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Pseudofinite structures

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Abstract: The present thesis is intended for students of logic that are interested in finite model theory. The thesis reports on a construction of structures that are limits of classes of finitely generated structures- the so-called pseudo-finite structures. We will explore namely Fraïssé's amalgamation method. This method has seen use in combinatorics and finite model theory and its generalisation, Hrushovski's method, has been used in geometric model theory. The first part of this thesis is theoretical. Key terms and definitions can be found there alongside formulations and proofs of theorems that describe Fraïssé's method and infer results from it. The second part gives several examples of how this method is used.

Keywords: Model theory, amalgamation, superstructures.

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Contents

Introduction

The theme of this thesis are limits of classes of finite, or more generally finitely generated, first-order structures. These are sometimes called by collective term "pseudo-finite", although that term may mean in some context structures satisfying some more conditions. In this work we study Fraïssé's amalgamation method. This method -informally speaking- takes a set of finitely generated structures that are "similar enough" and that "combine nicely" and then constructs a countable structure that is -in a sense- the limit of this class.

This structure (called Fraïssé limit) has many interesting properties, particularly when working with relational structures. Every Fraïssé limit is homogeneous. In the cases of relational structures the theory ofFraïssé limits is also ω-categorical and has quantifier elimination. With this properties in mind, one can prove ω-categoricity of a relational theory by constructing a countable model and then proving that the class of finite substructures of this model has the properties described in this thesis.

In contemporary model theory is Fraisse's method used extensively in a generalization invented by Hrushovski called Hrushovski's construction . This method has seen widespread use in geometric model theory. It was also used by Hrushovski himself to disprove a conjecture of B.Zilber Hrushovski [1993]. This method is, however, beyond the scope of this thesis.

Various other version of Fraïssé's method have yielded various results in finite model theory and combinatorics. Some of the applications can be seen in dissertations Bodirsky [2004] and Andrews [2010]

The first chapter of this thesis focuses on establishing the terminology and definitions of Fraïssé's amalgamation method. It also proves the essential theorems that lay the groundwork for the following chapters.

The second chapter focuses on more important results that are achieved by this method, namely the aforementioned ω-categoricity and quantifier elimination of particular relational theories.

The third chapter consists of examples of applications of this method on different classes of structures. It also contains an example showing that the conditions required by the method are necessary. We also give natural examples of classes that do not satisfy these conditions.

1. Fraïssé's construction

1.1 Ages

In this and the next chapter we will define properties and prove theorems that come from the Fraïssé's work Fraïssé [1953]. The form in which they are presented is described in Hodges [1997].

If we are given a set of finitely generated structures, and we want to construct their limit, we need them to be in some sense similar to each other. Fraïssé defines 3 properties that the set must satisfy for method to work.

Let us start with some informal definitions. Let L be a countable language.

- If we are given an L-structure D, the **age** of this structure is the class $\mathbf{K}$ of all *finitely generated* structures that can be embedded into D.
- If a class K is an **age** of any L-structure D, we say that K is an **Age** of some L-structure.

These definitions, although very specific, are a bit too strong for our needs. We will not actually use the structures in class **age** themselves, but rather their isomorphism types. Therefore we will call a class an **age** if it is an age of some structure *up to an isomorphism.* That means that the class contains structures that are representants of isomorphism classes. Thus **age** is a set and we can also talk about the size of the **age** .

Definition 1. *If we are given an L-structure D, the **age** of this structure is the set **K** of all (*up to isomorphism*)* finitely generated *structures that can be embedded into D.*

*If a class K is an **age** of an L-structure D, we say that K is an **age** .*

A good example of an age is a set of linear orderings of finitely many elements. This is the age of $(\mathbb{Q}, <)$ or the $(\mathbb{N}, <)$ or any infinite linear ordering for that matter.

Suppose K is an age. We can see that it is non-empty and that it has following important properties:

Hereditary property (HP) : If $A \in K$ and $B \subseteq A$ is a finitely generated substructure of A, then B is isomorphic to some structure in K.

Joint embedding property (JEP) : If $A, B \in K$ then there exists $C \in K$ such that A and B are both embeddable into C.

An example of a class that fails to satisfy (JEP) is the class of all finite fields. However, if the class contains only finite fields of the same characteristic, we can see that (JEP) is indeed satisfied.

We see that if a class were to be an age of some structure, (HP) and (JEP) are a necessary conditions. The next theorem states that they are also sufficient.

Theorem 1. *Suppose that K is a finite or countable class of finitely generated L-structures that satisfies (HP) and (JEP) . Then K is an age of some countable or finite L-structure.*

Proof. We begin by listing all the elements in K as $(A_i|i \in \mathbb{N})$. Next we define a sequence $(B_i|i \in \mathbb{N})$ of structures in K by induction as follows: We put $B_1 = A_1$. If we know B_i, we construct B_{i+1} by using (JEP) on structures B_i and A_{i+1}. We can treat B_{i+1} as a superstructure of B_i, because B_i is embeddable into it.

Therefore we can define

$$C = \bigcup_{i \in \mathbb{N}} B_i \ .$$

Since C is a countable union of at most countable structures, it is at most countable itself.

If we are given any finitely generated substructure X of C, it is (isomorphic to a structure) in K. We know this, because all generators of X are in some B_i, X is (isomorphic to) a substructure of B_i and thus is by (HP) contained by K. So K is indeed an age of C. □

Such structure C is *not* unique. With this construction we may arrive to different structures, as C depends on both the exact ordering of A_i elements of K and on the exact embeddings $B_i \hookrightarrow B_{i+1}$. For example from the class of all the finite orderings we can construct orderings of $\mathbb{Q}$ or $\mathbb{N}$ (or any countable ordering, for that matter). We will discuss this in *section 3.1*.

In order to construct a structure that more deeply reflects the structures in K, we need another property.

Amalgamation property (AP) If A, B, C are structures in K, and $f : A \longrightarrow B$ and $g : A \longrightarrow C$ are embeddings, then there exists a structure D in K and embeddings $t : B \longrightarrow D$ and $s : C \longrightarrow D$ such that $tf = sg$.

