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Homogenization of flows of non-Newtonian fluids and strongly nonlinear elliptic systems

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Homogenizace toků nenewtonovských tekutin a silně nelineárních eliptických rovnic

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Abstrakt: Teorie homogenizace umožňuje nalézt pro zadaný systém parciálních diferenciálních rovnic popisující model s komplikovanou vnitřní strukturou systém popisující model bez této struktury, jehož řešení je v jistém smyslu aproximací řešení původního systému. V této práci jsou metody teorie homogenizace aplikovány na tři systémy parciálních diferenciálních rovnic, z nichž první popisuje proudění jisté třídy nenewtonowských tekutin porézním prostředím. Druhý se používá pro modelování proudění tekutin v elektrickém poli, jejichž viskozita se výrazně mění v závislosti na intenzitě elektrického pole. Ve třetím systému je uvažován eliptický operátor, jehož růst a koercivita jsou určeny obecnou anizotropní nehomogenní $\mathcal{N}$–funkcí.

Klíčová slova: Elektrorheologická tekutina, homogenizace, nelineární eliptický systém, nenewtonovská tekutina, $\mathcal{N}$–funkce, proudění porézním prostředím.

Title: Homogenization of flows of non-Newtonian fluids and strongly nonlinear elliptic systems

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Abstract: The theory of homogenization allows to find for a given system of partial differential equations governing a model with a very complicated internal structure a system governing a model without this structure, whose solution is in a certain sense an approximation of the solution of the original problem. In this thesis, methods of the theory of homogenization are applied to three systems of partial differential equations. The first one governs a flow of a class of non-Newtonian fluid through a porous medium. The second system is utilized for modeling of a flow of a fluid through an electric field wherein the viscosity depends significantly on the intensity of the electric field. For the third system is considered an elliptic operator having growth and coercivity indicated by a general anisotropic inhomogeneous $\mathcal{N}$–function.

Keywords: Electrorheological fluid, homogenization, flow through porous media, $\mathcal{N}$–function, nonlinear elliptic system, non-Newtonian fluids.

Contents

Introduction

The theory of homogenization was developed for the study of mathematical models for physical systems that feature several different length scales. In the case of two length scales we distinguish between the so-called macroscale and microscale. Plainly speaking the macroscale is the length scale on which the system interacts with its environment. The microscale is determined by a recurring property with the distance of recurrence much smaller than the size on the macroscale level. The example of a material with such an arrangement on the microscale level-the microstructure, are composites made of components with significantly different physical properties. The components of the composite remain separate and distinct in the resulting microstructure. The composite exhibits characteristics that differ from the individual components. The porous medium, which is a material with the microstructure formed by a solid matrix with interconnected void-the pores, also represents the example of a material possessing the two-scale nature.

The benefit of the theory of homogenization is that it provides methods that allow to establish models involving macroscopic quantities of the physical system that can be easily measured experimentally. Many strategies have been developed over the last decades for the passage from a model incorporating different length scales to a model incorporating the coarse scale only. In essence, all of these strategies follow the same idea. It consists in the identification of a parameter $0 < \varepsilon \ll 1$ that represents the distance of recurrence on the microscopic level of the physical system with two distinguished length scales. Then the asymptotics of the corresponding mathematical model is studied for ε tending to zero that involves finding an appropriate topology in which the two–scale model converges to a limit model. Moreover, apriori it is not guaranteed that any information about the microscale transfers into the limit model. In this context, it is appropriate to call the procedure of the transition from the model involving the parameter ε to a limit model as homogenization. Indeed, passing to the limit $\varepsilon \to 0$ causes shrinking of the microstructure and the limit model describes a physical system that is homogeneous from this point of view.

To illustrate the general description one can consider the derivation of celebrated Darcy's law which states that the velocity of the flow of a Newtonian fluid through a porous medium is proportional to the difference of an external force and pressure gradient. The idea of using the homogenization approach for deriving Darcy's law goes back to Tartar, see [6]. In this situation the physical model that incorporates two scales consists of a Newtonian fluid that occupies a porous medium. The flow of the fluid is modeled by the Stokes system with the homogeneous Dirichlet boundary condition in a domain whose microstructure is formed by periodically repeated cells consisting of a fluid and solid part with the size being described by a parameter $\varepsilon \ll 1$ and the fluid parts of all cells form a connected and open set. The homogenization process consists in two steps. In the first step one finds a suitable extension of the velocity and the pressure to the whole domain (to the part which is assumed to be solid). In the second step the parameter ε is let to tend to zero using the weak topology of spaces L^2 and $W^{1,2}$.

Dissertation summary

This thesis consists of three chapters that are based on original articles completed during my doctoral studies. In each of the chapter the homogenization process for a specific mathematical model is investigated. In the background of all three papers one finds the following scheme. A system of partial differential equations that describes the behavior of physical model incorporating both macro and micro scale, which is represented by a parameter $0 < \varepsilon \ll 1$, is considered at the beginning. After establishing the existence of a weak solution of the system for an arbitrary but fixed ε, the uniform estimate of the weak solution with respect to ε are obtained. In all three papers it is assumed that the arrangement on the micro scale level of the physical system is periodic. It is then natural to apply two-scale convergence method for the passage to the limit $\varepsilon \to 0$ to obtain the homogenized system. Especially, the weak compactness with respect to the topology of two-scale convergence ensures the existence of a weakly two-scale convergent sequence of weak solutions. In order to identify the limit of nonlinear terms involving the stress tensor a variant of Minty's trick is employed. The details of the application of Minty's trick depend on the fact whether one can test the weak formulation of the system with two scales as well as the homogenized system by a solution or not.

The results from the following articles are included in the thesis:

1. Martin Kalousek: Homogenization of incompressible generalized Stokes flows through a porous medium, *Nonlinear Anal.*, 136:1-39, 2016

   This article is devoted to the derivation of the Darcy-type law for both stationary and nonstationary flow of a non–Newtonian fluid via the homogenization of the Stokes system in the domain with a microstructure. It is assumed that the dependence of the viscosity of a fluid on the shear rate is determined by an $\mathcal{N}$–function with certain restrictions on growth.
   In order to be able to adopt the approach that was successfully used in order to derive Darcy's law, several facts that are known in the context of standard Lebesgue spaces are justified also in the context of Orlicz spaces. As particular results, the existence of restriction operator and its estimates in an Orlicz space as well as the characterization of annihilators of subspaces of Sobolev–Orlicz space of solenoidal functions are established.
   The assumption that the $\mathcal{N}$–function satisfies the $\Delta'-$condition appearing in the published paper was removed from the first chapter since it turned out to be redundant.

2. Miroslav Bulíček, Martin Kalousek, Petr Kaplický: Homogenization of an incompressible stationary flow of an electrorheological fluid, *Ann. Mat. Pura Appl.*, online first, 2016

   In this paper a mathematical model describing a flow of an electrorheological fluid through an electric field is considered. The dependence of the tensor on the shear rate is assumed to be given by a power function with a constant exponent and to capture significant changes in viscosity depending on the electric field. The stress tensor is also assumed to depend on a spatial variable.
   The assumed lower bound on the exponent causes that a weak solution of the two-scale system possesses the regularity which does not allow testing the weak

formulation of the system by its solution. This excludes the possibility of the direct application of Minty's trick. In order to overcome this inconvenience the pressure is decomposed into parts one of which is bounded in the same Lebesgue space as the stress tensor and the second part is precompact in the bigger Lebesgue space. This decomposition is combined with the method of Lipschitz truncation of Sobolev functions and the Div-Curl lemma to determine the limit of nonlinear term that is required for the application of Minty's trick. Let us point out that due to the structure of the stress tensor velocity gradients corresponding to different values of the parameter ε belong to the same Lebesgue space on which the maximal operator is bounded.

3. Miroslav Bulíček, Piotr Gwiazda, Martin Kalousek, Agnieszka Świerczewska-Gwiazda: Homogenization of nonlinear elliptic systems in nonreflexive Musielak-Orlicz spaces
   In the paper is studied the homogenization process for families of strongly nonlinear elliptic systems with the homogeneous Dirichlet boundary condition. The growth and the coercivity of the elliptic operator is assumed to be indicated by a general inhomogeneous anisotropic $\mathcal{N}$–function M, which which may be possibly also dependent on the spatial variable and in this case it is supposed to satisfy a condition of log–Hölder type.
   The general form of the $\mathcal{N}$–function and the generality of corresponding function spaces bring two challenges one has to face. First, the density of smooth functions in the strong topology is not available in considered function spaces. However, this issue is overcome by using the density of smooth functions in the so–called modular topology, which was recently exploited in obtaining results concerning the existence of solutions of nonlinear elliptic equations in Musielak-Orlicz spaces. The second issue is the lack of reflexivity of function spaces involved. Accordingly, the crucial property of the two–scale convergence, the compactness of bounded sets in the weak sense, is not available and the two–scale convergence method cannot be directly adopted. Nevertheless, the function spaces involved possess separable preduals. Consequently, they provide the compactness of bounded sets in the weak* sense. Therefore it is reasonable to establish the weak* two–scale convergence. Moreover, except for the weak* two–scale compactness of bounded sets also other properties of this notion of convergence analogous to the standard weak two–scale convergence are derived and used for the identification of the limit of the nonlinear term.

Conclusions

In Chapter 1 the two–scale convergence method was applied in modeling a non–Newtonian fluid flow through a porous medium. Starting from the stationary and also nonstationary generalized Stokes system governing the flow of a fluid with shear dependent viscosity the homogenized systems were derived. These systems do not involve the complicated structure of the porous medium. However, they possess an intrinsic two–scale coupling. Let us point out that the coupling has a nonlinear character, which comes from the assumed nonlinear dependence of the viscosity on the shear rate. In contrast with the case of constant viscosity, for which the homogenized system was derived in [2], when the linearity of the stress tensor allows to separate the scales, there is no simple way how to separate scales in the homogenized problems. One should not regard the structural complexity of the homogenized problems as a drawback. It cannot be expected that the homogenized systems describing complex phenomena have a simple form. Although the structure of the homogenized systems does not pose any complication in proving the existence of weak solutions of these systems, for which the theory of monotone operators is successfully applied, it poses challenges if the homogenized systems would be treated numerically.

In Chapter 2 the method of Lipschitz approximation of Sobolev functions was combined with the two–scale convergence in order to verify the following hypothesis. The homogenization of a system of partial differential equations governing a stationary flow of an electrorheological fluid can be performed under the same assumption on the exponent p, which ensures the existence of a weak solution of this system, i.e., $p > p_0 := \frac{2d}{d+2}$. Outcomes of this chapter have to be regarded as the first step on the way of the complete verification of this hypothesis. In fact, in the model considered in this chapter it was assumed that the growth of the viscous stress tensor is indicated by the power function with a constant exponent, which simplifies the model of the flow of an electrorheological fluid originated by Rajagopal and Růžička involving a power function with a variable exponent for the estimate of the growth of the stress tensor. Let us note that throughout the homogenization process the existence of the weak solution of the homogenized system, which is of the generalized Navier-Stokes type, was also shown.

In Chapter 3 the modified two–scale convergence method was applied to perform the homogenization process for families of elliptic systems, where the coercivity and growth of the nonlinear term is indicated by a general inhomogeneous anisotropic $\mathcal{N}$–function, which may depend also on the spatial variable, i.e., the homogenization process will change the characteristic function spaces at each step. Known results concerning homogenization of elliptic equations were obtained assuming the case of L^p–setting with restrictions on constant exponent, see e.g. [1] and [4], or variable exponent that is additionally log-Hölder continuous, e.g. [7], which correspond to the case of a very particular $\mathcal{N}$–function satisfying along with its conjugate $\mathcal{N}$–function the Δ_2 condition. Accordingly, all known results are restricted to reflexive spaces only. The result in this chapter was obtained without any assumptions on Δ_2 condition for an $\mathcal{N}$–function and its conjugate, which implies that a Musielak–Orlicz space corresponding to it is not reflexive. However, the key ingredient, density of smooth function in modular topology, requires a condition of log-Hölder type on the $\mathcal{N}$–function and a certain estimate of the quotient of the $\mathcal{N}$–function and the convex biconjugate of the infimum of the $\mathcal{N}$–function over the closed cube covering the spatial domain.

Finally, let us mention possible extensions of obtained results. It seems that the result on homogenization of generalized Stokes flows in a porous medium can be extended to the case of a stationary generalized Navier-Stokes flow assuming a suitable smallness of the external body force by adopting the approach from [3]. Nevertheless, the nonstationary case is completely open. The result of Chapter 2 could be extended also to the case of nonstationary system because the method of parabolic Lipschitz truncations has been developed already. Concerning the results of Chapter 3, again the extension to the parabolic case is possible since all necessary ingredients are available also for function spaces on the time–space domains. More challenging task would be to consider oscillations in time.

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Homogenization of incompressible generalized Stokes flows through a porous medium

Martin Kalousek

Abstract

We study the homogenization for families of steady and also unsteady incompressible generalized Stokes systems in a periodic porous medium. We assume that the stress tensor possesses an Orlicz growth and the size of solid parts of the porous medium is comparable to the size of the period. Homogenized systems are established using the two-scale convergence method adopted to Orlicz space setting. We prove the existence and uniqueness of weak solutions of the homogenized systems.

Keywords

Porous media flow, non-Newtonian fluid, shear-dependent viscosity, Orlicz space, periodic homogenization, two-scale convergence method

1.1 Description of the model and statement of the results

1.1.1 Introduction

In many areas, including oil recovery, biomechanical processes and civil engineering, it is important to study the low speed flow in porous media. Difficulties arise if such problems are treated numerically since rapid variations appear on the microscale level. One way to avoid these difficulties is using the homogenization theory to obtain a system of equations describing the macroscopic behavior of the fluid flow whose solution is in a certain sense an approximation of a solution of an initial system.

When modeling these processes, it is sometimes sufficient to consider a flow of a Newtonian fluid whose viscosity does not depend on the shear rate $\mathbf{Du}$, where $\mathbf{u}$ is a velocity of the flow. The simplest example of a system, which describes the macroscopic behavior of the incompressible flow of the Newtonian fluid in a porous medium, is Darcy's law. The rigorous derivation of this law based on the homogenization of the Stokes system with homogeneous Dirichlet data in a domain with a periodic microstructure was given in [34] by Tartar who introduced an abstract pressure extension. The explicit formula for pressure extension was derived by Lipton and Avellaneda in [22]. The influence of the asymptotic size of an element of the microstructure on a form of the homogenized system was studied by Allaire in [1],[2]. The homogenization of the nonstationary Stokes system was studied by Allaire in [4].

If the dependence of the viscosity of the fluid on the shear rate cannot be neglected, one can consider a generalized Newtonian fluid. The study of properties of weak solutions of systems governing the flow of such fluids started in Ladyzhenskaya's work [21] and was later developed in e.g. [24], [25] and [15]. The dependence of the viscosity on the shear rate can be given by a superlinear convex function, see e.g. [7], not only by a variant of a power function as is assumed in previous papers.

The homogenization process for a flow of a polymeric fluid, which has the shear dependent viscosity, was studied in [6].

In this paper we assume that the dependence of the viscosity on the shear rate is given by the formula $\eta(\mathbf{Du}) = \frac{\varphi'(|\mathbf{Du}|)}{|\mathbf{Du}|}$ for a function φ specified later. Using methods from some of the above mentioned works, the systems approximating the law governing a steady as well as unsteady generalized Newtonian fluid flow through a porous medium are derived. Let us denote by $\mathbf{u}$ the velocity, p the pressure, $\mathbf{f}$ the external force, $\nabla\mathbf{u}$ the velocity gradient, $\mathbf{Du} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ the symmetric part of $\nabla\mathbf{u}$.

The nondimensional incompressible stationary Stokes system can be written in the form

$$
\begin{aligned}
\mathbf{u}_t - \frac{1}{\mathscr{Re}} \operatorname{div}\left(\eta(\mathbf{Du})\mathbf{Du}\right) + \mathscr{Eu}\nabla p &= \frac{1}{\mathscr{Fr}}\mathbf{f} \text{ in } \Omega,\\
\operatorname{div}\mathbf{u} &= 0 \quad\quad \text{in } \Omega,\\
\mathbf{u} &= 0 \quad\quad \text{on } \partial\Omega.
\end{aligned}
$$

The viscosity function η characterizes the considered fluid. Fluids, for which η is non-constant, are called non-Newtonian fluids. Examples of fluids with such a behavior of η are very dilute polymeric liquids or low molecular weight biological liquids. In order to introduce characteristic numbers $\mathscr{Re}, \mathscr{Eu}$ and $\mathscr{Fr}$ we first mention the macroscopic characteristics of the porous media Ω: L_{ref} the reference length, V_{ref} the reference velocity, P_{ref} the reference pressure, F_{ref} the reference volume force and the characteristics of fluid: ρ the density, η_{ref} the reference viscosity. Then characteristic numbers are defined as follows:
Reynolds number $\mathscr{Re} = \frac{V_{ref}L_{ref}}{\eta_{ref}}$, Euler number $\mathscr{Eu} = \frac{P_{ref}}{V_{ref}^2\rho}$ and Froude number $\mathscr{Fr} = \frac{V_{ref}^2}{F_{ref}L_{ref}}$.

Let us consider that a microscopic scale l is small compared to the reference length L_{ref}. We denote $\varepsilon = \frac{l}{L_{ref}}$ and suppose that $\mathscr{Re}$ behaves as ε^{-1}, $\mathscr{Eu}$ and $\mathscr{Fr}$ behave as ε^0.

First, we assume the case of a steady flow. Assumptions on $\mathscr{Re}, \mathscr{Fr}, \mathscr{Eu}$ lead us to the stationary generalized Stokes system for the rescaled velocity $\mathbf{u}^\varepsilon = \varepsilon^{-1}\mathbf{u}$ and the pressure p^ε using the notation $\mathbf{S}(\mathbf{Du}^\varepsilon) = \eta(\mathbf{Du}^\varepsilon)\mathbf{Du}^\varepsilon$

$$
\begin{aligned}
-\varepsilon \operatorname{div}\mathbf{S}(\varepsilon\mathbf{Du}^\varepsilon) + \nabla p^\varepsilon &= \mathbf{f} \text{ in } \Omega^\varepsilon,\\
\operatorname{div}\mathbf{u}^\varepsilon &= 0 \text{ in } \Omega^\varepsilon,\\
\mathbf{u}^\varepsilon &= 0 \text{ on } \partial\Omega^\varepsilon,\\
\int_{\Omega^\varepsilon} p^\varepsilon &= 0,
\end{aligned}
\tag{SGS$_\varepsilon$}
$$

for which we establish the homogenized problem: Find a triplet $(\mathbf{u}^0, p, \pi) \in X^{1,\varphi}_{y,0}(\Omega \times Y)^d \times W^{1,\varphi^*}(\Omega) \times L^{\varphi^*}(\Omega \times Y)$ satisfying

$$
\begin{aligned}
-\operatorname{div}_y \mathbf{S}(\mathbf{D}_y\mathbf{u}^0(x,y)) + \nabla_y\pi(x,y) &= \mathbf{f}(x) - \nabla p(x) && \text{in } \Omega \times Y_F,\\
\operatorname{div}_y \mathbf{u}^0(x,y) &= 0 && \text{in } \Omega \times Y,\\
\operatorname{div}_x \left(\int_Y \mathbf{u}^0(x,y)\mathrm{d}y\right) &= 0 && \text{in } \Omega,\\
\mathbf{u}^0(x,y) &= 0 && \text{in } \Omega \times Y_S,\\
\left(\int_Y \mathbf{u}^0(x,y)\mathrm{d}y\right)\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega,\\
\int_{Y_F} \pi(x,y)\mathrm{d}y &= 0 && \text{for a.a. } x \in \Omega,\\
\int_\Omega p(x)\mathrm{d}x &= 0.
\end{aligned}
\tag{HSS}
$$

Even though authors in [6] assumed as an initial problem the Navier-Stokes system for a generalized Newtonian fluid, the scaling, which they used, caused that the form of their homogenized system coincides with the form of (HSS) in the case of the viscosity

with a power dependence on the shear rate.
We state the results concerning (HSS):

Theorem 1.1.1 *Let Ω, Y, Y_S, Y_F fulfill Assumption 1.1.5, φ fulfill Assumption 1.2.2 and $\mathbf{f} \in L^{\varphi^*}(\Omega)^d$. Then the problem* (HSS) *admits a unique weak solution.*

Theorem 1.1.2 *Let Ω, Y, Y_S, Y_F fulfill Assumption 1.1.5, φ fulfill Assumption 1.2.2, $\mathbf{f} \in L^{\varphi^*}(\Omega)^d$ and $(\mathbf{u}^0, p, \pi)$ be the unique weak solution of* (HSS)*. Let for $\varepsilon > 0$ $(\mathbf{u}^\varepsilon, p^\varepsilon)$ be the unique weak solution of* (SGS$_\varepsilon$)*. We extend $\mathbf{u}^\varepsilon$ by zero in $\Omega \setminus \Omega^\varepsilon$ and let P^ε be the extension of p^ε to Ω from Lemma 1.5.4. Then as $\varepsilon \to 0$*

$$\begin{aligned} \mathbf{u}^\varepsilon &\xrightarrow{2-s} \mathbf{u}^0 \ \text{in } L^\varphi(\Omega \times Y)^d, \\ \varepsilon \mathbf{D}\mathbf{u}^\varepsilon &\xrightarrow{2-s} \mathbf{D}_y \mathbf{u}^0 \ \text{in } L^\varphi(\Omega \times Y)^{d\times d}, \\ P^\varepsilon &\rightharpoonup p \ \text{in } L^{\varphi^*}(\Omega). \end{aligned}$$

If the flow is unsteady, we deal with the nonstationary generalized Stokes system

$$\begin{aligned} \mathbf{u}^\varepsilon_t - \varepsilon \operatorname{div} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon) + \nabla p^\varepsilon = \mathbf{f} \quad & \text{in } (0,T) \times \Omega^\varepsilon, \\ \operatorname{div} \mathbf{u}^\varepsilon = 0 \quad & \text{in } (0,T) \times \Omega^\varepsilon, \\ \mathbf{u}^\varepsilon = 0 \quad & \text{in } (0,T) \times \partial\Omega^\varepsilon, \\ \mathbf{u}^\varepsilon(0) = \mathbf{a}^\varepsilon \ & \text{in } \Omega^\varepsilon, \\ \int_\Omega p^\varepsilon = 0 \quad & \text{a.e. in } (0,T), \end{aligned} \tag{NGS$_\varepsilon$}$$

for which we establish a homogenized problem: Find a triplet $(\mathbf{u}^0, p, \pi) \in \Big(X^{1,\varphi}_{y,0}(Q_T \times Y) \cap L^\infty(0,T; L^2(\Omega \times Y)^d)\Big) \times W^{1,\varphi^*}_x(Q_T) \times L^{\varphi^*}(Q_T \times Y)$, here $Q_T = (0,T) \times \Omega$, satisfying

$$\begin{aligned} \mathbf{u}^0_t(t,x,y) - \operatorname{div}_y \mathbf{S}(\mathbf{D}_y \mathbf{u}^0(t,x,y)) + \nabla_y \pi(t,x,y) = \mathbf{f}(t,x) - \nabla p(t,x) \ & \text{in } Q_T \times Y_F, \\ \operatorname{div}_y \mathbf{u}^0(t,x,y) = 0 \quad & \text{in } Q_T \times Y, \\ \operatorname{div}_x \left(\int_Y \mathbf{u}^0(t,x,y) \mathrm{d}y \right) = 0 \quad & \text{in } Q_T, \\ \mathbf{u}^0(t,x,y) = 0 \quad & \text{in } Q_T \times Y_S, \\ \left(\int_Y \mathbf{u}^0(t,x,y) \mathrm{d}y \right) \cdot \mathbf{n} = 0 \quad & \text{in } (0,T) \times \partial\Omega, \\ \mathbf{u}^0(0,x,y) = \mathbf{a}^0(x,y) \quad & \text{in } \Omega \times Y, \\ \int_{Y_F} \pi(t,x,y) \mathrm{d}y = 0 \quad & \text{for a.a.} (t,x) \in Q_T, \\ \int_\Omega p(t,x) \mathrm{d}x = 0 \quad & \text{for a.a. } t \in (0,T). \end{aligned} \tag{HNS}$$

We state the results concerning (HNS):

Theorem 1.1.3 *Let Ω, Y, Y_S, Y_F fulfill Assumption 1.1.5, φ fulfill Assumption 1.2.2, $\mathbf{f} \in L^{\varphi^*}(Q_T)^d$, $\mathbf{a}^0 \in L^2_{y,0}(\Omega \times Y)^d$, $\operatorname{div}_y \mathbf{a}^0 = 0$ in $\Omega \times Y$. Then the problem* (HNS) *admits a unique weak solution $(\mathbf{u}^0, p, \pi)$.*

Theorem 1.1.4 *Let Ω, Y, Y_S, Y_F fulfill Assumption 1.1.5, φ fulfill Assumption 1.2.2 and*

$$\mathbf{f} \in L^\infty\left(0,T;L^{\varphi^*}(\Omega)^d\right),\ \ \mathbf{f}_t \in L^2(Q_T)^d. \tag{1.1}$$

Let the embedding

$$L^\varphi(\Omega) \hookrightarrow L^2(\Omega), \tag{1.2}$$

as well as

$$W_0^{m,2}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \tag{1.3}$$

for $m > 1 + \frac{d}{2}$ hold. Let an extension of an initial condition $\mathbf{a}^\varepsilon \in W^{m,2}_{0,\mathrm{div}}(\Omega^\varepsilon)^d$ by zero to $\Omega \setminus \Omega^\varepsilon$ satisfy the uniform bound

$$\int_\Omega |\mathbf{a}^\varepsilon|^2 + \int_\Omega |\varepsilon^{m-\frac{d}{2}}\nabla^m \mathbf{a}^\varepsilon|^2 \le c \tag{1.4}$$

and there exist a function $\mathbf{a}^0 \in X^{1,\varphi}_{y,0}(\Omega \times Y)$ such that

$$\mathbf{a}^\varepsilon \xrightarrow{2-s} \mathbf{a}^0 \ \textit{in}\ L^2(\Omega\times Y)^d \ \textit{and}\ \|\mathbf{a}^\varepsilon\|_{L^2(\Omega)} \to \|\mathbf{a}^0\|_{L^2(\Omega\times Y)}. \tag{1.5}$$

Let $(\mathbf{u}^0, p, \pi)$ be the weak solution of (HNS) *and for any $\varepsilon > 0$ $(\mathbf{u}^\varepsilon, p^\varepsilon)$ be the weak solution of* (NGS$_\varepsilon$)*. We extend $\mathbf{u}^\varepsilon$ by zero in $\Omega \setminus \Omega^\varepsilon$ and let P^ε be the extension of p^ε to Ω from Lemma 1.6.4. Then as $\varepsilon \to 0$*

$$\begin{aligned}
\mathbf{u}^\varepsilon &\xrightarrow{2-s} \mathbf{u}^0 && \textit{in}\ L^\varphi(Q_T\times Y)^d,\\
\mathbf{u}^\varepsilon_t &\xrightarrow{2-s} \mathbf{u}^0_t && \textit{in}\ L^2(Q_T\times Y)^d,\\
\varepsilon\mathbf{D}\mathbf{u}^\varepsilon &\xrightarrow{2-s} \mathbf{D}_y\mathbf{u}^0 && \textit{in}\ L^\varphi(Q_T\times Y)^{d\times d},\\
P^\varepsilon &\longrightarrow p && \textit{in}\ L^{\varphi^*}(Q_T).
\end{aligned}$$

For clarity, we recall the meaning of differential operators appearing in the paper. Let $\mathbf{u} : (0,T)\times\Omega\times Y \to \mathbb{R}^d$ then

$$\mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}, \nabla_x \mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j=1}^d, \mathrm{div}_x\,\mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i}, \nabla_y \mathbf{u} = \left(\frac{\partial u_i}{\partial y_j}\right)_{i,j=1}^d, \mathrm{div}_y\,\mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial y_i}.$$

We omit the subscript x for $\mathbf{u} : (0,T)\times\Omega \to \mathbb{R}^d$, i.e.,

$$\nabla \mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j=1}^d,\ \ \mathrm{div}\,\mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i}.$$

More detailed description of the pressure extension P^ε as well as assumptions on a sequence of initial values are given in Sections 1.5 and 1.6. Function spaces are introduced in Section 1.2.

1.1.2 Geometry of a porous medium

Let $d = 2$ or 3, we summarize assumptions on the unit cell $Y = (0,1)^d$, its solid part Y_S, fluid part Y_F and the domain Ω.

Assumption 1.1.5

$$Y_S \subset B_{r_0}(c_Y), r_0 \in \left(0, \frac{2}{5}\right) \textit{ is closed }, |Y_S| > 0, \partial Y_S \in C^k \;\; k > 1 + \frac{d}{2},$$
$$Y_F = Y \setminus Y_S \textit{ is open and connected},$$
$$\Omega \subset \mathbb{R}^d \textit{ is bounded}, \partial\Omega \in C^1,$$

where $B_{r_0}(c_Y)$ denotes the ball with the radius r_0 and the center c_Y, which stands for the center of Y.

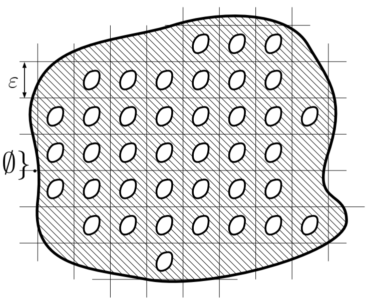
A periodic porous medium consists of a set Ω and an associated microstructure. The domain Ω is covered by a regular grid created by the periodic repetition of the cell arising from Y by rescaling its edge to the $\varepsilon-$length, $\varepsilon \in (0,1)$. An element of this grid is denoted Y_i^ε. Solid and fluid parts of Y_i^ε are rescaled in the same way and are denoted $Y_{S_i}^\varepsilon$ and $Y_{F_i}^\varepsilon$ respectively.

Let us denote

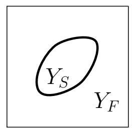
$I^\varepsilon = \{i : Y_{S_i}^\varepsilon \subset \Omega\}$, $H^\varepsilon = \{i : Y_{S_i}^\varepsilon \cap (\mathbb{R}^d \setminus \Omega) \neq \emptyset\}$.

Finally, we define the fluid part Ω^ε of a porous medium as

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{i \in I^\varepsilon} Y_{S_i}^\varepsilon. \tag{1.6}$$



the porous medium Ω^ε



the unit cell Y

1.2 Function spaces and preliminaries

This section contains the definition and properties of the $\mathcal{N}$–function. Proofs of statements are postponed to the appendix.

Definition 1.2.1 A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be an $\mathcal{N}$–function if φ is continuous, convex and satisfies

$$\lim_{t\to 0_+} \frac{\varphi(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty, \qquad \varphi(t) > 0 \text{ if } t > 0.$$

An $\mathcal{N}$–function φ^* defined as

$$\varphi^*(t) = \sup_{s\geq 0}\{st - \varphi(s)\},$$

is called the complementary $\mathcal{N}$–function to φ.
We say that φ satisfies Δ_2-condition if there is $K > 0$ such that for all $t > 0$ $\varphi(2t) \leq K\varphi(t)$. We denote $\Delta_2(\varphi)$ the smallest constant K having this property.

The generic constants are denoted by c. When circumstances require it, we may also include quantities on which the constant depends, e.g. $c(d)$ for the dependence on the dimension d. If we want to distinguish between different constants in one formula, we utilize subscripts, e.g. c_1, c_2 etc.

We write $f \sim g$ if there are positive constants c_1, c_2 such that $c_1 f \leq g \leq c_2 f$. As a consequence of $\Delta_2(\varphi) < \infty$ we have uniformly in $t \geq 0$:

$$\varphi'(t)t \sim \varphi(t), \tag{1.7}$$

$$\varphi^*(\varphi'(t)) \sim \varphi(t). \tag{1.8}$$

Moreover, if $\Delta_2(\varphi^*) < \infty$ then there are positive constants c_1, c_2 and $1 < q_1 \leq q_2 < \infty$ such that for $s, t \geq 0$

$$c_1 \min\{s^{q_1}, s^{q_2}\}\varphi(t) \leq \varphi(s,t) \leq c_2 \max\{s^{q_1}, s^{q_2}\}\varphi(t), \tag{1.9}$$

Young's inequality holds: For all $\delta > 0$ there is c_δ depending on $\Delta_2(\varphi, \varphi^*)$ such that for all $s, t \geq 0$

$$st \leq \delta\varphi(t) + c_\delta \varphi^*(s). \tag{1.10}$$

As a consequence of $\Delta_2(\varphi) < \infty$ and monotonicity of φ on $[0, \infty)$, we have the existence of $c > 0$ such that for any $s, t > 0$

$$\varphi(s+t) \leq c\left(\varphi(s) + \varphi(t)\right). \tag{1.11}$$

We summarize assumptions on the $\mathcal{N}$–functions φ and φ^*.

Assumption 1.2.2 *The $\mathcal{N}$–functions φ, φ^* satisfy*

$$\Delta_2(\varphi) < \infty,$$
$$\Delta_2(\varphi^*) < \infty,$$
$$\varphi \in C^2((0,\infty)), \tag{1.12}$$
$$\varphi'(t) \sim t\varphi''(t) \text{ uniformly in } t \geq 0. \tag{1.13}$$

As a prototype of an $\mathcal{N}$–function with such a behavior, we can consider

$$\varphi(t) = \frac{2}{p}\left((1+t^2)^{\frac{p}{2}} - 1\right), \tag{1.14}$$

$$\varphi(t) = \int_0^t s^{1-q}\left(\operatorname{arcsinh}(s)\right)^q \, \mathrm{d}s \tag{1.15}$$

with $p \in (1,\infty), q \in (0,1)$. For more details see [19, Theorem 4.3., Theorem 5.2.]. The experimental study of the dependence of the viscosity of generalized Newtonian fluids on the shear rate yielded several models, see [5]. The $\mathcal{N}$–function (1.14) induces the well known power-law model

$$\mathbf{S}(\mathbf{D}) = (1 + |\mathbf{D}|^2)^{\frac{q-2}{2}}\mathbf{D}.$$

Another example is the Sutterby model

$$\mathbf{S}(\mathbf{D}) = \left(\frac{\operatorname{arcsinh}(|\mathbf{D}|)}{|\mathbf{D}|}\right)^q \mathbf{D}$$

induced by the $\mathcal{N}$–function (1.15).

Lemma 1.2.3 *Let $\mathcal{N}$–function φ satisfies* (1.12) *and* (1.13) *then there is $c > 0$ such that for any $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times d}_{sym}$*

$$(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q}))(\mathbf{P} - \mathbf{Q}) \geq c\varphi''(|\mathbf{P}| + |\mathbf{Q}|)|\mathbf{P} - \mathbf{Q}|^2,$$
$$|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| \leq c\varphi''(|\mathbf{P}| + |\mathbf{Q}|)|\mathbf{P} - \mathbf{Q}|.$$

Proof. See [9, Lemma 21]. □

For an open set $\Omega \subset \mathbb{R}^d$ we define the Orlicz space $L^\varphi(\Omega)$ as

$$L^\varphi(\Omega) = \left\{ v \in L^1_{loc}(\Omega) : \int_\Omega \varphi\left(|v(x)|\right)\, \mathrm{d}x < \infty \right\}.$$

The mapping $\mathbf{v} \mapsto \int_\Omega \varphi\left(|v(x)|\right)\, \mathrm{d}x : L^\varphi(\Omega)^d \to [0, \infty)$ is called a modular. $L^\varphi(\Omega)$ equipped with the Luxemburg norm

$$\|v\|_{L^\varphi(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \varphi\left(\frac{|v(x)|}{\lambda}\right)\, \mathrm{d}x \leq 1 \right\}$$

is Banach space. Some of its properties are summarized here.

Lemma 1.2.4 *Let $\Delta_2(\varphi) < \infty$ then*

1. *$\mathcal{D}(\Omega)$ is dense in $L^\varphi(\Omega)$,*
2. *$L^\varphi(\Omega)$ is separable,*
3. *$L^\varphi(\Omega)$ is reflexive whenever $\Delta_2(\varphi^*) < \infty$,*
4. *for given $f \in L^\varphi(\Omega)$, $g \in L^{\varphi^*}(\Omega)$, $fg \in L^1(\Omega)$ and the generalized Hölder's inequality holds*
$$\int_\Omega fg \leq 2\|f\|_{L^\varphi(\Omega)}\|g\|_{L^{\varphi^*}(\Omega)}.$$
5. *For $\{f^n\} \subset L^\varphi(\Omega), f \in L^\varphi(\Omega)$ we have $\lim_{n\to\infty} \int_\Omega \varphi(|f^n - f|) = 0$ if and only if $\lim_{n\to\infty} \|f^n - f\|_{L^\varphi(\Omega)}$.*

Proof. See [28]. □

We define Sobolev-Orlicz space $W^{1,\varphi}(\Omega)$ as

$$W^{1,\varphi}(\Omega) = \left\{ v \in L^\varphi(\Omega) : \nabla v \in L^\varphi(\Omega)^d \right\}.$$

equipped with the norm $\|v\|_{W^{1,\varphi}(\Omega)} = \|v\|_{L^\varphi(\Omega)} + \|\nabla v\|_{L^\varphi(\Omega)^d}$.

We introduce the following function spaces

$\mathcal{D}(\Omega)$ smooth functions compactly supported in Ω,

$L^{\varphi}(\Omega)/\mathbb{R}$ the set of classes of functions from $L^{\varphi}(\Omega)$, functions from a class differ from each other for an additive constant,

$W_0^{1,\varphi}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,\varphi}}}$,

$W_{0,\mathrm{div}}^{1,\varphi}(\Omega) = \overline{\{v \in \mathcal{D}(\Omega); \operatorname{div} v = 0 \text{ in } \Omega\}}^{\|\cdot\|_{W^{1,\varphi}}}$,

$W_x^{1,\varphi}(Q_T) = \{v \in L^{\varphi}(Q_T) : \nabla_x v \in L^{\varphi}(Q_T)^d\}$ with the norm $\|\cdot\|_{W_x^{1,\varphi}(Q_T)} = \|\cdot\|_{L^{\varphi}(Q_T)} + \|\nabla\cdot\|_{L^{\varphi}(Q_T)}$,

$W_{x,0}^{1,\varphi}(Q_T) = \overline{C_0^{\infty}(Q_T)}^{\|\cdot\|_{W_x^{1,\varphi}}}$,

$W_{x,0,\mathrm{div}}^{1,\varphi}(Q_T) = \overline{\{v \in C_0^{\infty}(Q_T); \operatorname{div} v = 0 \text{ in } Q_T\}}^{\|\cdot\|_{W_x^{1,\varphi}}}$,

$C^{\infty}(Y)$ smooth, $Y-$periodic functions on Y,

$W^{1,\varphi}(Y) = \left\{v \in W_{loc}^{1,\varphi}\left(\mathbb{R}^d\right) : v, \nabla v \ Y\text{-periodic}\right\}$,

$\mathcal{D}(\Omega; C^{\infty}(Y))$ smooth functions compactly supported in Ω with values in $C^{\infty}(Y)$,

$L_{y,0}^2(\Omega \times Y) = \{v \in L^2(\Omega \times Y) : v = 0 \text{ in } \Omega \times Y_S\}$,

$W_{0,\mathrm{div}_y}^{m,2}(\Omega \times Y) = \overline{\{v \in \mathcal{D}(\Omega; C^{\infty}(Y)) : v = 0 \text{ in } \Omega \times Y_S, \operatorname{div}_y v = 0 \text{ in } \Omega \times Y\}}^{\|\cdot\|_{W^{m,2}}}$,

$L^{\varphi}(\Omega \times Y) = \left\{v \in L_{loc}^{\varphi}(\Omega \times \mathbb{R}^d) : y \mapsto v(x,y) \text{ is } Y\text{-periodic for a.a. } x \in \Omega\right\}$,

$L^{\varphi}(Q_T \times Y) = \left\{v \in L_{loc}^{\varphi}(Q_T \times \mathbb{R}^d) : y \mapsto v(t,x,y) \text{ is } Y\text{-periodic for a.a. } (t,x) \in Q_T\right\}$,

$W_{y,0}^{1,\varphi}(\Omega \times Y) = \{\mathbf{v} \in L^{\varphi}(\Omega \times Y)^d : \nabla_y \mathbf{v} \in L^{\varphi}(\Omega \times Y)^{d\times d}, \mathbf{v} = 0 \text{ in } \Omega \times Y_S\}$ equipped with the norm $\|\cdot\|_{W_{y,0}^{1,\varphi}(\Omega\times Y)} = \|\cdot\|_{L^{\varphi}(\Omega\times Y)} + \|\nabla_y\cdot\|_{L^{\varphi}(\Omega\times Y)}$,

$W_{y,0}^{1,\varphi}(Q_T \times Y) = \{\mathbf{v} \in L^{\varphi}(Q_T \times Y)^d : \nabla_y \mathbf{v} \in L^{\varphi}(Q_T \times \mathbb{R}^d)^{d\times d}, \mathbf{v} = 0 \text{ in } Q_T \times Y_S\}$,

$$X_{y,0}^{1,\varphi}(\Omega \times Y) = \Bigg\{\mathbf{v} \in W_{y,0}^{1,\varphi}(\Omega \times Y) : \operatorname{div}_y \mathbf{v} = 0 \text{ in } \Omega \times Y, \operatorname{div}_x\left(\int_Y \mathbf{v}\right) = 0 \text{ in } \Omega, \left(\int_Y \mathbf{v}\right)\cdot \mathbf{n} = 0 \text{ on } \partial\Omega\Bigg\},$$

$$X_{y,0}^{1,\varphi}(Q_T \times Y) = \Bigg\{\mathbf{v} \in W_{y,0}^{1,\varphi}(Q_T \times Y) : \operatorname{div}_y \mathbf{v} = 0 \text{ in } Q_T \times Y, \operatorname{div}_x\left(\int_Y \mathbf{v}\right) = 0 \text{ in } Q_T, \left(\int_Y \mathbf{v}\right)\cdot \mathbf{n} = 0 \text{ on } (0,T)\times\partial\Omega\Bigg\},$$

$$X_{y,0}^{m,2}(\Omega \times Y) = \Bigg\{\mathbf{v} \in W^{m,2}(\Omega \times Y)^d : \mathbf{v} = 0 \text{ in } \Omega \times Y_S, \operatorname{div}_y \mathbf{v} = 0 \text{ in } \Omega \times Y, \operatorname{div}_x\left(\int_Y \mathbf{v}\right) = 0 \text{ in } \Omega, \left(\int_Y \mathbf{v}\right)\cdot \mathbf{n} = 0 \text{ on } \partial\Omega\Bigg\}.$$

Remark 1.2.5

(i) Further, whenever a space of functions on Y, $\Omega\times Y$ or $Q_T\times Y$ appears, it is always assumed that functions are $Y-$periodic. Whenever the subscript $_C$ appears in the description of a function space, it stands for functions with a compact support in a considered set.

(ii) The divergence of function from $X^{1,\varphi}$ averaged over Y with respect to x is understood as a functional on $W^{1,\varphi^*}(\Omega)$. The normal component of a function from $X^{1,\varphi}$ on $\partial\Omega$ is understood as an extension of a functional defined for smooth functions on $\overline{\Omega} \times Y$, for details see [17, Chapter III, sect. 2].

(iii) The properties of N-function φ listed at the beginning of the section are not sufficient for the identification $L^\varphi(A; L^\varphi(B)) = L^\varphi(A\times B)$, where A, B are measurable subsets of $\mathbb{R}^d$. The exact condition on φ that ensures this identification is stated in [14, Proposition 1.3]. It implies that $\mathcal{N}$–function satisfying this condition has to be a power function and therefore one looses the generality. That is the reason why Bochner–Orlicz spaces do not appear throughout the paper. The embedding $L^\varphi(A \times B) \subset L^1(A; L^\varphi(B))$ always holds, see [14, Corollary 1.1.0].

Lemma 1.2.6 *Let $\Delta_2(\varphi, \varphi^*) < \infty$ and Ω^ε be defined by (1.6). There exist $c_1, c_2, c_3, c_4 > 0$ such that for any $\mathbf{v} \in W_0^{1,\varphi}(\Omega^\varepsilon)$ and for any $\varepsilon > 0$*

$$\|\mathbf{v}\|_{L^\varphi(\Omega^\varepsilon)} \leq c_1\varepsilon\|\nabla\mathbf{v}\|_{L^\varphi(\Omega^\varepsilon)} \leq c_2\varepsilon\|\mathbf{D}\mathbf{v}\|_{L^\varphi(\Omega^\varepsilon)} \tag{1.16}$$

$$\int_{\Omega^\varepsilon} \varphi(|\mathbf{v}|) \leq c_3 \int_{\Omega^\varepsilon} \varphi(\varepsilon|\nabla\mathbf{v}|) \leq c_4 \int_{\Omega^\varepsilon} \varphi(\varepsilon|\mathbf{D}\mathbf{v}|). \tag{1.17}$$

Remark 1.2.7 One shows $\|\mathbf{v}\|_{L^\varphi(\Omega\times Y)} \leq c\|\nabla_y\mathbf{v}\|_{L^\varphi(\Omega\times Y)}$ for any $\mathbf{v} \in W^{1,\varphi}_{y,0}(\Omega \times Y)$ using similar consideration as in the proof of Lemma 1.2.6. Thus $\|\nabla_y \cdot \|_{L^\varphi(\Omega\times Y)}$ is the equivalent norm on $W^{1,\varphi}_{y,0}(\Omega \times Y)$. Similarly $\|\nabla_y \cdot \|_{L^\varphi(Q_T\times Y)}$ is the equivalent norm on $W^{1,\varphi}_{y,0}(Q_T \times Y)$.

Lemma 1.2.8 *Let Ω, Y, Y_S, Y_F fulfill Assumption 1.1.5 and φ fulfill Assumption 1.2.2. Then there is a family of linear continuous operators $\{T^\delta\}$,*

$$T^\delta : W^{1,\varphi}_{y,0}(\Omega \times Y) \to \left\{\mathbf{v} \in \mathcal{D}\left(\Omega; C^\infty(Y)^d\right) : \mathbf{v} = 0 \text{ in } \Omega \times Y_S\right\}$$

such that as $\delta \to 0$

$$\|\nabla_y(T^\delta\mathbf{u} - \mathbf{u})\|_{L^\varphi(\Omega\times Y)} \to 0.$$

Furthermore, $\operatorname{div}_x \left(\int_Y T^\delta\mathbf{u}\right) = 0$ in Ω if $\operatorname{div}_x \left(\int_Y \mathbf{u}\right) = 0$ in Ω and $\operatorname{div}_y T^\delta\mathbf{u} = 0$ in $\Omega\times Y$ if $\operatorname{div}_y \mathbf{u} = 0$ in $\Omega \times Y$. We get the existence of $\{T^\delta\}$ having corresponding properties also for the time-dependent case. Consequently, we obtain

(1) $\mathcal{C} = \left\{\mathbf{v} \in \mathcal{D}\left(\Omega; C^\infty(Y)^d\right) : \mathbf{v} = 0 \text{ in } \Omega \times Y_S\right\}$ is dense in $W^{1,\varphi}_{y,0}(\Omega \times Y)$.

(2) $\mathcal{C}_D = \left\{\mathbf{v} \in \mathcal{D}\left(\Omega; C^\infty(Y)^d\right) : \mathbf{v} = 0 \text{ in } \Omega \times Y_S, \operatorname{div}_y \mathbf{v} = 0 \text{ in } \Omega \times Y,\right.$ $\left.\operatorname{div}_x \left(\int_Y \mathbf{v}\right) = 0\right\}$ is dense in $X^{1,\varphi}_{y,0}(\Omega \times Y)$.

(3) $\mathcal{C}^T = \left\{\mathbf{v}\in C^\infty\left([0,T]; \mathcal{D}\left(\Omega; C^\infty(Y)^d\right)\right) : \mathbf{v} = 0 \text{ in } \Omega\times Y_S\right\}$ is dense in $W^{1,\varphi}_{y,0}(Q_T\times Y)$.

(4) $\mathcal{C}^T_D = \left\{\mathbf{v}\in C^\infty\left([0,T]; \mathcal{D}\left(\Omega; C^\infty(Y)^d\right)\right) : \mathbf{v} = 0 \text{ in } \Omega\times Y_S, \operatorname{div}_y \mathbf{v} = 0 \text{ in } \Omega \times Y,\right.$ $\left.\operatorname{div}_x \left(\int_Y \mathbf{v}\right) = 0\right\}$ is dense in $X^{1,\varphi}_{y,0}(Q_T \times Y)$.

Lemma 1.2.9 *Let $\Delta_2(\varphi,\varphi^*)<\infty$ and $\Sigma\subset\mathbb{R}^d$ be a bounded Lipschitz domain. Let p be a distribution on Σ such that $\nabla p\in (W_0^{1,\varphi}(\Sigma))^*$. Then $p\in L^{\varphi^*}(\Sigma)/\mathbb{R}$ and there is $c=c(\Delta(\varphi,\varphi^*),\Sigma)>0$ such that*

$$\|p\|_{L^{\varphi^*}(\Sigma)/\mathbb{R}}\le c\|\nabla p\|_{(W_0^{1,\varphi}(\Sigma))^*}.$$

Lemma 1.2.10 *Let $\Delta_2(\varphi,\varphi^*)<\infty$, $\Sigma\subset\mathbb{R}^d$ be a bounded domain, $\partial\Sigma\in C^2$ and $G\in L^\varphi(\Sigma)^{d\times d}$. There is a unique $\mathbf{u}\in W_0^{1,\varphi}(\Sigma)^d$ satisfying*

$$\begin{aligned}-\Delta\mathbf{u}+\nabla r&=-\operatorname{div}G \ \textit{in } \Sigma,\\ \operatorname{div}\mathbf{u}&=0 \qquad\quad \textit{in } \Sigma.\end{aligned}\tag{1.18}$$

Moreover, there exists $c>0$ such that

$$\int_\Sigma\varphi(|\nabla\mathbf{u}|)\le\int_\Sigma\varphi(c|G|).\tag{1.19}$$

Let us recall definitions of annihilators:
Let X be a Banach space and M be a subspace of X. The annihilator $M^\perp$ of M is defined as

$$M^\perp=\{f\in X^*:\forall m\in M\ f(m)=0\}.$$

Let X^* be a dual space to X and U be a subspace of X^*. The annihilator $U_\perp$ of U is defined as

$$U_\perp=\{x\in X:\forall u\in U\ u(x)=0\}.$$

The following characterization is needed to recover pressure in Theorems 1.1.1 and 1.1.3, compare [4].

Lemma 1.2.11 *Let Ω,Y,Y_S,Y_F fulfill Assumption 1.1.5 and $\mathcal{N}$–function φ fulfill Assumption 1.2.2 then*

$$\begin{aligned}\left(X^{1,\varphi}_{y,0}(\Omega\times Y)\right)^\perp&=\{\nabla_x q+\nabla_y\tilde q: q\in W^{1,\varphi^*}(\Omega);\tilde q\in L^{\varphi^*}(\Omega\times Y),\ \textit{for a.a. } x\in\Omega\\ &\qquad\int_Y\tilde q=0\},\end{aligned}\tag{1.20}$$

$$\begin{aligned}\left(X^{1,\varphi}_{y,0}(Q_T\times Y)\right)^\perp&=\{\nabla_x q+\nabla_y\tilde q: q\in W^{1,\varphi^*}_x(Q_T);\tilde q\in L^{\varphi^*}(Q_T\times Y),\\ &\qquad \textit{for a.a. } (t,x)\in Q_T\int_Y\tilde q=0\}.\end{aligned}\tag{1.21}$$

1.3 Two-scale convergence and its basic properties

The concept of the two-scale convergence first appeared in [26] and was later developed in [3]. Statements presented below are adopted from [33], where the two-scale convergence for Orlicz setting was introduced. Some of the statements were modified for our purposes.

Throughout this section we assume $\Delta_2(\varphi,\varphi^*)<\infty$. We use the expression $w_\varepsilon(x)=w\left(x,\frac{x}{\varepsilon}\right)$ for a function $w\in L^\varphi(\Omega\times Y)$.

Definition 1.3.1 We say that a sequence $\{v^\varepsilon\}\subset L^\varphi(\Omega)$ converges in $L^\varphi(\Omega)$

(1) weakly two-scale to some $v^0 \in L^\varphi(\Omega \times Y)$ if for any $w \in L^{\varphi^*}(\Omega; C(Y))$

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x) w_\varepsilon(x) \mathrm{d}x = \int_\Omega \int_Y v^0(x,y) w(x,y) \mathrm{d}y \mathrm{d}x,$$

which we express $v^\varepsilon \xrightarrow{2-s} v^0$.

(2) strongly two-scale to some $v^0 \in L^\varphi(\Omega \times Y)$ if for any $\kappa > 0$ and $w \in L^\varphi(\Omega; C(Y))$ with $\|v^0 - w\|_{L^\varphi(\Omega\times Y)} \leq \frac{\kappa}{2}$ there exists $\alpha > 0$ such that for any $\varepsilon \in (0, \alpha)$ $\|v^\varepsilon - w_\varepsilon\|_{L^\varphi(\Omega)} \leq \kappa$, which we express $v^\varepsilon \xrightarrow{2-s} v^0$.

Remark 1.3.2 Every weakly two-scale convergent sequence $\{v^\varepsilon\}$ in $L^\varphi(\Omega)$ converges weakly in $L^\varphi(\Omega)$. Indeed, the choice of the test function w independent of y yields $v^\varepsilon \rightharpoonup \int_Y v^0$ in $L^\varphi(\Omega)$. Thus $\{v^\varepsilon\}$ has properties of weakly convergent sequences, especially $\{v^\varepsilon\}$ is bounded in $L^\varphi(\Omega)$.

