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Pricing of FRA and IRS under OIS discounting

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Abstract: The subject of this thesis is to review the pricing and valuation of forward rate agreements and fixed-for-floating interest rate swaps. Firstly, we describe a pricing and valuation model that was used before the financial crisis of 2007/2008. The model is based on one curve which is used for both estimating the derivative's payoff and discounting, thus we call the model a single-curve model. After the financial crisis some of the single-curve's model assumptions were impaired and the model had to be reviewed. We call the reviewed model a multi-curve model as we nowadays need a different curve for discounting and estimating the payoffs. Both models are compared on a numerical example where we value fixed-for-floating swaps.

Keywords: IRS FRA OIS discounting interest rates

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Contents

Introduction

Forward rate agreements (FRA) and interest rate swaps (IRS) constitute a large part of the market of interest rate derivatives. They are used by big financial institutions to manage their interest rate risk as well as by speculators to achieve some profit. The main goal of this thesis is to describe and review how FRAs and IRSs were priced and valued before and after the financial crisis of 2007/2008.

After introducing a basic interest rate theory and the nature of FRAs and IRSs in the first two sections, we proceed with section 3, which shows how pricing and valuation formula were derived before the financial crisis. This section demonstrates how it was possible to move between a discount, spot and forward curve when pricing and valuing the derivatives.

Section 4 revisits pricing and valuation in the aftermath of the financial crisis. We focus on the the introduction of OIS discounting and its implications.

The final section 5 is devoted to a numerical example which compares pricing and valuation before and after the crisis. It also discusses the implication of OIS discounting on forward interest rates and the derivatives' value.

To make the text more clear, the thesis avoids technical details of the derivatives' contracts. We focus on the underlying principles of the pricing and valuation. The reader should be able to deal with the technical details once he understands the principles.

It should be also noted that the author wrote bachelor thesis on the same topic in the past, [Rolák, 2013], which focuses more on the economic perspective of the pricing and valuation. On the contrary to that thesis, in the following text we confine our attention on the financial mathematics nature of the topic.

1. Interest rates: definitions and notations

This section provides basic definitions and notations necessary for working with interest rate derivatives. We start with more general definitions and then gradually transition to the interest rate setup used for our FRA and IRS pricing purposes. These foundations of interest rates can be found for example in chapter 1 [Brigo and Mercurio, 2006], from where we freely take over the basic terms and definitions.

1.1 Spot rates and zero coupon bonds

The concept of the time value of money claims the unit of money available at the present time to be worth more than the same amount in the future.

Definition 1. *A continuously compounded yield of a risk-free investment over an infinitesimal period of time is called the short rate. It is a stochastic process and we shall denote it by $r(t)$, where $t \geq 0$.*

Definition 2. *Let $B(t)$ represent the value of a bank account at time $t \geq 0$. The bank account's interest is accrued continuously at $r(t)$ available for any infinitesimal period of time t. Assuming $B(0) = 1$, we can write*

$$B(t) = \exp\left(\int_0^t r(u)du\right). \tag{1.1}$$

Defintion 2 says that the future value of invested a unit amount of cash at 0 is worth at time t the amount from (1.1). The question is then how much an investor should invest at t to achieve a unit amount of cash at $T > t$. The answer provides a discount factor.

Definition 3. *The discount factor $D(t,T)$ between two time instants t and T is the amount one would have to invest at t into the bank account to collect a unit amount of cash at T. We shall denote it*

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r(u)du\right).$$

It is important to note that since the interest rate $r(t)$ is stochastic, $D(t,T)$ also becomes a random process at time t dependent on the evolution of the future rate $r(t)$ between t and T. The discount factor is in a close relationship with a zero-coupon bond price.

Definition 4. *A zero-coupon bond is a security promising to pay a unit amount of cash at its maturity time T. There are no payments prior to the maturity. We shall denote the zero-coupon bond price $P(t,T)$. Let $P(T,T)$=1 for every T.*

Unlike $D(t,T)$ the zero-coupon bond price $P(t,T)$ is deterministic at time t. It can be shown, see e.g. [Brigo and Mercurio, 2006], that $P(t,T)$ can be viewed as an expectation of the random variable $D(t,T)$ under a certain probability measure. We will not further focus on the stochastic nature of interest rate process $r(t)$ and set attention to its deterministic version. If we assume the interest rate process to be a function of time, both the bank account and discount factor also become deterministic functions of time. The following definition enable us measure the time between two time instants.

Definition 5. *The time difference between $T-t$ (in years) is denoted by $\tau(t,T)$. The method to measure $\tau(t,T)$ is called a day-count convention.*

For example the 30/360 day-count convention assumes that months are 30 days long and years 360 days long. The time difference between dates $D_1 = (d_1, m_1, y_1)$ and $D_2 = (d_2, m_2, y_2)$ in the 30/360 day-count convention is as follows

$$\tau(D_1, D_2) = \frac{max(30-d_1, 0) + min(d_2, 30) + 360(y_2 - y_1) + 30(m_2 - m_1 - 1)}{360}.$$

We can now formulate the deterministic version of definition 1.

Definition 6. *Let $R(t,T)$ be the continuously-compounded spot interest rate available at time t with maturity T and $P(t,T)$ price of the zero-coupon bond at t. $R(t,T)$ is such a rate at which $P(t,T)$ continuously accrues to yield a unit of cash at T*

$$R(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)}.$$

While $r(t)$ constantly changes over the period, $R(t,T)$ is a constant for the whole period (t,T). This fact enable us to transition between $R(t,T)$ and $P(t,T)$ and vice versa

$$\begin{aligned} P(t,T) &= e^{-R(t,T)\tau(t,T)}, \\ e^{R(t,T)\tau(t,T)} P(t,T) &= 1. \end{aligned} \tag{1.2}$$

We can look at this transition between $R(t,T)$ and $P(t,T)$ from a different point of view. Let us consider two investing strategies, one is to buy $P(t,T)$ and hold it to maturity T, the other is to lend money for $(T-t)$ at the spot rate $R(t,T)$. Assuming the same risk profile of both investments, they should provide the same payoff at T, i.e. (1.2) should hold. It is needed to note that this concept of thinking is theoretical as $P(t,T)$ are theoretical quantities which are not directly quoted on financial markets [Brigo and Mercurio, 2006]. We usually derive $P(t,T)$ from a given set of interest rates. The following definitions introduce other types of compounding which work with discrete time and compounding frequency.

Definition 7. *Let $L(t,T)$ be the simply-compounded spot interest rate known at time t with maturity T and $P(t,T)$ price of the zero-coupon bond at t. $L(t,T)$ is such a rate at which $P(t,T)$ is invested at t and produces a unit of cash at maturity T*

$$L(t,T) = \frac{1 - P(t,T)}{P(t,T)\tau(t,T)}. \tag{1.3}$$

Similarly to continuous-compounding, the relationship between $L(t,T)$ and $P(t,T)$ is clear

$$P(t,T)(1+L(t,T)\tau(t,T))=1,$$
$$P(t,T)=\frac{1}{1+L(t,T)\tau(t,T)}.$$

Finally we have k-times-per-year compounding.

Definition 8. *Let $Y^k(t,T)$ be the k-times-per-year compounded spot interest rate known at time t with maturity T and P(t,T) price of the zero-coupon bond at t. $Y^k(t,T)$ is such a rate at which P(t,T) is invested at t and produces a unit of cash at maturity T. The interest is always reinvested after each k-th of the year at $Y^k(t,T)$*

$$Y^k(t,T)=\frac{k}{[P(t,T)]^{1/(k\tau(t,T))}}-k.$$

For $k=1$, we obtain annually-compounded spot interest rate

$$Y(t,T)=\frac{1}{[P(t,T)]^{1/(\tau(t,T))}}-1. \tag{1.4}$$

Similarly to (1.2) we can write

$$P(t,T)\left(1+\frac{Y^k(t,T)}{k}\right)^{k\tau(t,T)}=1,$$
$$P(t,T)=\left(\frac{1}{1+\frac{Y^k(t,T)}{k}}\right)^{k\tau(t,T)}.$$

It can be shown that the continuously-compounded interest rate is the limiting case of k-times-per-year compounded interest rate

$$\begin{aligned}
\lim_{k\to\infty} Y^k(t,T) &= \lim_{k\to\infty}\frac{k}{[P(t,T)]^{1/(k\tau(t,T))}}-k,\\
&= \lim_{k\to\infty}\frac{-k\tau(t,T)([P(t,T)]^{1/(k\tau(t,T))}-1)}{\tau(t,T)[P(t,T)]^{1/(k\tau(t,T))}},\\
&= -\frac{\ln P(t,T)}{\tau(t,T)}=R(t,T).
\end{aligned}$$

In the final part of this section, we define a spot curve and zero-bond curve. The latter will be further reffered to as a discount curve.