For example, the class of linear orderings has (AP) : Let A be such ordering, a substructure of both C and B (f, g are inclusions). Ordering D can be obtained by taking C and "inserting it" with elements of B to their appropriate places (so that they keep their position relative to elements of A).

The amalgamation property looks very similar to (JEP) but they are not special cases of each other. Let us see why.

Suppose we are given a class of all finite fields. As we discussed earlier it does not have (JEP) . It has, however, (AP) because if we are given A, B, C and f, g as in the definition, we know that A, B, and C have the same characteristic, therefore they can be embedded into common D.

Let us take a class L consisting of two linear orders L_1 and L_2, where L_1 and L_2 are linear orderings of 1 and 2 elements respectively. This class has (JEP) (both are embeddable to L_2) but does not have (AP) . Let $L_1 = \{e\}$ and $L_2 = \{a, b\}; a < b$. Now let $f : A \longrightarrow B$ be an embedding that assigns e to a, and $g : A \longrightarrow B$ assign e to b. Any common extension of f and g must have at least three elements (the image of e, at least one element to either side of e).

(AP) becomes a special case of (JEP) if there is a one more condition satisfied.

Theorem 2. *Suppose K is a class of finitely generated L-structures that contains a structure E that is embeddable into every structure in K. Then if (AP) is satisfied, also (JEP) is satisfied.*

Proof. Let B and C be structures in K. E is embeddable to both of them, so by (AP) there exists D such that B and C are both embeddable to it. □

Groups, for example, satisfy this condition with E being the trivial group.

1.2 Fraïssé's theorem

From Category theory point of view, direct limits are very interesting objects. How can we use the properties described above to construct similar objects?

Definition 2. *Let A be a structure. We say that A is a* ***homogeneous*** *structure iff any isomorphism between finitely generated substructures of A can be extended to an automorphism of the whole A.*

With this property defined we can finally formulate Fraïssé's theorem.

Theorem 3. ***(Fraïssé's theorem)*** *Assume that L is a countable language and K a countable or finite set of finitely generated L-structures that has (HP) , (JEP) , and (AP) . Then there exists at most countable homogeneous structure D, such that K is the age of D. Such D is unique (up to an isomorphism).*

We call such a structure the **Fraïssé limit** of K. This theorem will be proved later.

Homogeneity is very hard to verify so we will use a weaker condition.

Definition 3. *We say that a structure D is **weakly homogeneous**, iff for every two finitely generated substructures $A \subseteq B \subseteq D$ and an embedding $f : A \longrightarrow D$ there exists an embedding $g : B \longrightarrow D$ that extends f.*

Theorem 4. *Let C and D be at most countable, weakly homogeneous structures, with the same age. Let C_1 be a finitely generated substructure of C embedded into D by $f : C_1 \longrightarrow D$. Then the following statements hold:*

1. *There exists an isomorphism between C and D, that is an extension of f.*
2. *C is homogeneous.*

Proof. We will find a chain $(C_n | n \in \mathbb{N})$ of finitely generated substructures of C such that $\bigcup_{n\in\mathbb{N}} C_n = C$, and similarly a chain $(D_n | n \in \mathbb{N})$ that $\bigcup_{n\in\mathbb{N}} D_n = D$. Since C and D have the same age we may assume that C_k are also substructures of D and D_k are substructures of C.

We shall now produce a chain of partial isomorphisms $(f_n : C_n \longrightarrow D_n | n \in \mathbb{N})$ such that $\bigcup_{n\in\mathbb{N}} f_n = f_\omega$ is an isomorphism between C and D. We will do it by ensuring that for every odd $n = 2 - 1$ the domain of f_n will contain C_{k+1} an for every even $n = 2k$ the image will contain D_k

Let $f_1 = f$. Suppose we know f_{2k}. It is an embedding of $dom(f_{2k})$ into D. By weak homogeneity of D it extends to an embedding of $\langle dom(f_{2k}) \cup C_{k+1} \rangle_C$ into D. We will take this extension as f_{2k+1}. The symbol $\langle X \rangle_S$ where S is a structure and X is a set of elements in S denotes the substructure in S generated by elements in X. Note that if X is finitely generated set then $\langle X \rangle_S$ is a finitely generated structure.

Similarly, suppose we know f_{2k-1}. Its inverse is an embedding of $Im(f_{2k-1})$ into C. So by weak homogeneity we can extend the inverse to an embedding of $\langle Im(f_{2k-1}) \cup D_k \rangle_D \longrightarrow C$. We take f_{2k} to be inverse of this embedding.

To prove that C and D are both homogeneous, we use the first part of this theorem with $D = C$. □

We have thus proved the uniqueness part of Fraïssé's theorem. Now onto the existence.

Suppose we are given a class K that has (HP) , (JEP) and (AP) . We are going to construct an increasing chain $(D_n | n \in \mathbb{N})$ of structures from K that has following property:

(*) If $A \subseteq B$ are two structures in K, and for some i there exists an embedding $f : A \longrightarrow D_i$, then there is a $j > i$ and embedding $g : B \longrightarrow D_j$ that extends f.

We will take D to be $\bigcup_{n\in\mathbb{N}} D_n$. D then has the age K and is weakly homogeneous (so by **Theorem 4** it is also homogeneous).

To prove that D has the age K, we must first note the easy fact that age of D is included in K. As for the other inclusion: suppose we have an arbitrary $A \in K$. Then by (JEP) on A and D_1 there exists a structure $B \in K$ where $A \subseteq B$ and D_1 is embeddable in B. By (*) the identity on D_1 (that is also an embedding into D_1) extends to an embedding of B into D_j for some j. So by (HP) B and also A lie in the age of D.
Weak homogeneity is also given by (*) because any embeddings f, g are also embeddings into D, and g is indeed an extension of f.