Lemma 1.3.3

(1) Let $v \in L^\varphi(\Omega, C(Y))$ then as $\varepsilon \to 0$

$$v_\varepsilon \xrightarrow{2-s} v \text{ in } L^\varphi(\Omega), \ \lim_{\varepsilon \to 0} \|v_\varepsilon\|_{L^\varphi(\Omega)} = \|v\|_{L^\varphi(\Omega\times Y)}.$$

(2) Let $\boldsymbol{\psi} \in \mathscr{D}\left(\Omega, C^\infty(Y)^d\right)$. Then

$$\begin{aligned} \varepsilon \mathbf{D}\boldsymbol{\psi}_\varepsilon &\xrightarrow{2-s} \mathbf{D}_y \boldsymbol{\psi} \qquad \text{in } L^\varphi(\Omega), \\ \mathbf{S}(\varepsilon \mathbf{D}\boldsymbol{\psi}_\varepsilon) &\xrightarrow{2-s} \mathbf{S}(\mathbf{D}_y\boldsymbol{\psi}) \ \text{in } L^{\varphi^*}(\Omega). \end{aligned}$$

Proof. For the proof of the first assertion see [33, Proposition 4.3 and 4.4].
Let us show 2. The definition of $\boldsymbol{\psi}_\varepsilon$ implies

$$\varepsilon \mathbf{D}\boldsymbol{\psi}_\varepsilon = \varepsilon \mathbf{D}_x \boldsymbol{\psi}\left(x, \frac{x}{\varepsilon}\right) + \mathbf{D}_y \boldsymbol{\psi}\left(x, \frac{x}{\varepsilon}\right).$$

The first term on the right hand side converges to zero in $L^\varphi(\Omega)^{d\times d}$, whereas the second term converges strongly two-scale to $\mathbf{D}_y\boldsymbol{\psi}$ in $L^\varphi(\Omega)^{d\times d}$ by 1.
We denote $R = 2\max\{\|D_x\boldsymbol{\psi}\|_{L^\infty(\Omega\times Y)}, \|D_y\boldsymbol{\psi}\|_{L^\infty(\Omega\times Y)}\}$. Clearly, the restriction of S on $B_R = \{\mathbf{D} \in \mathbb{R}^{d\times d}_{sym} : |\mathbf{D}| \leq R\}$ is Lipschitz continuous with the constant denoted by $c(S)$ thanks to Lemma 1.2.3 and $\mathbf{S}(0) = 0$. We note that $\{\mathbf{S}(\varepsilon\mathbf{D}\boldsymbol{\psi}_\varepsilon)\} \subset L^{\varphi^*}(\Omega)$. We pick $\kappa > 0$ and $W \in L^{\varphi^*}\left(\Omega; C(Y)^{d\times d}_{sym}\right)$ such that

$$\|\mathbf{S}(\mathbf{D}_y\boldsymbol{\psi}) - W\|_{L^{\varphi^*}(\Omega\times Y)} \leq \frac{\kappa}{2}. \tag{1.22}$$

Since $\varepsilon\mathbf{D}\boldsymbol{\psi}_\varepsilon \xrightarrow{2-s} \mathbf{D}_y\boldsymbol{\psi}$ in $L^\varphi(\Omega)^{d\times d}_{sym}$, we find for κ and $\tilde{W} \in L^\varphi\left(\Omega; C(Y)^{d\times d}_{sym}\right)$ such that

$$\|\mathbf{D}_y\boldsymbol{\psi} - \tilde{W}\|_{L^\varphi(\Omega\times Y)} \leq \frac{\kappa}{8c(S)} \tag{1.23}$$

$\alpha_1 > 0$ such that for any $\varepsilon \in (0, \alpha_1)$

$$\|\varepsilon\mathbf{D}\boldsymbol{\psi}_\varepsilon - \tilde{W}_\varepsilon\|_{L^\varphi(\Omega)} \leq \frac{\kappa}{4c(S)}. \tag{1.24}$$

In fact we take $\|\tilde{W}\|_{L^\infty(\Omega\times Y)} \leq R$, which is possible since $\|\mathbf{D}_y\boldsymbol{\psi}\|_{L^\infty(\Omega\times Y)} \leq R$. Next, we estimate

$$\begin{aligned}\|\mathbf{S}(\varepsilon\mathbf{D}\boldsymbol{\psi}_\varepsilon) - W_\varepsilon\|_{L^{\varphi^*}(\Omega)} &\leq \|\mathbf{S}(\varepsilon\mathbf{D}\boldsymbol{\psi}_\varepsilon) - \mathbf{S}(\tilde{W}_\varepsilon)\|_{L^{\varphi^*}(\Omega)} + \|\mathbf{S}(\tilde{W}_\varepsilon) - W_\varepsilon\|_{L^{\varphi^*}(\Omega)} \\ &\leq c(S)\|\varepsilon\mathbf{D}\boldsymbol{\psi}_\varepsilon - \tilde{W}_\varepsilon\|_{L^\varphi(\Omega)} + \|\mathbf{S}(\tilde{W}_\varepsilon) - W_\varepsilon\|_{L^{\varphi^*}(\Omega)}.\end{aligned} \tag{1.25}$$

It remains to treat the second term on the right hand side of the latter inequality. To this end, we apply (1.22) and (1.23) to obtain

$$\begin{aligned}\|\mathbf{S}(\tilde{W}) - W\|_{L^{\varphi^*}(\Omega\times Y)} &\leq \|\mathbf{S}(\tilde{W}) - \mathbf{S}(\mathbf{D}_y\boldsymbol{\psi})\|_{L^{\varphi^*}(\Omega\times Y)} + \|\mathbf{S}(\mathbf{D}_y\boldsymbol{\psi}) - W\|_{L^{\varphi^*}(\Omega\times Y)} \\ &\leq c(S)\|\tilde{W} - \mathbf{D}_y\boldsymbol{\psi}\|_{L^\varphi(\Omega\times Y)} + \frac{\kappa}{2} \leq \frac{5\kappa}{8}.\end{aligned} \tag{1.26}$$

Since $\mathbf{S}(\tilde{W}) \in L^{\varphi^*}(\Omega; C(Y))$, we see that $\mathbf{S}(\tilde{W}_\varepsilon) - W_\varepsilon \xrightarrow{2-s} \mathbf{S}(\tilde{W}) - W$ in $L^{\varphi^*}(\Omega)$ by 1. Furthermore, it follows from (1.26) that we can find $\alpha_2 > 0$ such that for $\varepsilon \in (0, \alpha_2)$ $\|\mathbf{S}(\tilde{W}_\varepsilon) - W_\varepsilon\|_{L^{\varphi^*}(\Omega)} \leq \frac{3\kappa}{4}$. Finally, using the previous observation and (1.24) in (1.25), we infer that for $\varepsilon \in (0, \alpha)$ with $\alpha = \min\{\alpha_1, \alpha_2\}$

$$\|\mathbf{S}(\varepsilon\mathbf{D}\boldsymbol{\psi}_\varepsilon) - W_\varepsilon\|_{L^{\varphi^*}(\Omega\times Y)} \leq \kappa.$$

□

Lemma 1.3.4 *Let $v^\varepsilon \xrightarrow{2-s} v^0$ in $L^\varphi(\Omega)$ and $z^\varepsilon \xrightarrow{2-s} z^0$ in $L^{\varphi^*}(\Omega)$ then*

$$\lim_{\varepsilon\to 0}\int_\Omega v^\varepsilon z^\varepsilon = \int_\Omega\int_Y v^0 z^0.$$

Proof. Let us fix $\kappa > 0$ and $w \in L^\varphi(\Omega; C(Y))$ with $\|v^0 - w\|_{L^\varphi(\Omega\times Y)} \leq \frac{\kappa}{2}$. Then there exists $\alpha_1 > 0$ such that $\|v^\varepsilon - w_\varepsilon\|_{L^\varphi(\Omega\times Y)} \leq \kappa$ for each $\varepsilon \in (0, \alpha_1)$ due to the strong two-scale convergence of $\{v^\varepsilon\}$. Moreover, there is $\alpha_2 > 0$ such that $\left|\int_\Omega w_\varepsilon z^\varepsilon - \int_\Omega\int_Y wz^0\right| \leq \kappa$ for any $\varepsilon \in (0, \alpha_2)$ due to the weak two-scale convergence of $\{z^\varepsilon\}$. We decompose and estimate

$$\begin{aligned}D =&\left|\int_\Omega v^\varepsilon z^\varepsilon - \int_\Omega\int_Y v^0 z^0\right| \leq \left|\int_\Omega (v^\varepsilon - w_\varepsilon) z^\varepsilon\right| + \\ &\left|\int_\Omega z^\varepsilon w_\varepsilon - \int_\Omega\int_Y z^0 w\right| - \left|\int_\Omega\int_Y (v^0 - w) z^0\right| = I + II + III.\end{aligned}$$

Let us take $\alpha = \min(\alpha_1, \alpha_2)$. Then for any $\varepsilon \in (0, \alpha)$ obviously $II \leq \kappa$. According to Remark 1.3.2 the weakly two-scale convergent sequence $\{z^\varepsilon\}$ is bounded. Hence we have

$$\begin{aligned}I &\leq 2\|v^\varepsilon - w_\varepsilon\|_{L^\varphi(\Omega)}\|z^\varepsilon\|_{L^{\varphi^*}(\Omega)} \leq c\kappa, \\ III &\leq 2\|v^0 - w\|_{L^\varphi(\Omega\times Y)}\|z^0\|_{L^{\varphi^*}(\Omega\times Y)} \leq c\kappa.\end{aligned}$$

Hence $D \leq c\kappa$ and the proof is finished. □

Theorem 1.3.5 *From any bounded sequence in $L^\varphi(\Omega)$ one can extract a subsequence, which converges weakly two-scale in $L^\varphi(\Omega)$.*

Proof. See [33, Theorem 4.1]. □

Lemma 1.3.6 *Let $\{\mathbf{v}^\varepsilon\}$ and $\{\varepsilon\nabla_x\mathbf{v}^\varepsilon\}$ be bounded sequences in $L^\varphi(\Omega)^d$ and $L^\varphi(\Omega)^{d\times d}$ respectively. Then there exists $\mathbf{v}\in W^{1,\varphi}_y(\Omega\times Y)^d$ and a subsequence $\{\mathbf{v}^{\varepsilon'}\}$ such that as $\varepsilon'\to 0$*

$$\mathbf{v}^{\varepsilon'}\xrightarrow{2-s}\mathbf{v},\qquad \varepsilon'\nabla\mathbf{v}^{\varepsilon'}\xrightarrow{2-s}\nabla_y\mathbf{v}.$$

Proof. As an immediate consequence of Theorem 1.3.5, we obtain the existence of the subsequence $\{\mathbf{v}^{\varepsilon'}\}$ and functions $\mathbf{v},\zeta$ such that

$$\lim_{\varepsilon'\to 0}\int_\Omega \mathbf{v}^{\varepsilon'}(x)\cdot\sigma\left(x,\frac{x}{\varepsilon'}\right)\mathrm{d}x=\int_{\Omega\times Y}\mathbf{v}(x,y)\cdot\sigma(x,y)\mathrm{d}y\mathrm{d}x, \tag{1.27}$$

$$\lim_{\varepsilon'\to 0}\int_\Omega \varepsilon'\nabla\mathbf{v}^{\varepsilon'}(x):\psi\left(x,\frac{x}{\varepsilon'}\right)\mathrm{d}x=\int_{\Omega\times Y}\zeta(x,y):\psi(x,y)\mathrm{d}y\mathrm{d}x$$

for any $\sigma\in\mathcal{D}\left(\Omega;C^\infty(Y)^d\right)$ and $\psi\in\mathcal{D}(\Omega;C^\infty(Y)^{d\times d})$ respectively. Integration by parts, boundedness of $\{\mathbf{v}^\varepsilon\}$, (1.27) and disintegration by parts yield

$$\lim_{\varepsilon'\to 0}\int_\Omega\varepsilon'\nabla\mathbf{v}^{\varepsilon'}:\psi_{\varepsilon'}=-\lim_{\varepsilon'\to 0}\int_\Omega\varepsilon'\mathbf{v}^{\varepsilon'}\cdot\left[(\operatorname{div}_x\psi)_{\varepsilon'}+\frac{1}{\varepsilon'}(\operatorname{div}_y\psi)_{\varepsilon'}\right]=$$
$$-\lim_{\varepsilon'\to 0}\int_\Omega\mathbf{v}^{\varepsilon'}\cdot(\operatorname{div}_y\psi)_{\varepsilon'}=-\int_{\Omega\times Y}\mathbf{v}\cdot\operatorname{div}_y\psi=\int_{\Omega\times Y}\nabla_y\mathbf{v}:\psi.$$

Hence one concludes $\zeta=\nabla_y\mathbf{v}$. □

The following lemma states that L^2-norm is lower semicontinuous with respect to the weak two-scale convergence in L^2.

Lemma 1.3.7 *Let $v^\varepsilon\xrightarrow{2-s}v^0$ in $L^2(\Omega)$ then*

$$\liminf_{\varepsilon\to 0}\|v^\varepsilon\|_{L^2(\Omega)}\geq\|v^0\|_{L^2(\Omega\times Y)}.$$

Proof. See [3, Proposition 1.6.]. □

In the case of time dependent functions, we formulate the definition of two scale convergence in the following way. The expression $w_\varepsilon(t,x)=w\left(t,x,\frac{x}{\varepsilon}\right)$ is used for a function $w\in L^\varphi(Q_T\times Y)$.

Definition 1.3.8 We say that a sequence $\{v^\varepsilon\}\subset L^\varphi(Q_T)$ converges in $L^\varphi(Q_T)$

(1) weakly two-scale to some $v^0\in L^\varphi(Q_T\times Y)$ if for any $w\in L^{\varphi^*}(Q_T;C(Y))$

$$\lim_{\varepsilon\to 0}\int_{Q_T}v^\varepsilon(t,x)w_\varepsilon(t,x)\mathrm{d}x\mathrm{d}t=\int_{Q_T}\int_Y v^0(t,x,y)w(t,x,y)\mathrm{d}y\mathrm{d}x\mathrm{d}t.$$

(2) strongly two-scale to some $v^0\in L^\varphi(Q_T\times Y)$ if for any $\kappa>0$ and $w\in L^\varphi(Q_T;C(Y))$ with $\|v^0-w\|_{L^\varphi(Q_T\times Y)}\leq\frac{\kappa}{2}$ there exists $\alpha>0$ such that for any $\varepsilon\in(0,\alpha)$ $\|v^\varepsilon-w_\varepsilon\|_{L^\varphi(Q_T)}\leq\kappa$.

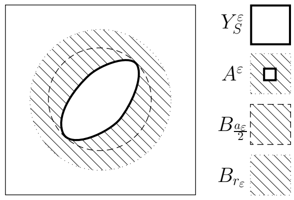
With this definition of the weak and strong two-scale convergence, we can formulate analogues of Lemma 1.3.3, Lemma 1.3.4, Theorem 1.3.5 and Lemma 1.3.6 with Q_T instead of Ω.

1.4 Restriction operator

The idea of the construction and using of the restriction operator comes from [34]. We adopt it for the Orlicz setting.

Lemma 1.4.1 *Let $\Delta_2(\varphi, \varphi^*) < \infty$. Consider the period Y^ε. Put $r_\varepsilon = \frac{\varepsilon}{2}\left(r_0 + \frac{1}{2}\right)$ and denote $A^\varepsilon = B_{r_\varepsilon} \setminus Y_S^\varepsilon$ (B_{r_ε} denotes the ball having the radius r_ε and the same center as Y^ε, see the figure below). For any $\mathbf{u} \in W^{1,\varphi}(Y^\varepsilon)$ there exists $\mathbf{v} \in W^{1,\varphi}(A^\varepsilon), q \in L^{\varphi^*}(A^\varepsilon)$ satisfying*

$$\begin{aligned}
-\Delta \mathbf{v} &= -\Delta \mathbf{u} + \nabla q && \text{in } A^\varepsilon,\\
\operatorname{div} \mathbf{v} &= \operatorname{div} \mathbf{u} + \frac{1}{|A^\varepsilon|}\int_{Y_S^\varepsilon} \operatorname{div} \mathbf{u} \, \mathrm{d}x && \text{in } A^\varepsilon,\\
\mathbf{v} = \mathbf{u} \ \text{on } \partial B_{r_\varepsilon} \quad \mathbf{v} &= 0 && \text{on } Y_S^\varepsilon.
\end{aligned}$$



the cell Y^ε

Furthermore, there is $c > 0$ independent of $\mathbf{u}$ and ε such that

$$\int_{A^\varepsilon} \varphi\left(|\nabla \mathbf{v}|\right) \le \int_{Y^\varepsilon} \varphi\left(c(|\nabla \mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|)\right). \tag{1.28}$$

Proof. The function $\mathbf{v}$ is constructed as the sum $\mathbf{v} = \mathbf{w} + \boldsymbol{\alpha} + \boldsymbol{\beta}$.
Let us describe the construction of the function $\boldsymbol{\alpha} \in W^{1,\varphi}(A^\varepsilon)$. We choose a cut-off function $\theta \in C^\infty(Y^\varepsilon)$ with properties:

$$\theta \in [0,1], \theta \equiv 0 \text{ in } B_{\varepsilon r_0}, \theta \equiv 1 \text{ in } Y^\varepsilon \setminus B_{r_\varepsilon}, |\nabla \theta| \le \frac{c}{\varepsilon}.$$

For $\boldsymbol{\alpha} = \theta \mathbf{u}$ we then have

$$\begin{aligned}
&\boldsymbol{\alpha} = \mathbf{u} \text{ on } \partial B_{r_\varepsilon}, \ \boldsymbol{\alpha} = 0 \text{ on } \partial Y_S^\varepsilon,\\
&|\boldsymbol{\alpha}| \le |\mathbf{u}| && \text{in } A^\varepsilon,\\
&|\nabla \boldsymbol{\alpha}| \le |\nabla \mathbf{u}| + \frac{c}{\varepsilon}|\mathbf{u}| && \text{in } A^\varepsilon.
\end{aligned}$$

The last inequality implies

$$\int_{A^\varepsilon} \varphi(|\nabla \boldsymbol{\alpha}|) \le \int_{A^\varepsilon} \varphi(c(|\nabla \mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|)). \tag{1.29}$$

The function $\boldsymbol{\beta} \in W_0^{1,\varphi}(A^\varepsilon)$ is a solution of

$$\operatorname{div} \boldsymbol{\beta} = -\operatorname{div} \boldsymbol{\alpha} + \operatorname{div} \mathbf{u} + \frac{1}{|A^\varepsilon|}\int_{Y_S^\varepsilon} \operatorname{div} \mathbf{u} \, \mathrm{d}y = F \text{ in } A^\varepsilon.$$

[13, Theorem 6.6] ensures the existence of such $\boldsymbol{\beta}$ together with the estimate

$$\int_{A^\varepsilon} \varphi(|\nabla \boldsymbol{\beta}|) \le \int_{A^\varepsilon} \varphi(c|F|)$$

because the compatibility condition $\int_{A^\varepsilon} F = 0$ is satisfied. We continue with the estimate of F. We estimate with the help of Jensen's inequality, the fact $|Y_S^\varepsilon| \sim |A^\varepsilon|$

and $\Delta_2(\varphi) < \infty$

$$\begin{aligned}
&\int_{A^\varepsilon} \varphi\left(\left|-\operatorname{div}\boldsymbol{\alpha} + \operatorname{div}\mathbf{u} + \frac{1}{|A^\varepsilon|}\int_{Y_S^\varepsilon}\operatorname{div}\mathbf{u}\right|\right) \\
&\leq c\left(\int_{A^\varepsilon}\varphi\left(|\nabla\mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|\right) + \int_{A^\varepsilon}\varphi\left(\frac{1}{|Y_S^\varepsilon|}\int_{Y_S^\varepsilon}\frac{|Y_S^\varepsilon|}{|A^\varepsilon|}|\operatorname{div}\mathbf{u}|\right)\right) \qquad (1.30)\\
&\leq c\left(\int_{A^\varepsilon}\varphi\left(|\nabla\mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|\right) + \int_{Y_S^\varepsilon}\varphi\left(|\nabla\mathbf{u}|\right)\right) \leq \int_{Y^\varepsilon}\varphi\left(c\left(|\nabla\mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|\right)\right).
\end{aligned}$$

Finally, the function $\mathbf{w} \in W_0^{1,\varphi}(A^\varepsilon)$ is a solution of the problem

$$\begin{aligned}
-\Delta\mathbf{w} - \nabla q &= -\Delta\left(\mathbf{u} - \boldsymbol{\alpha} - \boldsymbol{\beta}\right) && \text{in } A^\varepsilon,\\
\operatorname{div}\mathbf{w} &= 0 && \text{in } A^\varepsilon.
\end{aligned}$$

We employ Lemma 1.2.10 to show the existence and uniqueness of $\mathbf{w}$ and the estimate

$$\int_{A^\varepsilon}\varphi(|\nabla\mathbf{w}|) \leq \int_{A^\varepsilon}\varphi(c|\nabla(\mathbf{u} - \boldsymbol{\alpha} - \boldsymbol{\beta})|). \qquad (1.31)$$

Clearly, (1.31), (1.30) and (1.29) yield (1.28). □

Lemma 1.4.2 *Let $\Delta_2(\varphi, \varphi^*) < \infty$. There exists a restriction operator $R_\varepsilon : W_0^{1,\varphi}(\Omega)^d \to W_0^{1,\varphi}(\Omega^\varepsilon)^d$ with properties:*

$$\begin{aligned}
&R_\varepsilon \textit{ is linear}, && (1.32a)\\
&R_\varepsilon(\mathbf{w}) = \mathbf{w} \textit{ for } \mathbf{w} \in W_0^{1,\varphi}(\Omega^\varepsilon)^d \textit{ extended by } 0 \textit{ on } \Omega\setminus\Omega^\varepsilon, && (1.32b)\\
&\operatorname{div}\mathbf{w} = 0 \textit{ in } \Omega \Rightarrow \operatorname{div}R_\varepsilon(\mathbf{w}) = 0 \textit{ in } \Omega^\varepsilon, && (1.32c)\\
&\|R_\varepsilon(\mathbf{w})\|_{L^\varphi(\Omega)} \leq c\left(\|\mathbf{w}\|_{L^\varphi(\Omega)} + \varepsilon\|\nabla\mathbf{w}\|_{L^\varphi(\Omega)}\right), && (1.32d)\\
&\|\nabla R_\varepsilon(\mathbf{w})\|_{L^\varphi(\Omega)} \leq c\left(\frac{1}{\varepsilon}\|\mathbf{w}\|_{L^\varphi(\Omega)} + \|\nabla\mathbf{w}\|_{L^\varphi(\Omega)}\right). && (1.32e)
\end{aligned}$$

Proof. We define R_ε on the cell $Y_i^\varepsilon \in I^\varepsilon$ as

$$R_\varepsilon\mathbf{u}(x) = \begin{cases} \mathbf{u}(x) & x \in Y_i^\varepsilon \setminus ({A^\varepsilon}_i \cup Y_{Si}^\varepsilon)\\ \mathbf{v}(x) & x \in A_i^\varepsilon\\ 0 & x \in Y_{Si}^\varepsilon \end{cases}$$

where $\mathbf{u}, \mathbf{v}$ are from Lemma 1.4.1 and $R_\varepsilon\mathbf{u} = \mathbf{u}$ on the cell $Y_i^\varepsilon \in H^\varepsilon$. It is easy to see that R_ε satisfies (1.32a),(1.32b) and (1.32c). We have the estimate

$$\int_{Y_{F_i}^\varepsilon}\varphi(|\nabla R_\varepsilon\mathbf{u}|) \leq \int_{Y_i^\varepsilon\setminus\left({A^\varepsilon}_i\cup Y_{Si}^\varepsilon\right)}\varphi(|\nabla\mathbf{u}|) + \int_{A_i^\varepsilon}\varphi(|\nabla\mathbf{v}|) \leq \int_{Y_i^\varepsilon}\varphi\left(c\left(|\nabla\mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|\right)\right) \qquad (1.33)$$

Summing up over $i \in I^\varepsilon \cup H^\varepsilon$ in (1.33) yields

$$\int_{\Omega^\varepsilon}\varphi(|\nabla R_\varepsilon\mathbf{u}|) \leq \int_\Omega\varphi\left(c\left(|\nabla\mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|\right)\right).$$

Since R_ε is linear, we obtain from the latter inequality for $\lambda > 0$

$$\int_{\Omega^\varepsilon} \varphi\left(\frac{|\nabla R_\varepsilon \mathbf{u}|}{\lambda}\right) \leq \int_\Omega \varphi\left(\frac{c\left(|\nabla \mathbf{u}| + \frac{1}{\varepsilon}|\mathbf{u}|\right)}{\lambda}\right),$$

which implies

$$\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{c}{\lambda}\left(|\nabla \tilde{\mathbf{u}}| + \frac{1}{\varepsilon}|\tilde{\mathbf{u}}|\right)\right) \leq 1\right\} \subset \left\{\lambda > 0 : \int_{\Omega^\varepsilon} \varphi\left(\frac{|\nabla R_\varepsilon \tilde{\mathbf{u}}|}{\lambda}\right) \leq 1\right\}.$$

We apply infimum over λ on both sides to obtain

$$\|\nabla R_\varepsilon \tilde{\mathbf{u}}\|_{L^\varphi(\Omega^\varepsilon)} \leq \left\| c\left(|\nabla \tilde{\mathbf{u}}| + \frac{1}{\varepsilon}|\nabla \tilde{\mathbf{u}}|\right)\right\|_{L^\varphi(\Omega)} \leq c\left(\|\nabla \tilde{\mathbf{u}}\|_{L^\varphi(\Omega)^{d\times d}} + \frac{1}{\varepsilon}\|\tilde{\mathbf{u}}\|_{L^\varphi(\Omega)^d}\right).$$

Hence we get (1.32e), which together with (1.16) implies (1.32d). □

1.5 Homogenization of the stationary generalized Stokes system

Definition 1.5.1 A triplet $(\mathbf{u}^0, p, \pi) \in X^{1,\varphi}_{y,0}(\Omega \times Y) \times W^{1,\varphi^*}(\Omega) \times L^{\varphi^*}(\Omega \times Y)$ is said to be a weak solution of the problem (HSS) if for any $\mathbf{w} \in W^{1,\varphi}_{y,0}(\Omega \times Y)$

$$\int_\Omega \int_Y \mathbf{S}(\mathbf{D}_y \mathbf{u}^0) : \mathbf{D}_y \mathbf{w} + \int_\Omega \int_Y \nabla_x p \cdot \mathbf{w} - \int_\Omega \int_Y \pi \operatorname{div}_y \mathbf{w} = \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{w}. \tag{1.34}$$

Proof of Theorem 1.1.1. We use the theory of pseudomonotone mappings from [29]. First, we observe that $X^{1,\varphi}_{y,0}(\Omega \times Y)$ is a separable, reflexive Banach space. A mapping

$$\mathcal{A} : X^{1,\varphi}_{y,0}(\Omega \times Y) \to \left(X^{1,\varphi}_{y,0}(\Omega \times Y)\right)^* \text{ defined by } \mathcal{A}(\mathbf{w})(\mathbf{v}) = \int_\Omega \int_Y \mathbf{S}(\mathbf{D}_y \mathbf{w}) : \mathbf{D}_y \mathbf{v}$$

is strictly monotone by Lemma 1.2.3. Next, we show that $\mathcal{A}$ is hemicontinuous, i.e., the function $t \mapsto \mathcal{A}(\mathbf{w} + t\mathbf{v})(\mathbf{z})$ is continuous. Let us pick $\mathbf{w}, \mathbf{v}, \mathbf{z} \in X^{1,\varphi}_{y,0}(\Omega \times Y)$. Using Lemma 1.2.8, we find a sequence $\{\mathbf{z}^n\} \in \mathcal{D}\left(\Omega; C^\infty(Y)^d\right)$, $\mathbf{z}^n = 0$ in $\Omega \times Y_S$ such that $\|\nabla_y(\mathbf{z} - \mathbf{z}^n)\|_{L^\varphi(\Omega\times Y)} \to 0$. Let us fix $n \in \mathbb{N}$. As a consequence of Vitali's convergence theorem, we obtain as $t^k \to t$

$$\mathcal{A}(\mathbf{w} + t^k \mathbf{v})(\mathbf{z}^n) \to \mathcal{A}(\mathbf{w} + t\mathbf{v})(\mathbf{z}^n).$$

Realizing that the limit in n is uniform for all k, which allows to change the order of limits, one concludes the hemicontinuity of $\mathcal{A}$. Strict monotonicity and hemicontinuity of $\mathcal{A}$ imply pseudomonotonicity of $\mathcal{A}$, see [29, Lemma 2.9]. For $\mathbf{v} \in X^{1,\varphi}_{y,0}(\Omega \times Y)$ we have $\mathbf{v}(x, \cdot) \in L^\varphi(Y)^d, \nabla_y \mathbf{v}(x, \cdot) \in L^\varphi(Y)^{d\times d}$ for $x \in \Omega \setminus N$ for some $N \subset \Omega, |N| = 0$ by Remark 1.2.5 (iii). Then Lemma 1.2.6 imply

$$\int_\Omega \int_Y \varphi(|\mathbf{v}|) \leq \int_\Omega \int_Y \varphi(c|\nabla_y \mathbf{v}|),$$

from which $\|\mathbf{v}\|_{L^\varphi(\Omega\times Y)} \leq c\|\nabla_y \mathbf{v}\|_{L^\varphi(\Omega\times Y)}$ follows by repeating the procedure from the proof of (1.32e) in Lemma 1.4.2. Then the proof of coercivity of $\mathcal{A}$ is the same as in

[11, Lemma 3.1.]. Thus for $\mathcal{A}$ pseudomonotone and coercive, we obtain according to [29, Theorem 2.6] the existence of a $\mathbf{u}^0 \in X^{1,\varphi}_{y,0}(\Omega \times Y)$, which is unique due to the strict monotonicity of $\mathcal{A}$.
It remains to obtain a pressure in the form $\nabla_x p + \nabla_y \pi$. Let us define functional F on $W^{1,\varphi}_{y,0}(\Omega \times Y)$ by

$$\langle F, \mathbf{w} \rangle = \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{w} - \int_\Omega \int_Y \mathbf{S}(\mathbf{D}_y \mathbf{u}^0) : \mathbf{D}_y \mathbf{w}.$$

Obviously, $F \in (X^{1,\varphi}_{y,0}(\Omega \times Y))^\perp$. Thus F has the required form according to Lemma 1.2.11. $\square$

Definition 1.5.2 A pair $(\mathbf{u}^\varepsilon, p^\varepsilon) \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega^\varepsilon)^d \times L^{\varphi^*}(\Omega^\varepsilon)$ is said to be a weak solution of the problem (SGS_ε) if for any $\mathbf{w} \in W^{1,\varphi}_0(\Omega^\varepsilon)^d$

$$\varepsilon \int_{\Omega^\varepsilon} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon) : \mathbf{D}\mathbf{w} - \int_{\Omega^\varepsilon} p^\varepsilon \ \operatorname{div} \mathbf{w} = \int_{\Omega^\varepsilon} \mathbf{f} \cdot \mathbf{w}. \tag{1.35}$$

Lemma 1.5.3 *Let Ω^ε be defined by* (1.6), $\mathbf{f} \in L^{\varphi^*}(\Omega)^d$, $\Delta_2(\varphi, \varphi^*) < \infty$ *and* $\varepsilon > 0$ *be fixed. Then there exits a unique* $\mathbf{u}^\varepsilon \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega^\varepsilon)^d$ *satisfying* (1.35) *for all solenoidal* $\mathbf{w}$. *Moreover, there are* $c_1, c_2 > 0$ *independent of* ε *such that*

$$\int_{\Omega^\varepsilon} \varphi(|\varepsilon \mathbf{D}\mathbf{u}^\varepsilon|) \le c_1 \int_{\Omega^\varepsilon} \varphi^*(|\mathbf{f}|), \tag{1.36}$$

$$\int_{\Omega^\varepsilon} \varphi^*(|\mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon)|) \le c_2 \int_{\Omega^\varepsilon} \varphi^*(|\mathbf{f}|). \tag{1.37}$$

Proof. When proving the existence of a weak solution, we show that a mapping

$$\mathcal{A} : W^{1,\varphi}(\Omega^\varepsilon)^d \to \left(W^{1,\varphi}(\Omega^\varepsilon)^d\right)^* \ \text{ defined as } \mathcal{A}(\mathbf{w})(\mathbf{v}) = \int_{\Omega^\varepsilon} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{w}) : \mathbf{D}\mathbf{v}.$$

is pseudomotone and coercive. Since one proceed in an analogous manner to the proof of Theorem 1.1.1, details are omitted here.
Testing (SGS_ε) by $\mathbf{u}^\varepsilon$, applying Young's inequality (1.10) with α small and Korn's inequality (1.17) leads to

$$\begin{aligned}\int_{\Omega^\varepsilon} \varphi'\left(|\varepsilon \mathbf{D}\mathbf{u}^\varepsilon|\right)|\varepsilon \mathbf{D}\mathbf{u}^\varepsilon| = \int_{\Omega^\varepsilon} \mathbf{f} \cdot \mathbf{u}^\varepsilon &\le c_\alpha \int_{\Omega^\varepsilon} \varphi^*\left(|\mathbf{f}|\right) + \alpha \int_{\Omega^\varepsilon} \varphi\left(|\mathbf{u}^\varepsilon|\right) \\ &\le c_\alpha \int_{\Omega^\varepsilon} \varphi^*\left(|\mathbf{f}|\right) + c\alpha \int_{\Omega^\varepsilon} \varphi\left(|\varepsilon \mathbf{D}\mathbf{u}^\varepsilon|\right).\end{aligned}$$

Applying the property (1.7) and absorbing the last term on the right hand side to the left yield (1.36). (1.37) follows from the property (1.8) and (1.36). $\square$

The function $\mathbf{u}^\varepsilon$ can be extended by zero in $\Omega \setminus \Omega^\varepsilon$ because of its zero trace on $\partial\Omega^\varepsilon$. Then all estimates from Lemma 1.5.3 are valid for Ω replacing Ω^ε. To find a uniformly bounded extension of a pressure, we follow ideas of Tartar from [34].

Lemma 1.5.4 *Let the assumptions of Lemma 1.5.3 be fulfilled. There is a pressure function* $p^\varepsilon \in L^{\varphi^*}(\Omega^\varepsilon)$ *corresponding to* $\mathbf{u}^\varepsilon$ *from Lemma 1.5.3. Moreover, there is an extension* P^ε *of* p^ε, *which satisfies for any* $\mathbf{w} \in \mathcal{D}(\Omega)^d$

$$\int_\Omega \varepsilon \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon) : \mathbf{D}R_\varepsilon \mathbf{w} - \int_\Omega P^\varepsilon \operatorname{div} \mathbf{w} = \int_\Omega \mathbf{f} \cdot R_\varepsilon \mathbf{w}. \tag{1.38}$$

Finally, there is $c>0$ independent of ε such that

$$\|P^\varepsilon\|_{L^{\varphi^*}(\Omega)/\mathbb{R}} \leq c. \tag{1.39}$$

Proof. We reconstruct the pressure in a standard way, i.e., according to de Rham's theorem there is a distribution p^ε such that for any $\mathbf{w} \in \mathscr{D}(\Omega^\varepsilon)$

$$\int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} \mathbf{w} = \varepsilon \int_{\Omega^\varepsilon} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon) : \mathbf{D}\mathbf{w} + \int_{\Omega^\varepsilon} \mathbf{f} \cdot \mathbf{w}.$$

We use the apriori estimates (1.36), (1.37) and the inequality (1.17) to estimate the right hand side of the latter inequality

$$\begin{aligned}\int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} \mathbf{w} &\leq \varepsilon \|\mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon)\|_{L^{\varphi^*}(\Omega)} \|\mathbf{D}\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)} + \|\mathbf{f}\|_{L^{\varphi^*}(\Omega)} \|\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)} \\ &\leq c(\|\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)} + \varepsilon \|\mathbf{D}\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)}) \leq c\varepsilon \|\mathbf{D}\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)} \leq c\varepsilon \|\mathbf{w}\|_{W_0^{1,\varphi}(\Omega^\varepsilon)}.\end{aligned} \tag{1.40}$$

It follows that $\nabla p^\varepsilon \in (W_0^{1,\varphi}(\Omega^\varepsilon))^*$. The constant c appearing in Lemma 1.2.9 depends on the domain, which is Ω^ε in this case, but the exact dependence of c on ε is not known. So we obtain $p^\varepsilon \in L^\varphi(\Omega^\varepsilon/\mathbb{R})$ but not a uniform estimate of p^ε with respect to ε. That is the reason why it is necessary to deal with the extension P^ε. Let us define a functional F_ε by setting

$$\langle F_\varepsilon, \mathbf{w} \rangle = -\int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div} R_\varepsilon \mathbf{w}, \ \ \mathbf{w} \in W_0^{1,\varphi}(\Omega).$$

Then the estimates (1.40), (1.32d) and (1.32e) imply $\langle F_\varepsilon, \mathbf{w}\rangle \leq c\|\mathbf{w}\|_{W_0^{1,\varphi}(\Omega)}$ because $\varepsilon < 1$ is considered, thus $F_\varepsilon \in \left(W_0^{1,\varphi}(\Omega)\right)^*$. Due to the property (1.32c) we obtain $\langle F_\varepsilon, \mathbf{w}\rangle = 0$ for $\mathbf{w}$ with $\operatorname{div} \mathbf{w} = 0$ in Ω. In accordance with de Rham's theorem there is a distribution, which we denote P^ε, such that $F_\varepsilon = \nabla P^\varepsilon$. The estimate (1.39) is a consequence of (1.40) and Lemma 1.2.9. (1.38) obviously follows and integrals in this equality can be considered over Ω instead of Ω^ε since $R_\varepsilon \mathbf{w} = 0$ in $\Omega \setminus \Omega^\varepsilon$. □

Remark 1.5.5 In the same way as in [22] we can find the explicit formula for the extension P^ε, i.e.,

$$P^\varepsilon = p^\varepsilon \text{ in } \Omega^\varepsilon, \ P^\varepsilon = \frac{1}{|Y^\varepsilon_{F_i}|} \int_{Y^\varepsilon_{F_i}} p^\varepsilon \text{ in each } Y^\varepsilon_{S_i}.$$

Lemma 1.5.6 *Let the assumptions of Lemma 1.5.3 be fulfilled. Let for any $\varepsilon > 0$ $\mathbf{u}^\varepsilon$ be from Lemma 1.5.3 extended by zero in $\Omega \setminus \Omega^\varepsilon$ and P^ε be the corresponding extended pressure from Lemma 1.5.4. Then from arbitrary sequences $\{\mathbf{u}^\varepsilon\}$, $\{P^\varepsilon\}$ one can extract subsequences $\{\mathbf{u}^{\varepsilon'}\}$, $\{P^{\varepsilon'}\}$ and find functions $\mathbf{u}^0 \in W^{1,\varphi}_{y,0}(\Omega \times Y), \mathbf{S}^0 \in L^{\varphi^*}(\Omega \times Y)^{d\times d}$ and $p \in L^{\varphi^*}(\Omega)$ such that as $\varepsilon' \to 0$*

$$\mathbf{u}^{\varepsilon'} \xrightarrow{2-s} \mathbf{u}^0 \quad \textit{in } L^\varphi(\Omega)^d, \tag{1.41}$$

$$\varepsilon' \nabla \mathbf{u}^{\varepsilon'} \xrightarrow{2-s} \nabla_y \mathbf{u}^0 \quad \textit{in } L^\varphi(\Omega)^{d\times d}, \tag{1.42}$$

$$\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} \xrightarrow{2-s} \mathbf{D}_y \mathbf{u}^0 \quad \textit{in } L^\varphi(\Omega)^{d\times d}, \tag{1.43}$$

$$\mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) \xrightarrow{2-s} \mathbf{S}^0 \quad \textit{in } L^{\varphi^*}(\Omega)^{d\times d}, \tag{1.44}$$

$$P^{\varepsilon'} \longrightarrow p \quad \textit{in } L^{\varphi^*}(\Omega). \tag{1.45}$$

Proof. Let us consider a sequence $\{\mathbf{u}^\varepsilon\}$. According to Lemma 1.5.3 $\{\varepsilon \mathbf{D}\mathbf{u}^\varepsilon\}$ and thus $\{\mathbf{u}^\varepsilon\}$, due to the inequality (1.16), are bounded sequences in $L^\varphi(\Omega)^{d\times d}$ and $L^\varphi(\Omega)^d$ respectively. Then Lemma 1.3.6 ensures the existence of subsequence $\{\mathbf{u}^{\varepsilon_j}\}$ and a function $\mathbf{u}^0 \in W^{1,\varphi}_{y,0}(\Omega\times Y)$ such that (1.41), (1.42) hold with $\varepsilon' = \varepsilon_j$. (1.43) is a direct consequence of (1.42).
By Lemma 1.5.3 together with Theorem 1.3.5, we may assume the existence of $\mathbf{S}^0 \in L^{\varphi^*}(\Omega\times Y)^{d\times d}$ satisfying (1.44) with $\varepsilon' = \varepsilon_{j_k}$ and also the existence of the existence of $P^0 \in L^{\varphi^*}(\Omega\times Y)$ such that

$$P^{\varepsilon'} \xrightarrow{2-s} P^0. \tag{1.46}$$

Now, we set $\varepsilon' = \varepsilon_{j_{k_l}}$. It remains to show that P^0 is independent of y. Let us choose $\mathbf{w} \in \mathscr{D}\left(\Omega, C^\infty(Y)^d\right)$ then $\mathbf{w}_\varepsilon(x) = \mathbf{w}\left(x, \frac{x}{\varepsilon}\right) \in W^{1,\varphi}_0(\Omega)^d$ and after using $\varepsilon\mathbf{w}_\varepsilon$ as a test function in (1.38) with $\varepsilon = \varepsilon'$, we obtain

$$\int_\Omega \mathbf{S}(\varepsilon'\mathbf{D}\mathbf{u}^{\varepsilon'}) : (\varepsilon')^2 \mathbf{D}R_{\varepsilon'}\mathbf{w}_{\varepsilon'} + \int_\Omega P^{\varepsilon'}\varepsilon' \operatorname{div}\mathbf{w}_{\varepsilon'} = \varepsilon' \int_\Omega \mathbf{f}\cdot R_{\varepsilon'}\mathbf{w}_{\varepsilon'}. \tag{1.47}$$

Using the estimates (1.32e) and (1.37) we infer

$$\begin{aligned}
&\left|\int_\Omega \mathbf{S}(\varepsilon'\mathbf{D}\mathbf{u}^{\varepsilon'})(\varepsilon')^2 : DR_{\varepsilon'}\mathbf{w}_{\varepsilon'}\right| \\
&\quad\le c(\varepsilon')^2 \|\mathbf{S}(\varepsilon'\mathbf{D}\mathbf{u}^{\varepsilon'})\|_{L^{\varphi^*}(\Omega)} \left(\frac{1}{\varepsilon'}\|\mathbf{w}_{\varepsilon'}\|_{L^\varphi(\Omega)} + \|\nabla\mathbf{w}_{\varepsilon'}\|_{L^\varphi(\Omega)}\right) \\
&\quad\le c\varepsilon' \left(\left\|\mathbf{w}(\cdot,\frac{\cdot}{\varepsilon'})\right\|_{L^\varphi(\Omega)} + \varepsilon'\left\|\nabla_x\mathbf{w}(\cdot,\frac{\cdot}{\varepsilon'})\right\|_{L^\varphi(\Omega)} + \left\|\nabla_y\mathbf{w}(\cdot,\frac{\cdot}{\varepsilon'})\right\|_{L^\varphi(\Omega)}\right).
\end{aligned}$$

We estimate the right hand side of (1.47) using (1.32d)

$$\left|\varepsilon'\int_\Omega \mathbf{f}\cdot R_{\varepsilon'}\mathbf{w}_{\varepsilon'}\right| \le c\varepsilon' \left(\left\|\mathbf{w}(\cdot,\frac{\cdot}{\varepsilon'})\right\|_{L^\varphi(\Omega)} + \varepsilon'\left\|\nabla_x\mathbf{w}(\cdot,\frac{\cdot}{\varepsilon'})\right\|_{L^\varphi(\Omega)} + \left\|\nabla_y\mathbf{w}(\cdot,\frac{\cdot}{\varepsilon'})\right\|_{L^\varphi(\Omega)}\right).$$

Finally, we have

$$\int_\Omega P^{\varepsilon'}\varepsilon' \operatorname{div}\mathbf{w}_{\varepsilon'} = \int_\Omega P^{\varepsilon'}(x)\varepsilon' \operatorname{div}_x \mathbf{w}\left(x,\frac{x}{\varepsilon'}\right) \mathrm{d}x + \int_\Omega P^{\varepsilon'}(x) \operatorname{div}_y \mathbf{w}\left(x,\frac{x}{\varepsilon'}\right)\mathrm{d}x$$

and using (1.39)

$$\left|\int_\Omega P^{\varepsilon'}(x)\varepsilon' \operatorname{div}_x \mathbf{w}\left(x,\frac{x}{\varepsilon'}\right)\mathrm{d}x\right| \le c\varepsilon' \left\|\operatorname{div}_x \mathbf{w}\left(\cdot,\frac{\cdot}{\varepsilon'}\right)\right\|_{L^\varphi(\Omega)}.$$

Since $\left\{\mathbf{w}\left(x,\frac{x}{\varepsilon'}\right)\right\}, \left\{\nabla_x\mathbf{w}\left(x,\frac{x}{\varepsilon'}\right)\right\}, \left\{\nabla_y\mathbf{w}\left(x,\frac{x}{\varepsilon'}\right)\right\}, \left\{\operatorname{div}_x\mathbf{w}\left(x,\frac{x}{\varepsilon'}\right)\right\}$ are bounded with respect to ε' in $L^\varphi-$norm, we obtain after limit passage $\varepsilon'\to 0$ in (1.47) using (1.46)

$$\forall \mathbf{w} \in \mathscr{D}\left(\Omega, C^\infty(Y)^d\right) \int_\Omega\int_Y P^0 \operatorname{div}_y \mathbf{w}(x,y) = 0.$$

Hence we deduce that $P^0(x,y) = p(x)$. □

Lemma 1.5.7 *The limit function* $\mathbf{u}^0$ *from Lemma 1.5.6 satisfies*

$$\operatorname{div}_y \mathbf{u}^0 = 0 \text{ in } \Omega \times Y, \tag{1.48}$$

$$\operatorname{div}_x \left(\int_Y \mathbf{u}^0 \right) = 0 \text{ in } \Omega, \tag{1.49}$$

$$\left(\int_Y \mathbf{u}^0 \right) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \tag{1.50}$$

$$\mathbf{u}^0 = 0 \text{ in } \Omega \times Y_S. \tag{1.51}$$

Proof. Let us show (1.48). We choose arbitrary $w \in \mathcal{D}(\Omega; C^\infty(Y))$. We denote $w_{\varepsilon'}(x) = w\left(x, \frac{x}{\varepsilon'}\right)$, apply in (1.41) a test function $\varepsilon' \nabla w_{\varepsilon'}$, integrate by parts and use the periodicity of $\mathbf{u}^0$ and w

$$\lim_{\varepsilon' \to 0} \int_\Omega \mathbf{u}^{\varepsilon'} \cdot \varepsilon' \nabla w_{\varepsilon'} = \int_\Omega \int_Y \mathbf{u}^0 \cdot \nabla_y w = - \int_\Omega \int_Y \operatorname{div}_y \mathbf{u}^0 w.$$

The first term of the latter chain of equalities is obviously zero due to the solenoidality of $\mathbf{u}^{\varepsilon'}$. Hence we conclude (1.48).
To show (1.49), we use the solenoidality of $\mathbf{u}^{\varepsilon'}$ and apply in (1.41) a test function ∇w for $w \in \mathcal{D}(\Omega)$ similarly as in the previous case.
To show (1.50), we use the zero trace of $\mathbf{u}^{\varepsilon'}$, the solenoidality of $\mathbf{u}^{\varepsilon'}$ and apply in (1.41) a test function ∇w for $w \in C^\infty(\bar{\Omega})$ and integrate by parts to obtain

$$0 = \lim_{\varepsilon' \to 0} \int_\Omega \mathbf{u}^{\varepsilon'} \cdot \nabla_x w = \int_\Omega \int_Y \mathbf{u}^0 \cdot \nabla_x w = - \left\langle \operatorname{div}_x \left(\int_Y \mathbf{u}^0 \right), w \right\rangle + \int_{\partial\Omega} \left(\int_Y \mathbf{u}^0 \right) \cdot \mathbf{n} w.$$

To conclude (1.50), it remains to employ (1.49). Moreover, (1.49) ensures that (1.50) has a good meaning in the sense of traces.
To show (1.51), we choose arbitrary $\mathbf{w} \in C^\infty(Y)^d$ such that $\mathbf{w} = 0$ in Y_F and $z \in \mathcal{D}(\Omega)$. Then using $z\mathbf{w}_{\varepsilon'}$ as a test function in (1.41), we have

$$\lim_{\varepsilon' \to 0} \int_\Omega \mathbf{u}^{\varepsilon'} \cdot z\mathbf{w}_{\varepsilon'} = \int_\Omega \int_{Y_S} \mathbf{u}^0 \cdot z\mathbf{w}.$$

We decompose the integral under the limit as

$$\int_{\Omega^\varepsilon} \mathbf{u}^{\varepsilon'} \cdot z\mathbf{w}_{\varepsilon'} + \int_{\Omega \setminus \Omega^\varepsilon} \mathbf{u}^{\varepsilon'} \cdot z\mathbf{w}_{\varepsilon'},$$

remind that $\mathbf{u}^{\varepsilon'}$ is extended by zero in $\Omega \setminus \Omega^\varepsilon$ and realize that $\mathbf{w}_{\varepsilon'} = 0$ in Ω^ε to get

$$\int_\Omega \int_{Y_S} \mathbf{u}^0 \cdot z\mathbf{w} = 0.$$

Hence (1.51) immediately follows. □

Proof of Theorem 1.1.2. Let us choose $\mathbf{w} \in \mathcal{D}\left(\Omega, C^\infty(Y)^d\right)$ with $\mathbf{w} = 0$ in $\Omega \times Y_S$ and $\operatorname{div}_y \mathbf{w} = 0$ in $\Omega \times Y$. Then we have $\mathbf{w}_{\varepsilon'}(x) = \mathbf{w}\left(x, \frac{x}{\varepsilon'}\right) \in W_0^{1,\varphi}(\Omega)^d$ and $\mathbf{w}_{\varepsilon'} = 0$ in $\Omega \setminus \Omega^\varepsilon$. Taking $\mathbf{w}_{\varepsilon'}$ as a test function in (1.38) with $\varepsilon = \varepsilon'$ yields

$$\int_\Omega \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{w}_{\varepsilon'} - \int_\Omega P^{\varepsilon'} \operatorname{div} \mathbf{w}_{\varepsilon'} = \int_\Omega \mathbf{f} \cdot \mathbf{w}_{\varepsilon'}$$

due to (1.32b). When passing to the limit $\varepsilon' \to 0$, we apply the convergences (1.43) and (1.45). We get the limit system for any $\mathbf{w} \in \mathscr{D}\left(\Omega, C^\infty(Y)^d\right)$ with $\mathbf{w} = 0$ in $\Omega \times Y_S$ and $\operatorname{div}_y \mathbf{w} = 0$ in $\Omega \times Y$

$$\int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \mathbf{w} - \int_\Omega \int_Y p \operatorname{div}_x \mathbf{w} = \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{w}.$$

Using de-Rham's theorem yields the existence of a distribution $\pi(x, y)$ such that for any $\mathbf{w} \in \mathscr{D}\left(\Omega, C^\infty(Y)^d\right)$ with $\mathbf{w} = 0$ in $\Omega \times Y_S$

$$\int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \mathbf{w} - \int_\Omega \int_Y p \operatorname{div}_x \mathbf{w} - \int_\Omega \int_Y \pi \operatorname{div}_y \mathbf{w} = \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{w}, \tag{1.52}$$

which is equivalent to the weak formulation of (HSS) due to Lemma 1.2.8. This turns out if we identify the two-scale limit $\mathbf{S}^0$. To this end, we use the monotonicity of S and Minty's trick. We test the system (SGS_ε) by $\mathbf{u}^{\varepsilon'}$ and obtain

$$\int_\Omega \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} = \int_\Omega \mathbf{f} \cdot \mathbf{u}^{\varepsilon'}.$$

We pass to the limit as $\varepsilon' \to 0$ in the latter equality using the convergence (1.41)

$$\lim_{\varepsilon' \to 0} \int_\Omega \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} = \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{u}^0. \tag{1.53}$$

Putting $\mathbf{w} = \mathbf{u}^0$ in (1.52), which is allowed due to Lemma 1.2.8, yields

$$\int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \mathbf{u}^0 = \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{u}^0. \tag{1.54}$$

Let us choose $\boldsymbol{\psi} \in \mathscr{D}\left(\Omega, C^\infty(Y)^d\right)$. The monotonicity of S allows us to write

$$\begin{aligned} 0 &\le \int_\Omega (\mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) - \mathbf{S}(\varepsilon' \mathbf{D}\boldsymbol{\psi}_{\varepsilon'})) : \varepsilon' \mathbf{D}(\mathbf{u}^{\varepsilon'} - \boldsymbol{\psi}_{\varepsilon'}) \\ &= \int_\Omega \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} - \mathbf{S}(\varepsilon' \mathbf{D}\boldsymbol{\psi}_{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} - \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\boldsymbol{\psi}_{\varepsilon'} \\ &\quad + \mathbf{S}(\varepsilon' \mathbf{D}\boldsymbol{\psi}_{\varepsilon'}) : \varepsilon' \mathbf{D}\boldsymbol{\psi}_{\varepsilon'} = I_{\varepsilon'} + II_{\varepsilon'} + III_{\varepsilon'} + IV_{\varepsilon'} \end{aligned} \tag{1.55}$$

We want to pass to the limit $\varepsilon' \to 0$. Using (1.53) and (1.54), we obtain

$$\lim_{\varepsilon \to 0} I_{\varepsilon'} = \int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \mathbf{u}^0.$$

Lemma 1.3.3 1. and the weak two-scale convergence of $\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}$ to $\mathbf{D}_y \mathbf{u}^0$ yields

$$\lim_{\varepsilon' \to 0} II_{\varepsilon'} = \int_\Omega \int_Y \mathbf{S}(\mathbf{D}_y \boldsymbol{\psi}) : \mathbf{D}_y \mathbf{u}^0.$$

The weak two-scale convergence of $\mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'})$ to $\mathbf{S}^0$ together with Lemma 1.3.3 1. yields

$$\lim_{\varepsilon' \to 0} III_{\varepsilon'} = \int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \boldsymbol{\psi}.$$

Finally, the Lemma 1.3.3 1. and 2. yield

$$\lim_{\varepsilon' \to 0} IV_{\varepsilon'} = \int_\Omega \int_Y \mathbf{S}(\mathbf{D}_y \boldsymbol{\psi}) : \mathbf{D}_y \boldsymbol{\psi}.$$

This together with the estimate (1.53) allows us to pass to the limit in (1.55) and we arrive at

$$0 \le \int_\Omega \int_Y (\mathbf{S}^0 - \mathbf{S}(\mathbf{D}_y \boldsymbol{\psi})) : \mathbf{D}_y(\mathbf{u}^0 - \boldsymbol{\psi}).$$

Since $\boldsymbol{\psi}$ was arbitrary, we can set $\boldsymbol{\psi} = \mathbf{u}^0 + \lambda \mathbf{z}$, where $\lambda > 0, \mathbf{z} \in \mathscr{D}\left(\Omega; C^\infty(Y)^d\right)$ and Minty's trick ensures that $\mathbf{S}^0(x,y) = \varphi'(|\mathbf{D}_y \mathbf{u}^0(x,y)|)\frac{\mathbf{D}_y\mathbf{u}^0(x,y)}{|\mathbf{D}_y\mathbf{u}^0(x,y)|}$ almost everywhere in $\Omega \times Y$.
Since we know that the weak solution $(\mathbf{u}^0, p, \pi)$ of (HSS) is unique and we can extract from every subsequence of $\{(\mathbf{u}^\varepsilon, P^\varepsilon)\}$ a convergent subsequence with the limit $(\mathbf{u}^0, p)$, we have $\{(\mathbf{u}^\varepsilon, P^\varepsilon)\} \to (\mathbf{u}^0, p)$ as $\varepsilon \to 0_+$. □

1.6 Homogenization of the nonstationary generalized Stokes system

Definition 1.6.1 A triplet $(\mathbf{u}^0, p, \pi) \in \left(L^\infty\left(0,T; L^2(\Omega\times Y)^d\right) \cap X^{1,\varphi}_{y,0}(Q_T \times Y)\right) \times W^{1,\varphi^*}_x(Q_T) \times L^{\varphi^*}(Q_T \times Y)$ with $\mathbf{u}^0_t \in (W^{1,\varphi}_{y,0}(Q_T\times Y))^*$ is said to be a weak solution of the problem (HNS) if for any $\mathbf{w} \in W^{1,\varphi}_{y,0}(Q_T \times Y)$

$$\langle \mathbf{u}^0_t, \mathbf{w}\rangle + \int_{Q_T}\int_Y \mathbf{S}(\mathbf{D}_y\mathbf{u}^0) : \mathbf{D}_y \mathbf{w} + \int_{Q_T}\int_Y \nabla_x p \cdot \mathbf{w} - \int_{Q_T}\int_Y \pi \operatorname{div} \mathbf{w} = \int_{Q_T}\int_Y \mathbf{f}\cdot\mathbf{w}$$

and

$$\lim_{t\to 0} \|\mathbf{u}^0(t) - \mathbf{a}^0\|_{L^2(\Omega\times Y)} = 0. \tag{1.56}$$

Proof of Theorem 1.1.3. Let $\{\mathbf{w}^i\} \subset X^{m,2}_{y,0}(\Omega \times Y)$ be such that for

$$V_n = \operatorname{span}\{\mathbf{w}^1, \dots \mathbf{w}^n\}$$

$\bigcup_{n=1}^\infty V_n$ is dense in $X^{m,2}_{y,0}(\Omega\times Y)$. Here $m > 1 + \frac{d}{2}$, which ensures the validity of the embedding $W^{m,2}(\Omega\times Y)^d \hookrightarrow C^1(\overline{\Omega\times Y})^d$. Furthermore, we suppose that $\{\mathbf{w}^i\}$ is orthonormal in $L^2(\Omega\times Y)^d$. Proceeding similarly as in [23, Theorem 4.11], one shows that $\{\mathbf{w}^i\}$ is formed by solutions of the spectral problem

$$(\mathbf{w}^i, \mathbf{v})_{W^{m,2}(\Omega\times Y)} = \lambda_i(\mathbf{w}^i, \mathbf{v})_{L^2(\Omega\times Y)} \text{ for any } \mathbf{v} \in X^{m,2}_{y,0}(\Omega\times Y),$$

where $(\cdot,\cdot)_{W^{m,2}(\Omega\times Y)}, (\cdot,\cdot)_{L^2(\Omega\times Y)}$ stands for the scalar product on $W^{m,2}(\Omega\times Y)$ and $L^2(\Omega\times Y)$ respectively. We note that $X^{m,2}_{y,0}(\Omega\times Y)$ is dense in $X^{1,\varphi}_{y,0}(\Omega\times Y)$ since $X^{m,2}_{y,0}(\Omega\times Y) \supset \mathscr{C}_D$, see Lemma 1.2.8. As $W^{1,\varphi}_{y,0}(\Omega\times Y)\cap L^2(\Omega\times Y)^d \supset \mathscr{C}$, one shows the density of $W^{1,\varphi}_{y,0}(\Omega\times Y)\cap L^2(\Omega\times Y)^d$ in $L^2_{y,0}(\Omega\times Y)^d$ by repeating some of considerations from the proof of Lemma 1.2.8. Let us denote $\mathscr{V}_n = C^1([0,T], V_n)$ and $\mathscr{X} = \overline{\bigcup_{n=1}^\infty \mathscr{V}_n}^{\|\cdot\|_{C^1(\overline{Q_T\times Y})}}$. Then $\mathscr{X}$ is dense in $X^{1,\varphi}_{y,0}(Q_T\times Y)$. We consider a sequence $\{\mathbf{f}^n\} \subset C^\infty\left([0,T]; \mathscr{D}(\Omega)^d\right)$ such that $\mathbf{f}^n \to \mathbf{f}$ in $L^{\varphi^*}(Q^T)^d$. For fixed $n \in \mathbb{N}$ we define an approximation $\mathbf{u}^{0,n}$ by

$$\mathbf{u}^{0,n}(t,x,y) = \sum_{i=1}^n d_i^n(t)\mathbf{w}^i(x,y).$$

Coefficients $d_i^n : [0,T] \to \mathbb{R}^d$ are chosen in such a way that for all $i = 1, \dots, n$

$$\begin{aligned} d_i^n(0) &= \int_{\Omega\times Y} \mathbf{a}^0 \cdot \mathbf{w}^i, \\ \int_{\Omega\times Y} \mathbf{u}_t^{0,n} \cdot \mathbf{w}^i + \int_{\Omega\times Y} \mathbf{S}(\mathbf{D}_y \mathbf{u}^{0,n}) : \mathbf{D}_y \mathbf{w}^i &= \int_{\Omega\times Y} \mathbf{f}^n \cdot \mathbf{w}^i,\ t \in [0,T]. \end{aligned} \tag{1.57}$$

Remark that $(1.57)_1$ means that $\mathbf{u}^{0,n}(0) = \mathbf{a}^n$ is the projection of $\mathbf{a}^0$ on V_n in $L^2(\Omega\times Y)^d$ and thus $\mathbf{a}^n \to \mathbf{a}^0$ in $L^2(\Omega \times Y)^d$. We denote for $i = 1, \dots, n$

$$(g_n(\mathbf{d}^n(t)))_i = \int_{\Omega\times Y} \mathbf{f}^n \cdot \mathbf{w}^i - \mathbf{S}\left(\sum_{k=1}^n d_k^n(t) \mathbf{D}_y \mathbf{w}^k\right) : \mathbf{D}_y \mathbf{w}^i$$

and rewrite $(1.57)_2$ using the orthogonality of $\{\mathbf{w}^i\}$ in $L^2(\Omega \times Y)^d$ as

$$\dot{\mathbf{d}}^n = \mathbf{g}_n(\mathbf{d}^n).$$

Since the right hand side of the latter system is continuous in t and $\mathbf{d}^n$, there exists $t^* \in (0,T)$ and a solution $\mathbf{d}^n \in C^1[0,t^*)$ of the above system with the initial condition $(1.57)_1$ according to the theory for ordinary differential equations.
Let us derive first apriori estimates. We multiply the system (1.57) by $d_i^n(t)$, sum over i and obtain for any $t \in (0,T)$

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{u}^{0,n}(s)\|^2_{L^2(\Omega\times Y)} + \int_0^t \int_{\Omega\times Y} \mathbf{S}(\mathbf{D}_y \mathbf{u}^{0,n}) : \mathbf{D}_y \mathbf{u}^{0,n} = \int_0^t \int_{\Omega\times Y} \mathbf{f}^n \cdot \mathbf{u}^{0,n}. \tag{1.58}$$

We apply (1.7), Young's inequality (1.10) with small δ and Korn's inequality (1.17) to get

$$\begin{aligned} &\|\mathbf{u}^{0,n}(t)\|^2_{L^2(\Omega\times Y)} + c \int_0^t \int_{\Omega\times Y} \varphi(|\mathbf{D}_y \mathbf{u}^{0,n}|) \\ &\le \|\mathbf{a}^n\|^2_{L^2(\Omega\times Y)} + c_\delta \int_0^t \int_{\Omega\times Y} \varphi^*(|\mathbf{f}^n|) + \delta \int_0^t \int_{\Omega\times Y} \varphi(|\mathbf{u}^{0,n}|) \\ &\le \|\mathbf{a}^n\|^2_{L^2(\Omega\times Y)} + c_\delta \int_0^t \int_{\Omega\times Y} \varphi^*(|\mathbf{f}^n|) + c\delta \int_0^t \int_{\Omega\times Y} \varphi(|\mathbf{D}_y \mathbf{u}^{0,n}|). \end{aligned}$$

Hence we have the apriori estimate

$$\sup_{t\in[0,t^*]} \|\mathbf{u}^{0,n}(t)\|^2_{L^2(\Omega\times Y)} + \int_0^T \int_{\Omega\times Y} \varphi(|\mathbf{D}_y \mathbf{u}^{0,n}|) \le c. \tag{1.59}$$

Since the constant on the right hand side of (1.59) is independent of t^*, we obtain from the estimate of the first term on the left hand side that the coefficients d_i^n exist on the interval $(0,T)$ and (1.59) holds with $t^* = T$. From the estimate (1.59), we also deduce

$$\int_{Q_T\times Y} \varphi^*\left(\mathbf{S}(\mathbf{D}_y \mathbf{u}^{0,n})\right) \le c$$

thanks to (1.8). We deduce from $(1.57)_2$ that for any $\mathbf{w} \in \mathfrak{X}$, in fact for any $\mathbf{w} \in X^{1,\varphi}_{y,0}(Q_T \times Y)$ because of the density of $\mathfrak{X}$ in $X^{1,\varphi}_{y,0}(Q_T \times Y)$

$$\int_{Q_T\times Y} \mathbf{u}_t^{0,n} \cdot \mathbf{w} + \int_{Q_T\times Y} \mathbf{S}(\mathbf{D}_y \mathbf{u}^{0,n}) : \mathbf{D}_y \mathbf{w} = \int_{Q_T\times Y} \mathbf{f}^n \cdot \mathbf{w}. \tag{1.60}$$

Hence the apriori estimates imply the uniform estimate

$$\|\mathbf{u}_t^{0,n}\|_{\left(X_{y,0}^{1,\varphi}(Q_T\times Y)\right)^*} \le c.$$

In fact we have the uniform estimate of $\mathbf{u}_t^{0,n}$ in $W_{y,0}^{1,\varphi}(Q_T\times Y)$. Since $X_{y,0}^{1,\varphi}(Q_T\times Y)$ is a subspace of $W_{y,0}^{1,\varphi}(Q_T\times Y)$, we find an extension of $\mathbf{u}_t^{0,n}$, for which we keep the notation, on $W_{y,0}^{1,\varphi}(Q_T\times Y)$ with the same norm as $\mathbf{u}_t^{0,n}$ employing Hahn-Banach theorem. As a consequence of the uniform estimate of $\mathbf{u}_t^{0,n}$ and (1.59), we obtain that as $n\to\infty$, up to subsequences,

$$\begin{aligned}
\mathbf{u}^{0,n} &\rightharpoonup^* \mathbf{u}^0 && \text{in } L^\infty\left(0,T;L^2(\Omega\times Y)^d\right),\\
\mathbf{D}_y\mathbf{u}^{0,n} &\rightharpoonup \mathbf{D}_y\mathbf{u}^0 && \text{in } L^\varphi(Q_T\times Y),\\
\mathbf{S}(\mathbf{D}_y\mathbf{u}^{0,n}) &\rightharpoonup \overline{S} && \text{in } L^{\varphi^*}(Q_T\times Y),\\
\mathbf{u}_t^{0,n} &\rightharpoonup \mathbf{u}_t^0 && \text{in } \left(W_{y,0}^{1,\varphi}(Q_T\times Y)\right)^*.
\end{aligned} \tag{1.61}$$

Having these convergences, we can perform the limit passage $n\to\infty$ in (1.60)

$$\langle \mathbf{u}_t^0,\mathbf{w}\rangle + \int_{Q_T\times Y}\overline{\mathbf{S}}:\mathbf{D}_y\mathbf{w} = \int_{Q_T\times Y}\mathbf{f}\cdot\mathbf{w} \tag{1.62}$$

for any $\mathbf{w}\in X_{y,0}^{1,\varphi}(Q_T\times Y)$.

