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Faculty of Mathematics and Physics

BACHELOR THESIS

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Martingale Approach to Roulette

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Study programme: Mathematics
Study branch: General Mathematics

Prague 2016

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Abstract: This main aim of this thesis is to compare two different strategies in Roulette – betting on a color and betting on a single number. Betting on a color represents a conservative strategy with diversified asset and betting on a number represents a more risky strategy without diversification. Distribution of the Maximum, the Last Exit Time and the Number of Visits of zero will be given for each strategy using Martingales or Markov Chains. The theoretical results will be supported by Monte Carlo simulations.

Keywords: Martingales, Optional Sampling Theorem, Distribution of the Maximum of the Drifted Random Walk, Last Exit Time, Visits of Zero

I would like to thank doc. RNDr. Jan Večeř, Ph.D. for supervising and helping me with the thesis.

Contents

Introduction

Roulette is a casino game which is played since the 18th century. Nowadays, there are 2 basic types of roulette - European and American. They have the same number of red (18) and black (18) numbers. They differ in number of zeros which are neither red nor black. The European has 1 zero and the American has 2 zeros. This thesis focuses only on the European type.

It is obvious that roulette is not a fair game. Therefore, the player knows that he/she will eventually lose his/her money. But some strategies can be better than others in some specific sense. Let us consider only \$1 bets for purposes of this thesis. We can regard the roulette as 37 single numbers and our question is whether and how to diversify our asset. It is straight-forward to show that the more numbers we bet on, the smaller the variance is. Let us assume we spread our bet on k numbers. X_k is a random variable representing our profit/loss defined as

$$X_k = \begin{cases} \frac{36-k}{k}, & \frac{k}{37} \\ -1, & \frac{37-k}{37}. \end{cases}$$

Then the variance of X_k is

$$\mathrm{var}(X_k) = \mathsf{E}\left[X_k^2\right] - (\mathsf{E}\ X_k)^2 = \frac{1296 - 35k}{375} - \left(-\frac{1}{37}\right)^2 = \frac{1296(37-k)}{1369k}$$

If k=1, the variance is approximately 34.8. Obviously, if k=37, the variance is 0.

The main goal of this thesis is to compare 2 strategies – the Red-Black strategy (the variance is approximately 0.99) and the Single Number strategy. On one hand, the betting on a color represents a conservative strategy – the asset is diversified, the variance is small, but the winning payout is also small (1 to 1). On the other hand, the betting on a number represents a more risky strategy – the bet is not diversified at all, but the winning payout is larger (35 to 1). Three statistics of the cumulative profit/loss will be studied in each chapter - the Maximum, the Last Exit Time and the Number of Visits of Zero. Comparing these statistics will give us a more detailed view on the two strategies and it will answer the question whether it is sensible to diversify or not.

The theoretical results are supported by Monte Carlo simulations at the end of each chapter.

1. Red-Black Strategy

There are 18 black numbers, 18 red numbers and 0 that is neither black nor red in European roulette. There are 18 numbers we bet on and 19 we do not in Red-Black strategy. Probability of winning is $\frac{18}{37}$ and probability of losing is $\frac{19}{37}$. Note that this strategy is equivalent to betting on Odd/Even and Low/High numbers.

This betting strategy leads to *a discrete stochastic process* $Y = \{Y(n), n \in \mathbb{N}\}$, where n represents time and $Y(n)$ is the resulting profit/loss at time n. For purposes of this thesis, we will consider only simple \$1 bets on Red or Black. As a result, $\{Y(1), Y(2), Y(3), ...\}$ is a sequence of independent and identically distributed random variables. The random variable $Y(n)$ has the following distribution

$$Y(n) = \begin{cases} +1, & 1 - p = \frac{18}{37} \\ -1, & p = \frac{19}{37}. \end{cases}$$

It is slightly unusual that p denotes probability of loss. This convention comes from the theory of the Markov Chains – it is more typical to study a loss process as it walks to $+\infty$ rather than $-\infty$. The expected value of $Y(n)$ is

$$\mathsf{E}\left[Y(n)\right] = 1 \cdot \tfrac{18}{37} + (-1) \cdot \tfrac{19}{37} = -\tfrac{1}{37}.$$

