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MASTER THESIS

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Finitely Related Algebras

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Algebry konečného relačního stupně

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Abstrakt: Algebraická struktura je konečného relačního stupně, pokud je její klon určen konečným počtem finitárních relací. V této práci zkoumáme grafové algebry s cílem určit, které z nich mají tuto vlastnost. Představujeme stručný souhrn základních teoretických poznatků a uvádíme již známé výsledky o algebrách konečného relačního stupně, zejména klademe důraz na souvislost s Mal'cevskými podmínkami. Dále pak ukazujeme základní poznatky o struktuře grafových algeber. Těžiště této spočívá v částečné klasifikaci grafových algeber konečného relačního stupně. Provádíme důkazy pro různé třídy grafových algeber, například algebry určené souvislými bipartitními grafy či grafy obsahujícími určité podgrafy, avšak několik případů zůstává nerozhodnutých.

Klíčová slova: grafové algebry, konečný relační stupeň, termové operace, klony

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Abstract: An algebraic structure is finitely related if its clone is determined by a finite set of finitary relations. In this thesis we examine graph algebras in order to determine which of them have this property. We provide a brief summary of a background theory and we present an overview of known results, in particular, we emphasize the relation between finitely related algebras and Mal'cev conditions. Further we present basic results about the structure of graph algebras. The main part of this thesis is a partial classification of finitely related graph algebras. We provide proofs for various classes of graph algebras, for example for algebras defined by connected bipartite graphs or algebras defined by graphs containing certain subgraphs, although several cases are missing to complete the classification.

Keywords: graph algebras, finitely related, term operations, clones

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Contents

Introduction

Clone theory is a branch of algebra that studies clones, that is, sets of operations closed under composition. Important and typical examples are clones of algebras: a *clone* of an algebra **A** is the set of all term operations of **A**. Historically, this field started to develop in 1941 when Post[28] described all clones on two element-set. Clones are studied extensively because properties of an algebra, for example the structure of subuniverses, congruences and automorphisms, depend only on its term operations, that is, on its clone.

From the Galois correspondence between sets of operations and sets of relations we know that every clone is determined by a set of relations. An algebra is called *finitely related* if its clone is determined by a finite set of relations. Finitely related algebras are particularly important for computational problems parametrized by a finite set of relations, e.g., the constraint satisfaction problem (CSP).

It is because of to CSP that finitely related algebras receive an attention in recent years. Probably the most important result is a characterization of finite finitely related algebras in congruence modular varieties: a finite algebra **A** in a congruence modular variety is finitely related if and only if **A** has few subpowers. We say that an algebra **A** has *few subpowers* if the number of subalgebras of $\mathbf{A}^n$ is less than $2^{p(n)}$ for some polynomial p. Roughly, an algebra has few subpowers if there is not too many relations compatible with the clone of the algebra. The aforementioned result was proved by a combination of results of Aichinger, Mayr and McKenzie [2] and Barto [5]. This results show that many well-known classes of algebras are finitely related, groups, rings and lattices belong among the most important examples.

Several articles about various classes of semigroups were published recently. For example Davey, Jackson, Pitkethly and Szabó [12] showed that commutative or nilpotent semigroups are finitely related. Dolinka [17] proved that idempotent semigroups enjoy that property as well.

However, there are uncountably many non-finitely related algebras and no general characterization is available. Therefore there is an endeavor to characterize all finitely related algebras.

We study *graph algebras* in this thesis, that is, groupoids whose binary operation is defined by a graph. Those were introduced in 1979 by Shallon and were studied quite heavily since that time. Although there are results concerning many properties of graph algebras, for example in [30], it is not known which of them are finitely related. We aim to find out which graphs algebras are finitely related. Although partial results were announced in personal communication by Kazda and Bulín, the proof techniques and methods were brought up independently.

In the first chapter we recall some definitions and basic universal algebraic theory, which will be needed later. We also present an overview of known results with emphasis on a connection between finitely related algebras and Mal'cev conditions, which characterize many algebraic properties of varieties. In the second chapter we introduce the concept of graph algebras and we present several basic observations. The core of the thesis is in chapters 3 and 4. In Chapter 3 we present two classes of graph algebras which are non-finitely related, whereas in Chapter 4 are introduced classes of algebras which are finitely related. In the conclusion we discuss few examples of graph algebras which we can not decide to have or not to have the property and we present several ideas on possible future research of those algebras.

1 Preliminaries

In this chapter we recall basic universal algebraic definitions and results that we will need later. Most of the topics are covered in [7]. Furthermore, we present a comprehensive overview of known results about finitely related algebras. We also discuss the connection between finitely related algebras and Mal'cev conditions. Several theorems and observations about proof techniques used in the thesis are presented as well.

1.1 Introductory theory

Let A be a set, an *n-ary operation* on A is a mapping $f : A^n \to A$. An algebra $\mathbf{A}$ is a pair $(A, \mathcal{F})$, such that A is a set, called the *universe*, and $\mathcal{F}$ is a possibly infinite set of operations on A, called the *basic operations*. In this thesis we will consider only finite algebras, that is, algebras which universe is a finite set. Let $(A, \mathcal{F})$ be an algebra with an independent set of operations $\mathcal{F} = (f_i | i \in I)$, the function $\tau : I \to \mathbf{N} \cup \{0\}$ which assigns to every $i \in I$ the arity of the operation f_i is called the *similarity type* of $\mathbf{A}$. We say that two algebras are *similar* if the have the same similarity type.

A *variety* is a class of similar algebras closed under forming homomorphic images, subalgebras and products. Let $\mathbf{A} = (A, \mathcal{F})$ and $\mathbf{B} = (B, \mathcal{F}')$. We say that $\mathbf{B}$ is a *reduct* of $\mathbf{A}$ if $A = B$ and $\mathcal{F}' \subset \mathcal{F}$, that is, we get $\mathbf{B}$ by omitting operations from $\mathbf{A}$. Similarly we call $\mathbf{B}$ an *expansion* of $\mathbf{A}$ if $A = B$ and $\mathcal{F} \subset \mathcal{F}'$, i.e., we add new operations to $\mathbf{A}$.

By π_k^n we denote the k-th n-ary *projection*, that is, the operation defined by $\pi_k^n(x_1, \dots, x_n) = x_k$. A set of operations on A is a *clone* if it contains all the projections and it is closed under composition. The smallest clone containing all the basic operations of algebra $\mathbf{A}$ is denoted by $\mathrm{Clo}(\mathbf{A})$ and it is equal to the set of all term operations of $\mathbf{A}$, that is, operations obtained by composition of the basic operations and the projections.

We provide an example of a clone to illustrate the definition. We call an operation f on a set A *idempotent* if $f(a, \dots, a) = a$ for every $a \in A$. Since all the projections have the property and a composition of idempotent operations is an idempotent operation, all idempotent operations forms a clone.

In 1941 Post [28] completely described the lattice of clones on two-element set, in particular, he showed that the clone lattice is countable. However, in 1959 Janov and Mučnik [20] proved that for any A such that $|A| \geq 3$, the lattice of clones is uncountable. The full description of the clone lattices seems hopeless.

Clones play an important role in universal algebra because many structural properties of $\mathbf{A}$ depends only on $\mathrm{Clo}(\mathbf{A})$. In particular, a subset B of A is a subuniverse of $\mathbf{A}$ (a universe of a subalgebra) if and only if it is closed under all term operations.

By a k-ary *relation* on a set A we mean any subset of A^k. We say that an n-ary operation f *preserves* a k-ary relation R if

$$(x_{11}, x_{12}, \ldots, x_{1k}), (x_{21}, \ldots, x_{2k}), \ldots, (x_{n1}, x_{n2}, \ldots, x_{nk}) \in R$$
$$\Rightarrow (f(x_{11}, x_{21}, \ldots, x_{n1}), \ldots, f(x_{1k}, x_{2k}, \ldots, x_{nk})) \in R.$$

In other words, f preserves k-ary relation R if and only if R is a subuniverse of $(A, f)^k$. We also say that R is *invariant* under f or f is *compatible* with R. In this situation we write $f \triangleright R$. Let us illustrate the notion of being preserved by few examples. An equivalence R on $\mathbf{A}$ is a congruence if and only if $\mathrm{Clo}(\mathbf{A}) \triangleright R$. As we mentioned before $\mathbf{B} \leq \mathbf{A}$ if and only if $\mathrm{Clo}(\mathbf{A}) \triangleright B$, where B is regarded as a unary relation on A.

A *relational structure* $\mathbb{A}$ is a pair $(A, \mathcal{R})$, where A is a set and $\mathcal{R}$ is a set of relations. By $\mathrm{Rel}(A)$ we denote the set of all relations on A. The notion of being preserved can be extended to sets of operations and sets of relations. We say that $F \subseteq \mathrm{Op}(A)$ preserves $\mathcal{R} \subseteq \mathrm{Rel}(A)$ if $f \triangleright \theta$ for all $f \in F$ and all $\theta \in \mathcal{R}$. An operation which preserves all relations of a relational structure $\mathbb{A}$ is called a *polymorphism* of $\mathbb{A}$. We denote the set of all polymorphism of a relational structure $\mathbb{A}$ by $\mathrm{Pol}(\mathbb{A})$. The set of all relations preserved by a set of operations F is denoted by $\mathrm{Inv}(F)$.

Theorem 1. *[9, 18] For any relational structure $\mathbb{A}$, $\mathrm{Pol}(\mathbb{A})$ is a clone on A. If A is finite, then every clone on A is of the form $\mathrm{Pol}(\mathbb{A})$ for some $\mathbb{A}$.*

Since all clones on A are determined by a certain set of relations, we can ask whether the set is finite or not.

Definition 1. *We say algebra $\mathbf{A}$ is* finitely related *if and only if there exists a relational structure $\mathbb{A}$ with finitely many relations such that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathbb{A})$.*

Informally, finitely related algebras can be described by finitely many relations. Finitely related algebras have been also called *predicately describable*, *finitely definable*, *finite relational degree* or *of finite degree*.

The set of all clones on a finite set A ordered by inclusion is a lattice. The minimal element is the clone containing only the projections, we denote it by $\mathrm{Proj}(A)$, the maximal element is the clone of all operations on A, we denote it by $\mathrm{Op}(A)$. The operators Inv and Pol forms a Galois connection between sets of operations and sets of relations on A. From the theory of Galois connections we obtain a closure operator on sets of operations and Theorem 1 implies that the closed subsets of $\mathrm{Op}(A)$ are exactly the clones. In particular, the clone lattice is complete. Moreover, the lattice of clones is algebraic [7], where the compact elements are finitely generated clones.

A relation R is *primitively positively definable* or *pp-definable* from $\mathbb{A}$ if it can be defined by a first order formula using only conjunction, existential quantification and the equality relation. A set closed under pp-definitions is called a *relational clone*. Relational clones are counterparts of clones in $\mathrm{Rel}(A)$, there is an anti-isomorphism between the lattice of clones on A and the lattice of relational clones on A [7]. As a consequence of the mentioned facts the clone lattice is dually algebraic, where the co-compact elements are finitely related clones.

In the thesis we use basic graph theory. All the notions we use are covered by [16]. By *graph* we mean a pair (G, E), where G is a set of *vertices* and $E \subseteq G^2$ is a symmetric relation, called the set of *edges*. In other words, we consider undirected graphs which may have *loops*, that is, edges that connect a vertex to itself. All graphs will be denoted by blackboard bold letters. To avoid a confusion we will denote the set of a edges of a graph $\mathbb{G}$ by $E(\mathbb{G})$. By $L(\mathbb{G})$ we will denote the set of all vertices of $\mathbb{G}$ with a loop. By a complete graph we mean a graph such that any two distinct vertices are adjacent (regardless of loops). The *degree* $\deg(v)$ of a vertex v is the number of edges at v, that is, $\deg(v) = |\{(v, x)|(v, x) \in E\}|$.

1.2 Examples

1.2.1 Finitely related algebras

We aim to summarize all the known results on finitely related algebras, therefore even partial or later generalized results are listed. Let us start with the oldest examples of finitely related algebras. We say that an algebra $\mathbf{A}$ has a *near-unanimity* operation, if there is a term operation $f : A^n \to A$ of arity at least 3 such that $f(x, \dots, x, y, x, \dots, x) = x$ for all possible pairs x, y and all positions of y. In 1975 Baker and Pixley [3] showed that each algebra with near-unanimity term is finitely related. An example of a class of algebras with such terms are lattices and all their expansions. Indeed, every lattice has the term operation $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$, which satisfies $m(y, x, x) = m(x, y, x) = m(x, x, y) = x$, and therefore has a near-unanimity operation. Obviously each expansion of a lattice has the same term operation.

Pöschel and Kalužnin [29] proved in 1979 that all unary algebras are finitely related. An algebra is called a *unary algebra* if its basic operations are only unary operations. The most natural examples of unary algebras arise from sets of automorphism.

Finitely related algebras have recently received an attention due to their link to computational complexity of the constraint satisfaction problem (CSP). An overview of the CSP can be found for example in [6].

Many new results come from the theory of Mal'cev conditions. Although we discuss the link between finitely related algebras and Mal'cev conditions in the following subsection, for the sake of completeness we list the most important examples: Boolean algebras, Heyting algebras, groups, quasigroups, rings and Lie algebras.

There are several other modern results, most of them concern semigroups. Some of those results come out of the theory of natural dualities, for example all finite rectangular bands, those are semigroups which satisfy $xyx = x$ for every x, y, are finitely related [14]. It is shown in [11] that for every set F of compatible operations with $\wedge$ on a finite semilattice $(A, \wedge)$ the enriched semilattice $\mathbf{A} = (A, \{\wedge \cup F\})$ is finitely related. We say that an n-ary operation f is *compatible with* $\wedge$, for some semilattice $(A, \wedge)$, if it is a homomorphism from $(A, \wedge)^n \to (A, \wedge)$. Another such example are quasi-primal algebras. An algebra $\mathbf{A}$ is called *quasi-*

primal if the ternary discriminator $t : A^3 \to A$ given by

$$t(a,b,c) = \begin{cases} a & \text{if } a \neq b \\ c & \text{if } a = b, \end{cases}$$

is in Clo($\mathbf{A}$). A proof of an even slightly stronger result by Davey, Pitkethly and Willard is in [13].

There is an interesting chain of results concerning various classes of semigroups. In 2008 Mašulović and Pöschel [26] showed that every finite commutative monoid is finitely related. This result was generalized in 2011 by Davey, Jackson, Pitkethly and Szabó [12]. They proved that every finite commutative semigroups has the property. In the same article they also showed that every semigroup with a bounded p_n-sequence is finitely related. The *p_n-sequence* of a finite algebra $\mathbf{A}$ is the sequence $p_1(\mathbf{A}), p_2(\mathbf{A}), p_3(\mathbf{A}), \ldots$, where $p_n(\mathbf{A})$ denotes the number of n-ary term operations of $\mathbf{A}$ that depends on all n coordinates. Probably the most important example of semigroups with bounded p_n-sequence are semigroups which satisfy, for some $l \in \mathbf{N}$, the equation $x_1 \ldots x_l \approx y_1 \ldots y_l$, so called l-nilpotent semigroups. In that case each term function depends on at most $l-1$ coordinates and therefore has bounded p_n-sequence.

Several open problems formulated in [12] were answered by Mayr in 2012 [25]. He proved that the following classes of semigroups are finitely related: Clifford semigroups, 3-nilpotent monoids, regular bands and semigroups with a single idempotent. The definitions are as follows, a semigroup is *completely regular* if every one of its elements lies in a subgroup. A completely regular semigroup in which all idempotents are central is called a *Clifford semigroup*. Natural examples of Clifford semigroups are groups and commutative inverse semigroups. We say a semigroup is *inverse* if for each element x there exists y such that $x = xyx$ and $y = yxy$. A semigroups is a *regular band* if it is idempotent, that is for each x is $xx = x$ satisfied, and $zxzyz = zxyz$ for all x, y, z.

In the article [25] the author asks whether all idempotent semigroups (bands) are finitely related. This question was answered affirmatively in 2015 by Dolinka [17] As we will see in the next section there exists a semigroup which is not finitely related. However a complete characterization is not available.

1.2.2 Finitely related algebras and Mal'cev conditions

A *Mal'cev condition* is, roughly, a characterization of properties in varieties by the existence of certain terms involved in certain identities. The research of Mal'cev conditions was started in 1954 when Mal'cev showed a connection between permutability of congruences of algebras in a variety V and existence of a certain ternary term. Since then numerous interesting characterizations of various varietal properties appeared.

We say that a variety of algebras V *satisfies* an identity $p(x_1, \ldots, x_n) \approx q(x_1, \ldots, x_n)$, written $V \models p \approx q$, if every algebra $\mathbf{A}$ in V does, that is for every choice of $\mathbf{a} \in A^n$ we have $p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n)$.

We will list the most relevant properties and the respective Mal'cev conditions in this subsection. Let us start with the aforementioned condition describing congruence permutable algebras. A variety V is *congruence permutable* if congruence of each algebra in V permute with respect to the relational product.

In [23] Mal'cev showed that a variety V is congruence permutable if and only if there is a term $p(x,y,z)$ in such that $V \models p(y,y,x) \approx x$ and $V \models p(x,y,y) \approx x$. This result concerns a wide class of well known algebras, for example the term $t(x,y,z) = x \cdot y^{-1}z$ is a Mal'cev term for groups and its expansions. Therefore varieties of groups, rings and Lie groups are congruence permutable. Also observe that $t(x,y,z) = (x/(y\backslash y))(y\backslash z))$ is a Mal'cev term in any quasigroup.

