behavior of $\bar{\mathrm{r}}_1(K_n)$.

Corollary 4.2. *It holds that* $\lim_{n\to\infty} \bar{\mathrm{r}}_1(K_n) = 2$.

Finally, we will prove that complete graphs have the minimum average range among all n-vertex graphs.

Theorem 4.6. *For any connected $G \in \mathcal{G}_n$ which is not isomorphic to the complete graph K_n, $\bar{\mathrm{r}}_1(K_n) < \bar{\mathrm{r}}_1(G)$ holds.*

Proof. We recall Chapter 3 and Theorem 3.1 which states that for every connected graph G, $\mathrm{r}^{\max}(G) = \mathrm{diam}(G) + 1$.

We see that graphs where the diameter is equal to one are precisely complete graphs. Thus any connected G with the diameter strictly greater than one has a Lipschitz mapping of range at least 3. Since there is always exactly one Lipschitz mapping of range 1 (set all to zero), we get that such G has an average range greater or equal to 2. Combining this with Theorem 4.5 yields the proof. □

4.6 Complete bipartite graphs

We prove an exact formula for another well-known class of graphs, *complete bipartite graphs.*

Theorem 4.7. *For every $p, q \in \mathbb{N}$, a complete bipartite graph $K_{p,q}$ satisfies*

$$|\mathcal{L}_1(K_{p,q})| = 3^p + 3^q + 2^{p+q} - 2^{p+1} - 2^{q+1} + 1,$$

and

$$\bar{\mathrm{r}}_1(K_{p,q}) = 3 - 2^{p+q} \cdot (3^p + 3^q + 2^{p+q} - 2^{p+1} - 2^{q+1} + 1)^{-1}.$$

Proof. We use Theorem 3.1 that implies that the possible ranges of $K_{p,q}$ form a subset of $\{1, 2, 3\}$. We analyze separate cases of possible ranges and count how many such mappings exist. Let us denote the part of size p by P and the other one, with the size q, as Q. Without loss of generality, assume that all 1-Lipschitz mappings are rooted in some fixed vertex of P.

- *Range equal to 1:* Clearly, there is exactly one such mapping, sending everything to zero.
- *Range equal to 2:* A homomorphic image of a 1-Lipschitz mapping is some closed interval, as we observed earlier in preliminary chapter. Thus the possibilities for the homomorphic image of the range 2 are $\{0, 1\}$ and $\{-1, 0\}$. These cases are symmetric to each other, so let us analyze, without loss of generality, the case $\{0, 1\}$.

range	homomorphic image	number of such mappings
1	$\{0\}$	1
2	$\{0,1\}$	$2^{p+q-1}-1$
	$\{0,-1\}$	dtto
3	$\{0,1,2\}$	$3^{p-1}-2^{p-1}$
	$\{-2,-1,0\}$	dtto
	$\{-1,0,1\}$	$3^{p-1}+3^q-2^p-2^{q+1}+2$

Table 4.1: Table for Theorem 4.7.

There are 2^{p+q-1} possibilities how to assign 0 and 1 to the vertices excluding the root. However, one of these possibilities is the trivial mapping of the range 1 (everything mapped to zero). The result is that there are $2^{p+q-1}-1$ mappings with the homomorphic image $\{0,1\}$.

- *Range equal to 3:* Again we have multiple cases. The cases $\{0,1,2\}$ and $\{-2,-1,0\}$ are symmetric, the third is $\{-1,0,1\}$.

  Let us solve the case $\{0,1,2\}$ first. Clearly, 0 and 2 cannot be in different parts, otherwise there would exist an edge with endpoints mapped to 0 and to 2, violating the definition of 1-Lipschitz mapping. That further implies the impossibility of $v_q \in Q$ mapped to 2. By a similar argument we get that only in the case that all vertices of Q are mapped to one we can get the homomorphic image $\{0,1,2\}$. We can then place any of the numbers from $\{0,1,2\}$ on the part P. However, we must exclude assignments with no 2 on the part P. That yields $3^{p-1}-2^{p-1}$ possibilities.

  The remaining case is $\{-1,0,1\}$. Again, we see that 1 and -1 cannot be in different parts. Thus, either the part P has all vertices mapped to zero and on Q we can choose for every vertex an image from the set $\{-1,0,1\}$, or vice versa. That gives us $3^{p-1}+3^q$ choices from which we must exclude those that use only some propper subset of $\{-1,0,1\}$. Finally, we get the formula

  $$3^{p-1}+3^q-2^p-2^{q+1}+2$$

  for this case.

Table 4.1 summarize all the cases. The number of 1-Lipschitz mappings of $K_{p,q}$ is equal to

$$3^p+3^q+2^{p+q}-2^{p+1}-2^{q+1}+1,$$

i.e. the sum of the third column of Table 4.1. By straightforward calculations we get

$$\bar{\mathrm{r}}_1(K_{p,q}) = 3 - 2^{p+q}\cdot(3^p+3^q+2^{p+q}-2^{p+1}-2^{q+1}+1)^{-1}.$$

$\square$

We conclude this section with the observation on the limiting behavior of $\bar{\mathrm{r}}_1(K_{p,q})$ as $(p+q) \to \infty$. Clearly, the average range is 3 in limit.

4.7 Stars

Definition 4.3. A *star graph* S_n is a tree with n vertices; one vertex of degree $n-1$ and $n-1$ leaves (vertices of degree one). Or, alternatively, it is a complete bipartite graph $K_{1,n-1}$.

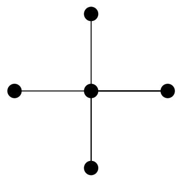

Figure 4.3: A star S_5 with five vertices and four leaves.

Theorem 4.8. *A star S_n satisfies*

$$\bar{\mathrm{r}}_1(S_n) = 3 - \frac{2^n}{3^{n-1}},$$

and

$$\bar{\mathrm{r}}_{\pm 1}(S_n) = 3 - 2^{2-n}.$$

Proof. We will use the definition of stars as a special case of complete bipartite graphs. We can then use Theorem 4.7 for the case of 1-Lipschitz mappings with $p := 1$ and $q := n-1$. The desired claim follows.

We will now prove the second formula. Without loss of generality, we will root our graph in the central vertex. Observe that all leafs will get either $+1$ or -1 and only cases in which range is equal to 2 are the cases of either all leaves mapped to 1 or to -1. The rest of the cases have the range equal to 3. Totally, there are 2^{n-1} of strong 1-Lipschitz mappings. That concludes our claim. □

4.8 Hypercubes

Hamming cubes, also known as *hypercubes* or *d-dimensional cubes*, were extensively studied. We use the definition by Diestel [14].

Definition 4.4. [14] Let $d \in \mathbb{N}$ and $V := \{0,1\}^d$; thus, V is the set of all binary sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the *d-dimensional cube* or *hypercube*. Sometimes, we write Q_d.

Galvin [19] proved the conjecture of Kahn [29] that as d goes to infinity, every hypercube Q_d has the average range of strong 1-Lipschitz mapping equal to 6. This inspired us to present the following open problem.

Problem 4.1. What is the limit of the average range of 1-Lipschitz mapping of Q_d as d goes to infinity?