Let us consider another random variable – the cumulative profit-loss $S(n)$ defined as

$$S(n) = \sum_{i=1}^{n} Y(n).$$

This variable represents how much money we made or lost at time n. $S(n)$ can also be recognized as a *one-dimensional random walk*. Since the expected value of $Y(n)$ is less than 0, $S(n)$ is *a random walk with a negative drift*. Assume S at time $n = 0$ is zero. It is clear that

$$S(n) = S(n-1) + Y(n), \ \forall n \in \mathbb{N}. \tag{1.1}$$

Theorem 1 (The strong law of large numbers for i.i.d.). *Let* $Y(1), Y(2), Y(3), \cdots$ *be i.i.d. random variables. Then*

$$\overline{Y(n)} \to \mathsf{E}[Y(1)] \ a.s.$$

if and only if $\mathsf{E}[|Y(1)|] < \infty$.

Proof. See Dupač and Hušková [2013] page 79. □

$\mathsf{E}\left[|Y(1)|\right] = 1 \cdot \frac{18}{37} + 1 \cdot \frac{19}{37} = 1$. From Theorem 1 we get that $S(n) = n \cdot \overline{Y(n)}$ converges to $-\infty$ almost surely for $n \to \infty$.

We see that our game will ultimately end up negative. In particular, this means that the Last Exit Time τ from non-negative values defined as

$$\tau = \max\{n : S(n) = 0\}$$

is finite: $\mathsf{P}(\tau < \infty) = 1$. Let us assume an example from Dubins and Savage [1965]. We are at a casino with \$1,000 and we desperately need \$10,000 the next day. Even though we know that our chances of losing are higher than chances of winning, gambling is our only option. To better understand the game, some important statistics will be described in the next few sections.

1.1 Maximum

We would like to know the distribution of the Maximum that can be reached while playing the Red-Black strategy. Let M be a random variable defined as

$$M = \max_n S(n),\ n \in \mathbb{N}_0.$$

Maximum is a discrete variable and, as a result of the definition of $S(n)$, it can reach only integer values greater or equal to *zero*. We need to find the distribution of M.

In the Red-Black strategy, $Y(n)$ can only reach 1 or -1 and from Equation 1.1 we see that S cannot "skip" any value from $\mathbb{Z}$. Saying it in more appropriate way: if there exists $n \in \mathbb{N}$ such that $S(n) = k > 0$, then there exists $m \in \mathbb{N}$ such that $S(m) = k - 1$. That leads us to

$$\begin{aligned}\mathsf{P}(M = k) &= \mathsf{P}(M \geq k) - \mathsf{P}(M \geq k+1) = \\ &= \mathsf{P}(S(n) \text{ hits } k \text{ eventually}) - \mathsf{P}(S(n) \text{ hits } k+1 \text{ eventually}).\end{aligned}$$

To find this probability, we can use several approaches. In this thesis we use a Martingale approach and a Markov Chain approach with absorbing boundary described in detail in the following text.

1.1.1 Martingale

Let us start with definition of martingale.

Definition 1. $X = \{X(n), n \in \mathbb{N}_0\}$ *that satisfies for any time* n*:*
a) $\mathsf{E}(|X(n)|) < \infty$*,*
b) $\mathsf{E}[X(n+1)|X(1), \cdots, X(n)] = X(n)$
is a discrete-time martingale.