Now how will we construct this chain?
Suppose we are given a class K of finitely generated structures that has (HP) , (JEP) , and (AP) . Let P be a set of all such pairs of structures (A, B) from K where $A \subseteq B$. Let π be a bijection between $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that $\pi(i, j) \geq i$. We take D_1 to be any structure in K. The rest will be constructed by induction: suppose we are given a D_k. We list all the triples $(A_{kj}, B_{kj}, f_{kj})|j \in \mathbb{N}$ where $(A_{kj}, B_{kj}) \in P$ and f_{kj} is an embedding of A into D_k. The list is countable, because there are at most countably many (A_{kj}, B_{kj}) and for every A_{kj} there is only a countable amount of embeddings into D_k (we can see this by considering all possible images of generators of A_{kj}).
Take i,j that $k = \pi(i, j)$ and (A_{ij}, B_{ij}, f_{ij}). We use (AP) on A_{ij} with embeddings $f_{ij} : A_{ij} \longrightarrow D_i$ and inclusion into D_k. Their amalgamation will be our D_{k+1}.
Thus we ensure that for this pair $A \subseteq B$ and embedding $f : A \longrightarrow D_i$ there does indeed exist D_{k+1} and an embedding of B into D that extends f. So condition (*) is indeed satisfied (because every triple (A, B, f) such as in (*) has its assigned ij). So the existence part of Fraïssé's theorem is proven.

Also note that every one of the three conditions is necessary:

Theorem 5. *Let D be at most countable structure that is homogeneous. Then the age K of D has (HP) , (JEP) and (AP) .*

Proof. Any age has (HP) and (JEP) , so we only need to prove (AP) . Let A, B and C be structures in K with embeddings $f : A \longrightarrow B$ and $g : A \longrightarrow C$. Since A is in the age of D, there also exists an embedding $e_A : A \longrightarrow D$. By weak homogeneity on $f(A) \subseteq B$ there exists an extension of embedding $e_A f^{-1}$ into an embedding $e_B : B \longrightarrow D$. Similarly there is an extension of $e_A g^{-1}$ and that is $e_C : C \longrightarrow D$. We take the structure generated by $e_C(C)$ and $e_B(B)$. It is finitely generated, so it is in K. It is also an amalgamation of A, B, C and f, g. □

2. Important results

The Fraïssé's construction has been an invaluable tool used to produce ω-categorical structures. In order to get even more interesting results from the Fraïssé's construction we need to set apart another property that guarantees that the finitely generated substructures are manageable.

Definition 1. Structure A is **uniformly locally finite** if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that following holds: whenever B is a substructure of A with less than n generators ($n \in \mathbb{N}$), B has at most $f(n)$ elements.

It is easy to see that any structure in a finite language that does not have any function symbols is uniformly locally finite.

We say that a class of structures K is uniformly locally finite if every structure in K is uniformly locally finite, with the same $f : \mathbb{N} \to \mathbb{N}$. With this property we may achieve the results described below. First we will note a useful lemma.

Lemma 6. *Let $L = (C, F, R)$ be a finite language (C being the set of constants, F being the set of function symbols and R being the set of relation symbols). Let A be a finite L-structure generated by a n-tuple $\bar{a}$. Then there exists a* quantifier-free *formula $\psi_{A,\bar{a}}(x_1, \ldots, x_n)$ such that for every L-structure B such that $B \models \psi_{A,\bar{a}}(\bar{b})$ for some tuple $\bar{b}$ in B there exists an embedding of A into B.*

Proof. We will construct $\psi_{A,\bar{a}}(x_1, \ldots, x_n)$ as follows: For every element $q \in A$ we fix a term $t_q(\bar{x})$ such that $q = t_q(\bar{a})$ (this is possible, because A is generated by $\bar{a}$). Note that t_q need not depend on every element of $\bar{a}$.

First we would like to ensure that the values of these $t_q(\bar{x})$ are distinct. In order to do that we construct the conjunction

$$\bigwedge_{q_1, q_2 \in A; q_1 \neq q_2} t_{q_1}(\bar{x}) \neq t_{q_2}(\bar{x}) \ .$$

Next we provide all the necessary information about how do constants, functions, and relations work. For constants we add the statement

$$\bigwedge_{c_k \in C} c_k = t_q(\bar{x})$$

where q is chosen such that in A holds that $c_k = q$ $(= t_q(\bar{a}))$.

Similarly, we add for every function f (of arity k) the conjunction

$$\bigwedge_{q_1 \ldots q_k \in A} f(t_{q_1}(\bar{x}) \ldots t_{q_k}(\bar{x})) = t_p(\bar{x})$$

where $p = t_p(\bar{a})$ is the element such that $f(t_{q_1}(\bar{a}) \dots t_{q_k}(\bar{a})) = t_p(\bar{a})$.

Finally, for every relation R (of arity k) and every tuple $(q_1 \dots q_k)$ we look at literals $R(q_1 \cdots q_n)$ and $\neg R(q_1 \cdots q_n)$. We mark the true one and move on. Then we construct the conjunction using these marked literals

$$\bigwedge_{q_1 \dots q_k \in A} (\neg) R(t_{q_1}(\bar{x}) \dots t_{q_k}(\bar{x}))$$

(with negation at appropriate places).

We take $\psi_{A,\bar{a}}(x_1, \dots, x_n)$ to be the conjunction of all these big conjunctions. If B satisfies $\psi_{A,\bar{a}}(\bar{b})$ for some tuple $\bar{b}$ then we take the embedding to be the map that assigns $a_i \mapsto b_i$ for every i. Indeed, this map is obviously an isomorphism between A $(= \langle \bar{a} \rangle_A)$ and $\langle \bar{b} \rangle_B$. □

Theorem 7. *Let L be a language as in lemma 6. Let K be an uniformly locally finite set of finitely generated L-structures that satisfies (HP) , (JEP) , and (AP) . Let M be the Fraïssé's limit of K and let T be the first order theory of M: $T = Th(M)$. Then the following holds:*

1. *T is ω-categorical.*
2. *T has quantifier elimination.*

Proof. Without loss of generality we may assume that no two structures in K are isomorphic to each other (so that K only contains isomorphism types). Note that for every n there are only finitely many structures in K generated by n elements. This follows from the finiteness of L and from the fact that K is uniformly locally finite. Each of n-generated structures has at most $f(n)$ elements (therefore there is only a finite amount of functions and relations that may be the interpretations of L).