Another class defined by a term satisfying certain identities are algebras with a near-unanimity term. Although near-unanimity terms do not describe any varietal property of congruences, they play a key role in research of finitely related algebras. As we mentioned before, every lattice has a near-unanimity term.

An algebra is called *arithmetical* if it is congruence permutable and if it admits a near-unanimity term at once. A variety is *arithmetical* if its every member is arithmetical. In 1963 Pixley [27] showed that arithmetical varieties are characterized by so called *Pixley terms*: a variety V is arithmetical if and only if there is a term such that $V \models p(x,y,x) \approx p(x,y,y) \approx p(y,y,x) \approx x$. The most notable examples of arithmetical varieties are Heyting algebras and Boolean algebras.

Another example of a property characterized by a Mal'cev condition is varietal congruence distributivity. A variety is *congruence distributive* if the congruence lattice of every algebra in V is distributive. In 1967 Jónsson [21] showed that a variety V is congruence distributive if and only if there is a positive integer n and ternary terms $p_0, p_1, \dots, p_n$ (those terms are called, not surprisingly, *Jónsson terms*) such that V satisfies the identities

(i) $p_i(x,y,x) \approx x$, for $0 \leq i \leq n$,

(ii) $p_0(x,y,z) \approx x$,

(iii) $p_n(x,y,z) \approx z$,

(iv) $p_i(x,x,y) \approx p_{i+1}(x,x,y)$, for i even,

(v) $p_i(x,y,y) \approx p_{i+1}(x,y,y)$, for i odd.

The last classical variety property we mention here is congruence modularity. We say that a variety V is *congruence modular* if congruence lattice of every algebra in V is modular. This property was characterized by Day [15] in 1969, just two years after the result of Jónsson. A variety V is congruence modular if and only if there exists a positive integer n and quaternary terms $p_0, p_1, \dots, p_n$ such that V satisfies the identities

(i) $p_i(x,y,y,x) \approx x$, for $0 \leq i \leq n$,

(ii) $p_0(x,y,z,u) \approx x$,

(iii) $p_n(x,y,z,u) \approx u$,

(iv) $p_i(x,x,y,y) \approx p_{i+1}(x,x,y,y)$, for i even,

(v) $p_i(x,y,y,z) \approx p_{i+1}(x,y,y,z)$, for i odd.

The most important class of algebras in our context is the class of algebras with few subpowers. We say that an algebra $\mathbf{A}$ has *few subpowers* if, for some polynomial p, the number of subalgebras of $\mathbf{A}^n$ is less than $2^{p(n)}$. The condition having of few subpowers could be informally interpreted as not having too many compatible relations. The concept of few subpowers was introduced in [8].

A useful combinatorial characterization of algebras with few subpowers was given in [8]. An algebra has few subpowers if and only if it has a k-edge term which happens if and only if it has a cube term. An operation $f : A^{k+1} \to A$ is a k - *edge operation* if for all $x, y \in A$ we have $f(y, y, x, \dots, x) = f(y, x, y, x, \dots, x) = x$ and for all $i \in \{4, \dots, k+1\}$ and for all $x, y \in A$, we have $f(x, \dots, x, y, x, \dots, x) = x$, with y in position i. An n-cube term is $(2^n - 1)$-ary term t satisfying the identities

$$t\begin{pmatrix} y & x & y & \dots & y \\ x & y & y & \dots & y \\ x & x & x & \dots & y \\ \vdots & & & \ddots & y \\ x & x & x & \dots & y \end{pmatrix} \approx \begin{pmatrix} x \\ x \\ x \\ \vdots \\ x \end{pmatrix},$$

where the columns of the matrix on the left are all the elements of $\{x, y\}^n - \{x\}^n$. Note that a cube term trivially implies an edge term.

We would like to point out various relations between aforementioned classes of algebras. Note that a Pixley term is a special case of both Mal'cev and near-unanimity terms. Indeed, it follows from the definition that a Pixley term is a special case of a Mal'cev term and the term $m(x, y, z) = p(x, p(x, y, z), z)$ satisfies $m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x$ and thus $m(x, y, z)$ is a ternary near-unanimity term.

Also observe that if an algebra has a near-unanimity term, then it has a cube-term. Note that near-unanimity term satisfies similar set of equations as a cube term. Indeed, if we omit columns of matrix X which contain more than one y, we get exactly a set of equations defining a near-unanimity term. Another interesting observation is that cube term generalizes Mal'cev term. Indeed, for $k = 2$ we get

$$t\begin{pmatrix} y & y & x \\ x & y & y \end{pmatrix} \approx \begin{pmatrix} x \\ x \end{pmatrix}$$

and so $t(x, y, z)$ is both a Mal'cev term and a ternary cube term. Therefore the class of finite algebras with few subpowers contains both Mal'cev algebras and algebras with near-unanimity term operations.

There are several other interesting results describing the relations between classes of algebras. It is trivial to observe that the class of congruence distributive varieties is included in the class of congruence modular varieties. A proof that near-unanimity term implies Jónsson terms is in [22]. The last piece of our jigsaw was proved by Aichinger, Mar and McKenzie [2]. They showed in their influential paper that every algebra with few subpowers is congruence modular. All the mentioned classes and its characterization via Mal'cev conditions are depicted in Figure 1.1

Let us now investigate the link between Mal'cev conditions and the property of being finitely related. In 2011 Aichinger, Mayr and McKenzie [2] proved that all algebras with few subpowers are finitely related. Together with the already mentioned characterization of algebras with few subpowers, we get that every algebra with a cube term is finitely related. This result was preceded by Aichinger [1]. He showed that a clone containing a Mal'cev term is finitely related if it contains all constants.

In 2011 Barto proved so-called Zádori conjecture. He showed that every finite, finitely related algebra in a congruence distributive variety has near-unanimity term [4]. However Barto generalized his result in 2015 [5] by proving much stronger Valeriote conjecture, that is, any finite, finitely related algebra in congruence modular variety has few subpowers.

A combination of already mentioned results gives us an important theorem:

Theorem 2. *[2, 5] The following are equivalent for a finite algebra* $\mathbf{A}$

(1) $\mathbf{A}$ *is finitely related and is in a congruence modular variety*

(2) $\mathbf{A}$ *has few subpowers*

Maróti, Markovič and McKenzie [24] showed a very useful characterization of idempotent algebras with a cube term using cube-term blockers: A finite idempotent algebra has a cube term if and only if has no cube term-blockers. A *cube-term blocker* in $\mathbf{A}$ is any pair (D, S) of subuniverses of $\mathbf{A}$ such that $\varnothing < D < S \leq \mathbf{A}$ and such that for every term operation $t(x_1, \ldots, x_n)$ of $\mathbf{A}$ there is i, $1 \leq i \leq n$, so that whenever $\mathbf{s} \in S^n$ and $s_i \in D$ then $t(\mathbf{s}) \in D$.

As a corollary of [2] Maróti, Markovič and McKenzie [24] showed the following result.

Theorem 3. *[2, 24] Let* A *be a nonempty finite set, then maximal non-finitely related idempotent clones are precisely of form* $\mathrm{Pol}(A; \{\{a\}, \forall a \in A, S^m \backslash (S \backslash D)^m, m \in \mathbb{N}\})$, *where* (D, S) *is a cube-term blocker.*

Another contribution of cube-term blockers is that it is algorithmically easy to decide whether an idempotent algebras has few subpowers or not.

For sake of readability we illustrate some of the mentioned results in the following figure.

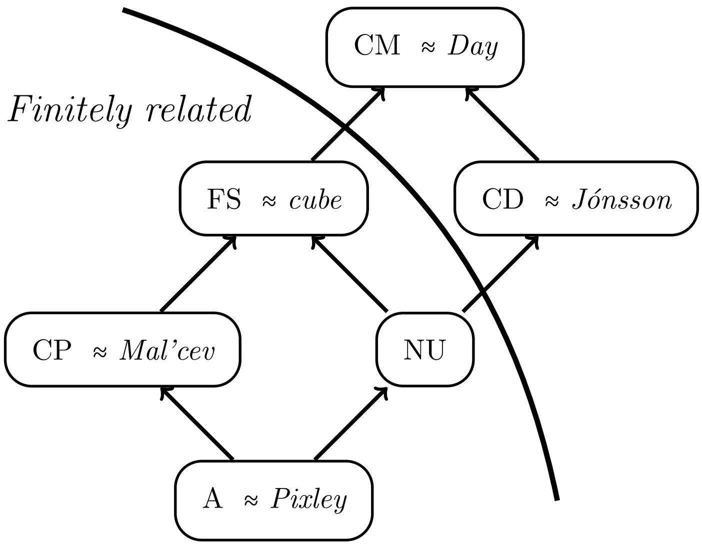


Figure 1.1: Comparison of some classes of algebras. Arrows depict inclusions.

The letter "A" stays for algebras which are in an arithmetical varieties, "CP" stays for algebras in congruence permutable varieties, "CM" denotes all algebras in congruence modular varieties. Analogously we define "CD". "FS" stays for the class of all algebras that have few subpowers and finally "NU" denotes the class of algebras which have a near-unanimity term.

1.2.3 Non-finitely related algebras

Although there are uncountably many non-finitely related algebras, concrete examples are quite rare. We start with a general result, then we proceed to examples. As we can see in the previous subsection, Barto [4] showed that a finite, finitely related, congruence distributive algebra has a near-unanimity term. This means that a an algebra in congruence distributive variety which does not have a near-unanimity term is not finitely related.

Probably the most well known non-finitely related algebra is the two-element implication algebra $\mathbf{I} = (\{0,1\}, \rightarrow)$, defined in the following figure. An elementary proof that $\mathbf{I}$ is not finitely related can be found for example in [12].

$\rightarrow$	0	1
0	1	1
1	0	1

Figure 1.2: The implication algebra $\mathbf{I}$

There are other clones on the two-element set which are not finitely related. For example it is the ternary algebra $\mathbf{J} = (\{0,1\}, x \wedge (y \vee z))$. This and few other examples can be found in [10]

An example of an unary semigroup which is not finitely related is Rees matrix semigroup $\mathbf{A}_2$ enriched with the natural involution. This is a result by Davey, Jackson, Pitkethly and Szabó in [12]. The universe of $\mathbf{A}_2'$ is $\{0\} \cup \{1,2\}^2$, the multiplication is defined so that 0 is a zero element and

$$(i,j)(k,l) = \begin{cases} 0 & \text{if } j = k = 1 \\ (i,l) & \text{otherwise,} \end{cases}$$

for all $i,j,k,l \in \{1,2\}$ and the involution is given by $0' = 0$ and $(i,j)' = (j,i)$ for all $i,j \in \{1,2\}$.

Another example is so called Murskiĭ's groupoid $\mathbf{M} = (\{0,1,2\}, *)$. The groupoid is defined in the following figure. Actually, it is an example of a graph algebra (see Chapter 2 for definition). A proof that $\mathbf{M}$ is non-finitely related is for example in [12]. Also it is proved by Theorem 7.

$*$	0	1	2
0	0	0	0
1	0	0	2
2	0	1	2

Figure 1.3: Murskiĭ's groupoid $\mathbf{M}$

Davey, Jackson, Pitkethly and Szabó posed in [12] open problem whether all semigroups are finitely related. The question was answered negatively by Mayr in [25], by showing that 6-element Brandt monoid $\mathbf{B}_2^1 = (B_2^1, \cdot, i)$ is not finitely related. We define $\mathbf{B}_2^1$ as follows:

$$B_2^1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

multiplication $\cdot$ is defined as the usual matrix multiplication and the unit element is the identity matrix. The proof that $\mathbf{B_2^1}$, and even some of its expansions, is not finitely related can be found in [25].

1.3 Proof techniques

In general it is very difficult, for an arbitrary algebra $\mathbf{A}$, to find a relational structure $\mathbb{A}$ such that Clo($\mathbf{A}$)=Pol($\mathbb{A}$). There are several techniques to show that an algebra is finitely related.

Let n be in $\mathbb{N}$, we will denote the set $\{1, 2, \dots, n\}$ by $\underline{n}$. For every equivalence on $\underline{n}$ we define a map $\hat{\alpha} : \underline{n} \to \underline{n}$ so $\hat{\alpha}(i)$ is the maximal element in the α-block of i. By $\mathrm{Eq}_k(n)$ we mean the set of all congruences on $\underline{n}$ with k blocks. Every equivalence on $\underline{n}$ produces a new term from any term t by identifying certain coordinates.

Definition 2. *Let $f(x_1, \dots, x_n)$ be an n-ary operation on a set A and let α be an equivalence on $\underline{n}$. We define an operation $f_\alpha(x_1, \dots, x_n)$ on A as follows:*

$$f_\alpha(a_1, \dots, a_n) = f(a_{\hat{\alpha}(1)}, \dots, a_{\hat{\alpha}(n)})$$

for all $\mathbf{a} \in A^n$. We say that $f_\alpha(x_1, \dots, x_n)$ is a polymer *of f.*

Note that if f is an n-ary operation on A, for $n \geq |A| + 1$, then it is uniquely determined by its polymers.

For an equivalence with only one block $\{i, j\}$ such that $1 \leq i < j \leq n$ we put $f_{ij} = f_\alpha$. Observe that we can obtain f_{ij} from f by identifying the variable x_i with the variable x_j.

We say that an operation $f(x_1, \dots, x_n)$ has *term polymers* if for all pairs (i, j) such that $1 \leq i < j \leq n$ is $f_{ij}(x_1, \dots, x_n)$ a term operation.

An operation f *depends on* its i-th coordinate if there exist $\mathbf{a}$,$\mathbf{b}$ such that $a_j = b_j$ for all $j \in \underline{n} \setminus i$ and $f(\mathbf{a}) \neq f(\mathbf{b})$. Note the i-th projection fails to depend on any of its coordinates but the i-th one. Let $\alpha \in \mathrm{Eq}_k(n)$, observe that f_α depends on at most k coordinates, as the congruence α has k blocks. For example f_{ij} does not depend on x_i.

There is a very useful characterization of finite relatedness using polymers of f. It is due to Jablonskiĭ[19] and Rosenberg and Szendrei[31].

Theorem 4. *For a finite algebra $\mathbf{A}$ are the following conditions equivalent*

(1) $\mathbf{A}$ is finitely related

(2) there exists k such that for all $n > k$, an operation $f : A^n \to A$ is a term operation of $\mathbf{A}$ provided that $f_\alpha : A^n \to A$ is a term operation of $\mathbf{A}$, for all $\alpha \in Eq_k(n)$

(3) there exists k such that for all $n > k$, an operation $f : A^n \to A$ is a term operation of $\mathbf{A}$ provided that $f_{ij} : A^n \to A$ is a term operation of $\mathbf{A}$, for all $1 \leq i < j \leq n$

We use extensively this theorem, our main technique to prove that algebra $\mathbf{A}$ is finitely related is to show that condition (3) is satisfied.

One can also characterize finitely related clones with descending chains of clones as showed Pöschel and Kalužnin in [29].

Theorem 5. *Let C be a clone on a finite set A. Then C is not finitely related if and only if there is a descending chain of clones $C_1 \supset C_2 \supset C_3 \ldots$, such that $C = \bigcap_{i \in \mathbb{N}} C_i$*

There are several interesting preservation and non-preservation results. For example, Marković, Maróti and McKenzie [24] proved the following theorem.

Theorem 6. *Let $\mathbf{R}$ and $\mathbf{C}$ be finite algebras such that $\mathbf{R} \leq \mathbf{C}^n$ and for some i the i-the projection of $\mathbf{R}$ equals $\mathbf{C}$. Then $\mathbf{C}$ is finitely related if and only if $\mathbf{R}$ is finitely related.*

From that theorem follows a corollary that the property being finitely related is a varietal property. The same was independently showed by Davey, Jackson, Pitkethly and Szabó in [12].

Corollary 1. *If $\mathbf{A}$ and $\mathbf{B}$ are similar finite algebras such that $V(\mathbf{A}) = V(\mathbf{B})$, then $\mathbf{A}$ is finitely related if and only if $\mathbf{B}$ is finitely related.*

However, in general the property of being finitely related is not preserved by many algebraic constructions. For example, neither of the following preserves the property being finitely related: taking subalgebras, homomorphic images, direct products or subdirect factors.

We provide examples that any mentioned algebraic construction does not preserve the property. Let $\mathbf{A}$ be an algebra defined by the table in the following figure.

$\cdot$	0	1	2
0	1	1	2
1	0	1	2
2	0	1	2

Figure 1.4: Groupoid $\mathbf{A}$

Although the algebra $\mathbf{A}$ is finitely related the two-element implication algebra $\mathbf{I}$ is its subalgebra. Also the two-element implication algebra is a homomorphic image of $\mathbf{A}$.

Let $\mathbf{I}$ be the two-element implication algebra defined above and define $\mathbf{I}_{ab} = (\{0,1\}, \to, a, b)$, where a, b are nullary operations. Then both $\mathbf{I}_{0,1}$ and $\mathbf{I}_{1,0}$ are finitely related but the product $\mathbf{I}_{0,1} \times \mathbf{I}_{1,0}$ is not. Detailed proofs of all the aforementioned examples can be found in [12].

Note that there are positive preservation result for special cases. For example it is shown in [24] that if $\mathbf{A}$ and $\mathbf{B}$ are two finite idempotent algebras of the same signature and both of them have a cube term, then $\mathbf{A} \times \mathbf{B}$ has a cube term as well.