Next, we identify $\overline{S}$. We observe that $\mathbf{u}^{0,n}(T)\rightharpoonup\mathbf{u}^0(T)$ in $L^2(\Omega\times Y)^d$, which immediately follows recalling that $\mathbf{u}^{0,n}(T)=\mathbf{a}^n+\int_0^T\mathbf{u}_t^{0,n}$, the weak convergence of $\{\mathbf{u}_t^{0,n}\}$ in $\left(X_{y,0}^{1,\varphi}(Q_T\times Y)\right)^*$ and $\{\mathbf{a}^n\}$ in $L^2(\Omega\times Y)^d$. Hence using the weak lower semicontinuity of L^2-norm, we obtain from (1.58)

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{Q_T}\int_Y \mathbf{S}(\mathbf{D}_y\mathbf{u}^{0,n}):\mathbf{D}_y\mathbf{u}^{0,n}\\
&\quad\le \lim_{n\to\infty}\int_{Q_T}\int_Y \mathbf{f}^n\cdot\mathbf{u}^{0,n} + \lim_{n\to\infty}\|\mathbf{a}^n\|^2_{L^2(Q_T\times Y)} + \limsup_{n\to\infty}\{-\|\mathbf{u}^{0,n}(T)\|^2_{L^2(Q_T\times Y)}\}\\
&\quad\le \int_{Q_T}\int_Y \mathbf{f}\cdot\mathbf{u}^0 + \|\mathbf{a}^0\|^2_{L^2(Q_T\times Y)} - \|\mathbf{u}^0(T)\|^2_{L^2(Q_T\times Y)}.
\end{aligned}$$

Comparing this with (1.62) tested by $\mathbf{u}^0$ yields

$$\limsup_{n\to\infty}\int_{Q_T}\int_Y \mathbf{S}(\mathbf{D}_y\mathbf{u}^{0,n}):\mathbf{D}_y\mathbf{u}^{0,n} \le \int_{Q_T}\int_Y\overline{\mathbf{S}}:\mathbf{D}_y\mathbf{u}^0,$$

from which $\overline{\mathbf{S}}=\mathbf{S}(\mathbf{D}_y\mathbf{u}^0)$ follows by Minty's trick.

The last task is to show the attainment of an initial value. Integrating (1.57) over $(0,t)$, we get for almost all $t\in(0,T)$

$$\int_{\Omega\times Y}(\mathbf{u}^{0,n}(t)-\mathbf{u}^{0,n}(0))\cdot\mathbf{w}^i + \int_0^t\int_{\Omega\times Y}\mathbf{S}(\mathbf{D}_y\mathbf{u}^{0,n}):\mathbf{D}_y\mathbf{w}^i = \int_0^t\int_{\Omega\times Y}\mathbf{f}^n\cdot\mathbf{w}^i.$$

Applying convergences (1.61), we perform the limit passage $n\to\infty$ and then use density of $\bigcup_{n=1}^\infty V_n$ in $X_{y,0}^{m,2}(\Omega\times Y)$ to obtain for any $\mathbf{w}\in X_{y,0}^{m,2}(\Omega\times Y)$

$$\int_{\Omega\times Y}(\mathbf{u}^0(t)-\mathbf{a}^0)\cdot\mathbf{w} + \int_0^t\int_{\Omega\times Y}\overline{\mathbf{S}}:\mathbf{D}_y\mathbf{w} = \int_0^t\int_{\Omega\times Y}\mathbf{f}\cdot\mathbf{w},$$

which implies for any $\mathbf{w} \in X^{m,2}_{y,0}(\Omega \times Y)$

$$\int_{\Omega\times Y} (\mathbf{u}^0(t) - \mathbf{a}^0) \cdot \mathbf{w} \to 0 \text{ as } t \to 0_+. \tag{1.63}$$

Using the density of $X^{m,2}_{y,0}(\Omega \times Y)$ in $L^2(\Omega \times Y)^d \cap X^{1,\varphi}_{y,0}(\Omega \times Y)$, we see that the latter convergence holds for any $\mathbf{w} \in L^2(\Omega \times Y)^d \cap X^{1,\varphi}_{y,0}(\Omega \times Y)$. Since for $t \in (0,T)$ $\mathbf{u}^0(t) - \mathbf{a}^0 \in L^2(\Omega \times Y)^d \cap X^{1,\varphi}_{y,0}(\Omega \times Y)$, which is a closed subspace of Hilbert space $L^2(\Omega \times Y)^d \cap W^{1,\varphi}_{y,0}(\Omega \times Y)$, we have

$$\int_{\Omega\times Y} (\mathbf{u}^0(t) - \mathbf{a}^0) \cdot \mathbf{w}_\perp = 0$$

for $\mathbf{w}_\perp \in \left(L^2(\Omega \times Y)^d \cap X^{1,\varphi}_{y,0}(\Omega \times Y)\right)^\perp$. Thus (1.63) holds for any $\mathbf{w} \in L^2(\Omega\times Y)^d \cap W^{1,\varphi}_{y,0}(\Omega \times Y)$, which is dense in $L^2_{y,0}(\Omega \times Y)^d$. Since $\mathbf{u}^0(t) - \mathbf{a}^0 = 0$ in $\Omega \times Y_S$, (1.63) holds for any $\mathbf{w} \in L^2(\Omega \times Y)^d$, i.e., $\mathbf{u}^0(t) \rightharpoonup \mathbf{a}^0$ in $L^2(\Omega \times Y)^d$ as $t \to 0_+$, which implies due to the weak lower semicontinuity of the norm

$$\liminf_{t\to 0} \|\mathbf{u}^0(t)\|_{L^2(\Omega\times Y)} \geq \|\mathbf{a}^0\|_{L^2(\Omega\times Y)}. \tag{1.64}$$

Employing convergences (1.61) and the weak lower semicontinuity of the L^2−norm in (1.58), where the second term is neglected since it is nonnegative, yields

$$\|\mathbf{u}^0(t)\|^2_{L^2(\Omega\times Y)} - \|\mathbf{a}^0\|^2_{L^2(\Omega\times Y)} \leq 2\int_{\Omega\times Y} \mathbf{f} \cdot \mathbf{u}^0,$$

which implies

$$\limsup_{t\to 0} \|\mathbf{u}^0(t)\|_{L^2(\Omega\times Y)} \leq \|\mathbf{a}^0\|_{L^2(\Omega\times Y)}. \tag{1.65}$$

Clearly, (1.64) and (1.65) imply $\lim_{t\to 0_+} \|\mathbf{u}^0(t)\|^2_{L^2(\Omega\times Y)} = \|\mathbf{a}^0\|^2_{L^2(\Omega\times Y)}$, from which (1.56) follows.
It remains to find a pressure in the form $\nabla_x p(t,x) + \nabla_y \pi(t,x,y)$. Let us define functional F on $W^{1,\varphi}_{y,0}(Q_T \times Y)$ by

$$\langle F, \mathbf{w}\rangle = \int_0^T \int_\Omega \int_Y \mathbf{f}\cdot\mathbf{w} - \int_\Omega\int_Y \mathbf{S}(\mathbf{D}_y\mathbf{u}^0) : \mathbf{D}_y\mathbf{w} - \int_0^T\int_\Omega\int_Y \mathbf{u}^0_t \cdot \mathbf{w}.$$

Obviously, $F \in \left(X^{1,\varphi}_{y,0}(Q_T \times Y)\right)^\perp$. Thus F has the required form according to Lemma 1.2.11. $\square$

Definition 1.6.2 Let us denote $Q^\varepsilon_T = (0,T)\times\Omega^\varepsilon$. A pair $(\mathbf{u}^\varepsilon, p^\varepsilon) \in \Big(L^\infty\left(0,T; L^2(\Omega^\varepsilon)^d\right)$ $\cap W^{1,\varphi}_{x,0,\mathrm{div}}(Q^\varepsilon_T)^d\Big) \times L^{\varphi^*}(Q^\varepsilon_T)$ with $\mathbf{u}^\varepsilon_t \in \left(W^{1,\varphi}_{x,0}(Q^\varepsilon_T)^d\right)^*$ is said to be a weak solution of the problem (NGS$_\varepsilon$) if for any $\mathbf{w} \in W^{1,\varphi}_{x,0}(Q^\varepsilon_T)$

$$\langle \mathbf{u}^\varepsilon_t, \mathbf{w}\rangle + \varepsilon\int_0^T\int_{\Omega^\varepsilon} \mathbf{S}(\varepsilon\mathbf{D}\mathbf{u}^\varepsilon) : \mathbf{D}\mathbf{w} - \int_0^T\int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div}\mathbf{w} = \int_0^T\int_{\Omega^\varepsilon} \mathbf{f}\cdot\mathbf{w} \tag{1.66}$$

and $\lim_{t\to 0} \|\mathbf{u}^\varepsilon(t) - \mathbf{a}^\varepsilon\|_{L^2(\Omega^\varepsilon)} = 0$.

Lemma 1.6.3 *Let $\Delta_2(\varphi,\varphi')<\infty$, Ω^ε be defined by (1.6), $\mathbf{f}\in L^{\varphi^*}((0,T)\times\Omega)^d$, $\mathbf{a}^\varepsilon$ be the same as in Theorem 1.1.3 and $\varepsilon>0$ be fixed. Then there exists a unique $\mathbf{u}^\varepsilon\in L^\infty\left(0,T;L^2(\Omega^\varepsilon)^d\right)\cap W^{1,\varphi}_{x,0,\mathrm{div}}(Q^\varepsilon_T)$ satisfying (1.66) for any $\mathbf{w}$ solenoidal. Moreover, there are constants $c_1,c_2,c_3,c_4>0$ independent of ε such that*

$$\sup_{t\in(0,T)}\|\mathbf{u}^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon)}+\int_0^T\int_{\Omega^\varepsilon}\varphi(|\varepsilon\mathbf{D}\mathbf{u}^\varepsilon|)\le c_1, \tag{1.67}$$

$$\int_0^T\int_{\Omega^\varepsilon}\varphi^*(\varphi'(|\varepsilon\mathbf{D}\mathbf{u}^\varepsilon|))\le c_2, \tag{1.68}$$

$$\|\mathbf{u}^\varepsilon_t\|^2_{L^2((0,T)\times\Omega^\varepsilon)}+\sup_{t\in(0,T)}\int_{\Omega^\varepsilon}\varphi(|\varepsilon\mathbf{D}\mathbf{u}^\varepsilon(t)|)\le c_3, \tag{1.69}$$

$$\|\mathbf{u}^\varepsilon_t\|_{L^\infty(0,T;L^2(\Omega^\varepsilon))}\le c_4. \tag{1.70}$$

Proof. To show the existence of a weak solution $\mathbf{u}^\varepsilon$, we can follow the lines of the proof of Theorem 1.1.3 with minor changes. Let $\{\mathbf{w}^i\}\subset W^{m,2}_{0,\mathrm{div}}(\Omega^\varepsilon)^d$ be such that for

$$V_n=\operatorname{span}\{\mathbf{w}^1,\dots\mathbf{w}^n\}$$

$\bigcup_{n=1}^\infty V_n$ is dense in $W^{m,2}_{0,\mathrm{div}}(\Omega^\varepsilon)^d$, again $m>1+\frac{d}{2}$. Moreover, we suppose that $\{\mathbf{w}^i\}$ is orthonormal in $L^2(\Omega^\varepsilon)^d$ and the projection $P_n: W^{m,2}_{0,\mathrm{div}}(\Omega^\varepsilon)^d\to V_n$ is bounded uniformly in $n\in\mathbb{N}$ and $\varepsilon>0$, see [23, Theorem 4.11]. Let us denote $\mathscr{V}_n=C^1([0,T],V_n)$ and $\mathfrak{X}=\overline{\bigcup_{n=1}^\infty\mathscr{V}_n}^{\|\cdot\|_{C^1(\overline{Q_T\times Y})}}$. Then $\mathfrak{X}$ is dense in $W^{1,\varphi}_{x,0,\mathrm{div}}(Q^\varepsilon_T)$. We consider sequences $\{\mathbf{f}^n\},\{\mathbf{f}^n_t\}\subset C^\infty\left([0,T];\mathscr{D}(\Omega^\varepsilon)^d\right)$ such that $\mathbf{f}^n\to\mathbf{f}$ in $L^{\varphi^*}(Q^T_\varepsilon)^d$, $\mathbf{f}^n_t\to\mathbf{f}_t$ in $L^2(Q^T_\varepsilon)^d$ and $\|\mathbf{f}^n(0)\|_{L^2(\Omega^\varepsilon)}\le c$.

For fixed $n\in\mathbb{N}$ we define an approximation $\mathbf{u}^{\varepsilon,n}$ by

$$\mathbf{u}^{\varepsilon,n}(t,x)=\sum_{i=1}^n d^{\varepsilon,n}_i(t)\mathbf{w}^i(x).$$

Coefficients $d^{\varepsilon,n}_i:[0,T]\to\mathbb{R}^d$ are chosen in such a way that for all $i=1,\dots,n$

$$\begin{aligned} &d^{\varepsilon,n}_i(0)=\int_{\Omega^\varepsilon}\mathbf{a}^\varepsilon\cdot\mathbf{w}^i,\\ &\int_{\Omega^\varepsilon}\partial_t\mathbf{u}^{\varepsilon,n}\cdot\mathbf{w}^i+\int_{\Omega^\varepsilon}\mathbf{S}(\varepsilon\mathbf{D}\mathbf{u}^{\varepsilon,n}):\varepsilon\mathbf{D}\mathbf{w}^i=\int_{\Omega^\varepsilon}\mathbf{f}^n\cdot\mathbf{w}^i\ t\in[0,T]. \end{aligned} \tag{1.71}$$

Note that $(1.71)_1$ means $\mathbf{u}^{\varepsilon,n}(0)=P_n\mathbf{a}^\varepsilon$. Similarly as in the proof of Theorem 1.1.3, one deduces

$$\begin{aligned} \mathbf{u}^{\varepsilon,n}&\rightharpoonup^*\mathbf{u}^\varepsilon &&\text{in } L^\infty\left(0,T;L^2(\Omega^\varepsilon)^d\right),\\ \mathbf{u}^{\varepsilon,n}&\rightharpoonup\mathbf{u}^\varepsilon &&\text{in } L^\varphi(Q^\varepsilon_T)^d,\\ \mathbf{D}\mathbf{u}^{\varepsilon,n}&\rightharpoonup\mathbf{D}\mathbf{u}^\varepsilon &&\text{in } L^\varphi(Q^\varepsilon_T)^{d\times d},\\ \mathbf{u}^{\varepsilon,n}_t&\rightharpoonup\mathbf{u}^\varepsilon_t &&\text{in } \left(W^{1,\varphi}_{x,0}(Q_T)\right)^*. \end{aligned}$$

Due to the weak* lower semicontinuity of the norm in $L^\infty\left(0,T;L^2(\Omega^\varepsilon)^d\right)$ and the weak lower semicontinuity of the modular, see [10, Theorem 2.2.8], we can conclude (1.67). The fact that $\mathbf{u}^\varepsilon$ fulfills the weak formulation as well as attaining the initial value is

shown similarly as in the proof of Theorem 1.1.3.
The estimate (1.68) is an immediate consequence of (1.67) due to the property (1.8).
To derive (1.69), we multiply (1.71) by $\dot{d}_i^{\varepsilon,n}(t)$, sum over i and obtain for any $t \in (0,T)$

$$\int_0^t \int_{\Omega^\varepsilon} |\partial_t \mathbf{u}^{\varepsilon,n}|^2 + \int_0^t \int_{\Omega^\varepsilon} \partial_t \varphi(|\varepsilon \mathbf{D}\mathbf{u}^{\varepsilon,n}|) = \int_0^t \int_{\Omega^\varepsilon} \mathbf{f}^n \partial_t \mathbf{u}^{\varepsilon,n}.$$

Young's inequality yields

$$\int_0^t \int_{\Omega^\varepsilon} |\partial_t \mathbf{u}^{\varepsilon,n}|^2 + \int_{\Omega^\varepsilon} \varphi(|\varepsilon \mathbf{D}\mathbf{u}^{\varepsilon,n}(t)|) \leq \int_{\Omega^\varepsilon} \varphi(|\varepsilon \mathbf{D} P_n \mathbf{a}^\varepsilon|) + \int_0^t \int_{\Omega^\varepsilon} c_\delta |\mathbf{f}^n|^2 + \delta |\partial_t \mathbf{u}^{\varepsilon,n}|^2,$$

from which (1.69) follows with the help of (1.4) after the limit passage $n \to \infty$. To show (1.70), we first differentiate the system (1.71) with respect to t to obtain

$$\int_{\Omega^\varepsilon} \partial_{tt} \mathbf{u}^{\varepsilon,n} \cdot \mathbf{w}^i + \int_{\Omega^\varepsilon} \varphi''(\varepsilon \mathbf{D}\mathbf{u}^{\varepsilon,n}) \varepsilon^2 \mathbf{D} \partial_t \mathbf{u}^{\varepsilon,n} : \mathbf{D}\mathbf{w}^i = \int_{\Omega^\varepsilon} \partial_t \mathbf{f}^n \cdot \mathbf{w}^i$$

then we multiply the latter system by $\dot{d}_i^{\varepsilon,n}(t)$, sum over i and apply Hölder's inequality to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\partial_t \mathbf{u}^{\varepsilon,n}\|^2_{L^2(\Omega^\varepsilon)} + \int_{\Omega^\varepsilon} \varphi''(|\varepsilon \mathbf{D}\mathbf{u}^{\varepsilon,n}|) \varepsilon^2 |\mathbf{D} \partial_t \mathbf{u}^{\varepsilon,n}|^2 \leq \frac{1}{2} \|\partial_t \mathbf{f}^n\|^2_{L^2(\Omega^\varepsilon)} + \frac{1}{2} \|\partial_t \mathbf{u}^{\varepsilon,n}\|^2_{L^2(\Omega^\varepsilon)}.$$

The second term on the left hand side is nonnegative since $\varphi \in C^2((0,\infty))$ is convex thus Gronwall's inequality implies

$$\|\partial_t \mathbf{u}^{\varepsilon,n}(t)\|^2_{L^2(\Omega^\varepsilon)} \leq c(\|\partial_t \mathbf{f}^n\|^2_{L^2(\Omega^\varepsilon)} + \|\partial_t \mathbf{u}^{\varepsilon,n}(0)\|^2_{L^2(\Omega^\varepsilon)}).$$

Finally, we need to estimate the last term on the right hand side. To this end, we take (1.71) for $t = 0$, multiply it by $\dot{d}_i^{\varepsilon,n}(0)$, sum over i and get

$$\int_{\Omega^\varepsilon} |\partial_t \mathbf{u}^{\varepsilon,n}(0)|^2 = \int_{\Omega^\varepsilon} \mathbf{f}^n(0) \cdot \partial_t \mathbf{u}^{\varepsilon,n}(0) - \varepsilon \int_{\Omega^\varepsilon} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^{\varepsilon,n}(0)) : \mathbf{D} \partial_t \mathbf{u}^{\varepsilon,n}(0).$$

We employ the integration by parts, Young's and Korn's inequality to obtain

$$\begin{aligned} \int_{\Omega^\varepsilon} |\partial_t \mathbf{u}^{\varepsilon,n}(0)|^2 \leq{}& c \int_{\Omega^\varepsilon} |\mathbf{f}^n(0)|^2 + \delta \int_{\Omega^\varepsilon} |\partial_t \mathbf{u}^{\varepsilon,n}(0)|^2 \\ &+ c\varepsilon^2 \int_{\Omega^\varepsilon} |\operatorname{div} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^{\varepsilon,n}(0))|^2 + \delta \int_{\Omega^\varepsilon} |\partial_t \mathbf{u}^{\varepsilon,n}(0)|^2. \end{aligned}$$

The final step is to estimate the third term on the right hand side. Since we have (1.9), (1.13), the embedding (1.3), precisely $\|\cdot\|_{L^\infty(\Omega^\varepsilon)} \leq c\varepsilon^{m-1-\frac{d}{2}} \|\nabla^{m-1} \cdot\|_{L^2(\Omega^\varepsilon)}$, the inequality $\|\cdot\|_{L^2(\Omega^\varepsilon)} \leq c\varepsilon^{m-1} \|\nabla^{m-1} \cdot\|_{L^2(\Omega^\varepsilon)}$, which are derived in the same way as Lemma 1.2.6, and the uniform boundedness of the projection on $W^{m,2}(\Omega^\varepsilon)$ with respect to n and ε, we infer

$$\begin{aligned} &\int_{\Omega^\varepsilon} |\varepsilon \operatorname{div} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^{\varepsilon,n}(0))|^2 \\ &\leq c\varepsilon^2 \int_{\Omega^\varepsilon} (\varphi''(|\varepsilon \mathbf{D} P_n \mathbf{a}^\varepsilon|))^2 |\varepsilon \nabla \mathbf{D} P_n \mathbf{a}^\varepsilon|^2 \leq c\varepsilon^2 (\varphi''(\|\varepsilon \mathbf{D} P_n \mathbf{a}^\varepsilon\|_{L^\infty(\Omega^\varepsilon)}))^2 \|\varepsilon \nabla \mathbf{D} P_n \mathbf{a}^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \\ &\leq c\varepsilon^{2m} \max\{\|\varepsilon \mathbf{D} P_n \mathbf{a}^\varepsilon\|^{2(q_1-2)}_{L^\infty(\Omega^\varepsilon)}, \|\varepsilon \mathbf{D} P_n \mathbf{a}^\varepsilon\|^{2(q_2-2)}_{L^\infty(\Omega^\varepsilon)}\} \|\nabla^{m-1} \mathbf{D} P_n \mathbf{a}^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \\ &\leq c\varepsilon^{2m} \max\{\varepsilon^{(2m-d)(q_1-2)} \|\nabla^{m-1} \mathbf{D} P_n \mathbf{a}^\varepsilon\|^{2(q_1-1)}_{L^2(\Omega^\varepsilon)}, \varepsilon^{(2m-d)(q_2-2)} \|\nabla^{m-1} \mathbf{D} P_n \mathbf{a}^\varepsilon\|^{2(q_2-1)}_{L^2(\Omega^\varepsilon)}\} \\ &\leq c \max\{\|\varepsilon^{m-\frac{d}{2}} \nabla^m \mathbf{a}^\varepsilon\|^{2(q_1-1)}_{L^2(\Omega^\varepsilon)}, \|\varepsilon^{m-\frac{d}{2}} \nabla^m \mathbf{a}^\varepsilon\|^{2(q_2-1)}_{L^2(\Omega^\varepsilon)}\} \leq c. \end{aligned}$$

We also used $\varepsilon < 1$ and (1.4). We apply the assumption (1.1), the estimate (1.69) and pass to the limit $n \to \infty$ to conclude (1.70). $\square$

As in the stationary case, the zero boundary condition for $\mathbf{u}^\varepsilon$ in $(0,T)\times\partial\Omega^\varepsilon$ allows us to extend functions $\mathbf{u}^\varepsilon$ by zero to $(0,T)\times(\Omega\setminus\Omega^\varepsilon)$ and all estimates from Lemma 1.6.3 hold with Ω replacing Ω^ε. When extending pressure, we follow again the approach, which comes from [34] and was modified for unsteady situation in [4].

Lemma 1.6.4 *Let the assumptions of Lemma 1.6.3 be fulfilled and the embedding* (1.2) *hold. There is a pressure function* $p^\varepsilon \in L^\infty(I; L^{\varphi^*}(\Omega^\varepsilon)/\mathbb{R})$ *corresponding to* $\mathbf{u}^\varepsilon$ *from Lemma 1.6.3. Moreover, there is an extension* P^ε *of* p^ε *in* $(\mathrm{NGS}_\varepsilon)$*, which satisfies for a.a.* $t \in (0,T)$ *and any* $\mathbf{w} \in \mathcal{D}(\Omega)^d$

$$\int_\Omega \mathbf{u}_t^\varepsilon \cdot R_\varepsilon \mathbf{w} + \int_\Omega \varepsilon \mathbf{S}(\varepsilon D\mathbf{u}^\varepsilon) : \mathbf{D} R_\varepsilon \mathbf{w} - \int_\Omega P^\varepsilon \operatorname{div} \mathbf{w} = \int_\Omega \mathbf{f} \cdot R_\varepsilon \mathbf{w} \tag{1.72}$$

Finally, there is $c > 0$ *independent of* ε *such that*

$$\|P^\varepsilon\|_{L^\infty(I;L^{\varphi^*}(\Omega)/\mathbb{R})} \leq c. \tag{1.73}$$

Proof. Since we know $\mathbf{u}_t^\varepsilon \in L^\infty\left(0,T; L^2(\Omega)^d\right)$, we can reconstruct the pressure at almost every time level $t \in (0,T)$ as in the steady case. We obtain for any $\mathbf{w} \in W_0^{1,\varphi}(\Omega^\varepsilon)$ and almost all $t \in (0,T)$

$$\int_{\Omega^\varepsilon} p^\varepsilon(t) \operatorname{div} \mathbf{w} = \int_{\Omega^\varepsilon} \mathbf{u}_t^\varepsilon(t) \cdot \mathbf{w} - \varepsilon \int_{\Omega^\varepsilon} \mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon(t)) : \mathbf{D}\mathbf{w} + \int_{\Omega^\varepsilon} \mathbf{f}(t) \cdot \mathbf{w}.$$

We estimate the right hand side of the previous inequality using the apriori estimates (1.67)-(1.70)

$$\begin{aligned}\int_{\Omega^\varepsilon} p^\varepsilon(t) \operatorname{div} \mathbf{w} \leq{}& \|\mathbf{u}_t^\varepsilon\|_{L^\infty(I,L^2(\Omega))} \|\mathbf{w}\|_{L^2(\Omega^\varepsilon)} + \varepsilon \|\mathbf{S}(\varepsilon \mathbf{D}\mathbf{u}^\varepsilon)\|_{L^\infty(I,L^{\varphi^*}(\Omega))} \|\mathbf{D}\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)} \\ &+ \|\mathbf{f}\|_{L^\infty(I,L^{\varphi^*}(\Omega))} \|\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)}.\end{aligned}$$

We have due to the assumption (1.2)

$$\int_{\Omega^\varepsilon} p^\varepsilon(t) \operatorname{div} \mathbf{w} \leq c(\|\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)} + \varepsilon \|\mathbf{D}\mathbf{w}\|_{L^\varphi(\Omega^\varepsilon)}) \tag{1.74}$$

and similarly as in the stationary case, we get the regularity of p^ε.
Using the restriction operator as in the stationary case, we define an extension P^ε for almost all $t \in (0,T)$ and $\mathbf{w} \in W_0^{1,\varphi}(\Omega)$ by

$$\int_\Omega P^\varepsilon(t) \operatorname{div} \mathbf{w} = \int_{\Omega^\varepsilon} p^\varepsilon(t) \operatorname{div} R_\varepsilon \mathbf{w}. \tag{1.75}$$

Then we use (1.74) to conclude the uniform estimate (1.73).
(1.72) easily follows from (1.75). $\square$

Lemma 1.6.5 *Let the assumptions of Lemma 1.6.3 be fulfilled. Let for any* $\varepsilon > 0$ $\mathbf{u}^\varepsilon$ *be from Lemma 1.6.3 extended by zero in* $(0,T)\times(\Omega\setminus\Omega^\varepsilon)$ *and* P^ε *be the corresponding extended pressure from Lemma 1.6.4. Then from arbitrary sequences* $\{\mathbf{u}^\varepsilon\}$, $\{P^\varepsilon\}$ *one* *can extract subsequences* $\{\mathbf{u}^{\varepsilon'}\}$, $\{P^{\varepsilon'}\}$ *and find functions* $\mathbf{u}^0 \in W^{1,\varphi}_{y,0}(Q_T \times Y), \mathbf{S}^0 \in L^{\varphi^*}(Q_T \times Y)^{d\times d}$ *and* $p \in L^{\varphi^*}(Q_T)$ *such that as* $\varepsilon \to 0$

$$\mathbf{u}^{\varepsilon'} \xrightarrow{2-s} \mathbf{u}^0 \quad \text{in } L^{\varphi}(Q_T)^d, \tag{1.76}$$
$$\varepsilon'\mathbf{D}\mathbf{u}^{\varepsilon'} \xrightarrow{2-s} \mathbf{D}_y\mathbf{u}^0 \quad \text{in } L^{\varphi}(Q_T)^{d\times d}, \tag{1.77}$$
$$\mathbf{u}^{\varepsilon'}_t \xrightarrow{2-s} \mathbf{u}^0_t \quad \text{in } L^{2}(Q_T)^d, \tag{1.78}$$
$$\mathbf{u}^{\varepsilon'}(T) \xrightarrow{2-s} \mathbf{u}^0(T) \quad \text{in } L^{2}(\Omega)^d, \tag{1.79}$$
$$\mathbf{S}(\varepsilon'\mathbf{D}\mathbf{u}^{\varepsilon'}) \xrightarrow{2-s} \mathbf{S}^0 \quad \text{in } L^{\varphi^*}(Q_T)^{d\times d}, \tag{1.80}$$
$$P^{\varepsilon'} \longrightarrow p \quad \text{in } L^{\varphi^*}(Q_T). \tag{1.81}$$

Proof. The proof of (1.76),(1.77) and (1.80) is a complete analogue to the stationary case using the estimate (1.67), the version of Theorem 1.3.5 and Lemma 1.3.6 respectively, for time-dependent functions. One can show (1.78) in the similar manner as in Lemma 1.3.6. When showing (1.79), we recall the identity

$$\mathbf{u}^{\varepsilon'}(T) = \mathbf{u}^{\varepsilon'}(0) + \int_0^T \mathbf{u}^{\varepsilon'}_t = \mathbf{a}^{\varepsilon'} + \int_0^T \mathbf{u}^{\varepsilon'}_t, \tag{1.82}$$

which implies for any $\mathbf{w} \in L^2(\Omega \times Y)^d$

$$\int_\Omega \mathbf{u}^{\varepsilon'}(T)\cdot \mathbf{w}\left(x, \frac{x}{\varepsilon'}\right) = \int_\Omega \left[\mathbf{a}^{\varepsilon'} + \int_0^T \mathbf{u}^{\varepsilon'}_t\right]\cdot \mathbf{w}\left(x, \frac{x}{\varepsilon'}\right).$$

Using the assumption (1.5) and the convergence (1.78) the right hand side of the latter inequality converges as $\varepsilon' \to 0$ to

$$\int_{\Omega\times Y} \left[\mathbf{a}^0 + \int_0^T \mathbf{u}^0_t\right]\cdot \mathbf{w}\left(x, y\right) = \int_{\Omega\times Y} \mathbf{u}^0(T)\cdot \mathbf{w}(x,y),$$

where we applied (1.82) for $\mathbf{u}^0$. Hence one conclude (1.79). We obtain (1.81) as an immediate consequence of Theorem 1.3.5 and the estimate (1.73), which implies the uniform estimate of P^ε in $L^{\varphi^*}(Q_T)$. To show that the weak two-scale limit P^0 is independent of y we proceed analogously as in the proof of Lemma 1.5.6. □

Lemma 1.6.6 *The limit function* $\mathbf{u}^0$ *from Lemma 1.6.5 satisfies*

$$\begin{aligned} \operatorname{div}_y \mathbf{u}^0 &= 0 \ \text{in } Q_T \times Y,\\ \operatorname{div}_x\left(\int_Y \mathbf{u}^0\right) &= 0 \ \text{in } Q_T,\\ \left(\int_Y \mathbf{u}^0\right)\cdot \mathbf{n} &= 0 \ \text{on } (0,T)\times\partial\Omega,\\ \mathbf{u}^0 &= 0 \ \text{in } Q_T \times Y_S. \end{aligned}$$

Proof. The proof is analogous to the proof of (1.48)-(1.51). □

Proof of Theorem 1.1.4. In (1.72) with $\varepsilon = \varepsilon'$, we choose the same test function $\mathbf{w}^{\varepsilon'}$ as in the proof of Theorem 1.1.2, multiply by $z \in C^\infty([0,T])$ and integrate by parts to obtain

$$\int_0^T\int_\Omega \mathbf{u}^{\varepsilon'}_t \cdot z\mathbf{w}^{\varepsilon'} + \int_0^T\int_\Omega \varepsilon'\mathbf{S}(\varepsilon' D\mathbf{u}^{\varepsilon'}) : z\mathbf{D}\mathbf{w}^{\varepsilon'} - \int_0^T\int_\Omega P^{\varepsilon'} z \operatorname{div}\mathbf{w}^{\varepsilon'} = \int_0^T\int_\Omega \mathbf{f}\cdot z\mathbf{w}^{\varepsilon'}.$$

We employ the convergences (1.76)-(1.81) when passing to the limit $\varepsilon' \to 0$ to infer that for $\mathbf{w} \in \mathscr{D}\left(\Omega; C^\infty(Y)^d\right)$ with $\operatorname{div}_y \mathbf{w} = 0$ in $\Omega \times Y$ and $\mathbf{w} = 0$ in $\Omega \times Y_S$, $z \in C^\infty([0,T])$

$$\int_0^T \int_\Omega \int_Y \mathbf{u}_t^0 \cdot z\mathbf{w} + \int_0^T \int_\Omega \int_Y \mathbf{S}^0 : z\mathbf{D}_y \mathbf{w} - \int_0^T \int_\Omega \int_Y pz \operatorname{div}_x \mathbf{w} = \int_0^T \int_\Omega \int_Y \mathbf{f} \cdot z\mathbf{w}, \tag{1.83}$$

which is equivalent to the validity of

$$\int_\Omega \int_Y \mathbf{u}_t^0 \cdot \mathbf{w} + \int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \mathbf{w} - \int_0^T \int_\Omega \int_Y p \operatorname{div}_x \mathbf{w} = \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{w}$$

for a.a. $t \in (0,T)$ and $\mathbf{w} \in \mathscr{D}\left(\Omega; C^\infty(Y)^d\right)$ with $\operatorname{div}_y \mathbf{w} = 0$ in $\Omega \times Y$ and $\mathbf{w} = 0$ in $\Omega \times Y_S$. Hence we deduce $\lim_{t \to 0_+} \|\mathbf{u}^0(t) - \mathbf{a}^0\|_{L^2(\Omega \times Y)} = 0$ in a standard way.

As in the stationary case, we need to identify the limit function $\mathbf{S}^0$. The crucial information for this identification is

$$\limsup_{\varepsilon' \to 0} \int_0^T \int_\Omega \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} \le \int_0^T \int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \mathbf{u}^0, \tag{1.84}$$

which we prove now. We test the system (NGS_ε) by $\mathbf{u}^\varepsilon$ to obtain

$$\|\mathbf{u}^{\varepsilon'}(T)\|^2_{L^2(\Omega)} - \|\mathbf{a}^{\varepsilon'}\|^2_{L^2(\Omega)} + \int_0^T \int_\Omega \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \mathbf{f}\mathbf{u}^{\varepsilon'} \mathrm{d}x\mathrm{d}t$$

then pass to the limit as $\varepsilon' \to 0$, apply (1.5) and Lemma 1.3.7 on $\left\{\mathbf{u}^{\varepsilon'}(T)\right\}$. We arrive at

$$\|\mathbf{u}^0(T)\|^2_{L^2(\Omega \times Y)} - \|\mathbf{a}^0\|^2_{L^2(\Omega \times Y)} + \limsup_{\varepsilon' \to 0} \int_0^T \int_\Omega \mathbf{S}(\varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'}) : \varepsilon' \mathbf{D}\mathbf{u}^{\varepsilon'} \le \int_0^T \int_\Omega \int_Y \mathbf{f}\mathbf{u}^0. \tag{1.85}$$

Conversely, Lemma 1.2.8 allows to test the limit system (1.83) by $\mathbf{u}^0$. Thus we infer

$$\|\mathbf{u}^0(T)\|^2_{L^2(\Omega \times Y)} - \|\mathbf{a}^0\|^2_{L^2(\Omega \times Y)} + \int_0^T \int_\Omega \int_Y \mathbf{S}^0 : \mathbf{D}_y \mathbf{u}^0 = \int_0^T \int_\Omega \int_Y \mathbf{f} \cdot \mathbf{u}^0.$$

Comparing this with (1.85) yields (1.84). Then we proceed in the similar way as in the stationary case.

Using de Rham's theorem we get the existence of a distribution $\pi(t,x,y)$ such that for $\mathbf{w} \in \mathscr{D}\left(\Omega; C^\infty(Y)^d\right)$ with $\mathbf{w} = 0$ in $\Omega \times Y_S$, $z \in \mathscr{D}((0,T))$

$$\begin{aligned}
&\int_0^T \int_\Omega \int_Y \mathbf{u}_t^0 \cdot z\mathbf{w} + \int_0^T \int_\Omega \int_Y \mathbf{S}(\mathbf{D}_y \mathbf{u}^0) : z\mathbf{D}_y \mathbf{w} - \int_0^T \int_\Omega \int_Y pz \operatorname{div}_x \mathbf{w} \\
&- \int_0^T \int_\Omega \int_Y \pi z \operatorname{div}_y \mathbf{w} = \int_0^T \int_\Omega \int_Y \mathbf{f} \cdot z\mathbf{w},
\end{aligned}$$

which is equivalent to the weak formulation of the problem (HNS) due to Lemma 1.2.8. Again, the uniqueness of the solution of the homogenized problem implies $(\mathbf{u}^\varepsilon, P^\varepsilon) \to (\mathbf{u}^0, p)$ as $\varepsilon \to 0_+$. □

1.7 Appendix

Before we present the proof of Lemma 1.2.8, we state modified versions of [31, Chapter 7, Proposition 2.2], [3, Lemma 2.10] and some auxiliary lemmas.

Lemma 1.7.1 *Let D be a closed subset of Y with a positive measure having no intersection with ∂Y, $\partial D \in C^k$, $k > 1 + \frac{d}{2}$. Then there exists a unique solution $(\mathbf{w}^i, r^i) \in W^{1,2}_{per}(Y)^d \times L^2_{per}(Y)/\mathbb{R}$ of*

$$\begin{aligned} -\Delta \mathbf{w}^i + \nabla r^i &= \mathbf{e}_i \ \textit{in } Y \setminus D, \\ \operatorname{div} \mathbf{w}^i &= 0 \ \ \textit{in } Y, \\ \mathbf{w}^i &= 0 \ \ \textit{on } D. \end{aligned}$$

Moreover, $\mathbf{w}^i \in W^{1,\varphi}(Y)^d$ and a permeability matrix $K = (k_{ij})^d_{i,j=1}$ defined via $k_{ij} = \int_Y \nabla_y \mathbf{w}^i \nabla_y \mathbf{w}^j \mathrm{d}y$ is symmetric and positive definite.

Proof. One shows existence of the solution $(\mathbf{w}^i, r^i) \in W^{1,2}_{per}(Y)^d \times L^2_{per}(Y)$ in a standard way. Applying [17, Theorem IV.5.1] yields $\mathbf{w}^i \in W^{k,2}(Y \setminus D)^d$. Hence the embedding implies $\mathbf{w}^i \in W^{1,\infty}(Y \setminus D)^d$. We see $\nabla \mathbf{w}^i \in L^\infty(D)^{d\times d}$. Therefore we have $\nabla \mathbf{w}^i \in L^\infty(Y)^{d\times d}$, which implies $\mathbf{w}^i \in W^{1,\varphi}(Y)^d$.
The symmetry and positive definitness of K was showed in [31, Chapter 7, Proposition 2.2]. □

Lemma 1.7.2 *Let D, K and $\{\mathbf{w}^i\}$ be as in Lemma 1.7.1. We define an operator $P : L^\varphi(\Omega)^d \to \{\mathbf{u} \in W^{1,\varphi}_y(\Omega \times Y) : \mathbf{u} = 0 \textit{ in } \Omega \times D\}$ as*

$$P(\boldsymbol{\theta}) = \sum_{i=1}^d \left(\left(K^{-1}\boldsymbol{\theta}(x)\right) \cdot \mathbf{e}^i\right) \mathbf{w}^i.$$

Then P is linear and

$$\begin{aligned} \int_Y P(\boldsymbol{\theta})\, \mathrm{d}y &= \boldsymbol{\theta}, \\ \operatorname{div}_y P(\boldsymbol{\theta}) &= 0 \ \textit{in } \Omega \times Y, \\ P(\boldsymbol{\theta}) &= 0 \ \textit{in } \Omega \times D, \\ \|\nabla_y P(\boldsymbol{\theta})\|_{L^\varphi(\Omega\times Y)} &\le c\|\boldsymbol{\theta}\|_{L^\varphi(\Omega)}. \end{aligned} \tag{1.86}$$

Proof. The properties $(1.86)_1$-$(1.86)_3$ immediately follow from the definition of P. To show $(1.86)_4$, we use the regularity of $\mathbf{w}^i$. □

Lemma 1.7.3 *Let $\Delta(\varphi, \varphi^*) < \infty$ and D be as in Lemma 1.7.1. There is a linear continuous operator $Q : L^\varphi(\Omega \times Y) \to W^{1,\varphi}_y(\Omega \times Y)$ such that for any $g \in L^\varphi(\Omega \times Y)$, $\int_Y g(x,y) = 0$ for almost all $x \in \Omega$ $Q(g)$ satisfies*

$$\begin{aligned} \operatorname{div}_y Q(g) &= g \ \textit{in } \Omega \times (Y \setminus D), \\ Q(g) &= 0 \ \textit{in } \Omega \times D, \\ \|\nabla_y Q(g)\|_{L^\varphi(\Omega\times Y)} &\le c\|g\|_{L^\varphi(\Omega\times Y)}. \end{aligned} \tag{1.87}$$

Proof. To solve the latter problem, we first consider that for a.a. $x \in \Omega$ $g(x, \cdot) \in L^\varphi(Y)$ and the compatibility condition $\int_Y g(x,y)\mathrm{d}y = 0$ is satisfied. Then the existence of $Q(g)$ with

$$\int_Y \varphi\left(|\nabla_y Q(g)(x,y)|\right)\mathrm{d}y \le \int_Y \varphi\left(c|g(x,y)|\right)\mathrm{d}y$$

follows from [13, Theorem 6.6]. The latter inequality implies

$$\int_\Omega \int_Y \varphi\left(|\nabla_y Q(g)(x,y)|\right)\mathrm{d}y\mathrm{d}x \le \int_\Omega \int_Y \varphi\left(c|g(x,y)|\right)\mathrm{d}y\mathrm{d}x,$$

from which one derives $(1.87)_3$ similarly as (1.32e) was derived. □

Lemma 1.7.4 *Let $\Delta(\varphi,\varphi^*) < \infty$, Ω be a bounded set with a Lipschitz boundary and $\lambda > 1$. Then there is a linear continuous operator $S^\lambda : L^\varphi(\Omega)^d \to L^\varphi_C(\Omega)^d$, more precisely there is $c(\Omega) > 0$ such that for any $\mathbf{h} \in L^\varphi(\Omega)^d$ $\operatorname{dist}(\operatorname{supp}(S^\lambda\mathbf{h}), \partial\Omega) \ge c(\Omega)(\lambda - 1)$. Moreover, $\operatorname{div} S^\lambda(\mathbf{h}) = 0$ in Ω for $\mathbf{h}$ such that $\operatorname{div}\mathbf{h} = 0$ in Ω and as $\lambda \to 1_+$*

$$\|S^\lambda \mathbf{h} - \mathbf{h}\|_{L^\varphi(\Omega)} \to 0. \tag{1.88}$$

Proof. Let us assume for a moment that Ω is star-shaped with respect to a point x_S. We consider a mapping $J^\lambda : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ defined by $J^\lambda(x, x_S) = \lambda(x - x_S) + x_S$. Then we define $S^\lambda_{x_S}(\mathbf{h})(x) = \mathbf{h}\left(J^\lambda(x,x_S)\right)$ for a function $\mathbf{h} \in L^\varphi(\Omega)^d$ extended by zero outside of Ω. We immediately see that $S^\lambda_{x_S}$ is linear and bounded since for $\lambda > 1$ $\Omega \subset J^\lambda(\Omega, x_S)$ and $S^\lambda_{x_S}(\mathbf{h})$ has a compact support in Ω. We have

$$\int_\Omega \varphi(|S^\lambda_{x_S}(\mathbf{h})(x)|) = \int_{J^\lambda(\Omega,x_S)} \varphi(|S^\lambda(\mathbf{h})(x)|) = \lambda^{-d}\int_\Omega \varphi(|\mathbf{h}(x)|).$$

To prove (1.88), we first realize that there is a positive constant $c(\Omega)$ depending on the domain Ω such that for any $\mathbf{h} \in L^\varphi(\Omega)^d$ $\operatorname{supp}(S^\lambda_{x_S}\mathbf{h}) \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega \ge c(\Omega)(\lambda - 1))\}$. Then as $\lambda \to 1_+$

$$\int_\Omega \varphi(|S^\lambda_{x_S}\mathbf{h}(x) - \mathbf{h}(x)|) = \int_\Omega \varphi(|\mathbf{h}(\lambda(x - x_S) + x_S) - \mathbf{h}(x)|) \to 0$$

follows for $\mathbf{h} \in C(\overline{\Omega})^d$ because $\mathbf{h}$ is uniformly continuous on $\overline{\Omega}$. Using the density of $C(\overline{\Omega})^d$ in $L^\varphi(\Omega)^d$ we conclude (1.88). Since $\operatorname{div} S^\lambda_{x_S}\mathbf{h} = \lambda S^\lambda_{x_S}(\operatorname{div}\mathbf{h})$ in a sense of functionals, see Remark 1.2.5 (ii), we obtain that $\operatorname{div}\mathbf{h} = 0$ in Ω implies $\operatorname{div} S^\lambda_{x_S}\mathbf{h} = 0$ in Ω.