Definition 9. *The annually compounded spot curve is the graph of the function which maps maturities T expressed in years to their respective spot rates*

$$T\mapsto\begin{cases} L(t,T) & t<T\leq t+1,\\ Y(t,T) & t+1<T.\end{cases}$$

Similarly we can build a spot curve for different compounding conventions. For example a curve with semiannual compounding, a 6M (6-month) spot curve, fits the following

$$T \mapsto \begin{cases} L(t,T) & t < T \leq t + 0.5, \\ Y^2(t,T) & t + 0.5 < T. \end{cases}$$

The interbank offered rate (IBOR) is the estimated average rate at which reference banks could obtain unsecured funds for a given period and currency from other reference banks in the interbank money market. This interbank offer rate is called EURIBOR, PRIBOR, LIBOR in the Euro, Prague, London interbank money market respectively. However, as there is only a finite number of spot rates available, one has to interpolate between these quotes. To construct the following PRIBOR spot curve, we used linear interpolation.

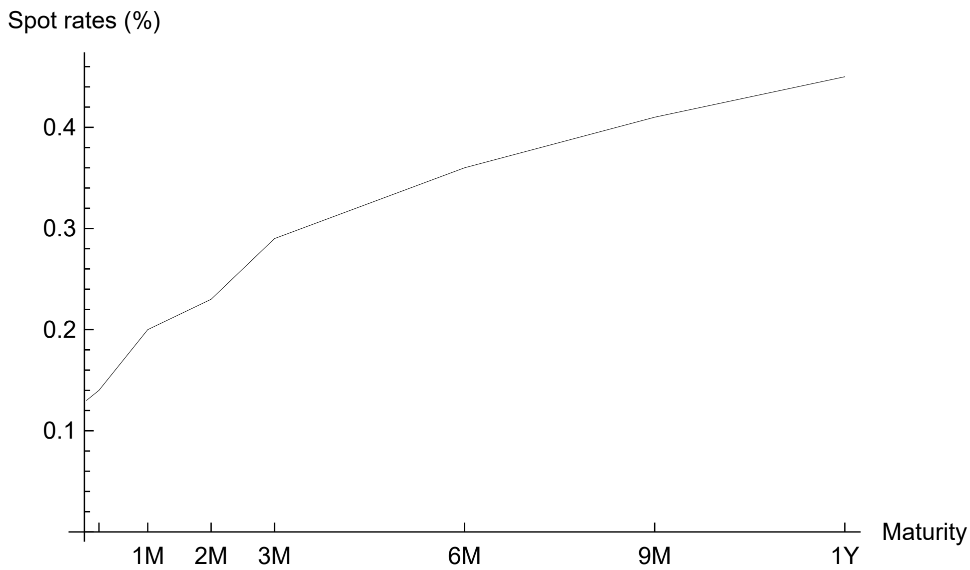

Figure 1.1: PRIBOR spot curve as of 15.4.2016.

With given spot rates, we can calculate discount factors as per (1.3), (1.4) and construct the discount curve.

Definition 10. *The discount curve is the graph of the function mapping maturities T expressed in years to their respective discount factors*

$$T \mapsto \begin{cases} P(t,T) & T > t, \\ 1 & T = t. \end{cases}$$

We can now construct a PRIBOR discount curve to our PRIBOR spot curve.

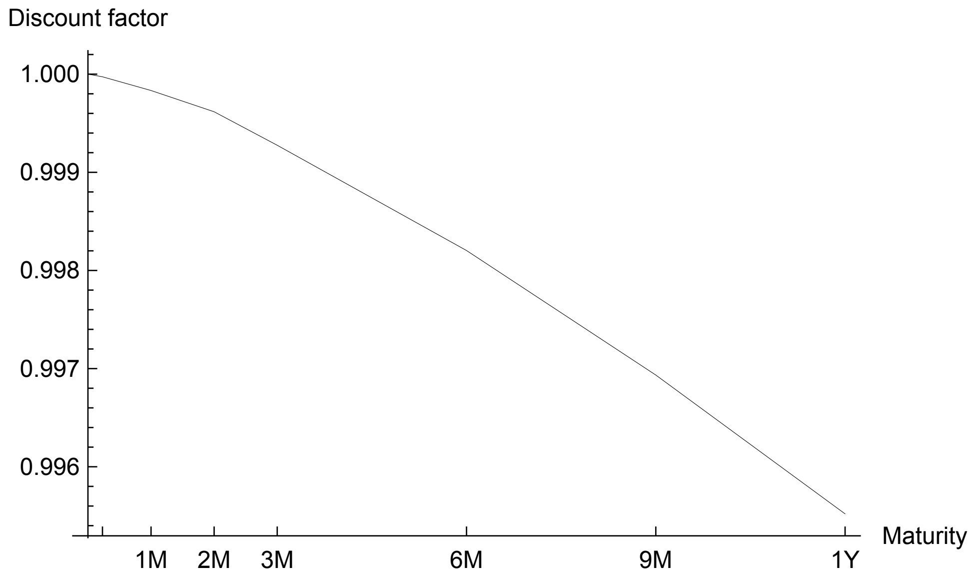

Figure 1.2: PRIBOR discount curve as of 15.4.2016.

1.2 Forward rates

With given spot rates, we can calculate at t the present value of a future cash-flow occurring at T by multiplying it by $P(t,T)$. However, as we approach T, the changes of spot rates will result in variability of the present value of the cash-flow. Thus we might be interested in the expected future spot rates known as forward rates. The forward rates can be inferred from the spot rates. An implied forward interest rate $F(t;T,S)$ is a rate known at t with an interest rate period starting at T and maturing S, where $t \leq T < S$. We will refer to T as the expiry. We can calculate $F(t;T,S)$ through the following equation

$$1 + L(t,S)\tau(t,S) = (1 + L(t,T)\tau(t,T))(1 + F(t;T,S)\tau(T,S)), \tag{1.5}$$

$$F(t,T,S) = \frac{1}{\tau(T,S)}\left(\frac{1 + L(t,S)\tau(t,S)}{1 + L(t,T)\tau(t,T)} - 1\right). \tag{1.6}$$

Similarly to (1.5), we would calculate a forward rate for k-times-per-year compounding. The equality in (1.5) in other words says that investing cash at $L(t,S)$ should provide the same payoff as investing at $L(t,T)$ and then rolling the investment at $F(t;T,S)$. This equality is discussed in more detail in Section 3.1. Since we can calculate discount factors from spot rates, we can use discount factors to derive forward rates.

Definition 11. *The simply-compounded forward interest rate F(t;T,S) known at time t for the expiry T and maturity S, $t \leq T < S$, is defined by*

$$F(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right),$$

the k-times-per-year compounded forward interest rate F(t;T,S) is then

$$F(t;T,S) = k\left[\left(\frac{P(t,T)}{P(t,S)}\right)^{1/(k\tau(T,S))} - 1\right].$$

For our purposes, we will only need to build a simply-compounded forward curve, which we fill further refer to as a forward curve.

Definition 12. *Let F(t;T,S) be a simply-compounded forward interest rate. A simply-compounded forward curve at time t is the graph of the function mapping expiry T expressed in years to its forward rate with tenor S-T.*

$$T \mapsto F(t;T,S) \quad t \leq T,$$

clearly $F(t;T,S) = L(t,S)$ *for* $T = t$*.*

2. Interest rate derivatives: FRA and IRS

This chapter provides a short description of four interest rate derivatives: forward rate agreements, fixed-for-floating interest rate swaps, basis swaps and overnight index swaps. For more extensive description of these financial derivatives see e.g. popular textbook [Hull, 2014].

2.1 Forward rate agreements

Forward rate agreements (FRAs) are over-the-counter derivative contracts between two parties who agree at time t to exchange interest rate payments at the future time $T > t$. The FRA buyer makes an interest rate payment at T based on a fixed rate K and a notional N negotiated at t. The other party, the FRA seller, makes an interest payment based on a floating reference rate $L(T, S)$ and on the same notional N at T, where S is the maturity of the FRA contract. $\tau(T, S) = S - T$ is then the interest rate period of the FRA contract. The interest rate period is commonly up to one-year-long and is subject to simple compounding. When we work with the interest rate derivatives in the thesis, the reference rate is IBOR.

The FRA contract parties do not settle the whole interest rate payments but exchange only the difference between them. If $L(T, S) > K$ at T, then the FRA buyer receives the interest rate payment from the FRA seller at time T

$$N(L(T, S) - K)\tau(T, S)P(T, S). \tag{2.1}$$

The direction of the payment in (2.1) is opposite if $L(T, S) < K$. There is no payment at time T from either party in case of $L(T, S) = K$.