For every structure $A \in K$ and every n-tuple $\bar{a}$ of generators we construct the formulas $\psi_{A,\bar{a}}(x_1, \dots, x_n)$ as described in lemma 6. We then take U to be the set of all the sentences of the type

$$\forall \bar{x} \ (\psi_{A,\bar{a}}(\bar{x}) \to \exists y \ \psi_{B,\bar{a}b}(\bar{x}, y))$$

where A is any structure in K generated by a tuple $\bar{a}$ and B is a structure generated by $\bar{a}$ and b. This set of sentences says that any structure generated by n-tuple $\bar{x}$ can be extended to *any* $n+1$-generated structure B that extends A by adding one generator y. The Fraïssé's limit M obviously satisfies U (because of the weak homogeneity).

Next we take V to be the set of sentences of the form

$$\forall \bar{x} \bigvee_{A,\bar{a}} \psi_{A,\bar{a}}(\bar{x})$$

with A again being all the structures in K generated by some n-tuple $\bar{a}$. As noted above the disjunction is finite. If a structure satisfies these sentences, it means that any substructure generated by n-tuple of distinct elements $\bar{a}$ is isomorphic to some A in K with the same generator set $\bar{a}$. This means that the age of any structure that satisfies them is exactly K. M obviously satisfies V.

We take W to be $W = U \cup V$. M is a model of W. We will next prove that any countable model $D \models W$ is weakly homogeneous structure of the age K.

The second part follows from $D \models V$.

The weak homogeneity can be proven by induction on the difference of number of generators using that $D \models U$. First, we note that if $\bar{a}$ is empty, then sentences in U say that any 1-generated structure is embeddable in D. Suppose we have given A,B structures in K where $A \subseteq B$. Suppose that $f : A \longrightarrow D$ is an embedding. We want to extend it to the embedding $g : B \longrightarrow D$. Let n be the size of $gen(B) \setminus gen(A)$ where $gen(A)$ is a set of generators of A and $gen(B)$ is the set of generators of B extending $gen(A)$. $gen(B)$ is always finite because by uniform local finiteness the whole B is finite. If $n = 1$, the existence of g is given by a formula in U- we map the extra element to the witness of y.

In an induction step we assume that there exists an embedding $g' : B' \longrightarrow D$ extending f, where B' is a substructure of B obtained by excluding one generator. Then similarly to the first step, there exists $g : B \longrightarrow D$ that extends g' and therefore also extends f.

Now we know that D is weakly homogeneous, countable, and has the age K. It is therefore isomorphic to M. Therefore W is ω-categorical and it axiomatizes $T = Th(M)$.

Now onto the quantifier elimination: Let $\phi(\bar{x})$ be an L-formula with non-empty set of parameters $\bar{x}$. We take X to be the set of all the tuples $\bar{a}$ in M such that $\phi(\bar{a})$ holds in M. Suppose $\bar{b}$ is another tuple of elements in M such that there exists isomorphism $\langle\bar{a}\rangle_M \to \langle\bar{b}\rangle_M$ that takes $\bar{a}$ to $\bar{b}$. Then this isomorphism extends to an isomorphism of the entire M (M is homogeneous). Therefore $\phi(\bar{b})$ also holds. Therefore $\bar{b} \in X$. And so the formula $\phi(\bar{x})$ is equivalent (in T) to the formula

$$\bigvee_{(\langle\bar{a}\rangle_M,\bar{a})|\bar{a}\in X} \psi_{\langle\bar{a}\rangle_M,\bar{a}}(\bar{x})$$

which is quantifier-free. Note that the uniform local finiteness guarantees that this formula is finite because it ensures that there are only finitely many different (up to an isomorphism) $\langle\bar{a}\rangle_M$ with a generator set $\bar{a}$.

If $\bar{x}$ is empty, then ϕ is a sentence. It is therefore equivalent (in T) to a propositional constant $\top$ or $\bot$ because T is complete. □

3. Examples

3.1 Naïve construction

This is an example showing why does the amalgamation property matter. Let us assume that K is a set of all finite linear orderings. Following the construction described in **Theorem 1** we will construct two structures $(\mathbb{N}, <)$ and $(\mathbb{Q}, <)$.

We start with the ordering of $\mathbb{N}$. Consider the A_i as in **Theorem 1** to be the ordering of i elements. Every B_i is exactly A_i with A_i being embedded into it by an identity map and A_{i-1} to be embedded onto the first $i-1$ elements.

As for the ordering of $\mathbb{Q}$ we consider A_i-s to be same as above. The B_i will now be a chain of orderings of $2^n - 1$ elements, with n being the smallest integer such that $2^n - 1 > i$. When the sizes of B_i and B_{i-1} are the same, the embedding of B_{i-1} onto B_i will be an identity, and the embedding of A_i will be arbitrary. In the steps where $B_i \neq B_{i-1}$ we choose the embedding of B_{i-1} onto B_i as a map to every even element. The embedding of A_i onto B_i will be the map to every odd element. This way no element in the final structure will be maximal or minimal and every pair of elements will have another element between them. This structure therefore satisfies the theory of dense linear ordering (DLO) and therefore is isomorphic to the ordering of $\mathbb{Q}$.

3.2 Dense linear ordering

Now we will show how does the weak homogeneity affect the outcome of the construction in Section 3.1.

Let K be the class of all finite linear orderings. K has (HP) , (JEP) , and (AP) as shown in chapter 1. K is also universally locally finite because the language has no function symbols. K therefore has a Fraïssé limit. We will call this Fraïssé limit **(DLO)** - the dense linear ordering.

Theorem 8. *Let a structure $Q = (Q, \leq)$ be a countable linear ordering. Then Q is the Fraïssé limit of K iff Q has no minimal nor maximal element and for all $x, y \in Q$ such that $x \leq y$ there exists $z \in Q$ such that $x < z < y$*

Q is then isomorphic to the ordering of rationals.