For Ω having a Lipschitz boundary we find sets $\{O_j\}_{j=1}^M$ such that $\bigcup_{j=1}^M O_j \supset \partial\Omega$ and $O'_j = \Omega \cap O_j$ are star-shaped with respect to points x_{S_j} and a set O'_{M+1} such that $O'_{M+1} \subset \overline{O_M + 1'} \subset \Omega$. Let $\{\psi_j\}_{j=1}^{M+1}$ be a partition of unity subordinated to $\{O'_j\}_{j=1}^{M+1}$. We put $c(\Omega) = \min_j\{c(O'_j)\}$, denote $\Omega^\lambda = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > c(\Omega)(\lambda - 1)\}$ and define S^λ as $S^\lambda(\mathbf{h})(x) = \sum_{j=1}^M \psi_j(x) S^\lambda_{x_{S_j}}(\mathbf{h})(x) + \psi_{M+1}(x)\mathbf{h}(x) - \mathbf{g}^\lambda(x)$, where $\mathbf{g}^\lambda$ is a solution of the problem

$$\begin{aligned}
\operatorname{div}\mathbf{g}^\lambda &= \sum_{j=1}^M \nabla\psi_j(x)\mathbf{H}_{\lambda,j}(x) - \frac{1}{|\Omega|}\int_\Omega \sum_{j=1}^M \nabla\psi_j(x)\mathbf{H}_{\lambda,j}(x) \text{ in } \Omega^\lambda,\\
\mathbf{g}^\lambda &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega^\lambda,\\
\|\mathbf{g}^\lambda\|_{W^{1,\varphi}(\Omega)} &\le c\sum_{j=1}^M \|\nabla\psi_j\|_{L^\infty(\Omega)}\|\mathbf{H}_{\lambda,j}\|_{L^\varphi(\Omega)},
\end{aligned}$$

where $\mathbf{H}_{\lambda,j}(x) = \mathbf{h}(J^\lambda(x, x_{S_j})) - \mathbf{h}(x)$. Such $\mathbf{g}^\lambda$ exists according to [13, Theorem 6.6]. Then $S^\lambda : L^\varphi(\Omega)^d \to L^\varphi_C(\Omega)^d$ is linear, bounded, $\operatorname{div}\left(S^\lambda \mathbf{h}\right) = 0$ in Ω for any $\mathbf{h}$ satisfying $\operatorname{div}\mathbf{h} = 0$ in Ω and (1.88) holds. □

Lemma 1.7.5 *Let $\Sigma \subset \mathbb{R}^n, n \geq 1$ be open, $\Delta_2(\varphi) < \infty$, $f^\delta \in L^\varphi(\Sigma), f \in \varphi(\Sigma)$ be such that as $\delta \to 0$*

$$f^\delta \to f \ \text{ in } L^\varphi(\Sigma). \tag{1.89}$$

We extend f^δ, f by zero in $\mathbb{R}^n \setminus \Sigma$ and define $(f^\delta)_\delta$ as a convolution of f^δ with a mollifier ω_δ having a support in the ball with the diameter proportional to δ. Then $(f^\delta)_\delta \in \mathcal{D}(\mathbb{R}^n)$ and as $\delta \to 0$

$$(f^\delta)_\delta \to f \ \text{ in } L^\varphi(\mathbb{R}^n). \tag{1.90}$$

Proof. We modify the approach from [20, Theorem 3.18.1.1]. We pick a function $g \in L^{\varphi^*}(\mathbb{R}^n)$ with $\|g\|_{L^{\varphi^*}(\mathbb{R}^n)} \leq 1$. Then using Fubini's theorem and properties of the mollifier, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} ((f^\delta)_\delta(x) - f(x)) g(x) \mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f^\delta(x-y) - f(x) \right) \omega_\delta(y) g(x) \mathrm{d}y \mathrm{d}x \right| \\
&\leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (f^\delta(x-y) - f(x)) g(x) \mathrm{d}x \right| \omega_\delta(y) \mathrm{d}y \\
&\leq \int_{\mathbb{R}^n} \|f^\delta(\cdot - \delta y) - f\|_{L^\varphi(\mathbb{R}^n)} \omega(y) \mathrm{d}y \leq \sup_{|z| \leq \delta} \|f^\delta(\cdot - \delta z) - f\|_{L^\varphi(\mathbb{R}^n)}.
\end{aligned} \tag{1.91}$$

Since

$$\|f^\delta(\cdot - \delta z) - f\|_{L^\varphi(\mathbb{R}^n)} \leq \|f^\delta(\cdot - z) - f(\cdot - z)\|_{L^\varphi(\mathbb{R}^n)} + \|f(\cdot - z) - f\|_{L^\varphi(\Sigma)},$$

where the first term on the right hand side tends to zero uniformly in z due to (1.89), whereas the second term tends to zero by the $L^\varphi-$ mean continuity property of f, see [20, 3.15.3]. Going back to (1.91), we obtain (1.90) using the dual definition of the norm on L^φ. □

Proof of Lemma 1.2.8. We want to use the mollification technique to obtain a smooth approximation of $\mathbf{u} \in W^{1,\varphi}_{y,0}(\Omega \times Y)$. But before it is necessary to modify $\mathbf{u}$ in such a way that ensures that the mollified function is zero in Y_S. Since ∂Y_S is Lipschitz, in fact better quality of ∂Y_S is assumed, we find sets $\{E_j\}_{j=1}^N$, $E_j \subset \overline{E_j} \subset Y$ with $\min_j \operatorname{dist}(E_j, \partial Y) \geq c > 0$ such that $\bigcup_{j=1}^N E_j \supset \partial Y_S$, $E'_j = E_j \cap Y_F$ are star-shaped with respect to points y_{S_j} and a set $E'_{N+1} \subset Y_F$ such that $\operatorname{dist}(E'_{N+1}, \partial Y_S) \geq c > 0$ and $\bigcup_{j=1}^{N+1} E'_j \supset Y_F$. Let $\{\phi_j\}$ be a partition of unity subordinated to $\{E'_j\}$ consisting of $Y-$periodic functions. We define for $x \in \Omega, y \in Y$ considering $\delta \in (0,1)$

$$\mathbf{v}^\delta(x,y) = \sum_{j=1}^N \phi_j(y) \mathbf{u}(x, J^{1-\delta}(y, y_{S_j})) + \phi_{N+1}(y) \mathbf{u}(x,y),$$

where the mapping J was introduced in the proof of Lemma 1.7.4. We extend $\mathbf{v}^\delta$ $Y-$periodically. We also obtain the existence of $c(Y_S)$ such that $\mathbf{v}^\delta = 0$ in $\Omega \times Y_F^\delta$, where $Y_F^\delta = \{y \in Y_F : \operatorname{dist}(\partial Y_S, y) \le c(Y_S)\delta\}$. Similar considerations as in Lemma 1.7.4 yield

$$\|\nabla_y \mathbf{v}^\delta\|_{L^\varphi(\Omega\times Y)} \le c(1-\delta)\|\nabla_y \mathbf{u}\|_{L^\varphi(\Omega\times Y)}. \tag{1.92}$$

We define a function $\mathbf{z}^\delta$ as a solution of the problem

$$\begin{aligned}
\operatorname{div}_y \mathbf{z}^\delta &= \sum_{j=1}^N \nabla_y \phi_j(y)\mathbf{U}_{\delta,j}(x,y) - \int_Y \sum_{j=1}^N \nabla_y\phi_j(y)\mathbf{U}_{\delta,j} \text{ in } \Omega\times(Y\setminus Y_F^\delta),\\
\mathbf{z}^\delta &= 0 \qquad\qquad \text{on } \Omega\times Y_F^\delta,\\
\|\nabla_y \mathbf{z}^\delta\|_{L^\varphi(\Omega\times Y)} &\le c\sum_{j=1}^N \|\nabla_y\phi_j\|_{L^\infty(Y)}\|\mathbf{U}_{\delta,j}\|_{L^\varphi(\Omega\times Y)},\\
\mathbf{z}^\delta &\text{ is } Y-\text{periodic},
\end{aligned} \tag{1.93}$$

where $\mathbf{U}_{\delta,j}(x,y) = \mathbf{u}(x, J^{1-\delta}(y, y_{S_j})) - \mathbf{u}(x,y)$. Such $\mathbf{z}^\delta$ exists according to Lemma 1.7.3 and we see that

$$\operatorname{div}_y(\mathbf{v}^\delta - \mathbf{z}^\delta) = \int_Y \sum_{j=1}^N \nabla_y\phi_j(y)\left(\mathbf{u}(x, J^{1-\delta}(y, y_{S_j})) - \mathbf{u}(x,y)\right) dy \text{ in } \Omega\times(Y\setminus Y_F^\delta),$$

which vanishes if $\operatorname{div}_y \mathbf{u} = 0$ in $\Omega\times Y$ after integration by parts. We define an approximation $\mathbf{u}^\delta$ of $\mathbf{u}$ using operator P from Lemma 1.7.2 and S from Lemma 1.7.4 as

$$\mathbf{u}^\delta = \left(\mathbf{v}^\delta - \mathbf{z}^\delta - P\left(\int_Y (\mathbf{v}^\delta - \mathbf{z}^\delta) - S^{1+\delta}\left(\int_Y \mathbf{u}\right)\right)\right)\chi_{\Omega_\delta},$$

where $\Omega_\delta = \{x\in\Omega : \operatorname{dist}(x,\Omega) > c(\Omega)\delta\}$, $c(\Omega)$ coming from Lemma 1.7.4. Then we see that $\operatorname{supp}\mathbf{u}^\delta \subset \Omega\times Y$ and we also have

$$\|\nabla_y \mathbf{u}^\delta\|_{L^\varphi(\Omega\times Y)} \le c\|\nabla_y\mathbf{u}\|_{L^\varphi(\Omega\times Y)}. \tag{1.94}$$

We obtain using $(1.86)_1$ and $(1.86)_4$

$$\begin{aligned}
&\|\nabla_y(\mathbf{u}-\mathbf{u}^\delta)\|_{L^\varphi(\Omega\times Y)}\\
&\le \|\nabla_y\mathbf{u}(1-\chi_{\Omega^\delta})\|_{L^\varphi(\Omega\times Y)} + \|\nabla_y(\mathbf{u}-\mathbf{v}^\delta)\chi_{\Omega^\delta}\|_{L^\varphi(\Omega\times Y)} + \|\nabla_y\mathbf{z}^\delta\|_{L^\varphi(\Omega\times Y)}\\
&\quad + c\left\|\int_Y \mathbf{z}^\delta\right\|_{L^\varphi(\Omega)} + c\left\|\int_Y(\mathbf{v}^\delta-\mathbf{u})\right\|_{L^\varphi(\Omega)} + \left\|\int_Y \mathbf{u} - S^{1+\delta}\left(\int_Y\mathbf{u}\right)\right\|_{L^\varphi(\Omega)}\\
&\le \|\nabla_y\mathbf{u}\chi_{\Omega\setminus\Omega^\delta})\|_{L^\varphi(\Omega\times Y)} + c\|\nabla_y(\mathbf{u}-\mathbf{v}^\delta)\|_{L^\varphi(\Omega\times Y)} + c\|\nabla_y\mathbf{z}^\delta\|_{L^\varphi(\Omega\times Y)}\\
&\quad + c\left\|\int_Y \mathbf{v}^\delta - S^{1+\delta}\left(\int_Y\mathbf{u}\right)\right\|_{L^\varphi(\Omega)}\\
&= I^\delta + II^\delta + III^\delta + IV^\delta.
\end{aligned}$$

We proceed in the same way as in the proof of (1.88) when showing that $II^\delta, IV^\delta \to 0$ as $\delta\to 0$ and similarly using (1.93) also $III^\delta \to 0$. Next, we observe that by the Lebesgue monotone convergence theorem

$$\int_{\Omega\times Y} \varphi(|\nabla_y\mathbf{u}\chi_{\Omega\setminus\Omega^\delta}|) \to 0$$

as $\delta \to 0$, which is equivalent to $I^\delta \to 0$ since $\Delta_2(\varphi) < \infty$. $\operatorname{div}_x \left(\int_Y \mathbf{u}\right) = 0$ in Ω implies $\operatorname{div}_x \left(\int_Y \mathbf{u}^\delta\right) = 0$ in Ω since $S^{1+\delta}\left(\int_Y \mathbf{u}\right)$ is zero in Ω_δ and $\operatorname{div}_y \mathbf{u} = 0$ in $\Omega \times Y$ implies $\operatorname{div}_y \mathbf{u}^\delta = 0$ in $\Omega \times Y$. Finally, we define $T^\delta \mathbf{u} = (\mathbf{u}^\delta)_{\frac{1}{3}\min(c(\Omega),c(Y_S))\delta}$ as a convolution of $\mathbf{u}^\delta$ with a standard mollifier, which has a support contained in a ball with the diameter $\frac{1}{3}\min(c(\Omega), c(Y_S))\delta$ in $\mathbb{R}^{2d}$. This ensures that $\operatorname{supp} T^\delta \mathbf{u} \subset \Omega \times Y$. T^δ is obviously linear and performing similar computations as in the proof of Lemma 1.7.5, we obtain continuity of T^δ thanks to (1.94). Lemma 1.7.5 implies that $\|\nabla_y (T^\delta \mathbf{u} - \mathbf{u})\|_{L^\varphi(\Omega\times Y)} \to 0$. The proof for the time-dependent case is analogous. □

Proof of Lemma 1.2.9. Using the dual norm in the space $L^\varphi(\Sigma)$, we have

$$\|p\|_{L^{\varphi^*}(\Sigma)/\mathbb{R}} = \inf_{k\in\mathbb{R}} \|p+k\|_{L^{\varphi^*}(\Sigma)} \le \left\| p - \int_\Sigma p \right\|_{L^{\varphi^*}(\Sigma)} = \sup_{\|g\|_{L^\varphi(\Sigma)}\le 1} \left| \int_\Sigma \left(p - \int_\Sigma p\right) g \right|.$$

For $g \in L^\varphi(\Sigma)$ consider $\mathbf{v} \in W_0^{1,\varphi}(\Sigma)$, which is solution of the problem

$$\begin{aligned} \operatorname{div} \mathbf{v} &= g - \int_\Sigma g \text{ in } \Sigma, \\ \|\mathbf{v}\|_{W_0^{1,\varphi}(\Sigma)} &\le c\|g\|_{L^\varphi(\Sigma)}. \end{aligned}$$

Such $\mathbf{v}$ exists according to [13, Theorem 6.6.] and $c = c(\Delta(\varphi, \varphi^*), \Sigma)$. Having this $\mathbf{v}$, we proceed further

$$\begin{aligned} \sup_{\|g\|_{L^\varphi(\Sigma)}\le 1} \left| \int_\Sigma \left(p - \int_\Sigma p\right)\left(g - \int_\Sigma g\right) \right| &\le c \sup_{\|\mathbf{v}\|_{W_0^{1,\varphi}(\Sigma)}\le 1} \left| \int_\Sigma \left(p - \int_\Sigma p\right) \operatorname{div} \mathbf{v} \right| \\ &= c\|\nabla p\|_{(W_0^{1,\varphi}(\Sigma))^*}. \end{aligned}$$

□

Before we present the proof of Lemma 1.2.10, we mention the real interpolation theorem of Peetre [27, Theorem 5.1], which helps us in several cases to obtain estimates in the setting of Orlicz spaces.

Theorem 1.7.6 *Let $\Delta_2(\varphi, \varphi^*) < \infty$ and q_1, q_2 be as in (1.9). Let $S : L^{q_j}(\Omega) \to L^{q_j}(\Omega), j = 1, 2$ be a linear and bounded operator. Then there exists $K > 0$ depending on $\Delta_2(\varphi, \varphi^*) < \infty$ and operator norms of S such that for any $f \in L^\varphi(\Omega)$*

$$\begin{aligned} \|Sf\|_{L^\varphi(\Omega)} &\le K\|f\|_{L^\varphi(\Omega)}, \\ \int_\Omega \varphi\left(\frac{|Sf|}{K}\right) &\le \int_\Omega \varphi(|f|). \end{aligned}$$

Proof of Lemma 1.2.10. To show the existence, uniqueness of $\mathbf{u}$ and estimate of $\nabla \mathbf{u}$, we apply the interpolation technique from [12, Theorem 18]. According to [17, Theorem IV.6.1 b)], there exists for any $q \in (1, \infty)$ a unique weak solution $\mathbf{u}$ of (1.18) in $W_0^{1,q}(\Sigma)^d$, for which the estimate $\|\nabla \mathbf{u}\|_{L^q(\Sigma)} \le c\|G\|_{L^q(\Sigma)}$ holds. In fact in the formulation of [17, Theorem IV.6.1 b)], the right hand side is considered to be an element g of $W^{-1,q}$. But clearly

$$\langle g, \mathbf{v}\rangle_{W_0^{-1,q}(\Sigma), W_0^{1,q}(\Sigma)} = \int_\Sigma G : \nabla \mathbf{v}$$

satisfies $g \in W^{-1,q}(\Sigma)$ and $\|g\|_{W^{-1,q}(\Sigma)} \leq c\|G\|_{L^q(\Sigma)}$. Let us consider a solution operator $S : L^q(\Sigma)^{d\times d} \to L^q(\Sigma)^{d\times d}$ to (1.18) defined by $\mathbf{S}(G) = \nabla \mathbf{u}$. Due to the uniqueness of $\mathbf{u}$, S does not depend on q. Hence S is uniquely defined from $\bigcup_{1<q<\infty} L^q(\Sigma)^{d\times d} \to \bigcup_{1<q<\infty} L^q(\Sigma)^d$. Then the estimate (1.19) is a consequence of the Therorem 1.7.6. The uniqueness of $\mathbf{u}$ in $W_0^{1,q_1}(\Sigma)^d$ and the embedding $L^\varphi(\Sigma) \hookrightarrow L^{q_1}(\Sigma)$ yields the uniqueness of $\mathbf{u}$ in $W_0^{1,\varphi}(\Sigma)^d$. □

Lemma 1.7.7 *Let $Y = (0,1)^d$. There is $c > 0$ such that for any $\mathbf{v} \in W^{1,1}(Y)^d, \mathbf{v} = 0$ on ∂D in the sense of traces, where $D \subset Y$ is a Lipschitz domain with positive $d-$dimensional measure, such that the following inequality holds*

$$\|\mathbf{v}\|_{L^1(Y)} \leq c\|\nabla \mathbf{v}\|_{L^1(Y)}. \tag{1.95}$$

Proof. In fact (1.95) is equivalent to

$$\|\mathbf{v}\|_{W^{1,1}(Y)} \leq c\|\nabla \mathbf{v}\|_{L^1(Y)}. \tag{1.96}$$

Let us assume for a contradiction that (1.96) does not hold. Then we can find for each $n \in \mathbb{N}$ a function $\mathbf{v}^n$ such that

$$\|\mathbf{v}^n\|_{W^{1,1}(Y)} \geq n\|\nabla \mathbf{v}^n\|_{L^1(Y)}.$$

Without loss of generality we can take $\|\mathbf{v}^n\|_{W^{1,1}(Y)} = 1$ for each $n \in \mathbb{N}$. Then as $n \to \infty$ $\nabla \mathbf{v}^n \to 0$ in $L^1(Y)^d$. Since the sequence $\{\mathbf{v}^n\}$ is bounded in $W^{1,1}(Y)^d$, the embedding to $L^{\frac{d}{d-1}}(Y)^d$ and the reflexivity of $L^{\frac{d}{d-1}}(Y)^d$ yield the existence of a weakly convergent subsequence $\{\mathbf{v}^n\}$ in $L^{\frac{d}{d-1}}(Y)^d$ with a limit $\mathbf{v}$. We preserve the notation for simplicity because we can assume that $\{\mathbf{v}^n\}$ itself has this property. Obviously $\mathbf{v}^n \rightharpoonup \mathbf{v}$ in $L^1(Y)^d$ and $\nabla \mathbf{v}^n \rightharpoonup 0$ in $L^1(Y)^{d\times d}$. Hence $\mathbf{v}^n \rightharpoonup \mathbf{v}$ in $W^{1,1}(Y)^d$. The compact imbedding of $W^{1,1}(Y)^d$ in $L^1(Y)^d$ yields $\|\mathbf{v}\|_{L^1(Y)} = 1$. Since $\nabla \mathbf{v} = 0$ almost everywhere in Y, $\mathbf{v}$ has to be equal to a constant in Y and this constant is zero because $\mathbf{v} = 0$ on ∂D and we have the contradiction. □

Proof of Lemma 1.2.6. First, we show the modular inequality

$$\int_{\Omega^\varepsilon} \varphi(|\mathbf{v}|) \leq \int_{\Omega^\varepsilon} \varphi(c\varepsilon|\nabla \mathbf{v}|), \tag{1.97}$$

from which the norm version follows. Let us extend $\mathbf{v}$ by zero outside of Ω^ε. Let us consider a cell Y_i^ε with $i \in I^\varepsilon \cup H^\varepsilon$. The first task is to show the inequality

$$\int_{Y_i^\varepsilon \cap \Omega^\varepsilon} \varphi(|\mathbf{v}|) \leq \int_{Y_i^\varepsilon \cap \Omega^\varepsilon} \varphi(c\varepsilon|\nabla \mathbf{v}|). \tag{1.98}$$

We present it only for the case $Y_i^\varepsilon, i \in I^\varepsilon$. The other case is treated similarly. Using the inequality (1.11) and Poincaré's inequality [13, Theorem 6.5.], we obtain

$$\begin{aligned}\int_{Y_i^\varepsilon} \varphi(|\mathbf{v}|) &\leq c\left(\int_{Y_i^\varepsilon} \varphi(|\mathbf{v} - \langle \mathbf{v}\rangle_{Y_i^\varepsilon}|) + \int_{Y_i^\varepsilon} \varphi(|\langle \mathbf{v}\rangle_{Y_i^\varepsilon}|)\right)\\ &\leq \int_{Y_i^\varepsilon} \varphi(c\varepsilon|\nabla \mathbf{v}|) + c\int_{Y_i^\varepsilon} \varphi(|\langle \mathbf{v}\rangle_{Y_i^\varepsilon}|),\end{aligned}$$

where $c = c(\Delta(\varphi, \varphi^*))$. To conclude (1.98), we need to show

$$\int_{Y_i^\varepsilon} \varphi(|\langle \mathbf{v} \rangle_{Y_i^\varepsilon}|) \le \int_{Y_i^\varepsilon} \varphi(c\varepsilon |\nabla \mathbf{v}|).$$

By rescaling to $\varepsilon-$length and taking $D^\varepsilon = Y_{S_i}^\varepsilon$ for $i \in I^\varepsilon$ or $D^\varepsilon = Y^\varepsilon \cap (\mathbb{R}^d \setminus \Omega^\varepsilon)$ for $i \in H^\varepsilon$ in the inequality (1.96) we obtain

$$\int_{Y_i^\varepsilon} |\mathbf{v}| \le c\varepsilon \int_{Y_i^\varepsilon} |\nabla \mathbf{v}|.$$

We use the latter inequality and Jensen's inequality to get

$$\int_{Y_i^\varepsilon} \varphi(|\langle \mathbf{v} \rangle_{Y_i^\varepsilon}|) \le \int_{Y_i^\varepsilon} \varphi\left(\frac{1}{|Y_i^\varepsilon|}\int_{Y_i^\varepsilon} |\mathbf{v}|\right) \le \int_{Y_i^\varepsilon} \varphi\left(\frac{c\varepsilon}{|Y_i^\varepsilon|}\int_{Y_i^\varepsilon} |\nabla\mathbf{v}|\right) \le \int_{Y_i^\varepsilon} \varphi\left(c\varepsilon|\nabla \mathbf{v}|\right).$$

Thus we have shown (1.98). To conclude (1.97), we just sum over all $i \in I^\varepsilon \cup H^\varepsilon$ in (1.98). Now, we are prepared to show the norm estimate

$$\|\mathbf{v}\|_{L^\varphi(\Omega^\varepsilon)} \le c\varepsilon \|\nabla \mathbf{v}\|_{L^\varphi(\Omega^\varepsilon)}. \tag{1.99}$$

With the help of (1.97) one obtains

$$\{\lambda > 0 : \int_{\Omega^\varepsilon} \varphi\left(\frac{c\varepsilon|\nabla \mathbf{v}|}{\lambda}\right) \le 1\} \subset \{\lambda > 0 : \int_{\Omega^\varepsilon} \varphi\left(\frac{|\mathbf{v}|}{\lambda}\right) \le 1\},$$

from which (1.99) follows using properties of infimum and the definition of $L^\varphi-$norm. To show (1.16) and (1.17) it remains to apply a variant of Korn's inequality [13, Theorem 6.10] in (1.99) and (1.97) because $\mathbf{v} = 0$ on $\partial\Omega^\varepsilon$. □

The rest of the appendix is devoted to the proof Lemma 1.2.11 but several auxiliary statements precedes the proof itself.

Lemma 1.7.8 *Let X be a Banach space and M be a subspace of X and U, V be subspaces of X^*. Then*

$$(M^\perp)_\perp = \overline{M}, \tag{1.100}$$

$$(U_\perp)^\perp = \overline{U} \text{ if } X \text{ is reflexive}, \tag{1.101}$$

$$(U + V)_\perp = U_\perp \cap V_\perp. \tag{1.102}$$

Poof of Lemma 1.7.8. See [30, Theorem 4.7] for the proof of (1.100).
According to [30, Theorem 4.7 (a)], $(U_\perp)^\perp$ is the closure of U in the weak* topology of X^*. The reflexivity of X implies that $(U_\perp)^\perp$ is the closure of U in the weak topology of X^*. Since $(U_\perp)^\perp$ is a subspace of X^*, it is also a convex subset of X^*. Hence its closures in the norm and weak topology coincide, see [30, Theorem 3.12].
To show (1.102), we suppose $x \in (U + V)_\perp$ then we have $u(x) = 0$ $\forall u \in U$ because $U \subset U + V$ hence $x \in U_\perp$. We obtain in the same way that $x \in V_\perp$ thus $x \in U_\perp \cap V_\perp$. On the other hand, if we suppose $x \in U_\perp \cap V_\perp$ then $\forall u \in U, \forall v \in V$ $u(x) = v(x) = 0$. Hence $\forall u \in U, \forall v \in V$ $(u + v)(x) = 0$ thus $x \in (U + V)_\perp$ and (1.102) holds. □

The following theorem generalizes [8, Theorem 9.5], which was formulated for a Hilbert space setting.

Theorem 1.7.9 *Let X be a reflexive Banach space. Let M and N be closed subspaces of X such that $M+N$ is closed. Then*

$$(M \cap N)^{\perp} = M^{\perp} + N^{\perp}. \tag{1.103}$$

Proof of Theorem 1.7.9. At first, we show that

$$(M \cap N)^{\perp} = \overline{M^{\perp} + N^{\perp}}. \tag{1.104}$$

The closedness of M, N together with (1.100) and (1.102) yields

$$M \cap N = (M^{\perp})_{\perp} \cap (N^{\perp})_{\perp} = (M^{\perp} + N^{\perp})_{\perp}.$$

Using (1.101), we conclude (1.104). Let us show (1.103). Obviously, $(M \cap N)^{\perp} \supset M^{\perp} + N^{\perp}$. Thus it suffices to show

$$(M \cap N)^{\perp} \subset M^{\perp} + N^{\perp}. \tag{1.105}$$

Fix $g \in (M \cap N)^{\perp}$. We define $Z = M + N$, which is a closed subspace of X. Therefore Z is a Banach space. Whenever $z \in Z$ has two representations $z = m_1 + n_1 = m_2 + n_2$, where $m_1, m_2 \in M$ and $n_1, n_2 \in N$, then $m_1 - m_2 = n_2 - n_1 \in M \cap N$. Hence we have

$$g(m_1 - m_2) = 0, \text{ i.e., } g(m_1) = g(m_2). \tag{1.106}$$

Then it follows that the functional f, defined on Z by

$$f(m + n) = g(m) \ \forall m \in M, n \in N \tag{1.107}$$

is linear, well-defined due to (1.106) and satisfies

$$f(n) = 0 \ \forall n \in N. \tag{1.108}$$

Next, we show that f is bounded on Z. Let us observe that the mapping $A : M \times N \to M + N$ defined by $A(u, v) = u + v$ is clearly linear and surjective. Moreover, for each $(m, n) \in M \times N \setminus \{(0, 0)\}$ we get

$$\frac{\|A(m,n)\|_Z}{\|(m,n)\|_{M \times N}} = \frac{\|m+n\|_X}{\|m\|_X + \|n\|_X} \leq 1$$

and A is bounded. Now, since $M + N$ is closed thus complete in X, the open mapping theorem implies the existence of positive δ such that ($B(W)$ stands for the unit ball in W) $A(B(M \times N)) \supset \delta B(M + N)$. Especially for any $m \in M, n \in N$ with $\|m+n\|_Z \leq 1$ there exist $(\tilde{m}, \tilde{n}) \in B(M \times N)$ such that $\tilde{m} + \tilde{n} = A(\tilde{m}, \tilde{n}) = \delta(m + n)$, which implies according to (1.107)

$$f(m+n) = \frac{f(\tilde{m} + \tilde{n})}{\delta} = \frac{g(\tilde{m})}{\delta} \leq \frac{\|g\|_{X^*} \|\tilde{m}\|_X}{\delta} \leq \frac{\|g\|_{X^*}}{\delta}.$$

Hence we obtain boundedness of f on Z. Hahn-Banach theorem yields the existence of the extension $\bar{f} \in X^*$ of f such that $\bar{f} = f$ on Z. (1.108) immediately implies $\bar{f} \in N^{\perp}$. We decompose $g = g - \bar{f} + \bar{f}$ and it remains to show that $g - \bar{f} \in M^{\perp}$. Since $M \hookrightarrow Z$ and $\bar{f} = f$ on Z, we obtain for each $m \in M$ that $(g - \bar{f})(m) = f(m) - f(m) = 0$, which implies $g \in M^{\perp} + N^{\perp}$, i.e., we proved (1.105). □

For the proof of Lemma 1.2.11 we need also the characterization of annihilators of the sets:

$$X_1 = \{\mathbf{v} \in W^{1,\varphi}_{y,0}(\Omega \times Y) : \operatorname{div}_y \mathbf{v} = 0 \text{ in } \Omega \times Y\},$$

$$X_2 = \{\mathbf{v} \in W^{1,\varphi}_{y,0}(\Omega \times Y) : \operatorname{div}_x \left(\int_Y \mathbf{v}\right) = 0 \text{ in } \Omega, \left(\int_Y \mathbf{v}\right) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

$$X_1^T = \{\mathbf{v} \in W^{1,\varphi}_{y,0}(Q_T \times Y) : \operatorname{div}_y \mathbf{v} = 0 \text{ in } Q_T \times Y\},$$

$$X_2^T = \{\mathbf{v} \in W^{1,\varphi}_{y,0}(Q_T \times Y) : \operatorname{div}_x \left(\int_Y \mathbf{v}\right) = 0 \text{ in } Q_T, \left(\int_Y \mathbf{v}\right) \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \partial\Omega\}.$$

To obtain this characterization, we proceed in two steps. First, we show that annihilators of above spaces are in fact annihilators of certain Sobolev-Bochner spaces. Thus it is necessary to characterize these annihilators. Let us denote

$$M_1 = \{\mathbf{v} \in \mathcal{D}(\Omega; C^\infty(Y))^d : \mathbf{v} = 0 \text{ in } \Omega \times Y_S, \operatorname{div}_y \mathbf{v} = 0 \text{ in } \Omega \times Y\},$$

$$M_2 = \{\mathbf{v} \in \mathcal{D}(\Omega; C^\infty(Y))^d : \mathbf{v} = 0 \text{ in } \Omega \times Y_S, \operatorname{div}_x \left(\int_Y \mathbf{v} \mathrm{d}y\right) = 0 \text{ in } \Omega\},$$

$$M_1^T = \{\mathbf{v} \in C^\infty \left([0,T]; \mathcal{D}(\Omega; C^\infty(Y))^d\right) : \mathbf{v} = 0 \text{ in } Q_T \times Y_S, \operatorname{div}_y \mathbf{v} = 0 \text{ in } Q_T \times Y\},$$

$$M_2^T = \{\mathbf{v} \in C^\infty \left(\mathcal{D}(\Omega; C^\infty(Y))^d\right) : \mathbf{v} = 0 \text{ in } Q_T \times Y_S, \operatorname{div}_x \int_Y \mathbf{v} = 0 \text{ in } Q_T\},$$

$$N_i = \overline{M_i}^{\|\cdot\|_{L^s(\Omega;W^{1,s}(Y))^d}}, \; N_i^T = \overline{M_i^T}^{\|\cdot\|_{L^s(Q_T;W^{1,s}(Y))}} \; i = 1, 2.$$

Since $\Delta_2(\varphi) < \infty$, we have $X_i = \overline{M_i}^{\|\nabla_y \cdot\|_{L^\varphi(\Omega\times Y)}}, \; X_i^T = \overline{M_i^T}^{\|\nabla_y \cdot\|_{L^\varphi(Q_T\times Y)}}$.

Lemma 1.7.10 *Let Ω, Y, Y_S, Y_F fulfill the Assumption 1.1.5, $\Delta_2(\varphi, \varphi^*) < \infty$. We denote $\tilde{X} = W^{1,\varphi}_{y,0}(\Omega \times Y), \tilde{X^T} = W^{1,\varphi}_{y,0}(Q_T \times Y)$. If $L^\varphi \hookrightarrow L^s$ then*

$$X_i^\perp = \overline{N_i^\perp}^{\|\cdot\|_{\tilde{X}^*}},$$

$$(X_i^T)^\perp = \overline{(N_i^T)^\perp}^{\|\cdot\|_{\tilde{X^T}^*}} \; i = 1, 2.$$

Proof. To avoid misunderstaning, we recall that $X_i^\perp \subset \tilde{X}^*$, $N_i^\perp \subset \tilde{N}^*$, where $\tilde{N} = \left(L^s(\Omega; W^{1,s}(Y)^d\right)$.

We take $F \in \overline{N_i}^{\|\cdot\|_{\tilde{X}^*}}$. Then there is a sequence $\{F^n\} \subset N_i^\perp$ such that $F^n \to F$ in $\tilde{X}^*$ as $n \to \infty$. For arbitrarily chosen $u \in X_i$, we obtain $\langle F, \mathbf{u}\rangle = \lim_{n\to\infty} \langle F^n, \mathbf{u}\rangle = 0$, i.e., $F \in X_i^\perp$.

To show the opposite inclusion, we observe that $N_i^\perp$ is a subspace of $\tilde{N}^* \subset \tilde{X}^*$ thus it is a convex subset of $\tilde{X}^*$ and its closures in the strong and weak topology on $\tilde{X}^*$ coincide. Therefore it suffices to show that for $F \in X_i^\perp$ there is a sequence $\{F^n\} \subset N_i^\perp$ such that $F^n(\mathbf{u}) \to F(\mathbf{u})$ for any $\mathbf{u} \in \tilde{X}$ since $\tilde{X}$ is a reflexive Banach space and we can identify $\tilde{X}^{**}$ with $\tilde{X}$. Let us describe the construction of $\{F^n\}$. We define $F^n(\mathbf{u}) = F(T^{\frac{1}{n}}(\mathbf{u}))$ for $\mathbf{u} \in \tilde{N}, \mathbf{u} = 0$ in $\Omega \times Y_S$, where the linear continuous operator $T^{\frac{1}{n}} : \left\{\mathbf{u} \in \tilde{N}, \mathbf{u} = 0 \text{ in } \Omega \times Y_S\right\} \to \left\{\mathbf{u} \in \mathcal{D}(\Omega; C^\infty(Y))^d : \mathbf{u} = 0 \text{ in } \Omega \times Y_S\right\}$ comes from Lemma 1.2.8. Then $F^n \in \tilde{X}^*$ and moreover, $F^n \in X_i^\perp$. Finally, we pick $\mathbf{u} \in \tilde{X}$ and obtain as $n \to \infty$

$$|F^n(\mathbf{u}) - F(\mathbf{u})| = |F(T^n\mathbf{u} - \mathbf{u})| \le \|F\|_{\tilde{X}^*} \|T^n\mathbf{u} - \mathbf{u}\|_{\tilde{X}} \to 0.$$

The second assertion is shown in a similar way. □

As mentioned, we need to characterize annihilators of N_i, N_i^T $i = 1, 2$ to proceed with the characterization of annihilators of X_i, X_i^T $i = 1, 2$. Before doing so, we state two auxiliary lemmas, which we use in the proof of variant of [17, Theorem III.5.3.].

Lemma 1.7.11 *Let X, Z be reflexive Banach spaces, $A : X \to Z$ be a continuous linear operator with a domain $D(A) = X$ and $R(A)$ be closed in Z. Then range of A^* is characterized as*

$$R(A^*) = (kerA)^{\perp}.$$

Proof. Since [30, Theorem 4.12] asserts that $\ker A = R(A^*)_{\perp}$, it suffices to show that $R(A^*)$ is closed in X^* if we take into account (1.101). But this closedness is equivalent to the assumption $R(A)$ is closed in Z, see [30, Theorem 4.14]. □

Lemma 1.7.12 *Let $s \in (1, \infty)$ and Ω, Y, Y_S, Y_F fulfill Assumption 1.1.5. We denote $X = \{ \mathbf{v} \in L^s\left(\Omega; W^{1,s}(Y)^d\right) : \mathbf{v} = 0 \text{ in } \Omega \times Y_S \}$. Let $F \in N_1^{\perp}$. Then there is a unique $q \in L^{s'}(\Omega; L^{s'}(Y)), \int_Y q = 0$ such that for all $\mathbf{v} \in X$*

$$\langle F, \mathbf{v} \rangle = \int_\Omega \int_Y q \operatorname{div}_y \mathbf{v}.$$

Proof. We denote $Z = \left\{ r \in L^s(\Omega; L^s(Y)), \int_Y r = 0 \right\}$. Spaces X, Z are reflexive Banach spaces. We define an operator $A : X \to Z$ as $A(\mathbf{v}) = \operatorname{div}_y \mathbf{v}$. According to Lemma 1.7.3 considered for $L^s\left(\Omega; W^{1,s}(Y)^d\right)$ setting, $R(A) = Z$, which is obviously closed. Lemma 1.7.11 yields $(\ker A)^{\perp} = R(A^*)$. Hence we infer that for $F \in (\ker A)^{\perp}$ there is $q \in Z^* = \left\{ q \in L^{s'}(\Omega; L^{s'}(Y)), \int_Y q = 0 \right\}$ such that $F = A^*(q)$. Then we obtain

$$\langle F, \mathbf{v} \rangle_{X^*, X} = \int_\Omega \int_Y A^*(q) \mathbf{v} = \int_\Omega \int_Y q A(\mathbf{v}) = \int_\Omega \int_Y q \operatorname{div}_y \mathbf{v}.$$

□

Lemma 1.7.13 *Let Ω, Y, Y_S, Y_F fulfill Assumption 1.1.5. Then*

$$N_1^{\perp} = \left\{ \nabla_y q; q \in L^{s'}(\Omega; L^{s'}(Y)), \int_Y q = 0 \right\}, \tag{1.109}$$

$$N_2^{\perp} = \{ \nabla_x \tilde{q}; \tilde{q} \in W^{1,s'}(\Omega) \}, \tag{1.110}$$

$$(N_1^T)^{\perp} = \left\{ \nabla_y q; q \in L^{s'}(Q_T; L^{s'}(Y)), \int_Y q = 0 \right\}, \tag{1.111}$$

$$(N_2^T)^{\perp} = \{ \nabla_x \tilde{q}; \tilde{q} \in L^{s'}(0, T; W^{1,s'}(\Omega)) \}. \tag{1.112}$$

Proof. Let us pick $q \in L^{s'}(\Omega; L^{s'}(Y)), \int_Y q = 0$. Then we have for any $\mathbf{v} \in N_1$ $\langle \nabla_y q, \mathbf{v} \rangle = 0$. To show the opposite inclusion we pick $F \in N_1^{\perp}$. According to the Lemma 1.7.12, the functional F can be uniquely represented as

$$\langle F, \mathbf{v} \rangle = \int_\Omega \int_Y \tilde{q} \operatorname{div}_y \mathbf{v}$$

for some $\tilde{q} \in L^{s'}(\Omega; L^{s'}(Y)), \int_Y \tilde{q} = 0$, which means in fact $F = \nabla_y q$ for $q = -\tilde{q}$. Clearly,

$$N_2^{\perp} \supset \{ \nabla_x \tilde{q}, \tilde{q} \in W^{1,s'}(\Omega) \}$$

Moreover, from [16, Lemma 7], we have for

$$M = \{\mathbf{v} \in L^s(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

that $M^\perp = \{\nabla_x \tilde{q}, \tilde{q} \in W^{1,s'}(\Omega)\}$. Since $M \subset N_2$ if we identify M with functions from N_2, which are constant with respect to $y-$variable, it follows that $M^\perp \supset N_2^\perp$.
The equalities (1.111) and (1.112) follow from the obvious observation $L^s(0,T;X)^\perp = L^{s'}(0,T;X^\perp)$, (1.109) and (1.110). □

Lemma 1.7.14 *Let Ω, Y, Y_S, Y_F fulfill the Assumption 1.1.5, $\Delta_2(\varphi, \varphi^*) < \infty$. Then*

$$\begin{aligned}
X_1^\perp &= \{\nabla_y q : q \in L^{\varphi^*}(\Omega \times Y), \int_Y q = 0\},\\
X_2^\perp &= \{\nabla_x r : r \in W^{1,\varphi^*}(\Omega)\},\\
(X_1^T)^\perp &= \{\nabla_y q : q \in L^{\varphi^*}(Q_T \times Y), \int_Y q = 0\},\\
(X_2^T)^\perp &= \{\nabla_x r : r \in W_x^{1,\varphi^*}(Q_T)\}.
\end{aligned}$$

Proof. To apply Lemma 1.7.10, it suffices to consider $s \in (1,\infty)$ such that $L^\varphi \hookrightarrow L^s$ and realize that

$$\begin{aligned}
\overline{N_1^\perp}^{\|\cdot\|_{(W^{1,\varphi}_{y,0}(\Omega\times Y))^*}} &= \left\{\nabla_y q : q \in L^{\varphi^*}(\Omega \times Y), \int_Y q = 0\right\},\\
\overline{N_2^\perp}^{\|\cdot\|_{(W^{1,\varphi}_{y,0}(\Omega\times Y))^*}} &= \{\nabla_x r : r \in W^{1,\varphi^*}(\Omega)\},\\
\overline{(N_1^T)^\perp}^{\|\cdot\|_{(W^{1,\varphi}_{y,0}(Q_T\times Y))^*}} &= \left\{\nabla_y q : q \in L^{\varphi^*}(Q_T \times Y), \int_Y q = 0\right\},\\
\overline{(N_2^T)^\perp}^{\|\cdot\|_{(W^{1,\varphi}_{y,0}(Q_T\times Y))^*}} &= \{\nabla_x r : r \in W_x^{1,\varphi^*}(Q_T)\}.
\end{aligned}$$

□

Before we prove Lemma 1.2.11, we state one more auxiliary lemma.

Lemma 1.7.15 *Let A be a symmetric and positive definite matrix. Let $\Sigma \in C^1$ and $\mathbf{f} \in L^\varphi(\Sigma)^d$ and $\Delta_2(\varphi, \varphi^*) < \infty$. Then there exists a unique weak solution $v \in W^{1,\varphi}(\Sigma)/\mathbb{R}$ to the problem*

$$\begin{aligned}
-\operatorname{div}(A\nabla v - \mathbf{f}) &= 0 \text{ in } \Sigma,\\
(A\nabla v - \mathbf{f}) \cdot \mathbf{n} &= 0 \text{ on } \partial\Sigma.
\end{aligned} \tag{1.113}$$

Moreover, there exists $c > 0$ such that

$$\int_\Sigma \varphi(|\nabla v|) \le \int_\Sigma \varphi(c|\mathbf{f}|). \tag{1.114}$$

Proof. Let us pick $r \in (1,\infty)$, consider $\mathbf{f} \in L^r(\Sigma)^d$. We use the transformation described in [18, Lemma 6.1]. There is an orthogonal matrix P such that the matrix P^TAP has on the diagonal eigenvalues of the matrix A, which we denote $\lambda_1, \dots, \lambda_d$. We denote $D = (\lambda_i^{-\frac{1}{2}}\delta_{ij})$ and set $Q = PD$. Under the transformation $\tilde{x} = Qx$ the equation (1.113) is changed to

$$\begin{aligned}
-\operatorname{div}(\nabla\tilde{v} - \tilde{\mathbf{f}}) &= 0 \text{ in } \tilde{\Sigma},\\
(\nabla\tilde{v} - \tilde{\mathbf{f}}) \cdot \mathbf{n} &= 0 \text{ on } \partial\tilde{\Sigma}.
\end{aligned}$$

for the functions $\tilde{v}(\tilde{x}) = v(Q^{-1}\tilde{x})$ and $\tilde{f}(\tilde{x}) = f(Q^{-1}\tilde{x})$. The existence of a unique $\tilde{v} \in W^{1,r}(\Sigma)/\mathbb{R}$ satisfying the system above in the weak sense together with

$$\|\nabla\tilde{v}\|_{L^r(\tilde{\Sigma})} \le c\|\tilde{\mathbf{f}}\|_{L^r(\tilde{\Sigma})} \tag{1.115}$$

is ensured by [32, Theorem 5.4. and Theorem 1.3.], where $c = c(d, r, \tilde{\Omega})$. Now, we come back to $x-$variable via the transformation $x = Q^{-1}\tilde{x}$. We have, denoting $\lambda_{max}, \lambda_{min}$ the maximal and minimal eigenvalues of A, for any $z \in \mathbb{R}^d$

$$\lambda_{min}^{\frac{1}{2}}|z| \le |Q^{-1}z| \le \lambda_{max}^{\frac{1}{2}}|z| \tag{1.116}$$

since P^{-1} is orthogonal. Change of variables $\tilde{x} \to x$ in (1.115) together with (1.116) yield $\|\nabla v\|_{L^r(\Sigma)} \le c\|\mathbf{f}\|_{L^r(\Sigma)}$ with $c = c(d, r, A, \Omega)$ thus we have also $v \in W^{1,r}(\Sigma)/\mathbb{R}$. Finally, we apply the interpolation technique in the same way as in the proof Lemma 1.2.10 to finish the proof. □

Proof of Lemma 1.2.11. Using the previous notation, we write $X^{1,\varphi}_{y,0}(\Omega\times Y) = X_1 \cap X_2$. According to Theorem 1.7.9, it suffices to show that $X_1 + X_2$ is closed to conclude (1.20). We prove that $X_1 + X_2 = W^{1,\varphi}_{y,0}(\Omega \times Y)$, which is clearly closed.
Let us fix $\mathbf{v} \in W^{1,\varphi}_{y,0}(\Omega\times Y)$ and take the matrix K defined in Lemma 1.7.1 with $D = Y_S$. Then according to Lemma 1.7.15 there is a unique (up to an additive constant) weak solution $p \in W^{1,\varphi}(\Omega)$ of

$$\begin{aligned} -\operatorname{div}_x(K\nabla_x p(x) - \int_Y \mathbf{v}(x,y)\mathrm{d}y) &= 0 \text{ in } \Omega,\\ (K\nabla_x p(x) - \int_Y \mathbf{v}(x,y)\mathrm{d}y)\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.117}$$

Having the function p, we decompose

$$\mathbf{v}(x,y) = \sum_{i=1}^{d} \mathbf{w}^i(y)\partial_{x_i}p(x) + \left(\mathbf{v}(x,y) - \sum_{i=1}^{d}\mathbf{w}^i(y)\partial_{x_i}p(x)\right) =: \mathbf{v}_1 + \mathbf{v}_2$$

and employing the inequality (1.11), and the regularity of $\mathbf{w}^i$, we obtain $\mathbf{v}_1 \in X_1$, $\mathbf{v}_2 \in X_2$, which finishes the proof of (1.20).
To show (1.21), we prove in an analogous way that $X_1^T + X_2^T$ is closed ($X^{1,\varphi}_{y,0}(Q_T\times Y) = X_1^T \cap X_2^T$). We fix $\mathbf{v} \in W^{1,\varphi}_{y,0}(Q_T \times Y)$ and consider for almost all $t \in (0,T)$ the equation (1.117) with $p(x) = \tilde{p}(t,x)$. Thus thanks to (1.114) $\tilde{p} \in W^{1,\varphi}_x(Q_T)$. Then the decomposition analogous to the previous case is

$$\mathbf{v}(t,x,y) = \sum_{i=1}^{d}\mathbf{w}^i(y)\partial_{x_i}\tilde{p}(t,x) + \left(\mathbf{v}(t,x,y) - \sum_{i=1}^{d}\mathbf{w}^i(y)\partial_{x_i}\tilde{p}(t,x)\right) =: \mathbf{v}_1 + \mathbf{v}_2$$

with $\mathbf{v}_1 \in X_1^T$ and $\mathbf{v}_2 \in X_2^T$. □

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Homogenization of an incompressible stationary flow of an electrorheological fluid

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Abstract

We combine two scale convergence, theory of monotone operators and results on approximation of Sobolev functions by Lipschitz functions to prove a homogenization process for an incompressible flow of a generalized Newtonian fluid. We avoid the necessity of testing the weak formulation of the initial and homogenized systems by corresponding weak solutions, which allows optimal assumptions on lower bound for a growth of the elliptic term. We show that the stress tensor for homogenized problem depends on the symmetric part of the velocity gradient involving the limit of a sequence selected from a family of solutions of initial problems.

Keywords

Non-Newtonian incompressible fluids, two-scale convergence, homogenization, Lipschitz approximation

2.1 Introduction

Electrorheological fluids are special liquids characterized by their ability to change significantly the mechanical properties when an electric field is applied. This behavior has been extensively investigated for the development of smart fluids, which are currently exploited in technological applications, e.g. brakes, clutches or shock absorbers. Results of the ongoing research indicate their possible applications also in electronics. One approach for modeling of the flow of electrorheological fluids is the utilization of a system of partial differential equations derived by Rajagopal and Růžička, for details see [14]. This system in the case of an isothermal, homogeneous (with density equal to one), incompressible electrorheological fluid reads

$$\partial_t \mathbf{u} - \operatorname{div} \mathbf{S} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0, \tag{2.1}$$

in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3, \ldots$. The symbol $\mathbf{u}$ denotes the velocity, $\mathbf{S}$ the extra stress tensor, $\operatorname{div}(\mathbf{u} \otimes \mathbf{u})$ is the convective term with $\mathbf{u} \otimes \mathbf{u}$ denoting the tensor product of the vector $\mathbf{u}$ with itself defined as $(u_i u_j)_{i,j=1,\ldots,d}$, π is the pressure and $\mathbf{f}$ the external body force. The stress tensor $\mathbf{S}$ is assumed to depend on the symmetric part $\mathbf{Du}$ of the velocity gradient $\nabla \mathbf{u}$. The presence of an electric field is captured by the supposed dependence of $\mathbf{S}$ on the spatial variable in such a way that the growth of $\mathbf{S}$ corresponds to $|\mathbf{Du}|^{p(\cdot)-1}$ for some variable exponent p.

For this setting assuming additionally a periodic variable exponent with a small period ε, it was shown by Zhikov in [18] that as $\varepsilon \to 0$ a subsequence of solutions of initial problems converges to a solution of the homogenized problem having the extra stress tensor independent of the spatial variable. Zhikov's approach is based on the fact that the regularity of solutions of the initial as well as homogenized problem allows to use these solutions as a test function. In fact, this sufficient regularity is ensured by the value of the lower bound for the variable exponent $p \geq p_0 := \max((d + \sqrt{3d^2 + 4d})/(d + 2), 3d/(d+2))$.

In the seminal article [11] a method of Lipschitz approximation of Sobolev functions is developed that allows to decrease the lower bound for p. In the article [11] the method is applied to the problem of existence of a weak solution to the stationary generalized Navier-Stokes model. The proof of this result is simplified in [7]. Moreover, it is extended in this article to a flow of an electrorheological fluid, i.e., to the stationary variant of the model (2.1) with a variable exponent p. It took lot of work till the approach was modified in such a way that it is applicable to evolutionary problems. See

[8], where the existence of a weak solution to the evolutionary generalized Navier Stokes problem is studied. The method is used to an evolutionary problem in Orlicz setting in the article [5], compare also with [4], where the approximation is constructed to be divergence free. The optimal lower bound for the exponent p assuring the existence of weak solutions is given by the requirement that $p \geq p_1$ such that $W^{1,p_1}(\Omega)$ compactly embeds to $L^2(\Omega)$ that allows to treat well the convective term. This lower bound is $p_1 > 2d/(d+2)$, see [8] for the problem (2.1) with constant p and [7] for the stationary variant of (2.1) with variable p. It is natural to ask: "Can one proceed with the homogenization process also if the lower bound for p is between p_0 and $2d/(d+2)$?" This paper should be regarded as the first step on the way for the answer to this question. To concentrate on the interplay between method of Lipschitz approximations and two scale convergence we start with the stationary problem first.

We consider the model (2.1) derived in [14] to describe the electrorheological fluids, i.e., fluids that change their properties due to application of an electric field. For more details on modelling of the fluids and physical motivation see [14]. The general form of the stress tensor $\mathbf{S}$ is

$$
\begin{aligned}
\mathbf{S} =& \alpha_1 \mathbf{E}\otimes\mathbf{E} + \alpha_2 \mathbf{Du} + \alpha_3(\mathbf{Du})^2 + \alpha_4((\mathbf{Du})\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes(\mathbf{Du})\mathbf{E}) + \alpha_5((\mathbf{Du})^2\mathbf{E}\otimes\mathbf{E} \\
&+ \mathbf{E}\otimes(\mathbf{Du})^2\mathbf{E})
\end{aligned}
$$

where $\mathbf{E}$ is the applied electric field and α_i , $i = 1, ..., 5$ are functions of

$$
|\mathbf{E}|^2, \mathrm{tr}(\mathbf{Du}), \mathrm{tr}(\mathbf{Du})^2, \mathrm{tr}(\mathbf{Du})^3, \mathrm{tr}((\mathbf{Du})\mathbf{E}\otimes\mathbf{E}), \mathrm{tr}((\mathbf{Du})^2\mathbf{E}\otimes\mathbf{E}).
$$

We assume that the electric field is given and oscillates rapidly. Moreover we set $\alpha_i = 0$ for $i \in \{1, 3, 4, 5\}$. We start with a model problem given by

$$
\alpha_2(\mathbf{E}, \mathbf{Du}) = \alpha(\delta + |\mathbf{Du}|^2)^{\frac{p-2}{2}} + \beta(\delta + |\mathbf{Du}|^2)^{\frac{\gamma-2}{2}},
$$

with constants $p > 1$, $\delta \geq 0$. The functions α, β, γ depend on the given electric field $\mathbf{E}$. This model agrees with [14, Lemma 4.46]. Since we assume that the electric field is periodic and we are interested in what happens when the period of $\mathbf{E}$ tends to zero, we further suppress this dependence and will assume that $\alpha, \beta, \gamma : \mathbb{R}^d \to \mathbb{R}$ are periodic with respect to $Y = (0,1)^d$ and continuous. Moreover we assume that there exist $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ such that for all $y \in \mathbb{R}^d$, $0 < \alpha_0 \leq \alpha(y) \leq \alpha_1$, $0 \leq \beta(y) \leq \beta_0$, $1 \leq \gamma_0 \leq \gamma(y) \leq \gamma_1 < p$. Altogether we consider for $\varepsilon \in (0,1)$ scale of models

$$
\mathbf{S}^\varepsilon(x, \boldsymbol{\xi}) = \big(\alpha(\frac{x}{\varepsilon})(\delta + |\boldsymbol{\xi}|^2)^{\frac{p-2}{2}} + \beta(\frac{x}{\varepsilon})(\delta + |\boldsymbol{\xi}|^2)^{\frac{\gamma(\frac{x}{\varepsilon})-2}{2}}\big)\boldsymbol{\xi}, \quad x \in \mathbb{R}^d, \, \boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{sym}, \tag{2.2}
$$

and study behavior of solutions to

$$
\begin{aligned}
-\operatorname{div}(\mathbf{S}^\varepsilon(x, \mathbf{Du}^\varepsilon) - \mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon) + \nabla\pi^\varepsilon = -\operatorname{div}\mathbf{F}, \quad \operatorname{div}\mathbf{u}^\varepsilon = 0 \quad \text{in } \Omega, \\
\mathbf{u}^\varepsilon = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega \pi^\varepsilon = 0.
\end{aligned} \tag{2.3}
$$

as ε tends to 0. It appears that the solutions converge as $\varepsilon \to 0_+$ up to a subsequence to a solution of a homogenized problem.

Although the mathematically precise formulation of our main result requires some preparatory work, we believe that the main result can be presented already here with some simplifications and imprecisions. The main theorem in full detail is formulated in Section 2.3, see Theorem 2.3.3.

Theorem 2.1.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $p > 2d/(d+2)$, $\mathbf{S}^\varepsilon$ be given by (2.2) and $\mathbf{F} \in L^{p'}(\Omega; \mathbb{R}^{d\times d}_{sym})$. Let $\{(\mathbf{u}^\varepsilon, \pi^\varepsilon)\}_{\varepsilon\in(0,1)}$ be a family of weak solutions of the system (2.3) satisfying energy inequality. Then there exists a sequence $\{\varepsilon_k\}_{k=1}^{+\infty}$, $s > 0$ such that as $k \to +\infty$*

$$\varepsilon_k \to 0, \quad \mathbf{u}^{\varepsilon_k} \rightharpoonup \mathbf{u} \text{ in } W^{1,p}_0(\Omega; \mathbb{R}^d), \quad \pi^{\varepsilon_k} \rightharpoonup \pi \text{ in } L^s(\Omega),$$

and $(\mathbf{u}, \pi)$ is a weak solution of a homogenized system. This limit system is of the generalized Navier Stokes type.

The homogenized stress tensor is independent of the space variable and is presented in subsection 2.2.3.

We want to emphasize that in our setting we allow that $p < 3d/(d+2)$ and so the term $\int_\Omega \mathbf{u}\otimes\mathbf{u} : \mathbf{D}\mathbf{u}dx$ is not defined. In this situation we are not allowed to test weak formulations of the problems (2.3) by their weak solutions. We are not aware of any other result on homogenization that would treat this situation.

The main obstacle to verification of the homogenization process is the fact that we allow the optimal lower bound for growth of $\mathbf{S}^\varepsilon$. The situation is similar to the limit passage in the stress tensor in the proof of the existence of weak solutions of generalized Navier-Stokes equations. However, one cannot straightforwardly adopt the methods, which are successfully applied for existence proofs, because of oscillations which occur in the spatial variable of the stress tensor.

The method is nevertheless based on the two scale convergence combined with the theory of monotone operators and the approximation of Sobolev functions by Lipschitz functions (so called Lipschitz truncation method). This leads us to the assumption that the part of the tensor $\mathbf{S}^\varepsilon$ that contains the variable exponent is of a lower order, i.e., $\gamma_1 < p$. This assumption allows us to use the Lipschitz truncation method in the space $W^{1,p}$ where it is well known. Without this assumption, for example if the term with power $p-2$ is not present in (2.2), we would need to know that the method works uniformly in spaces $W^{1,\gamma_\varepsilon(\cdot)}$, $\gamma_\varepsilon(\cdot) = \gamma(\cdot/\varepsilon)$. This requires that the maximal operator is continuous on $L^{\gamma_\varepsilon(\cdot)}$ uniformly with respect to $\varepsilon \in (0,1)$. For a particular ε the continuity of Maximal operator is known if γ_ε is log-Hölder continuous, see [6]. However since the family γ_ε is not uniformly log-Hölder continuous for $\varepsilon \in (0,1)$, we cannot use this result.