2.2 Fixed-for-floating interest rate swaps

Fixed-for-floating interest rate swaps are over-the-counter derivative contracts between two parties similar in design to FRAs. FRAs can actually be viewed as the simplest form of a fixed-for-floating interest rate swap. The buyer of such a swap is called the fixed-rate payer and pays at times $T_1 = \{T_{1,1}, \ldots, T_{1,n}\}$ the following

$$NK_n\tau_{1,i},$$

where K_n is the swap rate, $\tau_{1,i} \in \tau_1 = \{\tau_{1,1}, \ldots, \tau_{1,n}\}$ and is the year fraction between $T_{1,i-1}$ and $T_{1,i}$.

The swap buyer in return receives from the swap seller floating payments

$$NL(T_{2,i-1}, T_{2,i})\tau_{2,i}$$

at times $T_2 = \{T_{2,1}, \ldots, T_{2,m}\}$ and have in general a different year fraction $\tau_2 = \{\tau_{2,1}, \ldots, \tau_{2,m}\}$. The floating rate resets at times $T_{2,0}, \ldots, T_{2,m-1}$. We call times $T_{1,0}, T_{2,0}$ the start dates of the fixed and floating leg. The start dates can be the same as origin t of the contract or can be assigned to future time $T > t$. If $T_1 = T_2$, only the difference between the payments is settled. If $L(T_{2,i-1}, T_{2,i}) > K_n$, the fixed-rate payer receives at times $T_1 = T_2$

$$N(L(T_{2,i-1}, T_{2,i})\tau_{2,i} - K_n \tau_{1,i}). \tag{2.2}$$

The direction of the payment in (2.2) is opposite when $L(T_{2,i-1}, T_{2,i}) < K_n$.

2.3 Basis swaps

The difference between basis swaps and fixed-for-floating is that both parties exchange regular floating payments. Thus we can call the basis swap the floating-for-floating swap. One party makes regular floating payments linked to a floating reference rate $L_x(T_{2,i-1}, T_{2,i})$ at times T_1

$$NL_x(T_{1,i-1}, T_{1,i})\tau_{1,i},$$

where x is the underlying interest rate tenor, typically $x \in \{1M, 3M, 6M, 12M\}$. Other notation is consistent with the one of the fixed-for-floating swaps. The other party makes payments at T_2

$$NL_y(T_{2,i-1}, T_{2,i})\tau_{2,i},$$

based on a floating reference rate $L_y(T_{2,i-1}, T_{2,i})$, where $y \in \{1M, 3M, 6M, 12M\}$ and $y \neq x$. The party of the shorter tenor usually adds a basis spread bs to its reference floating rate. This clarifies the name basis swap.

2.4 Overnight index swaps

The overnight index swap (OIS) is in design a fixed-for-floating interest rate swap defined in 2.2. The buyer of the swap makes fixed payments

$$NK_{\text{OIS},n}\tau_{1,i},$$

where $K_{\text{OIS},n}$ is an OIS rate. The floating floating-rate payments are

$$NL(T_{2,i-1}, T_{2,i})\tau_{2,i},$$

where each $L(T_{2,i-1}, T_{2,i})$ is linked to the annual effective interest rate of individual overnight interest rates, which we will further refer as ONIA[1] rates, included in the interest rate period between $T_{2,i-1}$ and $T_{2,i}$ and is calculated as per

$$L(T_{2,i-1}, T_{2,i})\tau_{2,i} + 1 = \prod_{j=1}^{n_i}(1 + L_j\delta_j),$$

$$L(T_{2,i-1}, T_{2,i}) = \frac{1}{\tau_{2,i}}\left[\prod_{j=1}^{n_i}(1 + L_j\delta_j) - 1\right],$$

where $j = 1$ ($j = n_i$) is the first (last) day of the interest rate period between $T_{2,i-1}$ and $T_{2,i}$, L_j is the ONIA rate valid at the j-th day and δ_j is the year fraction for which L_j is valid. For $T_1 = T_2$, the direction and the amount of the payment is the same as in (2.2).

[1] OverNight index average. It becomes Euro OverNight Index Average (EONIA) in the euro interbank market.

3. Pricing before the crisis

The valuation of financial derivatives generally comes down to projecting expected future derivatives payoffs and discounting them to the date of valuation. The price of a FRA (interest rate swap) is a fixed rate (swap rate) at which we enter the contract. The fair price is the one which secures a fair starting position for both parties entering the contract. By fair, we mean that the value of the derivative is equal zero for both parties at the origin t of the contract. The process of pricing is then understood as calculating the fair price at t. The process of valuing is then determining the value of the contract at $T \geq t$.

3.1 FRA price

We will show that there is just one fair rate that provides the zero value of the contract at time t. Such rate is an implied forward rate by the IBOR spot rates available at t, we will further refer to IBOR spot rates as spot rates. We can derive it through a no-arbitrage principle. By arbitrage we understand an investment strategy with a zero initial investment at t and of a positive return with positive probability. In the figures

$$\psi(t) = 0 \text{ and}$$
$$P[\psi(T) > 0] > 0,$$
$$P[\psi(T) \geq 0] = 1, T > t,$$

where $\psi(t)$ is the value of the investment strategy at time $t \geq 0$. The simply-compounded implied forward rate can be derived by two investment strategies. The first one is to deposit a unit of cash at t for S at rate $L(t, S)$, whose payoff at S is

$$C_1(S) = 1 + L(t, S)\tau(t, S).$$

The other strategy is to deposit a unit of cash at t for a shorter period $T < S$ at $L(t, T)$ and then reinvest the proceeds $C_2(T)$ at T at K locked in at t

$$C_2(T) = 1 + L(t, T)\tau(t, T),$$
$$C_2(S) = (1 + L(t, T)\tau(t, T))(1 + K\tau(T, S)).$$

Assuming the same risk profile of these two investment strategies, both of them should generate the same payoff at S

$$C_1(S) = C_2(S),$$
$$1 + L(t, S)\tau(t, S) = (1 + L(t, T)\tau(t, T))(1 + K\tau(T, S)).$$

Seperating K, we get

$$K = \frac{1}{\tau(T, S)}\left(\frac{1 + L(t, S)\tau(t, S)}{1 + L(t, T)\tau(t, T)} - 1\right). \tag{3.1}$$

We can see that K exactly matches $F(t; T, S)$ in (1.6). In case of no match, there would be a possibility of an arbitrage. For example $K > F(t; T, S)$, we would be able to set-up $\psi(t) = 0$ consisting of borrowing a unit of cash at $L(t, S)$ and investing the unit at $L(t, T)$ and entering the FRA as a fixed-rate receiver at K. At T we would reinvest the deposit plus the accrued interest plus the FRA payoff at $L(T, S)$, which produces the yield of K for the period $S - T$. We would end up with a positive payoff at S

$$C_2(S) - C_1(S) > 0.$$

The profit-driven market participants would execute the strategy which involves selling the FRA. This would increase the supply of the FRA at t and in turn drive down the FRA price K to its respective implied forward rate $F(t; T, S)$. Similarly, we would show $K < F(t; T, S)$ is not possible in an arbitrage-free market. The same logic can be used to derive arbitrage-free k-times-per-year compounded K.

3.2 FRA value

The fair FRA rate K is then the respective implied forward rate which secures the zero value of the FRA contract for both parties and is free of arbitrage. However, as the time approaches its maturity the expected spot rates change which results in the change of the value of the FRA contract. If we bought a FRA at time $t' < t$ with the fixed rate K' and our bought FRA is traded on the market, the FRA value at t is

$$\mathrm{FRA}(t, T, S, \tau(T, S), N, K) = N(K - K')\tau(T, S)P(t, S), \tag{3.2}$$

where K is the price of the FRA traded on the market at time t. Since the FRA contract is a zero-sum game, the FRA buyer's profit from (3.2) is the loss for the FRA seller[1]. In case there is no market quote of our FRA, we would have to replace K with the respective implied forward rate

$$\mathrm{FRA}(t, T, S, \tau(T, S), N, K) = N(F(t; T, S) - K')\tau(T, S)P(t, S). \tag{3.3}$$

The no-arbitrage argument produces an objective and fair price as well as value of FRAs. From now on, we price and value the interest rate derivatives with the assumption that the market is free of such arbitrage.