4.9 Paths

In [61], authors compute several values of $\bar{\mathrm{r}}_1(P_n)$ (see Table 4.2) and claim that no explicit formula for an average range of a path is known. We fill this gap and present such formula, exploiting the tool used in the random walk analysis called *reflection principle*.

n	2	3	4	5	6	7	8	9	10	11	12
$\bar{\mathrm{r}}_1(P_n)$	$\frac{5}{3}$	$\frac{19}{9}$	$\frac{67}{27}$	$\frac{227}{81}$	$\frac{751}{243}$	$\frac{2445}{729}$	$\frac{7869}{2187}$	$\frac{25107}{6561}$	$\frac{78767}{19683}$	$\frac{250793}{59049}$	$\frac{786985}{177147}$

Table 4.2: Table of values of $\bar{\mathrm{r}}_1(P_n)$ for $2 \le n \le 12$.

We will define auxiliary random variables and we will speak for a while also in the language of standard random walks which are naturally encoded in 1-Lipschitz mapping of P_n. We remind the reader Section 2.4 where we discussed this connection (bijection) in detail.

- M_n^+ is a random variable corresponding to the maximum non-negative number in the image of a 1-Lipschitz mapping f.
- M_n^- is a random variable corresponding to the minimum non-positive number in the image of a 1-Lipschitz mapping f.
- X_n denotes the number $f(v_n)$, i.e. image of the second endpoint of P_n.

Theorem 4.9. *For a path P_n we have*

$$\bar{\mathrm{r}}_1(P_n) = 1 + 3^{-n+1} \cdot 2 \cdot \sum_{k=0}^{n-1} k \cdot \sum_{i=0}^{\lfloor \frac{n-1-k}{2} \rfloor} \left(\binom{n-1}{k+i} \binom{n-k-i-1}{i} + \binom{n-1}{k+1+i} \binom{n-k-i-2}{i} \right).$$

Proof. The average range of path P_n can be formulated as:

$$\bar{\mathrm{r}}_1(P_n) = E[M_n^+ - M_n^- + 1].$$

From the symmetry of M_n^+ and M_n^- and from the linearity of expectation, one gets:

$$\bar{\mathrm{r}}_1(P_n) = E[M_n^+ + M_n^+ + 1] = E[M_n^+] + E[M_n^+] + 1 = 2E[M_n^+] + 1.$$

Set $M_n := M_n^+$. Now let us prove that $P(M_n \geq r) = P(X_n \geq r) + P(X_n \geq r+1)$.

The walks with $M_n \geq r$ fit into two groups. Either such walks end in $s \geq r$ or in $s < r$. In the second case, we can reflect the section of the path after the first time we get to r and we get a new walk which now ends in $s' > r$. See Figure 4.4 for an illustration. Since this process is invertible and every path that reaches $s \geq r$ must have $M_n \geq r$, we get:

$$P(M_n \geq r) = P(X_n \geq r) + P(X_n \geq r+1).$$

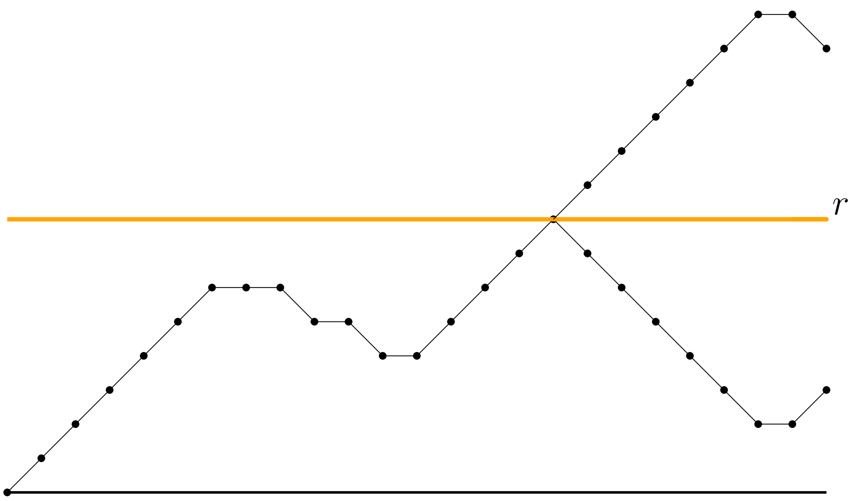

Figure 4.4: An illustration of the reflection principle.

Next we will prove that: $P(M_n = r) = P(X_n = r) + P(X_n = r+1)$.

$$\begin{aligned} P(M_n = r) &= P(M_n \geq r) - P(M_n \geq r+1) \\ &= P(X_n \geq r) + P(X_n \geq r+1) - P(X_n \geq r+1) - P(X_n \geq r+2) \\ &= P(X_n = r) + P(X_n = r+1). \end{aligned}$$

Now we need to determine $P(X_n = r)$. Recall the aforementioned bijection between $\{1, -1, 0\}$-sequences and walks from Section 2.4.

We have $n-1$ edges so if we want to attain some fixed k, we need to sum up our sequence to k. Thus we need to pick k additional 1's over -1's. Summing up through the all possible values of the number of -1's we get:

$$P(X_n = k) = 3^{-n+1} \cdot \sum_{i=0}^{\lfloor \frac{n-1-k}{2} \rfloor} \binom{n-1}{k+i} \binom{n-k-i-1}{i}. \tag{4.1}$$

And for $P(X_n = k+1)$ analogously:

$$P(X_n = k+1) = 3^{-n+1} \cdot \sum_{i=0}^{\lfloor \frac{n-1-k}{2} \rfloor} \binom{n-1}{k+1+i} \binom{n-k-i-2}{i}. \tag{4.2}$$

We are now ready to combine all of this together and we get:

$$\bar{\mathrm{r}}_1(P_n) = 1 + 2 \cdot \sum_{k=0}^{n-1} k \cdot \Big(P(X_n = k) + P(X_n = k+1) \Big).$$

We note that $\binom{a}{b}$ is defined as zero if $b > a$. Substituting $P(X_n = k)$ by (4.1) and $P(X_n = k+1)$ by (4.2) we get the desired claim.

□

Besides the exact formula for the average range of a path, we prove the following relation between the $\bar{\mathrm{r}}_1$ of paths P_n and P_{n+1}.

Lemma 4.1. *For every $n \in \mathbb{N}$, $\bar{\mathrm{r}}_1(P_{n+1}) - \bar{\mathrm{r}}_1(P_n) \leq 2/3$.*

Proof. Let us write $v_1, v_2, \ldots, v_n$ for vertices of P_n consecutively and let

$$E(P_n) := \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}.$$

For P_{n+1}, set $V(P_{n+1}) := V(P_n) \cup \{v_{n+1}\}$ and $E(P_{n+1}) = E(P_n) \cup \{v_nv_{n+1}\}$.

Pick v_1 as the root of P_n and P_{n+1} as well and consider all 1-Lipschitz mappings $\mathcal{L}(P_n)$ and $\mathcal{L}(P_{n+1})$. Choose an arbitrary f from $\mathcal{L}(P_n)$. Now $f(v_n) = r$ for some $r \in \mathbb{Z}$. If we want to extend this f to a 1-Lipschitz mapping f' of P_{n+1}, we see that we can set $f'(v_{n+1})$ to either r, $r+1$ or $r-1$. Choosing r does not increase the range. Since we want to do an upper estimate, let us presume that choosing $r+1$ or $r-1$ always increases the range. Thus we get:

$$\bar{\mathrm{r}}(P_{n+1}) \leq \bar{\mathrm{r}}(P_n) + 2/3.$$

Which is only a different form of the desired claim. □

This simple upper bound has two corollaries.