Define a random variable $X(n)$ as

$$X(n) = \left(\frac{p}{1-p}\right)^{S(n)}.$$

It is a discrete-time martingale, since

$$\begin{aligned}\mathsf{E}\left[X(n+1)|S(0), \cdots, S(n)\right] &= \mathsf{E}_n \left(\frac{p}{1-p}\right)^{S(n+1)} = \mathsf{E}_n \left(\frac{p}{1-p}\right)^{S(n)+Y(n+1)} = \\ &= \left(\frac{p}{1-p}\right)^{S(n)} \cdot \mathsf{E} \left(\frac{p}{1-p}\right)^{Y(n+1)} = \\ &= \left(\frac{p}{1-p}\right)^{S(n)} \cdot \left[\left(\frac{p}{1-p}\right)^{1} \cdot (1-p) + \left(\frac{p}{1-p}\right)^{-1} \cdot p\right] = \\ &= \left(\frac{p}{1-p}\right)^{S(n)} \cdot (p + 1 - p) = \\ &= \left(\frac{p}{1-p}\right)^{S(n)} = X(n).\end{aligned}$$

Theorem 2 (Optional sampling theorem)**.** *Let $X = \{X(n), n \in \mathbb{N}\}$ be a discrete-time martingale and τ a stopping time with values in $\mathbb{N} \cup \{\infty\}$, both with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Assume that one of the following three conditions holds:*
a) The stopping time τ is almost surely bounded.
b) $\mathsf{E}[\tau] < \infty$ and there exists a constant c such that $\mathsf{E}[|X(n+1) - X(n)|\,|\mathcal{F}_n] \le c$ a.s. for all $n \in \mathbb{N}$.
c) There exists a constant c such that $\left|X_{\min\{n,\tau\}}\right| \le c$ a.s. for all $n \in \mathbb{N}$.
Then $X(n)$ is almost surely well defined random variable and $\mathsf{E}[X(\tau)] = \mathsf{E}[X(1)]$.

Let us define stopping time $\tau = \min\{n : \ S(n) = 1 \text{ or } S(n) = b\}$. Then $X(n)$ is a discrete-time martingale that satisfies Theorem 2. Expected value of $X(1)$ is

$$\begin{aligned}\mathsf{E}[X(1)] &= \mathsf{E}\left(\frac{p}{1-p}\right)^{S(1)} = \mathsf{E}\left(\frac{p}{1-p}\right)^{Y(1)} = \\ &= \left(\frac{p}{1-p}\right)^{1} \cdot (1-p) + \left(\frac{p}{1-p}\right)^{-1} \cdot p = 1.\end{aligned}$$

Using Theorem 2 we get

$$1 = \mathsf{E}\left(\frac{p}{1-p}\right)^{S(\tau)} = \left(\frac{p}{1-p}\right)^{1} \cdot \mathsf{P}[S(\tau) = 1] + \left(\frac{p}{1-p}\right)^{b} \cdot \mathsf{P}[S(\tau) = b].$$

If we use limit $b \to -\infty$, the second summand in the previous equation disappears because $\frac{p}{1-p} > 0$. Thus,

$$\begin{aligned}1 &= \frac{p}{1-p} \cdot \mathsf{P}[S(\tau) = 1], \\ \mathsf{P}[S(\tau) = 1] &= \frac{1-p}{p} = \frac{18}{19}.\end{aligned}$$

This is the probability that the random walk $S(n)$ will eventually reach 1; or equivalently, the probability that M is greater than 0.

1.1.2 Markov Chain

We will use the following transformation to map the wealth process to a simple random walk:

$$W(n) = k - S(n).$$

The process W has a fixed starting point k and drifts to $+\infty$. The process $W(n)$ hitting zero at some point is equivalent to $S(n)$ hitting k. Therefore, for solving of our problem we only need to compute probability

$$\begin{aligned}\alpha(k) &= \mathsf{P}(W(n) \text{ hits } 0 \text{ eventually} | W(0) = k) = \\ &= \mathsf{P}(S(n) \text{ hits } k \text{ eventually} | S(0) = 0).\end{aligned}$$

Definition 2 (Discrete Markov chain)**.** *A sequence of random variables $\{X_n, n \in \mathbb{N}_0\}$ is called discrete Markov chain with state space T if*

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0) = P(X_{n+1} = j | X_n = i)$$

for all $n \in \mathbb{N}_0$ and for all $j, i, i_{n-1}, \cdots, i_0 \in T$, if $\mathsf{P}(X_n = i, \cdots, X_0 = i_0) > 0$.