Proof. " $\Rightarrow$ "
We assume that Q is the Fraïssé limit of all the finite linear ordering therefore Q is weakly homogeneous. For the sake of contradiction assume that Q has a maximal element m. Let A be a linear ordering with only one element a. Let B be a linear

ordering with only two elements a, b where $a < b$. Let $f : A \to Q$ be an embedding such that $f(a) = m$. This embedding can not be extended into any embedding of B. This is a contradiction with G being weakly homogeneous. The non-existence of minimal element can be proven analogically.

Let us have any two elements $x, y \in Q$ such that $x < y$. The substructure $A =$ "$x < y$" is a finite linear ordering. Let B be a finite linear ordering with elements x, y, z' where $x < z' < y$. A is embedded into Q by identity map therefore there exists an embedding f of B that extends this identity. In Q there must hold that $x < f(z') < y$. $f(z')$ is the desired z described in 2.

" $\Leftarrow$ "
Assume Q has the properties described. We want to prove that Q is weakly homogeneous. Given two finite linear orderings $A \subset B$ and an embedding $f : A \to G$ we want to show that f extends to some embedding $g : B \to G$. We can show this by induction on the number of elements in B that are not in A. We only need to focus on the case where B has only one element b that is not in A.

If b is the largest/smallest element of B then we can map it to any element larger/smaller than any element in $f(A)$. Such element always exists.

What if b is between two elements $a, c \in A$? We know that there exists an element $z \in Q$ between $f(a)$ and $f(c)$. We map b onto z.

By taking A to be the empty structure we see that any finite linear ordering can be embedded into Q. Therefore Q has the age K and thus is (isomorphic to) the (DLO). □

3.3 Dense partial ordering

A binary relation $\prec$ is a partial order if it is antireflexive, transitive and antisymmetric. A partial ordering is any structure in the language $\{\prec\}$ where $\prec$ is a partial order. If $x \prec y$ then we say that "x is smaller than y" and "y is bigger than x." There may be elements x, y such that $\neg(x \prec y)$ and $\neg(y \prec x)$. We will call any such elements incomparable.

Let K be a class of all the finite partial orderings. K then has (HP) , (JEP) and (AP) .

(HP) is trivial because any substructure of a partial ordering is also a partial ordering (since $\prec$ does not lose its antireflexivity, transitivity, nor antisymmetry).

(JEP) follows from (AP) because a single-element structure is embeddable in any partial ordering (**Theorem 2**). To show (AP) let A, B, C be partial orderings and let $f : A \to B$ and $g : A \to C$ be embeddings. Then the amalgamation will be the structure D with the universe $A \cup (B \setminus A) \cup (C \setminus A)$. The partial order $\prec$ of the structure will be as follows:

1. If $x, y \in A$ then $x \prec y$ if and only if it holds in A.

2. If $x \in (B \setminus A)$ and $y \in A$ then $x \prec y$ if and only if $x \prec f(y)$ in B. Also $y \prec x$ if and only if $f(y) \prec x$ in B. Analogously for $x \in (C \setminus A)$

3. If $x \in (B \setminus A)$ and $y \in (C \setminus A)$ then $\neg(x \prec y)$ and $\neg(y \prec x)$.

We will define embedding $f' : B \to D$ as identity on $B \setminus A$ and as f^{-1} on $Im(f)$. Analogously for g'. It follows that these are embeddings and that $f'f = g'g = id_A$.

K is also obviously uniformly locally finite.

Theorem 9. *Let $W = (W, \prec)$ be a countable partial ordering. Then the following are equivalent:*

1. W is the Fraïssé limit of K.

2. For every three finite (or possibly empty) disjunct sets $A, B, C \subset W$ where

$$\forall a \in A\ \forall b \in B\ \forall c \in C\ (a \prec b) \wedge \neg(c \prec a) \wedge \neg(b \prec c)$$

there exists an element x such that

$$\forall a \in A\ \forall b \in B\ \forall c \in C\ (a \prec x \prec b) \wedge \neg(c \prec x) \wedge \neg(c \prec x)$$

(x is between any two elements of A and B and is incomparable to C).

Proof. 1.⇒2.

Suppose we are given A, B, C as in 2. Together they form a finite partial order. We extend this structure to a structure X by adding one element x' for which

$$\forall a \in A\ \forall b \in B\ \forall c \in C\ (a \prec x' \prec b) \wedge \neg(c \prec x') \wedge \neg(c \prec x')$$

holds. X is a finite partial ordering. x' obviously does not violate antireflexivity, transitivity nor antisymmetry. Antireflexivity is obvious. As for transitivity, x' has smaller elements only in A and bigger elements only in B. Any element smaller than x' is therefore automatically smaller than all of elements bigger than x'. Antisymmetry is not violated either because naturally $A \cap B = \emptyset$. By weak homogeneity of W there exists an embedding of X that extends the embedding of $A \cup B \cup C$. We take x in the theorem to be the image of x'.

2.⇒1. We will prove the weak homogeneity. Let us have two partial orderings $U \subseteq V$ and an embedding $f :\to W$. We need to show that there exists $g : V \to W$ extending f. We will proceed by induction on the size of $V \setminus U$. It suffices to show this only for size of 1.

Suppose that v is the only element in $V \setminus U$. We take A' to be the set of all the elements in V smaller than v, B' to be the set of all elements of V bigger than v, and C' to be the set of every element in V incomparable with v. We take $A = f(A')$, $B = f(B')$, $C = f(C')$. There exists an element x that is bigger that every element of A, precedes every element of B, and is incomparable to any element of C. We take $g(v) = x$. This is the desired extension.

Note that this construction also works for U being empty. Every finite partial ordering is therefore embeddable to W. Any finite substructure of W is also a finite partial ordering because antisymmetry and transitivity of $\prec$ is retained. Therefore the age of W is precisely K. □

From this property we can see that every finite set X has countably many incomparable upper/lower bounds (elements bigger/smaller than every element of X). For the sake of the argument assume that X has only finitely many incomparable lower bounds. If we would take Y to be the set of these bounds then the condition would fail for $B = X$ and $C = Y$.