Corollary 4.3. *For every $r, q \in \mathbb{N}$, $r > q$, $\bar{\mathrm{r}}_1(P_r) \leq \bar{\mathrm{r}}_1(P_q) + (r-q) \cdot \frac{2}{3}$ holds.*

Proof. Use Lemma 4.1 $(r-q)$ times. □

Corollary 4.4. *For every P_n, $\bar{\mathrm{r}}_1(P_n) \leq \frac{2n+1}{3}$ holds.*

Proof. Choose $r := n$ and $q := 1$. Then use the previous lemma and observe that $\bar{\mathrm{r}}_1(P_1)$ is equal to one. □

We remark that for all paths in general we cannot get a better upper bound by a constant than in Lemma 4.1 since $\bar{\mathrm{r}}_1(P_2) - \bar{\mathrm{r}}_1(P_1) = \frac{2}{3}$.

4.10 Wired regular trees

Wired regular trees are k-regular trees (every non-leaf vertex has the same degree k) which are "wired" by adding one isolated vertex which is then connected by edge with every leaf. See Figure 4.5 for an example.

Benjamini et al. [1] proved that for any k-regular wired tree W on n vertices, $\bar{\mathrm{r}}_{\pm 1}(W) = O(\log n)$, i.e. the average range of a strong Lipschitz mapping is asymptotically logarithmic.

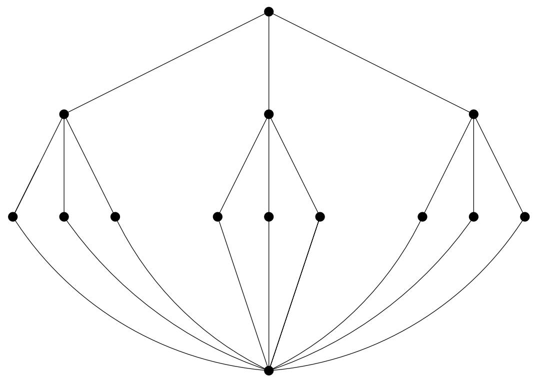

Figure 4.5: An example of a 3-regular tree with wired leaves.

4.11 Trees

The most recent result in the area of graph-indexed random walks is the result of Wu, Xhu and Zhu from 2016 [61]. The authors tried to attack the LNR and BHM conjecture and got the following partial result.

Theorem 4.10. *[61] For any tree T_n on n vertices holds the following,*

1. $\bar{\mathrm{r}}_1(T_n) \leq \bar{\mathrm{r}}_1(P_n)$,
2. $\bar{\mathrm{r}}_{\pm 1}(T_n) \leq \bar{\mathrm{r}}_{\pm 1}(P_n)$.

Their approach is to use a special transformation called KC-transformation, named by Kelmans [30], which we already mentioned in Section 4.2. Csikvári [10, 11] proved that this transformation induces a partially ordered set on the class of all n-vertex trees with the path P_n as the maximum element and the star S_n as the minimum element. By carefully choosing a right chain in this poset they prove Theorem 4.10.

We note that by proving Theorem 4.8 and Theorem 4.9 we showed the precise formulas for the minimum and the maximum possible average range of trees on n vertices for the case of 1-Lipschitz mappings.

4.12 Cycles

In this section, more specifically in Theorem 4.12, we will show a formula for the average range of cycle graphs C_n.

See Table 4.12 for values of $\mathrm{r}_1(C_n)$ of smaller cycles computed with code from Appendix A.

n	3	4	5	6	7	8	9	10	11	12
$\bar{\mathrm{r}}_1(C_n)$	$\frac{13}{7}$	$\frac{41}{19}$	$\frac{121}{51}$	$\frac{365}{141}$	$\frac{1093}{393}$	$\frac{3281}{1107}$	$\frac{9841}{3139}$	$\frac{29525}{8953}$	$\frac{88573}{25653}$	$\frac{265721}{73789}$

Table 4.3: Table of values of $\bar{\mathrm{r}}_1(C_n)$ for $3 \le n \le 12$.

First, let us introduce what the *trinomial triangle* is.

Trinomial triangle

The trinomial triangle is similar to the Pascal (binomial) triangle of binomial coefficients. One can similarly define trinomial coefficients in a recursive way.

Definition 4.5. *(Trinomial triangle and central trinomial coefficient)* Trinomial numbers (coefficients) $\binom{n}{k}_2$ are defined as:

$$\binom{0}{0}_2 = 1$$

$$\binom{n+1}{k}_2 = \binom{n}{k-1}_2 + \binom{n}{k}_2 + \binom{n}{k+1}_2 \text{ for } n \ge 0,$$

where $\binom{n}{k}_2 = 0$ for $k < -n$ and $k > n$.

Central trinomial coefficients are the numbers $\binom{n}{0}_2$, where $n \in \mathbb{N}_0$.

$$\begin{array}{ccccccccc}
 & & & & \color{blue}{1} & & & & \\
 & & & 1 & \color{blue}{1} & 1 & & & \\
 & & 1 & 2 & \color{blue}{3} & 2 & 1 & & \\
 & 1 & 3 & 6 & \color{blue}{7} & 6 & 3 & 1 & \\
1 & 4 & 10 & 16 & \color{blue}{19} & 16 & 10 & 4 & 1
\end{array}$$

Figure 4.6: The trinomial triangle with central trinomial coefficients in blue color.

The sequence for central trinomial coefficients in OEIS is A123456 [48]. See Figure 4.6 for a visualization of the trinomial triangle with highlighted central trinomial coefficients. Trinomial coefficients appear quite often in enumerative combinatorics. Let us show one particular example.

Example. Suppose you have a king on a chessboard (it does not have to be the usual 8×8 one). Each entry of the triangle corresponds to the number of paths using the minimum number of steps between some cells of the chessboard. See Figure 4.7.

Useful fact is that central trinomial coefficients satisfy the following identity (for its derivation, see for example [4]):

$$\binom{n}{0}_2 = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-2k+1)}{(k!)^2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}. \tag{4.3}$$

1	3	6	7	6	3	1
3	1	2	3	2	1	3
6	2	1	1	1	2	6
7	3	1	♔	1	3	7
6	2	**1**	**1**	**1**	2	6
3	**1**	**2**	**3**	**2**	**1**	3
1	**3**	**6**	**7**	**6**	**3**	**1**

Figure 4.7: Each number represents the number of ways how to get to that cell with the minimum number of step with the figure of king. [60]

Motzkin numbers

For the proof of the formula for $\bar{\mathrm{r}}_1(C_n)$ we need to define *generalized Motzkin number and paths.* We will further write only Motzkin numbers and Motzkin paths.

Definition 4.6. Consider a lattice path, beginning at $(0, 0)$, ending at (n, k) and satisfying that y-coordinate of every point is non-negative. Furthermore, every two consecutive steps (i, a) and $(i + 1, b)$ must satisfy $|a - b| \leq 1$. Such lattice path are called Motzkin paths.

The set of all the possible paths ending in (n, k) is denoted by $m(n, k)$ and the cardinality of this set is denoted by $M(n, k)$. We call $M(n, k)$ the Motzkin number.