2 Graph algebras

In this chapter we introduce graph algebras and prove few basic results about them. Most importantly we show that each term of a graph algebra can be represented as graph with a significant vertex. We also show that graph algebras do not satisfy any aforementioned criteria for deciding finite relatedness and neither they belong to any class of algebras where is finite relatedness already decided.

The concept of graph algebras was introduced in a dissertation of Shallon in 1979. Most of the results in this chapter were brought up before, for example see an article by Pöschel[30]. However, all the presented results were proved independently.

Definition 3. *The* graph algebra *of a graph* $\mathbb{G} = (G, E)$ *with* $0 \notin G$ *is an algebra* $\mathbf{A} = (A, *)$, *where* $A = G \cup \{0\}$ *and, for all* $x, y \in A$,

$$x * y = \begin{cases} x & \textit{if } (x, y) \in E \\ 0 & \textit{otherwise.} \end{cases}$$

In the whole chapter we will keep the notation from the definition. We show that every term of $\mathbf{A}$ can be represented by a certain graph. This reduces evaluation of terms to checking whether a mapping is a graph homomorphism or not.

Since $\mathbf{A}$ has one basic binary operation, each term can be expressed as a binary tree with variables in leaves. We denote by $l(t)$ the leftmost vertex in the tree representing t.

Definition 4. *An n-ary* T-graph *is a pair* $(\mathbb{H}, x_i)$, *where* $\mathbb{H}$ *is a connected graph with vertex set* $H \subseteq \{x_1, \ldots, x_n\}$ *and* $x_i \in H$ *is called the* significant vertex.

We say that an n-ary operation f on A is represented *by an n-ary T-graph* $(\mathbb{H}, x_i)$ *if for any* $\mathbf{a} \in A^n$,

$$f(a_1, \ldots, a_n) = \begin{cases} a_i & \textit{if } a_j \in G \textit{ for every } j \textit{ s.t. } x_j \in H \textit{ and the mapping } x_j \mapsto a_j, \\ & x_j \in H \textit{ is a homomorphism from } \mathbb{H} \textit{ to } \mathbb{G}, \\ 0 & \textit{otherwise.} \end{cases}$$

We say that a term t of $\mathbf{A}$ is represented by a T-graph if $t^{\mathbf{A}}$ is.

Lemma 1. *Each term t can be represented by a T-graph* $(\mathbb{H}, x_i)$, *where* $x_i = l(t)$.

Proof. Let $t(x_1, \ldots, x_n)$ be an arbitrary term over $X = \{x_1, \ldots, x_n\}$. First we recursively define a relation $R(t)$ as follows:

- $R(t) = \varnothing$ if $t = x_i$
- $R(t) = (l(s_1), l(s_2)) \cup R(s_1) \cup R(s_2)$ if $t = s_1 * s_2$.

Then we define the vertex set $H = \{x_i\}$ if $R(t) = \varnothing$ and $H = \{x_i | \exists y : (x_i, y) \in R(t)\}$ otherwise. That is, each vertex corresponds to a variable which appears in the term. We will show that the graph $\mathbb{H} = (H, R(t))$ together with the vertex $l(t)$ is a T-graph that represents the term t.

We prove the claim by induction on the depth of t. If $t = x_i$, then $R(t)$ is empty and therefore $\mathbb{H}$ is a single vertex with no edge. Let $\mathbf{a} \in A^n$ be such that $a_i \in G$, then $x_i \mapsto a_i$ is a homomorphism from $\mathbb{H}$ to $\mathbb{G}$, and $t(\mathbf{a}) = a_i$. Otherwise $t(\mathbf{a}) = 0$. Obviously $x_i = l(t)$.

Assume now that $t = s_1 \star s_2$ and that $(\mathbb{H}_1, l(s_1)) = ((H_1, R(s_1)), l(s_1))$ and $(\mathbb{H}_2, l(s_2)) = ((H_2, R(s_2)), l(s_2))$ are T-graphs representing s_1 and s_2, respectively. First we observe that $(\mathbb{H}, l(t))$ is a T-graph. Since obviously $l(t) \in X$, it is enough to show that $\mathbb{H}$ is connected. Since $\mathbb{H}_1$ and $\mathbb{H}_2$ are connected, $l(s_1) \in H_1$, $l(s_2) \in H_2$ and $(l(s_1), l(s_2)) \in R(t)$, then $\mathbb{H}$ is connected as well and therefore $(\mathbb{H}, l(t))$ is a T-graph.

It remains to show that $(\mathbb{H}, l(t))$ represents t. Let $\mathbf{a}$ be an arbitrary n-tuple such that $a_j \in G$ for every j such that $x_j \in H$. Assume that $\varphi : H \to V$ such that $x_j \mapsto a_j$ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Since $\mathbb{H}_1$, $\mathbb{H}_2$ are subgraphs of $\mathbb{G}$, then φ is a graph homomorphism from $\mathbb{H}_1$ to $\mathbb{G}$ and from $\mathbb{H}_2$ to $\mathbb{G}$. Hence $s(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(l(s_1))$ and $s(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(l(s_2))$. Since φ is a homomorphism, $(\varphi(l(s_1)), \varphi(l(s_2))$ is an edge in $\mathbb{G}$. Therefore, $t(\varphi(x_1), \dots, \varphi(x_n)) = s_1(\varphi(x_1), \dots, \varphi(x_n)) * s_2(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(l(s_1)) * \varphi(l(s_2)) = \varphi(l(s_1)) = l(t)$. Apparently $l(t) = l(s_1) = x_i$.

Now suppose that φ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Then at least one of the following possibilities occur: φ is not a homomorphism from $\mathbb{H}_1$ to $\mathbb{G}$ or φ is not a homomorphism from $\mathbb{H}_2$ to $\mathbb{G}$ or φ does not map $(l(s_1), l(s_2))$ to an edge in $\mathbb{G}$. Therefore $s(\varphi(x_1), \dots, \varphi(x_n)) = 0$ or $s(\varphi(x_1), \dots, \varphi(x_n)) = 0$ or $\varphi(l(s_1)) * \varphi(l(s_2)) = 0$ and thus $t(\varphi(x_1), \dots, \varphi(x_n)) = 0$.

Let $\mathbf{a}$ be an n-tuple such that $a_j = 0$ for some j such that $x_j \in H$. Then $x_j \in H_1$ or $x_j \in H_2$ and thus $t(\mathbf{a}) = 0$.

□

The construction of a T-graph $(\mathbb{H}, x_2)$ representing term $t(x_1, x_2, x_3, x_4) = (x_2 * x_4) * (x_2 * (x_1 * x_4))$ is illustrated in the figure below.

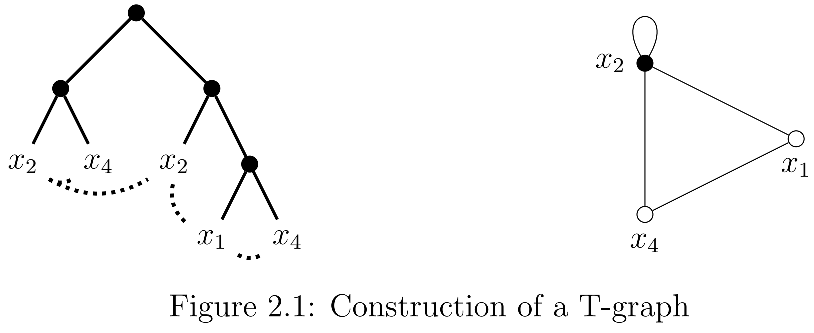


Figure 2.1: Construction of a T-graph

Lemma 2. *Each T-graph* $(\mathbb{H}, x_i)$ *represents a term* t *with* $l(t) = x_i$*.*

Proof. We will use the induction on the number of edges. Let $(\mathbb{H}, x_i)$ be a T-graph with vertex set $H \subseteq \{x_1, \dots, x_n\}$. If $\mathbb{H}$ has no edges, then $(\mathbb{H}, x_i)$ represents the term $t(x_1, \dots, x_n) = x_i$. If $\mathbb{H}$ has only one edge, then it represents the term $t(x_1, \dots, x_n) = x_i * x_j$, where x_j is the vertex adjacent to x_i (possibly $x_i = x_j$).

Assume that the statement is satisfied for all graphs with m edges, where $m \geq 2$. Let $(\mathbb{H}, x_i)$ be an arbitrary T-graph such that $\mathbb{H}$ has $m + 1$ edges. Note that we can always find an edge $e = (x_j, x_k)$ in $\mathbb{H}$ such that $\mathbb{H} \setminus e$ has one nontrivial component and the significant vertex is contained in it. Indeed, if $\mathbb{H}$ contains a cycle, then we can pick any edge from that cycle. In that case we define $H_1 = H$ Otherwise, $\mathbb{H}$ is a tree and we can pick an edge containing a leaf x_j which is not the significant vertex. In that case we define $H_1 = H \setminus x_j$. We denote $(H_1, E(\mathbb{H}) \setminus e)$ by $\mathbb{H}_1$. The graph $\mathbb{H}_1$ has m edges and therefore the T-graph $(\mathbb{H}_1, x_i)$ represents a term $s(x_1, \dots, x_n)$ with $l(s) = x_i$. Let us substitute an occurrence of x_i in s by $x_i * x_j$ and denote this new term by t. Clearly, $l(t) = l(s) = x_i$.

Let $\mathbf{a}$ be an n-tuple such that $a_j \in G$ for every j such that $x_j \in H$. Assume that $\varphi : H \to G$, such that $x_j \mapsto a_j$, is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Thus $\varphi : \mathbb{H}_1 \to \mathbb{G}$ is a homomorphism as well and we have $s(a_1, \dots, a_n) = x_i$. Because φ maps (x_j, x_k) to an edge in $\mathbb{G}$, we have $a_j * a_k = a_j$. Therefore, $t(a_1, \dots, a_n) = s(a_1, \dots, a_n) = a_i$.

Let us assume that $\varphi : H \to V$ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Then either φ is not a homomorphism from $\mathbb{H}_1$ to $\mathbb{G}$, or φ does not map (x_i, x_j) to an edge in $\mathbb{G}$. Since we substituted an appearance of x_i in s by $x_i * x_j$, the former case implies that $t(a_1, \dots, a_n) = s(a_1, \dots, a_n) = 0$. The latter case implies that (a_j, a_k) is not an edge in $\mathbb{G}$ and therefore $a_j * a_k = 0$. Thus we have $t(a_1, \dots, a_n) = 0$.

Let $\mathbf{a}$ be an n-tuple such that $a_j = 0$ for some j such that $x_j \in H$. If $x_j \in H_1$ then we have $s(a_1, \dots, a_n) = 0$ and thus $t(a_1, \dots, a_n) = 0$. Otherwise $x_j \in H \setminus H_1$ and therefore there exists an edge (x_j, x_k) such that $(x_j, x_k) \in \mathbb{H}$ and $(x_j, x_k) \notin \mathbb{H}_1$. Hence $a_j * a_k = 0$ and it follows from the definition of t that $t(a_1, \dots, a_n) = 0$. Thus $(\mathbb{H}, x_i)$ represents the term t. □

We have proved that an operation f is a term operation if and only if it is representable by a T-graph. Note, however, that there can be more T-graphs representing the same term.

Let us examine some properties of the class of graph algebras. The class of graph algebras does not form a variety, e.g., the product of two graph algebras is not a graph algebra. In general, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two graph algebras defined by $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively and let $z \in G_2$. Then we have $(x, z) *_{\mathbf{A}_1 \times \mathbf{A}_2} (y, 0) = (x *_{\mathbf{A}_1} y, 0)$ for all $x, y \in G_1$. It follows from the definition of graph algebra that $a * b = a$ or $a * b = 0$. Therefore a necessary condition for $\mathbf{A}_1 \times \mathbf{A}_2$ to be a graph algebra is such that $x *_{\mathbf{A}_1} y = 0_{\mathbf{A}_1}$, for every $x, y \in A_1$, which happens if $\mathbb{G}_1$ has no edges.

In contrast, we show that there is a subdirect product of $\mathbf{A}_1$,$\mathbf{A}_2$ which is a graph algebra. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two graph algebras defined by $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively. The algebra $\mathbf{B} = (\{(x, 0_{\mathbf{A}_2}) | x \in A_1\} \cup \{(0_{\mathbf{A}_1}, y) | y \in A_2, *)$, where $*$ is defined componentwise, is a subdirect product of $\mathbf{A}_1$,$\mathbf{A}_2$. Note that $\mathbf{B}$ is the graph algebra defined by $\mathbb{G} = \mathbb{G}_1 \cup \mathbb{G}_2$.

Let $\mathbf{A}$ be a graph algebra defined by a graph $\mathbb{G}$. Observe that $\mathbb{B}$ is a subalgebra of $\mathbf{A}$ if and only if it is defined by an induced subgraph of $\mathbb{G}$.

The structure of congruences of a graph algebra defined by a connected graph is described in the following lemma.

Lemma 3. *Let $\mathbf{A}$ be a graph algebra defined by a connected graph $\mathbb{G}$ and let $a, b \in G$. If $\{x | (a,x) \in E(\mathbb{G})\} = \{x | (b,x) \in E(\mathbb{G})\}$, that is, a and b have the same set of adjacent vertices in $\mathbb{G}$, then the congruence generated by the pair (a,b) is equal to $\{(x,x) | \forall x \in A\} \cup (a,b) \cup (b,a)$. Otherwise it is trivial.*

Proof. Recall that a relation $\sim$ is a congruence on $\mathbf{A}$ if and only if $x \sim y \Rightarrow x * z \sim y * z$ and $z * x \sim z * y$ for every $y \in A$.

Let $a, b \in G$ have the same set of adjacent vertices and let c be an arbitrary element of A. If $(a,c) \in E(\mathbb{G})$ then we have $a * c = a \sim b = b * c$ and $c * a = c = c * b$. Otherwise $a * c = b * c = c * a = c * b = 0$ and therefore a congruence generated by a pair (a,b) is equal to $\{(x,x) | \forall x \in A\} \cup (a,b) \cup (b,a)$.

Conversely, let $a, b \in G$ and assume that there is $c \in G$ such that $a * c = a$ and $b * c = 0$. Then we have $a * c = a \sim 0 = b * c$ and therefore a is congruent to 0. Thus all vertices adjacent to a are in congruence with 0 since $d * 0 = 0 \sim d = d * a$. It follows that the congruence generated by the pair (a,b) is equal to $A \times A$ □

An observation about congruences of all graph algebras follows from the proof of the lemma above. Let $a, b \in \mathbb{G}$ and by G_a, G_b denote the components of $\mathbb{G}$ containing a,b respectively. If a and b are adjacent to the same vertices then the congruence generated by a and b is equal to $\{(x,x) | x \in G \cup \{0\}\} \cup (a,b) \cup (b,a)\}$. Otherwise the congruence generated by (a,b) is equal to $\{(x,x) | x \in G\} \cup \{(G_a \cup G_b \cup \{0\}) \times (G_a \cup G_b \cup \{0\})\}$.

Let α be a nontrivial congruence on $\mathbf{A}$ defined by $\mathbb{G}$, then the blocks of α looks as follows: every block $[a]_\alpha \neq [0]_\alpha$ contains a set of vertices of $\mathbb{G}$, possibly a single vertex, such that all of them have the same set of adjacent vertices and block $[0]_\alpha$ contains 0 and all the components $\mathbb{G}_i$ of $\mathbb{G}$ such that there exists $(a,b) \in \alpha$ such that $a \in \mathbb{G}_i$ and a, b have different set of neighbors.

Lemma 4. *Every factor of a graph algebra is a graph algebra.*

Proof. Let $\mathbf{A}$ be an algebra defined by a graph $\mathbb{G}$ and let α be a congruence on $\mathbf{A}$. If α is equal to $A \times A$, then $\mathbf{A}/\alpha$ is defined by the empty graph. If α is equal to $\{(a,a) | a \in A\}$ then $\mathbf{A}/\alpha$ is defined by $\mathbb{G}$.

Otherwise α is a nontrivial and so its blocks are as described in the observation above. Let α has $k+1$ blocks and let us denote the blocks by $[0]_\alpha, [a_1]_\alpha, \dots, [a_k]_\alpha$. Note that we observed before that if there is an edge between two vertices in blocks $[a_i]_\alpha, [a_j]_\alpha$, possibly $[a_i]_\alpha = [a_j]_\alpha$, then every two vertices a, b such that $a \in [a_i]_\alpha$ and $b \in [a_j]_\alpha$ are connected by an edge.

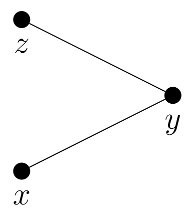
We define a graph $\mathbb{H}$ and we show that the algebra defined by $\mathbb{H}$ is isomorphic to $\mathbf{A}/\alpha$. We define $H = \{b_1, b_2, \dots, b_k\}$ and $(b_i, b_j) \in E(\mathbb{H})$ if there is an edge between $[a_i]_\alpha, [a_j]_\alpha$. The algebra defined by $\mathbb{H}$ is denoted by $\mathbf{B}$ and we define a homomorphism $\varphi : \mathbf{B} \to \mathbf{A}/\alpha$ as follows: $0 \mapsto [0]_\alpha$ and $b_i \mapsto [a_i]_\alpha$ for all $b_i \in H$.