3.4 Random Graph

A **graph** is any structure in a language $\{E\}$ where E is a binary relation that is non-reflexive and symmetric. We usually denote it as $G = (V_G, E_G)$ where V_G is the universe of G and E_G is its particular interpretation of the relation symbol E. Any element of V_G is called a **vertex**. If we have two vertexes $x, y \in V_G$ such that $E_G(x, y)$ we say that x and y are **adjacent**. Such pair x, y may be called an **edge**. Any substructure of a graph is called a subgraph.

Let K be a class of all finite graphs. We see that K has (HP) , (JEP) and (AP) . (HP) is obvious since any subgraph of a finite graph is also a graph. Trivial graph $T = (\{x\}, \emptyset)$ can be embedded into any graph therefore by **Theorem 2** we only need to prove (AP) .

Let G_0 be a graph with embeddings $f : G_0 \to H_1$ and $g : G_0 \to H_2$. We want to construct the amalgamation $G_1 = (V_{G_1}, E_{G_1})$. Let V_{G_1} be $V_{H_1} \cup V_{H_2} \setminus Im(g)$. We define E for every two elements $x, y \in V_{G_1}$ as follows:

1. If both $x, y \in V_{H_1}$-part of V_{G_1} or if both $x, y \in V_{H_2}$-part of V_{G_1} then they are adjacent if and only if they are adjacent in H_1 or H_2 respectively.

2. If $x \in Im(f)$ and $y \in V_{H_2}$ then they are adjacent if $gf^{-1}(x)$ is adjacent to y in H_2.

3. If $x \in V_{H_1} \setminus Im(f)$ and $y \in V_{H_2} \setminus Im(g)$ then they are adjacent arbitrarily.

We define $k : H_1 \to G_1$ as a default injection of V_{H_1} into V_{G_1}. k is an embedding because it respects the adjacency. Then we define $l : H_2 \to G_1$ to be such function that if $x \in Im(g)$ then $l(x) = kfg^{-1}(x)$ and if $x \in H_2 \setminus Im(g)$ then $l(x) = x$ (default injection). This is also an embedding.

K is also uniformly locally finite with f being an identity. Therefore K satisfies the assumptions of Fraïssé's theorem and thus has a Fraïssé limit. We'll denote this Fraïssé limit as Γ and call it **the random graph**.

Theorem 10. *Let a structure $G = (V_G, E_G)$ be a countable graph. Then the following is equivalent:*

1. *G is (isomorphic to) the random graph Γ*

2. *For every two distinct sets $X, Y \subset V_G$ there exists a vertex $v \in V_G$ such that $\forall x \in X\ E(x, v)$ and $\forall y \in Y\ \neg E(y, v)$*

Proof. First we prove the implication 1. $\Rightarrow$ 2.
If G is the random graph then it is a Fraïssé limit of a set. Therefore G is weakly homogeneous. Suppose we are given distinct sets $X, Y \subset V_G$. $A = (V_A, E_A)$ where $V_A = X \cup Y$ and E_A is the restriction of E_G on V_A is a graph. If we extend A by one vertex v' that is adjacent to all the vertexes of X and not adjacent to any of the vertexes of Y we get a graph that we will denote as B. A is embedded into G by the identity map. By weak homogeneity there exists an embedding of $f : B \to G$ that keeps X and Y in their place. We take v to be $f(v')$.

Now onto 2. $\Rightarrow$ 1.
We will prove that G is weakly homogeneous. Suppose we are given two graphs $A \subseteq B$ and an embedding $f : A \to G$. We want to extend it to an embedding $g : B \to G$. We can do this by induction on the size of $V_B \setminus V_A$. It suffices to prove for $|V_B \setminus V_A| = 1$ because induction steps will have the same proof. We take w to be the vertex that is in $V_B \setminus V_A$. We take X to be the set of vertexes adjacent to w in B and Y to be the set of vertexes not adjacent to w in B. We then apply the property 2. on $f(X)$ and $f(Y)$ to produce a vertex v. We then map w onto v and we have the desired extension.

If we take A to be the empty structure we see that any finite graph is embeddable into G. G has therefore the age K and therefore is isomorphic to the random graph. □

Random graph is regularly used as an example of Fraïssé's construction. W. Hodges also uses it in his book Hodges [1997]. He also shows an extension of the **Theorem 8** to any finite language L that has no function symbols. The Fraïssé limit is then called **Random structure of** L.

3.5 Random triangle-free graph

Let K be the class of all graphs to which cannot be embedded a triangle (K_3). We will call them triangle-free. They can be characterised by the sentence

$$\forall x, y, z\ [((x \neq y \wedge y \neq z \wedge z \neq x) \wedge\ E(x, y)) \rightarrow (\neg E(x, z) \wedge \neg E(y, z))]$$

K has (HP) , (JEP) and (AP) .

(HP) is trivial since if a triangle can not be embedded into a graph then it cannot be embedded into any substructure.

(JEP) is a consequence of (AP) since a single-vertex graph can be embedded into any graph in K.

To show (AP) we do the same construction as in section about Random graph above except we change the condition 3. to be that if given $x \in V_{H_1} \setminus Im(f)$ and $y \in V_{H_2} \setminus Im(g)$ then they are *not* adjacent.

The amalgamation is once again triangle-free. This is because there is no triangle with all the edges inside the $Im(k)$ nor $Im(l)$. Since there are no edges between their difference, there are also no triangles with some edges in one but not the other.

K is also uniformly locally finite with f being an identity. It has a Fraïssé limit. We will call this limit **Random triangle-free graph**.