Let us outline the structure of the paper. In Section 2.2 we introduce function spaces appearing in the paper, collect several fundamental lemmas and show some facts about two-scale convergence. In the last subsection we define the homogenized tensor $\hat{\mathbf{S}}$ and show its needed properties. Section 2.3 is devoted to the formulation of the general main result and its proof.

2.2 Preliminaries

We recall that a domain $\Omega \subset \mathbb{R}^d$, $Y = (0,1)^d$. The following function spaces appear further: $C^\infty_{0,\mathrm{div}}(\Omega) = \{\mathbf{u} \in C^\infty_0(\Omega; \mathbb{R}^d) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\}$, $C^\infty_{per}(Y) = \{u \in C^\infty(\mathbb{R}^d) : u\ Y\text{-periodic}\}$, $C^\infty_{per,div}(Y) = \{\mathbf{u} \in C^\infty_{per}(\mathbb{R}^d) : \operatorname{div}\mathbf{u} = 0 \text{ in } Y\}$, $W^{1,p}_{per}(Y, \mathbb{R}^d)$ is a closure of $\{\mathbf{u} \in C^\infty_{per}(Y), \int_Y \mathbf{u} = 0\}$ in the classical Sobolev norm, $\mathcal{D}\left(\Omega; C^\infty_{per}(Y)\right)$ is the space of smooth functions $u : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that $u(x,\cdot) \in C^\infty_{per}(Y)$ for any $x \in \Omega$ and there is $K \Subset \Omega$ such that for any $x \in \Omega \setminus K$: $u(x,\cdot) = 0$ in $\mathbb{R}^d$.

We introduce a closed subspace of $L^p(Y;\mathbb{R}^{d\times d}_{sym})$ and its annihilator in $L^{p'}(Y;\mathbb{R}^{d\times d}_{sym})$ by

$$G(Y) = \left\{ \mathbf{D}\mathbf{w} : \mathbf{w} \in W^{1,p}_{per}(Y;\mathbb{R}^d), \operatorname{div}\mathbf{w} = 0 \text{ in } Y \right\},$$
$$G^{\perp}(Y) = \left\{ \mathbf{V}^* \in L^{p'}(Y;\mathbb{R}^{d\times d}_{sym}) : \forall \mathbf{V} \in G(Y) \ \int_Y \mathbf{V}^*(y)\cdot\mathbf{V}(y)\mathrm{d}y = 0 \right\}.$$

Note that $C^{\infty}_{per,div}(Y)$ is dense in $G(Y)$. If we consider the set $\mathbb{R}^{d\times d}_{sym}$ as a subset of constant functions of $L^p(Y;\mathbb{R}^{d\times d}_{sym})$ then $\mathbb{R}^{d\times d}_{sym} \cap G(Y) = \emptyset$.

For the sake of clarity, we recall the meaning of differential operators appearing in the paper. Let us consider $\mathbf{u} : \Omega \times Y \to \mathbb{R}^d$ then

$$\nabla_x \mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)^d_{i,j=1}, \ \operatorname{div}_x \mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i}, \ \nabla_y \mathbf{u} = \left(\frac{\partial u_i}{\partial y_j}\right)^d_{i,j=1}, \ \operatorname{div}_y \mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial y_i}.$$

We omit the subscript if the function depends on the variable from one domain only. Throughout the paper the identity matrix is denoted by $\mathbf{I}$, the zero matrix by $\mathbf{O}$. The generic constants are denoted by c. When circumstances require it, we may also include quantities, on which the constant depend, e.g. $c(d)$ for the dependence on the dimension d. If we want to distinguish between different constants in one formula, we utilize subscripts, e.g. c_1, c_2 etc.

Let M, N be open subsets of $\mathbb{R}^d$. $M \Subset N$ means that $M \subset \overline{M} \subset N$, $\overline{M}$ being compact.

2.2.1 Auxiliary tools

Lemma 2.2.1 (Biting lemma, [3]) *Let $E \subset \mathbb{R}^d$ be a bounded domain and $\{v^n\}$ be a sequence of functions bounded in $L^1(E)$. Then there exists a subsequence $\{v^{n_k}\} \subset \{v^n\}$, a function $v \in L^1(E)$ and a sequence of measurable sets $\{E_j\}, E \supseteq E_1 \supseteq E_2 \supseteq \cdots$ with $|E_j| \to 0$ as $j \to \infty$ such that for each j: $v^{n_k} \rightharpoonup v$ in $L^1(E \setminus E_j)$ as $k \to \infty$.*

Lemma 2.2.2 (Dunford, [9, Section III.2 Theorem 15]) *Let $\Sigma \in \mathbb{R}^d$ be a measurable set. A subset M of $L^1(\Sigma)$ is relatively weakly compact if and only if it is bounded and uniformly integrable, i.e., for any $\theta > 0$ there is $\delta > 0$ such that for any $f \in M$ and a measurable $K \subset \Sigma$ with $|K| < \delta$ we have $\int_K |f| < \theta$.*

Lemma 2.2.3 [10, Theorem 10.11] *Let $\Sigma \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $q \in (1,\infty)$ and denote $L^q_0(\Sigma) = \{h \in L^q : \int_\Sigma h = 0\}$. There exists a continuous linear operator $\mathcal{B} : L^q_0(\Sigma) \to W^{1,q}_0(\Sigma;\mathbb{R}^d)$ such that $\operatorname{div}\mathcal{B}h = h$ for any $h \in L^q_0(\Sigma)$.*

Lemma 2.2.4 [10, Theorem 10.21] *Let $\Sigma \subset \mathbb{R}^d$ be an open set, $p, q, r > 1$. Assume*

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \text{ in } L^p(\Sigma;\mathbb{R}^d), \mathbf{v}^n \rightharpoonup \mathbf{v} \text{ in } L^q(\Sigma;\mathbb{R}^d) \text{ as } n \to \infty \text{ and } \tfrac{1}{p} + \tfrac{1}{q} < \tfrac{1}{r} \le 1.$$

In addition, let for a certain $s > 1$

$$\{\operatorname{div}\mathbf{u}^n\} \text{ be precompact in } \left(W^{1,s}_0(\Sigma;\mathbb{R}^d)\right)^*,$$
$$\{\operatorname{curl}\mathbf{v}^n\} = \{\nabla(\mathbf{v}^n) - (\nabla(\mathbf{v}^n))^T\} \text{ be precompact in } \left(W^{1,s}_0(\Sigma;\mathbb{R}^d)^d\right)^*.$$

Then

$$\mathbf{u}^n \cdot \mathbf{v}^n \rightharpoonup \mathbf{u}\cdot\mathbf{v} \text{ in } L^r(\Sigma).$$

The history of this lemma goes back to the works [12] and [16].

For $f \in L^1(\mathbb{R}^d)$, we define the Hardy-Littlewood maximal function as

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \mathrm{d}y,$$

where $B_r(x)$ stands for a ball having a center at x and radius r.

Lemma 2.2.5 *Let $\Omega \subset \mathbb{R}^d$ be open and bounded with a Lipschitz boundary and $\alpha \geq 1$. Then there is $c > 0$ such that for any $\mathbf{v} \in W_0^{1,\alpha}(\Omega; \mathbb{R}^d)$ and every $\lambda > 0$ there is $\mathbf{v}^\lambda \in W_0^{1,\infty}(\Omega; \mathbb{R}^d)$ satisfying*

$$\begin{aligned} \|\mathbf{v}^\lambda\|_{W^{1,\infty}(\Omega)} &\leq \lambda, \\ |\{x \in \Omega : \mathbf{v}(x) \neq \mathbf{v}^\lambda(x)\}| &\leq c \frac{\|\mathbf{v}\|^\alpha_{W^{1,\alpha}(\Omega)}}{\lambda^\alpha}. \end{aligned} \tag{2.4}$$

Proof. The similar assertion, formulated for functions that do not vanish on $\partial\Omega$, appeared in [1]. For our purposes we refer to [7, Theorem 2.3], which for any $\mathbf{v} \in W_0^{1,\alpha}(\Omega; \mathbb{R}^d)$ and any numbers $\theta, \lambda > 0$ ensures the existence of $\mathbf{v}_{\theta,\sigma} \in W_0^{1,\infty}(\Omega; \mathbb{R}^d)$ such that

$$\|\mathbf{v}_{\theta,\sigma}\|_{L^\infty(\Omega)} \leq \theta, \ \|\nabla \mathbf{v}_{\theta,\sigma}\|_{L^\infty(\Omega)} \leq c(d,\Omega)\sigma$$

and up to a set of Lebesgue measure zero

$$|\{\mathbf{v}_{\theta,\sigma} \neq \mathbf{v}\}| \subset \Omega \cap (\{M(\mathbf{v}) > \theta\} \cup \{M(\nabla \mathbf{v}) > \sigma\}).$$

We pick $\mathbf{v} \in W_0^{1,\alpha}(\Omega; \mathbb{R}^d)$ and $\lambda > 0$. We apply [7, Theorem 2.3] with $\lambda, \frac{\lambda}{c(d,\Omega)}$ and denote $\mathbf{v}^\lambda = \mathbf{v}_{\lambda, \frac{\lambda}{c(d,\Omega)}}$ to conclude $(2.4)_1$. Moreover, since we have for any $f \in L^\alpha(\mathbb{R}^d)$ and $\sigma > 0$

$$|\{|f| > \sigma\}| \leq \int_{\mathbb{R}^d} \left(\frac{|f|}{\sigma}\right)^\alpha = \frac{\|f\|^\alpha_{L^\alpha(\mathbb{R}^d)}}{\sigma^\alpha},$$

we obtain for $\alpha > 1$ using the strong type estimate for the maximal function, see [15, Theorem 1]

$$|\{\mathbf{v}_{\theta,\sigma} \neq \mathbf{v}\}| \leq \frac{\|M(\mathbf{v})\|^\alpha_{L^\alpha(\mathbb{R}^d)}}{\lambda^\alpha} + c\frac{\|M(\nabla\mathbf{v})\|^\alpha_{L^\alpha(\mathbb{R}^d)}}{\lambda^\alpha} \leq c\frac{\|\mathbf{v}\|^\alpha_{W^{1,\alpha}(\Omega)}}{\lambda}.$$

For $\alpha = 1$ the estimate $(2.4)_2$ is a direct consequence of the weak type estimate of the maximal function, see again [15, Theorem 1]. □

2.2.2 Two-scale convergence

The following concept of convergence was introduced by Nguetseng in his seminal paper [13]: a sequence $\{u^\varepsilon\}$ bounded in $L^2(\Omega)$ is said weakly two-scale convergent to $u^0 \in L^2(\Omega \times Y)$ if for any smooth function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, which is $Y-$periodic in the second argument,

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\Omega \times Y} u^0(x,y)\psi(x,y) \mathrm{d}x\mathrm{d}y. \tag{2.5}$$

Properties of this notion of convergence were investigated and applied to a number of problems, see [2], and the concept was also extended to $L^p, p \geq 1$. It was shown later that there is an alternative approach, so called periodic unfolding, for the introduction of the weak two-scale convergence, which allows to represent the two-scale convergence by means of the standard weak convergence in a Lebesgue space on the product $\Omega \times Y$. In the same manner the strong two-scale convergence is introduced. Since it is known that both presented notions of the weak two-scale convergence are equivalent, see [17], all properties known for the weak two-scale convergence introduced via (2.5) hold also for the second approach. We introduce the weak two-scale convergence via periodic unfolding.

Definition 2.2.6 We define functions $n : \mathbb{R} \to \mathbb{Z}$, $N : \mathbb{R}^d \to \mathbb{Z}^d$ as

$$n(x) = \max\{n \in \mathbb{Z} : n \leq x\},$$
$$N(x) = (n(x_1), \dots, n(x_d)),$$

Then we have for any $x \in \mathbb{R}^d, \varepsilon > 0$, a two-scale decomposition $x = \varepsilon\left(N\left(\frac{x}{\varepsilon}\right) + R\left(\frac{x}{\varepsilon}\right)\right)$. We also define for any $\varepsilon > 0$ a two-scale composition function $T_\varepsilon : \mathbb{R}^d \times Y \to \mathbb{R}^d$ as $T_\varepsilon(x, y) = \varepsilon\left(N\left(\frac{x}{\varepsilon}\right) + y\right)$.

Remark 2.2.7 It follows that $T_\varepsilon(x, y) \to x$ uniformly in $\mathbb{R}^d \times Y$ as $\varepsilon \to 0$ since $T_\varepsilon(x, y) = x + \varepsilon\left(y - R\left(\frac{x}{\varepsilon}\right)\right)$.

Definition 2.2.8 We say that a sequence of functions $\{v^\varepsilon\} \subset L^r(\mathbb{R}^d)$

1. converges to v^0 weakly two-scale in $L^r(\mathbb{R}^d \times Y)$, $v^\varepsilon \xrightharpoonup{2-s} v^0$, if $v^\varepsilon \circ T_\varepsilon$ converges to v^0 weakly in $L^r(\mathbb{R}^d \times Y)$,

2. converges to v^0 strongly two-scale in $L^r(\mathbb{R}^d \times Y)$, $v^\varepsilon \xrightarrow{2-s} v^0$, if $v^\varepsilon \circ T_\varepsilon$ converges to v^0 strongly in $L^r(\mathbb{R}^d \times Y)$.

Remark 2.2.9 We define two-scale convergence in $L^r(\Omega \times Y)$ as two-scale convergence in $L^r(\mathbb{R}^d \times Y)$ for functions extended by zero to $\mathbb{R}^d \setminus \Omega$.

Lemma 2.2.10 *Let $g \in L^1(\mathbb{R}^d; C_{per}(Y))$. Then, for any $\varepsilon > 0$, the function $(x, y) \mapsto g(T_\varepsilon(x, y), y)$ is integrable and*

$$\int_{\mathbb{R}^d} g\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\mathbb{R}^d} \int_Y g(T_\varepsilon(x, y), y) \mathrm{d}y \mathrm{d}x.$$

Proof. See [17, Lemma 1.1] □

Lemma 2.2.11

i) Let $v \in L^r(\Omega; C_{per}(Y)), r \in [1, \infty)$, v be $Y-$periodic, define $v^\varepsilon(x) = v(\frac{x}{\varepsilon}, x)$ for $x \in \Omega$. Then $v^\varepsilon \xrightarrow{2-s} v$ in $L^r(\Omega \times Y)$ as $\varepsilon \to 0$.

ii) Let $v^\varepsilon \xrightharpoonup{2-s} v^0$ in $L^r(\Omega \times Y)$ then $v^\varepsilon \rightharpoonup \int_Y v^0(\cdot, y)\mathrm{d}y$ in $L^r(\Omega)$.

iii) Let $\{v^\varepsilon\}$ be a bounded sequence in $L^r(\Omega), r \in (1, \infty)$. Then there is $v_0 \in L^r(\Omega \times Y)$ and a sequence $\varepsilon_k \to 0$ as $k \to +\infty$ such that $v^{\varepsilon_k} \xrightharpoonup{2-s} v_0$ in $L^r(\Omega \times Y)$ as $k \to +\infty$.

iv) Let $\{v^\varepsilon\}$ *converge weakly to* v *in* $W^{1,r}(\Omega), r \in (1,\infty)$ *as* $\varepsilon \to 0$. *Then there is* $v_0 \in L^r(\Omega; W^{1,r}_{per}(Y))$ *and a sequence* $\varepsilon_k \to 0$ *as* $k \to +\infty$ *such that* v^{ε_k} *converges strongly to* v *in* $L^r(\Omega)$ *and* ∇v^{ε_k} *converges weakly two-scale to* $\nabla_x v + \nabla_y v_0$ *in* $L^r(\Omega \times Y)^d$ *as* $k \to +\infty$.

v) Let $v^\varepsilon \xrightharpoonup{2-s} v^0$ *in* $L^r(\Omega \times Y)$ *and* $w^\varepsilon \xrightarrow{2-s} w^0$ *in* $L^{r'}(\Omega \times Y)$ *then* $\int_\Omega v^\varepsilon w^\varepsilon \to \int_\Omega \int_Y v^0 w^0$.

Proof. The equalities

$$v^\varepsilon \circ T_\varepsilon(x,y) = v\left(T_\varepsilon(x,y), \frac{T_\varepsilon(x,y)}{\varepsilon}\right) = v(T_\varepsilon(x,y),y)$$

hold by definition of T_ε and $Y-$periodicity of v. If $v \in C(\overline{\Omega \times Y})$, Remark 2.2.7 immediately implies

$$\int_{\Omega\times Y} |v(T_\varepsilon(x,y),y) - v(x,y)|^r \,\mathrm{d}x\mathrm{d}y \to 0 \text{ as } \varepsilon \to 0.$$

For general $v \in L^r(\Omega; C_{per}(Y))$ we need to approximate v by a continuous function and then proceed as in the proof of mean continuity of Lebesgue integrable functions.

We obtain (ii) if functions independent of y-variable are considered in the definition (2.5) of the weak convergence in $L^r(\Omega \times Y)$.

The assertion (iii) is a direct consequence of Lemma 2.2.10, the weak compactness of bounded sets in $L^r(\Omega \times Y)$ and Definition 2.2.8$_1$.

For the proof of (iv) with $r = 2$ see [2, Proposition 1.14. (i)], the proof for general $r \neq 2$ is analogous.

Statement (v) follows immediately from definition of the weak and strong two-scale convergence and Lemma 2.2.11 applied to function $g = v^\varepsilon w^\varepsilon$ independent of y, see [17, Proposition 1.4]. □

2.2.3 Definition and properties of the homogenized stress tensor

Inspired by [18] we introduce a tensor $\hat{\mathbf{S}} : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ as

$$\hat{\mathbf{S}}(\boldsymbol{\xi}) = \int_Y \mathbf{S}(y, \boldsymbol{\xi} + \mathbf{V}(y))\mathrm{d}y, \tag{2.6}$$

where the function $\mathbf{V}$ is a solution of the cell problem: Let $\boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{sym}$ be fixed. We seek $\mathbf{V} \in G(Y)$ such that for any $\mathbf{W} \in G(Y)$

$$\int_Y \mathbf{S}(y, \boldsymbol{\xi} + \mathbf{V}(y)) : \mathbf{W}(y)\mathrm{d}y = 0. \tag{2.7}$$

Since $G(Y)$ is reflexive and the tensor $\mathbf{S}$ is strictly monotone, the existence and uniqueness of $\mathbf{V}$ follows using the theory of monotone operators. In the next section we show that the tensor $\hat{\mathbf{S}}$ arises when the homogenization process $\varepsilon \to 0_+$ is performed in (2.14). Properties of $\hat{\mathbf{S}}$ are listed in the following lemma.

Lemma 2.2.12 *There are constants* $\hat{c}_1, \hat{c}_2 > 0$ *such that for any* $\boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{sym}$

$$\begin{aligned} \hat{\mathbf{S}}(\boldsymbol{\xi}) : \boldsymbol{\xi} &\geq c_1|\boldsymbol{\xi}|^p - \hat{c}_1, \\ |\hat{\mathbf{S}}(\boldsymbol{\xi})|^{p'} &\leq \hat{c}_2(|\boldsymbol{\xi}|^p + 1). \end{aligned} \tag{2.8}$$

Moreover, $\hat{\mathbf{S}}$ *is strictly monotone and continuous on* $\mathbb{R}^{d\times d}_{sym}$.

Proof. Let $\mathbf{V}$ be a weak solution of the cell problem corresponding to $\boldsymbol{\xi}$. Then using (S4), Jensen's inequality and the fact that $\mathbf{V}$ is a symmetric gradient of $Y-$periodic function we obtain

$$\begin{aligned}\hat{\mathbf{S}}(\boldsymbol{\xi}):\boldsymbol{\xi} &= \int_Y \mathbf{S}(y,\boldsymbol{\xi}+\mathbf{V}(y))\mathrm{d}y:\boldsymbol{\xi} = \int_Y \mathbf{S}(y,\boldsymbol{\xi}+\mathbf{V}(y)):(\boldsymbol{\xi}+\mathbf{V}(y))\mathrm{d}y\\ &\geq \int_Y c_1|\boldsymbol{\xi}+\mathbf{V}(y)|^p\mathrm{d}y-\tilde{c}_1 \geq c_1\left|\boldsymbol{\xi}+\int_Y \mathbf{V}(y)\mathrm{d}y\right|^p-\tilde{c}_1 = c_1|\boldsymbol{\xi}|^p-\tilde{c}_1,\end{aligned}$$

which is $(2.8)_1$. Using (S4), the Hölder and Young inequalities yields

$$\begin{aligned}|\hat{\mathbf{S}}(\boldsymbol{\xi})|^{p'} &= \left|\int_Y \mathbf{S}(y,\boldsymbol{\xi}+\mathbf{V}(y))\mathrm{d}y\right|^{p'} \leq \int_Y |\mathbf{S}(y,\boldsymbol{\xi}+\mathbf{V}(y))|^{p'}\mathrm{d}y \leq \int_Y c_2\big(|\boldsymbol{\xi}+\mathbf{V}(y)|^p\mathrm{d}y+1\big)\\ &\leq \int_Y \frac{c_2}{c_1}\mathbf{S}(y,\boldsymbol{\xi}+\mathbf{V}(y))\cdot(\boldsymbol{\xi}+\mathbf{V}(y))\mathrm{d}y+c_2(1+\frac{\tilde{c}_1}{c_1})\\ &= \frac{c_2}{c_1}\hat{\mathbf{S}}(\boldsymbol{\xi})\cdot\boldsymbol{\xi}+c_2(1+\frac{\tilde{c}_1}{c_1}) \leq \frac{1}{2}|\hat{\mathbf{S}}(\boldsymbol{\xi})|^{p'}+\hat{c}_2(|\boldsymbol{\xi}|^p+1),\end{aligned}$$

for suitable (large) $\hat{c}_2>0$. Hence we obtain $(2.8)_2$. Let $\boldsymbol{\xi}^1,\boldsymbol{\xi}^2\in\mathbb{R}^{d\times d}_{sym}$,$\boldsymbol{\xi}^1\neq\boldsymbol{\xi}^2$ and $\mathbf{V}^1,\mathbf{V}^2$ be corresponding weak solutions of the cell problem. Then we infer

$$(\hat{\mathbf{S}}(\boldsymbol{\xi}^1)-\hat{\mathbf{S}}(\boldsymbol{\xi}^2)):(\boldsymbol{\xi}^1-\boldsymbol{\xi}^2) = \int_Y(\mathbf{S}(y,\boldsymbol{\xi}^1+\mathbf{V}^1(y))-\mathbf{S}(y,\boldsymbol{\xi}^2+\mathbf{V}^2)):(\boldsymbol{\xi}^1+\mathbf{V}^1-\boldsymbol{\xi}^2-\mathbf{V}^2)\mathrm{d}y.$$

The strict monotonicity of $\hat{\mathbf{S}}$ then follows since the integrand on the right hand side of the latter identity is positive due to (S3).

In order to obtain the continuity of $\hat{\mathbf{S}}$ we first show that the mapping $\boldsymbol{\xi}\mapsto\mathbf{S}(\cdot,\boldsymbol{\xi}+\mathbf{V})$, where $\mathbf{V}$ is the solution to the corresponding cell problem, is weakly continuous with values in $L^{p'}(Y;\mathbb{R}^{d\times d}_{sym})$. Let us choose $\{\boldsymbol{\xi}^n\}_{n=1}^\infty$ such that $\boldsymbol{\xi}^n\to\boldsymbol{\xi}$ in $\mathbb{R}^{d\times d}_{sym}$ as $n\to\infty$. Let $\{\mathbf{V}^n\}_{n=1}^\infty\subset G(Y)$ be a sequence of solutions of corresponding cell problems and denote $\mathbf{S}^n:=\mathbf{S}(\cdot,\boldsymbol{\xi}^n+\mathbf{V}^n)$. Since $\{\mathbf{S}^n\}_{n=1}^\infty$ and $\{\mathbf{V}^n\}_{n=1}^\infty\}$ are bounded in $L^{p'}(Y;\mathbb{R}^{d\times d}_{sym})$ and $L^p(Y;\mathbb{R}^{d\times d}_{sym})$ by (S4), they contain weakly convergent subsequences in $L^{p'}(Y;\mathbb{R}^{d\times d}_{sym})$ and $L^p(Y;\mathbb{R}^{d\times d}_{sym})$. Let us assume without loss of generality that

$$\mathbf{V}^n\rightharpoonup\mathbf{V}^* \text{ in } L^p(Y;\mathbb{R}^{d\times d}_{sym}),\quad \mathbf{S}^n\rightharpoonup\mathbf{S}^* \text{ in } L^{p'}(Y;\mathbb{R}^{d\times d}_{sym}) \text{ as } n\to\infty. \tag{2.9}$$

We show that $\mathbf{V}=\mathbf{V}^*$ and $\mathbf{S}^*=\mathbf{S}(\cdot,\boldsymbol{\xi}+\mathbf{V})$. As we have by the definition of the weak solution of the cell problem

$$\forall\mathbf{W}\in G(Y):\int_Y\mathbf{S}^n:\mathbf{W}=0, \tag{2.10}$$

it follows from (2.9) that

$$\lim_{n\to\infty}\int_Y\mathbf{S}^n:\mathbf{V}^n=0=\int_Y\mathbf{S}^*:\mathbf{V}^*. \tag{2.11}$$

Clearly, (S3) implies that

$$\forall\mathbf{W}\in L^{p'}(Y;\mathbb{R}^{d\times d}_{sym}):0\leq\int_Y(\mathbf{S}^n(y)-\mathbf{S}(y,\boldsymbol{\xi}^n+\mathbf{W}(y))):(\mathbf{V}^n(y)-\mathbf{W}(y))\mathrm{d}y. \tag{2.12}$$

We want to perform the limit passage $n \to \infty$. Let us observe that for all fixed $\mathbf{W} \in L^p(Y;\mathbb{R}^{d\times d}_{sym})$

$$\mathbf{S}(\cdot,\boldsymbol{\xi}^n+\mathbf{W}) \to \mathbf{S}(\cdot,\boldsymbol{\xi}+\mathbf{W}) \text{ in } L^{p'}(Y;\mathbb{R}^{d\times d}_{sym}) \text{ as } n\to\infty. \tag{2.13}$$

Indeed, $\mathbf{S}(\cdot,\boldsymbol{\xi}^n+\mathbf{W}) \to \mathbf{S}(\cdot,\boldsymbol{\xi}+\mathbf{W})$ a.e. in Y. Moreover, we have for $K\subset Y$ by (S4) that

$$\begin{aligned}\int_K |\mathbf{S}(y,\boldsymbol{\xi}^n+\mathbf{W}(y))-\mathbf{S}(y,\boldsymbol{\xi}+\mathbf{W}(y))|^{p'}\mathrm{d}y &\le \int_K c_2\left(|\boldsymbol{\xi}^n+\mathbf{W}|^p+|\boldsymbol{\xi}+\mathbf{W}|^p+2\right)\\ &\le c|K|(|\boldsymbol{\xi}^n|^p+|\boldsymbol{\xi}|^p+2)+c\int_K|\mathbf{W}|^p,\end{aligned}$$

which implies that $|\mathbf{S}(y,\boldsymbol{\xi}^n+\mathbf{W})-\mathbf{S}(y,\boldsymbol{\xi}+\mathbf{W})|^{p'}$ is equiintegrable and the Vitali convergence theorem yields (2.13). Hence we obtain from (2.12) using (2.11) and (2.13) that

$$\forall \mathbf{W}\in L^{p'}(Y;\mathbb{R}^{d\times d}_{sym}): 0\le \int_Y (\mathbf{S}^*(y)-\mathbf{S}(y,\boldsymbol{\xi}+\mathbf{W}(y))):(\mathbf{V}^*(y)-\mathbf{W}(y))\mathrm{d}y.$$

Minty's trick gives that $\mathbf{S}^*(y)=\mathbf{S}(y,\boldsymbol{\xi}+\mathbf{V}^*(y))$ a.e. in Y. Passing to the limit $n\to+\infty$ in (2.10) we get that $\mathbf{V}^*$ is a solution to the cell problem corresponding to $\boldsymbol{\xi}$. Since this solution is unique we get $\mathbf{V}=\mathbf{V}^*$. Up to now we showed that from $\{\mathbf{S}^n\}$ we can extract a subsequence weakly convergent towards $\mathbf{S}(\cdot,\boldsymbol{\xi}+\mathbf{V})$ in $L^{p'}(Y;\mathbb{R}^{d\times d}_{sym})$. Since this limit is unique, the whole sequence must converge to it.

One easily obtains due to the weak continuity of $\boldsymbol{\xi}\mapsto\mathbf{S}(\cdot,\boldsymbol{\xi}+\mathbf{V})$ for any $\boldsymbol{\eta}\in\mathbb{R}^{d\times d}_{sym}$

$$\left(\hat{\mathbf{S}}(\boldsymbol{\xi}^n)-\hat{\mathbf{S}}(\boldsymbol{\xi})\right):\boldsymbol{\eta}=\int_Y(\mathbf{S}(y,\boldsymbol{\xi}^n+\mathbf{V}^n(y))-\mathbf{S}(y,\boldsymbol{\xi}+\mathbf{V}(y))):\boldsymbol{\eta}\mathrm{d}y\to 0$$

as $n\to+\infty$. Since the space $\mathbb{R}^{d\times d}_{sym}$ is finite dimensional, we have the continuity of $\hat{\mathbf{S}}$. □

2.3 Statement and proof of the main theorem

Let us recall the problem, which we deal with. The domain $\Omega\subset\mathbb{R}^d$, $d=2,3,\dots$ is supposed to be bounded and Lipschitz, $Y=(0,1)^d$. For $\varepsilon\in(0,1)$ we consider the following stationary version of the problem (2.1)

$$\begin{aligned}-\operatorname{div}(\mathbf{S}^\varepsilon(x,\mathbf{D}\mathbf{u}^\varepsilon)-\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon)+\nabla\pi^\varepsilon=-\operatorname{div}\mathbf{F},\quad \operatorname{div}\mathbf{u}^\varepsilon=0\quad\text{in }\Omega,\\ \mathbf{u}^\varepsilon=0\quad\text{on }\partial\Omega,\quad \int_\Omega\pi^\varepsilon=0.\end{aligned}\tag{2.14}$$

The function $\mathbf{S}^\varepsilon$ is for any $x\in\mathbb{R}^d$ and $\mathbf{D}\in\mathbb{R}^{d\times d}_{sym}$ given by $\mathbf{S}^\varepsilon(x,\mathbf{D})=\mathbf{S}(x/\varepsilon,\mathbf{D})$, where the tensor $\mathbf{S}:\mathbb{R}^d\times\mathbb{R}^{d\times d}_{sym}\to\mathbb{R}^{d\times d}_{sym}$ satisfies in general the assumption

Assumption 2.3.1

(S1) $\mathbf{S}$ *is* $Y-$*periodic in the first variable, i.e., periodic in each argument* $y_i, i=1,\dots,d$ *with the period 1, and continuous in the first variable,*

(S2) $\mathbf{S}$ *is a Carathéodory function, i.e.,* $\mathbf{S}(\cdot,\boldsymbol{\xi})$ *is measurable for all* $\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{sym}$, $\mathbf{S}(y,\cdot)$ *is continuous for almost all* $y\in\mathbb{R}^d$,

(S3) *for* $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^{d\times d}_{sym}$, $\boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$ *and a.a.* $y \in \mathbb{R}^d$, $(\mathbf{S}(y,\boldsymbol{\xi}_1) - \mathbf{S}(y,\boldsymbol{\xi}_2)) : (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) > 0$,

(S4) *there are* $p > 1$, $p' = p/(p-1)$, $c_1, \tilde{c}_1, c_2 > 0$ *that for all* $y \in \mathbb{R}^d$, $\boldsymbol{\xi} \in \mathbb{R}^{d\times s}_{sym}$

$$\mathbf{S}(y,\boldsymbol{\xi}) : \boldsymbol{\xi} \geq c_1|\boldsymbol{\xi}|^p - \tilde{c}_1, \ |\mathbf{S}(y,\boldsymbol{\xi})|^{p'} \leq c_2(|\boldsymbol{\xi}|^p + 1).$$

A typical example of a stress tensor $\mathbf{S}$ satisfying Assumption 2.3.1 is given in (2.2). Let us begin with the existence of a weak solution of (2.14).

Lemma 2.3.2 *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain,* $\varepsilon > 0$ *be fixed,* $\mathbf{F} \in L^{p'}(\Omega; \mathbb{R}^{d\times d}_{sym})$, $p > 2d/(d+2)$, *Assumption 2.3.1 be fulfilled and* s *be determined by*

$$s = \begin{cases} \min\left\{\frac{dp}{2(d-p)}, p'\right\} & p < d, \\ p' & p \geq d. \end{cases} \tag{2.15}$$

Then there exists a weak solution $(\mathbf{u}^\varepsilon, \pi^\varepsilon)$ *of* (2.14), *which is a pair* $(\mathbf{u}^\varepsilon, \pi^\varepsilon) \in W^{1,p}_{0,\mathrm{div}}(\Omega; \mathbb{R}^d) \times L^s(\Omega)$ *such that for any* $\mathbf{w} \in C^\infty_0(\Omega; \mathbb{R}^d)$

$$\int_\Omega (\mathbf{S}^\varepsilon - \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon - \pi^\varepsilon \mathbf{I}) : \mathbf{D}\mathbf{w} = \int_\Omega \mathbf{F} : \mathbf{D}\mathbf{w}. \tag{2.16}$$

Moreover, there is $c > 0$ *independent of* ε *such that*

$$\|\mathbf{D}\mathbf{u}^\varepsilon\|_{L^p(\Omega)} \leq c, \quad \|\pi^\varepsilon\|_{L^s(\Omega)} \leq c. \tag{2.17}$$

Proof. Due to Assumptions 2.3.1 we can adopt the technique used for the proof in [7, Theorem 3.1]. □

Now we can formulate the general version of the main theorem.

Theorem 2.3.3 *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain,* $p > 2d/(d+2)$, $\mathbf{S}$ *satisfy Assumption 2.3.1 and* $\mathbf{F} \in L^{p'}(\Omega; \mathbb{R}^{d\times d}_{sym})$. *Let* $\{(\mathbf{u}^\varepsilon, \pi^\varepsilon)\}_{\varepsilon\in(0,1)}$ *be a family of weak solutions of the system* (2.14) *constructed in Lemma 2.3.2. Then there exists a sequence* $\{\varepsilon_k\}_{k=1}^{+\infty}$ *such that as* $k \to +\infty$

$$\varepsilon_k \to 0, \quad \mathbf{u}^{\varepsilon_k} \rightharpoonup \mathbf{u} \text{ in } W^{1,p}_0(\Omega; \mathbb{R}^d), \quad \pi^{\varepsilon_k} \rightharpoonup \pi \text{ in } L^s(\Omega),$$

where s *is determined in* (2.15) *and* $(\mathbf{u}, \pi)$ *is a weak solution of the system*

$$\begin{gathered} -\operatorname{div}\left(\widehat{\mathbf{S}}(\mathbf{D}\mathbf{u}) - \mathbf{u} \otimes \mathbf{u}\right) + \nabla\pi = -\operatorname{div}\mathbf{F} \quad \text{in } \Omega, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega \pi = 0. \end{gathered} \tag{2.18}$$

with $\hat{\mathbf{S}}$ *given by* (2.6).

Let us outline main steps of the proof. First, we note that from Lemma 2.3.2 there is for a fixed $\varepsilon \in (0,1)$ a weak solution of the problem (2.14), which is bounded uniformly with respect to ε. This uniform boundedness implies the existence of a sequence $\{\mathbf{u}^{\varepsilon_k}\}_{k=1}^{+\infty}$ that converges weakly in $W^{1,p}(\Omega; \mathbb{R}^d)$ to a limit $\mathbf{u}$. We can also assume that the sequence $\{\mathbf{S}^{\varepsilon_k}\}_{k=1}^{+\infty}$ converges weakly in $L^{p'}(\Omega)$ to a limit $\bar{\mathbf{S}}$. To be able to use the Minty trick to identify the limit function $\bar{\mathbf{S}}$ we would require

$$\lim_{k\to\infty}\int_\Omega \mathbf{S}^{\varepsilon_k} : \mathbf{D}\mathbf{u}^{\varepsilon_k} = \int_\Omega \bar{\mathbf{S}} : \mathbf{D}\mathbf{u}. \tag{2.19}$$

The assumption $p > \frac{2d}{d+2}$ ensures the precompactness of the term $\mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon$ in $L^q(\Omega;\mathbb{R}^{d\times d}_{sym})$ for some q that might be less than p'. Then one cannot test the weak formulation involving the limit $\bar{\mathbf{S}}$ with the limit $\mathbf{u}$ to obtain (2.19). Actually in this situation we are not allowed to test (2.14) with $\mathbf{u}^\varepsilon$ as well. To overcome this inconvenience we decompose the pressure π^ε into three parts. The first one corresponds to $\mathbf{S}^\varepsilon$ and is bounded in $L^{p'}$, the second one corresponds to $\mathbf{F} + \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon$ and is precompact in L^q for any $q \in (1, s)$, where s might be less than p' and is determined by (2.15) and the last part is harmonic. Then we employ the div-curl lemma to obtain the identity of the type (2.19). In fact, we show (2.19) for certain subsets of Ω determined by Biting lemma 2.2.1.

At this moment we decompose the pressure into a part that is bounded in $L^{p'}$, a part that is precompact but in a bigger space and a harmonic part.

Lemma 2.3.4 *Let s be given by* (2.15) *and the functions $\mathbf{u}^\varepsilon, \pi^\varepsilon, \mathbf{S}^\varepsilon, \mathbf{F}$ be extended by zero on $\mathbb{R}^d \setminus \Omega$ for $\varepsilon > 0$. Let functions $\pi^{\varepsilon,1} \in L^{p'}(\mathbb{R}^d), \pi^{\varepsilon,2} \in L^{\frac{p^*}{2}}(\mathbb{R}^d), \pi^{\varepsilon,3} \in L^{p'}(\Omega)$ be defined as*

$$\begin{aligned}\pi^{\varepsilon,1} &= \operatorname{div}\operatorname{div}\mathcal{N}\left(\mathbf{S}^\varepsilon\right),\\ \pi^{\varepsilon,2} &= -\operatorname{div}\operatorname{div}\mathcal{N}\left(\mathbf{F} + \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon\right),\\ \pi^{\varepsilon,3} &= \pi^\varepsilon - \pi^{\varepsilon,1} - \pi^{\varepsilon,2}.\end{aligned}$$

Here $\mathcal{N}$ denotes the componentwise Newton potential and $p^* = dp/(d-p)$ if $d > 2$ and $p^* > 1$ if $d = 2$. Then

$$\begin{aligned}&\{\pi^{\varepsilon,1}\} \textit{ is bounded in } L^{p'}(\mathbb{R}^d),\\ &\{\pi^{\varepsilon,2}\} \textit{ is precompact in } L^q(\mathbb{R}^d) \textit{ for any } q \in [1, s),\\ &\{\pi^{\varepsilon,3}\} \textit{ is precompact in } L^{p'}(O) \textit{ for any } O \Subset \Omega.\end{aligned} \tag{2.20}$$

Proof. Applying the theory of Calderon-Zygmund operators, see [6, Section 6.3], yields the estimates

$$\|\pi^{\varepsilon,1}\|_{L^{p'}(\mathbb{R}^d)} \le c\|\mathbf{S}^\varepsilon\|_{L^{p'}(\mathbb{R}^d)}, \quad \|\pi^{\varepsilon,2}\|_{L^s(\mathbb{R}^d)} \le c, \tag{2.21}$$

and the precompactness of $\{\pi^{\varepsilon,2}\}$ in $L^q(\mathbb{R}^d), q \in [1, s)$ since $\{\mathbf{F}+\mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon\}$ is precompact in $L^q(\mathbb{R}^d;\mathbb{R}^{d\times d}_{sym})$ by $(2.26)_2$. It follows from (2.16) and (2.17) that $\{\pi^{\varepsilon,3}\}_{\varepsilon\in(0,1)}$ are harmonic functions in Ω and bounded in $L^s(\Omega)$. Hence we can extract from $\{\pi^{\varepsilon,3}\}_{\varepsilon\in(0,1)}$ a subsequence that converges uniformly in any $O \Subset \Omega$. Thus $\{\pi^{\varepsilon,3}\}_{\varepsilon\in(0,1)}$ is precompact in $L^{p'}(O)$. □

We want to find a sequence $\{\varepsilon_k\}_{k=1}^{+\infty} \subset (0, 1)$ such that $\varepsilon_k \to 0$ as $k \to +\infty$ so that sequences of functions $\{\mathbf{S}^{\varepsilon_k}\}_{k=1}^{+\infty}$, $\{\mathbf{u}^{\varepsilon_k}\}_{k=1}^{+\infty}$ and $\{\pi^{\varepsilon_k}\}_{k=1}^{+\infty}$ has some additional properties. To abbreviate the notation we will write F^k for F^{ε_k}.

We start with extracting convergent subsequences.

Lemma 2.3.5 *Let s be given by (2.15). For any $N \subset (0,1)$, $0 \in \partial N$ there is a sequence $\{\varepsilon_k\}_{k=1}^{+\infty} \subset N$ such that $\varepsilon_k \to 0$ as $k \to +\infty$ and functions $\mathbf{u} \in W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^d)$, $\mathbf{u}^0 \in L^p\left(\Omega; W^{1,p}_{per}(Y)^d\right)$, $\overline{\mathbf{S}^0} \in L^{p'}(\Omega\times Y;\mathbb{R}^{d\times d}_{sym})$, $\overline{\pi} \in L^s(\Omega\times Y)$ and $\overline{\pi^1} \in L^{p'}(\Omega\times Y)$ such that as $k \to +\infty$*

$$\begin{aligned}
\mathbf{D}\mathbf{u}^k &\xrightharpoonup{2-s} \mathbf{D}\mathbf{u} + \mathbf{D}_y\mathbf{u}^0 && \text{in } L^p(\Omega\times Y;\mathbb{R}^{d\times d}_{sym}), && (2.22)\\
\mathbf{S}^k &\xrightharpoonup{2-s} \overline{\mathbf{S}^0} && \text{in } L^{p'}(\Omega\times Y;\mathbb{R}^{d\times d}_{sym}), && (2.23)\\
\pi^k &\xrightharpoonup{2-s} \overline{\pi} && \text{in } L^s(\Omega\times Y), && (2.24)\\
\pi^{k,1} &\xrightharpoonup{2-s} \overline{\pi^1} && \text{in } L^{p'}(\Omega\times Y), && (2.25)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{u}^k &\rightharpoonup \mathbf{u} && \text{in } W^{1,p}_0(\Omega;\mathbb{R}^d),\\
\mathbf{u}^k &\to \mathbf{u} && \text{in } L^{p^*}(\Omega;\mathbb{R}^d),\\
\pi^k &\rightharpoonup \pi := \int_Y \overline{\pi}(\cdot,y)\mathrm{d}y && \text{in } L^s(\Omega),\\
\pi^{k,1} &\rightharpoonup \pi^1 := \int_Y \overline{\pi^1}(\cdot,y)\mathrm{d}y && \text{in } L^{p'}(\Omega),\\
\mathbf{S}^k &\rightharpoonup \overline{\mathbf{S}} := \int_Y \overline{\mathbf{S}^0}(\cdot,y)\mathrm{d}y && \text{in } L^{p'}(\Omega;\mathbb{R}^{d\times d}_{sym}).
\end{aligned} \tag{2.26}$$

Moreover, the limit functions satisfy

$$\begin{aligned}
&\text{for almost all } x\in\Omega: \ \mathbf{D}_y\mathbf{u}^0(x,\cdot) \in G(Y), && (2.27)\\
&\text{for almost all } x\in\Omega: \ \overline{\mathbf{S}^0}(x,\cdot) \in G^\perp(Y), && (2.28)\\
&\text{for almost all } x\in\Omega: \ \overline{\pi^1}(x,\cdot)\mathbf{I} \in G^\perp(Y), && (2.29)
\end{aligned}$$

and for any $\mathbf{w} \in C_0^\infty(\Omega)$

$$\int_\Omega (\overline{\mathbf{S}} - \mathbf{u}\otimes\mathbf{u} - \pi\mathbf{I}) : \mathbf{D}\mathbf{w} = \int_\Omega \mathbf{F} : \mathbf{D}\mathbf{w}. \tag{2.30}$$

Proof. In the proof we subsequently extract a subsequences. We will not explicitly refer to this fact.

The convergences in (2.26) follow in a standard way from (2.17), (2.20), Sobolev embedding theorem. Validity of (2.30) follows from (2.26) and (2.16).

Convergence $(2.26)_1$ and Lemma 2.2.11 (*iv*) imply (2.22). Statements (2.23), (2.24), (2.25) and identification of weak limits in (2.26) follow from (2.17), $(2.20)_1$, the assumption on growth of $\mathbf{S}$ and Lemma 2.2.11 (*ii*) and (*iii*).

Let us show (2.27). The convergence (2.22) means that for any $\boldsymbol{\psi} \in \mathcal{D}\left(\Omega; C^\infty_{per}(Y)^{d\times d}\right)$

$$\lim_{k\to+\infty} \int_\Omega \mathbf{D}\mathbf{u}^k(x) : \boldsymbol{\psi}\left(x, \frac{x}{\varepsilon_k}\right)\mathrm{d}x = \int_\Omega\int_Y \left(\mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y)\right) : \boldsymbol{\psi}(x,y)\mathrm{d}x\mathrm{d}y. \tag{2.31}$$

We pick $a \in \mathcal{D}(\Omega), b \in C^\infty_{per}(Y)$ and put $\boldsymbol{\psi}(x,y) = a(x)b(y)\mathbf{I}$ in (2.31). Obviously, we get using the weak convergence of $\{\mathbf{u}^k\}$ in $W_0^{1,p}(\Omega;\mathbb{R}^d)$

$$
\begin{aligned}
0 &= \lim_{k\to+\infty}\int_\Omega \operatorname{div}\mathbf{u}^k(x)a(x)b\left(\frac{x}{\varepsilon_k}\right)\mathrm{d}x = \lim_{k\to+\infty}\int_\Omega \mathbf{D}\mathbf{u}^k(x)a(x)b\left(\frac{x}{\varepsilon_k}\right):\mathbf{I}\mathrm{d}x\\
&= \int_\Omega\int_Y \left(\mathbf{D}\mathbf{u}(x)+\mathbf{D}_y\mathbf{u}^0(x,y)\right)a(x)b(y):\mathbf{I}\mathrm{d}y\mathrm{d}x\\
&= \int_\Omega \operatorname{div}\mathbf{u}(x)a(x)\int_Y b(y)\mathrm{d}y\mathrm{d}x + \int_\Omega\int_Y \operatorname{div}_y\mathbf{u}^0(x,y)b(y)\mathrm{d}y\ a(x)\mathrm{d}x\\
&= \int_\Omega\int_Y \operatorname{div}_y\mathbf{u}^0(x,y)b(y)\mathrm{d}y\ a(x)\mathrm{d}x.
\end{aligned}
$$

Hence for a.a. $x\in\Omega$ $\operatorname{div}_y\mathbf{u}^0(x,\cdot)=0$ a.e. in Y, i.e. we conclude (2.27).

We show that for any $\sigma\in C_0^\infty(\Omega)$ and $\mathbf{h}\in C^\infty_{per,\operatorname{div}}(Y;\mathbb{R}^d)$

$$
\int_\Omega\int_Y \overline{\mathbf{S}^0}(x,y):\mathbf{D}\mathbf{h}(y)\mathrm{d}y\sigma(x)\mathrm{d}x = 0. \tag{2.32}
$$

Since $\varepsilon_k\sigma(x)\mathbf{h}\left(\frac{x}{\varepsilon_k}\right)$ is not solenoidal, the correction $\mathbf{B}^k(x)=\mathscr{B}\left(\varepsilon_k\mathbf{h}\left(\frac{x}{\varepsilon_k}\right)\nabla\sigma(x)\right)$ that satisfies

$$
\begin{gathered}
\operatorname{div}\mathbf{B}^k(x)=\varepsilon_k\mathbf{h}(\frac{x}{\varepsilon_k})\nabla\sigma(x) \text{ in } x\in\Omega,\quad \mathbf{B}^k=0 \text{ on } \partial\Omega,\\
\|\nabla\mathbf{B}^k\|_{L^\gamma(\Omega)}\le c(\gamma,\|\mathbf{h}\|_{L^\infty(Y)},\|\nabla\sigma\|_{L^\infty(\Omega)})\varepsilon_k
\end{gathered}
$$

with an arbitrary $\gamma\in(1,\infty)$, is introduced to allow using $\varepsilon_k\sigma(x)\mathbf{h}\left(\frac{x}{\varepsilon_k}\right)-\mathbf{B}^k$ as a test function in (2.16). We note that the existence of $\mathbf{B}^k$ is ensured by Lemma 2.2.3. Then we employ convergences as $k\to+\infty$

$$
\begin{aligned}
\mathbf{B}^k &\to 0 && \text{in } L^\gamma(\Omega;\mathbb{R}^d),\\
\mathbf{D}\mathbf{B}^k &\to 0 && \text{in } L^\gamma(\Omega;\mathbb{R}^{d\times d}_{sym}),\\
\sigma\mathbf{u}^k\otimes\mathbf{u}^k &\to \sigma\mathbf{u}\otimes\mathbf{u} && \text{in } L^1(\Omega;\mathbb{R}^{d\times d}_{sym}),\\
\mathbf{S}^k &\xrightarrow{2-s} \overline{\mathbf{S}^0} && \text{in } L^{p'}(\Omega;\mathbb{R}^{d\times d}_{sym}),
\end{aligned}
$$

to obtain from (2.16) by (2.5) and Lemma $2.2.11_1$ that

$$
\int_\Omega\int_Y (\overline{\mathbf{S}^0}(x,y)-\mathbf{u}(x)\otimes\mathbf{u}(x)-\mathbf{F}(x)):\mathbf{D}_y\mathbf{h}(y)\mathrm{d}y\sigma(x)\mathrm{d}x = 0.
$$

Hence (2.32) and thus (2.28) follow due to an obvious fact $\int_Y \mathbf{D}_y\mathbf{h}(y)=0$.

Finally, we infer that for any $\mathbf{W}=\mathbf{D}\mathbf{w}\in G(Y)$ and almost all $x\in\Omega$

$$
\int_Y \overline{\pi^1}(x,y)\mathbf{I}:\mathbf{W}(y)\mathrm{d}y = \int_Y \overline{\pi^1}(x,y)\operatorname{div}_y\mathbf{w}(y)\mathrm{d}y = 0,
$$

which concludes (2.29). □

Further we also utilize the following lemma concerning the equiintegrability property of sequences $\{\mathbf{S}^k\}_{k=1}^\infty$, $\{\pi^{k,1}\}_{k=1}^\infty$.

Lemma 2.3.6 *For any $N \subset (0,1)$, $0 \in \partial N$ there is a sequence $\{\varepsilon_k\}_{k=1}^{+\infty} \subset N$ such that $\varepsilon_k \to 0$ as $k \to +\infty$ and a sequence of measurable sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots \subset \Omega$ with $|\Omega \setminus \Omega_n| \to 0$ as $n \to +\infty$ such that for any $n \in \mathbb{N}$ and $\theta > 0$ there is $\delta > 0$ such that for any $k \in \mathbb{N}$ and $K \subset \Omega_n$ with $|K| < \delta$*

$$\|\mathbf{S}^k\|_{L^{p'}(K)} + \|\pi^{k,1}\|_{L^{p'}(K)} < 2\theta^{\frac{1}{p'}}. \tag{2.33}$$

Proof. Let us consider an arbitrary sequence $\{\varepsilon_k\}_{k=1}^{+\infty} \subset N$, $\varepsilon_k \to 0$ as $k \to +\infty$ and denote $G^k = |\mathbf{S}^k|^{p'} + |\pi^{k,1}|^{p'}$. The apriori estimate $(2.17)_1$, the growth condition on $\mathbf{S}$ and $(2.20)_1$ imply the boundedness of $\{G^k\}_{k=1}^{\infty}$ in $L^1(\Omega)$. The application of Chacon's biting lemma 2.2.1 on $\{G^k\}_{k=1}^{\infty}$ yields the existence of sets $\Omega_n \subset \Omega$ with $|\Omega \setminus \Omega_n| \to 0$ as $n \to \infty$ and the existence of a subsequence $\{G^k\}_{k=1}^{\infty}$ (that will not be relabeled) and a function $G \in L^1(\Omega)$ such that $G^k \rightharpoonup G$ in $L^1(\Omega_n)$ as $k \to \infty$. According to Dunford theorem 2.2.2 we obtain the equiintegrability of $\{G^k\}_{k=1}^{\infty}$ on Ω_n, i.e., for any $\theta > 0$ there is $\delta > 0$ such that for any $k \in \mathbb{N}$ and $K \subset \Omega_n$ with $|K| < \delta$ we have

$$\int_K |\mathbf{S}^k|^p + |\pi^{k,1}|^p < \theta,$$

which implies (2.33). □

From now on we assume that a sequence $\{\varepsilon_k\}_{k=1}^{+\infty} \subset N$, $\varepsilon_k \to 0$ as $k \to +\infty$ is chosen in such a way that all conclusion of Lemmas 2.3.4, 2.3.5 and 2.3.6 hold. In particular we fix the corresponding sequence $\{\Omega_n\}_{n=1}^{+\infty}$ of sets from Lemma 2.3.6. The rest of the paper is devoted to finding for this particular sequence the relation between $\overline{\mathbf{S}}$ and $\mathbf{Du}$.

In the following lemma we construct for any element of a subsequence of $\{\mathbf{u}^k\}_{k=1}^{+\infty}$ a sequence $\{\mathbf{u}^{k,\lambda}\}_{\lambda=1}^{+\infty}$, whose elements have the symmetric gradient bounded in $L^\infty(\Omega)$ independently of k. Hence a subsequence that converges weakly-star in $L^\infty(\Omega)$ can be selected. Moreover, limit functions form a sequence that contains a weakly convergent subsequence in $L^p(\Omega)$.

Lemma 2.3.7 *There is $c > 0$ and a subsequence of $\{\varepsilon_k\}_{k=1}^{+\infty}$ (that will not be relabeled) such that*

$$\forall k, \lambda \in \mathbb{N} : \|\mathbf{Du}^{k,\lambda}\|_{L^p(\Omega)} \le c, \tag{2.34}$$

$$\forall \lambda \in \mathbb{N} : \mathbf{Du}^{k,\lambda} \rightharpoonup^* \mathbf{Du}^\lambda \text{ as } k \to +\infty \text{ in } L^\infty(\Omega;\mathbb{R}^d), \tag{2.35}$$

where we denoted by $\mathbf{u}^{k,\lambda}$ functions constructed to $\mathbf{u}^k$ by Lemma 2.2.5. Moreover, a subsequence $\{\mathbf{u}^{\lambda_l}\}_{l=1}^{+\infty}$ can be selected such that

$$\mathbf{Du}^{\lambda_k} \rightharpoonup \mathbf{Du} \text{ as } k \to +\infty \text{ in } L^p(\Omega;\mathbb{R}^{d\times d}_{sym}),. \tag{2.36}$$

Proof. The application of Lemma 2.2.5 to the sequence $\{\mathbf{u}^k\}$ yields the existence of functions $\mathbf{u}^{k,\lambda} \in W^{1,\infty}(\Omega;\mathbb{R}^d)$, $k, \lambda \in \mathbb{N}$ satisfying

$$\|\mathbf{u}^{k,\lambda}\|_{W^{1,\infty}(\Omega)} \le \lambda, \quad |\{x \in \Omega : \mathbf{u}^k(x) \ne \mathbf{u}^{k,\lambda}(x)\}| \le c\frac{\|\mathbf{u}^k\|^p_{W^{1,p}(\Omega)}}{\lambda^p}. \tag{2.37}$$

Utilizing (2.37), Friedrichs and Korn's inequalities, we obtain

$$\int_\Omega |\mathbf{Du}^{k,\lambda}|^p = \int_{\{\mathbf{u}^k=\mathbf{u}^{k,\lambda}\}} |\mathbf{Du}^{k,\lambda}|^p + \int_{\{\mathbf{u}^k\ne\mathbf{u}^{k,\lambda}\}} |\mathbf{Du}^{k,\lambda}|^p \le \int_\Omega |\mathbf{Du}^k|^p$$
$$+ \lambda^p|\{x \in \Omega : \mathbf{u}^k(x) \ne \mathbf{u}^{k,\lambda}(x)\}| \le c\|\mathbf{Du}^k\|^p_{L^p(\Omega)},$$

which implies (2.34) due to (2.17).