Since φ is obviously both injection and surjection, it is enough to show that for every pair b_i, b_j is $\varphi(b_i *_\mathbf{B} b_j) = \varphi(b_i) *_\mathbf{A} \varphi(b_j)$ satisfied. If b_i and b_j connected by an edge in $\mathbb{H}$ we have $\varphi(b_i *_\mathbf{B} b_j) = \varphi(b_i) = a_i$ and $\varphi(b_i) *_\mathbf{A} \varphi(b_j) = a_i *_\mathbf{A} a_j = a_i$, since there is an edge between any two vertices in $[a_i]_\alpha$ and $[a_j]_\alpha$. If b_i and b_j are not connected by an edge, but $b_i, b_j \in H$, then we have $\varphi(b_i *_{\mathbf{B}} b_j) = \varphi(0) = 0$ and $\varphi(b_i) *_{\mathbf{A}} \varphi(b_j) = a_i *_{\mathbf{A}} a_j = 0$ since there are no edges between vertices of $[a_i]_\alpha$ and $[a_j]_\alpha$. Finally let $b_i = 0$, then we have $\varphi(b_i *_{\mathbf{B}} b_j) = \varphi(0) = 0$ and $\varphi(b_i) *_{\mathbf{A}} \varphi(b_j) = 0 *_{\mathbf{A}} a_j = 0$.

So we have shown that every quotient of a graph algebra is a graph algebra as well. Therefore the class of graph algebras is closed under taking homomorphic images. ☐

Note that the class of graph algebras is not included in any class of algebras mentioned in the previous chapter. First, observe that a graph algebras does not have a cube term. Indeed, each term t can be represented as a T-graph $(\mathbb{H}, x_i)$. Therefore to satisfy the condition $t(X) = \mathbf{x}$ from the definition of a cube term, there must be a column of all x's in the matrix X. However this is not possible and thus no graph algebra has a cube term. Therefore the result of Aichinger, Mayr and McKenzie [2] can not be used to characterize finitely related graph algebras.

The following example illustrates that, in general, graph algebras are not associative, commutative, or idempotent. Let $\mathbb{G}$ be a graph such that $G = \{x, y, z\}$ and $E = \{(x, y), (y, z)\}$ and let $\mathbb{G}$ define algebra $\mathbf{C}$. Observe that $x * (y * z) = x$ and $(x * y) * z = 0$. Thus graph algebras are not associative and therefore are not semigroups. They are neither commutative, since $x = x * y \neq y * x = y$, nor idempotent, because $x * x = 0$. So no result concerning semigroups can be applied.



$\cdot$	0	x	y	z
0	0	0	0	0
x	0	0	x	0
y	0	y	0	y
z	0	0	z	0

Figure 2.2: Graph $\mathbb{G}$ defining algebra $\mathbf{C}$ and its multiplication table

3 Non-finitely related algebras

In this section we will show two classes of graph algebras which are not finitely related, those are all graph algebras defined by a graph containing $\mathbb{R}_1$ as an induced subgraph and not containing $\mathbb{R}_2$ and all graph algebras defined by a graph containing $\mathbb{R}_2$ such that its every vertex with a loop has an edge to every vertex. The graphs $\mathbb{R}_1$ and $\mathbb{R}_2$ are shown in the following figures. Observe that so called Murskiĭ's groupoid from Figure 1.3 is an example of a graph which belongs to the first mentioned class.

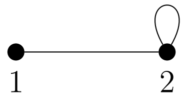


Figure 3.1: $\mathbb{R}_1$

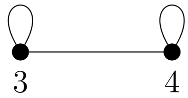


Figure 3.2: $\mathbb{R}_2$

We start with the class of algebras defined by a graph which has $\mathbb{R}_1$ as an induced subgraph and does not contain $\mathbb{R}_2$. We will prove that there exists an operation f such that condition (3) from Theorem 4 is not satisfied, that is, we will show that for every big enough n there exists an operation f which has term polymers, but $f \notin Clo(\mathbf{A})$.

Theorem 7. *All graph algebras defined by a graph containing graph $\mathbb{R}_1$ as an induced subgraph and not containing graph $\mathbb{R}_2$ as a subgraph are not finitely related.*

Proof. Let $\mathbf{A}$ be an algebra defined by a graph $\mathbb{G} = (G, E)$ satisfying the assumptions. Let $n \geq |A| + 1$. We define f as follows:

$$f(a_1, a_2, \ldots, a_n) = \begin{cases} a_i & \text{if } a_i \in L(\mathbb{G}) \text{ and } (a_j, a_k) \in E(\mathbb{G}) \text{ for every } j, k \in \underline{n} \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for all $\mathbf{a} \in A^n$, is $f(\mathbf{a})$ well defined. Indeed, if there exist $a_1, \ldots, a_n$ such that a_i,$a_j \in L(\mathbb{G})$ and $f(a_1, \ldots, a_n) \neq 0$, then (a_i, a_j) must be an edge in $\mathbb{G}$. Therefore a_i, a_j are vertices with a loop connected by an edge in $\mathbb{G}$. Then necessarily $a_i = a_j$, otherwise $\mathbb{G}$ contains $\mathbb{R}_2$, contrary to our assumptions.

Now we prove that for all i, j f_{ij} is a term operation, by showing that each operation f_{ij} is representable by a T-graph. Let $X = \{x_1, \ldots, x_n\}$. Let us fix an arbitrary pair $i, j \in \underline{n}$. Let $\mathbf{a}$ be an arbitrary n-tuple. From the definition of f_{ij} we get:

$$f_{ij}(a_1, a_2, \ldots, a_n) = \begin{cases} a_j & \text{if } a_j \in L(\mathbb{G}) \text{ and } (a_k, a_l) \in E(\mathbb{G}) \text{ for any } k, l \in \underline{n} \setminus i, \\ 0 & \text{otherwise.} \end{cases}$$

Claim. *The operation f_{ij} is represented by the T-graph $(\mathbb{H}, x_j)$, where $\mathbb{H}$ is the complete graph with vertices $X \setminus \{x_i\}$ together with a loop at x_j (see Figure 3.3).*

Proof. Note that the pair $(\mathbb{H}, x_j)$ is indeed a T-graph. Let $\mathbf{a}$ be an n-tuple such that $a_k \in G$ for all k such that $x_k \in H$. Assume that $\varphi : H \to V$ such that $x_k \mapsto a_k$ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Then each edge of $\mathbb{H}$ is mapped to an edge in $\mathbb{G}$. In particular, each loop is mapped to a loop. Therefore, $(a_k, a_l) \in E(\mathbb{G})$ for every k, l such that $x_k, x_l \in H$, and $a_j \in L(\mathbb{G})$. Thus $f_{ij}(a_1, \ldots, a_n) = a_j$.

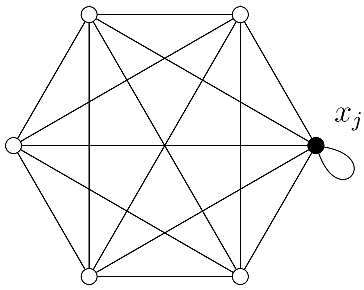


Figure 3.3: A graph representing f_{ij} for $n = 7$

Otherwise there is an edge $e = (x_k, x_l)$ in $\mathbb{H}$, possibly $k = l = j$, such that $(a_k, a_l) \notin E$. Therefore we have $f_{ij}(a_1, \ldots, a_n) = 0$.

Let $\mathbf{a}$ be an n-tuple such that $a_j = 0$ for some j such that $x_j \in H$. Obviously $f(a_1, \ldots, a_n) = 0$. ■

We claim that f is not a term operation of $\mathbf{A}$. Otherwise f is represented by a T-graph $(\mathbb{H}, v)$. Since $f(1, 2, \ldots, 2) = 2 \neq 0$, the mapping $x_1 \mapsto 1$, $x_2, \ldots, x_n \mapsto 2$ is a homomorphism from $\mathbb{G}$ to $\mathbb{R}_1$ and $2 = \varphi(v)$, so $v \neq x_1$. Similarly, v cannot be equal to any other variable $x_2, \ldots, x_n$ since $f(2, 1, 2, \ldots, 2) = \cdots = f(2, \ldots, 2, 1) = 2$.

We have shown that there is a term operation f for all $n \geq |A| + 1$, which does not satisfy the condition (3) from Theorem 4 . Therefore $\mathbf{A}$ is not finitely related. □

Note that Theorem 7 covers a broad class of graph algebras. In particular, it can be applied on disconnected graphs, for example graph $\mathbb{G} = \mathbb{G}_1 \dot\cup \mathbb{G}_2$ in Figure 3.4 is not finitely related, although we show later that graph $\mathbb{G}_1$ defines a finitely related algebra.

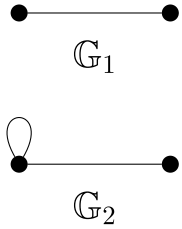


Figure 3.4: A graph defining non-finitely related algebra

Let us now prove that every graph $\mathbb{G}$, which contains $\mathbb{R}_2$ and such that every vertex with a loop is adjacent to every vertex in $\mathbb{G}$, defines a non-finitely related algebra. Although we use the same proof technique as before, the principle is slightly different. Last time, we defined an operation such that its significant vertex can not be determined uniquely. Now we will define an operation which can not be represented by any connected graph.

Theorem 8. *Let $\mathbb{G}$ be a connected graph containing $\mathbb{R}_2$ as an induced subgraph such that* $\deg(v) = |G|$ *for every $v \in L(\mathbb{G})$ and $L(\mathbb{G}) \neq G$. Then the algebra defined by $\mathbb{G}$ is non-finitely related.*

Proof. We will show that condition (3) from Theorem 4 is not satisfied, that is, that for each big enough n there exists an operation with term polymers which does not belong to $\mathrm{Clo}(\mathbf{A})$. Let $\mathbb{H}$ be a disjoint union of two complete graphs without loops, such that $H = \{x_1, \dots, x_n\}$. For every $\mathbf{a} \in A^n$, we define $f(a_1, \dots, a_n)$ as follows:

$$f(a_1,\dots,a_n) = \begin{cases} a_n & \text{if } \mathbf{a} \in A^n \text{ and the mapping } x_i \mapsto a_i \\ & \text{is a homomorphism from } \mathbb{H} \text{ to } \mathbb{G}, \\ 0 & \text{otherwise.} \end{cases}$$

First, we will show that the operation f can not be represented by a T-graph. Although there is a candidate for the significant vertex, the vertex x_n, there is no connected graph $\mathbb{H}'$ such that a T-graph $(\mathbb{H}', v)$ could represent f. Denote the components of $\mathbb{H}$ by X_1 and X_2 respectively. Assume that there is a T-graph $(\mathbb{H}', x_n)$ which has edge (x_i, x_j) such that $x_i \in X_1$ and $x_j \in X_2$. Observe that any graph $\mathbb{G}$ satisfying the assumptions has $\mathbb{R}_3$, which is shown in Figure 3.5, as an induced subgraph. Then the mapping $x_i \mapsto 5$, $x_j \mapsto 5$, $x_k \mapsto 4$ for every $x_k \in X_1 \setminus \{x_i\}$ and $x_k \mapsto 3$ for every $x_k \in X_2 \setminus \{x_j\}$ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. However, the mapping is not a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$, because there are two vertices connected by an edge mapped on a vertex without a loop, and therefore it follows from the definition of a T-graph that any connected graph can not represent f.

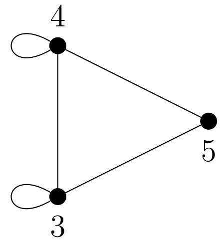


Figure 3.5: $\mathbb{R}_3$

We will show that every polymer of f can be represented by a T-graph. Let $1 \le 1 < j \le n$. We have to distinguish between two cases, whether x_i, x_j are in the same component or not.

Claim. *Let x_i and x_j be in different components, then the operation f_{ij} is a term operation.*

Proof. For simplicity assume that $x_i \in X_1$ and $x_j \in X_2$. We claim that f_{ij} can be represented by a T-graph $(\mathbb{H}', v)$ defined as follows: $H' = X \setminus x_i$, $E(\mathbb{H}') = \{E(\mathbb{H}) \cup \{(x_k, x_j) | \forall x_k \in X_1\} \setminus \{(x_k, x_i) | \forall x_k \in X_1\}$ and $v = x_n$. Note that $(\mathbb{H}', x_n)$

is a T-graph, because $X_1 \setminus x_i$ and X_2 induce complete subgraphs and the edges (x_k, x_j), where $x_k \in X_1$, are connecting them.

Let $\mathbf{a} \in A^n$, such that $a_k \in G$ for every $k \in \underline{n} \setminus i$ and let $\varphi' : H' \to G$, such that $x_k \mapsto a_k$ for every $x_k \in H'$, be a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$. We define a mapping $\varphi : H \to G$ by $x_i = a_j$ and $x_k = a_k$ for all the other coordinates. We show that if φ' is a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$, then φ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Let $(x_k, x_l) \in E(\mathbb{H})$, where x_k and x_l are both different from x_i. Since φ' maps (x_k, x_l) on edge in $\mathbb{G}$, φ maps (x_k, x_l) on edge in $\mathbb{G}$ as well. Let (x_k, x_i) be an edge in $E(\mathbb{G})$. The mapping φ' maps edge (x_k, x_j) on an edge in $\mathbb{G}$. Since $\varphi(x_k) = \varphi'(x_k)$ and $\varphi(x_i) = \varphi(x_j) = \varphi'(x_j)$ we see that φ maps the edge (x_k, x_i) on an edge in $\mathbb{G}$. So φ maps every edge in $\mathbb{H}$ on an edge in $\mathbb{G}$ and thus it is a homomorphism from $\mathbb{H}$ to $\mathbb{G}$. We have $f_{ij}(\mathbf{a}) = f(a_1, \ldots, a_j, \ldots, a_j, \ldots, a_n) = a_n$.

If φ' is not a graph homomorphism, there is edge (x_k, x_l) such that $(a_k, a_l) \notin E(\mathbb{G})$. If $k, l \neq j$ then (x_k, x_l) is in $E(\mathbb{H})$ and the mapping φ is not a homomorphism from $\mathbb{H}$ to $\mathbb{G}$. If $k = j$ then (x_i, x_l) is in $E(\mathbb{H})$, φ maps (x_i, x_l) on (a_j, a_l) and therefore φ is not a homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Similarly for $l = j$. In all the cases we have $0 = f(a_1, \ldots, a_j, \ldots, a_j, \ldots) = f_{ij}(\mathbf{a})$. Let $\mathbf{a} \in A^n$ such that $a_k = 0$ for some $k \in \underline{n} \setminus i$. Then obviously $f_{ij}(\mathbf{a}) = 0$.

It follows from the definition of a T-graph, that f_{ij} is represented by $(\mathbb{H}', x_n)$ and therefore it is a term operation. ∎

Since every vertex with a loop of $\mathbb{G}$ has an edge to every vertex, then every term can be represented by a T-graph $(\mathbb{H}', v)$, where $\mathbb{H}'$ has the same property.

Claim. *Let x_i and x_j be in the same component, then the operation f_{ij} is a term operation.*

Proof. We define a candidate T-graph for representing f_{ij} and we show that it indeed represents f_{ij}. For simplicity assume that x_i, x_j are in X_1. Let $H' = X \setminus x_i$, $E(\mathbb{H}') = \{E(\mathbb{H}') \cup \{(x_k, x_j) | x_k \in X_2\} \cup (x_j, x_j)\} \setminus \{(x_k, x_i) | x_k \in X_1\}$ and $v = x_n$. Observe that the pair $(\mathbb{H}', x_n)$ is a T-graph since x_j has an edge to every vertex and $x_n \in H'$.

Let $\mathbf{a} \in A^n$, such that $a_k \in G$ for every $k \in \underline{n} \setminus i$ and let $\varphi' : H' \to G$, such that $x_k = a_k$ for every $x_k \in H'$, be a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$. We define $\varphi : H \to G$ as follows: $x_i = a_j$ and $x_k = a_k$ for all the other coordinates. We show that if φ' is a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$, then φ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Let (x_k, x_l) be an edge in $\mathbb{H}$, where both x_k and x_l are different from x_i. Because φ' maps (x_k, x_l) on an edge in $\mathbb{G}$, $\varphi'(x_k) = \varphi(x_k)$ and $\varphi'(x_l) = \varphi(x_l)$, we see that $\varphi((x_k, x_l))$ is an edge in $E(\mathbb{G})$ as well. Let (x_k, x_i) be an edge in $\mathbb{H}$. Since φ' is a homomorphism, we have that $\varphi'((x_k, x_j))$ is in $E(\mathbb{G})$. Because $\varphi(x_k) = \varphi'(x_k)$ and $\varphi(x_i) = \varphi(x_j) = \varphi'(x_j)$ we see that φ maps the edge (x_k, x_i) to an edge in $\mathbb{G}$. Hence φ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and we have $f_{ij}(\mathbf{a}) = f(a_1, \ldots, a_j, \ldots, a_j, \ldots, a_n) = a_n$.