3.3 IRS price

The plain vanilla fixed-for-floating interest rate swap can be decomposed into two legs: the coupon-bearing bond (the fixed leg) and the floating-rate bond (the floating leg). The fixed leg with a swap rate K_n pays at times $T_1 = \{T_{1,1}, \ldots, T_{1,n}\}$ the following

$$NK_n\tau_{1,i},$$

[1]From now on, we will value the FRAs and swaps from the perspective of the fixed-rate payer (the buyer) throughout the thesis without explicitly saying that. Since the derivative contracts are a zero-sum game, the positive value for the fixed-rate payer is the same value but negative for the fixed-rate receiver (the seller) and vice versa.

where $\tau_{1,i} \in \tau_1 = \{\tau_{1,1}, \dots, \tau_{1,n}\}$ and is the year fraction between $T_{1,i-1}$ and $T_{1,i}$. The present value of the fixed leg is then

$$\mathrm{FIX}(t, T_1, \tau_1, N, K_n) = \sum_{i=1}^{n} N K_n \tau_{1,i} P(t, T_{1,i}).$$

The floating leg pays $NL(T_{2,i-1}, T_{2,i})\tau_{2,i}$ at times $T_2 = \{T_{2,1}, \dots, T_{2,m}\}$ and has in general a different year fraction $\tau_2 = \{\tau_{2,1}, \dots, \tau_{2,m}\}$. The present value of the floating leg is

$$\mathrm{FLT}(t, T_2, \tau_2, N) = \sum_{i=1}^{m} NL(T_{2,i-1}, T_{2,i})\tau_{2,i} P(t, T_{2,i}),$$

the interest rate $L(T_{2,i-1}, T_{2,i})$ resets at $T_{2,0}, \dots, T_{2,m-1}$. To ensure the initial zero value of the swap at t, the following should hold

$$\sum_{i=1}^{n} N K_n \tau_{1,i} P(t, T_{1,i}) = \sum_{i=1}^{m} NL(T_{2,i-1}, T_{2,i})\tau_{2,i} P(t, T_{2,i}). \tag{3.4}$$

As we do not know the future spot rates $L(T_{2,i-1}, T_{2,i})$ for $i = 1, \dots, m$, we can estimate them with implied forward rates $F(t; T_{2,i-1}, T_{2,i})$. Then we can solve (3.4) for K_n, we get

$$K_n = \frac{\sum_{i=1}^{m} F(t; T_{2,i-1}, T_{2,i})\tau_{2,i} P(t, T_{2,i})}{\sum_{i=1}^{n} \tau_{1,i} P(t, T_{1,i})}. \tag{3.5}$$

However, $F(t, T_{2,i-1}, T_{2,i})$ are not directly quoted on the market and we need to extract them from the available quoted swap rates K_n. We will take advantage of the relationship between forward rates and discount factors

$$K_n = \frac{\sum_{i=1}^{m} \left(\frac{P(t, T_{2,i-1})}{P(t, T_{2,i})} - 1 \right) P(t, T_{2,i})}{\sum_{i=1}^{n} \tau_{1,i} P(t, T_{1,i})} = \frac{1 - P(t, T_{2,m})}{\sum_{i=1}^{n} \tau_{1,i} P(t, T_{1,i})}. \tag{3.6}$$

Thus, if we can extract discount factors, we are able to calculate forward rates. The swap rates K_n quoted on the market are said to be par, meaning that the respective swap values are zero. Since the floating leg of the swap is worth par at inception, it actually resets to par at every resetting date T_2, the fixed leg must have the par value at inception as well

$$1 = K_n \sum_{i=1}^{n} \tau_{1,i} P(t, T_{1,i}) + P(t, T_{1,n}),$$

separating $P(t, T_{1,n})$, we get

$$P(t, T_{1,n}) = \frac{1 - K_n \sum_{i=1}^{n-1} \tau_{1,i} P(t, T_{1,i})}{1 + \tau_{1,n} K_n}. \tag{3.7}$$

Through equation (3.7) we can recursively calculate discount factors. This recursive process is called bootstrapping[2]. The bootstrapped discount factors can be used to calculate IBOR spot rates and extend the IBOR spot curve beyond maturities of one year. [Ametran and Bianchetti, April 2013] mention that the IBOR spot curve is usually build and bootstrapped from the following market instruments.

1. IBOR spot rates available at the interbank market, covering the window from 1D up to 1Y.
2. FRA contracts, covering the window from 1M up to 2Y.
3. Short term interest rate Futures contracts[3], covering the window from 3M up to 2Y and more.
4. IRS contracts, covering the window from 2Y-3Y up to 60Y.

The previous market instruments often have underlying interest rates of different tenors. Let us demonstrate how to bootstrap discount factors from swap rate quotes on a simple example.

Example 1. Let us have the following quotes of par swap rates:

Table 3.1: Swap rate quotes (% p.a.).

12M	18M	24M
1.65%	2.29%	2.81%

The payments of the floating legs are indexed to 6M LIBOR, the current quote of 6M LIBOR is 1.4% p.a. For the sake of simplicity assume the 30/360 day-count convention and that the payments of both legs are settled semiannually. What are then the implied semiannually compounded discount factors of 12M, 18M and 24M LIBOR?

In our notation, we have payment settlements at $T_1 = T_2 = \{0.5, 1, 1.5, 2\}$ and try to calculate $(P(0, 0.5), P(0, 1), P(0, 1.5), P(0, 2))^T$. Zero-coupon bond price $P(0, 0.5)$ can be calculated from the 6M LIBOR quote. We know that the present value of the fix leg of the 12M swap is worth par, then

$$1 = 0.0165 \cdot 0.5 \cdot P(0, 0.5) + (0.0165 \cdot 0.5 + 1) \cdot P(0, 1),$$

$$P(0, 1) = 0.96776.$$

The same is true of the 12M swap:

$$1 = 0.0229 \cdot 0.5 \cdot P(0, 0.5) + 0.0229 \cdot 0.5 \cdot P(0, 1) + (0.0229 \cdot 0.5 + 1) \cdot P(0, 1.5),$$

[2]The meaning of this term should not be confused with the bootstrapping technique in statistics.

[3]Futures contracts are standardized derivative contracts traded on exchanges similar in principle to FRA. However, their pricing and valuation is different to FRA, see chapter 6 in [Hull, 2014].

$$P(0, 1.5) = 0.933869.$$

This can be done recursively until we reach $P(0, 2) = 0.893738$. However, we can notice that this recursive calculations lead to the following system of linear equations

$$\begin{pmatrix} K_{0.5}+1 & 0 & 0 & 0 \\ K_1 & K_1+1 & 0 & 0 \\ K_{1.5} & K_{1.5} & K_{1.5}+1 & 0 \\ K_2 & K_2 & K_2 & K_2+1 \end{pmatrix} \begin{pmatrix} P(0,0.5) \\ P(0,1) \\ P(0,1.5) \\ P(0,2) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

where the 4x4 matrix represents the payments of the fixed legs of the respective interest rate swaps. Even though $K_{0.5}$ is not quoted, it must be equal to the 6M LIBOR to ensure the equality of the both swap's legs. We look for such a vector of discount factors

$$(P(0, 0.5), P(0, 1), P(0, 1.5), P(0, 2))^T$$

which secures that the fixed leg of each swap is worth par. Clearly, there exists only one such solution given the regularity of the matrix of payments. With given discount factors we can build the 6M IBOR spot curve.

3.4 IRS value

If we assume that both fixed payments and floating payments occur at the same time, i.e $T_1 = T_2$, $\tau_1 = \tau_2$, the swap can be valued as a portfolio of FRAs

$$\mathrm{IRS}(t, T_1, T_2, \tau_1, \tau_2, N, K_n) = \sum_{i=1}^{n} \mathrm{FRA}(t, T_{1,i-1}, T_{1,i}, \tau_{1,i}, N, K_n).$$

In more general cases when $T_1 \neq T_2$ and $\tau_1 \neq \tau_2$, we can decompose the swap into a portfolio of two bonds. The value from the fixed rate payer's perspective is

$$\mathrm{IRS}(t, T_1, T_2, \tau_1, \tau_2, N, K_n) = \mathrm{FLT}(t, T_2, \tau_2, N) - \mathrm{FIX}(t, T_1, \tau_1, N, K_n). \tag{3.8}$$

The value for the fixed rate receiver is the opposite of (3.8). If we have a swap that is actively traded on the market, we can compare its swap rate $K_n^{'}$ negotiated in the past date $t^{'} < t$ to the quoted par swap rate K_n available at t. If the fixed rate payer entered the new swap which matches the terms of his old one, he would eliminate the effect of the floating leg on the swap's value as the floating rate payments would cancel out. His two positions in the swaps would form an annuity, a series of equal payments at regular intervals. The present value of the annuity at t after j past fixed payments is

$$A(t, K_n^{'}) = \sum_{i=j+1}^{n} N(K_n - K_n^{'})\tau_{1,i} P(t, T_{1,i}).$$

The impact of $K_n - K_n^{'}$ on the swap's value can also be interpreted as follows. $K_n^{'}$ contains the information of the forward rates prevailing at the past date when the swap was negotiated. However the forward rates change as a result of the changes of spot rates. The up-to-date information of the prevailing forward rates is reflected in K_n. Thus if $K_n > K_n^{'}$, then the prevailing implied forward rates are higher than the ones prevailing when the swap was negotiated in the past. This fact means that the floating leg of the swap is worth more and the swap is ultimately more valuable for the fixed rate payer. $K_n < K_n^{'}$ follows the same logic.