The convergence (2.35) follows from $(2.37)_1$ by a diagonal procedure. Moreover, the estimate (2.34), (2.35) and the weak lower semicontinuity of the L^p−norm imply the existence of a positive constant c such that

$$\forall \lambda \in \mathbb{N} : \|\mathbf{D}\mathbf{u}^\lambda\|_{L^p(\Omega)} \leq c.$$

Hence we can pick a function $\tilde{\mathbf{u}} \in W_0^{1,p}(\Omega;\mathbb{R}^d)$ and a subsequence $\{\lambda_l\}_{l=1}^{+\infty}$ such that

$$\mathbf{u}^{\lambda_l} \rightharpoonup \tilde{\mathbf{u}} \text{ as } l \to +\infty \text{ in } W^{1,p}(\Omega;\mathbb{R}^d).$$

It remains to show $\tilde{\mathbf{u}} = \mathbf{u}$. Using the boundedness of the sequences $\{\mathbf{u}^k\}, \{\mathbf{u}^{k,\lambda}\}$ in $W^{1,p}(\Omega;\mathbb{R}^d)$ and the estimate $(2.37)_2$, we obtain

$$\int_\Omega |\mathbf{u}^{k,\lambda} - \mathbf{u}^k| = \int_{\{\mathbf{u}^{k,\lambda}\neq\mathbf{u}^\varepsilon\}} |\mathbf{u}^{k,\lambda} - \mathbf{u}^k| \leq \|\mathbf{u}^{k,\lambda} - \mathbf{u}^k\|_{L^p(\Omega)} |\{\mathbf{u}^{k,\lambda} \neq \mathbf{u}^k\}|^{\frac{1}{p'}} \leq \frac{c}{\lambda^{p-1}}.$$

Moreover, the compact embedding $W^{1,p}(\Omega;\mathbb{R}^d) \hookrightarrow L^1(\Omega;\mathbb{R}^d)$ implies

$$\|\mathbf{u}^\lambda - \mathbf{u}\|_{L^1(\Omega)} = \lim_{k\to+\infty} \|\mathbf{u}^{k,\lambda} - \mathbf{u}^k\|_{L^1(\Omega)}.$$

Therefore $\mathbf{u}^\lambda \to \mathbf{u}$ in $L^1(\Omega)$ and we conclude $\tilde{\mathbf{u}} = \mathbf{u}$ a.e. in Ω. □

In the rest of the paper we denote for any $k, l \in \mathbb{N}$ the function $\mathbf{u}^{k,l} := \mathbf{u}^{\varepsilon_k,\lambda_l}$, where $\{\lambda_l\}$ and $\{\varepsilon_k\}$ are sequences constructed in Lemma 2.3.7. Now, we are prepared to show that for certain subsets $\tilde{\Omega}_n$ of Ω we can identify $\lim_{k\to\infty} \int_{\tilde{\Omega}_n} \mathbf{S}^k : \mathbf{D}\mathbf{u}^k \mathrm{d}x$.

Lemma 2.3.8 *Let $O \Subset \Omega$ be arbitrary open and denote $\tilde{\Omega}_n = \Omega_n \cap O$. Then for each $n \in \mathbb{N}$*

$$\lim_{k\to+\infty} \int_{\tilde{\Omega}_n} \mathbf{S}^k : \mathbf{D}\mathbf{u}^k = \int_{\tilde{\Omega}_n} \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}. \tag{2.38}$$

Proof. For fixed $n \in \mathbb{N}$ and any $k, l \in \mathbb{N}$ we decompose using the solenoidality of $\mathbf{u}^k$

$$\begin{aligned}\int_{\tilde{\Omega}_n} \mathbf{S}^k : \mathbf{D}\mathbf{u}^k &= \int_{\tilde{\Omega}_n} (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^k = \int_{\tilde{\Omega}_n} (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\left(\mathbf{u}^k - \mathbf{u}^{k,l}\right) \\ &+ \int_{\tilde{\Omega}_n} (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^{k,l} = I^{k,l} + II^{k,l}.\end{aligned}$$

We want to perform the limit passage $k \to +\infty$ and then $l \to +\infty$ in both terms on the right hand side of the latter equality. We denote $\tilde{\Omega}_n^{k,l} = \tilde{\Omega}_n \cap \{\mathbf{u}^k \neq \mathbf{u}^{k,l}\}$ and estimate using Hölder's inequality, (2.21), $(2.17)_1$ and (2.34)

$$|I^{k,l}| \leq c\|\mathbf{S}^k - \pi^{k,1}\mathbf{I}\|_{L^{p'}(\tilde{\Omega}_n^{k,l})} \|\mathbf{D}(\mathbf{u}^k - \mathbf{u}^{k,l})\|_{L^p(\tilde{\Omega}_n^{k,l})} \leq c\left(\|\mathbf{S}^k\|_{L^{p'}(\tilde{\Omega}_n^{k,l})} + \|\pi^{k,1}\|_{L^{p'}(\tilde{\Omega}_n^{k,l})}\right).$$

As $|\tilde{\Omega}_n^{k,l}| \leq c\lambda_l^{-p}$ by $(2.37)_2$, we get by Lemma 2.3.6 that for any $\theta > 0$ there exists $l_0 \in \mathbb{N}$ such that for any $l > l_0$ and $k \in \mathbb{N}$ we have $|I^{k,l}| < \theta$ and therefore

$$\lim_{l\to+\infty} \lim_{k\to+\infty} I^{k,l} = \lim_{k\to+\infty} \lim_{l\to+\infty} I^{k,l} = 0.$$

Note that for this estimate it is essential that $\{\pi^{k,1}\}$ is bounded in $L^{p'}(\Omega)$. The terms $\pi^{k,2}$ and $\pi^{k,3}$ cannot be included to $I^{k,l}$.

For the limit passage $k \to +\infty$ in $II^{k,l}$ we employ Lemma 2.2.4. Let us pick $q \in (1, s)$, where s is determined by (2.15). We have for any $\mathbf{w} \in W_0^{1,q'}(O; \mathbb{R}^d)$ in the sense of distributions

$$\Big\langle \operatorname{div}(\mathbf{S}^k - \pi^{k,1}\mathbf{I}), \mathbf{w} \Big\rangle = -\int_O \Big(\mathbf{F} + \mathbf{u}^k \otimes \mathbf{u}^k + (\pi^{k,2} + \pi^{k,3})\mathbf{I} \Big) : \mathbf{D}\mathbf{w}. \tag{2.39}$$

It follows from Lemma 2.3.4 that $\{\mathbf{F} + \mathbf{u}^k \otimes \mathbf{u}^k + (\pi^{k,2} + \pi^{k,3})\mathbf{I}\}$ is precompact in $L^q(O; \mathbb{R}^{d\times d}_{sym})$. Therefore we obtain that $\{\operatorname{div}(\mathbf{S}^k + \pi^{k,1}\mathbf{I})\}$ is precompact in $W^{-1,q}(O; \mathbb{R}^d)$. Here it is necessary that part of the pressure corresponding to $\mathbf{S}^k$, i.e. $\pi^{k,1}$, that is not precompact in any Lebesgue space, does not appear on the right hand side of (2.39). We observe that $\operatorname{curl}(\nabla \mathbf{u}^{k,l}) = 0$. Then Lemma 2.2.4 and the convergences $(2.26)_{4,5}$ and (2.35) imply

$$(\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^{k,l} = (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \nabla \mathbf{u}^{k,l} \rightharpoonup (\overline{\mathbf{S}} - \pi^1\mathbf{I}) : \nabla \mathbf{u}^l = (\overline{\mathbf{S}} - \pi^1\mathbf{I}) : \mathbf{D}\mathbf{u}^l$$

in $L^1(O)$. Hence we deduce using (2.36) and the solenoidality of $\mathbf{u}$

$$\begin{aligned}\lim_{l\to+\infty}\lim_{k\to+\infty} II^{k,l} &= \lim_{l\to+\infty}\lim_{k\to+\infty}\int_O (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^{k,l}\chi_{\tilde{\Omega}_n} \\ &= \lim_{l\to+\infty}\int_O (\overline{\mathbf{S}} - \pi^1\mathbf{I}) : \mathbf{D}\mathbf{u}^l\chi_{\tilde{\Omega}_n} = \int_{\tilde{\Omega}_n} \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}.\end{aligned}$$

□

Having all preliminary claims shown we justify the limit passage $\varepsilon \to 0$ in the weak formulation of (2.14).

Proof of Theorem 2.3.3. It remains to show the relation

$$\overline{\mathbf{S}^0}(x,y) = \mathbf{S}(x, \mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y)) \text{ for almost all } x \in \Omega, y \in Y. \tag{2.40}$$

This equality namely immediately implies that $\mathbf{D}_y\mathbf{u}^0(x,\cdot)$ is the solution of the cell problem (2.7) with $\boldsymbol{\xi} = \mathbf{D}\mathbf{u}(x)$ for a.a. $x \in \Omega$ by (2.27) and (2.28). Consequently, integrating (2.40) over Y we obtain $\overline{\mathbf{S}}(x) = \int_Y \mathbf{S}(x, \mathbf{D}\mathbf{u} + \mathbf{D}_y\mathbf{u}^0)\mathrm{d}y = \hat{\mathbf{S}}(\mathbf{D}\mathbf{u})$ and (2.18) holds.

Finally, we prove (2.40). We fix $n \in \mathbb{N}$, a corresponding Ω_n from Lemma 2.3.6 and $O \Subset \Omega$. Keeping the notation of Lemma 2.3.8, using (2.26), (2.27) and (2.28), it follows from (2.38) that

$$\lim_{k\to+\infty}\int_{\tilde{\Omega}_n} \mathbf{S}^k : \mathbf{D}\mathbf{u}^k = \int_{\tilde{\Omega}_n}\int_Y \overline{\mathbf{S}^0} : (\mathbf{D}\mathbf{u} + \mathbf{D}_y\mathbf{u}^0). \tag{2.41}$$

We choose $\mathbf{U} \in L^p(\tilde{\Omega}_n; C_{per}(Y; \mathbb{R}^{d\times d}_{sym}))$. The monotonicity of $\mathbf{S}$ implies

$$\begin{aligned}0 &\le \int_{\tilde{\Omega}_n} \Big(\mathbf{S}^k(x) - \mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right)\Big) : \Big(\mathbf{D}\mathbf{u}^k(x) - \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\Big)\,\mathrm{d}x \\ &= \int_{\tilde{\Omega}_n} \mathbf{S}^k(x) : \mathbf{D}\mathbf{u}^k(x)\mathrm{d}x - \int_{\tilde{\Omega}_n} \mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right) : \mathbf{D}\mathbf{u}^k(x)\mathrm{d}x \\ &- \int_{\tilde{\Omega}_n} \mathbf{S}^k(x) : \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\mathrm{d}x + \int_{\tilde{\Omega}_n} \mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right) : \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\mathrm{d}x \\ &= I^k - II^k - III^k + IV^k.\end{aligned}$$

We want to pass to the limit as $k \to +\infty$ in I^k, II^k, III^k, IV^k. We use (2.41) for the passage in I^k. Applying Lemma 2.2.11 (i) to $\mathbf{S}(y, \mathbf{U}(x,y))$ and $\mathbf{U}$ yields

$$\begin{aligned} \mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right) &\xrightarrow{2-s} \mathbf{S}(y, \mathbf{U}(x,y)) \text{ in } L^{p'}(\Omega \times Y; \mathbb{R}^{d\times d}_{sym}), \\ \mathbf{U}\left(x, x\varepsilon_k^{-1}\right) &\xrightarrow{2-s} \mathbf{U}(x,y) \text{ in } L^{p}(\Omega \times Y; \mathbb{R}^{d\times d}_{sym}) \end{aligned}$$

as $k \to +\infty$. Employing these convergences and (2.22) we infer

$$\begin{aligned} \lim_{k\to+\infty} II^k &= \int_{\tilde{\Omega}_n}\int_Y \mathbf{S}(y, \mathbf{U}(x,y)) : (\mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y)\mathrm{d}y\mathrm{d}x, \\ \lim_{k\to+\infty} III^k &= \int_{\tilde{\Omega}_n}\int_Y \overline{\mathbf{S}^0}(x,y) : \mathbf{U}(x,y)\mathrm{d}y\mathrm{d}x, \\ \lim_{k\to+\infty} IV^k &= \int_{\tilde{\Omega}_n}\int_Y \mathbf{S}(y, \mathbf{U}(x,y)) : \mathbf{U}(x,y)\mathrm{d}y\mathrm{d}x. \end{aligned}$$

Thus one obtains for any $n \in \mathbb{N}$ and $\mathbf{U} \in L^p(\tilde{\Omega}_n; C_{per}(Y; \mathbb{R}^{d\times d}_{sym}))$

$$\int_{\tilde{\Omega}_n}\int_Y \left(\overline{\mathbf{S}^0}(x,y) - \mathbf{S}(y, \mathbf{U}(x,y))\right) : \left(\mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y) - \mathbf{U}(x,y)\right)\mathrm{d}y\mathrm{d}x \geq 0. \quad (2.42)$$

To be able to apply Minty's trick, we need (2.42) to be satisfied for any $\mathbf{U} \in L^p(\tilde{\Omega}_n \times Y; \mathbb{R}^{d\times d}_{sym})$. In order to obtain that we consider $\mathbf{U} \in L^p(\tilde{\Omega}_n \times Y; \mathbb{R}^{d\times d}_{sym})$ and $\{\mathbf{U}^k\} \subset L^p(\tilde{\Omega}_n; C_{per}(Y; \mathbb{R}^{d\times d}_{sym}))$ such that $\mathbf{U}^k \to \mathbf{U}$ in $L^p(\tilde{\Omega}_n \times Y; \mathbb{R}^{d\times d}_{sym})$. Then we have due to the growth of $\mathbf{S}$ and theory of Nemytskii operators that $\mathbf{S}(y, \mathbf{U}^k) \to \mathbf{S}(y, \mathbf{U})$ in $L^{p'}(\tilde{\Omega}_n \times Y; \mathbb{R}^{d\times d}_{sym})$. Therefore one deduces the accomplishment of (2.42) for any $\mathbf{U} \in L^p(\tilde{\Omega}_n \times Y; \mathbb{R}^{d\times d}_{sym})$. Minty's trick yields that $\overline{\mathbf{S}^0}(x,y) = \mathbf{S}(y, \mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y))$ for almost all $(x,y) \in \tilde{\Omega}_n \times Y$. Since $|\Omega \setminus \Omega_n| \to 0$, $\{\Omega \setminus \Omega_n\}_{n=1}^{+\infty}$ is a decreasing sequence of measurable sets and $O \Subset \Omega$ was arbitrary, we have for almost all $(x,y) \in \Omega \times Y$ $\overline{\mathbf{S}^0}(x,y) = \mathbf{S}(y, \mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y))$. □

Let us note that we have simultaneously proven the following lemma concerning the existence of a weak solution of the problem (2.18).

Lemma 2.3.9 *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $\mathbf{F} \in L^{p'}(\Omega; \mathbb{R}^{d\times d}_{sym})$ and $p > \frac{2d}{d+2}$, Assumption 2.3.1 be fulfilled and s be determined by* (2.15)*. Then there exists a weak solution $(\mathbf{u}, \pi)$ of the problem* (2.18)*, which is a pair $(\mathbf{u}, \pi) \in W^{1,p}_{0,\mathrm{div}}(\Omega; \mathbb{R}^d) \times L^s(\Omega)$ such that for any $\mathbf{w} \in C^\infty_0(\Omega; \mathbb{R}^d)$*

$$\int_\Omega (\hat{\mathbf{S}}(\mathbf{D}\mathbf{u}) - \mathbf{u} \otimes \mathbf{u} - \pi\mathbf{I}) : \mathbf{D}\mathbf{w} = \int_\Omega \mathbf{F} : \mathbf{D}\mathbf{w}.$$

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Homogenization of nonlinear elliptic systems in nonreflexive Musielak-Orlicz spaces

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Abstract

We study the homogenization process for families of strongly nonlinear elliptic systems with the homogeneous Dirichlet boundary conditions. The growth and the coercivity of the elliptic operator is assumed to be indicated by a general inhomogeneous anisotropic $\mathcal{N}$–function M, which may be possibly also dependent on the spatial variable, i.e., the homogenization process will change the characteristic function spaces at each step. Such a problem is well known and there exist many positive results for the L^p–setting with restrictions on constant exponent or variable exponent that is assumed to be additionally log-Hölder continuous. These situations correspond to a very particular case of $\mathcal{N}$–functions satisfying Δ_2 and ∇_2–conditions. We shall show that for M satisfying a condition of log-Hölder type one can avoid all difficulties and provide a rather general theory without any assumption on the validity of Δ_2 or ∇_2 conditions.

Keywords

nonlinear elliptic problems, Musielak–Orlicz spaces, periodic homogenization, two-scale convergence method

3.1 Introduction

Our primary interest is to study the behaviour of the following system as $\varepsilon \to 0_+$:

$$\begin{aligned} \operatorname{div} \mathbf{A}\left(\frac{x}{\varepsilon}, \nabla \mathbf{u}^\varepsilon\right) &= \operatorname{div} \mathbf{F} \text{ in } \Omega, \\ \mathbf{u}^\varepsilon &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d, d \geq 2$ is a bounded Lipschitz domain and $\mathbf{u}^\varepsilon : \Omega \to \mathbb{R}^N$ with $N \in \mathbf{N}$ is an unknown. The data of problem (3.1) are $\mathbf{F} : \Omega \to \mathbb{R}^{d\times N}$ and $\mathbf{A} : \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$. The operator $\mathbf{A}$ is strongly nonlinear with the growth prescribed by a spatially inhomogeneous anisotropic $\mathcal{N}$–function

The studies on homogenization of elliptic equations go back to the works of Oleinik and Zhikov [11] and Allaire [1]. The second one is in particular worth of recalling here as it uses the notion of two-scale convergence. The setting of non-standard growth conditions of the opeartor $\mathbf{A}$ appeared in [16]. The authors considered the growth prescribed by means of variable exponent $p(x)$. To identify an elliptic operator in the homogenized problem, i.e., after letting $\varepsilon \to 0_+$ in (3.1), the authors applied a variant of the compensated compactness argument. For this approach to work one needs that a decomposition of Helmholtz type holds for function spaces involved. It has to be pointed out that a decomposition of a similar type is unknown for our structure of function spaces. The current formulation, however, transfers the problem to different functional setting, where such a decomposition is not valid anymore.

Known results from the theory of homogenization of elliptic equations, c.f. [16], suggest that the limit $\mathbf{u}$ of a sequence of solutions of (3.1) satisfies a problem, in which the nonlinear operator is independent of a spatial variable, i.e., the problem possesses the form

$$\begin{aligned} \operatorname{div} \hat{\mathbf{A}}(\nabla \mathbf{u}) &= \operatorname{div} \mathbf{F} \text{ in } \Omega, \\ \mathbf{u} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

where, denoting $Y := (0,1)^d$, the operator $\hat{\mathbf{A}}$ is defined as

$$\hat{\mathbf{A}}(\boldsymbol{\xi}) := \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}(y)) \mathrm{d}y,$$

and $\mathbf{w} : \mathbb{R}^d \to \mathbb{R}^N$ is the solution of the cell problem, i.e., $\mathbf{w}$ is Y-periodic and solves

$$\operatorname{div} \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}(y)) = 0 \text{ in } Y.$$

Our effort will be spent on establishing the convergence of solutions $\mathbf{u}^\varepsilon$ as $\varepsilon \to 0_+$, finding the connection between the operators $\hat{\mathbf{A}}$ and $\mathbf{A}$ and establishing solvability of (3.2). These results are summarized in Theorem 3.1.2.

It is worth noticing that the above mentioned results are known in "subcritical" cases, i.e., cases where the underlying $\mathcal{N}$–function satisfies Δ_2 and ∇_2–condition. In our result we do not require validity of any of these two conditions. However, we have to compensate it (and one can naturally expect it) by the assumption on log-Hölder continuity with respect to the spatial variable of the $\mathcal{N}$–function M. It also seems that the "borderline" cases are mostly untouched and the main purpose of the paper is to develop sufficiently robust theory, that would allow us to talk about the limit $\varepsilon \to 0_+$ in (3.1) also for these borderline cases.

We first formulate certain minimal assumptions on the operator $\mathbf{A}$, that will be used in what follows:

(A1) $\mathbf{A}$ is a Carathéodory mapping, i.e., $\mathbf{A}(\cdot, \xi)$ is measurable for any $\xi \in \mathbb{R}^{d\times N}$ and $\mathbf{A}(y, \cdot)$ is continuous for a.a. $y \in \mathbb{R}^d$,

(A2) $\mathbf{A}$ is $Y-$periodic, i.e., periodic in each argument $y_i, i = 1, \dots, d$ with the period 1,

(A3) There exists an $\mathcal{N}$–function $M : \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, \infty)$ and a constant $c > 0$ such that for a.a. $y \in Y$ and all $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$ there holds[1]

$$\mathbf{A}(y, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq c(M(y, \boldsymbol{\xi}) + M^*(y, \mathbf{A}(y, \boldsymbol{\xi}))),$$

(A4) For all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{d\times N}$ such that $\boldsymbol{\xi} \neq \boldsymbol{\eta}$ and a.a. $y \in Y$, we have

$$(\mathbf{A}(y, \boldsymbol{\xi}) - \mathbf{A}(y, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) > 0.$$

Concerning the $\mathcal{N}$–function M and the corresponding function spaces, we shall impose the following conditions on M:

(M1) M is $Y-$periodic in the first variable,

(M2) there exist $\mathcal{N}$–functions $m_1, m_2 : [0, \infty) \to [0, \infty)$ such that

$$m_1(|\boldsymbol{\xi}|) \leq M(y, \boldsymbol{\xi}) \leq m_2(|\boldsymbol{\xi}|) \text{ on } Y,$$

(M3) there exist constants $A > 0$ and $B \geq 1$ such that for all $y_1, y_2 \in Y$ with $|y_1 - y_2| \leq \frac{1}{2}$ and all $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$ we have

$$\frac{M(y_1, \boldsymbol{\xi})}{M(y_2, \boldsymbol{\xi})} \leq \max\{|\boldsymbol{\xi}|^{-\frac{A}{\log|y_1 - y_2|}}, B^{-\frac{A}{\log|y_1 - y_2|}}\}. \tag{3.3}$$

Let us consider a family of d–dimensional cubes covering the set Y. Namely, a family $\{Q_j^\delta\}_{j=1}^{N^\delta}$ consists of closed cubes of edge 2δ such that $\operatorname{int} Q_j^\delta \cap \operatorname{int} Q_i^\delta = \emptyset$

[1] Note that the condition could be formulated more generally, i.e., $\mathbf{A}(y, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq c(M(y, \boldsymbol{\xi}) + M^*(y, \mathbf{A}(y, \boldsymbol{\xi}))) - k(y)$ for some integrable function k. For readability we omit this generality here, however such case could easily be treated, see e.g. [6].

for $i \neq j$ and $Y \subset \bigcup_{j=1}^{N^\delta} Q_j^\delta$. Moreover, for each cube Q_j^δ we define the cube $\tilde{Q}_j^\delta$ centered at the same point and with parallel corresponding edges of length 4δ. Assume that there are constants $C, D, E > 0$ and $G \geq 1$ such that for all $y \in Q_j^\delta$ and all $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$ we have

$$\frac{M(y,\boldsymbol{\xi})}{(M_j^\delta)^{**}(\boldsymbol{\xi})} \leq C \max\{|\boldsymbol{\xi}|^{-\frac{D}{\log(E\delta)}}, G^{-\frac{D}{\log(E\delta)}}\} \tag{3.4}$$

where $\delta < \delta_0$ (δ_0 is chosen in such a way that $D\delta_0 \leq \frac{1}{2}$),

$$M_j^\delta(\boldsymbol{\xi}) := \inf_{y\in\tilde{Q}_j^\delta} M(y,\boldsymbol{\xi}). \tag{3.5}$$

and $(M_j^\delta)^{**}$ is the biconjugate of M_j^δ.

Remark 3.1.1 In the case of a certain special form of M we relax condition (M3) via Lemma 3.4.5. Consider an $\mathcal{N}$–function M possessing the form

$$M(y,\boldsymbol{\xi}) = \sum_{i=1}^{K} k_i(y)M_i(\boldsymbol{\xi}) + \tilde{M}(y,|\boldsymbol{\xi}|) \text{ for some } K \in \mathbb{N}$$

where M_i, $i = 1,\dots,K$, are spatially independent $\mathcal{N}$–functions and k_i are nonnegative functions. In this case it is sufficient to assume that $\tilde{M}$ is continuous on $\mathbb{R}^d \times [0,\infty)$ and satisfies (3.3). Concerning the functions k_i we assume that there exist constants $C_i > 1$ such that $\frac{k_i(y_1)}{k_i(y_2)} \leq C_i^{-\frac{1}{\log|y_1-y_2|}}$ for $i = 1,\dots,K$ and any $y_1, y_2 \in Y$ with $|y_1 - y_2| \leq \frac{1}{2}$. Then, keeping the notation from (M3) and considering an arbitrary $\delta < \delta_0 \leq \frac{1}{6\sqrt{d}}$, we have

$$\begin{aligned} M_\delta^j(\boldsymbol{\xi}) &= \inf_{\tilde{Q}_j^\delta}\left(\sum_{i=1}^{K} k_i(y)M_i(\boldsymbol{\xi}) + \tilde{M}(y,|\boldsymbol{\xi}|)\right) \geq \sum_{i=1}^{K} \inf_{\tilde{Q}_\delta^j} k_i(y)M_i(\boldsymbol{\xi}) + \inf_{\tilde{Q}_\delta^j} \tilde{M}(y,|\boldsymbol{\xi}|) \\ &\geq \sum_{i=1}^{K} \inf_{\tilde{Q}_\delta^j} k_i(y)M_i(\boldsymbol{\xi}) + (\tilde{M}_j^\delta)^{**}(|\boldsymbol{\xi}|) =: \bar{M}_\delta^j(\boldsymbol{\xi}). \end{aligned}$$

Obviously, due to the continuity of functions k_i and $\tilde{M}$ there are points $\bar{y}_i \in \tilde{Q}_j^\delta$ such that $\bar{M}_\delta^j(\boldsymbol{\xi}) = \sum_{i=1}^{K} k_i(\bar{y}_i)M_i(\boldsymbol{\xi}) + (\tilde{M}_j^\delta)^{**}(|\boldsymbol{\xi}|)$. Moreover, the function $\bar{M}_\delta^j$ is convex with respect to $\boldsymbol{\xi}$. Hence we obtain $\bar{M}_\delta^j(\boldsymbol{\xi}) \leq (M_\delta^j)^{**}(\boldsymbol{\xi})$ and since for any $y \in Q_\delta^j$ it follows that $|y - \bar{y}_i| \leq 3\delta\sqrt{d}$ for all $i = 1,\dots,K$, we get

$$\begin{aligned} \frac{M(y,\boldsymbol{\xi})}{(M_\delta^j)^{**}(\boldsymbol{\xi})} \leq& \sum_{i=1}^{K} \frac{k_i(y)}{k_i(\bar{y}_i)} + \frac{\tilde{M}(y,|\boldsymbol{\xi}|)}{(\tilde{M}_j^\delta)^{**}(|\boldsymbol{\xi}|)} \leq K \max_{i=1,\dots,K}\{C_i^{-\frac{1}{\log|y-\bar{y}_i|}}\} \\ &+ C \max\{|\boldsymbol{\xi}|^{-\frac{D}{\log(E\delta)}}, G^{-\frac{D}{\log(E\delta)}}\} \\ \leq& \tilde{C} \max\{|\boldsymbol{\xi}|^{-\frac{D}{\log(E\delta)}}, \max\{\max_{i=1,\dots,K}\{C_i\}, G\}^{-\frac{\max\{D,1\}}{\log(\max\{3\sqrt{d},E\}\delta)}}\} \end{aligned}$$

for some constants $\tilde{C}, D, E > 0$ and $G \geq 1$ if $\delta < \delta_0 \leq \min\{\frac{1}{6\sqrt{d}}, \frac{1}{2E}\}$ is considered.

To finish the introduction, we formulate the main result of the paper.

Theorem 3.1.2 *Let $\mathbf{A}$ satisfy (A1)–(A4), the $\mathcal{N}$–functions M satisfy (M1)–(M3),*

$$\mathbf{F} \in L^{\infty}(\Omega; \mathbb{R}^{d\times N}) \tag{3.6}$$

and for any $\varepsilon > 0$ let $\mathbf{u}^{\varepsilon}$ be a unique solution of the problem (3.1)*. Then for an arbitrary sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that $\varepsilon_j \to 0$ as $j \to \infty$, we have the following convergence result*

$$\mathbf{u}^{\varepsilon_j} \rightharpoonup \mathbf{u} \textit{ in } W_0^{1,1}(\Omega; \mathbb{R}^N),$$

where $\mathbf{u}^{\varepsilon_j}$ is the sequence of solutions solving (3.1) *with $\varepsilon = \varepsilon_j$ and $\mathbf{u}$ is a unique solution to* (3.2)*, provided that one of the following conditions holds:*

(C1) The set Ω is star–shaped.

(C2) The set Ω is Lipschitz and we have the single equation, i.e., $N = 1$.

3.2 Preliminaries

Since we deal with rather general function spaces and growth conditions imposed on the nonlinearity $\mathbf{A}$, we recall in the appendix several facts about the corresponding function spaces and their properties and we refer the interested reader to [12, 13] for more details. In the forthcoming section we concentrate already on the specific spaces related to the considered problem.

3.2.1 Function spaces related to the problem

Once the general Musielak–Orlicz spaces are introduced in the appendix, we now focus on the specific spaces related to the problem. Here, Ω will be the Lipschitz domain and Y the set $(0,1)^d$. For $M : Y \times \mathbb{R}^{d\times N} \to \mathbb{R}_+$ the $\mathcal{N}$-function, we use the subscript y to underline the role of y for the spaces $L^{M_y}(\Omega \times Y; \mathbb{R}^{d\times N})$ and similarly E^{M_y} endowed with the norm

$$\|\mathbf{v}\|_{L^{M_y}} = \|\mathbf{v}\|_{E^{M_y}} := \inf\left\{\lambda > 0 : \int_{\Omega}\int_{Y} M\left(y, \frac{\mathbf{v}(x,y)}{\lambda}\right) \mathrm{d}y\mathrm{d}x \le 1\right\}.$$

We note that whenever a function dependent on a variable from Y appears, it is always $Y-$periodic although the $Y-$periodicity might not be stressed. We further denote the spaces of smooth periodic or compactly supported functions as

$$C^{\infty}_{per}(Y; \mathbb{R}^N) := \{\mathbf{v} \in C^{\infty}(\mathbb{R}^d; \mathbb{R}^N) : \mathbf{v} \text{ is } Y\text{-periodic}\},$$
$$C^{\infty}_{c}(\Omega; \mathbb{R}^N) := \{\mathbf{v} \in C^{\infty}(\mathbb{R}^d; \mathbb{R}^N) : \operatorname{supp} \mathbf{v} \text{ is compact in } \Omega\}$$

and naturally also the corresponding Bochner spaces $C^{\infty}_c\left(\Omega; C^{\infty}_{per}(Y)\right)$. Then the standard Sobolev spaces are defined as

$$W_0^{1,1}(\Omega; \mathbb{R}^N) := \overline{\{\mathbf{v} \in C^{\infty}_c(\Omega; \mathbb{R}^N)\}}^{\|\cdot\|_{1,1}},$$
$$W_{per}^{1,1}(Y; \mathbb{R}^N) := \overline{\{\mathbf{v} \in C^{\infty}_{per}(Y; \mathbb{R}^N); \int_Y \mathbf{v} = 0\}}^{\|\cdot\|_{1,1}}.$$

Moreover, due to the Poincaré inequality, we always choose an equivalent norm on $W_0^{1,1}$ and $W_{per}^{1,1}$ as $\|\mathbf{v}\|_{1,1} := \|\nabla \mathbf{v}\|_1$. We shall define the Sobolev–Musielak–Orlicz space

$$W^1_{per}E^M(Y; \mathbb{R}^N) := \overline{\{\mathbf{v} \in C^{\infty}_{per}(Y; \mathbb{R}^N); \int_Y \mathbf{v} = 0\}}^{\|\cdot\|_{W^1_{per}(Y;\mathbb{R}^N)}}.$$

where $\|\mathbf{v}\|_{W^1_{per}L^M(Y)} := \|\nabla \mathbf{v}\|_{L^M(Y)}$ and the following spaces

$$V_0^M := \left\{ \mathbf{v} \in W_0^{1,1}(\Omega; \mathbb{R}^N) : \nabla \mathbf{v} \in L^M(\Omega; \mathbb{R}^{d\times N}) \right\},$$
$$V_{per}^M := \left\{ \mathbf{v} \in W_{per}^{1,1}(Y; \mathbb{R}^N) : \nabla \mathbf{v} \in L^M(Y; \mathbb{R}^{d\times N}) \right\}.$$

In addition, we utilize the following closed subspace of $E^M(Y; \mathbb{R}^{d\times N})$ and its annihilator

$$G(Y) := \{\nabla \mathbf{w} : \mathbf{w} \in W^1_{per}E^M(Y; \mathbb{R}^N)\},$$
$$G^{\perp}(Y) := \{\mathbf{W}^* \in L^{M^*}_{per}(Y; \mathbb{R}^{d\times N}) : \int_Y \mathbf{W}^*(y) \cdot \mathbf{W}(y)\mathrm{d}y = 0 \text{ for all } \mathbf{W} \in G(Y)\}.$$

In our situation, the $\mathcal{N}$–function possesses the property of log-Hölder continuity and the following theorem ensures the approximation of every function from V^M_{per} and V^M_0 in the sense of modular topology by functions that are smooth and periodic, with compact support respectively.

Theorem 3.2.1 *Let $\Sigma \subset \mathbb{R}^d$ be a bounded domain and an $\mathcal{N}$–function M satisfy (M2) and (M3) with Σ replacing Y. Then we have the following modular convergence results:*

1) Let Σ be Lipschitz then for any scalar function $v \in V_0^M \cap L^\infty(\Sigma)$ there exists a sequence $\{v^k\}_{k=1}^\infty \subset C_c^\infty(\Sigma)$ such that $\nabla v^k \xrightarrow{M} \nabla v$.

2) Let Σ be star–shaped then for any function $\mathbf{v} \in V_0^M$ there exists a sequence $\{\mathbf{v}^k\}_{k=1}^\infty \subset C_c^\infty\left(\Sigma; \mathbb{R}^N\right)$ such that $\nabla \mathbf{v}^k \xrightarrow{M} \nabla \mathbf{v}$.

3) Let $\Sigma = Y$ then for any function $\mathbf{v} \in V_{per}^M$ there exists a sequence $\{\mathbf{v}^k\}_{k=1}^\infty \subset C_{per}^\infty\left(\Sigma; \mathbb{R}^N\right)$ such that $\nabla \mathbf{v}^k \xrightarrow{M} \nabla \mathbf{v}$.

Proof. The assertion 1) is [5, Theorem 2.2]. To prove the assertions 2) and 3) one follows the common scheme:

1. Construction of the mollification $\nabla \mathbf{v}^{\delta_k}$ of $\nabla \mathbf{v}$.
2. Showing that the family $\{\nabla \mathbf{v}^{\delta_k}\}_{k=1}^\infty$ is uniformly bounded on $L^M(\Sigma; \mathbb{R}^{d\times N})$.
3. Showing that $\nabla \mathbf{v}^{\delta_k} \xrightarrow{M} \nabla \mathbf{v}$.

The detailed proof can be performed by repeating Steps 1-3 from the proof of [5, Theorem 2.2]. □

We state several technical lemmas.

Lemma 3.2.2 *[6, Lemma 2.1.] Let $N \geq 1$, M be an $\mathcal{N}$–function and $\{\mathbf{v}^k\}_{k=1}^\infty$ be a sequence of measurable $\mathbb{R}^N$–valued functions on Σ. Then $\mathbf{v}^k \xrightarrow{M} \mathbf{v}$ in $L^M(\Sigma; \mathbb{R}^N)$ if and only if $\mathbf{v}^k \to \mathbf{v}$ in measure and there exists some $\lambda > 0$ such that $\{M(\cdot, \lambda \mathbf{v}^k)\}_{k=1}^\infty$ is uniformly integrable, i.e.,*

$$\lim_{R\to\infty} \left(\sup_{k\in\mathbb{N}} \int_{\{x: |M(x,\lambda \mathbf{v}^k(x))| > R\}} M(x, \lambda \mathbf{v}^k(x))\mathrm{d}x \right) = 0.$$

Lemma 3.2.3 *[6, Lemma 2.2.] Let M be an $\mathcal{N}$–function and assume that there is $c > 0$ such that $\int_\Sigma M(x, \mathbf{v}^k)\mathrm{d}x \le c$ for all $k \in \mathbb{N}$. Then $\{\mathbf{v}^k\}_{k=1}^\infty$ is uniformly integrable.*

Lemma 3.2.4 *Let M be an $\mathcal{N}$–function and Σ be a bounded domain. Then for any $v \in V_0^M(\Sigma)$ we have $\nabla T_k(v) \xrightarrow{M} \nabla v$ as $k \to \infty$, where*

$$T_k(v) = \begin{cases} v & \text{if } |v| \le k \\ k\frac{v}{|v|} & \text{if } |v| > k. \end{cases}$$

Proof. Clearly, $\nabla T_k(u) \to \nabla u$ a.e. in Σ, which has finite measure. Hence the sequence $\{\nabla T_k(u)\}_{k=1}^\infty$ converges to ∇u in measure. Moreover, as $M(\cdot, \nabla T_k(u)) \le M(\cdot, \nabla u)$ a.e. in Σ by the definition of T_k, Lemma 3.2.3 implies that $\{\nabla T_k(u)\}_{k=1}^\infty$ is uniformly integrable. These two facts are equivalent to $\nabla T_k(u) \xrightarrow{M} \nabla u$ according to Lemma 3.2.2. □

3.2.2 Standard tools used for homogenization

Lemma 3.2.5 *Let X be a Banach space, V be a subspace of X, g be a closed, convex functional on X that is continuous at some $x \in V$. Then*

$$\inf_{x\in V}\{g(x) - \langle \eta, x\rangle\} + \inf_{\xi \in V^\perp} g^*(\eta + \xi) = 0 \tag{3.7}$$

for all $\eta \in X^*$.

Proof. One deduces by definition of a convex conjugate that

$$\forall \xi \in X^* : (g-\eta)^*(\xi) = \sup_{x\in X}\{\langle \eta + \xi, x\rangle - g(x)\} = g^*(\eta + \xi). \tag{3.8}$$

According to [7, Theorem 14.2]

$$\inf_{x\in V} A(x) + \inf_{x^*\in V^\perp} A^*(x^*) = 0$$

for a closed, convex functional A that is continuous at some $x \in V$. We set $A(x) := (g-\eta)(x)$ and the expression for A^* determined by (3.8) in the latter equality to conclude (3.7). □

The rest of this section is devoted to the introduction of the two–scale convergence via periodic unfolding. This approach allows to represent the weak two–scale convergence by means of the standard weak convergence in a Lebesgue space on the product $\Omega \times Y$, details for the case of L^p spaces can be found in [15]. In the same manner the strong two–scale convergence is introduced. Since function spaces, which we are working with, provide only the weak* compactness of bounded sets, we introduce the weak two–scale compactness in the weak* sense. However, it turns out that this notion of convergence and some of its properties are sufficient for our purposes. We define functions $n : \mathbb{R} \to \mathbb{Z}$ and $N : \mathbb{R}^d \to \mathbb{Z}^d$ as

$$n(t) = \max\{n \in \mathbb{Z} : n \le t\}\ \forall t \in \mathbb{R}, N(x) = (n(x_1), \ldots, n(x_d))\ \forall x \in \mathbb{R}^d.$$

Then we have for any $x \in \mathbb{R}^d, \varepsilon > 0$, a two–scale decomposition $x = \varepsilon\left(N\left(\frac{x}{\varepsilon}\right) + R\left(\frac{x}{\varepsilon}\right)\right)$. We also define for any $\varepsilon > 0$ a two–scale composition function $S_\varepsilon : \mathbb{R}^d \times Y \to \mathbb{R}^d$ as $S_\varepsilon(x, y) := \varepsilon\left(N\left(\frac{x}{\varepsilon}\right) + y\right)$. It follows immediately that

$$S_\varepsilon(x, y) \to x \text{ uniformly in } \mathbb{R}^d \times Y \text{ as } \varepsilon \to 0 \tag{3.9}$$

since $S_\varepsilon(x,y) = x + \varepsilon\left(y - R\left(\frac{x}{\varepsilon}\right)\right)$. In the rest of the section we assume that m is an $\mathcal{N}$–function.
We say that a sequence of functions $\{v^\varepsilon\} \subset L^m(\mathbb{R}^d)$

1. converges to v^0 weakly* two–scale in $L^m(\mathbb{R}^d \times Y)$, $v^\varepsilon \xrightharpoonup{2-s}{}^* v^0$, if $v^\varepsilon \circ S_\varepsilon$ converges to v^0 weakly* in $L^m(\mathbb{R}^d \times Y)$,

2. converges to v^0 strongly two–scale in $E^m(\mathbb{R}^d \times Y)$, $v^\varepsilon \xrightarrow{2-s} v^0$, if $v^\varepsilon \circ S_\varepsilon$ converges to v^0 strongly in $E^m(\mathbb{R}^d \times Y)$.

We define two–scale convergence in $L^m(\Omega \times Y)$ as two–scale convergence in $L^m(\mathbb{R}^d \times Y)$ for functions extended by zero to $\mathbb{R}^d \backslash \Omega$. The following lemma will be utilized to express properties of two–scale convergence in terms of single-scale convergence.

Lemma 3.2.6 *[15, Lemma 1.1] Let g be measurable with respect to a $\sigma-$algebra generated by the product of the $\sigma-$algebra of all Lebesgue–measurable subsets of $\mathbb{R}^d$ and the $\sigma-$algebra of all Borel–measurable subsets of Y. Assume in addition that $g \in L^1(\mathbb{R}^d; L^\infty_{per}(Y))$ and extended it by $Y-$periodicity to $\mathbb{R}^d$ for a.a. $x \in \mathbb{R}^d$. Then, for any $\varepsilon > 0$, the function $(x,y) \mapsto g(S_\varepsilon(x,y),y)$ is integrable and*

$$\int_{\mathbb{R}^d} g\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\mathbb{R}^d} \int_Y g(S_\varepsilon(x,y),y) \mathrm{d}y \mathrm{d}x.$$

Several useful properties of the two–scale convergence are summarized in the following lemma.

Lemma 3.2.7 *Assume that $m : [0,\infty) \to [0,\infty)$ is an $\mathcal{N}$–function.*

(i) Let $v : \Omega \times Y \to \mathbb{R}$ be Carathéodory, $v \in E^m(\Omega \times Y)$, v be $Y-$periodic, define $v^\varepsilon(x) = v(x, \frac{x}{\varepsilon})$ for $x \in \Omega$. Then $v^\varepsilon \xrightarrow{2-s} v$ in $E^m(\Omega \times Y)$ as $\varepsilon \to 0$.

(ii) Let $v^\varepsilon \xrightharpoonup{2-s}{}^* v^0$ in $L^m(\Omega \times Y)$ then $v^\varepsilon \rightharpoonup^* \int_Y v^0(\cdot, y)\mathrm{d}y$ in $L^m(\Omega)$.

(iii) Let $v^\varepsilon \xrightharpoonup{2-s}{}^* v^0$ in $L^m(\Omega \times Y)$ and $w^\varepsilon \xrightarrow{2-s} w^0$ in $E^{m^*}(\Omega \times Y)$ then $\int_\Omega v^\varepsilon w^\varepsilon \to \int_\Omega \int_Y v^0 w^0$.

(iv) Let $v^\varepsilon \xrightharpoonup{2-s}{}^* v^0$ in $L^m(\Omega \times Y)$ then for any $\psi \in C^\infty_c\left(\Omega; C^\infty_{per}(Y)\right)$

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega \int_Y v^0(x,y)\psi(x,y)\mathrm{d}y\mathrm{d}x.$$

(v) Let $\{v^\varepsilon\}$ be a bounded sequence in $L^m(\Omega)$. Then there is $v^0 \in L^m(\Omega \times Y)$ and a sequence $\varepsilon_k \to 0$ as $k \to \infty$ such that $v^{\varepsilon_k} \xrightharpoonup{2-s}{}^* v^0$ in $L^m(\Omega \times Y)$ as $k \to \infty$.

(vi) Let $\{v^\varepsilon\} \subset V^m_0$ be such that

$$\begin{aligned} v^\varepsilon &\rightharpoonup^* v \quad \text{in } L^m(\Omega), \\ \nabla v^\varepsilon &\rightharpoonup^* \nabla v \ \text{ in } L^m(\Omega; \mathbb{R}^d). \end{aligned} \tag{3.10}$$

Then $v^\varepsilon \xrightharpoonup{2-s}{}^* v$ in $L^m(\Omega \times Y)$ and there is a sequence $\varepsilon_k \to 0$ as $k \to \infty$ and $\mathbf{v} \in L^m(\Omega \times Y; \mathbb{R}^d)$ such that $\nabla v^{\varepsilon_k} \xrightharpoonup{2-s}{}^* \nabla v + \mathbf{v}$ in $L^m(\Omega \times Y; \mathbb{R}^d)$ as $k \to \infty$ and

$$\int_Y \mathbf{v}(x,y) \cdot \boldsymbol{\psi}(y)\mathrm{d}y = 0$$

for a.a. $x \in \Omega$ and any $\boldsymbol{\psi} \in C^\infty_{per}(Y; \mathbb{R}^d), \operatorname{div} \boldsymbol{\psi} = 0$ in Y.

(vii) Let $\Phi : \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ *satisfy:*

(a) Φ *is Carathéodory,*

(b) $\Phi(\cdot, \boldsymbol{\xi})$ *is* $Y-$*periodic for any* $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$, $\Phi(y, \cdot)$ *is convex for almost all* $y \in Y$,

(c) $\Phi \geq 0$, $\Phi(\cdot, 0) = 0$.

Then for any sequence $\mathbf{U}^\varepsilon \xrightarrow{2-s}{}^{*} \mathbf{U}$ *in* $L^m(\Omega \times Y; \mathbb{R}^{d\times N})$ *it follows that*

$$\liminf_{\varepsilon\to\infty} \int_\Omega \Phi\left(\frac{x}{\varepsilon}, \mathbf{U}^\varepsilon(x)\right) \mathrm{d}x \geq \int_{\Omega\times Y} \Phi(y, \mathbf{U}(x,y))\mathrm{d}y\mathrm{d}x.$$

Proof. By Lemma 3.2.6 we have for v extended by zero on $(\mathbb{R}^d \setminus \Omega) \times Y$ that

$$v^\varepsilon(x) = v\left(x, \frac{x}{\varepsilon}\right) = v\left(S_\varepsilon(x,y), \frac{S_\varepsilon(x,y)}{\varepsilon}\right) = v\left(\varepsilon\left(N\left(\frac{x}{\varepsilon}\right) + y\right), y\right)$$

is an integrable function of (x, y). According to [8, Theorem 3.15.5] $v \in E^m(\Omega \times Y)$ is $m-$mean continuous, i.e., for given $\eta > 0$ there exists $\kappa > 0$ such that $\|v_h - v\|_{L^m} \leq \eta$ for $h = (h^1, h^2) \in \mathbb{R}^{2d}$ with $|h| < \kappa$, where

$$v_h(x,y) := \begin{cases} v(x + h^1, y + h^2) & \text{if } (x + h^1, y + h^2) \in \Omega \times Y, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for fixed $\eta > 0$ we find $\kappa > 0$ such that $\|v_h - v\|_{L^m(\Omega\times Y)} < \eta$ for all $|h| < \kappa$. Due to (3.9) we find $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ $\left\|\varepsilon(N(\frac{x}{\varepsilon}) + y) - x\right\|_{L^\infty(\mathbb{R}^d\times Y)} < \kappa$. For fixed η we found $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ we have $\|v^\varepsilon - v\|_{L^m(\Omega\times Y)} < \kappa$, which concludes (i).

We obtain (ii) once we use in the definition of the weak* two–scale convergence in $L^m(\Omega \times Y)$ test functions, which are independent of y-variable.

Assertion (iii) follows immediately from the definition of the weak* two–scale convergence in $L^m(\Omega)$, strong two–scale convergence in $E^{m^*}(\Omega)$ and Lemma 3.2.6 applied to the function $g = v^\varepsilon w^\varepsilon$ independent of y.

To show assertion (iv) we fix a weakly* two–scale convergent sequence $\{v^{\varepsilon_k}\}_{k=1}^\infty \subset L^m(\Omega)$ with a limit $v^0 \in L^m(\Omega \times Y)$ and $\psi \in C_c^\infty(\Omega; C_{per}^\infty(Y))$. Then we have $v^{\varepsilon_k}(x)\psi(x,y) \in L^1(\mathbb{R}^d; L_{per}^\infty(Y))$ provided that we set $v^{\varepsilon_k} = 0$ in $\mathbb{R}^d \setminus \Omega$, $\psi = 0$ in $(\mathbb{R}^d \setminus \Omega) \times Y$. Therefore by Lemma 3.2.6 we get

$$\int_\Omega v^{\varepsilon_k}(x)\psi\left(x, \frac{x}{\varepsilon_k}\right) \mathrm{d}x = \int_\Omega \int_Y v^{\varepsilon_k}(S_{\varepsilon_k}(x,y))\psi(S_{\varepsilon_k}(x,y), y)\mathrm{d}y\mathrm{d}x.$$

Combining this with the convergence results $v^{\varepsilon_k} \xrightarrow{2-s}{}^{*} v^0$ in $L^m(\Omega \times Y)$ and $\psi\left(x, \frac{x}{\varepsilon_k}\right) \xrightarrow{2-s} \psi(x,y)$ in $E^{m^*}(\Omega \times Y)$ as $k \to \infty$, which follows by assertion (i), we infer

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega v^{\varepsilon_k}(x)\psi\left(x, \frac{x}{\varepsilon_k}\right) \mathrm{d}x &= \lim_{k\to} \int_\Omega \int_Y v^{\varepsilon_k}(S_{\varepsilon_k}(x,y))\psi\left(S_{\varepsilon_k}(x,y), y\right) \mathrm{d}y\mathrm{d}x \\ &= \int_\Omega \int_Y v^0(x,y)\psi(x,y)\mathrm{d}y\mathrm{d}x \end{aligned}$$

by assertion (iii).
In order to show (v), we first realize that for any $\{v^\varepsilon\}$ bounded in $L^m(\Omega)$ Lemma 3.2.6 applied to a function $g = m\left(\frac{|v^\varepsilon|}{\lambda}\right)$ independent of y implies

$$c \geq \int_\Omega m\left(\frac{|v^\varepsilon(x)|}{\lambda}\right) \mathrm{d}x = \int_\Omega \int_Y m\left(\frac{|v^\varepsilon(S_\varepsilon(x,y))|}{\lambda}\right) \mathrm{d}y \mathrm{d}x$$

for some $\lambda > 0$. We deduce the existence of a selected subsequence $\{v^{\varepsilon_k} \circ S_{\varepsilon_k}\} \subset \{v^\varepsilon \circ S_\varepsilon\}$ and the limit function $v^0 \in L^m(\Omega \times Y)$ such that $v^{\varepsilon_k} \circ S_{\varepsilon_k} \xrightarrow{\ \ }^* v^0$ in $L^m(\Omega \times Y)$ as $k \to 0$ by the Banach-Alaoglu theorem for spaces with a separable predual. We recall that $L^m(\Omega \times Y) = \left(E^{m^*}(\Omega \times Y)\right)^*$. Assertion (v) obviously follows by the definition of weak* two–scale convergence.

In order to show (vi) we observe first that $\{v^\varepsilon\}_{\varepsilon \in (0,1)}$ is bounded in $L^m(\Omega)$. Thus by (v) there is a sequence $\varepsilon_k \to 0$ as $k \to \infty$ and $v^0 \in L^m(\Omega \times Y)$ such that $v^{\varepsilon_k} \xrightarrow{2-s}^* v^0$ in $L^m(\Omega \times Y)$. Then (iii) implies for all $\varphi \in C_c^\infty(\Omega, C_{per}^\infty(Y)^d)$ that

$$\begin{aligned}
0 &= -\lim_{k\to\infty} \varepsilon_k \int_\Omega \nabla v^{\varepsilon_k}(x) \cdot \left[\varphi\left(x, \frac{x}{\varepsilon_k}\right)\right] \mathrm{d}x = \lim_{k\to\infty} \varepsilon_k \int_\Omega v^{\varepsilon_k}(x) \operatorname{div} \left[\varphi\left(x, \frac{x}{\varepsilon_k}\right)\right] \mathrm{d}x \\
&= \lim_{k\to\infty} \int_\Omega \varepsilon_k v^{\varepsilon_k}(x) \operatorname{div}_x \varphi\left(x, \frac{x}{\varepsilon_k}\right) + v^{\varepsilon_k}(x) \operatorname{div}_y \varphi\left(x, \frac{x}{\varepsilon_k}\right) \mathrm{d}x \\
&= \int_\Omega \int_Y v^0(x,y) \operatorname{div}_y \varphi(x,y) \mathrm{d}y \mathrm{d}x,
\end{aligned}$$

which implies that v^0 is independent of y. As $v = \int_Y v^0$ by (ii), we see that for any weakly* two–scale convergent subsequence of $\{v^\varepsilon\}$ the limit is v. Hence v is the weak* two–scale limit of the entire sequence $\{v^\varepsilon\}$. Applying (iv) on the sequence $\{\nabla v^{\varepsilon_k}\}$ we get the subsequence $\{v^{\varepsilon_k}\}$ (that will not be relabeled) and $\mathbf{w} \in L^m(\Omega \times Y; \mathbb{R}^d)$ such that $\nabla v^{\varepsilon_k} \xrightarrow{2-s}^* \mathbf{w}$ in $L^m(\Omega \times Y; \mathbb{R}^d)$ as $k \to \infty$. Let us choose $z \in C_c^\infty(\Omega)$ and $\boldsymbol{\psi} \in C_{per}^\infty(Y; \mathbb{R}^d)$ with $\operatorname{div}_y \boldsymbol{\psi} = 0$ in Y. Then

$$\lim_{k\to\infty} \int_\Omega \nabla v^{\varepsilon_k}(x) \cdot z(x) \boldsymbol{\psi}\left(\frac{x}{\varepsilon_k}\right) \mathrm{d}x = \int_\Omega \int_Y \mathbf{w}(x,y) \cdot z(x) \boldsymbol{\psi}(y) \mathrm{d}y \mathrm{d}x$$

whereas the integration by parts yields

$$\begin{aligned}
\lim_{k\to\infty} \int_\Omega \nabla v^{\varepsilon_k}(x) \cdot z(x) \boldsymbol{\psi}\left(\frac{x}{\varepsilon_k}\right) \mathrm{d}x &= -\lim_{k\to\infty} \int_\Omega v^{\varepsilon_k}(x) \nabla z(x) \cdot \boldsymbol{\psi}\left(\frac{x}{\varepsilon_k}\right) \mathrm{d}x \\
= -\int_\Omega \int_Y v(x) \nabla z(x) \cdot \boldsymbol{\psi}(y) \mathrm{d}y \mathrm{d}x &= \int_\Omega \int_Y \nabla v(x) \cdot z(x) \boldsymbol{\psi}(y) \mathrm{d}y \mathrm{d}x.
\end{aligned}$$

Hence the function $\mathbf{v} = \mathbf{w} - \nabla v$ has all required properties.
Let us show (vii). It follows from Lemma 3.2.6 and Lemma 3.4.3 that for $\mathbf{U}^\varepsilon, \mathbf{U}$ extended by zero in $\mathbb{R}^d \setminus \Omega$

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_\Omega \Phi\left(\frac{x}{\varepsilon}, \mathbf{U}^\varepsilon(x)\right) \mathrm{d}x &= \liminf_{\varepsilon\to 0} \int_{\Omega\times Y} \Phi\left(y, \mathbf{U}^\varepsilon(S_\varepsilon(x,y)\right) \mathrm{d}x \mathrm{d}y \\
&\geq \int_{\Omega\times Y} \Phi(y, \mathbf{U}(x,y)) \mathrm{d}x \mathrm{d}y
\end{aligned}$$

since $\mathbf{U}^\varepsilon \xrightarrow{2-s}^* \mathbf{U}$ in $L^m(\Omega \times Y; \mathbb{R}^{d\times N})$ implies $\mathbf{U}^\varepsilon \longrightarrow \mathbf{U}$ in $L^1(\Omega \times Y; \mathbb{R}^{d\times N})$. Hence we conclude (vii). □

3.2.3 Properties of the mapping $\hat{\mathbf{A}}$

Let us define an operator $\hat{\mathbf{A}}\colon \mathbb{R}^{d\times N}\to\mathbb{R}^{d\times N}$ as

$$\hat{\mathbf{A}}(\boldsymbol{\xi}) = \int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w})\mathrm{d}y \tag{3.11}$$

where the $Y-$periodic function $\mathbf{w}$ is a solution of the following cell problem

$$\operatorname{div}\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}) = 0 \text{ in } Y. \tag{3.12}$$

In what follows, we show that this definition is meaningful and derive the essential properties of the operator $\hat{\mathbf{A}}$ needed later for the homogenization problem.