If φ' is not a homomorphism from $\mathbb{H}'$ to $\mathbb{G}$, then there is either x_j mapped on a vertex without a loop or an edge (x_k, x_l) such that $(a_k, a_l) \notin E(\mathbb{G})$. If $\varphi'(x_j)$ is a vertex without loop, then φ maps both x_i, x_j, connected by and edge, on the same vertex without a loop and therefore φ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Otherwise φ' maps x_j on a vertex with a loop and we assumed that all vertices with a loop in $\mathbb{G}$ are adjacent to every vertex, so we have $\deg(\varphi'(x_j)) = |G|$ and therefore if $\varphi'((x_k, x_l)) \notin E(\mathbb{G})$, then both x_k and x_l are distinct from x_j. If (x_k, x_l) such that $x_k \neq x_j$ and $x_l \neq x_j$ is an edge in $\mathbb{H}'$, then (x_k, x_l) is an edge in $\mathbb{H}$ as well. We have $\varphi(x_k) = \varphi'(x_k)$ and $\varphi(x_l) = \varphi'(x_l)$ and therefore $\varphi((x_k, x_l))$ is not an edge in $\mathbb{G}$. Hence φ is not a homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and so we have $f_{ij}(\mathbf{a}) = f(a_1, \dots, a_j, \dots, a_j, \dots, a_n) = 0$. Let $\mathbf{a} \in A^n$ such that $a_k = 0$ for some $k \in \underline{n} \setminus i$. Obviously we have $f_{ij}(\mathbf{a}) = 0$.

It follows from the definition of a T-graph that $(\mathbb{H}', x_n)$ represents f_{ij} and so it is a term operation. ■

The proof is now concluded. We showed that for every big enough n there exists a operation f with term polymers such that $f \notin \mathrm{Clo}(\mathbf{A})$. Therefore condition (3) from Theorem 4 is not satisfied and thus the algebra defined by $\mathbb{G}$ is non-finitely related. □

4 Finitely related algebras

In this chapter we present several new results about finitely related graph algebras. The chapter splits into two main parts: algebras defined by graphs containing $\mathbb{R}_2$ as an induced subgraph and algebras defined by bipartite graphs. Relational structure $\mathbb{R}_2$ is shown in Figure 3.2. In the both parts we use the same technique, the general idea is to show that condition (3) from Theorem 4 is satisfied: We will construct a T-graph $(\mathbb{H}, v)$ for an arbitrary operation f and show that operation $\overline{f}$ represented by the T-graph coincides with f.

We will use the following notation. By $\mathbf{a}b_i c_j$ we will denote the n-tuple such that the i-th coordinate is equal to b, the j-th coordinate is equal to c and the others are equal to a. By $\mathbf{a}^n\mathbf{b}^m c_i d_j$ we will denote the $(n+m)$-tuple such that the i-th coordinate is equal to c, the j-th coordinate is equal to d, all the other coordinates from the first to the n-the are equal to a and the others are equal to b.

Let us start with a lemma which is a reformulation of Corollary 1. We described how subalgebras and homomorphic images of a graph algebra look in Chapter 2, therefore is easy to observe that the following lemma is true.

Lemma 5. *Let* $\mathbf{A}$ *be a graph algebra defined by a graph* $\mathbb{G}$*. Let* $\mathbb{G}_1, \ldots, \mathbb{G}_n$ *be such that* $\mathbb{G}_i$ *is an induced subgraph of* $\mathbb{G}$ *or the algebra defined by* $\mathbb{G}_i$ *is a homomorphic image of* $\mathbf{A}$ *for every* i*. Then the algebra defined by the disjoint union of* $\mathbb{G}, \mathbb{G}_1, \ldots, \mathbb{G}_n$ *is finitely related if and only if* $\mathbf{A}$ *is.*

The previous lemma is illustrated by Figure 4.1. Denote by $\mathbf{A}$ the algebra defined by graph $\mathbb{G}_1$. By Theorem 9 is $\mathbf{A}$ finitely related. Obviously, $\mathbb{G}_2$ is an induced subgraph of $\mathbb{G}_1$. Although by Theorem 7 is the algebra defined by $\mathbb{G}_2$ non-finitely related, by the previous lemma is the algebra defined by $\mathbb{G}_1 \dot\cup \mathbb{G}_2$ finitely related. Note that Figure 3.4 provides an opposite example: $\mathbb{G}_1$ defines a non-finitely related algebra, $\mathbb{G}_2$ defines a finitely related algebra, but the algebra defined by $\mathbb{G}_1 \dot\cup \mathbb{G}_2$ is non-finitely related.

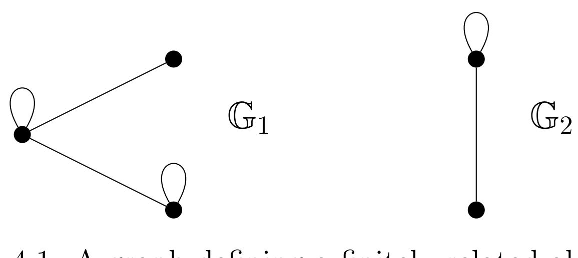


Figure 4.1: A graph defining a finitely related algebra

4.1 Algebras defined by a graph containing $\mathbb{R}_2$

We start this section with a useful observation about a significant vertex. The Lemma 6 says that every operation with term polymers of algebra $\mathbf{A}$ defined by a graph containing $\mathbb{R}_2$ has a uniquely determined significant vertex.

Lemma 6. *Let $\mathbb{G}$ be a graph containing $\mathbb{R}_2$ as a subgraph and $\mathbf{A}$ be the algebra defined by $\mathbb{G}$. Let $n \geq 4$ and let $f(x_1, \ldots, x_n)$ have term polymers. Then there is exactly one k such that $f(\mathbf{4}3_k) = 3$.*

Proof. Assume there is no k such that $f(\mathbf{4}3_k) = 3$. Then f_{12} satisfies $f_{12}(\mathbf{4}3_1 3_2) = f_{12}(\mathbf{4}) = f_{12}(\mathbf{4}3_3) = \ldots = f_{12}(\mathbf{4}3_n) = 4$. Since f_{12} is a term operation, it is representable by a T-graph $(\mathbb{H}, u)$. But u can not be equal to any of the variables $x_1, \ldots, x_n$, by the argument in the proof of Theorem 7.

Conversely, assume that there are two different indices k, l such that $f(\mathbf{4}3_k) = f(\mathbf{4}3_l) = 3$. For simplicity assume that $k = 1$ and $l = 2$. Then f_{34} satisfies $f_{34}(\mathbf{4}3_1) = 3$ and $f_{34}(\mathbf{4}3_2) = 3$. However, f_{34} is a term operation and therefore it is represented by a T-graph $(\mathbb{H}, u)$, The first equality implies $u = x_1$ and the second implies $u = x_2$, which is a contradiction. Thus there exists a single index k such that $f(\mathbf{4}3_k) = 3$. □

Note that when checking condition (3) from Theorem 4 we can assume that f depends on all of its coordinates. Indeed, if $f(x_1, \ldots, x_n)$ does not depend on the i-th coordinate, then $f(x_1, \ldots, x_{i-1}, x_i, \ldots, x_n) = f(x_1, \ldots, x_i, x_i, \ldots, x_n) = f_{i-1,i}(x_1, \ldots, x_n)$ and thus f is term operation as well.

This observation allows us to use the following lemma. It says that operations with term polymers which depend on all of its coordinates behave as term operations on n-tuples containing 0. Therefore we will be concerned only about n tuples such that $\mathbf{a} \in G^n$ while verifying that f coincides with $\overline{f}$.

Lemma 7. *Let $\mathbb{G}$ be a graph such that at least one vertex has a loop and let $f(x_1, \ldots, x_n)$ be an operation which depends on all of its coordinates and which has term polymers. Let $n \geq |A| + 2$. Then $f(\mathbf{a}) = 0$ whenever $a_k = 0$ for some k.*

Proof. Let $\mathbf{a}$ be an n-tuple such that $a_k = 0$ for some k. For simplicity assume that $k = 1$. Since f depends on the first coordinate, there exist two tuples $\mathbf{u}$ and $\mathbf{v}$ such that $v_1 \neq u_1$, $v_i = u_i$ for every $i \geq 2$ and $f(\mathbf{u}) \neq f(\mathbf{v})$. Note that we set n big enough to find a pair (i, j) such that $i \neq j$, i, j are distinct from 1 and $u_i = u_j$. So we have $f_{ij}(\mathbf{u}) \neq f_{ij}(\mathbf{v})$. Hence f_{ij} depends on x_1. Pick a vertex with a loop in $\mathbb{G}$ and denote it by 1. Then $f_{ij}(0, 1, \ldots, 1) = 0$. Thus we have $f(0, 1, \ldots, 1) = 0$ as well. Pick a pair (l, m) such that $l \neq m$, l, m are distinct from 1 and $a_l = a_m$. Then $f_{lm}(0, 1 \ldots, 1) = 0$ is satisfied and therefore f_{lm} depends on x_1. Hence $f_{lm}(\mathbf{a}) = 0$. From that it follows that $f(\mathbf{a}) = 0$. ■

Proposition 1. *Let $\mathbb{G} = (V, E)$ be a complete graph with loops at all vertices. Then the algebra $\mathbf{A}$ defined by $\mathbb{G}$ is finitely related.*

Proof. We will use the notation from Figure 3.2. Let $n \geq |A|+3$ and let $f(x_1, \ldots, x_n)$ have term polymers.

We will consider the case $|A| = 2$ separately. That is, $\mathbb{G}$ is a single vertex with a loop. We claim that f can be represented as the T-graph $(\mathbb{H}, x_1)$, where $\mathbb{H} = (X, X \times X)$. We need to show that $f(\mathbf{a}) = \overline{f}(\mathbf{a})$.

If $a_i = 0$ for some i, then $\overline{f}(\mathbf{a}) = 0$ by the definition of $\mathbb{H}$ and $f(\mathbf{a}) = 0$ by Lemma 7. Otherwise, $\mathbf{a} = (1, 1, \dots, 1)$, $\overline{f}(\mathbf{a}) = 1$ and $f(\mathbf{a}) = f_{12}(\mathbf{a}) = 1$.

Let us now assume that $|A| \geq 3$. By Lemma 6, there exists a unique k such that $f(\mathbf{43}_k) = 3$. We define the candidate T-graph representing f as $(\mathbb{H}, x_k)$, where $\mathbb{H} = (X, X \times X)$. Let, again, $\overline{f}$ is the operation represented by $(\mathbb{H}, x_k)$. We need to show that $f(\mathbf{a}) = \overline{f}(\mathbf{a})$.

If $a_i = 0$ for some i, then $f(\mathbf{a}) = \overline{f}(\mathbf{a})$ as before. So, assume $\mathbf{a} \in G^n$. By definition of $\overline{f}$, $\overline{f}(\mathbf{a}) = a_k$ and $\overline{f}(\mathbf{43}_k) = 3$. Since $n \geq |A| + 2$, we can find distinct i, j, both different from k, such that $a_i = a_j$. Then $f(\mathbf{a}) = f_{ij}(\mathbf{a})$. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}', u)$. We have $f_{ij}(\mathbf{43}_k) = 3$ and therefore x_k must be the significant vertex. Also note that any mapping $H' \to G$ is a homomorphism from $\mathbb{H}'$ to $\mathbb{G}$ as $\mathbb{G}$ is complete with a loop at every vertex. Thus $f(\mathbf{a}) = f_{ij}(\mathbf{a}) = a_k = \overline{f}(\mathbf{a})$ and the proof is concluded. □

The following theorem shows that every graph $\mathbb{G}$ containing $\mathbb{R}_4$ (see Figure 4.2) and such that the subgraph induced by $L(\mathbb{G})$ is complete graph defines a finitely related algebra. We can construct a candidate T-graph using relational structure $\mathbb{R}_4$. To learn whether there is an edge (x_i, x_j) in $\mathbb{H}$ is enough to check $f(\mathbf{43}_i\mathbf{5}_j)$, since it happens to be zero if (x_i, x_j) is an edge. Because the graph induced by $L(\mathbb{G})$ is a complete subgraph with loops at all vertices, we can assume that the graph induced by $L(\mathbb{H})$ is such.

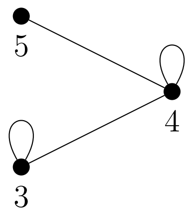


Figure 4.2: $\mathbb{R}_4$

Figure 4.3: $\mathbb{R}_1$

Theorem 9. *Let $\mathbb{G}$ be a graph containing $\mathbb{R}_4$ and let the subgraph induced by $L(\mathbb{G})$ be a complete graph with a loop at all vertices. Then $\mathbf{A}$ defined by $\mathbb{G}$ is finitely related.*

Proof. Let $n \geq |A| + 3$. Note that $|A|$ is at least 4, therefore $n \geq 7$. Let $f(x_1, \dots, x_n)$ be an arbitrary n-ary operation on A. Let f have term polymers and let f depend on all of its coordinates. Note that we set n big enough so that any n-tuple of elements of A contains two pairs (i_1, j_1), (i_2, j_2) of distinct indices such that $a_{i_1} = a_{j_1}$ and $a_{i_2} = a_{j_2}$. Let $X = \{x_1, \dots, x_n\}$.

Claim. *Let $x_i, x_j \in X$ such that $f(\mathbf{45}_i) = f(\mathbf{45}_j) \neq 0$. Then $f(\mathbf{43}_i\mathbf{5}_j) = 0 \Leftrightarrow f(\mathbf{43}_j\mathbf{5}_i) = 0$.*

Proof. Let x_i, x_j satisfy the assumption. Pick a pair (k, l) such that i, j, k, l are pairwise distinct. Then we have $f_{kl}(\mathbf{45}_i) = f_{kl}(\mathbf{45}_j)$. Let denote a T-graph representing f_{kl} by $(\mathbb{H}, v)$ and so there is no loop at x_i and x_j in the graph $\mathbb{H}$. Hence $f_{kl}(\mathbf{43}_i\mathbf{5}_j) = 0$ if and only if there is an edge $(x_i, x_j) \in \mathbb{H}$. Therefore

$f_{kl}(\mathbf{43}_i 5_j) = 0 \Leftrightarrow f_{kl}(\mathbf{43}_j 5_i) = 0$. From that follows that $f(\mathbf{43}_i 5_j) = 0 \Leftrightarrow f(\mathbf{43}_j 5_i) = 0$. ■

First, we construct a T-graph $(\mathbb{H}, v)$ representing f. We define $x_i \in L(\mathbb{H})$ if and only if $f(\mathbf{45}_i) = 0$. We define $E(\mathbb{H})$ as follows: $(x_i, x_j) \in E(\mathbb{H})$ if and only if $x_i, x_j \in L(\mathbb{H})$ or $x_j \in X \setminus L(\mathbb{H})$ and $f(\mathbf{43}_i 5_j) = 0$. Note that it follows from the previous claim that $E(\mathbb{H})$ is well defined. We define $H = X$.

Now we set the significant vertex v. All the assumptions of Lemma 6 are satisfied and therefore there is exactly one index k such that $f(\mathbf{43}_k) = 3$. We define $v = x_k$ if $v_k = 3$.

Claim. *The pair $(\mathbb{H}, x_k)$ is a T-graph.*

Proof. It is enough to show that $\mathbb{H}$ is connected. Observe that a graph on a vertex set V is connected if and only if there is an edge between every pair of disjoint subsets V_1, V_2 such that $V_1 \cup V_2 = V$.

Pick two disjoint subsets $V_1, V_2 \subset H$ such that $V_1 \cup V_2 = H$. If two vertices x_i,x_j with a loop are in different subsets, then there is edge (x_i, x_j) between V_1 and V_2 and we are done. Otherwise assume that $|V_1| \geq 2$. Pick $x_i, x_j \in V_1$. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}_1, u)$. Therefore $\mathbb{H}_1$ is connected and thus there is an edge between V_1 and V_2 in $\mathbb{H}_1$. That is, there are $x_k \in V_1$ and $x_l \in V_2$ such that $f_{ij}(\mathbf{43}_k 5_l) = f_{ij}(\mathbf{43}_l 5_k) = 0$. Assume that $x_l \in X \setminus L(\mathbb{H}_1)$. If $k \neq j$ we have $f(\mathbf{43}_k 5_l) = 0$ and from definition of $\mathbb{H}$ it follows that $(x_k, x_l) \in \mathbb{H}$. Otherwise we have $f(\mathbf{43}_i 3_j 5_l) = 0$. Pick a pair (p, q) such that both p, q are different from i, j, l. Then $f_{pq}(\mathbf{43}_i 5_l) = 0$ or $f_{pq}(\mathbf{43}_j 5_l) = 0$ is satisfied. Therefore the same is satisfied for f and it follows from the definition of $\mathbb{H}$ that there is either $(x_i, x_l) \in E(\mathbb{H})$ or $(x_j, x_l) \in E(\mathbb{H})$. Thus the graph $\mathbb{H}$ is connected. ■

We have shown that the pair $(\mathbb{H}, x_k)$ is a T-graph. Hence it represents a term operation $\overline{f}$. We will prove that these two operations coincide and therefore f is represented by the T-graph $(\mathbb{H}, v)$ as well.

Claim. *Let* $\mathbf{a}$ *be an n-tuple such that $f(\mathbf{a}) = 0$. Then $\overline{f}(\mathbf{a}) = 0$*

Proof. Let $\mathbf{a}$ be an n-tuple such that $f(\mathbf{a}) = 0$. If there is k such that $a_k = 0$ then $\overline{f}(\mathbf{a}) = 0$ because $\mathbb{H}$ is connected and $H = X$. Otherwise, pick a pair (i, j) such that $a_i = a_j$ and so $f_{ij}(\mathbf{a}) = 0$. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}', u)$. Therefore mapping $\varphi : H' \to V$ such that $x_k \mapsto a_k$ for every k such that $x_k \in H'$ is not a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$. This means there is either a vertex $x_l \in L(\mathbb{H}')$ mapped on a vertex a_l without a loop or an edge $(x_l, x_m) \in E(\mathbb{H}')$ such that $(a_l, a_m) \notin E(\mathbb{G})$.