Theorem 11. *Let $G = (V_G, E_G)$ be a countable graph. Then the following are equivalent:*

1. *G is the Random triangle-free graph*
2. *For every two finite sets $X, Y \subset V_G$ where $X \cap Y = \emptyset$ there exists a vertex z adjacent to every vertex in X and not adjacent to any vertex of Y if and only if no vertexes in X are adjacent to each other.*

Proof. 1.⇒2.
Let X, Y be disjunct subsets of V_G. There exists a graph A that is $X \cup Y$ extended by one element z' that is adjacent to whole X and not adjacent to any vertex of Y. If some two elements $x, y \in X$ are adjacent to each other then A has a triangle $\{x, y, z'\}$. Therefore it is not embeddable into G. If there would be such $z \in V_G$ that would be adjacent to all vertexes of X (and not adjacent to any of Y) then A would be embeddable into G. It is because we could take z' to z.

On the other hand, if no two elements of X are adjacent to each other then A is triangle-free and therefore is embeddable into G. We take z to be the image of z' in this embedding.

2.⇒1.
We need to prove that if given 2. then G is weakly homogeneous and every finite subgraph of G is triangle-free. Suppose we are given a subgraph of G. If it were to have a triangle on vertexes $\{a, b, c\}$ then the condition 2. would fail on $X = \{a, b\}$ because there would be $z = c$ that is adjacent to whole X. Therefore any finite graph embeddable into G is triangle-free.

Now suppose that we are given two triangle-free graphs $A \subseteq B$ and an embedding $f : A \to G$. We will produce the embedding $f' : B \to G$ that is an extension of f. We can do this by induction on the size of $V_B \setminus V_A$. It suffices to show for B having only one extra element. Let z' be this extra element. We mark X' to be the set of all the elements adjacent to z' in B and Y to be the set of elements not adjacent to it. We take $X = f(X')$ and $Y = f(Y')$. Since B is triangle-free we know that no elements of $f(X')$ are adjacent to each other. Therefore there exits an element $z \in V_G$ that is adjacent to whole X and not adjacent to any of Y. f' is the embedding that takes z' to z. □

Note that condition 2. does not work on *infinite* X. We can guarantee that if any two elements of it are adjacent then there does not exist any z that is adjacent to whole X. However we cannot be sure that even if they are not adjacent, such z does exist.

3.6 Prüfer group

A **group** is a structure $G = (G, \cdot_G, e_G)$ in the language $\cdot, e$ where $\cdot$ is a binary function and e is a constant. e_G is such element that $\forall g \in G\ g \cdot e_G = e_G \cdot g = g$ holds. " $\cdot$ " is an associative function such that $\forall g \in G \exists g'\ g \cdot g' = e_G$. We will use the following abbreviation:

$$\underbrace{g \cdot g \ldots g}_{k} = g^k$$

A p-**group** (if p is a prime) is a 1-generated (also called **cyclical**) group where there exists such n that $\forall g \in G\ g^{p^n} = e$ A p-group G is a p^n-group if n is the lowest possible where the sentence as above holds.

It follows that any p^n-group has exactly p^n elements that can be described as $\{a^k | k \in \{1, 2, \ldots p^n\}\}$ where a is the group's generator. From this it follows that any p-group is also commutative. It can also be seen that any subgroup of a p-group is also a p-group.

Let K be the set of all the p-groups. As noted above K has (HP) . It has the smallest structure (trivial group $\{e\}$) therefore we only need to show (by **Theorem 2**) that it has (AP) .

Let A be a p^n-group with embeddings $f : A \to B$ and $g : A \to C$ where B is a p^k-group and C is a p^l-group. Without loss of generality assume $k \geq l$. We define the amalgamation D to be exactly B with $f' : B \to B(= D)$ to be an identity on B. We want to show that there exists an embedding $g' : C \to B$ such that $g'(g(a)) = f(a)\ \forall a \in A$. Let a_0 be the generator of A. We put $g'(g(a_0)) = f(a_0)$. Let c_0 be the generator of C. There exists a r such that ${c_0}^r = g(a_0)$. There also exists exactly one $b \in B$ such that $b^r = f(a)$. This follows from the representation of the elements as ${b_0}^l$ where b_0 is the generator of B. We put $g'(c_0) = b$. For every other element in C we write it in the form ${c_0}^x$ and assign $g'({c_0}^x) = b^x$. Thus g' is an embedding. It is also true that for every element ${a_0}^x \in A:\ f({a_0}^x) = g'g({a_0}^x)$. Thus K has (AP) .

K is unfortunately not uniformly locally finite since there is no upper bound on the size of a p-group while they are all generated by a single element.

K has nonetheless a Fraïssé limit. We will call this limit a **Prüfer group** and denote it as $\mathbb{Z}_{p^\infty}$ Kaplanski [1976].

Theorem 12. *Let G be the group of complex numbers that are of the type $e^{2i\pi\frac{k}{p^n}}$ where $k, n \in \mathbb{N}$ with $\cdot_G$ being defined as the multiplication in the field $\mathbb{C}$.*
G is then the Prüfer group.

Proof. Let n be any natural number. A p^n-group A can be embedded into G by embedding $f : A \to G$ that takes the generator a_0 of A to the element $e^{2i\pi\frac{1}{p^n}}$ which is a generator of a p^n-subgroup. Since every substructure of G is a p-group G has the age exactly K.

Next we want to show that G is weakly homogeneous. Let $A \subseteq B$ be p-groups. It follows that if A is a p^n-group and B is a p^m-group then $m \geq n$. Let a_0 be a generator of A. Let $b_0 \in B$ be the element for which ${b_0}^{p^{m-n}} = a_0$. b_0 is a generator of B. Let $f : A \to G$ be an embedding. We define $g \in G$ as $g = f(a_0)$. It is easily seen that $g = e^{2i\pi\frac{k}{p^n}}$ where $\frac{k}{p^n}$ is in its simplest form. We then put $f' : B \to G$ to be such embedding that takes b_0 to $e^{2i\pi\frac{k}{p^m}}$. f' is an extension of f therefore G is weakly homogeneous. $\square$

3.7 Semisimple modules

Let R be a ring. $Mod - R$ is the class of structures called (right)R-modules in language $L = \{0, +, \{\cdot r | r \in R\}\}$ that follows the classic module definition as described in Wisbauer [1991]. Further we will only assume right modules. For left modules the following is analogous. In this whole example we will use definitions and facts about modules from Wisbauer [1991].