For more details we refer to the seminal paper [15], Motzkin numbers $M(n, 0)$ form the sequence A001006 in OEIS [46]. See Figure 4.8 for an example of a Motzkin path.

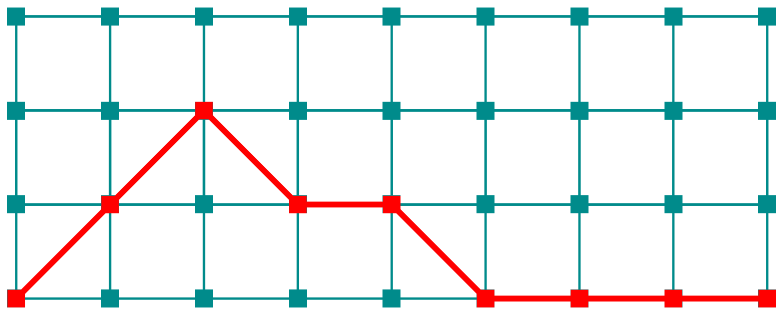

Figure 4.8: A Motzkin path from $(0, 0)$ to $(8, 0)$.

Main theorem

Theorem 4.11. *For any cycle graph C_n, $n \geq 3$, we have*

$$\bar{\mathrm{r}}_1(C_n) = \frac{3^n + (-1)^n}{2 \cdot \binom{n}{0}_2}.$$

We will prove Theorem 4.12 in series of lemmata that will be put together later.

Lemma 4.2. *There is a bijection between 1-Lipschitz mappings of C_n and the set of lattice paths starting at $(0, 0)$, ending at $(n, 0)$, and satisfying that for every two consecutive steps (i, a) and $(i + 1, b)$, $|a - b| \leq 1$.*

Proof. The proof is analogous to other bijections we made between 1-Lipschitz mappings of some type and some class of lattice walks. Take the sequence

$$v_1, v_2, \ldots, v_n, v_1$$

of the vertices of C_n such that v_1 is the root and the vertices appear consecutively on the cycle precisely as in this sequence. For every 1-Lipschitz mapping f of C_n we can define another sequence

$$(v_1, f(v_1)), (v_2, f(v_2)), \ldots, (v_n, f(v_n)), (v_1, f(v_1)).$$

The lemma follows easily. $\square$

Let us prove the formula for $|\mathcal{L}(C_n)|$.

Theorem 4.12. *For any C_n, $n \geq 3$, $|\mathcal{L}(C_n)| = \binom{n}{0}_2$.*

Proof. We will encode all 1-Lipschitz mappings of the cycle into the sequences $\{-1, 0, 1\}^n$. Consider the lattice walks constructed in Lemma 4.2. For each sequence

$$(v_1, f(v_1)), (v_2, f(v_2)), \ldots, (v_n, f(v_n)), (v_1, f(v_1)),$$

one can define the new sequence

$$f(v_2) - f(v_1), f(v_3) - f(v_2), \ldots, f(v_1) - f(v_n).$$

We know that these sequences must add up to 0. Thus for any total number k of ones in this sequence we must have k times -1 in this sequence as well. Furthermore, we have $k \leq \lfloor n/2 \rfloor$.

Summing over all possible k's we first pick $2k$ edges which have either $+1$ or -1. Then from these $2k$ edges, we choose k edges for placing 1. The rest of $n - 2k$ edges gets 0's and the rest of $2k - k = k$ edges gets (-1)'s. Formally:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}.$$

This coincides with identity (4.3) if we take into account that $\binom{a}{b}$ is defined to be equal to zero if $b > a$. □

Definition 4.7. We denote by $\mathcal{L}(C_n, -d)$ the set of 1-Lipschitz mappings f of C_n satisfying

$$\min_{v \in V(C_n)} f(v) = -d.$$

In other words, $\mathcal{L}(C_n, -d)$ denotes the set of all 1-Lipschitz mappings of C_n with $-d$ as the minimum value in their homomorphic images.

Another ingredient we need is the following theorem of Van Leeuwen.

Theorem 4.13. *[56] Within the class of walks on Z starting at 0 and with steps advancing by $+1$, 0 or -1, there is a bijection, conserving both the length of the walk and the number of steps 0, between on one hand the walks that end in 0, and on the other hand the walks that do not visit negative numbers. The bijection maps walks ending at 0 and whose minimal number visited is $-d$, to walks ending at $2d$, and is realized by reversing the direction of the d down-steps that first reach respectively the numbers $-1, -2, \ldots, -d$.*

Now we need to show a bijection between $\mathcal{L}(C_n, -d)$ and the set $m(n, 2d)$.

Lemma 4.3. *There exist a bijection from the set of Motzkin paths $m(n, 2d)$ to the set $\mathcal{L}(C_n, -d)$.*

Proof. The existence of such sequence follows straightforwardly from combining Theorem 4.13 and Lemma 4.2. □

For technical convenience, we will define the *irregular trinomial triangle* and *irregular trinomial coefficients*; see the sequence A027907 in OEIS [47]. See Figure 4.4, depicting a part of the irregular trinomial triangle.

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1

Table 4.4: A part of the irregular trinomial triangle. The entries of the table are numbers $T^*(n,k)$.

Definition 4.8. The *irregular trinomial coefficients* are defined as

$$T^*(n,k) = \binom{n}{k+n}_2.$$

The following lemmata, showing the relation of Motzkin paths and irregular trinomial coefficients will be crucial for the proof of the main theorem.

Remark 4.1. For every $n, k \in \mathbb{Z}$ the following identity holds:

$$T^*(n,k) = T^*(n-1,k) + T^*(n-1,k-1) + T^*(n-1,k-2).$$

Proof. This is easily verified from the definition of the trinomial coefficients. □

Lemma 4.4. *The following identity holds for every $n, k \in \mathbb{N}_0$, $k \leq n$,*

$$M(n,k) = T^*(n,n-k) - T^*(n,n-k-2). \tag{4.4}$$

Proof. We prove this theorem by induction on n. For $n = 0, 1$, the identity holds.

We divide the rest of the proof into two cases (the first case is needed because in case of $n = k$, we would not be able to use induction hypothesis):

Case 1: $n = k$. Then $1 = M(n,n) = T^*(n,0) + T^*(n,-2) = 1 + 0$, so this case is done.

Case 2: $n > k$. Now suppose the identity holds for all numbers up to $n-1$. By the definition of the generalized Motzkin numbers we have

$$M(n,k) = M(n-1,k) + M(n-1,k-1) + M(n-1,k+1). \tag{4.5}$$

And by induction hypothesis we can write

$$\begin{aligned} M(n,k) = T^*(n-1,n-k) &- T^*(n-1,n-k-2) \\ + T^*(n-1,n-k-1) &- T^*(n-1,n-k-3) \\ + T^*(n-1,n-k-2) &- T^*(n-1,n-k-4). \end{aligned}$$

From Remark 4.1 on the recurrence relation of irregular coefficients we get that the even summands and odd summands are equal to $T^*(n, n-k)$ and $-T^*(n, n-k-2)$, respectively. Our claim follows. □

Now we need the last lemma, concerning the sum of irregular coefficients.