Similarly to FRA valuation, in case of no swap rate quote K_n, we would calculate the theoretical swap rate as per (3.6) and substitute unknown K_n with it.

3.5 Model summary

We built our pricing and valuation model on the no-arbitrage argument which inherently involved the following assumption. When we inferred the implied forward rates from the IBOR spot rates in Section 1.2, we assumed that the credit and liquidity risk imbedded in both the investment scenarios represented by (1.5) is the same. [Mercurio, 2009], supports this assumption and claims that the market generally deemed the difference between two IBOR interest rates of the same length but with different compounding to be negligible before the crisis. We call this assumption IBOR interest rate homogeneity.

When valuing the FRAs (interest rate swaps), we compared our FRA rate (swap rate) with if-available its respective market quote. For example, If we bought a FRA at time $t^{'} < t$ with the fixed rate $K^{'}$ and our bought FRA is traded on the market, the FRA value at t is

$$\mathrm{FRA}(t, T, S, \tau(T, S), N, K) = N(K - K^{'})\tau(T, S)P(t, S), \tag{3.9}$$

If we realized the opposite transaction at t, i.e. sold the FRA at K at time t, we would created a cash flow $N\tau(T, S)(K - K^{'})P(T, S)$ available at time T irrespective of market conditions at T. The floating payments from our two positions in the FRAs would cancel out. Thus we are able to create a risk-free investment at t. If we assume that the counterparty is risk-free, the cash flow from the investment should then be discounted at a risk-free rate. When there is no quote of our respective FRAs, we can calculate the implied forward rates and speculate that if there was a respective FRA rate, it would equal the implied forward rate. Similarly to our case when the quotes were available, we would be able to create a risk-free investment and thus the payoff from the FRAs should be discounted at a risk-free rate. IBOR had been considered as a good proxy for a risk-free rate before the crisis, see e.g. chapter 9 in [Hull, 2014]. It also needs to be mentioned that the risk-free investment does not have to be set up for the valuation purposes.

Thanks to interest rate homogeneity and discounting at risk-free IBOR rates, pricing and valuation in the presented model was in principle convenient and straightforward. The IBOR discount factors were used for both building an IBOR spot curve and inferring an implied forward curve. This in turn enabled a seamless transition between the spot rates, discount factors and forward rates - from one set of the rates we can derive the other two. Thanks to this we call the described model *a single-curve model.* However, the new market conditions induced by the financial crisis of 2007/2008 impaired the aforementioned assumptions and the single-curve model has to be reviewed, which is the subject of the following section.

4. Pricing after the crisis

This chapter starts with a brief overview of the financial distress on financial markets that came with the outburst of the financial crisis in 2007/2008. Later we show that the single-curve model can no longer be used in the new market settings.

4.1 Financial crisis and its implications

The bankruptcy of one of the major banks Lehman Brothers and financial distress on financial markets shattered the confidence and trust in the interbank market. Banks started to hoard cash instead of lending each other and realized that a prime IBOR reference bank could exit the reference bank panel. Therefore IBOR banks started to price this risk in IBOR market quotes, which in turn now reflect the average credit and liquidity risk of the interbank money market [Bianchetti and Carlicchi, December 2012]. This fact can be seen on the difference between IBOR spot rates and OIS rates, which we will further refer to as the IBOR-OIS spread.

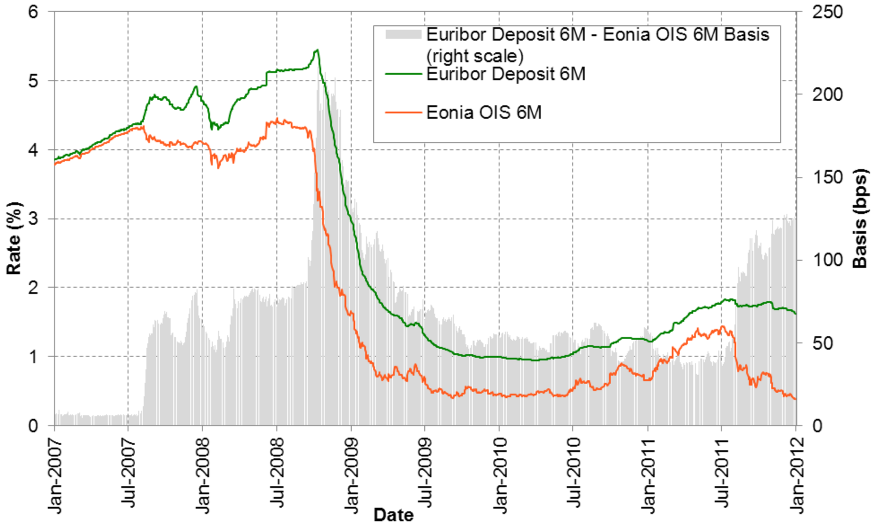

Figure 4.1: Daily quotations of 6M EURIBOR-OIS spread, the spread is displayed on the right scale. Source: [Bianchetti and Carlicchi, December 2012].

As IBOR market quotes now involve the average credit and liquidity risk of the interbank money market, OIS rates have become the new proxy for the risk-free rate. These rates have two sources of risk, both of them are relatively small. Firstly, the overnight rate involves the risk that a borrowing bank defaults, but this risk is negligible given the short time span. The second risk is that one party defaults on the OIS swap. [Hull, 2014] claims that the adjustment to an OIS swap rate to reflect default possibilities of either party is generally small. Moreover, the default risk can be and usually is minimized by posting collateral with respect to the change of the value of the swap. Market participants started to use collateral agreements more frequently to mitigate the credit risk. They present a bilaterally negotiated document governing the exchange of collateral between two derivatives parties. According to [ISDA, June 2013], 71% of all OTC derivatives transacted by respondent institutions were collateralized in 2012.

Another phenomenon important to our model is increased basis swap spreads. The party in the basis swap with payments indexed to a shorter interest rate tenor now adds a positive basis spread to the indexed interest rate. The increased credit risk perception resulted in the segmentation of the interest rate market into sub-areas according to the interest rate tenors. The difference between two interest rates of the same length but with different compounding is reflected in the basis swap spreads. Therefore the swap rates of the swaps with floating payments indexed to different IBOR tenors contain different credit and liquidity risk premium.

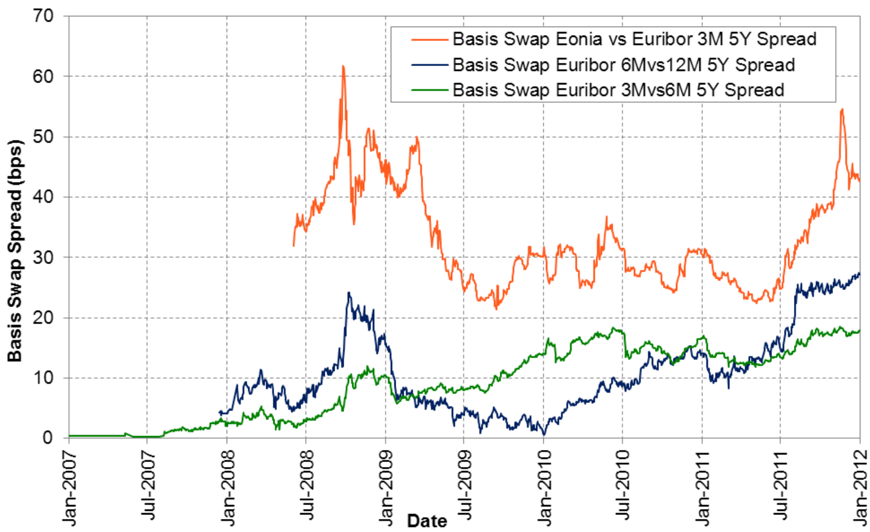

Figure 4.2: Daily quotations of 5Y basis swaps: 3M EURIBOR - 6M EURIBOR, 6M EURIBOR - 12M EURIBOR, EONIA - 3M EURIBOR. Source:[Bianchetti and Carlicchi, December 2012].

Taking into account the new market conditions, we revisit the single-curve model. We focus our attention on the 100% cash-collateralized forward rate agreements and interest rate swaps. By 100% cash-collateralized, we understand that the derivatives are priced and valued under OIS discounting. We take advantage of the fact that the FRA is a special case of the fixed-for-floating swap and thus start with the pricing and valuation of the swaps and then transition to the FRAs.