Assume that there is a vertex x_l with a loop mapped on a vertex a_l without a loop. If $l \neq j$, then $f_{ij}(\mathbf{45}_l) = f(\mathbf{45}_l) = 0$. From the definition of $\mathbb{H}$ it follows that vertex $x_l \in V(\mathbb{H})$ has a loop and therefore φ is neither a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Hence $\overline{f}(\mathbf{a}) = 0$. If $l = j$, then we have $f(\mathbf{45}_i 5_j) = 0$. Let (p, q) be a pair such that p, q and i, j are distinct. Therefore we have $f_{pq}(\mathbf{45}_i 5_j) = 0$. Since f_{pq} is a term operation, then there is a loop at x_i or a loop at x_j or an edge (x_i, x_j) in a graph representing f_{pq}. Hence $f_{pq}(\mathbf{45}_i) = 0$ or $f_{pq}(\mathbf{45}_j) = 0$ or $f_{pq}(\mathbf{43}_i 5_j) = 0$ is satisfied. It follows from the definition of f_{pq} that the same is satisfied for f. Note that $a_l = a_j = a_i$. It follows from the definition of $\mathbb{H}$ that $x_i \in L(\mathbb{H})$ or

$x_j \in L(\mathbb{H})$ or $(x_i, x_j) \in E(\mathbb{H})$ and therefore $\varphi : \mathbb{H} \to \mathbb{G}$ is not a homomorphism. Thus $\overline{f}(\mathbf{a}) = 0$.

Assume that $(x_l, x_m) \in E(\mathbb{H}')$ such that $a_l a_m \notin E$. In the previous paragraph we treated the case $l = m$. Because all the vertices with a loop in $\mathbb{G}$ form a complete subgraph of $\mathbb{G}$, both x_m and x_l can not be in $L(\mathbb{H}')$. Assume without loss of generality that $x_l \in X \setminus L(\mathbb{H}')$ and $l < m$.

We must distinguish between three cases. Let $l, m \neq j$. Because f_{ij} is a term function and (x_l, x_m) is an edge in $\mathbb{H}'$, we have $f_{ij}(\mathbf{4}5_l3_m) = f(\mathbf{4}5_l3_m) = 0$. It follows from the definition of $\mathbb{H}$ it that $(x_l, x_m) \in E(\mathbb{H})$. But we assumed (a_l, a_m) is not an edge in $\mathbb{G}$ and therefore φ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Thus we have $\overline{f}(\mathbf{a}) = 0$.

If $l = j$, then $f_{ij}(\mathbf{4}5_l3_m) = f(\mathbf{4}5_i5_j3_m) = 0$ is satisfied. We pick a pair (p, q) such that p, q and i, j are distinct. Then we have $f_{pq}(\mathbf{4}5_i3_m) = 0$ or $f_{pq}(\mathbf{4}5_j3_m) = 0$ and therefore $(x_i, x_m) \in E(\mathbb{H})$ or $(x_j, x_m) \in E(\mathbb{H})$. Note that $a_i = a_j = a_l$ and thus φ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Hence $\overline{f}(\mathbf{a}) = 0$. The proof for $m = j$ is analogical. ∎

Claim. *Let* $\mathbf{a}$ *be an* n*-tuple such that* $\overline{f}(\mathbf{a}) = 0$. *Then* $f(\mathbf{a}) = 0$.

Proof. Let $\mathbf{a}$ be an n-tuple such that $\overline{f}(\mathbf{a}) = 0$. At first assume there exists k such that $a_k = 0$. By Lemma 7 $f(\mathbf{a}) = 0$.

Assume all the elements of $\mathbf{a}$ are nonzero. Then $\overline{f}(\mathbf{a}) = 0$ if $\varphi : H \to V$, such that $x_k \mapsto a_k$, is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. That is a vertex $x_l \in L(\mathbb{H})$ is mapped on a vertex a_l without a loop or an edge (x_l, x_m) is mapped on $(a_l, a_m) \notin E(\mathbb{G})$.

Let x_l is a vertex with a loop such that its image is a vertex without loop. Since $\overline{f}$ is a term operation, we have $\overline{f}(\mathbf{4}5_l) = 0$ and therefore $f(\mathbf{4}5_l) = 0$. Pick a pair (i, j) such that $i \neq j$, i, j are different from l and $a_i = a_j$. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}', u)$. Then $f_{ij}(\mathbf{4}5_l) = 0$ and thus $x_l \in E(\mathbb{H}')$ has loop as well. Therefore $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = 0$.

Let edge $(x_l, x_m) \in E(\mathbb{H})$ be such that $\varphi(x_l, x_m) \notin E(\mathbb{G})$. As before we can assume that at least one vertex does not have a loop. Without loss of generality $x_l \in X \setminus L(\mathbb{H})$. Therefore we have $\overline{f}(\mathbf{4}5_l3_m) = f(\mathbf{4}5_l3_m) = 0$. We can pick a pair (i, j) such that $i \neq j$,$a_i = a_j$ and i, j are different from k, l. Then $f_{ij}(\mathbf{4}5_l3_m) = 0$, and since $f_{ij} \in Clo(\mathbf{A})$ we have $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = 0$. ∎

Both claims above combined give an equivalence $f(\mathbf{a}) = 0 \Leftrightarrow \overline{f}(\mathbf{a}) = 0$.

Claim. *Let* $\mathbf{a}$ *be an* n*-tuple such that* $\overline{f}(\mathbf{a}) = a_k$. *Then* $f(\mathbf{a}) = a_k$.

Proof. Let $\mathbf{a} \in A^n$ be an n-tuple such that $\overline{f}(\mathbf{a}) = a_k$. Because $\overline{f}$ is represented by the T-graph $(\mathbb{H}, x_k)$, $f(\mathbf{4}3_k) = 3$ is satisfied. As before we can find a pair (i, j) such that $a_i = a_j$, $i \neq j$ and both i, j are distinct from k. Since f_{ij} is a term operation, it is represented by a T-graph. We have $f_{ij}(\mathbf{4}3_k) = 3$ and therefore x_k must be the significant vertex. We have proved before that $\overline{f}(\mathbf{a}) = 0$ if and only if $f(\mathbf{a}) = 0$ and therefore $f(\mathbf{a}) \in G$. From that it follows that $f_{ij}(\mathbf{a}) = a_k$ and consequently $f(\mathbf{a}) = a_k$. ∎

The proof is now concluded. At first we constructed a pair $(\mathbb{H}, x_k)$. We showed that the pair is indeed a T-graph. Thus the operation $\overline{f}$ represented by $(\mathbb{H}, x_k)$ is a term operation. Then we proved that for all $\mathbf{a} \in A^n$ $f(\mathbf{a}) = \overline{f}(\mathbf{a})$ and therefore

f is a term operation as well. Since we assumed that f has term polymers, we showed that condition (3) from Theorem 4 is satisfied. $\square$

Note that there are disconnected graphs which are satisfying the assumptions of the previous theorem, An example of a such graph is in the following figure. Note that those graphs have vertices with loops in one component.

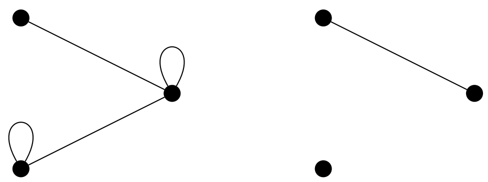


Figure 4.4: A graph defining a finitely related algebra

The next theorem shows that all graphs containing $\mathbb{R}_5$ as an induced subgraph define a finitely related algebra. We do not assume anything about $L(\mathbb{G})$ or about connectedness of $\mathbb{G}$, in other words, having $\mathbb{R}_5$ as an induced subgraph is enough to be finitely related. It is easy to construct a candidate T-graph $(\mathbb{H}, x_k)$. We use graph $\mathbb{R}_1$ to recognize which vertices in $\mathbb{H}$ have a loop. Intuitively $x_i \in L(\mathbb{H})$ if and only if $f(\mathbf{21}_i) = 0$. Similarly, we use graph $\mathbb{R}_4$ to determine whether (x_i, x_j) is an edge in $\mathbb{H}$ or not. We put (x_i, x_j) in $E(\mathbb{H})$ if and only if $f(\mathbf{43}_i5_j) = 0$.

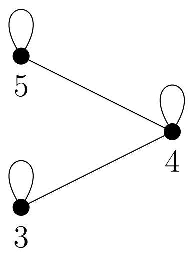


Figure 4.5: $\mathbb{R}_5$

Figure 4.6: $\mathbb{R}_1$

Theorem 10. *Let $\mathbb{G}$ be a graph containing $\mathbb{R}_5$ as as an induced subgraph. Then algebra $\mathbf{A}$ defined by graph $\mathbb{G}$ is finitely related.*

Proof. As before we will prove that condition (3) from Theorem 4 is satisfied. Let $n \geq |A| + 3$ and let $f(x_1, \ldots, x_n)$ be an arbitrary n-ary operation on A, which has term polymers and which depends on all of its coordinates. Note that we set n big enough so that any n-tuple of elements of A contains two pairs (i_1, j_1), (i_2, j_2) of distinct indices such that $a_{i_1} = a_{j_1}$ and $a_{i_2} = a_{j_2}$. Let $X = \{x_1, \ldots, x_n\}$.

We will define a T-graph $(\mathbb{H}, v)$. Then we prove that function represented by the T-graph, we denote it by $\overline{f}$, coincides with f and hence f is a term operation.

Claim. *We have $f(\mathbf{43}_i5_j) = 0$ if and only of $f(\mathbf{43}_j5_i) = 0$.*

Proof. Pick a pair (k, l) such that k, l are different from i, j and $k \neq l$. Then $f_{kl}(\mathbf{43}_i5_j) = 0$ if and only if there is an edge (x_i, x_j) in $\mathbb{H}'$, where $(\mathbb{H}', u)$ represents f_{kl}. Therefore $f_{kl}(\mathbf{43}_i5_j) = 0 \Leftrightarrow f_{kl}(\mathbf{43}_j5_i) = 0$ and thus the same holds for f. $\blacksquare$

First, we define $H = X$. Let us define $L(\mathbb{H})$. If there is a loop at every vertex of $\mathbb{G}$, then we define $L(\mathbb{H}) = X$, otherwise $\mathbb{G}$ contains $\mathbb{R}_1$ as an induced subgraph and $x_i \in L(\mathbb{H})$ if and only if $f(\mathbf{21}_i) = 0$. We define $(x_i, x_j) \in E(\mathbb{H})$ if and only if $i = j$ and $x_i \in L(\mathbb{H})$ or $f(\mathbf{43}_i 5_j) = 0$. Note that by the claim above the set $E(\mathbb{H})$ is well defined. Because all the assumptions of Lemma 6 are satisfied, there is only one index k such that $f(\mathbf{43}_k) = 3$. We say that the significant vertex v is equal to x_k if $v_k = 3$.

Claim. *The pair* $(\mathbb{H}, x_k)$ *is a T-graph*

Proof. It is enough to show that $\mathbb{H}$ is connected. Observe that a graph on a vertex set V is connected if and only if there is an edge between every pair of disjoint subsets V_1, V_2 such that $V_1 \cup V_2 = V$.

Let V_1, V_2 be such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = H$. Assume that $|V_1| \geq 2$ and pick two vertices x_i, x_j in V_1. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}', u)$. Since $\mathbb{H}'$ is connected, there is an edge (x_k, x_l) in $\mathbb{H}'$ between $V_1 \setminus \{x_i\}$ and V_2. So $f(\mathbf{43}_k 5_l) = 0$ is satisfied. If $k, l \neq j$ then we have $f(\mathbf{43}_k 5_l)$ and therefore $(x_k, x_l) \in E(\mathbb{H})$. If $k = j$ then (p, q) such that $p \neq q$ and k, l, i are distinct from p, q. Then we have $f_{ij}(\mathbf{43}_k 5_l) = f(\mathbf{43}_i 3_k 5_l) = f_{pq}(\mathbf{43}_i 3_k 5_l) = 0$. Since f_{pq} is a term operation we have $f_{pq}(\mathbf{43}_i 5_l) = 0$ or $f_{pq}(\mathbf{43}_k 5_l) = 0$. Hence the same is satisfied for f and from the definition of $\mathbb{H}$ it follows that (x_i, x_l) or (x_j, x_l) is an edge in $E(\mathbb{H})$. If $l = j$ we proceed similarly. ∎

We have shown that $(\mathbb{H}, x_k)$ is indeed a T-graph. Therefore $\overline{f}$ is a term operation. Now we prove that f and $\overline{f}$ coincide. That is, for every n-tuple $\mathbf{a}$ is $f(\mathbf{a}) = \overline{f}(\mathbf{a})$ satisfied.

Claim. *Let* $\mathbf{a}$ *be an* n*-tuple such that* $f(\mathbf{a}) = 0$. *Then* $\overline{f}(\mathbf{a}) = 0$.

Proof. Let $\mathbf{a}$ be an n-tuple such that $f(\mathbf{a}) = 0$. If there exists k such that a_k then $\overline{f}(\mathbf{a}) = 0$, since $\mathbb{H}$ is connected. Otherwise pick a pair (i, j) such that $a_i = a_j$. Hence $f_{ij}(\mathbf{a}) = 0$ as well. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}_1, u_1)$. So the mapping $x_k \mapsto a_k$, for every k such that $x_k \in H_1$, we denote it by φ, is not a graph homomorphism from $\mathbb{H}_1$ to $\mathbb{G}$. That is, there is either a vertex x_k with a loop mapped on a vertex without a loop or an edge (x_k, x_l) in $\mathbb{H}_1$, such that $(a_k, a_l) \notin E(\mathbb{G})$.

At first assume that there is a vertex $x_k \in L(\mathbb{H}_1)$ such that $a_k \notin L(\mathbb{G})$. Note that this case does not occur if $L(\mathbb{G}) = G$. So we have $f_{ij}(\mathbf{21}_k) = 0$. If $k \neq j$ then $f(\mathbf{21}_k) = 0$ is satisfied and it follows from the definition of $\mathbb{H}$ that there is a loop at vertex x_k in $\mathbb{H}$ as well. Hence $\overline{f}(\mathbf{a}) = 0$.

If $k = j$ then $f(\mathbf{21}_i 1_j) = 0$ holds. Pick a pair (p,q) such that p, q are distinct from i, j. Since f_{pq} is a term operation, it is represented by a T-graph $(\mathbb{H}_2, u_2)$ and there is a loop at x_i or a loop at x_j or an edge (x_i, x_j) in $\mathbb{H}_2$. Then $f_{pq}(\mathbf{21}_i) = 0$ or $f_{pq}(\mathbf{21}_j) = 0$ or $f_{pq}(\mathbf{43}_i 5_j) = 0$ is satisfied. Therefore $f(\mathbf{21}_i) = 0$ or $f(\mathbf{21}_k) = 0$ or $f(\mathbf{43}_i 5_j) = 0$ is satisfied and it follows from the definition of $\mathbb{H}$ that x_i or x_k in $\mathbb{H}$ has a loop or there is edge (x_i, x_k) in $E(\mathbb{H})$. Since x_i, x_j are mapped on vertex a_k, which does not have a loop, φ is neither a homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and thus $\overline{f}(\mathbf{a}) = 0$.

Assume that there is an edge $(x_k, x_l) \in E(\mathbb{H}_1)$ such that $(a_k, a_l) \notin E(\mathbb{G})$. Since f_{ij} is a term operation, we have $f_{ij}(\mathbf{43}_k 5_l) = 0$. If $k, l \neq j$ then $f(\mathbf{43}_k 5_l) = 0$ is satisfied and thus $(x_k, x_l) \in E(\mathbb{H})$. Therefore φ is not a homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and hence $\overline{f}(\mathbf{a}) = 0$.

If $k = j$ then we have $f(\mathbf{4}\mathbf{3}_i 3_j 5_l) = 0$. Pick a pair (p, q) such that p, q are distinct from i, j, l. Since f_{pq} is a term operation $f_{pq}(\mathbf{4}\mathbf{3}_i 5_l) = 0$ or $f_{pq}(\mathbf{4}\mathbf{3}_j 5_l) = 0$ is satisfied. Because p, q are distinct from i, j, l we have $f(\mathbf{4}\mathbf{3}_i 5_l) = 0$ or $f(\mathbf{4}\mathbf{3}_j 5_l) = 0$ and from the definition of $\mathbb{H}$ follows that either $(x_k, x_l) \in E(\mathbb{H})$ or $(x_i, x_l) \in E(\mathbb{H})$. Note that $a_i = a_j = a_k$ and therefore φ is not a homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Hence $\overline{f}(\mathbf{a}) = 0$. Similarly we can show that $\overline{f}(\mathbf{a}) = 0$ if $l = j$. The proof is analogous. ■

Claim. *Let* $\mathbf{a}$ *be an* n*-tuple such that* $\overline{f}(\mathbf{a}) = 0$*. Then* $f(\mathbf{a}) = 0$*.*

Proof. Let $\mathbf{a}$ be an n-tuple such that $\overline{f}(\mathbf{a}) = 0$. Assume that $a_i = 0$ for some i. Then by Lemma 7 we have $f(\mathbf{a}) = 0$. Assume that $a_i \neq 0$ for every $i \in \underline{n}$. Since $\overline{f}$ is represented by $\mathbb{H}$, the mapping $x_i \mapsto a_i$ for every i such that $x_i \in H$ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. We will denote the mapping by φ. Therefore φ maps a vertex x_k with a loop on a vertex without a loop or an edge (x_k, x_l) on $(a_k, a_l) \notin E(\mathbb{G})$.