Lemma 3.2.8 *Let $Y=(0,1)^d$, the operator $\mathbf{A}$ satisfy (A1)–(A4) and the $\mathcal{N}$–function M satisfy (M1)–(M3). Then the problem* (3.12) *admits a unique weak solution $\mathbf{w}_{\boldsymbol{\xi}}\in V^M_{per}$ satisfying for all $\boldsymbol{\varphi}\in V^M_{per}$*

$$\int_Y \mathbf{A}\left(y,\boldsymbol{\xi}+\nabla\mathbf{w}_{\boldsymbol{\xi}}(y)\right)\cdot\nabla\boldsymbol{\varphi}(y)\mathrm{d}y = 0. \tag{3.13}$$

Moreover,

$$\boldsymbol{\xi}^k\to\boldsymbol{\xi} \text{ in } \mathbb{R}^{d\times N} \text{ implies } \mathbf{A}(\cdot,\boldsymbol{\xi}^k+\nabla\mathbf{w}^k) \overset{*}{\rightharpoonup} \mathbf{A}(\cdot,\boldsymbol{\xi}+\nabla\mathbf{w}) \text{ in } L^{M^*}(Y;\mathbb{R}^{d\times N}), \tag{3.14}$$

where $\mathbf{w}^k$ is a solution of the cell problem corresponding to $\boldsymbol{\xi}^k$ and $\mathbf{w}$ to $\boldsymbol{\xi}$.

Proof. We omit existence and uniqueness proofs since it suffices to modify straightforwardly the methods used in the proofs of Theorem 3.4.7 in the appendix. Notice here that we do not have any restriction on the geometry since we deal only with spatially periodic setting.

Let us assume that $\{\boldsymbol{\xi}^k\}_{k=1}^\infty$ is such that $\boldsymbol{\xi}^k\to\tilde{\boldsymbol{\xi}}$ in $\mathbb{R}^{d\times N}$ as $k\to\infty$. We denote by $\mathbf{w}^k$ the solution of the cell problem corresponding to $\boldsymbol{\xi}^k$ and by $\tilde{\mathbf{w}}$ the solution corresponding to $\tilde{\boldsymbol{\xi}}$. We also denote $\mathbf{Z}^k(y):=\mathbf{A}(y,\boldsymbol{\xi}^k+\nabla\mathbf{w}^k(y))$. First, we show that

$$\int_Y M(y,\boldsymbol{\xi}^k+\nabla\mathbf{w}^k(y)) + M^*(y,\mathbf{Z}^k(y))\mathrm{d}y \le c. \tag{3.15}$$

Since $\mathbf{w}_k$ is always an admissible test function in (3.13) for $\boldsymbol{\xi}:=\boldsymbol{\xi}^k$, we directly obtain

$$\int_Y \mathbf{Z}^k(y)\cdot\nabla\mathbf{w}^k(y)\mathrm{d}y = 0. \tag{3.16}$$

Hence, using (A3), (3.16) and the Young inequality yields (assuming without loss of generality that $c\le 1$)

$$\begin{aligned} c\int_Y M^*(y,\mathbf{Z}^k(y)) + M(y,\boldsymbol{\xi}^k+\nabla\mathbf{w}^k)\mathrm{d}y &\le \int_Y \mathbf{Z}^k\cdot(\boldsymbol{\xi}^k+\nabla\mathbf{w}^k)\mathrm{d}y = \int_Y \mathbf{Z}^k\cdot\boldsymbol{\xi}^k\mathrm{d}y \\ &\le \frac{c}{2}\int_Y M^*(y,\mathbf{Z}^k)\mathrm{d}y + \int_Y M\left(y,\frac{2}{c}\boldsymbol{\xi}^k\right)\mathrm{d}y. \end{aligned}$$

The second integral on the right hand side is finite due to (M2) as $\{\boldsymbol{\xi}^k\}_{k=1}^\infty$ is bounded. Without loss of generality, we can assume that

$$\begin{aligned} \nabla\mathbf{w}^k &\overset{*}{\rightharpoonup} \nabla\bar{\mathbf{w}} \text{ in } L^M(Y;\mathbb{R}^N), \\ \mathbf{Z}^k &\overset{*}{\rightharpoonup} \mathbf{Z} \quad \text{in } L^{M^*}_{per}(Y;\mathbb{R}^{d\times N}) \end{aligned} \tag{3.17}$$

as $k \to \infty$. We show that $\bar{\mathbf{w}} = \tilde{\mathbf{w}}$ and $\mathbf{Z} = \mathbf{A}(\cdot, \tilde{\boldsymbol{\xi}} + \nabla \tilde{\mathbf{w}})$. We immediately obtain that

$$\lim_{k\to\infty} \int_Y \mathbf{Z}^k(y) \cdot \boldsymbol{\xi}^k \mathrm{d}y = \int_Y \mathbf{Z}(y) \cdot \tilde{\boldsymbol{\xi}} \mathrm{d}y. \tag{3.18}$$

Further, we also use the following identity

$$\int_Y \mathbf{Z}(y) \cdot \nabla \boldsymbol{\varphi}(y) \mathrm{d}y = 0 \qquad \text{for all } \boldsymbol{\varphi} \in V^M_{per}. \tag{3.19}$$

In order to show it, we observe that from (3.17) and the definition of $\mathbf{Z}^k$ the identity (3.19) follows for all $\boldsymbol{\varphi} \in W^1_{per}E^M(Y; \mathbb{R}^d)$. Since M satisfies (3.3), we can use the density of smooth functions in the modular topology, see Step 5 of Theorem 3.4.7, to deduce (3.19) for all $\boldsymbol{\varphi} \in V^M_{per}$. From (3.16), (3.18) and (3.19) we infer

$$\lim_{k\to\infty} \int_Y \mathbf{Z}^k(y) \cdot (\boldsymbol{\xi}^k + \nabla \mathbf{w}^k(y)) \mathrm{d}y = \int_Y \mathbf{Z}(y) \cdot (\tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}}(y)) \mathrm{d}y. \tag{3.20}$$

Since $\mathbf{A}(x, 0) = 0$ and $\mathbf{A}$ is monotone, the negative part of $\mathbf{Z}^k \cdot (\boldsymbol{\xi}^k + \nabla \mathbf{w}^k)$ is trivially weakly compact in $L^1(Y)$. Due to Lemma 3.4.6 and (3.20) we get

$$\begin{aligned} \int_Y \int_{\mathbb{R}^{d\times N}} \mathbf{A}(y, \boldsymbol{\zeta}) \cdot \boldsymbol{\zeta} \mathrm{d}\nu_y(\boldsymbol{\zeta}) \mathrm{d}y &\le \liminf_{k\to\infty} \int_Y \mathbf{Z}^k(y) \cdot (\boldsymbol{\xi}^k + \nabla \mathbf{w}^k(y)) \mathrm{d}y \\ &= \int_\Omega \mathbf{Z}(y) \cdot (\tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}}(y)) \mathrm{d}y, \end{aligned} \tag{3.21}$$

where ν_y is the Young measure generated by $\{\boldsymbol{\xi}^k + \nabla \mathbf{w}^k\}_{k=1}^\infty$. The monotonicity of $\mathbf{A}$ yields

$$\int_Y \int_{\mathbb{R}^{d\times N}} h(y, \boldsymbol{\zeta}) \mathrm{d}\nu_y(\boldsymbol{\zeta}) \mathrm{d}y \ge 0 \tag{3.22}$$

for $h(y, \boldsymbol{\zeta}) := (\mathbf{A}(y, \boldsymbol{\zeta}) - \mathbf{A}(y, \tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}})) \cdot (\boldsymbol{\zeta} - \tilde{\boldsymbol{\xi}} - \nabla \bar{\mathbf{w}})$. Since $\{\boldsymbol{\xi}^k + \nabla \mathbf{w}^k\}_{k=1}^\infty$ and $\{\mathbf{A}(\cdot, \boldsymbol{\xi}^k + \nabla \mathbf{w}^k)\}_{k=1}^\infty$ are weakly relatively compact due to (3.15) and $\mathbf{A}$ is a Carathéodory function, Lemma 3.4.6 implies

$$\begin{aligned} \tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}} &= \int_{\mathbb{R}^{d\times N}} \boldsymbol{\zeta} \mathrm{d}\nu_y(\boldsymbol{\zeta}) \qquad \text{a.e. in } Y, \\ \mathbf{Z} &= \int_{\mathbb{R}^{d\times N}} \mathbf{A}(\cdot, \boldsymbol{\zeta}) \mathrm{d}\nu_y(\boldsymbol{\zeta}) \text{ a.e. in } Y. \end{aligned} \tag{3.23}$$

Then we get

$$\int_Y \int_{\mathbb{R}^{d\times N}} h(y, \boldsymbol{\zeta}) \mathrm{d}\nu_y(\boldsymbol{\zeta}) \mathrm{d}y = \int_Y \int_{\mathbb{R}^{d\times N}} \mathbf{A}(y, \boldsymbol{\zeta}) \cdot \boldsymbol{\zeta} \mathrm{d}\nu_y(\boldsymbol{\zeta}) \mathrm{d}y - \int_Y \mathbf{Z} \cdot (\tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}}) \le 0$$

by (3.21). Combining this with (3.22) we obtain $\int_{\mathbb{R}^{d\times N}} h(y, \boldsymbol{\zeta}) \mathrm{d}\nu_y(\boldsymbol{\zeta}) = 0$ for a.a. $y \in Y$. As ν_y is a probability measure and $\mathbf{A}$ is strictly monotone, we infer that $\mathrm{supp}\{\nu_y\} = \{\tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}}\}$ a.e. in Y. Thus we have $\nu_y = \delta_{\tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}}(y)}$ a.e. in Y. Inserting this into $(3.23)_2$ yields $\mathbf{Z}(y) = \mathbf{A}(y, \tilde{\boldsymbol{\xi}} + \nabla \bar{\mathbf{w}}(y))$. Hence we infer due to (3.19) that $\bar{\mathbf{w}}$ is a weak solution to (3.12) corresponding to $\tilde{\boldsymbol{\xi}}$. Since this solution is unique, we obtain $\bar{\mathbf{w}} = \tilde{\mathbf{w}}$. Up to now we have shown that from $\{\mathbf{Z}^k\}_{k=1}^\infty$ there can be extracted a subsequence that converges weakly* to $\mathbf{A}(\cdot, \tilde{\boldsymbol{\xi}} + \nabla \tilde{\mathbf{w}})$ in $L^{M^*}(Y; \mathbb{R}^{d\times N})$. The uniqueness of this limit implies that the whole sequence $\{\mathbf{Z}^k\}_{k=1}^\infty$ must converge to $\mathbf{A}(\cdot, \tilde{\boldsymbol{\xi}} + \nabla \tilde{\mathbf{w}})$, which finishes the proof. □

Now, we investigate the properties of a functional $f : \mathbb{R}^{d\times N} \to [0,\infty)$ defined as

$$f(\boldsymbol{\xi}) = \inf_{\mathbf{W}\in G(Y)} \int_Y M(y, \boldsymbol{\xi} + \mathbf{W}(y))\mathrm{d}y. \tag{3.24}$$

Lemma 3.2.9 *Let $\mathcal{N}$–function M satisfy (M1)–(M2). Then the functional f defined in* (3.24) *is an $\mathcal{N}$–function, i.e., it satisfies:*

1) $f(\boldsymbol{\xi}) = 0$ *if and only if* $\boldsymbol{\xi} = \mathbf{0}$,

2) $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$,

3) f *is convex,*

4) $\lim_{|\boldsymbol{\xi}|\to 0} \frac{f(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = 0$, $\lim_{|\boldsymbol{\xi}|\to\infty} \frac{f(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty$.

Proof. First, we show that

$$m_1(|\boldsymbol{\xi}|) \le f(\boldsymbol{\xi}) \le m_2(|\boldsymbol{\xi}|). \tag{3.25}$$

Let us show the first inequality in the latter estimate. Using (M2), Jensen's inequality and the fact that the average over Y of the gradient of an $Y-$periodic function vanishes we have

$$\begin{aligned} f(\boldsymbol{\xi}) = \inf_{\mathbf{W}\in G(Y)} \int_Y M(y, \boldsymbol{\xi} + \mathbf{W}(y))\mathrm{d}y &\ge \inf_{\mathbf{W}\in G(Y)} \int_Y m_1(|\boldsymbol{\xi} + \mathbf{W}(y)|)\mathrm{d}y \\ &\ge \inf_{\mathbf{W}\in G(Y)} m_1\left(\left|\boldsymbol{\xi} + \int_Y \mathbf{W}(y)\mathrm{d}y\right|\right) \ge m_1(|\boldsymbol{\xi}|). \end{aligned}$$

On the other hand we get by (M2) that $f(\boldsymbol{\xi}) \le m_2(|\boldsymbol{\xi}|)$ since $\mathbf{0} \in G(Y)$, which follows from the fact that G is a subspace of $E^M(Y;\mathbb{R}^{d\times N})$.
Assertions 1) and 4) then follow immediately from (3.25).
Obviously, since M is even in the second argument and $G(Y)$ is a subspace of $E^M_{per}(Y;\mathbb{R}^{d\times N})$ we have 2).
In order to show the convexity of f we take $\lambda \in (0,1)$, $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^{d\times N}$ and $\mathbf{W}_1, \mathbf{W}_2 \in G(Y)$. Again the fact that $G(Y)$ is a subspace of $E^M_{per}(Y;\mathbb{R}^{d\times N})$ and the convexity of M yields

$$f(\lambda\boldsymbol{\xi}_1 + (1-\lambda)\boldsymbol{\xi}_2) \le \lambda \int_Y M(y, \boldsymbol{\xi}_1 + \mathbf{W}_1(y))\mathrm{d}y + (1-\lambda)\int_Y M(y, \boldsymbol{\xi}_2 + \mathbf{W}_2(y))\mathrm{d}y.$$

One obtains the desired conclusion by taking the infimum over $\mathbf{W}_1$ and $\mathbf{W}_2$ on the right hand side of the latter inequality. $\square$

Lemma 3.2.10 *Let $\mathcal{N}$–function M satisfy (M1)–(M2) and f be defined by* (3.24). *Then the conjugate $\mathcal{N}$–function f^* to f is given by*

$$f^*(\boldsymbol{\xi}) = \inf_{\substack{\mathbf{W}^*\in G^\perp(Y),\\ \int_Y \mathbf{W}^*(y)\mathrm{d}y = \boldsymbol{\xi}}} \int_Y M^*(y, \mathbf{W}^*(y))\mathrm{d}y. \tag{3.26}$$

Proof. Using the fact that the average over Y of a gradient of $Y-$periodic function vanishes we obtain defining a functional $\mathcal{F} : L^M(Y;\mathbb{R}^{d\times N}) \to \mathbb{R}$ as

$$\mathcal{F}(\mathbf{w}) = \int_Y M(y, \mathbf{w}(y))\mathrm{d}y.$$

that

$$
\begin{aligned}
f^*(\boldsymbol{\xi}) &= \sup_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}} \left\{ \boldsymbol{\xi}\cdot\boldsymbol{\eta} - \inf_{\mathbf{W}\in G(Y)} \mathscr{F}(\boldsymbol{\eta}+\mathbf{W}) \right\} \\
&= \sup_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}} \left\{ - \inf_{\mathbf{W}\in G(Y)} \left\{ \mathscr{F}(\boldsymbol{\eta}+\mathbf{W}) - \int_Y \boldsymbol{\xi}\cdot(\boldsymbol{\eta}+\mathbf{W}(y))\mathrm{d}y \right\} \right\} \\
&= - \inf_{\boldsymbol{\eta}\in\mathbb{R}^{d\times N}} \left\{ \inf_{\mathbf{W}\in G(Y)} \left\{ \mathscr{F}(\boldsymbol{\eta}+\mathbf{W}) - \int_Y \boldsymbol{\xi}\cdot(\boldsymbol{\eta}+\mathbf{W}(y))\mathrm{d}y \right\} \right\} \\
&= - \inf_{\mathbf{V}\in\mathbb{R}^{d\times N}\oplus G(Y)} \left\{ \mathscr{F}(\mathbf{V}) - \int_Y \boldsymbol{\xi}\cdot\mathbf{V}(y)\mathrm{d}y \right\}.
\end{aligned}
\tag{3.27}
$$

Expression (3.26) is a consequence of Lemma 3.2.5 applied on a functional $\mathscr{F}$. First, we observe that $\mathscr{F}$ is closed or equivalently, whenever $\mathbf{W}^k \to \mathbf{W}$ in $L^M(Y;\mathbb{R}^{d\times N})$ then

$$
\liminf_{k\to\infty} \mathscr{F}(\mathbf{W}^k) \geq \mathscr{F}(\mathbf{W}). \tag{3.28}
$$

Obviously $\mathbf{W}^k \to \mathbf{W}$ in $L^M_{per}(Y;\mathbb{R}^{d\times N})$ implies $\mathbf{W}^k \to \mathbf{W}$ in $L^1_{per}(Y;\mathbb{R}^{d\times N})$. In order to show (3.28) it suffices to apply the lower semicontinuity of integral functionals with a Carathéodory integrand, see [2, Theorem 4.2]. Moreover, $\mathscr{F}$ is continuous at $\mathbf{0}\in G$, which is a consequence of (3.67). The conjugate functional $\mathscr{F}^*$ to $\mathscr{F}$ is given by

$$
\mathscr{F}^*(\mathbf{W}^*) = \int_Y M^*(y, \mathbf{W}^*(y))\mathrm{d}y
$$

according to (3.68). Therefore by Lemma 3.2.5 we get from (3.27)

$$
f^*(\boldsymbol{\xi}) = \inf_{\mathbf{W}^*\in(\mathbb{R}^{d\times N}\oplus G(Y))^\perp} \int_Y M^*(y, \mathbf{W}^*(y)+\boldsymbol{\xi})\mathrm{d}y \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^{d\times N}.
$$

Finally, to conclude (3.26) we need to show that

$$
\left(\mathbb{R}^{d\times N}\oplus G(Y)\right)^\perp = \left\{ \mathbf{W}^*\in G^\perp(Y) : \int_Y \mathbf{W}^*(y)\mathrm{d}y = 0 \right\} =: (G^\perp(Y))_0.
$$

Obviously $(G^\perp(Y))_0 \subset \left(\mathbb{R}^{d\times N}\oplus G(Y)\right)^\perp$. In order to get the opposite inclusion, we choose $\mathbf{W}^*\in(\mathbb{R}^{d\times N}\oplus G(Y))^\perp$. Hence by the definition of the annihilator $\int_Y \mathbf{W}^*\cdot(\boldsymbol{\eta}+\mathbf{W})\mathrm{d}y = 0$ for any $\boldsymbol{\eta}\in\mathbb{R}^{d\times N}$ and $\mathbf{W}\in G(Y)$. We infer $\int_Y \mathbf{W}^* = 0$ by setting $\mathbf{W}=0$, $\boldsymbol{\eta} = \int_Y \mathbf{W}^*$ whereas $\mathbf{W}^*\in G^\perp(Y)$ follows by setting $\boldsymbol{\eta}=0$. ☐

The $\mathcal{N}$–functions f and f^* indicate the growth and coercivity properties of the operator $\hat{\mathbf{A}}$ as it is stated among other properties of $\hat{\mathbf{A}}$ in the following lemma.

Lemma 3.2.11 *Let the operator* $\mathbf{A}$ *satisfy (A1)–(A4) and the* $\mathcal{N}$*–function* M *satisfy (M1)–(M3). Then we have:*

($\hat{A}$1) There is a constant $c>0$ *such that for all* $\boldsymbol{\xi}\in\mathbb{R}^{d\times N}$

$$
\hat{\mathbf{A}}(\boldsymbol{\xi})\cdot\boldsymbol{\xi} \geq c(f(\boldsymbol{\xi}) + f^*(\hat{\mathbf{A}}(\boldsymbol{\xi}))).
$$

($\hat{A}$2) For all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^{d\times N}$, $\boldsymbol{\xi}\neq\boldsymbol{\eta}$

$$
(\hat{\mathbf{A}}(\boldsymbol{\xi}) - \hat{\mathbf{A}}(\boldsymbol{\eta}))\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) > 0.
$$

(Â3) $\hat{\mathbf{A}}$ *is continuous on* $\mathbb{R}^{d\times N}$.

Proof. Let $\mathbf{w}$ be a weak solution of cell problem (3.12) corresponding to $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$, which exists due to Lemma 3.2.8. Then it follows that

$$\begin{aligned}\hat{\mathbf{A}}(\boldsymbol{\xi})\cdot\boldsymbol{\xi} &= \int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}(y))\mathrm{d}y\cdot\boldsymbol{\xi} = \int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}(y))\cdot(\boldsymbol{\xi}+\nabla\mathbf{w}(y))\mathrm{d}y \\ &\geq c\int_Y M(y,\boldsymbol{\xi}+\nabla\mathbf{w}(y)) + M^*(y,\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}(y))\mathrm{d}y.\end{aligned} \tag{3.29}$$

Since $\mathbf{w}$ is the weak solution to (3.12), we get from (3.13) in a standard way using (A3) and the Young inequality that $\mathbf{A}(\cdot,\boldsymbol{\xi}+\nabla\mathbf{w}) \in L^{M^*}_{per}(Y;\mathbb{R}^{d\times N})$. Moreover, as identity (3.13) is satisfied for all $\boldsymbol{\varphi} \in V^M_{per}(Y;\mathbb{R}^N)$, it is obviously fulfilled for all $\boldsymbol{\varphi} \in W^1_{per}E^M(Y;\mathbb{R}^N)$. Therefore we have $\mathbf{A}(\cdot,\boldsymbol{\xi}+\nabla\mathbf{w}) \in G^{\perp}(Y)$. Consequently, regarding (3.11) we obtain by Lemma 3.2.10 that

$$\int_Y M^*(y,\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}(y)))\mathrm{d}y \geq f^*(\hat{\mathbf{A}}(\boldsymbol{\xi})). \tag{3.30}$$

This combined with (3.29) leads to the first part of the estimate in (Â1). It remains to justify that

$$\int_Y M(y,\boldsymbol{\xi}+\nabla\mathbf{w}(y))\mathrm{d}y \geq \inf_{\boldsymbol{\varphi}\in W^1_{per}E^M(Y;\mathbb{R})} \int_Y M(y,\boldsymbol{\xi}+\nabla\boldsymbol{\varphi}(y))\mathrm{d}y, \tag{3.31}$$

as the rest then follows from the definition of f and (3.29). However, here we have to face the density problem, which we overcome by using the constructive approach when dealing with the solution. Thus the remaining part of this paragraph will be devoted to the proof of (3.31).

We use the fact that $\mathbf{w}$ is in fact a modular limit of properly chosen sequence. Indeed, it follows from the construction of the solution in Theorem 3.4.7 that there exists a sequence $\{\mathbf{w}^k\}_{k=1}^{\infty} \subset W^1_{per}E^M(Y;\mathbb{R}^N)$ such that

$$\nabla\mathbf{w}^k \rightharpoonup^* \nabla\mathbf{w} \qquad \text{in } L^M(Y;\mathbb{R}^{d\times N}), \tag{3.32}$$

$$\nabla\mathbf{w}^k \longrightarrow \nabla\mathbf{w} \qquad \text{a.e. in } Y, \tag{3.33}$$

$$\mathbf{A}(\cdot,\boldsymbol{\xi}+\nabla\mathbf{w}^k) \rightharpoonup^* \mathbf{A}(\cdot,\xi+\nabla\mathbf{w}) \qquad \text{in } L^{M^*}(Y;\mathbb{R}^{d\times N}), \tag{3.34}$$

$$\lim_{k\to\infty}\int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}^k)\cdot\nabla\mathbf{w}^k\mathrm{d}y \leq \int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w})\cdot\nabla\mathbf{w}\mathrm{d}y. \tag{3.35}$$

Therefore, denoting $\mathbf{W}_\lambda := \nabla\mathbf{w}\chi_{\{|\nabla\mathbf{w}|\leq\lambda\}}$, we obtain that (thanks to monotonicity of $\mathbf{A}$, the fact that $\mathbf{W}_\lambda$ is bounded and (3.32)–(3.35))

$$\begin{aligned}&\lim_{\lambda\to\infty}\lim_{k\to\infty}\int_Y \left|(\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}^k)-\mathbf{A}(y,\boldsymbol{\xi}+\mathbf{W}_\lambda))\cdot(\nabla\mathbf{w}^k-\mathbf{W}_\lambda)\right|\mathrm{d}y \\ &\quad= \lim_{\lambda\to\infty}\lim_{k\to\infty}\int_Y (\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w}^k)-\mathbf{A}(y,\boldsymbol{\xi}+\mathbf{W}_\lambda))\cdot(\nabla\mathbf{w}^k-\mathbf{W}_\lambda)\mathrm{d}y \\ &\quad\leq \lim_{\lambda\to\infty}\int_Y (\mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w})-\mathbf{A}(y,\boldsymbol{\xi}+\mathbf{W}_\lambda))\cdot(\nabla\mathbf{w}-\mathbf{W}_\lambda)\mathrm{d}y \\ &\quad= \lim_{\lambda\to\infty}\int_Y \mathbf{A}(y,\boldsymbol{\xi}+\nabla\mathbf{w})\cdot\nabla\mathbf{w}\chi_{\{|\nabla\mathbf{w}|>\lambda\}}\mathrm{d}y = 0,\end{aligned}$$

where the last equality follows from the fact that $\mathbf{A}(\cdot, \boldsymbol{\xi} + \nabla \mathbf{w}) \cdot \nabla \mathbf{w} \in L^1(Y)$. Hence, evidently for any $\varphi \in L^\infty(Y)$ we deduce that

$$\left| \lim_{\lambda\to\infty} \lim_{k\to\infty} \int_Y (\mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}^k) - \mathbf{A}(y, \boldsymbol{\xi} + \mathbf{W}_\lambda)) \cdot (\nabla \mathbf{w}^k - \mathbf{W}_\lambda)\varphi \mathrm{d}y \right| = 0.$$

Hence, it follows from (3.32)–(3.34) that

$$\begin{aligned} 0 = &\left| \lim_{k\to\infty} \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}^k) \cdot \nabla \mathbf{w}^k \varphi \mathrm{d}y \right. \\ &\left. - \lim_{\lambda\to\infty} \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}) \cdot \mathbf{W}_\lambda \varphi + \mathbf{A}(y, \boldsymbol{\xi} + \mathbf{W}_\lambda)) \cdot (\nabla \mathbf{w} - \mathbf{W}_\lambda)\varphi \mathrm{d}y \right| \\ = &\left| \lim_{k\to\infty} \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}^k) \cdot \nabla \mathbf{w}^k \varphi \mathrm{d}y - \int_Y \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}) \cdot \nabla \mathbf{w} \varphi \mathrm{d}y \right|. \end{aligned}$$

Thus, we see that

$$\mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}^k) \cdot (\boldsymbol{\xi} + \nabla \mathbf{w}^k) \rightharpoonup \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}) \cdot (\boldsymbol{\xi} + \nabla \mathbf{w}) \text{ weakly in } L^1(Y). \tag{3.36}$$

Due to the equivalent characterization of the weak convergence in L^1, we see that the sequence $\{\mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w}^k) \cdot (\boldsymbol{\xi} + \nabla \mathbf{w}^k)\}_{k=1}^\infty$ is uniformly equi-integrable. Using also (A3), we see that also $\{M(y, \boldsymbol{\xi} + \nabla \mathbf{w}^k)\}_{k=1}^\infty$ is uniformly equi-integrable. Therefore, it follows from the Vitali theorem and (3.33) that

$$\lim_{k\to\infty} \int_Y M(y, \boldsymbol{\xi} + \nabla \mathbf{w}^k) \mathrm{d}y = \int_Y M(y, \boldsymbol{\xi} + \nabla \mathbf{w}) \mathrm{d}y.$$

Consequently, since $\mathbf{w}^k \in W^1_{per} E^M(Y; \mathbb{R}^N)$ we see that (3.31) holds, which finishes the proof of (Â1).

In order to show (Â2) we fix $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^{d\times N}, \boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$ and find corresponding weak solutions of the cell problem $\mathbf{w}_1$ and $\mathbf{w}_2$. One obtains (see also appendix)

$$\int_Y \mathbf{A}(y, \boldsymbol{\xi}_i + \nabla \mathbf{w}_i(y)) \cdot \nabla \mathbf{w}_j(y) \mathrm{d}y = 0 \text{ for } i, j = 1, 2$$

in the same way as (3.16) was shown. Then it follows that

$$\begin{aligned} &(\hat{\mathbf{A}}(\boldsymbol{\xi}_1) - \hat{\mathbf{A}}(\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) = \int_Y (\mathbf{A}(y, \boldsymbol{\xi}_1 + \nabla \mathbf{w}_1) - \mathbf{A}(y, \boldsymbol{\xi}_2 + \nabla \mathbf{w}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \mathrm{d}y \\ &= \int_Y (\mathbf{A}(y, \boldsymbol{\xi}_1 + \nabla \mathbf{w}_1) - \mathbf{A}(y, \boldsymbol{\xi}_2 + \nabla \mathbf{w}_2)) \cdot (\boldsymbol{\xi}_1 + \nabla \mathbf{w}_1 - \boldsymbol{\xi}_2 - \nabla \mathbf{w}_2) \mathrm{d}y > 0 \end{aligned}$$

by (A3).
To show (Â3) we consider $\{\boldsymbol{\xi}^k\}_{k=1}^\infty$ such that $\boldsymbol{\xi}^k \to \boldsymbol{\xi}$ in $\mathbb{R}^{d\times N}$ as $k \to \infty$, a corresponding sequence of weak solutions of the cell problems $\{\mathbf{w}^k\}_{k=1}^\infty$ and $\mathbf{w}$ corresponding to $\boldsymbol{\xi}$. Then we have for an arbitrary but fixed $\boldsymbol{\eta} \in \mathbb{R}^{d\times N}$ that

$$(\hat{\mathbf{A}}(\boldsymbol{\xi}^k) - \hat{\mathbf{A}}(\boldsymbol{\xi})) \cdot \boldsymbol{\eta} = \int_Y (\mathbf{A}(y, \boldsymbol{\xi}^k + \nabla \mathbf{w}^k) - \mathbf{A}(y, \boldsymbol{\xi} + \nabla \mathbf{w})) \cdot \boldsymbol{\eta} \mathrm{d}y \to 0$$

as $k \to \infty$ by (3.14). Since $\mathbb{R}^{d\times N}$ is finite dimensional, we conclude (Â3) from the latter convergence. □

3.3 Proof of the main theorem

We start this section by formulating and proving some lemmas that will be used in the proof of Theorem 3.1.2 which appears in subsection 3.3.1. Let us outline next steps. First, we derive estimates of a weak solution $\mathbf{u}^\varepsilon$ of (3.1) and corresponding $\mathbf{A}^\varepsilon(x) := \mathbf{A}\left(\frac{x}{\varepsilon}, \nabla\mathbf{u}^\varepsilon\right)$ that are uniform with respect to $\varepsilon \in (0,1)$. Then we extract a sequence $\{\mathbf{u}^{\varepsilon_k}\}_{k=1}^\infty$ such that $\{\nabla\mathbf{u}^{\varepsilon_k}\}_{k=1}^\infty$ converges weakly* to some $\nabla\mathbf{u}$ in $L^{m_1}(\Omega;\mathbb{R}^{d\times N})$ and a weakly* convergent sequence $\{\mathbf{A}^{\varepsilon_k}\}_{k=1}^\infty$ with a limit $\bar{\mathbf{A}} \in L^{m_2^*}(\Omega;\mathbb{R}^{d\times N})$. Then we show that the sequence $\{\nabla\mathbf{u}^{\varepsilon_k}\}_{k=1}^\infty$ converges weakly* two–scale to $\nabla\mathbf{u} + \mathbf{U}$ in $L^{m_1}(\Omega\times Y;\mathbb{R}^{d\times N})$ and $\{\mathbf{A}^{\varepsilon_k}\}_{k=1}^\infty$ converges weakly* two–scale to $\mathbf{A}^0$ in $L^{m_2^*}(\Omega\times Y;\mathbb{R}^{d\times N})$. Consequently, we apply the weak* two–scale semicontinuity of convex functionals to improve the regularity of limit functions, i.e., we obtain $\nabla\mathbf{u} \in L^f(\Omega;\mathbb{R}^{d\times N})$ and $\bar{\mathbf{A}} = \int_Y \mathbf{A}^0 \in L^{f^*}(\Omega;\mathbb{R}^{d\times N})$. This ensures that $\int_\Omega \bar{\mathbf{A}}\cdot\nabla\mathbf{u}\mathrm{d}x$ is meaningful. Then we employ a variant of the Minty trick for nonreflexive function spaces to identify the limit $\bar{\mathbf{A}}$.

First, we formulate the lemma concerning the existence and uniqueness of a solution to problem (3.1) for an arbitrary but fixed ε. The detailed proof in case (C1) is stated in the appendix, see Theorem 3.4.7. For the existence proof under condition (C2) we refer to [4]. We denote $M^\varepsilon(x,\boldsymbol{\xi}) = M\left(\frac{x}{\varepsilon},\boldsymbol{\xi}\right)$.

Lemma 3.3.1 *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain, the operator $\mathbf{A}$ satisfy (A1)–(A3) and the $N-$function M satisfy (M1)–(M3) and either (C1) or (C2) hold. Then for fixed $\varepsilon\in(0,1)$ there exists a unique weak solution of problem* (3.1)*, which is a function $\mathbf{u}^\varepsilon \in V_0^{M^\varepsilon}$ such that*

$$\int_\Omega \mathbf{A}\left(\frac{x}{\varepsilon}, \nabla\mathbf{u}^\varepsilon(x)\right)\cdot\nabla\boldsymbol{\varphi}(x)\mathrm{d}x = \int_\Omega \mathbf{F}(x)\cdot\nabla\boldsymbol{\varphi}(x)\mathrm{d}x \qquad \text{for all } \boldsymbol{\varphi}\in V_0^{M^\varepsilon}. \tag{3.37}$$

Lemma 3.3.2 *Let the assumptions of Lemma 3.3.1 be satisfied and $\mathbf{u}^\varepsilon$ be a weak solution of problem* (3.1)*. Then $\{\mathbf{A}^\varepsilon\}_{0<\varepsilon<1}$ is bounded in $L^{m_2^*}(\Omega;\mathbb{R}^{d\times N})$ and $\{\mathbf{u}^\varepsilon\}_{0<\varepsilon<1}$ is bounded in $V_0^{m_1}$ and we have the estimate*

$$\int_\Omega \frac{1}{2}m_1(|\nabla\mathbf{u}^\varepsilon|) + m_2^*(|\mathbf{A}^\varepsilon|) \le c\int_\Omega M_\varepsilon\left(x,\nabla\mathbf{u}^\varepsilon\right) + M_\varepsilon^*\left(x,\mathbf{A}^\varepsilon\right) \le C(\|\mathbf{F}\|_\infty, m_1^*). \tag{3.38}$$

Proof. We set $\boldsymbol{\varphi} := \mathbf{u}^\varepsilon$ in (3.37) to obtain

$$\int_\Omega \mathbf{A}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon = \int_\Omega \mathbf{F}\cdot\nabla\mathbf{u}^\varepsilon. \tag{3.39}$$

Using (3.39), (A3), the Young inequality, the convexity of M and the fact that the constant $c\le 1$, which is an obvious consequence of the Young inequality, it follows that

$$c\int_\Omega M_\varepsilon(x,\nabla\mathbf{u}^\varepsilon) + M_\varepsilon^*(x,\mathbf{A}^\varepsilon) \le \int_\Omega M_\varepsilon^*\left(x,\frac{2}{c}\mathbf{F}\right) + \frac{c}{2}M_\varepsilon(x,\nabla\mathbf{u}^\varepsilon).$$

Consequently, employing (M2) we obtain

$$c\int_\Omega \frac{1}{2}m_1(|\nabla\mathbf{u}^\varepsilon|) + m_2^*(|\mathbf{A}^\varepsilon|) \le c\int_\Omega \frac{1}{2}M_\varepsilon\left(x,\nabla\mathbf{u}^\varepsilon\right) + M_\varepsilon^*\left(x,\mathbf{A}^\varepsilon\right) \le \int_\Omega m_1^*\left(\frac{2}{c}|\mathbf{F}|\right).$$

Due to (3.6) the integral on the right hand side is finite and the desired conclusion (3.38) follows. ☐

Lemma 3.3.3 *Let the assumptions of Lemma 3.3.1 be satisfied. In addition and* $\mathbf{u}^\varepsilon$ *be a weak solution of problem* (3.1) *and* $\{\varepsilon_j\}_{j=1}^\infty$ *be an arbitrary sequence such that* $\varepsilon_j \to 0$ *as* $j \to \infty$*. Then there is a subsequence* $\{\varepsilon_{j_k}\}_{k=1}^\infty$*, functions* $\mathbf{u} \in V_0^{m_1}$*,* $\mathbf{U} \in L^{m_1}(\Omega \times Y; \mathbb{R}^{d\times N})$*,* $\bar{\mathbf{A}} \in L^{m_2^*}(\Omega; \mathbb{R}^{d\times N})$ *and* $\mathbf{A}^0 \in L^{m_2^*}(\Omega \times Y; \mathbb{R}^{d\times N})$ *such that as* $k \to \infty$ *we have the following weak convergence results (the sequences are denoted by* k *and not by* ε_{j_k} *for simplicity)*

$$\begin{aligned}
\mathbf{u}^k &\rightharpoonup^* \mathbf{u} && \text{in } L^{m_1}(\Omega; \mathbb{R}^N),\\
\nabla\mathbf{u}^k &\rightharpoonup^* \nabla\mathbf{u} && \text{in } L^{m_1}(\Omega; \mathbb{R}^{d\times N}),\\
\mathbf{A}^k &\rightharpoonup^* \bar{\mathbf{A}} && \text{in } L^{m_2^*}(\Omega; \mathbb{R}^{d\times N})
\end{aligned} \tag{3.40}$$

and the weak two–scale convergence results*

$$\begin{aligned}
\nabla\mathbf{u}^k &\xrightarrow{2-s}{}^* \nabla\mathbf{u} + \mathbf{U} && \text{in } L^{m_1}(\Omega \times Y; \mathbb{R}^{d\times N}),\\
\mathbf{A}^k &\xrightarrow{2-s}{}^* \mathbf{A}^0 && \text{in } L^{m_2^*}(\Omega \times Y; \mathbb{R}^{d\times N}).
\end{aligned} \tag{3.41}$$

Moreover, for a.a. $x \in \Omega$

$$\mathbf{U}(x,\cdot) \in \{\nabla\mathbf{w} : \mathbf{w} \in V_{per}^M\}, \tag{3.42}$$

$$\mathbf{A}^0(x,\cdot) \in G(Y)^\perp, \tag{3.43}$$

$$\int_Y \mathbf{A}^0(x,y) \cdot \mathbf{U}(x,y)\mathrm{d}y = 0. \tag{3.44}$$

Furthermore,

$$\mathbf{u} \in V_0^f(\Omega; \mathbb{R}^N), \tag{3.45}$$

$$\bar{\mathbf{A}} = \int_Y \mathbf{A}^0 \mathrm{d}y, \tag{3.46}$$

$$\bar{\mathbf{A}} \in L^{f^*}(\Omega; \mathbb{R}^{d\times N}) \tag{3.47}$$

where f *is given by* (3.24) *and* f^* *by* (3.26)*. The function* $\bar{\mathbf{A}}$ *satisfies*

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla\boldsymbol{\varphi} = \int_\Omega \mathbf{F} \cdot \nabla\boldsymbol{\varphi} \tag{3.48}$$

for all $\boldsymbol{\varphi} \in C_c^\infty\left(\Omega; \mathbb{R}^N\right)$*.*

Proof. The convergences in (3.40) are a direct consequence of the uniform estimates from Lemma 3.3.2 and the Poincaré type inequality, c.f. [3, Section 2.4]. The convergence $(3.41)_1$ is a consequence of $(3.40)_1$ and Lemma 3.2.7 (vi), which also yields for almost all $x \in \Omega$

$$\int_Y \mathbf{U}(x,y) \cdot \boldsymbol{\psi}(y)\mathrm{d}y = 0 \qquad \text{for all } \boldsymbol{\psi} \in C_{per}^\infty(Y; \mathbb{R}^{d\times N}), \operatorname{div}\boldsymbol{\psi} = 0, \tag{3.49}$$

whereas $(3.41)_2$ follows by Lemma 3.2.7 (v) due to Lemma 3.3.2. Moreover, (3.44) follows from Lemma 3.2.7 (ii), $(3.40)_2$ and $(3.41)_2$.

The convergence result $(3.41)_1$ and the uniqueness of weak* limit, the weak lower semicontinuity stated in Lemma 3.2.7 (vi) and the uniform estimate (3.38) imply

$$\begin{aligned}
&\int_\Omega \int_Y M(y, \nabla\mathbf{u} + \mathbf{U}) + M^*(y, \mathbf{A}^0)\mathrm{d}y\mathrm{d}x\\
&= \liminf_{k\to\infty} \int_\Omega M^{\varepsilon_k}(x, \nabla\mathbf{u}^k(x)) + (M^*)^{\varepsilon_k}(x, \mathbf{A}^k(x))\mathrm{d}x < \infty.
\end{aligned} \tag{3.50}$$

We obtain from (3.50) the existence of a measurable set $\bar{S} \subset \Omega$ such that $|\Omega \setminus \bar{S}| = 0$ and for all $x \in \bar{S}$ $\int_Y M(y, \nabla \mathbf{u}(x) + \mathbf{U}(x,y))\mathrm{d}y < \infty$, which implies $\mathbf{U}(x,\cdot) \in L^M(Y;\mathbb{R}^{d\times N})$. In addition, it follows from (3.49) that there exists $\mathbf{w}(x,\cdot) \in W^{1,1}_{per}(Y;\mathbb{R}^N)$ such that $\nabla_y \mathbf{w}(x,y) = \mathbf{U}(x,y)$. Therefore the estimate (3.50) gives $\nabla_y \mathbf{w}(x,\cdot) \in L^M(Y;\mathbb{R}^{d\times N})$. Accordingly, we have that $\mathbf{w}(x,\cdot) \in V^M_{per}$. Thus by Lemma 3.4.8 and the definition of function f, see (3.24), we conclude

$$\begin{aligned}\int_Y M(y, \nabla \mathbf{u}(x) + \nabla_y \mathbf{w}(x,y))\mathrm{d}y &\geq \inf_{\mathbf{v}\in W^1_{per}E^M(Y;\mathbb{R}^N)} \int_Y M(y, \nabla \mathbf{u}(x) + \mathbf{v}(y))\mathrm{d}y \\ &= f(\nabla \mathbf{u}(x)).\end{aligned}$$

Hence, integrating the result with respect to x over Ω and using the estimate (3.50), we obtain (3.45).

In order to show (3.43) we choose $z \in C^\infty_c(\Omega)$ and $\boldsymbol{\psi} \in C^\infty_{per}\left(Y;\mathbb{R}^N\right)$ and set $\boldsymbol{\varphi}(x) := \varepsilon z(x)\boldsymbol{\psi}\left(\frac{x}{\varepsilon}\right)$ in (3.37). Utilizing $(3.41)_2$ and $Y-$periodicity of $\boldsymbol{\psi}$ we arrive at

$$\begin{aligned}&\int_\Omega\int_Y \mathbf{A}^0(x,y)\cdot z(x)\nabla\boldsymbol{\psi}(y)\mathrm{d}y\mathrm{d}x = \lim_{k\to\infty}\int_\Omega \mathbf{A}^k(x)\cdot z(x)\nabla_y\boldsymbol{\psi}\left(\frac{x}{\varepsilon_{j_k}}\right)\mathrm{d}x \\ &= \lim_{k\to\infty}\int_\Omega \mathbf{A}^k(x)\cdot\nabla\left(\varepsilon_{j_k} z(x)\boldsymbol{\psi}\left(\frac{x}{\varepsilon_{j_k}}\right)\right)\mathrm{d}x \\ &\quad - \lim_{k\to\infty}\int_\Omega \mathbf{A}^k(x)\cdot\left(\varepsilon_{j_k}\nabla z(x)\otimes\boldsymbol{\psi}\left(\frac{x}{\varepsilon_{j_k}}\right)\right)\mathrm{d}x \\ &= \int_\Omega \mathbf{F}(x)\cdot z(x)\int_Y \nabla_y\boldsymbol{\psi}(y)\mathrm{d}y\mathrm{d}x = 0,\end{aligned}$$

which implies that there is a measurable set $\tilde{S} \subset \Omega$, $|\Omega\setminus\tilde{S}| = 0$ such that for all $x \in \tilde{S}$

$$\int_Y \mathbf{A}^0(x,y)\cdot\nabla_y\boldsymbol{\psi}(y)\mathrm{d}y = 0. \tag{3.51}$$

Using Theorem 3.2.1 we can find for any $\boldsymbol{\psi} \in W^1_{per}E^M\left(Y;\mathbb{R}^N\right)$ a sequence $\{\boldsymbol{\psi}^k\}_{k=1}^\infty \subset C^\infty_{per}\left(Y;\mathbb{R}^N\right)$ such that $\nabla\boldsymbol{\psi}^k \xrightarrow{M} \nabla\boldsymbol{\psi}$. Next, we observe that $\mathbf{A}^0(x,\cdot) \in L^{M^*}_{per}\left(Y;\mathbb{R}^N\right)$ for almost all $x \in \Omega$ due to (3.50). Then we set $\boldsymbol{\psi} = \boldsymbol{\psi}^k$ in (3.51) and employing Lemma 3.4.2 we perform the limit passage $k \to \infty$ to get (3.51) for any $\boldsymbol{\psi} \in W^1_{per}E^M\left(Y;\mathbb{R}^N\right)$, which implies (3.43). In a very similar manner, we use the approximation of $\mathbf{U}(x,\cdot) = \nabla_y\mathbf{w}(x,\cdot)$ in the modular topology of $L^M_{per}(Y;\mathbb{R}^{d\times N})$ to conclude (3.44) from (3.51).

Using the expression (3.26) for f^*, the estimate (3.50), (3.43) and (3.46), we get

$$\int_\Omega f^*(\bar{\mathbf{A}}(x))\mathrm{d}x \leq \int_\Omega\int_Y M^*(y,\mathbf{A}^0(x,y))\mathrm{d}y\mathrm{d}x < \infty,$$

which is (3.47).

The identity (3.48) is obtained by performing the limit passage $k \to \infty$ in (3.37) with $\varepsilon = \varepsilon_{j_k}$ using convergence $(3.40)_2$. □

The rest of the paper is devoted to the identification of $\bar{\mathbf{A}}$ in (3.48). Before doing so we state the last auxiliary result.

Lemma 3.3.4 *Let the assumption (A3) hold. Then*

1. *for any* $\mathbf{V} \in L^\infty(\Omega \times Y; \mathbb{R}^{d\times N})$ *we have* $\mathbf{A}(\cdot, \mathbf{V}) \in L^\infty(\Omega \times Y; \mathbb{R}^{d\times N})$,
2. *for any* $\mathbf{V} \in E^{M_y}(\Omega \times Y; \mathbb{R}^{d\times N})$ *we have* $\mathbf{A}(\cdot, \mathbf{V}) \in E^{M_y^*}(\Omega \times Y; \mathbb{R}^{d\times N})$ *provided (M2) holds.*

Proof. Let us observe that (A3) implies

$$|\mathbf{V}| \geq c\frac{M^*(\cdot, \mathbf{A}(\cdot, \mathbf{V}))}{|\mathbf{A}(\cdot, \mathbf{V})|}.$$

Assume that $\mathbf{V} \in L^\infty\left(\Omega \times Y; \mathbb{R}^{d\times N}\right)$ and $\|\mathbf{A}(\cdot, \mathbf{V})\|_{L^\infty} = \infty$, i.e., for any $K > 0$ there is a set $S_K \subset \Omega \times Y$, $|S_K| > 0$ such that $|\mathbf{A}(\cdot, \mathbf{V}(\cdot, \cdot))| > K$ on S_K. Since M^* is an $N-$function, for any $L > 0$ there is $K_L > 0$ such that we have $|\mathbf{V}| \geq c\frac{M^*(y,\mathbf{A}(y,\mathbf{V})}{|\mathbf{A}(y,\mathbf{V})|} > L$ on S_{K_L} with $|S_{K_L}| > 0$, which contradicts $\mathbf{v} \in L^\infty(\Omega \times Y; \mathbb{R}^{d\times N})$.
By (A3) and the Young inequality we obtain for any $t \geq 0$ and $\mathbf{V} \in E^{M_y}\left(\Omega \times Y; \mathbb{R}^{d\times N}\right)$ that

$$\begin{aligned} c\int_\Omega \int_Y M(y, \mathbf{V}) + M^*(y, t\mathbf{A}(y, \mathbf{V}))\mathrm{d}y\mathrm{d}x &\leq \int_\Omega \int_Y t\mathbf{A}(y, \mathbf{V})) \cdot \mathbf{V}\mathrm{d}y\mathrm{d}x \\ &\leq c\int_\Omega \int_Y M\left(y, \frac{2}{c}\mathbf{V}\right) + \frac{c}{2}M^*(y, t\mathbf{A}(y, \mathbf{V}))\mathrm{d}y\mathrm{d}x. \end{aligned}$$

Hence we infer $\int_\Omega \int_Y M^*(y, t\mathbf{A}(y, \mathbf{V}))\mathrm{d}y\mathrm{d}x \leq 2\int_\Omega \int_Y M\left(y, \frac{2}{c}\mathbf{V}\right)\mathrm{d}y\mathrm{d}x$ and the latter integral is finite by Lemma 3.4.4. We note that (3.69) holds since we assume (M2). We also utilize Lemma 3.4.4 to conclude that $\mathbf{A}(y, \mathbf{V}) \in E^{M_y^*}\left(\Omega \times Y; \mathbb{R}^{d\times N}\right)$. □

3.3.1 Proof of the theorem

In this final part we identify $\bar{\mathbf{A}}$. Through this section we always assume that all assumptions of Lemma 3.3.3 are satisfied and we consider the sequence of solutions $\mathbf{u}^k$ according to Lemma 3.3.3.
Step 1: We show the following identity

$$\lim_{k\to\infty} \int_\Omega \mathbf{A}^k \cdot \nabla\mathbf{u}^k\mathrm{d}x = \int_\Omega \bar{\mathbf{A}} \cdot \nabla\mathbf{u}\mathrm{d}x. \tag{3.52}$$

To show it, we first deduce the validity of the following identity

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla\mathbf{u}\mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla\mathbf{u}\mathrm{d}x. \tag{3.53}$$

If (C1) is fulfilled, we find a sequence $\{\mathbf{u}^n\}_{n=1}^\infty \subset C_c^\infty\left(\Omega; \mathbb{R}^N\right)$ such that $\mathbf{u}^n \xrightarrow{f} \mathbf{u}$ as $n \to \infty$. Then we set $\boldsymbol{\varphi} = \mathbf{u}^n$ in (3.48) and using Lemma 3.4.2 we conclude (3.53). Finally, if (C2) holds, we find for each $k \in \mathbb{N}$ a sequence $\{u^{k,n}\}_{n=1}^\infty \subset C_c^\infty(\Omega)$ such that $u^{k,n} \xrightarrow{f} T_k(u)$ as $n \to \infty$, where the truncation operator T_k was introduced in the proof of Lemma 3.2.2. Then we set $\varphi = u^{k,n}$ in (3.48) and using Lemma 3.4.2 we deduce

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla T_k(u)\mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla T_k(u)\mathrm{d}x.$$

Applying Lemma 3.2.4 we deduce (3.53). Then it follows from (3.39) using $(3.40)_1$ and (3.53) that

$$\lim_{k\to\infty} \int_\Omega \mathbf{A}^k \cdot \nabla\mathbf{u}^k\mathrm{d}x = \lim_{k\to\infty} \int_\Omega \mathbf{F} \cdot \nabla\mathbf{u}^k\mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla\mathbf{u}\mathrm{d}x = \int_\Omega \bar{\mathbf{A}} \cdot \nabla\mathbf{u}\mathrm{d}x,$$

which concludes (3.52).
Step 2: We show that the following inequality holds for all $\mathbf{V} \in C_c^\infty(\Omega; C_{per}^\infty(Y; \mathbb{R}^{d\times N}))$.

$$0 \leq \int_\Omega \int_Y (\mathbf{A}^0(x,y) - \mathbf{A}(y, \mathbf{V}(x,y))) \cdot (\nabla \mathbf{u}(x) + \mathbf{U}(x,y) - \mathbf{V}(x,y)) \mathrm{d}y \mathrm{d}x \tag{3.54}$$

Let us choose $\mathbf{V} \in C_c^\infty(\Omega; C_{per}^\infty(Y; \mathbb{R}^{d\times N}))$. Then according to Lemma 3.3.4 we obtain $\mathbf{A}(\cdot, \mathbf{V}) \in L^\infty(\Omega \times Y; \mathbb{R}^{d\times N}) \subset E^{m_1^*}(\Omega \times Y; \mathbb{R}^{d\times N}) \subset E^{m_2^*}(\Omega \times Y; \mathbb{R}^{d\times N})$. Moreover, $\mathbf{A}(\cdot, \mathbf{V})$ is obviously Carathéodory. Then for $\mathbf{V}^k(x) = \mathbf{V}(x, x\varepsilon_k^{-1})$ and $\tilde{\mathbf{A}}^k(x) := \mathbf{A}(x\varepsilon_k^{-1}, \mathbf{V}^k(x))$ we obtain

$$\begin{aligned} \mathbf{V}^k &\xrightarrow{2-s} \mathbf{V} && \text{in } E^{m_i}(\Omega \times Y; \mathbb{R}^{d\times N}), \\ \tilde{\mathbf{A}}^k &\xrightarrow{2-s} \mathbf{A}(\cdot, \mathbf{V}(\cdot,\cdot)) && \text{in } E^{m_i^*}(\Omega \times Y; \mathbb{R}^{d\times N}), i = 1,2, \end{aligned} \tag{3.55}$$

as $k \to \infty$ by Lemma 3.2.7 (i). From (A4) we get

$$\begin{aligned} 0 \leq &\int_\Omega (\mathbf{A}^k(x) - \tilde{\mathbf{A}}^k(x)) \cdot (\nabla \mathbf{u}^k(x) - \mathbf{V}^k(x)) \mathrm{d}x \\ = &\int_\Omega \mathbf{A}^k(x) \cdot \nabla \mathbf{u}^k(x) \mathrm{d}x - \int_\Omega \mathbf{A}^k(x) \cdot \mathbf{V}^k(x) \mathrm{d}x - \int_\Omega \tilde{\mathbf{A}}^k(x) \cdot \nabla \mathbf{u}^k(x) \mathrm{d}x \\ &+ \int_\Omega \tilde{\mathbf{A}}^k(x) \cdot \mathbf{V}^k(x) \mathrm{d}x \\ = &I_k - II_k - III_k + IV_k. \end{aligned}$$

Now, want to perform the passage $k \to \infty$. Using (3.52) we obtain that

$$\lim_{k\to\infty} I_k = \int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u} \mathrm{d}x.$$

Employing properties (3.46) and (3.44) yields

$$\lim_{k\to\infty} I_k = \int_\Omega \int_Y \mathbf{A}^0 \cdot \nabla \mathbf{u} \mathrm{d}y \mathrm{d}x = \int_\Omega \int_Y \mathbf{A}^0 \cdot (\nabla \mathbf{u} + \mathbf{U}) \mathrm{d}y \mathrm{d}x.$$