4.2 IRS price and value

The new risk-free curve for discounting is built from OIS rates $K_{\text{OIS},n}$. The bootstrapping of discount factors $P_{\text{OIS}}(t,T)$ is done analogically to (3.7)

$$P_{\text{OIS}}(t,T_{1,n}) = \frac{1 - K_{\text{OIS},n} \sum_{i=1}^{n-1} \tau_{1,i} P_{\text{OIS}}(t,T_{1,i})}{1 + \tau_{1,n} K_{\text{OIS},n}}.$$

We can still decompose the swap into two legs. However, the floating payments estimated by implied forward rates change under OIS discounting. Secondly, we have to take into account the market segmentation, so when building the forward curve of a certain interest rate tenor, we have to take swap rates with the floating leg indexed to the same underlying interest rate tenor. The present value of the fixed leg still equals the present value of the floating leg of the quoted swap rate

$$\sum_{i=1}^{n} NK_{x,n}\tau_{1,i}P_{\text{OIS}}(t,T_{1,i}) = \sum_{i=1}^{m} NF_x^*(t,T_{2,i-1},T_{2,i})\tau_{2,i}P_{\text{OIS}}(t,T_{2,i}), \tag{4.1}$$

where $K_{x,n}$ is the market quote of the swap rate whose the floating leg is indexed to IBOR with tenor x, $F_x^*(t;T_{2,i-1},T_{2,i})$ is the implied forward rate of the same tenor x under OIS discounting, typically $x \in \{1M, 3M, 6M, 12M\}$. With the bootstrapped OIS discount factors and the quoted swap rates, we can derive respective implied forward rates

$$F_x^*(t;T_{2,m-1},T_{2,m}) = \frac{\sum\limits_{i=1}^{n} K_{x,n}\tau_{1,i}P_{\text{OIS}}(t,T_{1,i}) - \sum\limits_{i=1}^{m-1} F_x^*(t;T_{2,i-1},T_{2,i})\tau_{2,i}P_{\text{OIS}}(t,T_{2,i})}{\tau_{2,m}P_{\text{OIS}}(t,T_{2,m})}.$$

Then with the OIS discount factors and derived forward rates, we can calculate the par swap rate of the collateralized interest rate swap

$$K_{x,n} = \frac{\sum\limits_{i=1}^{m} F_x^*(t;T_{2,i-1},T_{2,i})\tau_{2,i}P_{\text{OIS}}(t,T_{2,i})}{\sum\limits_{i=1}^{n} \tau_{1,i}P_{\text{OIS}}(t,T_{1,i})}.$$

The valuation of the swap follows the same logic of the single curve model. If we value the collateralized swap under OIS discounting as a portfolio of two bonds, we project the floating payments through $F_x^*(t;T_{2,i-1},T_{2,i})$ instead of $F(t;T_{2,i-1},T_{2,i})$. If we look at the swap valuation through the annuity approach, we can analyze the switch from IBOR to OIS discounting. The present value of the annuity at t after j past fixed payments under IBOR discounting equals

$$A(t,K'_{x,n}) = N(K_{x,n} - K'_{x,n})\tau_{1,i}\sum_{i=j+1}^{n} P(t,T_{1,i}),$$

the value of the annuity under OIS discounting is then

$$A_{\text{OIS}}(t,K'_{x,n}) = N(K_{x,n} - K'_{x,n})\tau_{1,i}\sum_{i=j+1}^{n} P_{\text{OIS}}(t,T_{1,i}).$$

The effect of the changed discounting can be analyzed through

$$A_{\text{OIS}}(t,K'_{x,n}) - A(t,K'_{x,n}) = N(K_{x,n} - K'_{x,n})\tau_{1,i}\sum_{i=j+1}^{n} [P_{\text{OIS}}(t,T_{1,i}) - P(t,T_{1,i})].$$

Given the positive IBOR-OIS spread, i.e. $P_{\text{OIS}}(t,T_{1,i}) - P(t,T_{1,i}) > 0$, the discounting switch results in increasing (decreasing) the swap value if $K_{x,n} > K'_{x,n}$

$(K_{x,n} < K'_{x,n})$. However, through this decomposition we lose the information of the impact of the change of forward rates on the swap value. Thus [Hull and White, March 2014] suggest defining the following swap values to analyze the separate discounting and forward-rate effect when switching from IBOR to OIS discounting

- IRS_{ID} value of the swap when IBOR discounting applied and projecting floating payments with $F_x(t; T_{2,i-1}, T_{2,i})$,

$$\text{IRS}_{\text{ID}} = \sum_{i=1}^{m} NF_x(t, T_{2,i-1}, T_{2,i})\tau_{2,i}(t, T_{2,i}) - \sum_{i=1}^{n} NK_{x,n}\tau_{1,i}P(t, T_{1,i}).$$

- IRS_{OD} value of the swap when OIS discounting applied and projecting floating payments with $F_x^*(t; T_{2,i-1}, T_{2,i})$,

$$\text{IRS}_{\text{OD}} = \sum_{i=1}^{m} NF_x^*(t, T_{2,i-1}, T_{2,i})\tau_{2,i}P_{\text{OIS}}(t, T_{2,i}) - \sum_{i=1}^{n} NK_{x,n}\tau_{1,i}P_{\text{OIS}}(t, T_{1,i}).$$

- IRS_{IO} value of the swap when OIS discounting applied and projecting floating payments with $F_x(t; T_{2,i-1}, T_{2,i})$,

$$\text{IRS}_{\text{IO}} = \sum_{i=1}^{m} NF_x(t, T_{2,i-1}, T_{2,i})\tau_{2,i}P_{\text{OIS}}(t, T_{2,i}) - \sum_{i=1}^{n} NK_{x,n}\tau_{1,i}P_{\text{OIS}}(t, T_{1,i}). \tag{4.2}$$

The total swap's value change from the discounting switch is $\text{IRS}_{\text{OD}} - \text{IRS}_{\text{ID}}$. The pure discounting effect is then $\text{IRS}_{\text{IO}} - \text{IRS}_{\text{ID}}$. The pure forward-rate effect is measured by $\text{IRS}_{\text{OD}} - \text{IRS}_{\text{IO}}$ and is equal to

$$\sum_{i=1}^{m} N[F_x^*(t, T_{2,i-1}, T_{2,i}) - F_x(t, T_{2,i-1}, T_{2,i})]\tau_{2,i}P_{\text{OIS}}(t, T_{2,i}). \tag{4.3}$$

If $F_x^*(t, T_{2,i-1}, T_{2,i}) < F_x(t, T_{2,i-1}, T_{2,i})$, then the forward-rate effect is negative and decreases total swap's value change $\text{IRS}_{\text{OD}} - \text{IRS}_{\text{ID}}$ and vice versa.

4.3 FRA price and value

The implied forward rate $F_x^*(t; T, S)$ is the new fair FRA rate which secures a zero equal starting position. A bought FRA at time t' with the FRA rate K' has value at $t > t'$

$$\text{FRA}(t, T, S, \tau(T, S), N, K) = N(K - K')\tau(T, S)P_{\text{OIS}}(t, S),$$

where K is the market quote of the FRA. If not available, we substitute it with $F_x^*(t; T, S)$. For example, the floating rate payer in the collateralized swap could enter the respective collateralized FRA as the fixed rate payer to hedge its floating payments.

4.4 Model summary

The single-curve model ceased to be legitimate in the presence of the IBOR-OIS spread and basis swap spreads. The latter was relatively easily solved. When we were deriving the implied forward rates of a certain tenor from the available market quotes of swap rates, in practice it can be also other market instruments, we used only those swaps whose floating payments are indexed to the same interest rate tenor.

The IBOR-OIS spread is indicative of the involved risk premium of the interbank market in IBOR rates. Thus OIS rates have become the new best proxy of the risk-free rate at which the derivatives payoffs are discounted. Another argument in favor for OIS discounting is that the cash collateral usually earns an overnight rate and OIS rates are derived from overnight rates. If the collateral earns the overnight rate and IBOR discounting was applied, given the positive IBOR-OIS spread the collateral would not be equal to the derivative's value at its maturity. The party with positive value would be exposed to the credit risk of the counterparty.

As opposed to the single-curve model, the new market conditions induced the separation of the discounting curve from the forward and spot curve. Nowadays, when deriving the implied forward rates consistent with OIS discounting, we need the quoted IBOR swap rates[1] and OIS discount factors. To calculate the IBOR swap rates, we need the OIS discount factors and implied forward rates consistent with OIS discounting. This in turn when valuing the collateralized derivatives breaks the seamless transition between the spot rates, discount factors and forward rates which we introduced in the single-curve model. It now requires two sets of the rates to derive the remaining set. Because of this we call the post crisis model *a multi-curve model*.