Assume that φ maps $x_k \in L(\mathbb{H})$ on $a_k \notin L(\mathbb{G})$. Note that this case does not occur if $L(\mathbb{G}) = G$. Therefore $\overline{f}(\mathbf{2}1_k) = f(\mathbf{2}1_k) = 0$. Pick a pair (i, j) such that $i, j \neq k$ and $a_i = a_j$. Then we have $f_{ij}(\mathbf{2}1_k) = 0$ as well. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}', u)$. Therefore there is loop at vertex x_k in $\mathbb{H}'$. From that follows that $\varphi \restriction_{H'}$ is neither a homomorphism from $\mathbb{H}'$ to $\mathbb{G}$ and thus $f_{ij}(\mathbf{a}) = 0$. From the definition of f_{ij} follows that $f(\mathbf{a}) = 0$.

Assume that φ maps an edge (x_k, x_l) in $E(\mathbb{H})$ on $(a_k, a_l) \notin E(\mathbb{G})$. Therefore we have $\overline{f}(\mathbf{4}\mathbf{3}_k 5_l) = f(\mathbf{4}\mathbf{3}_k 5_l) = 0$. Pick a pair (i, j) such that i, j are distinct from k, l and $a_i = a_j$. Therefore $f_{ij}((\mathbf{4}\mathbf{3}_k 5_l))$ is satisfied. Since f_{ij} is a term operation, it is represented by a T-graph $(\mathbb{H}', u)$. Therefore $(x_k, x_l) \notin E(\mathbb{H}')$. Thus $\varphi \restriction_{H'}$ is not a homomorphism from $\mathbb{H}'$ to $\mathbb{G}$ and hence $f_{ij}(\mathbf{a}) = 0$. Then $f(\mathbf{a}) = 0$ as well. ■

Claim. *Let* $\mathbf{a}$ *be an* n*-tuple such that* $\overline{f}(\mathbf{a}) = b$*, where* $b \neq 0$*. Then* $f(\mathbf{a}) = b$*.*

Proof. Let $\mathbf{a}$ be an n-tuple such that $\overline{f}(\mathbf{a}) = a_k$. Because $\overline{f}$ is represented by the T-graph $(\mathbb{H}, x_k)$, $f(\mathbf{4}\mathbf{3}_k) = 3$ is satisfied. As before we can find a pair (i, j) such that $a_i = a_j$, $i \neq j$ and both i, j are distinct from k. Since f_{ij} is a term operation, it is represented by a T-graph. We have $f_{ij}(\mathbf{4}\mathbf{3}_k) = 3$ and therefore x_k must be the significant vertex. We have proved before that $\overline{f}(\mathbf{a}) = 0$ if and only if $f(\mathbf{a}) = 0$ and therefore $f(\mathbf{a}) \in G$. From that it follows that $f_{ij}(\mathbf{a}) = a_k$ and consequently $f(\mathbf{a}) = a_k$. ■

The proof is now complete. At first we defined a pair $(\mathbb{H}, x_k)$. Then we proved that it is a T-graph. Finally we showed that $f(\mathbf{a}) = \overline{f}(\mathbf{a})$ for all n-tuples $\mathbf{a}$ in A^n. So these two operations coincide and thus f is represented by the T-graph as well. Hence f is a term operation and the condition (3) from Theorem 4 is satisfied. □

We showed few results concerning graphs with $\mathbb{R}_2$ as an induced subgraph in this section. As we could see in the previous chapter, there are algebras defined by graphs containing $\mathbb{R}_2$ which are non-finitely related. However, there are graphs containing $\mathbb{R}_2$ such that no theorem from this section or the previous chapter can be applied. We believe that all the others algebras defined by a graph containing $\mathbb{R}_2$ are finitely related, although we do not have any proof. For a discussion see the last section.

4.2 Algebras defined by bipartite graphs

In this section we will show that all algebras defined by connected bipartite graphs are finitely related, we will prove, as usual, that condition (3) from Theorem 4 holds.

First observe that every term operation of an algebra defined by a bipartite graph is either a constant zero operation or can be represented by a bipartite graph without loops. Indeed, let f be an n-ary term operation, $\mathbf{a} \in A^n$ an n-tuple such that $f(\mathbf{a}) \neq 0$ and let $(\mathbb{H}, v)$ be a T-graph representing f. A mapping such that $x_i \mapsto a_i$ for every $x_i \in H$ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$, therefore $\mathbb{H}$ does not contain a cycle of odd length, possibly a loop. In other words, $\mathbb{H}$ is a bipartite graph without loops.

Lemma 8. *Let* $\mathbf{A}$ *be an algebra defined by a bipartite graph. Let* $n \geq |A|^2 + 2$ *and assume that* $f(x_1, \ldots, x_n)$ *is an* n*-ary operation on* A *which has term polymers and which depends on all of its coordinates. Let* $\mathbf{a} \in A^n$ *be such that* $a_i = 0$ *for some* $i \in \underline{n}$*. Then* $f(\mathbf{a}) = 0$*.*

Proof. For simplicity assume that $a_1 = 0$. Since f depends on all of its coordinates, we have two tuples $\mathbf{u}, \mathbf{v}$, such that $u_1 \neq v_1$, $u_i = v_i$ for every $i \geq 2$ and $f(\mathbf{u}) \neq f(\mathbf{v})$. Observe that we set n big enough to find a pair of indices (i, j) such that $i < j$, both of i, j are distinct from 1, $a_i = a_j$ and $u_i = u_j$. So we have $f_{ij}(\mathbf{u}) \neq f_{ij}(\mathbf{v})$. Operation f_{ij} is not a constant zero operation, at least $f_{ij}(\mathbf{u})$ or $f_{ij}(\mathbf{v})$ is different from zero, and therefore it can be represented by a T-graph $(\mathbb{H}, v)$. Since we have $f_{ij}(\mathbf{u}) \neq f_{ij}(\mathbf{v})$, we see that x_1 is in H and thus $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = 0$. Similarly we can show that $f(\mathbf{a}) =$ if $a_i = 0$ for any $i \in \underline{n}$. □

Lemma 9. *Let* $\mathbf{A}$ *be an algebra defined by a connected bipartite graph* $\mathbb{G}$*. Let* f *be an operation with term polymers and which depends on all of its coordinates and* f_{ij} *be its polymer such that* f_{ij} *is not a constant zero operation. Then any T-graph* $(\mathbb{H}, v)$ *representing* f_{ij} *satisfies* $H = X \setminus \{x_i\}$*.*

Proof. Assume that there is x_k, for some k, and a T-graph $(\mathbb{H}, v)$, such that $x_k \notin H$. For simplicity assume that $i \neq 1$ and $x_1 \notin H$. Since f_{ij} is not a constant zero operation, there exists an n-tuple $\mathbf{a}$ such that $a_i = a_j$ and $f_{ij}(\mathbf{a}) \neq 0$. Because $x_1 \notin H$, the mapping such that $x_k \mapsto a_k$, for all $x_k \in H$, is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and thus $f_{ij}(0, a_2, \ldots) \neq 0$. Since $a_i = a_j$ we have $f(0, a_2, \ldots) \neq 0$. Observe, that all the assumptions of Lemma 8 are satisfied and therefore $f(0, a_2, \ldots) = 0$ which is an contradiction. Hence $x_1 \in H$ for any T-graph representing f_{ij}. □

Lemma 10. *Let* $\mathbf{A}$ *be an algebra defined by a connected bipartite graph* $\mathbb{G}$*,* $n > |A| + 1$ *and let* f *be an* n*-ary operation, which is not a constant zero operation. Assume that* f *has term polymers and* f *depends on all of its coordinates. Then there are two disjoint and nonempty sets of coordinates,* X_1 *and* X_2*, such that* $X_1 \cup X_2 = X$ *and* $f(\mathbf{a}) \neq 0$*, where* $\mathbf{a} \in A^n$ *is an* n*-tuple such that* $a_i = j$ *if* $x_i \in X_j$*, for* $j \in \{1, 2\}$*.*

Proof. Since f is not a constant zero operation, there exists an n-tuple $\mathbf{u}$ such that $f(\mathbf{u}) \neq 0$. We set n big enough to find a pair of indices (i, j) such that $i < j$ and $u_i = u_j$, so we have $f_{ij}(\mathbf{u}) \neq 0$. Operation f_{ij} is a term operation and by

Lemma 9 can be represented by a T-graph $(\mathbb{H}, v)$, where $\mathbb{H}$ is a bipartite graph and $H = X \setminus \{x_i\}$. Assume that $x_j \in H_1$. We define an n-tuple $\mathbf{a}$ as follows $a_i = 1$ and $a_k = l$ if $x_k \in H_l$. Obviously there is at least one index k such that $a_k = 1$ and one index k such that $a_k = 2$. The mapping defined by $x_k \mapsto a_k$, for every $x_k \in H$ is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and so we have $f(\mathbf{a}) \neq 0$. Since $a_i = a_j$, it follows from the definition of a polymer that $f_{ij}(\mathbf{a}) = f(\mathbf{a}) \neq 0$. □

Lemma 11. *Let X_1 and X_2 be the sets from Lemma 10 and let $x_i, x_j \in X_k$, for some $k \in \{1,2\}$. Let $(\mathbb{H}, v)$ be a T-graph representing f_{ij}, then parts of $\mathbb{H}$ are $X_1 \setminus \{x_i\}$ and $X_2 \setminus \{x_i\}$.*

Proof. From the Lemma 10 we know that there is an n-tuple $\mathbf{a}$ such that $a_i = k$ if $x_i \in X_k$ and $f(\mathbf{a}) \neq 0$. Since $x_i, x_j \in X_k$ we have $f_{ij}(\mathbf{a}) \neq 0$ and therefore f_{ij} is represented by a T-graph $(\mathbb{H}, v)$, where $\mathbb{H}$ is a bipartite graph. By Lemma 9 is $H = X \setminus \{x_i\}$. Because $f_{ij}(\mathbf{a}) \neq 0$, the mapping such that $x_k = a_k$ for every $x_k \in H$ is a graph homomorphism and thus parts of $\mathbb{H}$ are $X_1 \setminus \{x_i\}$ and $X_2 \setminus \{x_i\}$. □

Observe that every connected bipartite graph on three or more vertices contains graph $\mathbb{R}_6$ as an induced subgraph. The graph $\mathbb{R}_6$ is shown in Figure 4.7. We will use that substructure to recognize a significant vertex of an operation.

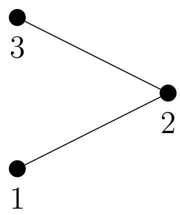


Figure 4.7: $\mathbb{R}_6$

Lemma 12. *Let $\mathbb{G}$ be a bipartite graph such that $|G| \geq 3$ and $\mathbf{A}$ be the algebra defined by $\mathbb{G}$. Let $n \geq |A|^2 + 2$ and let $f(x_1, \dots, x_n)$ be an operation with term polymers which depends on all of its coordinates. Assume that there is at least one $\mathbf{a} \in A^n$ such that $f(\mathbf{a}) \neq 0$. Let S be the of n-tuples such that one coordinate is equal to 3 and the others are equal to 1 or 2. Then there is only one tuple $\mathbf{v} \in S$ such that $f(\mathbf{v}) = 3$.*

Proof. Pick a pair (i,j) such that $i \neq j$ and f_{ij} is a nonzero operation. Since f_{ij} is a term operation, it is representable by a T-graph $(\mathbb{H}, x_k)$, where $\mathbb{H}$ is bipartite graph, and it follows from Lemma 11 that its parts are $H_1 = X_1 \setminus \{x_i\}$ and $H_2 = X_2 \setminus \{x_i\}$. For simplicity assume that $x_k \in H_1$. The mapping $\varphi : H \to G$, such that $x_k \mapsto 3$,$x_l \mapsto 2$ if $x_l \in H_2$ and $x_l \mapsto 1$ otherwise, is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. So we have $f_{ij}(\varphi(x_1), \dots, \varphi(x_n)) = 3$. Let $\varphi(x_i) = \varphi(x_j)$, then $f(\varphi(x_1), \dots, \varphi(x_n)) = 3$.

Assume that there are two n-tuples $\mathbf{u}, \mathbf{v} \in S$ such that $f(\mathbf{u}) = f(\mathbf{v}) = 3$. Since we set n big enough, there is a pair (i,j) such that $i \neq j$, $u_i = u_j$ and $v_i = v_j$. Therefore $f_{ij}(\mathbf{u}) = f_{ij}(\mathbf{v}) = 3$. Since f_{ij} is a term operation, there exists k such that $v_k = u_k = 3$. It follows from Lemma 11 that a T-graph $(\mathbb{H}, x_k)$ representing f_{ij} has parts $H_1 = X_1 \setminus \{x_i\}$ and $H_2 = X_2 \setminus \{x_i\}$. Again, for simplicity assume that $x_k \in H_1$. Then there is only mapping from $H \to G$ which is a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and the mapping is defined by $x_k \mapsto 3$, $x_l \mapsto 1$ if $x_l \in X_1$ and $x_l \mapsto 2$ otherwise. Therefore we have $\mathbf{u} = \mathbf{v}$. □

Theorem 11. *Let* $\mathbf{A}$ *be an algebra defined by a complete bipartite graph* $\mathbb{G}$*. Then* $\mathbf{A}$ *is finitely related.*

Proof. We will show that condition (3) from Theorem 4 is satisfied. Let $n \geq |A|^2+2$ and let f be an arbitrary n-ary operation which depends on all of its coordinates and which has term polymers.

We start with a definition of a T-graph $(\mathbb{H}, v)$ representing f. Let X_1 and X_2 be the sets from Lemma 10. For simplicity assume that $x_1 \in X_1$ and $x_n \in X_2$. We define $\mathbb{H} = (X, X_1 \times X_2)$.

Let $|G| = 2$, that is, $\mathbb{G}$ contains two vertices connected by an edge. Let $\mathbf{v}$ be the n-tuple such that $v_i = j$ if $x_i \in X_j$. We define $v = x_1$ if $f(\mathbf{v}) = 1$ and we define $x_k = x_n$ if $f(\mathbf{v}) = 2$.

Otherwise f satisfies all the assumptions of Lemma 12 and therefore there is exactly one n-tuple $\mathbf{v}$ such that one coordinate is equal to 3, the others are equal to 1 or 2 and $f(\mathbf{v}) = 3$. We define $v = x_i$ if $x_i = 3$.

The operation represented by the T-graph $(\mathbb{H}, x_k)$ will be denoted by $\overline{f}$. We need to show that $f(\mathbf{a}) = \overline{f}(\mathbf{a})$ for all $\mathbf{a} \in A^n$. Let $\mathbf{a}$ be an n-tuple such that $a_i = 0$ for some i. Then $f(\mathbf{a}) = 0$ by Lemma 8 and $\overline{f}(\mathbf{a}) = 0$ since $H = X$.

Let $a_i \neq 0$ for every $i \in \underline{n}$ and let $f(\mathbf{a}) = 0$. Then there is a pair (i,j) such that $i < j$ and $a_i = a_j$. Then we have $f_{ij}(\mathbf{a}) = 0$. Let us denote a T-graph representing f_{ij} by $(\mathbb{H}', v)$. The mapping $\varphi' : H' \to G$ defined by $x_k \mapsto a_k$ for every $x_k \in H'$ is not a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$ and it follows from Lemma 11 that mapping $\varphi : X \to G$ defined by $x_k \mapsto a_k$ for every $x_k \in X$ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and we have $\overline{f}(\mathbf{a}) = 0$.

Let $a_i \neq 0$ for all $i \in \underline{n}$ and let $\overline{f}(\mathbf{a}) = 0$. Therefore mapping defined by $x_i \mapsto a_i$, for every $x_i \in X$, is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Let us pick a pair of indices (i,j) such that $i < j$ and $a_i = a_j$. We denote a T-graph representing f_{ij} by $(\mathbb{H}', v)$. It follows from Lemma 11 that mapping defined by $x_i \mapsto a_i$, for every $x_i \in H'$ is not a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$ and so we have $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = 0$.

We have proved the equivalence $f(\mathbf{a}) = 0 \Leftrightarrow \overline{f}(\mathbf{a})$. Let $\mathbf{a}$ be a tuple such that $\overline{f}(\mathbf{a}) = a_k$, then we have $f(\mathbf{a}) \neq 0$. If $|G| = 2$, then there are exactly two n-tuples $\mathbf{u}, \mathbf{v}$ such that $f(\mathbf{u}) \neq 0$, $f(\mathbf{v}) \neq 0$. Those are $u_i = j$ if $x_i \in X_j$ and $u_i = j$ if $x_i \notin X_j$. We have $\overline{f}(\mathbf{u}) = f(\mathbf{u})$ from the definition of the significant vertex. Assume that $f(\mathbf{u}) = 1$. Pick a pair (i,j) such that $1, i, j, n$ are pairwise distinct, $i < j$ and $a_i = a_j$. Then x_1 is the significant vertex and $f_{ij}(\mathbf{u}) = 1$ as well. Hence $f_{ij}(\mathbf{v}) = f(\mathbf{v}) = \overline{f}(\mathbf{v}) = 2$. Similarly for the case $f(\mathbf{u}) = 2$.