A module is **simple** if it has no non-trivial submodules. We denote the class of simple R-modules as **Simp-**R. Every simple module is 1-generated. A module M is **semisimple** if there exists a set of simple modules $\{M_i | i \in I\}$ such that

$$M \simeq \bigoplus_{i \in I} M_i$$

where $\oplus$ is the direct sum.

Let R be a countable ring. Let K be the class of all semisimple R-modules that are a finite direct sum of simple modules (I is finite). Therefore every structure in K is finitely generated. K has (HP) because every submodule of a semisimple module is also semisimple. K has (JEP) because if M and N are semisimple modules then $M \oplus N$ is also a semisimple module.

In order to show that K has (AP) let us have M, K, and L semisimple R-modules and let $f : M \to K$ and $g : M \to L$ be embeddings. $K = Im(f) \oplus \widetilde{K}$ and $L = Im(g) \oplus \widetilde{L}$ where $\widetilde{K}$ and $\widetilde{L}$ are some semisimple modules. This holds because every submodule of a semisimple module has a complement. Let us have a module $N = M \oplus \widetilde{K} \oplus \widetilde{L}$. We define $f' : K \to N$ as f^{-1} on $Im(f)$ and as identity on $\widetilde{K}$. g' will be analogously g^{-1} on $Im(g)$ and identity on $\widetilde{L}$. N with f' and g' is the desired amalgamation.

K is not necessarily weakly homogeneous. Since every simple module is isomorphic to R/I for some maximal ideal I they are not necessarily bounded in size. For example with R being the ring of integers every simple module is isomorphic to $\mathbb{Z}_p$ for some prime p (and therefore has the size p).

If we, however, take R to be such a ring that has a finite amount of maximal ideals (for example Artinian rings) then K is indeed weakly homogeneous. The bounding function f is now $f(k) = k^n$ where n is the size of the largest simple module.

Theorem 13. *Let R be a ring. Let M be a Fraïssé limit of the set K of all finitely generated semisimple modules. Then*

$$M \simeq \bigoplus_{J \in Simp-R} \left(\bigoplus_{n \in \mathbb{N}} J \right)$$

Proof. We will prove that any M that has the described structure is weakly homogeneous of the age K. M is obviously semisimple therefore all of its substructures are also semisimple modules. Let S be any finitely generated semisimple R-module.

$$S \simeq \bigoplus_{J \in Simp-R} \left(\bigoplus_{i=1}^{n_J} J \right)$$

for some sequence n_J of natural numbers (i.e. all but finitely many are zero). It is easily seen that this is a submodule of M.

Let $A \subseteq B$ be two finitely generated semisimple modules. Let $f : A \to M$ be an embedding. We know that $B = A \oplus \widetilde{B}$ for some semisimple module $\widetilde{B}$. We need to find an embedding $f' : \widetilde{B} \to M$ whose image does not intersect the image of f. Luckily the structure of M allows us to write it as

$$M \simeq Im(f) \oplus \bigoplus_{J \in Simp-R} \left(\bigoplus_{n \in \mathbb{N}} J \right) \simeq Im(f) \oplus M$$

because $Im(f)$ contains only finite amount of simple submodules of M. Since $\widetilde{B}$ embeddable into M we may embed it into $Im(f) \oplus M$ such that $Im(f') \cap Im(f) = \{0\}$. Therefore g that is f on A and f' on $\widetilde{B}$ is the embedding of B into M that extends f. □

3.8 Classes that fail Fraïssé's theorem

In this section we will look at some classes that do not satisfy some of the conditions (HP) , (JEP) or (AP) .

The first example is the class of all **finite connected graphs**. Those are the graphs where for every two elements x, y there exists a path between them (path being a sequence of elements such that every two following elements are adjacent and elements x and y are one at start and one at the end).

This class has a minimal structure (single-vertex graph) therefore if it has (AP) then it has (JEP) . Amalgamation can be achieved by the same construction as in section 3.4. The amalgamation is a connected graph because if we were given any two vertexes $x, y \in V_{G_1}$ and a vertex $a \in kf(G_0)$ then there exists a path between x and a and also one between a and y. Together they form a path between x and y.

This class however does not have (HP) . Let us have a graph $P_2 = (\{x, y, z\}, E)$ where $E(x, y), E(y, z), \neg E(x, z)$. This is obviously a continuous graph. The subgraph $P' = (\{x, z\}, E')$ (where E' is the restriction of E) is not continuous (because there is no path between x and z).

The second example is the class of all **finite planar graphs**. Those are graphs that can be drawn on a two-dimensional Euclidean plane (proper definition can be found in Bondy and Murty [1976]). Kuratowski's theorem states that these graphs can be characterised as those not containing a subgraph reducible to K_5 (a graph on 5 elements all of which are adjacent to each other) or $K_{3,3}$ (a graph on 6 elements divided to two sets of three where each element is adjacent to all the elements in the other set). A precise statement and proof of this theorem can be found in Bondy and Murty [1976].

This class obviously has (HP) since K_5 nor $K_{3,3}$ can appear in any subgraph. It also has (JEP) because the graph produced by drawing these two graphs next to each other is still planar.

The class however does not have (AP) . Let us have $A = (\{a, b, c, k\}, E_A)$ where a, b, c are adjacent to k. Let us then have its extensions B, C where $B = \{a, b, c, k, \beta, E_B\}$ and $C = \{a, b, c, k, \gamma, E_C\}$. In B β is adjacent to a, b, c and to k. In C γ is also adjacent to a, b, c but is not adjacent to k. Suppose for the sake of argument that we have an amalgamation D and embeddings $f : B \to D$ and $g : C \to D$ such that $f = g$ on A. $f(k)$ is adjacent to all of $f(a), f(b), f(c)$. So is $f(\beta)$ and $g(\gamma)$. But since $f(\beta)$ is adjacent to $f(k)$ and $g(\gamma)$ is not, we know that $f(\beta)$ and $g(\gamma)$ are two distinct elements. Therefore D contains a subgraph $K_{3,3}$ and therefore is not planar.

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