[1]We can calculate IBOR spot rates as we are used to in the single-curve model. Firstly, we bootstrapped discount factors as per (3.7) and then translate them into IBOR spot rates.

5. Model comparison: a numerical example

We present the following example to compare the single-curve and multi-curve model. Since we set up the example not to violate the interest rate homogeneity, the model comparison breaks down to comparing IBOR and OIS discounting. Though the example being simplified, it in principle does not differ substantially from the ones with which market participants deal in practice.

Let us have the following market quotes of swaps whose floating leg is indexed to 1Y EURIBOR. Both legs settle annually and have the same market standard 30/360 and the same start dates which coincide with the origin of the contract $t = 0$. The next assumption is that we have OIS rates of the same maturities with the same conditions of the settlement available.

Table 5.1: Swap rate quotes (p.a.).

1Y	3.2%	8Y	5.46%
2Y	3.4%	9Y	5.64%
3Y	3.56%	10Y	5.8%
4Y	3.82%	12Y	5.95%
5Y	4.25%	15Y	6.2%
6Y	4.9%	20Y	6.4%
7Y	5.3%	30Y	7.1%

To have a swap rate for each year, the missing swap rate quotes were acquired through linear interpolation. For the purposes of our analysis, let us have three sets of OIS swap rates: OIS20, OIS60 and OIS200. The sets have the OIS rates lower 20 bps, 60 bps and 200 bps than the EURIBOR swap rates, respectively. We can now bootstrap the OIS and EURIBOR discount factors from the OIS and EURIBOR swap rates and build discount curves.

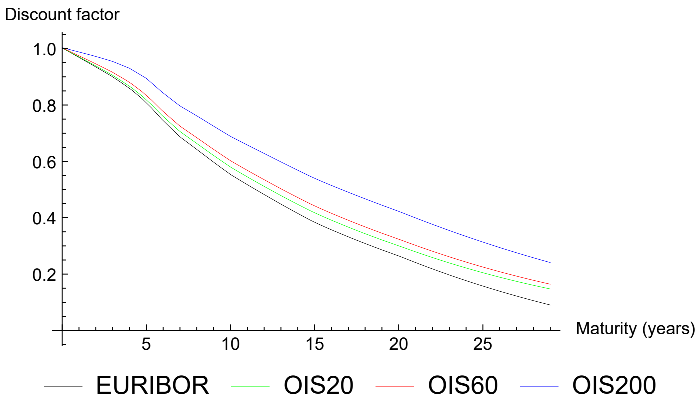

Figure 5.1: EURIBOR and OIS discounting curves.

From the quoted swap rates and bootstrapped discount factors, we can infer the 1Y EURIBOR forward curve under EURIBOR discounting (EURIBOR) and the 1Y EURIBOR forward curve consistent with OIS discounting (OIS20, OIS60, OIS200).

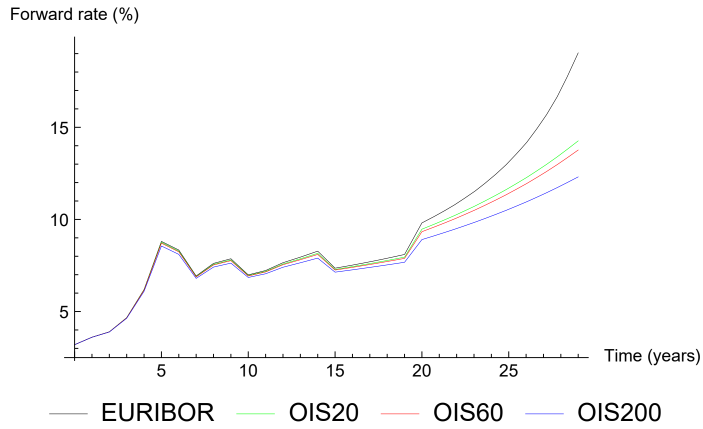

Figure 5.2: Forward curves under EURIBOR and OIS discounting.

The difference between EURIBOR and OIS forward rates becomes larger as we move forward along the forward curve. This can be explained in the following manner. If we value the swap as a portfolio of two bonds as per (4.1), swap rates $K_{x,n}$ as well as bootstrapped OIS discount factors $P_{\mathrm{OIS}}(t, T_{2,i})$ are given in our case. The OIS discount factors are higher than the EURIBOR ones. To offset this increase, the forward rates have to be lowered. Given the shape of the OIS discount curves (Figure 5.1), the forward rates under OIS discounting at the end of the forward curve has to be lowered the most.

We are now able to value and price a swap. To show the pure discounting and forward-rate effect, we value swaps of different maturities each with the swap rate 200 bps *higher* than its respective market quote and notional of 10 million euros (Table 5.2). The discount curves are OIS20, OIS60 and OIS200.

Table 5.2: Break-down of the impact of the discounting switch of swaps with swap rates 200 bps higher than market quotes.

OIS scenario	Swap maturity	Discounting Effect IRS_{IO} -IRS_{ID}	Forward-rate Effect IRS_{OD}-IRS_{IO}	Total change in swap's value IRS_{OD}-IRS_{ID}
OIS20	2Y	-1 075	-39	-1 114
	5Y	-4 055	-1 649	-5 704
	10Y	-8 064	-18 664	-26 728
	15Y	-21 739	-35 516	-57 255
	20Y	-39 387	-52 790	-92 177
	25Y	-7 422	-127 405	-134 827
	30Y	153 277	-342 604	-189 327
OIS60	2Y	-3 220	-115	-3 335
	5Y	-11 863	-4 076	-15 939
	10Y	-23 035	-34 185	-57 220
	15Y	-49 667	-62 045	-111 713
	20Y	-81 068	-89 621	-170 689
	25Y	-39 587	-195 380	-234 967
	30Y	161 225	-468 436	-307 211
OIS200	2Y	-10 904	-390	-11 294
	5Y	-40 106	-13 129	-53 235
	10Y	-77 434	-93 826	-171 261
	15Y	-153 707	-166 755	-320 462
	20Y	-240 101	-238 600	-478 701
	25Y	-151 842	-484 325	-636 167
	30Y	239 082	-1 027 320	-788 238

The absolute value of the forward-rate effect becomes bigger than the absolute value of the discounting effect as we increase the maturity of the swap. Given $F_x^*(t, T_{2,i-1}, T_{2,i}) < F_x(t, T_{2,i-1}, T_{2,i})$, the forward-rate effect is negative to the total change which can be seen from (4.3). The forward-rate effect becomes more negative with the increasing swap's maturity as the difference of the forward rates becomes larger as we move forward along the forward curve. The forward-rate effect also measures the error if we value the swap with OIS discounting and EURIBOR forward rates not consistent with OIS discounting.

The discounting effect increases the total change of the swap's value up to a certain maturity, 20Y in our case, and then starts to decrease it. The switch from IBOR to OIS discounting in IRS_{IO}, see (4.2), increases the total value change (it numerically decreases the discounting effect in our case). On the contrary, the use of $F(t, T_{2,i-1}, T_{2,i})$ instead of $F^*(t, T_{2,i-1}, T_{2,i})$ in IRS_{IO} decreases the total value change (it numerically increases the discounting effect). $F(t, T_{2,i-1}, T_{2,i})$ are similar to $F^*(t, T_{2,i-1}, T_{2,i})$ at the beginning of the forward curve, their difference grows as we move along the forward curve. Therefore swaps of long maturities with a higher swap rate than its respective market quote can have the discounting effect going contradictory to the forward-rate effect. In our case, it is apparent that $F(t, T_{2,i-1}, T_{2,i})$ becomes significantly higher than $F^*(t, T_{2,i-1}, T_{2,i})$ at 30 years

(Figure 5.2). Thus the 30Y swaps in all OIS scenarios have the discounting effect going contradictory to the forward-rate effect.

Similarly, we can analyze the pure discounting and forward-rate effect of the the same swaps but with the swap rates 200 bps *lower* than their respective market quotes (Table 5.3).

Table 5.3: Break-down of the impact of the discounting switch of swaps with swap rates 200 bps lower than market quotes.