If $|G| \geq 3$, then there is exactly one tuple $\mathbf{v}$ such that $v_k = 3$, the other coordinates are equal to 1 or 2 and $f(\mathbf{v}) = 3$. Pick a pair i, j such that $i < j$, both i, j are distinct from k and $a_i = a_j$. Because $f(\mathbf{v}) \neq 0$ we have $v_i = v_j$ and thus $f_{ij}(\mathbf{v}) = 3$. Since f_{ij} is a term operation, x_k is the significant vertex and therefore $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = a_k$.

The proof is now complete. We have shown for every operation f of arity higher than $|A|^2 + 2$ which has term polymers that f is a term operation as well and therefore condition (3) from Theorem 4 is satisfied. □

We continue with a similar theorem for general connected bipartite graphs. Observe, that every connected bipartite graph which is not complete contains subgraph $\mathbb{R}_7$ as an induced subgraph. The graph $\mathbb{R}_7$ is shown in Figure 4.8.

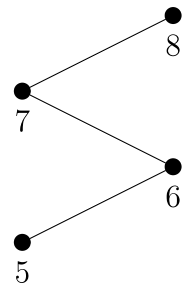


Figure 4.8: Relational structure $\mathbb{R}_7$

Theorem 12. *Let $\mathbb{G}$ be a connected bipartite graph without loops and let $\mathbf{A}$ be the algebra defined by $\mathbb{G}$. Then $\mathbf{A}$ is finitely related.*

Proof. If $\mathbb{G}$ is a complete bipartite graph, then all the assumption of the last proposition are satisfied.

As before we will show that condition (3) from Theorem 4 is satisfied. Let $n \geq |A|^2+2$, we will construct a pair $(\mathbb{H}, x_k)$ for an arbitrary n-ary operation f and we will show that the pair is a T-graph. Then we will prove that the operation defined by the T-graph coincides with f and therefore f itself is a term operation.

Let $\mathbf{u} \in A^n$, in the whole proof $\varphi_{\mathbf{u}}$ will denote a mapping from X to A such that $x_i \mapsto a_i$ for every $x_i \in X$.

Claim. *Let $\mathbf{v} = (\mathbf{6}^m \mathbf{7}^{n-m} 8_i 5_j)$ and let $\mathbf{u} = (\mathbf{7}^m \mathbf{6}^{n-m} 5_i 8_j)$. Then $f(\mathbf{u}) \neq 0$ if and only if $f(\mathbf{v}) \neq 0$.*

Proof. Let $f(\mathbf{u}) \neq 0$. Since we set n big enough, there is a pair (k,l) such that $k \neq l$ and $u_k = u_l = v_k = v_l$. Thus $f_{kl}(\mathbf{u}) \neq 0$ as well. Since f_{kl} is a nonzero term operation, it can be represented by a T-graph $(\mathbb{H}', u)$, where $\mathbb{H}'$ is a bipartite graph. Without loss of generality assume that $x_i \in H_1'$ and $x_j \in H_2'$. Therefore $\varphi : H' \to G$, defined by $\varphi(x_m) = u_m$ for every $x_m \in H'$ is a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$, that is, each edge in $\mathbb{H}'$ is mapped on an edge in $\mathbb{G}$. In particular $\varphi(H_1') = \{6,8\}$, $\varphi_{\mathbf{u}}(H_2') = \{5,7\}$ and $(x_i, x_j) \notin E(\mathbb{H}')$. Since $\mathbb{H}'$ is bipartite, we can observe that $\varphi' : H' \to G$, defined by $\varphi'(x_i) = v_i$ for every $x_i \in H'$, is a homomorphism as well. Thus $f_{ij}(\mathbf{v}) = f(\mathbf{v}) \neq 0$. ■

Let us start with a definition of $(\mathbb{H}, x_k)$. If there is no n-tuple $\mathbf{a}$ such that $f(\mathbf{a}) \neq 0$, then f is represented by any n-ary T-graph with a loop. Otherwise we define $H = X$. Let X_1 and X_2 be the sets from Lemma 10 and define $H_1 = X_1$ and $H_2 = X_2$. Moreover, we can rearrange the set X such that there is an index m such that $H_1 = \{x_1, \dots, x_m\}$ and $H_2 = \{x_{m+1}, \dots, x_n\}$. For each pair (i,j) such that $x_i \in H_l$ and $x_j \in H \setminus H_l$ consider n-tuple $\mathbf{a}$ such that $a_i = 8$, $a_j = 5$, $a_k = 6$ if $x_k \in H_l \setminus x_i$ and $a_k = 7$ otherwise. Then we define $(x_i, x_j) \in E(\mathbb{H})$ if and only if $f(\mathbf{a}) = 0$. The claim above shows that $E(\mathbb{H})$ is well defined.

It follows from Lemma 12 that there is exactly one n-tuple $\mathbf{v}$ such that one coordinate is equal to 3, the rest of the coordinates is equal to 1 or 2 and $f(\mathbf{v}) = 3$. We define $k = i$ if and only if $v_i = 3$.

Claim. *Pair $(\mathbb{H}, x_k)$ is a T-graph.*

Proof. Since obviously $x_k \in H$ it is enough to show that $\mathbb{H}$ is connected. Observe that graph (V, E) is connected if and only if for each pair of disjoint nonempty sets V_1, V_2 such that $V_1 \cup V_2 = V$ there is an edge between V_1 and V_2.

Let $V_1, V_2 \subseteq H$ be nonempty disjoint sets such that $V_1 \cup V_2 = X$. For simplicity assume that $|V_1| \geq 3$. Since we set n big enough, there are $x_i, x_j \in V_1$ such that x_i, x_j are in the same part. For simplicity assume that those are x_1 and x_2.Since f_{12} is a term operation, it is represented by a T-graph $(\mathbb{H}', u)$ and there is edge (x_i, x_j) between $V_1 \setminus x_1$ and V_2 in $\mathbb{H}'$.

Let $x_i \in X_1$ and let $\mathbf{a}$ be the n-tuple such that $a_i = 5$, $a_j = 8$, $a_k = 7$ if $x_k \in X_1 \setminus x_i$ and $a_k = 6$ otherwise. Then $f_{12}(\mathbf{a}) = 0$. If $j \neq 2$ we have $f(\mathbf{a}) = 0$ and therefore $(x_i, x_j) \in E(\mathbb{H})$. Assume that $j = 2$. Let $\mathbf{b}$ be such n-tuple that $b_1 = 8$ and $b_k = a_k$ for every $k \geq 2$. Then we have $f_{ij}(\mathbf{b}) = f(\mathbf{b}) = 0$. Pick a pair (p, q) such that $p \neq q$ and both p, q are different from $1, 2, i$. Since f_{pq} is a term operation $f_{pq}(\mathbf{6}^m \mathbf{7}^{m-n} 8_1 5_i) = 0$ or $f_{pq}(\mathbf{6}^m \mathbf{7}^{n-m} 8_2 5_i) = 0$ is satisfied. Therefore we have $f(\mathbf{6}^m \mathbf{7}^{m-n} 8_1 5_i) = 0$ or $f(\mathbf{6}^m \mathbf{7}^{n-m} 8_2 5_i) = 0$ and thus there is an edge between V_1 and V_2 in $\mathbb{H}$. Similarly we can show that there is an edge between V_1 and V_2 if $i = 2$. The proof is analogous. ■

Claim. $f(\mathbf{a}) = 0$ *if and only if* $\overline{f}(\mathbf{a}) = 0$

Proof. Let $\mathbf{a}$ be an n-tuple such that $a_i = 0$ for some $i \in \underline{n}$. Then $\overline{f}(\mathbf{a})$ by definition and by Lemma 8 we have that $f(\mathbf{a}) = 0$.

Assume that $\mathbf{a} \in G^n$ and $f(\mathbf{a}) = 0$. Pick a pair (i, j) such that $i \neq j$, $a_i = a_j$ and x_i, x_j are in X_k, for some $k \in \{1, 2\}$. So we have $f_{ij}(\mathbf{a}) = 0$. Let us pick a T-graph representing f_{ij} and denote it by $(\mathbb{H}', u)$. Hence a mapping from H' to G such that $x_k \mapsto a_k$, for every $x_k \in H'$ is not a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$. That is, there is edge (x_k, x_l) such that (a_k, a_l) is not in $E(\mathbb{G})$.

Let both a_k and a_l be in the same part of G. Then mapping φ', defined by $x_m \mapsto 1$ if $a_m \in G_1$ and $x_m \mapsto 2$ otherwise, is neither a graph homomorphism from $\mathbb{H}'$ to $\mathbb{G}$. It follows from Lemma 11 that φ, defined by $\varphi(x_i) = \varphi'(x_j)$ and $\varphi(x_m) = \varphi'(x_m)$ for all the other coordinates, is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$ and so $\overline{f}(\mathbf{a}) = 0$. Otherwise assume that $x_k \in X_1$ and $x_l \in X_2$. Therefore $f_{ij}(\mathbf{7}^m \mathbf{6}^{n-m} 5_k 8_l) = 0$. If $j \neq k, l$ then we have $f(\mathbf{7}^m \mathbf{6}^{n-m} 5_k 8_l) = 0$ and it follows from the definition of $\mathbb{H}$ that $(x_k, x_l) \in E(\mathbb{H})$. Since $(a_k, a_l) \notin E(\mathbb{G})$, we have $\overline{f}(\mathbf{a}) = 0$. If $j = k$ then we have $f(\mathbf{7}^m \mathbf{6}^{n-m} 5_i 5_k 8_l) = 0$. Pick a pair (p, q) such that $p \neq q$ and p, q, i, j, k are pairwise distinct. Then $f_{pq}(\mathbf{7}^m \mathbf{6}^{n-m} 5_k 8_l) = f(\mathbf{7}^m \mathbf{6}^{n-m} 5_k 8_l) = 0$ or $f_{pq}(\mathbf{7}^m \mathbf{6}^{n-m} 5_i 8_l) = (\mathbf{7}^m \mathbf{6}^{n-m} 5_i 8_l) = 0$ is satisfied. Therefore (x_k, x_l) or (x_i, x_l) is an edge in $E(\mathbb{G})$ and thus $\overline{f}(\mathbf{a}) = 0$. The proof is analogous if $j = l$.

Let $\mathbf{a} \in G^n$ be such that $\overline{f}(\mathbf{a}) = 0$. That is, $\varphi_{\mathbf{a}} : H \to G$ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Thus there is edge (x_k, x_l) such that $(a_k, a_l) \notin E(\mathbb{G})$. If a_k, a_l are in the same part, we define an n-tuple $\mathbf{b}$, such that $b_i = 1$ if $a_i \in G_1$ and $b_i = 2$ otherwise. Then the mapping $\varphi_{\mathbf{b}}$ is not a graph homomorphism from $\mathbb{H}$ to $\mathbb{G}$. Pick any pair (i, j) such that $b_i = b_j$ and x_i, x_j are in the same part of $\mathbb{H}$ and denote a T-graph which represents f_{ij} by $(\mathbb{H}', v)$. Then $\varphi_{\mathbf{b}} \restriction_{H'}$ is not a homomorphism from $\mathbb{H}'$ to $\mathbb{G}$ and we have $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = 0$.

Otherwise assume that $a_k \in X_1$ and $a_l \in X_2$. It follows from the definition that $f(\mathbf{7}^m \mathbf{6}^{n-m} 5_k 8_l) = 0$. Pick a pair (i, j) such that $i \neq j$, i, j is different from k, l and x_i, x_j are in the same part of $\mathbb{H}$. Denote a T-graph representing f_{ij} by $(\mathbb{H}', u)$. We have $f_{ij}(\mathbf{7}^m \mathbf{6}^{n-m} 5_k 8_l) = 0$ and therefore edge (x_k, x_l) is in $\mathbb{H}'$. Hence $\varphi_{\mathbf{a}} \restriction_{H'} : \mathbb{H}' \to \mathbb{G}$ is not a homomorphism and thus $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = 0$. ■

Claim. *$\overline{f}(\mathbf{a}) = a_k$ then $f(\mathbf{a}) = a_k$.*

Proof. Let $\mathbf{a}$ be an n-tuple such that $\overline{f}(\mathbf{a}) = a_k$ for some $k \in \underline{n}$. For simplicity assume that $x_k \in X_1$. It follows from the definition of the significant vertex that $f(\mathbf{1}^m\mathbf{2}^{n-m}3_k) = 3$. Pick a pair (i,j) such that $i \neq j$, i,j are different from k and $a_i = a_j$. We have proved in the claim above that $\overline{f}(\mathbf{a}) \neq 0$ if and only if $f(\mathbf{a}) \neq 0$ and thus $f_{ij}(\mathbf{a}) \neq 0$. Therefore x_i and x_j are in the same part and hence we have $f_{ij}(\mathbf{1}^m\mathbf{2}^{n-m}3_k) = 3$. Since f_{ij} is a term operation x_k must be a significant vertex. Thus $f_{ij}(\mathbf{a}) = f(\mathbf{a}) = a_k$. ■

The proof is now complete. At first we used a subgraph of $\mathbb{G}$ to construct a candidate pair $(\mathbb{H}, x_k)$ for a T-graph representing operation f. We showed that the pair is indeed a T-graph and that the operation defined by that T-graph coincides with f. Since the choice of operation f was arbitrary, we proved that the condition (3) from Theorem 4 is satisfied and therefore all algebras defined by a connected bipartite graph are finitely related. □

Conclusion

The aim of this thesis was to determine which graph algebras are finitely related and possibly to provide a complete characterization of finitely related graph algebras. Although we did not achieve the latter goal, we proved that two classes of algebras have the property and four classes of algebras do not have the property.

All proof techniques that we used were derived from the characterization of finitely related algebras using polymers of operations in Theorem 4. The technique to show that an algebra is non-finitely related is to find a counterexample to condition (3) from Theorem 4 , that is, to find, for every big enough n, an operation f such that it has term polymers but it is not a term operation itself. We did not find any systematic method for constructing such operations, in both cases we found a counterexample by studying properties of concrete classes of algebras.

We showed that every algebra defined by a graph containing $\mathbb{R}_1$ as an induced subgraph and not containing $\mathbb{R}_2$ as a subgraph is not finitely related. Roughly, algebras which belong to this class have *sparse loops*. Those algebras are, in a certain way, generalizations of so called Murskiĭ's groupoid and therefore it was no surprise that every algebra in the class is not finitely related. We constructed a counterexample to condition (3) from Theorem 4 such that for any T-graph representing f can not be uniquely determined a significant vertex.

On the other hand, the proof that every graph $\mathbb{G}$, which contains $\mathbb{R}_2$ and such that every vertex of $\mathbb{G}$ with a loop is adjacent to every vertex, defines a non-finitely related algebra, was rather surprising, because we had a conjecture that every graph containing $\mathbb{R}_2$ as a subgraph determines a finitely related algebra. However we constructed an operation f, for every big enough n, which is a counterexample to condition (3) from Theorem 4 such that every pair $(\mathbb{H}, x_k)$ representing f is such that $\mathbb{H}$ is a disconnected graph.

The technique to show that an algebra **A** defined by graph $\mathbb{G}$ is finitely related was based on using certain subgraphs of $\mathbb{G}$ to construct a suitable candidate T-graph for representing any operation satisfying all the assumptions.

We used graphs $\mathbb{R}_6$ and $\mathbb{R}_7$ to show that all algebras defined by a connected bipartite graph are finitely related. Every graph $\mathbb{G}$ containing $\mathbb{R}_2$ and satisfying one of the following conditions defines a finitely related algebra:

- $\mathbb{G}$ is a complete graph with a loop at every vertex,
- $\mathbb{G}$ contains $\mathbb{R}_4$ as an induced subgraph and the subgraph induced by $L(\mathbb{G})$ is a complete graph,
- $\mathbb{G}$ contains $\mathbb{R}_5$ as an induced subgraph.

We believe that the proof technique can not be applied to other classes of graph algebras. The subgraph used to construct a suitable T-graph $(\mathbb{H}, v)$ must be such that we can recognize at which vertices of $\mathbb{H}$ is a loop and which vertices are connected by an edge. Moreover, there are three types of edges: between vertices without loops, between vertices with loops and between a vertex with a loop and vertex without a loop.

There are algebras which might be finitely related, but we can not tell from any induced subgraph which vertices are adjacent. For example it is an algebra defined by a graph $\mathbb{G}$ containing $\mathbb{R}_2$ such that all the vertices in $L(\mathbb{G})$ have the same set of neighbors or an algebra defined by a graph $\mathbb{G}$ such that the subgraph induced by $L(\mathbb{G})$ is a disjoint union of complete graphs. For illustration see Figure 4.9. In both cases there is a subgraph such that we can recognize all the aforementioned properties but one. In the former case there is a subgraph such that we can not recognize an edges between vertices with a loop and vertices without a loops. In the latter case we can not recognize edges between vertices with a loop. It would be enough to distinguish components of the subgraph induced by $L(\mathbb{H})$.

Figure 4.9: Examples of graphs defining algebras about which we can not decide whether are finitely related

There are two other classes of graph algebras for which we can not determine whether they are finitely related or not. The first class are algebras defined by disconnected bipartite graphs. We believe that those are finitely related and that there is a proof very similar to the proof for connected bipartite graphs in Chapter 4. Probably the most challenging class of algebras are algebras defined by loopless graphs containing a cycle of odd length. We do not have have strong evidence supporting either option, we only believe that this problem is very likely tightly connected to coloring of graphs.

We presented several new interesting results, which may help with the complete characterization of finitely related graph. However the complete characterization still remains an open problem. This thesis concerns only undirected graphs, perhaps some insights can be obtained by studying the property being finitely related for directed graphs.

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