Lemma 4.5. *For every even $n \in N_0$ holds:*

$$\sum_{k=0}^{n} T^*(n, 2k) = (3^n + 1)/2, \tag{4.6}$$

and for odd $n \in N_0$ holds:

$$\sum_{k=1}^{n} T^*(n, 2k-1) = (3^n - 1)/2. \tag{4.7}$$

Proof. We will prove these identities by induction. For $n = 0, 1$, the respective identities hold. Now assume that both identities hold for all $n' < n$. By parity of n we distinguish two cases. We will prove the lemma for the case of n even. Odd case is very similar.

$$\begin{aligned}
\sum_{k=0}^{n} T^*(n, 2k) &= \sum_{k=1}^{n-1} T^*(n, 2k-1) + 2 \cdot \sum_{k=0}^{n-1} T^*(n, 2k) && (4.8)\\
&= 2 \cdot 3^{n-1} - \sum_{k=1}^{n-1} T^*(n, 2k-1) && (4.9)\\
&= 2 \cdot 3^{n-1} - (3^{n-1} - 1)/2 && (4.10)\\
&= (3^n + 1)/2. && (4.11)
\end{aligned}$$

- The first equation follows from Remark 4.1.
- The second equation follows from the fact that the sum of the n-th row of T^* is equal to 3^n. That can be easily proved by induction.
- The third equation follows from the induction hypothesis.
- The fourth equation is straightforward calculation.

□

We can finally prove the main theorem of this section and one of the main results of this thesis.

Proof of Theorem 4.12. We will first show the following identity for every $n \geq 3$.

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1)(T^*(n, n-2k) - T^*(n-1, n-2k-2)) = \begin{cases} \sum_{k=0}^{n} T^*(n, 2k), & n \text{ even} \\ \sum_{k=1}^{n} T^*(n, 2k-1), & n \text{ odd} \end{cases}$$

This identity follows from the straightforward calculations and from the observation that $T^*(n, n-k) = T^*(n, n+k)$ for every $n, k \in Z$.

For brevity, we will do the following calculation for n odd. The proof for n even is different in the last two equations but the only difference is the use of the different parts of lemma and identity 4.12, depending on the parity.

$$\begin{aligned}
\bar{\mathrm{r}}_1(C_n) \cdot |\mathcal{L}(C_n)| &= \sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1) \cdot M(n, 2k) \\
&\qquad\qquad \text{(by Lemma 4.3 and linearity of expectation)} \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1) \cdot \Big(T^*(n, n-2k) - T^*(n-1, n-2k-2)\Big) \\
&\qquad\qquad \text{(by Lemma 4.4)} \\
&= \sum_{k=1}^{n} T^*(n, 2k-1) \\
&= \frac{3^n - 1}{2}. \qquad\qquad \text{(by Lemma 4.5)}
\end{aligned}$$

Together with Theorem 4.12 taken into account we conclude the formula for $\bar{\mathrm{r}}_1(C_n)$. □

We present the following corollary regarding the asymptotics of $\bar{\mathrm{r}}_1(C_n)$.

Corollary 4.5. *For every $n \geq 3$, $\bar{\mathrm{r}}_1(C_n) \sim 2\sqrt{\frac{\pi}{3}n}$ holds.*

Proof. The asymptotics of central trinomial coefficients is known, see e.g. [17, p. 588]. Central trinomial coefficients satisfy

$$\binom{n}{0}_2 \sim \frac{3^{n+1/2}}{2\sqrt{\pi n}}.$$

The sign $\sim$ denotes the relation of two sequences. Two sequences a_n and b_n are in relation $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Using Theorem 4.12 and the mentioned asymptotics, we get: $\bar{\mathrm{r}}_1(C_n) \sim 2\sqrt{\frac{\pi}{3}n}$. □

4.13 Pseudotrees

Definition 4.9. We call a graph *unicyclic* if it contains exactly one cycle.

Definition 4.10. We call a graph *pseudotree* if it is a tree or a unicyclic graph. Equivalently, pseudotrees are graphs with at most one cycle.

Counting the number of 1-Lipschitz mappings

Lemma 4.6. *The number of* 1*-Lipschitz mappings of unicyclic graphs with order* n *and cycle size* c*,* $c \le n$*, is equal to*

$$\binom{c}{0}_2 \cdot 3^{n-c}.$$

Proof. Let us denote our unicyclic graph of order n and cycle size c by G and the subgraph induced by the vertices on its cycle by C. We use Theorem 4.12 to get the number of 1-Lipschitz mappings of the subgraph C. Now let us fix some f, a 1-Lipschitz mapping of C.

By deleting all the edges of the cycle C we get a forest $\mathcal{T}$ of trees $T_1, \dots, T_c$. In this forest, exactly one vertex in each tree T_i has an image under the mapping f. Thus we obtain $3^{|V(T_i)|-1}$ different 1-Lipschitz mappings for each of the tree in $\mathcal{T}$. Because we can choose all these mapping independently on each other, we obtain, summing over all possible mappings f, the following identity.

$$\mathcal{L}_1(G) = \binom{c}{0}_2 \cdot 3^{\sum_{i=1}^{c} |V(T_i)|-1} = \binom{c}{0}_2 \cdot 3^{n-c}.$$

□

Observe that Lemma 4.6 implies that two same-order unicyclic graphs with the same-length cycle have the same number of Lipschitz mappings.

KC-transformation

In this section it will be useful for us to give a name to one special subset of unicyclic graphs. See Figure 4.9 for an example.

Definition 4.11. A *corolla graph* is a unicyclic graph obtained by taking a cycle graph and joining some path graphs to it by identifying their endpoints with some vertex of that cycle. Every path is joined to exactly one vertex of the cycle. And every vertex of the cycle has at most one path attached.

We note that cycles form a subset of corolla graphs.

Now we are ready to introduce the *generalized KC-transformation* and the main result of [61].

Definition 4.12 (Generalized KC-transformation)**.** Take a connected graph G and pick $\{a, b\} \in \binom{V(G)}{2}$. Let $V_{a;b}(G)$ denote the set of those vertices which cannot reach b without passing by a in G. If it is satisfied the following condition that

$$\min(|V_{a;b}(G)|, |V_{b;a}(G)|) > 1,$$

then we can get a new graph $G_{a \to b}$ by modifying G in the following way.

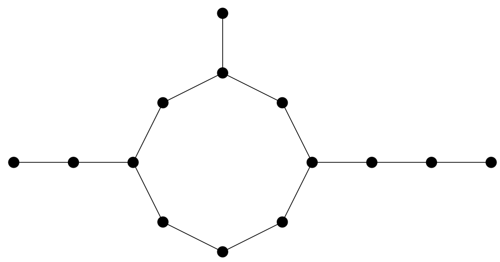

Figure 4.9: An example of corolla graph.

Remove the edges $bb_1, \ldots, bb_t$, where $b_1, \ldots, b_t$ are all the neighbors of b in $V_{b;a}(G)$ and add new edges $ab_1, \ldots, ab_t$.