OIS scenario	Swap maturity	Discounting Effect IRS_{IO} -IRS_{ID}	Forward-rate Effect IRS_{OD}-IRS_{IO}	Total change in swap's value IRS_{OD}-IRS_{ID}
OIS20	2Y	1 153	-39	1 114
	5Y	7 355	-1 649	5 704
	10Y	45 392	-18 664	26 728
	15Y	92 771	-35 516	57 255
	20Y	144 967	-52 790	92 177
	25Y	262 232	-127 405	134 827
	30Y	531 931	-342 604	189 327
OIS60	2Y	3 450	-115	3 335
	5Y	20 015	-4 076	15 939
	10Y	91 404	-34 185	57 220
	15Y	173 758	-62 045	111 713
	20Y	260 310	-89 621	170 689
	25Y	430 347	-195 380	234 967
	30Y	775 647	-468 436	307 211
OIS200	2Y	11 684	-390	11 294
	5Y	66 364	-13 129	53 235
	10Y	265 087	-93 826	171 261
	15Y	487 217	-166 755	320 462
	20Y	717 301	-238 600	478 701
	25Y	1 120 492	-484 325	636 167
	30Y	1 815 558	-1 027 320	788 238

Table 5.3 shows that the swaps with swap rates lower 200 bps than the market quotes have the same value as in the previous case but with the opposite sign. The forward-rate effect is the same and the reasoning behind it is the same as in the previous case. The difference is in the discounting effect which constantly increases the total change in the swap's value. This is down to the fact that both the switch from IBOR to OIS discounting and the use of $F(t, T_{2,i-1}, T_{2,i})$ instead of $F^*(t, T_{2,i-1}, T_{2,i})$ in IRS_{IO} increase the total value change.

Besides the swap's maturity, we can also look at the impact of the swap spread (the difference between the quoted par swap rate and our swap rate) on the swap's value when switching to OIS discounting. The following figure shows the difference in the 10Y swap's value between OIS and EURIBOR discounting for different swap spreads and for various OIS discounting curves.

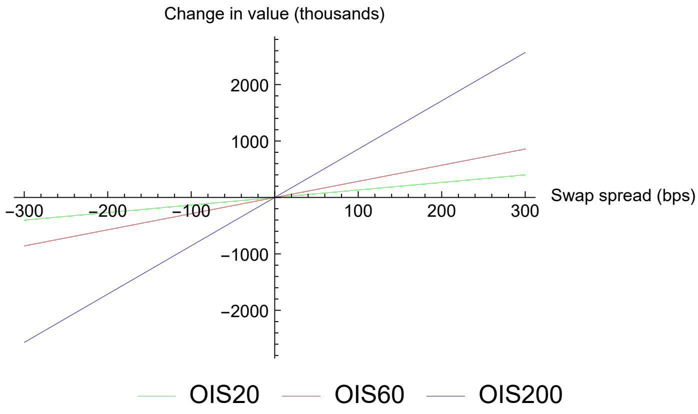

Figure 5.3: The change in value of the 10Y swap of 10 million euros notional under the discounting switch.

The discounting switch results in increasing (decreasing) the swap value if the swap spread > 0 (swap spread < 0). The larger the swap spread, the bigger the impact on the swap value under the discounting switch. With a given swap spread, the higher the IBOR swap rates than the OIS rates, the bigger the change in the value.

The value of the 10Y swap with the par swap rate is worth zero under both IBOR and OIS discounting as we can see in Figure 5.3. Even though the par swap is worth zero under OIS discounting, both legs of the swap are no longer worth par (Table 5.4).

Table 5.4: Value of the 10Y par swap under IBOR and OIS discounting.

Discounting	Fixed leg	Floating leg
IBOR	10 000 000	10 000 000
OIS20	10 341 771	10 341 771
OIS60	10 654 508	10 654 508
OIS200	11 848 756	11 848 756

It is evident that if we value a par swap under OIS discounting, the value of its respective legs increases compared to IBOR discounting. The bigger the difference between IBOR swap rates and OIS rates, the more significant increase.

In our settings, the difference of the models came down to comparing valuing swaps[1] under OIS and EURIBOR discounting. We set the example in the increasing term structure of swap rates (the swap rates increase with the swap's maturity) and a positive difference between IBOR swap rates and OIS rates. These market conditions are typical of the era after the financial crisis. The following points summarize the results:

- The swap with a par swap rate is still worth zero, though neither leg of the swap resets to par.

[1] We can take advantage of the fact that FRA can be considered as the swap with one interest rate period. The impact of OIS discounting on FRAs is marginal as they have maturity typically less than two years.

- The forward rates are lower under OIS discounting, they become lower when the difference between IBOR swap rates and OIS rates become larger and as we move along the forward curve.
- The switch to OIS discounting affects the value of swaps with a non-zero swap spread, the impact increases with the maturity of the swap.

Conclusion

The aim of this thesis was to describe how FRAs and fixed-for-floating interest rates swaps were priced and valued before and after the financial crisis of 2007/2008. In order to do this, we firstly introduced the basic interest rate theory in section 1, we focused on a spot, forward and discount curve in particular. Later in section 2, we described the nature of FRAs and interest rate swaps.

We described the single-curve model for pricing and valuation that had been used before the financial crisis in section 3. The name of the model is down to the fact IBOR discount factors were used for both building an IBOR spot curve and inferring implied forward curve. The formulas for pricing and valuation of the interest rate derivatives were derived and commented. The concluding part of the section reviewed the model and pointed out the underlying assumptions of the model. These were interest rate homogeneity and discounting at a risk-free rate. IBOR rates were considered as the best proxy for the risk-free rate before the crisis.

The IBOR-OIS and basis spread as well as derivative collateralization that reflects new market conditions during/after the financial crisis were introduce in section 4. The IBOR-OIS spread is indicative of the embedded credit and liquidity risk in IBOR rates. Therefore OIS rates have become the new proxy for the risk-free rate. This induced the separation of the discount curve from the forward and spot curve. Nowadays, when deriving the implied forward rates consistent with OIS discounting, we need quoted IBOR swap rates and OIS discount factors. This in turn when valuing the collateralized derivatives broke the seamless transition between the spot rates, discount factors and forward rates. Therefore we call the post-crisis model the multi-curve model. The basis spread invalidates interest rate homogeneity. When we derive the implied forward rates of a certain tenor from the available market quotes of swap rates, we now use only those swaps whose floating payments are indexed to the same interest rate tenor.

The numerical example in section 5 compared the single-curve and multi-curve model. We focused on the impact of the switch from IBOR to OIS discounting. The example is set to the positive difference between IBOR swap rates and OIS rates as it corresponds to the current market conditions. The discounting switch results in the different forward rates under OIS and IBOR discounting. The forward rates are lower under OIS discounting and this difference grows as we move along the forward curve and increase the difference between IBOR swap rates and OIS rates. Moreover, though the swap is worth zero at initiation, it is not worth par under OIS discounting. Overall, the discounting switch affects significantly only swaps with long maturities and with swap rates different to their respective par swap rates.

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List of Abbreviations

$B(t)$	Value of a bank account at time t
bps	Basis point (1 bps = 10^{-4})
bs	Basis spread
$D(t,T)$	Stochastic discount factor at time t for the maturity T
EONIA	Euro overnight index average
EURIBOR	Euro interbank offered rate
$F(t;T,S)$	Simply-compounded forward interest rate at time t for expiry T and maturity S consistent with IBOR discounting
$F^*_x(t;T,S)$	Simply-compounded forward interest rate at time t for expiry T and maturity S consistent with OIS discounting and of interest rate tenor x
FIX	Fixed leg of a swap
FLT	Floating leg of a swap
FRA	Forward rate agreement
IBOR	Interbank offered rate
IRS	Interest rate swap
IRS_{ID}	Value of the swap when IBOR discounting applied and projecting floating payments with $F^*_x(t;T,S)$
IRS_{IO}	Value of the swap when OIS discounting applied and projecting floating payments with $F_x(t;T,S)$
IRS_{OD}	Value of the swap when OIS discounting applied and projecting floating payments with $F^*_x(t;T,S)$
K	FRA rate
K_x	FRA rate of interest rate tenor x
K_n	Swap rate of a swap with maturity n
$K_{\text{OIS},n}$	OIS rate of a overnight index swap with maturity n
$K_{x,n}$	Swap rate of a swap with maturity n and of interest rate tenor x
$L(t,T)$	Simply-compounded spot interest rate at time t for the maturity T
LIBOR	London interbank offered rate
N	Notional of the derivative contract
ONIA	Over night index average
OIS	Overnight index swap
$\psi(t)$	Value of the investment strategy at time t
$P(t,T)$	Discount factor available at t for maturity T (Zero-coupon bond price at time t for the maturity T)
$P_{\text{OIS}}(t,T)$	OIS Discount factor available at t for maturity T (OIS Zero-coupon bond price at time t for the maturity T)
PRIBOR	Prague interbank offered rate
$r(t)$	Short rate at time t
$R(t,T)$	Continuously-compounded spot interest rate at time t for the maturity T
$\tau(t,T)$	Year fraction between t and T
$Y^k(t,T)$	K-times-per-year compounded spot interest rate at time t for the maturity T