It follows from $(3.41)_2$, $(3.55)_1$ and Lemma 3.2.7 (iii) that

$$\lim_{k\to\infty} II_k = \int_\Omega \int_Y \mathbf{A}^0 \cdot \mathbf{V} \mathrm{d}y \mathrm{d}x,$$

whereas $(3.41)_1$,$(3.55)_2$ and Lemma 3.2.7 (iii) imply

$$\lim_{k\to\infty} III_k = \int_\Omega \int_Y \mathbf{A}(y, \mathbf{V}(x,y)) \cdot (\nabla \mathbf{u}(x) + \mathbf{U}(x,y)) \mathrm{d}y \mathrm{d}x.$$

Finally, from (3.55) we deduce

$$\lim_{k\to\infty} IV_k = \int_\Omega \int_Y \mathbf{A}(y, \mathbf{V}(x,y)) \cdot \mathbf{V}(x,y) \mathrm{d}y \mathrm{d}x.$$

Hence one obtains (3.54).
Step 3: The goal is to show that $\mathbf{V} \in C_c^\infty(\Omega; C_{per}^\infty(Y; \mathbb{R}^{d\times N}))$ in (3.54) can be substituted by $\mathbf{V} \in L^\infty(\Omega \times Y; \mathbb{R}^{d\times N})$. Let us fix an arbitrary function $\mathbf{V} \in C_c^\infty(\Omega; C_{per}^\infty(Y; \mathbb{R}^{d\times N}))$. We first consider a sequence $\{K^m\}_{m=1}^\infty$ of compact subsets of Ω such that

$K^1 \subset K^2 \subset \ldots \Omega$ and $\bigcup_{m=1}^{\infty} K^m = \Omega$. Obviously, defining $\mathbf{V}^m := \mathbf{V}\chi_{K^m}$ for every $m \in \mathbb{N}$ we have that all $\mathbf{V}^m$'s are compactly supported in Ω and

$$\|\mathbf{V}^m\|_{L^\infty(\Omega\times Y)} \le \|\mathbf{V}\|_{L^\infty(\Omega\times Y)} \text{ for all } m \in \mathbb{N}. \tag{3.56}$$

Next, we observe that (3.56) implies the existence of a positive constant c such that

$$\|\mathbf{A}(\cdot,\mathbf{V}^m)\|_{L^\infty(\Omega\times Y)} \le c \ \text{ for all } m \in \mathbb{N}. \tag{3.57}$$

Assuming on the contrary that $\{\mathbf{A}(\cdot,\mathbf{V}^m)\}_{m=1}^\infty$ is unbounded, we have for arbitrary $K > 0$ the existence of $m_K > 0$ and $S_K \subset \Omega\times Y$ with $|S_K| > 0$ such that $|\mathbf{A}(\cdot,\mathbf{V}^{m_K})| > K$ on S_K. As M is an $N-$function, for a chosen $C > 0$ there is $R > 0$ such that $\frac{M^*(y,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} > C$ for any $|\boldsymbol{\xi}| \ge R$. Thus for the choice $C = \|\mathbf{V}\|_{L^\infty(\Omega\times Y)}$ we find m_R and $S_R \subset \Omega \times Y$ with $|S_R| > 0$ such that for $(x,y) \in S_R$ we obtain using (A3)

$$C < \frac{M^*(y,\mathbf{A}(y,\mathbf{V}^{m_R}))}{|\mathbf{A}(y,\mathbf{V}^{m_R})|} \le |\mathbf{V}^{m_R}| \le \sup_{m\in\mathbb{N}} \|\mathbf{V}^m\|_{L^\infty(\Omega\times Y)} \le C,$$

which is a contradiction and (3.57) is shown. Combining (M2) with (3.56) and (3.57) we get

$$\begin{aligned}\int_\Omega\int_Y M(y,\mathbf{V}^m) + M^*(y,\mathbf{A}(y,\mathbf{V}^m)\mathrm{d}y\mathrm{d}x &\le \int_\Omega\int_Y m_2(|\mathbf{V}^m|) + m_1^*(|\mathbf{A}(y,\mathbf{V}^m)|)\mathrm{d}y\mathrm{d}x \\ &\le \int_\Omega\int_Y m_2(\|\mathbf{V}^m\|_{L^\infty(\Omega\times Y)}) + m_1^*(\|\mathbf{A}(\cdot,\mathbf{V}^m)\|_{L^\infty(\Omega\times Y)}) \le c.\end{aligned}$$

Hence $\{\mathbf{V}^m\}_{m=1}^\infty$ and $\{\mathbf{A}(\cdot,\mathbf{V}^m)\}_{m=1}^\infty$ are uniformly integrable by Lemma 3.2.3. Furthermore, it follows from the definition of $\mathbf{V}^m$ and the properties of $\mathbf{A}$ that $\mathbf{V}^m \to \mathbf{V}$ and $\mathbf{A}(\cdot,\mathbf{V}^m) \to \mathbf{A}(\cdot,\mathbf{V})$ in measure as $m \to \infty$. Consequently, we get by Lemma 3.2.2 that

$$\begin{aligned}\mathbf{V}^m &\xrightarrow{M} \mathbf{V} && \text{in } L^{M_y}(\Omega\times Y;\mathbb{R}^{d\times N}),\\ \mathbf{A}(\cdot,\mathbf{V}^m) &\xrightarrow{M^*} \mathbf{A}(\cdot,\mathbf{V}) && \text{in } L^{M_y^*}(\Omega\times Y;\mathbb{R}^{d\times N}) \text{ as } m\to\infty.\end{aligned} \tag{3.58}$$

Let us consider a standard mollifier $\omega \in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. Since $\mathbf{V}^m$ is supported in $K^m \subset \Omega$ for all m, we can find for every m a sequence $\delta^n \to 0$ as $n \to \infty$ such that, defining $\mathbf{V}^{m,n} := \mathbf{V}^m * \omega^n$, where $\omega^n(z) = (\delta^n)^{-2d}\omega\left(\frac{z}{\delta^n}\right)$, we have $\mathbf{V}^{m,n} \in C_c^\infty(\Omega; C_{per}^\infty(Y))^{d\times N}$. We immediately observe that $\|\mathbf{V}^{m,n}\|_{L^\infty(\Omega\times Y)} \le \|\mathbf{V}^m\|_{L^\infty(\Omega\times Y)}$. In the same way as (3.57) was shown we get that $\|\mathbf{A}(\cdot,\mathbf{V}^{m,n})\|_{L^\infty(\Omega\times Y)} \le c(m)$. We also obtain that $\mathbf{V}^{m,n} \to \mathbf{V}^m$ and $\mathbf{A}(\cdot,\mathbf{V}^{m,n}) \to \mathbf{A}(\cdot,\mathbf{V}^m)$ in measure as $n \to \infty$ for every m. Moreover, for every m the sequences $\{\mathbf{V}^{m,n}\}_{n=1}^\infty$ and $\{\mathbf{A}(\cdot,\mathbf{V}^{m,n})\}_{n=1}^\infty$ are uniformly integrable, which can be shown analogously as above. Consequently, we have for every m that

$$\begin{aligned}\mathbf{V}^{m,n} &\xrightarrow{M} \mathbf{V}^m && \text{in } L^{M_y}(\Omega\times Y;\mathbb{R}^{d\times N}),\\ \mathbf{A}(\cdot,\mathbf{V}^{m,n}) &\xrightarrow{M^*} \mathbf{A}(\cdot,\mathbf{V}^m) && \text{in } L^{M_y^*}(\Omega\times Y;\mathbb{R}^{d\times N}) \text{ as } n\to\infty.\end{aligned} \tag{3.59}$$

Finally, employing (3.59), (3.58) and Lemma 3.4.2 we infer from (3.54) that

$$\begin{aligned}0 &\le \lim_{m\to\infty}\lim_{n\to\infty}\int_\Omega\int_Y (\mathbf{A}^0 - \mathbf{A}(y,\mathbf{V}^{m,n}))\cdot(\nabla\mathbf{u} + \mathbf{U} - \mathbf{V}^{m,n})\\ &= \int_\Omega\int_Y (\mathbf{A}^0 - \mathbf{A}(y,\mathbf{V}))\cdot(\nabla\mathbf{u} + \mathbf{U} - \mathbf{V}).\end{aligned}$$

Step 4: Let us denote for a positive k

$$S_k = \{(x,y) \in \Omega \times Y : |\nabla \mathbf{u}(x) + \mathbf{U}(x,y)| \le k\}$$

and χ_k be the characteristic function of S_k. We replace $\mathbf{V} \in L^\infty(\Omega\times Y; \mathbb{R}^{d\times N})$ in (3.54) by $(\nabla\mathbf{u}+\mathbf{U})\chi_j + h\mathbf{V}\chi_i$ where $0 < i < j$ and $h \in (0,1)$ to obtain

$$\begin{aligned}
0 \le &\int_\Omega \int_Y \mathbf{A}^0 \cdot (\nabla\mathbf{u}+\mathbf{U} - (\nabla\mathbf{u}+\mathbf{U})\chi_j)\mathrm{d}y\mathrm{d}x \\
&- \int_\Omega \int_Y \mathbf{A}(y, (\nabla\mathbf{u}+\mathbf{U})\chi_j + h\mathbf{V}\chi_i)) \cdot (\nabla\mathbf{u}+\mathbf{U} - (\nabla\mathbf{u}+\mathbf{U})\chi_j) \\
&- h \int_\Omega \int_Y (\mathbf{A}^0 - \mathbf{A}(y, (\nabla\mathbf{u}+\mathbf{U})\chi_j + h\mathbf{V}\chi_i)) \cdot \mathbf{V}\chi_i \mathrm{d}y\mathrm{d}x = I - II + III.
\end{aligned}$$

The term I disappears when performing the limit passage $j \to \infty$ by the Lebesgue dominated convergence theorem and the fact $|\Omega \times Y \setminus S_j| \to 0$ as $j \to \infty$. As $(\nabla\mathbf{u} + \mathbf{U})\chi_j + h\mathbf{V}\chi_i$ is zero in $\Omega\times Y \setminus S_j$, we see that $II = 0$ thanks to (A3). After dividing the resulting inequality by h and letting $j \to \infty$ we arrive with the help of Lebesgue dominated convergence theorem at

$$\int_{S_i} (\mathbf{A}^0 - \mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V})) \cdot \mathbf{V}\mathrm{d}y\mathrm{d}x \le 0. \tag{3.60}$$

By (M2) we obtain

$$\begin{aligned}
\int_{S_i} M^*(y, \mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V}))\mathrm{d}y\mathrm{d}x &\le \int_{S_i} m_1^*(|\mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V})|)\mathrm{d}y\mathrm{d}x \\
&\le |S_i| m_1^*(\|\mathbf{A}(\cdot, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V})\|_{L^\infty(S_i)}) \le c.
\end{aligned} \tag{3.61}$$

The fact that $\|\mathbf{A}(\cdot, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V})\|_{L^\infty(S_i)}$ is bounded independently of $h \in (0,1)$ is shown in the same way as (3.57) because

$$\|\nabla\mathbf{u}+\mathbf{U}+h\mathbf{V}\|_{L^\infty(S_i)} \le \|\nabla\mathbf{u}+\mathbf{U}\|_{L^\infty(S_i)} + \|\mathbf{V}\|_{L^\infty(\Omega\times Y)} \le i + \|\mathbf{V}\|_{L^\infty(\Omega\times Y)}.$$

Since $\mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V}) \to \mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U})$ a.e. in S_i and $\{\mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V})\}_{h\in(0,1)}$ is uniformly integrable on S_i due to (3.61) and Lemma 3.2.3, the Vitali theorem implies

$$\mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U}+h\mathbf{V}) \to \mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U}) \text{ in } L^1(S_i).$$

Therefore passing to the limit $h \to 0_+$ in (3.60) we arrive at

$$\int_{S_i} (\mathbf{A}^0 - \mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U})) \cdot \mathbf{V}\mathrm{d}y\mathrm{d}x \le 0.$$

Finally, setting

$$\mathbf{V} = \frac{\mathbf{A}^0 - \mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U})}{|\mathbf{A}^0 - \mathbf{A}(y, \nabla\mathbf{u}+\mathbf{U})| + 1}$$

yields

$$\mathbf{A}^0(x,y) = \mathbf{A}(y, \nabla\mathbf{u}(x) + \mathbf{U}(x,y)) \tag{3.62}$$

for a.a. $(x,y) \in S_i$. Since i was arbitrary and $|\Omega\times Y \setminus S_i| \to 0$ as $i \to \infty$, the equality (3.62) holds a.e. in $\Omega\times Y$. Moreover, due to the properties (3.42) and (3.43) we obtain that $\mathbf{U}(x,\cdot)$ is equal to the gradient of a weak solution of the cell problem (3.12) corresponding to $\boldsymbol{\xi} = \nabla\mathbf{u}(x)$. Finally, we get by (3.46) and (3.11) that

$$\bar{\mathbf{A}}(x) = \int_Y \mathbf{A}^0(x,y)\mathrm{d}y = \int_Y \mathbf{A}(y, \nabla\mathbf{u}(x) + \mathbf{U}(x,y))\mathrm{d}y = \hat{\mathbf{A}}(\nabla\mathbf{u}(x)). \tag{3.63}$$

Step 5: The existence of a unique weak solution of the problem (3.2), which is a function $\mathbf{u} \in V_0^f$ that satisfies

$$\int_\Omega \hat{\mathbf{A}}(\nabla\mathbf{u}) \cdot \nabla\boldsymbol{\varphi} = \int_\Omega \mathbf{F} \cdot \nabla\boldsymbol{\varphi} \; \forall\boldsymbol{\varphi} \in V_0^f. \tag{3.64}$$

We notice that the existence part has been proven in the previous steps. Indeed, in (3.63) we identified the function $\bar{\mathbf{A}}$, which arises in (3.48). Then using the density of smooth compactly supported functions in V_0^f we conclude (3.64). In order to show the uniqueness of a weak solution of (3.2) we can follow the proof of the uniqueness of a weak solution in Theorem 3.4.7.
Step 6: Since we know that (3.2) possesses a unique solution $\mathbf{u}$ and we can extract from any subsequence of $\{\mathbf{u}^j\}_{j=1}^\infty$ a subsequence that converges to $\mathbf{u}$ weakly in $W_0^{1,1}(\Omega;\mathbb{R}^N))$, the whole sequence $\{\mathbf{u}^j\}_{j=1}^\infty$ converges to $\mathbf{u}$ weakly in $W_0^{1,1}(\Omega;\mathbb{R}^N)$.

3.4 Appendix

3.4.1 Musielak–Orlicz spaces

Assume here that $\Sigma \subset \mathbb{R}^n$ is a bounded domain and $n \in \mathbf{N}$ is arbitrary. A function $M : \Sigma \times \mathbb{R}^n \to [0,\infty)$ is said to be an $\mathcal{N}-$function if it satisfies the following four requirements:

1. M is a Carathéodory function such that $M(x,\boldsymbol{\xi}) = 0$ if and only if $\boldsymbol{\xi} = \mathbf{0}$. In addition we assume that for almost all $x \in \Sigma$, we have $M(x,\boldsymbol{\xi}) = M(x,-\boldsymbol{\xi})$.
2. For almost all $x \in \Sigma$ the mapping $\boldsymbol{\xi} \mapsto M(x,\boldsymbol{\xi})$ is convex.
3. For almost all $x \in \Sigma$ there holds $\lim_{|\boldsymbol{\xi}|\to\infty} \frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty$.
4. For almost all $x \in \Sigma$ there holds $\lim_{|\boldsymbol{\xi}|\to 0} \frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = 0$.

The corresponding complementary $\mathcal{N}$–function M^* to M is defined for $\boldsymbol{\eta} \in \mathbb{R}^n$ and almost all $x \in \Sigma$ by

$$M^*(x,\boldsymbol{\eta}) := \sup_{\boldsymbol{\xi}\in\mathbb{R}^n} \{\boldsymbol{\xi}\cdot\boldsymbol{\eta} - M(x,\boldsymbol{\xi})\}$$

and directly from this definition, one obtains the generalized Young inequality

$$\boldsymbol{\xi}\cdot\boldsymbol{\eta} \le M(x,\boldsymbol{\xi}) + M^*(x,\boldsymbol{\eta}), \tag{3.65}$$

valid for all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^n$ and almost everywhere in Σ. In addition, for $\boldsymbol{\xi} := \nabla_{\boldsymbol{\eta}} M^*(x,\boldsymbol{\eta})$, we obtain the equality sign in (3.65), see [13, Section 5]. Finally, an $\mathcal{N}$-function M is said to satisfy the Δ_2–condition if there exists $c > 0$ and a nonnegative function $h \in L^1(\Sigma)$ such that for a.a. $x \in \Sigma$ and all $\boldsymbol{\xi} \in \mathbb{R}^n$

$$M(x,2\boldsymbol{\xi}) \le cM(x,\boldsymbol{\xi}) + h(x).$$

Having introduced the notion of N–function, we can define the generalized Musielak–Orlicz class $\mathscr{L}^M(\Sigma)$ as a set of all measurable functions $\mathbf{v} : \Sigma \to \mathbb{R}^n$ in the following way

$$\mathscr{L}^M(\Sigma) := \left\{ \mathbf{v} \in L^1(\Sigma;\mathbb{R}^n);\ \int_\Sigma M(x,\mathbf{v}(x))\mathrm{d}x < \infty \right\}.$$

In general the class $\mathscr{L}^M(\Sigma)$ does not form a linear vector space and therefore, we define the generalized Musielak–Orlicz space $L^M(\Sigma)$ as the the smallest linear space containing $\mathscr{L}^M(\Sigma)$. More precisely, we define

$$L^M(\Sigma) := \left\{ \mathbf{v} \in L^1(\Sigma;\mathbb{R}^n);\ \text{ there exists } \lambda > 0 \text{ such that } \int_\Sigma M\left(x, \frac{\mathbf{v}(x)}{\lambda}\right)\mathrm{d}x < \infty \right\}.$$

It can be shown that $L^M(\Sigma)$ is a Banach space with respect to the Orlicz norm

$$\|\mathbf{v}\|_{L^M} := \sup\left\{ \left|\int_\Sigma \mathbf{v}(x)\mathbf{w}(x)\mathrm{d}x\right| : \mathbf{w} \in L^{M^*}(\Sigma), \int_\Sigma M(x,\mathbf{w}(x))\mathrm{d}x \le 1 \right\}$$

or the equivalent Luxemburg norm

$$\|\mathbf{v}\|_{L^M} := \inf\left\{ \lambda > 0 : \int_\Sigma M\left(x, \frac{\mathbf{v}(x)}{\lambda}\right)\mathrm{d}x \le 1 \right\}.$$

Moreover, we have the following generalized Hölder inequality, see [14, Theorem 4.1.],

$$\left|\int_\Sigma \mathbf{u}\cdot\mathbf{v}\right| \le 2\|\mathbf{u}\|_{L^M}\|\mathbf{v}\|_{L^{M^*}}$$

valid for all $\mathbf{u} \in L^M(\Sigma)$ and all $\mathbf{v} \in L^{M^*}(\Sigma)$. It is not difficult to observe directly from the definition (or by Young inequality (3.65)), that

$$\|\mathbf{v}\|_{L^M} \le c\left(\int_\Sigma M(x,\mathbf{v}(x))\mathrm{d}x + 1\right), \tag{3.66}$$

with some $c > 0$, that can be set $c = 1$ if we work with the Orlicz norm. Similarly, for the functional $\mathscr{F} : L^M(\Sigma) \to \mathbb{R}$ defined as

$$\mathscr{F}(\mathbf{v}) := \int_\Sigma M(x,\mathbf{v}(x))\mathrm{d}x,$$

we can directly obtain from the definition and due to the convexity of M that if $\|\mathbf{v}\|_{L^M} \le 1$ and the Luxemburg norm is considered then

$$\mathscr{F}(\mathbf{v}) \le \|\mathbf{v}\|_{L^M}. \tag{3.67}$$

Finally, we also recall the definition of the conjugate functional $\mathscr{F}^* : L^{M^*}(\Sigma) \to \mathbb{R}$

$$\mathscr{F}^*(\mathbf{v}^*) := \sup_{\mathbf{v}\in L^M(\Sigma)} \left(\int_\Sigma \mathbf{v}\cdot\mathbf{v}^*\mathrm{d}x - \mathscr{F}(\mathbf{v}) \right)$$

and it is not difficult to observe by using the Young inequality that[2]

$$\mathscr{F}^*(\mathbf{v}^*) = \int_\Sigma M^*(x,\mathbf{v}^*(x))\mathrm{d}x. \tag{3.68}$$

[2]Young inequality (3.65) implies $\mathscr{F}^*(\mathbf{v}^*) \le \int_\Sigma M^*(x,\mathbf{v}^*(x))\mathrm{d}x$. On the other hand, we have $M^*(\cdot,\mathbf{v}^*) = \mathbf{v}^*\cdot\mathbf{w} - M(\cdot,\mathbf{w})$ for $\mathbf{w}(x) := \nabla_{\boldsymbol{\xi}} M^*(x, M^*(x,\mathbf{v}^*))$, which after integration leads to $\mathscr{F}^*(\mathbf{v}^*) \ge \int_\Sigma M^*(x,\mathbf{v}^*(x))\mathrm{d}x$ and (3.68) follows

We complete this subsection by recalling the basic functional-analytic facts about the generalized Musielak–Orlicz spaces. For this purpose we define an additional space

$$E^M(\Sigma) := \overline{\{L^\infty(\Sigma;\mathbb{R}^n)\}}^{\|\cdot\|_{L^M(\Sigma)}}.$$

The following key lemma summarizes the fundamental properties of the involved function spaces (see e.g. [12] for details).

Lemma 3.4.1 (separability, reflexivity) *Let M be an $\mathcal{N}$–function. Then*

1. *$E^M(\Sigma) = L^M(\Sigma)$ if and only if M satisfies the Δ_2–condition,*
2. *$(E^M(\Sigma))^* = L^{M^*}(\Sigma)$, i.e., $L^{M^*}(\Sigma)$ is a dual space to $E^M(\Sigma)$,*
3. *$E^M(\Sigma)$ is separable,*
4. *$L^M(\Sigma)$ is separable if and only if M satisfies the Δ_2–condition,*
5. *$L^M(\Sigma)$ is reflexive if and only if M, M^* satisfy the Δ_2–condition.*

We see from the above lemma that in some cases we need to face the problem with the density of bounded functions and also the lack of reflexivity and separability properties, that somehow excludes many analytical framework to be used. Thus, in addition to the strong/weak/weak* topology, we will also work with the modular topology. We say that a sequence $\{\mathbf{v}^k\}_{k=1}^\infty \subset L^M(\Sigma)$ converges modularly to $\mathbf{v}$ in $L^M(\Sigma)$ if there is $\lambda > 0$ such that as $k \to \infty$

$$\int_\Sigma M\left(x, \frac{\mathbf{v}^k(x) - \mathbf{v}(x)}{\lambda}\right) \mathrm{d}x \to 0.$$

We use the notation $\mathbf{v}^k \xrightarrow{M} \mathbf{v}$ for the modular convergence in $L^M(\Sigma)$. The key property of the modular convergence is stated in the following lemma.

Lemma 3.4.2 *[6, Proposition 2.2.] Let M be an $\mathcal{N}$–function and M^* be the conjugate $\mathcal{N}$–function to M. Suppose that sequences $\{\mathbf{v}^k\}_{k=1}^\infty$ and $\{\mathbf{w}^k\}_{k=1}^\infty$ are uniformly bounded in $L^M(\Sigma)$, $L^{M^*}(\Sigma)$ respectively. Moreover, let $\mathbf{v}^k \xrightarrow{M} \mathbf{v}$ and $\mathbf{w}^k \xrightarrow{M^*} \mathbf{w}$. Then $\mathbf{v}^k \cdot \mathbf{w}^k \to \mathbf{v} \cdot \mathbf{w}$ in $L^1(\Sigma)$ as $k \to \infty$.*

Finally, we also recall the weak* lower semicontinuity property of convex functionals. Since in our case, the $\mathcal{N}$–function M may not satisfy the Δ_2−−condition in general, the spaces do not have to be reflexive. However, due to Lemma 3.4.1, we see that any L^M always has a separable predual space and consequently any bounded sequence possesses a weakly* convergent subsequence. This motivates us to introduce the last convergence theorem, that can be obtained by standard weak lower semicontinuity properties of convex functionals, see e.g. [2, Theorem 4.5], namely:

Lemma 3.4.3 *Let $\Omega \subset \mathbb{R}^d$ be open, $n \in \mathbb{N}$ and $\Phi : Y \times \mathbb{R}^n \to \mathbb{R}$ satisfy:*

(a) Φ is Carathéodory,

(b) $\Phi(y, \cdot)$ is convex for almost all $y \in Y$,

(c) $\Phi \geq 0$.

Then we have the following semicontinuity property: $\mathbf{v}^k \rightharpoonup \mathbf{v}$ *in* $L^1(\Omega \times Y; \mathbb{R}^n)$ *as* $k \to \infty$ *implies*

$$\liminf_{k\to\infty} \int_\Omega \int_Y \Phi(y, \mathbf{v}^k(x,y))\mathrm{d}y\mathrm{d}x \geq \int_\Omega \int_Y \Phi(y, \mathbf{v}(x,y))\mathrm{d}y\mathrm{d}x.$$

We continue with the characterization of the space E^M.

Lemma 3.4.4 *Let* $\Sigma \subset \mathbb{R}^d$ *be bounded,* M *be an* $\mathcal{N}$*–function such that*

$$\forall R > 0 \ \int_\Sigma \sup_{|\boldsymbol{\xi}|\leq R} M(x, \boldsymbol{\xi})\mathrm{d}x < \infty. \tag{3.69}$$

Then

$$E^M(\Sigma) = \{\mathbf{v} \in L^M(\Sigma) : \ \forall t \geq 0 \ t\mathbf{v} \in \mathcal{L}^M(\Sigma)\}.$$

Proof. First, we observe that by (3.69) we obtain

$$\forall t \geq 0, \forall \mathbf{v} \in L^\infty(\Sigma) \ t\mathbf{v} \in \mathcal{L}^M(\Sigma). \tag{3.70}$$

For fixed $t \geq 0, \mathbf{v} \in L^\infty(\Sigma)$ we have

$$\int_\Sigma M\left(x, t\mathbf{v}\right)\mathrm{d}x < \infty$$

due to (3.70). Let $\mathbf{v} \in E^M(\Sigma)$ and a sequence $\{\mathbf{v}^k\}_{k=1}^\infty \subset L^\infty(\Sigma)$ be such that $\|\mathbf{v}^k - \mathbf{v}\|_{L^M} \to 0$ as $k \to \infty$ then the convexity of M in the second variable implies

$$\int_\Sigma M(x, t\mathbf{v})\mathrm{d}x \leq \frac{1}{2}\left(\int_\Sigma M(x, 2t\mathbf{v}^k)\mathrm{d}x + \int_\Sigma M(x, 2t(\mathbf{v} - \mathbf{v}^k))\mathrm{d}x\right).$$

The first integral on the right hand side is finite by (3.69) and the second one vanishes in the limit $k \to \infty$ by (3.67). Thus we showed $\forall t \geq 0 \ t\mathbf{v} \in \mathcal{L}^M(\Sigma)$.
Let $\mathbf{v} \in L^M(\Sigma)$ and assume that

$$\forall t \geq 0 \ t\mathbf{v} \in \mathcal{L}^M(\Sigma). \tag{3.71}$$

Defining $\mathbf{v}^k := \mathbf{v}\chi_{\{|\mathbf{v}|\leq k\}}$ we have $\{\mathbf{v}^k\}_{k=1}^\infty \subset L^\infty(\Sigma)$ and due to (3.71) we get for all $t \geq 0$

$$\int_\Sigma M(x, t(\mathbf{v}^k - \mathbf{v}))\mathrm{d}x = \int_{\{|\mathbf{v}|>k\}} M(x, t\mathbf{v})\mathrm{d}x \to 0$$

as $k \to \infty$. Hence for given $\delta > 0$ we find $n_0(\delta)$ such that for any $k \geq k_0$ $\int_\Sigma M(x, \delta^{-1}(\mathbf{v}^k - \mathbf{v}))\mathrm{d}x < 1$. Then we obtain $\|\mathbf{v}^k - \mathbf{v}\|_{L^M} \leq \delta$ by the definition of the Luxemburg norm and we conclude that $\mathbf{v} \in E^M(\Sigma)$. □

The last statement of this subsection concerns possible relaxing of assumption (M3). Namely, we show that for an $\mathcal{N}$–function M that is radially symmetric in the second variable the accomplishment of (3.3) implies the validity of (3.4).

Lemma 3.4.5 *Let $M : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$–function satisfying conditions (M2) and (M3). Assume that Q_j^δ with $\delta < \delta_0 := \frac{1}{8\sqrt{d}}$ is an arbitrary cube defined in (M3) and that there are constants $A > 0$ and $B \geq 1$ such that for all $y_1, y_2 \in \Sigma$ (where either Σ is a bounded Lipschitz domain or $\Sigma = Y$) with $|y_1 - y_2| \leq \frac{1}{2}$ and all $\xi \in [0,\infty)$ inequality* (3.3) *holds. Then for M_j^δ given by* (3.5) *and its biconjugate $(M_j^\delta)^{**}$ it follows that inequality* (3.4) *is satisfied.*

Proof. First, we fix an arbitrary $y \in Q_j^\delta$. Then we observe that

$$\frac{M(y,\xi)}{(M_j^\delta)^{**}(\xi)} = \frac{M(y,\xi)}{M_j^\delta(\xi)} \frac{M_j^\delta(\xi)}{(M_j^\delta)^{**}(\xi)}. \tag{3.72}$$

We estimate separately both quotients on the right hand side of the latter equality. By continuity of M we find $\bar{y} \in \tilde{Q}_j^\delta$ such that $M_j^\delta(\xi) = M(\bar{y},\xi)$. Then using condition (3.3) and the fact that $|y - \bar{y}| \leq 3\delta\sqrt{d} < \frac{1}{2}$ we get

$$\frac{M(y,\xi)}{M(\bar{y},\xi)} \leq \max\{\xi^{-\frac{A}{\log|y-\bar{y}|}}, B^{-\frac{A}{\log|y-\bar{y}|}}\} \leq \max\{\xi^{-\frac{A}{\log(3\delta\sqrt{d})}}, B^{-\frac{A}{\log(3\delta\sqrt{d})}}\}. \tag{3.73}$$

In order to estimate the second quotient in (3.72) we observe first that if $\xi \in [0,\infty)$ is such that $M_j^\delta(\xi) = (M_j^\delta)^{**}(\xi)$ then the statement is obvious. Therefore we assume that $M_j^\delta(\xi_0) > (M_j^\delta)^{**}(\xi_0)$ at some ξ_0. Due to continuity of M_j^δ and $(M_j^\delta)^{**}$ there is a neighborhood U of ξ_0 such that $M_j^\delta > (M_j^\delta)^{**}$ on U. Consequently, $(M_j^\delta)^{**}$ is affine on U. Moreover, (M2) implies that $m_1 \leq M_j^\delta \leq m_2$, where m_1 and m_2 are convex. Therefore there are ξ_1, ξ_2 such that $U \subset (\xi_1, \xi_2)$, $M_j^\delta > (M_j^\delta)^{**}$ on (ξ_1,ξ_2), $(M_j^\delta)^{**}(\xi_i) = M_j^\delta(\xi_i)$, $i = 1,2$ and $(M_j^\delta)^{**}$ is an affine function on $[\xi_1, \xi_2]$, i.e., for $t \in [0,1]$

$$(M_j^\delta)^{**}(t\xi_1 + (1-t)\xi_2) = tM_j^\delta(\xi_1) + (1-t)M_j^\delta(\xi_2). \tag{3.74}$$

We note that $\xi_1 > 0$ is always assumed because it follows that $0 = M_j^\delta(0) = (M_j^\delta)^{**}(0)$. Now, thanks to the continuity of M we find $y_i \in \tilde{Q}_j^\delta$ such that $M_j^\delta(\xi_i) = M(y_i, \xi_i)$, $i = 1, 2$. Consequently, it follows from (3.74) that

$$(M_j^\delta)^{**}(t\xi_1 + (1-t)\xi_2) = tM(y_1,\xi_1) + (1-t)M(y_2,\xi_2). \tag{3.75}$$

Denoting $\tilde{\xi} = t\xi_1 + (1-t)\xi_2$ we get

$$\frac{M_j^\delta\left(\tilde{\xi}\right)}{(M_j^\delta)^{**}\left(\tilde{\xi}\right)} \leq \frac{M\left(y_2,\tilde{\xi}\right)}{tM(y_1,\xi_1) + (1-t)M(y_2,\xi_2)} \leq \frac{tM(y_2,\xi_1) + (1-t)M(y_2,\xi_2)}{tM(y_1,\xi_1) + (1-t)M(y_2,\xi_2)}. \tag{3.76}$$

Next, we observe that the definition of M_j^δ implies $M(y_1,\xi_1) = M_j^\delta(\xi_1) \leq M(y_2,\xi_1)$. We can assume without loss of generality that

$$M(y_1,\xi_1) < M(y_2,\xi_1) \tag{3.77}$$

because for $M(y_1,\xi_1) = M(y_2,\xi_1)$ inequality (3.76) implies $M_j^\delta \leq (M_j^\delta)^{**}$ on $[\xi_1,\xi_2]$. Since we have always $M_j^\delta \geq (M_j^\delta)^{**}$ we arrive at $M_j^\delta = (M_j^\delta)^{**}$ on $[\xi_1,\xi_2]$.

Let us consider a function $h : [0,1] \to \mathbb{R}$ defined by

$$h(t) = \frac{tM(y_2,\xi_1) + (1-t)M(y_2,\xi_2)}{tM(y_1,\xi_1) + (1-t)M(y_2,\xi_2)}.$$

Then we compute

$$h'(t) = \frac{(M(y_2,\xi_1) - M(y_1,\xi_1))M(y_2,\xi_2)}{(t(M(y_1,\xi_1) - M(y_2,\xi_2)) + M(y_2,\xi_2))^2}.$$

Obviously, we have $h' > 0$ on $(0,1)$ due to (3.77). Therefore the maximum of h is attained at $t = 1$, which implies

$$\frac{M_j^\delta\left(\tilde{\xi}\right)}{(M_j^\delta)^{**}\left(\tilde{\xi}\right)} \le \frac{M(y_2,\xi_1)}{M(y_1,\xi_1)}. \tag{3.78}$$

Next, we apply condition (3.3) and $\xi_1 \le \tilde{\xi}$ to infer

$$\begin{aligned}\frac{M_j^\delta\left(\tilde{\xi}\right)}{(M_j^\delta)^{**}\left(\tilde{\xi}\right)} &\le \max\{\xi_1^{\frac{-A}{\log|y_2-y_1|}}, B^{\frac{-A}{\log|y_2-y_1|}}\} \le \max\{\xi^{\frac{-A}{\log|y_2-y_1|}}, B^{\frac{-A}{\log|y_2-y_1|}}\} \\ &\le \max\{\xi^{\frac{-A}{\log(4\delta\sqrt{d})}}, B^{\frac{-A}{\log(4\delta\sqrt{d})}}\}\end{aligned} \tag{3.79}$$

since $y_1, y_2 \in \tilde{Q}_j^\delta$ implies $|y_1 - y_2| \le 4\delta\sqrt{d} < \frac{1}{2}$. Combining (3.72) with (3.73) and (3.79) yields

$$\begin{aligned}\frac{M(y,\xi)}{(M_j^\delta)^{**}(\xi)} &\le \max\{\xi^{\frac{-A}{\log(3\delta\sqrt{d})}}, B^{\frac{-A}{\log(3\delta\sqrt{d})}}\} \cdot \max\{\xi^{\frac{-A}{\log(4\delta\sqrt{d})}}, B^{\frac{-A}{\log(4\delta\sqrt{d})}}\} \\ &\le \max\{\xi^{\frac{-2A}{\log(4\delta\sqrt{d})}}, B^{\frac{-2A}{\log(4\delta\sqrt{d})}}\}\end{aligned}$$

which is the desired conclusion. □

3.4.2 Young measures

We assume that basic facts on existence and properties of Young measures are known to the reader. The fundamental theorem on Young measures may be found in [9]. We only recall the lemma with properties of Young measures that will be used further. In the following $\mathcal{M}(\mathbb{R}^d)$ stands for the space of bounded Radon measures.

Lemma 3.4.6 *[9, Corollary 3.2] Let a Young measure $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ be generated by a sequence of measurable functions $\mathbf{z}^k : \Omega \to \mathbb{R}^d$. Let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function. Let also assume that the negative part $F^-(\cdot, \mathbf{z}^k)$ is weakly relatively compact in $L^1(\Omega)$. Then*

$$\liminf_{k\to\infty} \int_\Omega F(x, \mathbf{z}^k(x))\mathrm{d}x \ge \int_\Omega \int_{\mathbb{R}^d} F(x,\boldsymbol{\zeta})\mathrm{d}\nu_x(\boldsymbol{\zeta})\mathrm{d}x.$$

If, in addition, the sequence of functions $x \mapsto |F|(x, \mathbf{z}^k(x))$ is weakly relatively compact in $L^1(\Omega)$ then

$$F(\cdot, \mathbf{z}^k(\cdot)) \rightharpoonup \int_{\mathbb{R}^d} F(x,\boldsymbol{\zeta})\mathrm{d}\nu_x(\boldsymbol{\zeta})\mathrm{d}x.$$

3.4.3 Existence of solutions to elliptic problems

To the best of authors' knowledge only the result from [4] concerns the existence of weak solutions of elliptic problems in which the growth condition is given by an anisotropic inhomogeneous $\mathcal{N}$–function. In [4] the only scalar problem and an $\mathcal{N}$–function satisfying the condition (C2) are considered. In this part we show that the result in [4] can be extended also to the vector value problems provided we assume that the domain is star–shaped, i.e., the assumption (C1) holds.

Theorem 3.4.7 *Let $N \geq 1$, $\Omega \subset \mathbb{R}^d, d \geq 2$ be a star–shaped domain, an operator $\mathbf{A}$ satisfy (A1),(A3) and (A4). Let an $\mathcal{N}$–function $M : \Omega \times \mathbb{R}^{d\times N} \to [0, \infty)$ fulfill (M2) and (M3) with Ω replacing Y. Then the problem*

$$\begin{aligned} \operatorname{div} \mathbf{A}(x, \nabla \mathbf{u}(x)) &= \operatorname{div} \mathbf{F}(x) \ \textit{in } \Omega, \\ \mathbf{u} &= 0 \qquad\quad\ \textit{on } \partial\Omega \end{aligned}$$

possesses a unique weak solution, which is a function $\mathbf{u} \in V_0^M$ such that for all $\boldsymbol{\varphi} \in V_0^M$

$$\int_\Omega \mathbf{A}(x, \nabla \mathbf{u}) \cdot \nabla \boldsymbol{\varphi} \mathrm{d}x = \int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \mathrm{d}x. \tag{3.80}$$

Proof. The construction of a weak solution $\mathbf{u}$ will be performed in several steps following the approach from [4]. First, we consider for $\delta \in (0, 1)$ an auxiliary problem: to find $\mathbf{u}^\delta \in V_0^m$ such that

$$\int_\Omega \mathbf{B}(x, \nabla \mathbf{u}^\delta(x)) \cdot \nabla \boldsymbol{\varphi}(x) \mathrm{d}x = \int_\Omega \mathbf{F}(x) \cdot \nabla \boldsymbol{\varphi}(x) \mathrm{d}x \text{ for all } \boldsymbol{\varphi} \in C_c^\infty(\Omega; \mathbb{R}^N), \tag{3.81}$$

where we denoted $\mathbf{B}(x, \boldsymbol{\zeta}) := \mathbf{A}(x, \boldsymbol{\zeta}) + \delta \overline{\nabla} m(\boldsymbol{\zeta})$ and $\overline{\nabla} m(\boldsymbol{\zeta}) := \tilde{m}'(|\boldsymbol{\zeta}|) \frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}$. The $\mathcal{N}$–function $\tilde{m} : [0, \infty) \to [0, \infty)$ is such that $\tilde{m}^*$ satisfies Δ_2–condition and $\tilde{m}(|\boldsymbol{\zeta}|) \geq \sup_{x\in\Omega} M(x, \boldsymbol{\zeta})$. Moreover, the identity

$$\overline{\nabla} m(\boldsymbol{\zeta}) \cdot \boldsymbol{\zeta} = m(\boldsymbol{\zeta}) + m^*(\overline{\nabla} m(\boldsymbol{\zeta})) \tag{3.82}$$

holds. We show the existence of $\mathbf{u}^\delta$ and derive estimates of $\nabla \mathbf{u}^\delta$ and $\mathbf{A}(\cdot, \nabla \mathbf{u}^\delta)$ in $L^M(\Omega; \mathbb{R}^{d\times N})$, $L^{M^*}(\Omega; \mathbb{R}^{d\times N})$ respectively that are uniform with respect to $\delta \in (0, 1)$. Having the uniform estimates we pass to the limit $\delta \to 0_+$ to obtain a weak solution of the initial problem. The reason for such a modification is that from now the leading $\mathcal{N}$–function is independent of the spatial variable and its conjugate satisfies Δ_2–condition, which may not be the case in the original setting.

Step 1: In order to obtain the existence of $\mathbf{u}^\delta$ for fixed $\delta \in (0, 1)$ we employ the results on the so–called (S_m) class operators from [10]. It is necessary to verify assumption of [10, Theorem 4.3]. We omit the verification since it is performed in the same manner as in the proof of [4, Theorem 2.1]. The existence of a weak solution $\mathbf{u}^\delta$ of (3.81) then follows by [10, Theorem 5.1].

Step 2: Now, we derive estimates uniform with respect to δ. Since $\mathbf{u}^\delta \in V_0^m$, by Theorem 3.2.1 (claim 2) there is a sequence $\{\mathbf{u}^{\delta,k}\}_{k=1}^\infty \subset C_c^\infty(\Omega; \mathbb{R}^N)$ such that $\mathbf{u}^{\delta,k} \xrightarrow{m} \mathbf{u}^\delta$ as $k \to \infty$. As $\mathbf{u}^{\delta,k}$ for each k can be used as a test function in (3.81), Lemma 3.4.2 then implies

$$\int_\Omega (\mathbf{A}(x, \nabla \mathbf{u}^\delta(x)) + \delta \overline{\nabla} m(\nabla \mathbf{u}^\delta(x))) \cdot \nabla \mathbf{u}^\delta(x) \mathrm{d}x = \int_\Omega \mathbf{F}(x) \cdot \nabla \mathbf{u}^\delta(x) \mathrm{d}x. \tag{3.83}$$

We get by (A3), (3.82), the Young inequality using also the fact that $c \in (0,1]$ in (A3) together with the convexity of M with respect to the second variable that

$$\begin{aligned}\int_\Omega \frac{c}{2} M(x, \nabla \mathbf{u}^\delta(x)) + cM^*(x, \mathbf{A}(x, \nabla \mathbf{u}^\delta(x)) + \delta m(\nabla \mathbf{u}^\delta(x)) + \delta m^*(\overline{\nabla} m(\mathbf{u}^\delta(x))\mathrm{d}x \\ \leq \int_\Omega M^*\left(x, \frac{2}{c}\mathbf{F}(x)\right)\mathrm{d}x.\end{aligned}$$

Hence we have

$$\begin{aligned}\int_\Omega M(x, \nabla \mathbf{u}^\delta(x))\mathrm{d}x \leq c, \\ \int_\Omega M^*(x, \mathbf{A}(x, \nabla \mathbf{u}^\delta(x))\mathrm{d}x \leq c, \\ \int_\Omega \delta m^*(\overline{\nabla} m(\mathbf{u}^\delta(x))\mathrm{d}x \leq c.\end{aligned} \tag{3.84}$$

Consequently, we obtain the existence of a sequence $\{\delta_k\}_{k=1}^\infty$ such that $\delta_k \to 0$ as $k \to 0$ and denoting $\mathbf{A}^k = \mathbf{A}(\cdot, \nabla \mathbf{u}^{\delta_k})$, $\mathbf{u}^k = \mathbf{u}^{\delta_k}$ and $\mathbf{B}^k = \mathbf{B}(\cdot, \nabla \mathbf{u}^\delta(x))$ we have

$$\begin{aligned}\nabla \mathbf{u}^k &\overset{*}{\rightharpoonup} \nabla \mathbf{u} \text{ in } L^M(\Omega; \mathbb{R}^{d\times N}), \\ \mathbf{A}^k &\overset{*}{\rightharpoonup} \bar{\mathbf{A}} \quad \text{in } L^{M^*}(\Omega; \mathbb{R}^{d\times N}), \\ \mathbf{B}^k &\rightharpoonup \bar{\mathbf{A}} \quad \text{in } L^1(\Omega; \mathbb{R}^{d\times N})\end{aligned} \tag{3.85}$$

as $k \to \infty$.

Step 3: We shall show that

$$\limsup_{k\to\infty} \int_\Omega \mathbf{A}^k \cdot \nabla \mathbf{u}^k \leq \int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u}. \tag{3.86}$$

Performing the limit $k \to \infty$ in (3.83) with $\delta := \delta_k$ and $\boldsymbol{\varphi} := \mathbf{u}^k$ we get by using $(3.85)_1$ that

$$\limsup_{k\to\infty} \int_\Omega \mathbf{A}(x, \nabla \mathbf{u}^k) \cdot \nabla \mathbf{u}^k \leq \lim_{k\to\infty} \int_\Omega \mathbf{B}(x, \nabla \mathbf{u}^k) \cdot \nabla \mathbf{u}^k = \lim_{k\to\infty} \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}^k = \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}. \tag{3.87}$$

Employing $(3.85)_3$ we can pass to the limit $k \to \infty$ in (3.81) to obtain

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla \boldsymbol{\varphi} = \int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi} \mathrm{d}x \qquad \text{for all } \boldsymbol{\varphi} \in C_c^\infty(\Omega; \mathbb{R}^N). \tag{3.88}$$

Next, for any $\boldsymbol{\varphi} \in V_0^M$ we can use the assumptions on the domain Ω and M, and by Theorem 3.2.1 (claim 2) find a sequence $\{\boldsymbol{\varphi}^k\}_{k=1}^\infty \subset C_c^\infty(\Omega; \mathbb{R}^N)$ such that $\boldsymbol{\varphi}^k \xrightarrow{M} \boldsymbol{\varphi}$ as $k \to \infty$. Thus, we can use $\boldsymbol{\varphi}^k$ in (3.88) and by using Lemma 3.4.2, we deduce the identity

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla \boldsymbol{\varphi} = \int_\Omega \mathbf{F} \cdot \nabla \boldsymbol{\varphi}. \tag{3.89}$$

Inserting $\boldsymbol{\varphi} := \mathbf{u}$ into (3.89) yields

$$\int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u} = \int_\Omega \mathbf{F} \cdot \nabla \mathbf{u}. \tag{3.90}$$

We conclude (3.86) by comparing (3.87) and (3.90).
Step 4: To finish the existence proof it remains to show that

$$\bar{\mathbf{A}}(x) = \mathbf{A}(x, \nabla \mathbf{u}(x)) \text{ a.e. in } \Omega. \tag{3.91}$$

Indeed, once we have (3.91), we can combine it with (3.89) to obtain (3.80). Thus, we focus on (3.91). Since $\mathbf{A}(\cdot, 0) = 0$ and $\mathbf{A}$ is strictly monotone, one sees immediately that the negative part of $\mathbf{A}(x, \nabla \mathbf{u}^k) \cdot \nabla \mathbf{u}^k$ vanishes. Thus it is relatively weakly compact in $L^1(\Omega)$. By the second part of Lemma 3.4.6 we infer

$$\liminf_{k \to \infty} \int_\Omega \mathbf{A}(x, \nabla \mathbf{u}^k) \cdot \nabla \mathbf{u}^k \mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^{d \times N}} \mathbf{A}(x, \boldsymbol{\zeta}) \cdot \boldsymbol{\zeta} \mathrm{d}\nu_x(\boldsymbol{\zeta}) \mathrm{d}x,$$

where ν_x is the Young measure generated by $\{\nabla \mathbf{u}^k\}_{k=1}^\infty$. Comparing the latter inequality with (3.86) we have

$$\int_\Omega \int_{\mathbb{R}^{d \times N}} \mathbf{A}(x, \boldsymbol{\zeta}) \cdot \boldsymbol{\zeta} \mathrm{d}\nu_x(\boldsymbol{\zeta}) \mathrm{d}x \leq \int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u} \mathrm{d}x. \tag{3.92}$$

Let us define $h(x, \boldsymbol{\zeta}) := (\mathbf{A}(x, \boldsymbol{\zeta}) - \mathbf{A}(x, \nabla \mathbf{u})) \cdot (\boldsymbol{\zeta} - \nabla \mathbf{u})$. Then it follows from (A4) that

$$\int_\Omega \int_{\mathbb{R}^{d \times N}} h(x, \boldsymbol{\zeta}) \mathrm{d}\nu_x(\boldsymbol{\zeta}) \mathrm{d}x \geq 0. \tag{3.93}$$

As $\mathbf{A}$ is a Carathéodory function and the sequences $\{\nabla \mathbf{u}^k\}_{k=1}^\infty$ and $\{\mathbf{A}^k\}_{k=1}^\infty$ are weakly relatively compact in $L^1(\Omega)$ due to $(3.84)_{1,2}$, the second part of Lemma 3.4.6 implies

$$\begin{aligned} \nabla \mathbf{u} &= \int_{\mathbb{R}^{d \times N}} \boldsymbol{\zeta} \mathrm{d}\nu_x(\boldsymbol{\zeta}) \qquad \text{a.e. in } \Omega, \\ \bar{\mathbf{A}} &= \int_{\mathbb{R}^{d \times N}} \mathbf{A}(x, \boldsymbol{\zeta}) \mathrm{d}\nu_x(\boldsymbol{\zeta}) \text{ a.e. in } \Omega. \end{aligned} \tag{3.94}$$

Using these identities we deduce that

$$\int_\Omega \int_{\mathbb{R}^{d \times N}} h(x, \boldsymbol{\zeta}) \mathrm{d}\nu_x(\boldsymbol{\zeta}) \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{d \times N}} \mathbf{A}(x, \boldsymbol{\zeta}) \cdot \boldsymbol{\zeta} \mathrm{d}\nu_x(\boldsymbol{\zeta}) \mathrm{d}x - \int_\Omega \bar{\mathbf{A}} \cdot \nabla \mathbf{u} \mathrm{d}x \leq 0$$

by (3.92). It follows from (3.93) that $\int_{\mathbb{R}^{d \times N}} h(x, \boldsymbol{\zeta}) \mathrm{d}\nu_x(\boldsymbol{\zeta}) = 0$ a.e. in Ω. Since ν_x is a probability measure and $\mathbf{A}(x, \cdot)$ is strictly monotone, we conclude for a.a. $x \in \Omega$ that $\operatorname{supp}\{\nu_x\} = \{\nabla \mathbf{u}(x)\}$. Thus $\nu_x = \delta_{\nabla \mathbf{u}(x)}$ a.e. in Ω and inserting this into $(3.94)_2$ we conclude (3.91).
Step 5: In order to show uniqueness of a weak solution, we suppose that functions $\mathbf{u}_1, \mathbf{u}_2 \in V_0^M$ fulfill (3.80). Taking the difference of weak formulation with $\boldsymbol{\varphi} := \mathbf{u}_1 - \mathbf{u}_2$ yields

$$\int_\Omega (\mathbf{A}(x, \nabla \mathbf{u}_1) - \mathbf{A}(x, \nabla \mathbf{u}_2)) \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2) \mathrm{d}x = 0.$$

Hence we obtain by (A4) that $\nabla(\mathbf{u}_1 - \mathbf{u}_2) = 0$ a.e. in Ω and since the trace of $\mathbf{u}_1 - \mathbf{u}_2$ is zero on $\partial\Omega$ we conclude $\mathbf{u}_1 = \mathbf{u}_2$ a.e. in Ω. ☐

Similarly, as in the case of the monotone operator $\mathbf{A}$, we shall show certain properties of the minimizers to convex functional generated by the $\mathcal{N}$–function M. For simplicity, we state the following results only for spatially periodic setting, but they can be easily generalized also to the Dirichlet case. The main goal of the section is the following Lemma.

Lemma 3.4.8 *Let $M : Y \times \mathbb{R}^{d\times N} \to [0,\infty)$ be an $\mathcal{N}$–function. Then for arbitrary $\boldsymbol{\xi} \in \mathbb{R}^{d\times N}$ there exists $\tilde{\mathbf{u}} \in V^M_{per}$ such that for all $\mathbf{w} \in V^M_{per}$ there holds*

$$\int_Y M(y, \boldsymbol{\xi} + \nabla\tilde{\mathbf{u}}(y))\mathrm{d}y \le \int_Y M(y, \boldsymbol{\xi} + \nabla\mathbf{w}(y))\mathrm{d}y. \tag{3.95}$$

In addition, if M is strictly convex then the minimizer is unique. Furthermore, if M satisfies (M3) then

$$\int_Y M(y, \boldsymbol{\xi} + \nabla\tilde{\mathbf{u}}(y))\mathrm{d}y = \inf_{\mathbf{w}\in W^1_{per}E^M(Y;\mathbb{R}^N)} \int_Y M(y, \boldsymbol{\xi} + \nabla\mathbf{w}(y))\mathrm{d}y. \tag{3.96}$$

Proof. The existence of a function $\tilde{\mathbf{u}}$ solving (3.95) easily follows from the convexity of M and the fact that $\int_Y M(y, \boldsymbol{\xi})\mathrm{d}y < \infty$. The uniqueness in case of the strict convexity is also a standard task. Thus, we focus only on (3.96). We denote $\mathbf{A}(y, \boldsymbol{\xi}) := \nabla_{\boldsymbol{\xi}} M(y, \boldsymbol{\xi})$. Notice that due to the convexity $\mathbf{A}$ exists for almost all $\boldsymbol{\xi}$ and we extend it to the whole $\mathbb{R}^{d\times N}$ as a pseudodifferential. In addition, the operator $\mathbf{A}$ is a monotone mapping and there holds

$$M(y, \boldsymbol{\xi}) + M^*(y, \mathbf{A}(\boldsymbol{\xi})) = \mathbf{A}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi}.$$

Next, we use Theorem 3.4.7 to get an existence of $\mathbf{u} \in V^M_{per}$, which solves for all $\mathbf{w} \in C^\infty_{per}(Y; \mathbb{R}^N)$

$$\int_Y \mathbf{A}(y, \nabla\mathbf{u} + \boldsymbol{\xi}) \cdot \nabla\mathbf{w}\mathrm{d}y = 0. \tag{3.97}$$

Finally, due to the assumption on M (namely the log-Hölder continuity (3.3)), we see from Theorem 3.2.1 that for any $\mathbf{v} \in V^M_{per}$ we can find a sequence $\{\mathbf{v}^n\}_{n=1}^\infty \in C^\infty_{per}(Y; \mathbb{R}^N)$ that converges modularly to $\mathbf{v}$. Using the modular covergence we can set $\mathbf{w} := \mathbf{v}^n$ in (3.97), which after letting $n \to \infty$ leads to

$$\int_Y \mathbf{A}(y, \nabla\mathbf{u} + \boldsymbol{\xi}) \cdot \nabla\mathbf{v}\mathrm{d}y = 0 \qquad \text{for all } \mathbf{v} \in V^M_{per} \tag{3.98}$$

and in particular to

$$\int_Y \mathbf{A}(y, \nabla\mathbf{u} + \boldsymbol{\xi}) \cdot (\nabla\mathbf{u} - \nabla\tilde{\mathbf{u}})\mathrm{d}y = 0.$$

Hence, due to the convexity of M, we see that

$$\int_Y M(y, \boldsymbol{\xi} + \nabla\tilde{\mathbf{u}}) - M(y, \boldsymbol{\xi} + \nabla\mathbf{u})\mathrm{d}y \ge \mathbf{A}(y, \nabla\mathbf{u} + \boldsymbol{\xi}) \cdot (\nabla\tilde{\mathbf{u}} - \nabla\mathbf{u})\mathrm{d}y = 0.$$

Therefore, $\mathbf{u}$ is also a minimizer to (3.95). In addition, following step by step the proof of Lemma 3.2.11, we deduce that $\mathbf{u}$ can be constructed such that there is a sequence $\{\mathbf{u}^n\}_{n=1}^\infty \subset W^1_{per}E^M(Y; \mathbb{R}^N)$ such that

$$\int_Y M(y, \boldsymbol{\xi} + \nabla\mathbf{u})\mathrm{d}y = \lim_{n\to\infty} \int_Y M(y, \boldsymbol{\xi} + \nabla\mathbf{u}^n)\mathrm{d}y.$$

From this (3.96) directly follows. □

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