Definition 4.13. Let G be a connected graph. Take two different cut vertices a and b of G. We write $V(G; a, b)$ for the set

$$\Big(V(G) \setminus (V_{a;b}(G) \cup V_{b;a}(G))\Big) \cup \{a, b\}.$$

Theorem 4.14. *[61] Let G be a connected graph. Take two different cut vertices a and b of G. Let H be the subgraph of G induced by $V(G; a, b)$. Assume that H has an automorphism σ such that $\sigma(a) = b$ and $\sigma(b) = a$. Then $\bar{\mathrm{r}}_1(G) \geq \bar{\mathrm{r}}_1(G_{a \to b})$.*

It is worth noting that one of the corollaries of Theorem 4.14 is the aforementioned Theorem 4.10. We will use Theorem 4.14 to show that for every unicyclic graph that is not a corolla graph there exists some corolla graph of the same order and cycle size that has higher or equal $\bar{\mathrm{r}}_1$.

Theorem 4.15. *For every unicyclic graph U on n vertices that is not a corolla graph there exist a corolla graph R on n vertices such that $\bar{\mathrm{r}}_1(R) \geq \bar{\mathrm{r}}_1(U)$.*

Proof. Take an inclusion-wise maximal tree T rooted in r such that r is a vertex of the cycle of U and T is not isomorphic to a path graph. Furthermore, T must satisfy $V(U) \cap V(T) = \{r\}$. Since U is not a corolla graph, such tree must exist.

Consider a sequence $T_1, T_2, \ldots, T_s$ with $T_1 = T$ and T_s being a path graph such that for every $i \in \{2, \ldots, s\}$, $\bar{\mathrm{r}}_1(T_{i-1}) \leq \bar{\mathrm{r}}_1(T_i)$ holds. The existence of such sequence directly follows from Theorem 4.14.

We can easily extend this argument and define the sequence $U_1, U_2, \ldots, U_s$ such that U_i is the graph in which T is replaced by T_i. Clearly, $\bar{\mathrm{r}}_1(U_{i-1}) \leq \bar{\mathrm{r}}_1(U_i)$.

We can repeatedly find another tree T' in U_s, satisfying the same conditions as T (except that the root has to be of different of course) in U and proceed similarly until we cannot find some next T'. We get a corolla graph and our claim follows. □

We suspect that this theorem might be the first step to prove LNR and BHM conjectures for the class of pseudotrees.

Chapter 5

Extending partial Lipschitz mappings

While studying Lipschitz mappings we came up with an algorithmic problem which falls into widely studied paradigm of a partial structure extension. We give three examples of such problems to show a broader context.

5.1 Related problems

The following problems are briefly introduced together with some surveys and important references.

Precoloring extension. The following problem was introduced in the series of papers [3, 26, 27].

PROBLEM:	PRECOLORING EXTENSION
INPUT:	An integer $k \geq 2$, a graph $G = (V, E)$ with $\lvert V\rvert \geq k$, a vertex subset $W \subseteq V$, and a proper k-coloring of G_W.
QUESTION:	Can this k-coloring be extended to a proper k-coloring of the whole graph G?

To current date, more than twenty papers on the precoloring extension problem were published. No up-to-date survey is available, but Daniel Marx gathers an unofficial list of relevant papers on his webpage:

`http://www.cs.bme.hu/~dmarx/prext.php`.

The partial representation extension problem. The reader surely knows a planar drawing of graph. A particular drawing of the underlying graph can be seen as one of the possible representations. Studying the representations of various graph classes is a wide area of graph theory and we refer reader to the comprehensive monograph of Spinrad [54]. One can ask for a given graph G and some partial representation R' of G if it can be extended to some full representation R of G such that $R' \subseteq R$. This problem was studied for various graph classes, for example intersection graph classes [32, 33, 31] or planar graphs [8]. For a presentation of state of the art in partial representation extension problems, consult the PhD thesis of Klavík [34].

Homogeneous structures. Macpherson in his survey [42] defines the *homogeneous structure* as a countable first order structure M over a relational language (usually assumed finite) such that any isomorphism between finite substructures of M extends to an automorphism of M.

Homogeneity of structures is widely studied question so for brevity, we refer interested reader to the aforementioned survey of Macpherson [42] or to Lachlan's one [36].

5.2 Definition of our problem

We will define two similar problems in the setting of integer homomorphisms:

PROBLEM:	Partial M-Lipschitz mapping extension - M-LIPEXT
INPUT:	A connected graph $G = (V, E)$, a subset $V' \subseteq V$ with a function $f' : V' \to \mathbb{Z}$.
QUESTION:	Does there exist an M-Lipschitz mapping f of G such that $f' \subseteq f$?

PROBLEM:	Partial strong M-Lipschitz mapping extension - STRONG M-LIPEXT
INPUT:	A connected bipartite graph $G = (V, E)$, a subset $V' \subseteq V$ with a function $f' : V' \to \mathbb{Z}$.
QUESTION:	Does there exist a strong M-Lipschitz mapping f of G such that $f' \subseteq f$?

If the answer for a given instance of M-LIPEXT (or STRONG M-LIPEXT) is YES, we say that f' is *extendable* for the given G and the given type of problem. We often say only that f' is extendable when it is clear from the context which problem we are trying to solve.

See Figure 5.1 for an initial example. Clearly, this mapping cannot be extended to a 1-Lipschitz mapping but it can be extended to an L-Lipschitz mapping for every $L \geq 2$.

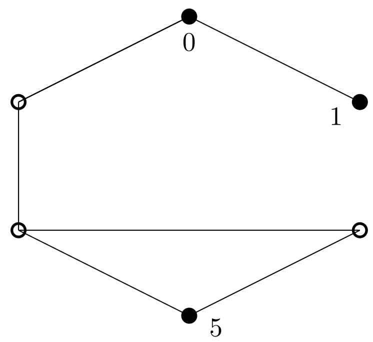

Figure 5.1: An example of a partial mapping with three prescribed vertices.

5.3 Partial strong M-Lipschitz mappings

We will show that Strong M-LipExt can be solved by a special *linear program (LP)* with the property that all its feasible solution are integral and a feasible solution exists if and only if f' is extendable.

Theorem 5.1. Strong M-LipExt *is solvable in polynomial time.*

Proof. We prove the theorem by constructing a linear program for the given instance with polynomially many inequalities of polynomial size. We will further show that we are interested in feasible solutions only and that if a feasible solution exists, it is always integral. As we know, LP can be solved in polynomial time with respect to the size of the program, as is explained e.g. in [43]. We note that it might be possible to simplify the following theorem by employing total unimodularity but we were unable to do that.

Step 1: Checking the mapping f'.

For the images of mapping f', we can easily check the necessary conditions on the difference of their images in quadratic time.

Note that if $V' = V(G)$ then it suffices to check the differences on every edge, check that at least one vertex maps to zero, and we are done.

So assume that $V(G) \setminus V'$ is nonempty and for all $u, v \in V'$, if $uv \in E(G)$ then $|f'(u) - f'(v)| = M$.

Step 2: Creating the LP.

We denote by $N_G(v)$ the set of vertices adjacent to v in G. We have the variables y_v for every vertex $v \in V(G) \setminus V'$. We need feasible solutions only, so the objective function is of no interest to us.

$$
\begin{array}{lrl}
\text{minimize} & 0 & \\
\text{subject to} & |f'(i) - y_j| = M & \forall i, j : i \in V', j \in N_G(i) \cap (V(G) \setminus V') \\
 & |y_k - y_l| = M & \forall k, l \in V(G) \setminus V', k \neq l
\end{array}
$$

However this is not a linear program yet. Absolute values violate the definition of LP. However, these can be changed to linear inequalities by the standard trick of adding two additional variables for each of inequalities (see for example [43]).

Step 3: Enforcing the existence of a root.

We still have a problem. We need to ensure that at least one vertex is mapped to zero, i.e. that we have some root. If there is no such vertex in V', i.e. $f'^{-1}(0) = \emptyset$, then, unfortunately, we do not have a strong Lipschitz mapping yet.

However, this can be fixed rather easily. We iterate over the vertices $r' \in V(G) \setminus V'$ and extend f' in the following way:

$$f' := f' \cup (r', 0).$$

Then we continue building our LP as previously. If there is no feasible solution, remove $(r', 0)$ from f' and try to add some other vertex $r'' \in V(G) \setminus V$.

Step 4: Complexity.

We will compute our LP program at most $O(V(G))$ times to ensure that we have G rooted. Every LP program consists of polynomially many inequalities, because we had one $O(E(G))$ inequalities with size bounded by a constant and by striping the absolute values away, we increased the number of inequalities only by some constant factor.

Thus the size of our program is polynomial in $V(G)$ and $E(G)$ and we indeed have a polynomial running time of our algorithm due to the polynomiality of linear programming.

Step 5: Correctness.

We claim that every output of our program is integral. Every f'' for which we build our LP has some vertex mapped to zero. Inequalities ensure that neighboring vertices get values either $+M$ or $-M$; an integer value. We can proceed inductively and integrality follows.

To finish, we need to prove the following. For a given instance of Strong M-LipExt, our algorithm outputs a complete M-Lipschitz mapping f, such that $f' \subseteq f$, if and only if f' is extendable.

Every output of our program has to satisfy that some vertex from $V(G)$ is mapped to zero and it also has to satisfy that for every $uv \in E(G), |f(u) - f(v)| = M$. Thus it an M-Lipschitz mapping of G. That finishes the proof of the if part.

If for every assignment of integers to the program variables there is no feasible solution, then some of the inequalities must be violated. Inequalities of our LP are precisely the conditions from the definition of strong M-Lipschitz mapping and therefore f' is not extendable. $\square$

5.4 Partial M-Lipschitz mappings

A simple generalization of the previous algorithm does not work for M-Lipschitz mappings, or at least we were unable to prove it. However, we were able to find a polynomial algorithm in the case that the input graph is a tree. We present Algorithm 1 for solving M-ParExt on trees.

Algorithm 1 Wave algorithm for M-ParExt on trees.

Require: A tree graph G, a vertex set $V' \subseteq V(G)$, and a partial M-Lipschitz mapping $f' : V' \to \mathbb{Z}$.

Check if $|f'(v) - f'(u)| \leq M$ for all $u, v \in V'$. If not, f' cannot be extended.
Set $P(v) := [f'(v), f'(v)]$ for every $v \in V'$.
Set $P(v) := [-\infty, \infty]$ for every $v \in V(G) \setminus V'$.

for every v' in V' **do**
  Start the DFS on G from v'.
  In DFS, whenever you process vertex v with $P(v) = [\underline{P}(v), \overline{P}(v)]$, do the following:
  for every $w \in N_G(v)$ **do**
    $P(w) := [\underline{P}(v) - M, \overline{P}(v) + M] \cap P(w)$.
  end for
end for

Find $r \in V(G)$ such that $0 \in P(v)$.
if no such r **then**
  return The mapping f' cannot be extended.
end if
Set $f(r) := 0$.

if $P(v) = \emptyset$ for some $v \in V(G)$ **then**
  return The mapping f' cannot be extended.
end if

Launch the BFS from r and for every visited vertex $v \neq r$, set $f(v)$ so that for parent vertex p, $f(v) \in [f(p) - M, f(p) + M]$ holds.

if the previous BFS could not be completed **then**
  return The mapping f' cannot be extended.
end if

return The mapping $f : V(G) \to \mathbb{Z}$.

We will now prove the correctness and complexity of this algorithm.

Lemma 5.1 (Correctness)**.** *Algorithm 1 is correct. It finds an M-Lipschitz mapping f that extends f' if and only if f' is extendable.*

Proof. We will write V for $V(G)$ and we will denote the iterations of code between the lines 4 and 10 *DFS phases* and the code executed between the lines 19 and 22 the *BFS phase.*

Suppose that the algorithm returns a mapping f. We claim that it is an M-Lipschitz mapping extending f'. Obviously, there exists a vertex mapped to zero under f – the vertex r. Furthermore, the condition

$$|f(u) - f(v)| \leq M, \forall uv \in E(G)$$

holds, otherwise the algorithm would stop on Line 19. Finally, we observe that for every $v' \in V'$, interval $P(v')$ is equal to $[f'(v'), f'(v')]$ at the end of the algorithm so f extends f'. That finishes the only if part of the equivalence.

Now let us prove the if part. We will prove that if the algorithm does not find an M-Lipschitz mapping f extending f', then f' is not extendable.

Algorithm can stop and fail to find such f exactly from the following reasons:

1. *Algorithm could not find a candidate for the root. (Line 13)*

   If at the end of the algorithm for every vertex $v \in V$, $0 \notin P(v)$, then for every $v \in V$ exists some vertex $v' \in V'$ such that $|f(v')| > M \cdot d(v, v')$. Clearly, f' is not extendable.

2. *There exists $v \in V$ such that $P(v) = \emptyset$. (Line 17)*

   If such v exists, then it implies that there exist two vertices $c, d \in V'$ such that the intersection $I = [f'(c) - M \cdot (c, v), f'(c) + M \cdot (c, v)] \cap [f'(d) - M \cdot (d, v), f'(d) + M \cdot (d, v)]$ is empty. However, I is exactly the set of all possible images that we can assign to v if c is set to $f'(c)$ and d is set to $f'(d)$. We conclude that f' is not extendable.

3. *Algorithm could not complete the BFS phase. (Line 21)*

   We will actually show that this case is not possible since the only possibility how 3) can happen is that some final interval $P(v)$ for some $v \in V$ is empty and the algorithm will halt even before the BFS phase can start (more precisely, the algorithm will stop at line 17).

   Assume that all intervals $P(v)$ are nonempty. Consider an edge $xy \in E(G)$. Assume further without loss of generality that in the last DFS phase (line 6), x was processed before y. Consider intervals $P'(x), P'(y)$ defined as the intervals $P(x), P(y)$, respectively, before the last DFS phase. Clearly, when x was processed in the last DFS phase, $P'(y) \cap [P'(x) - M, P'(x) + M]$ was set to a nonempty interval and therefore,

   $$\forall i \in P(x), \exists j \in P(y) : |i - j| \leq M.$$

   And conversely,

   $$\forall j \in P(y), \exists i \in P(x) : |i - j| \leq M.$$

   We conclude that the case 3) cannot occur.

This proves the if part and we are done. □

Lemma 5.2 (Complexity)**.** *Algorithm 1 is quadratic, i.e. its time complexity is $O(n^2)$, given that n is the number of vertices of the input graph.*

Proof. We are running $O(|V(G)|)$ times DFS on G plus we perform a constant number of linear traversals of data structure for G. That concludes that the algorithm runs in quadratic time. □

From these two lemmas we conclude the following theorem.

Theorem 5.2. *M-ParExt for trees is solvable in quadratic time.*

Of course, the natural open problem is to ask whether it is possible to find a polynomial algorithm for M-ParExt on general graphs. We strongly suspect that M-ParExt is indeed polynomial-time solvable for general graphs.

Problem 5.1. Decide the complexity of M-ParExt on general graphs.

Chapter 6

Conclusion

We presented a survey of results in the area of graph-indexed random walks, presented similar problems, showed new results and listed a couple of open problems.

We would like to summarize our own main results.

- Exact formulas for the average range of some classes of graphs. Namely,
  - complete graphs (Theorem 4.5),
  - complete bipartite graphs (Theorem 4.7) and stars (Theorem 4.8),
  - paths (Theorem 4.9),
  - cycles (Theorem 4.12).

  For all these cases except for paths we also showed the limiting behavior.
- We prove an exact formula for maximum range in Chapter 3.
- Chapter 5 introduces the problem of partial Lipschitz mapping extension. We show that the problem is:
  - polynomial for all graphs if we are extending strong M-Lipschitz mappings,
  - polynomial for trees if we are extending M-Lipschitz mappings.

6.1 Open problems

The area of graph-indexed random walks is full of unsolved and, by my humble opinion, both interesting and difficult problems. To stimulate research in the area, we conclude this thesis with a list of open problems. Together with the open problems mentioned in the previous text they make a modest collection of problems to solve. Some of these problems are work in progress.

We quickly remind the open problems that were already mentioned:

- Deciding if Conjecture 4.1 and Conjecture 4.2 are true or false. Do these conjectures hold when restricted to some superset of trees; for example pseudotrees or even cacti graphs? A *cactus graph* is a graph with the property that every edge belongs to at most one cycle.
- Problem 4.1 on the limiting behavior of the average range of 1-Lipschitz mappings of hypercubes.
- Finding a polynomial-time algorithm for M-ParExt on general graphs.

The following problems were not mentioned in the previous text. For the first one, we need to define the *average endomorphism range*. By $\mathrm{End}(G)$ we denote the set of all endomorphisms of G.

Definition 6.1. The *average endomorphism range* of a connected graph G is a function $\bar{\mathrm{r}}^{\mathrm{End}} : G \to \mathbb{R}$, such that

$$\bar{\mathrm{r}}^{\mathrm{End}}(G) := \frac{\sum_{\sigma \in \mathrm{End}(G)} \mathrm{r}(\sigma)}{|\mathrm{End}(G)|}.$$

Now we can state the problem, originally posed in [61].

Problem 6.1. Decide if the following holds for all $n \in \mathbb{N}$, and connected $G \in \mathcal{G}_n$,

$$\bar{\mathrm{r}}^{\mathrm{End}}(P_n) \geq \bar{\mathrm{r}}^{\mathrm{End}}(G) \geq \bar{\mathrm{r}}^{\mathrm{End}}(S_n).$$

As we saw in Section 4.2, Csikvári and Lin already studied this question for the number of endomorphisms and they proved [12, Theorem 1.8] that for every $G \in \mathcal{G}_n$, the inequality $|\mathrm{End}(P_n)| \geq |\mathrm{End}(G)| \geq |\mathrm{End}(S_n)|$ holds.

The second problem is connected to Section 4.3.

Problem 6.2. Find a characterization of graphs $G = (V, E)$ such that for every edge $e \in E$, and graph $G \setminus e$, holds that

$$\bar{\mathrm{r}}_1(G) \leq \bar{\mathrm{r}}_l(G \setminus e).$$

And finally, the following problem settled in affirmative would imply Conjecture 4.1 and 4.2 by taking Theorem 4.10 into account.

Problem 6.3. Is it true that for every connected graph G and its spanning tree T

$$\bar{\mathrm{r}}_1(T) \geq \bar{\mathrm{r}}_1(G) \quad \text{and} \quad \bar{\mathrm{r}}_{\pm 1}(T) \geq \bar{\mathrm{r}}_{\pm 1}(G)?$$

On the other hand, if Problem 6.3 does not hold, conjectures might still be true. As this problem seems to be quite difficult, at least by my opinion, it may be convenient to try to prove it for some restricted class of graphs first.

Appendix A

Code listings

To experimentally prove Conjecture 4.2 and Conjecture 4.1 for small-order graphs we have written a tester program in C++. All the following code can be downloaded, together with further instructions, from

`http://iuuk.mff.cuni.cz/~bok/master_thesis/`.

All the input graphs were generated using *nauty* and *Traces* by McKay and Piperno [44].

```
#include <iostream>
#include <vector>
#include <stdlib.h>

using namespace std;

int number = 0, rangesum = 0, onesidesum = 0, vertices, edges;
vector<int> distribution;
vector<int> sumwithzerosteps;

vector< vector<int> > graph;

void test(int remains, vector<bool> ismapped, vector<int> &mapping, vector<int> last, int rmin, int rmax, int next, int nextimage, int zeroes, bool strong)
{
  bool isnextset = false;

  for(int i=0; i<graph[next].size(); ++i)
    if(ismapped[graph[next][i]] && abs(mapping[graph[next][i]] - nextimage) > 1)
      return;

  ismapped[next] = true;
  mapping[next] = nextimage;
  rmin = min(rmin,nextimage);
  rmax = max(rmax,nextimage);
  last.push_back(next);

```

```
  if(!isnextset)
    for(int i=last.size()-1; i>=0; --i)
      if(ismapped[i])
        for(int j=0; j<graph[i].size(); ++j)
          if(!ismapped[graph[i][j]] && !isnextset)
          {
            next = graph[i][j];
            nextimage = mapping[i];
            isnextset = true;
          }

  if(remains == 1)
  {
    rangesum += rmax-rmin+1;
    onesidesum += rmax;
    ++distribution[rmax];
    ++number;
    sumwithzerosteps[zeroes] += rmax-rmin+1;
  }
  else
  {
    test(remains-1,ismapped,mapping,last,rmin,rmax,next,nextimage-1,zeroes,strong);
    test(remains-1,ismapped,mapping,last,rmin,rmax,next,nextimage+1,zeroes,strong);
    if(!strong)
      test(remains-1,ismapped,mapping,last,rmin,rmax,next,nextimage,zeroes+1,strong);
  }
};

int main()
{
  int a, b;
  vector<int> empty;

  cin >> vertices >> edges;

  vector<bool> ismapped(vertices,false);
  vector<int> mapping(vertices,0);
  distribution.resize(vertices+1,0);
  sumwithzerosteps.resize(vertices+1,0);

  for(int i=0; i<vertices; ++i)
    graph.push_back(empty);

  for(int i=1; i<=edges; ++i)
  {
    cin >> a >> b;
    graph[a].push_back(b);
    graph[b].push_back(a);
  }

  // the last parameter determines if the mappings will be strong or not
```

```
  test(vertices,ismapped,mapping,empty,0,0,0,0,0,true);

  cout << "The average range in fraction form: " << rangesum << " / " << number << endl;
  cout << "The average range: " << rangesum*1.0/number << endl;
  cout << "Distribution: " << endl;
  for(int i=0; i<=vertices; ++i)
    cout << i << "\t";
  cout << endl;
  for(int i=0; i<=vertices; ++i)
    cout << distribution[i] << "\t";
  cout << endl;

  return 0;
};
```

Listing A.1: Main tester `tester.cpp`

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