From Equation 1.1, it is clear that $S(n)$ depends only on the value of the previous $S(n-1)$ and the new value of $Y(n)$. As a result, $S(n)$ (and also $W(n)$) satisfies the Markov property and we can follow the Markov Chain Theory as explained for example in Prášková and Lachout [2012]. The most useful for us is the computation of probabilities.

$$\alpha(k) = \sum_{j=1}^{\infty} \alpha(j) p(k,j),$$

where $p(k,j)$ is probability of moving from a position k to j. In the Red-Black strategy, $S(n)$, $n \geq 1$, can only be reached from $S(n)-1$ with probability $1-p = \frac{18}{37}$ and from $S(n)+1$ with probability $p = \frac{19}{37}$. This is very useful, because the equation for computing $\alpha(k)$ reduces to

$$\alpha(k) = (1-p) \cdot \alpha(k-1) + p \cdot \alpha(k+1). \tag{1.2}$$

This is a difference equation and its solution can be found in the following way. First, find the particular solution in the form x^k and then find the general solution that satisfies boundary conditions.

Substituting $\alpha(k) = x^k$ ($x \neq 0$) to Equation 1.2 we get

$$\begin{aligned} x^k &= (1-p) \cdot x^{k-1} + p \cdot x^{k+1} \\ x &= 1 - p + p \cdot x^2 \\ p \cdot x^2 - x + 1 - p &= 0. \end{aligned}$$

This quadratic equation has two roots

$$x_{1,2} = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} = \frac{1 \pm (1-2p)}{2p} = \begin{cases} 1 \\ \frac{1-p}{p} = \frac{18}{19}. \end{cases}$$

This is the particular solution. The general solution is in the form

$$\alpha(k) = \sum_{i=1}^{2} c_i x_i^k = c_1 \left(\frac{18}{19}\right)^k + c_2 1^k.$$

Boundaries are needed to find c_i. The first condition comes from $S(0) = 0$ and thus $\alpha(0) = 1$. The second condition comes from that $S(n)$ converges to $-\infty$ almost surely for $n \to \infty$, so the probability of reaching $S(n) = \infty$ at some point converges to zero a.s. This gives us $\lim_{k\to\infty} \alpha(k) = 0$.

$$\begin{aligned} 1 =& \alpha(0) = c_1 \left(\frac{18}{19}\right)^0 + c_2 = c_1 + c_2 \\ 0 =& \lim_{k\to\infty} \alpha(k) = \lim_{k\to\infty} c_1 \left(\frac{18}{19}\right)^k + c_2 = c_2. \end{aligned}$$

From these equations we have $c_2 = 0$ and $c_1 = 1$ and thus the probability $\alpha(k)$ is

$$\alpha(k) = \left(\frac{18}{19}\right)^k.$$

Therefore,

$$\mathsf{P}(M = k) = \alpha(k) - \alpha(k+1) = \left(\frac{18}{19}\right)^k - \left(\frac{18}{19}\right)^{k+1} = \frac{1}{19} \cdot \left(\frac{18}{19}\right)^k$$

for all $k \geq 0$. The distribution of the Maximum is geometric with parameter $\lambda = \frac{1}{19}$. We can use that to derive some facts. First of all, the expected value, the variance and the standard deviation of geometric distribution is

$$\mathsf{E}[M] = \frac{1-\lambda}{\lambda} = \frac{\frac{18}{19}}{\frac{1}{19}} = 18, \tag{1.3}$$

$$\text{var}[M] = \frac{1-\lambda}{\lambda^2} = \frac{\frac{18}{19}}{\frac{1}{19^2}} = 18 \cdot 19 = 342, \tag{1.4}$$

$$\sigma[M] = \sqrt{\text{var}[M]} = \sqrt{342} \doteq 18.49. \tag{1.5}$$

Second of all, the probability that the Maximum is zero, or equivalently; that we never reach a positive value, is

$$\mathsf{P}(M = 0) = \tfrac{1}{19} \doteq 0.0526316. \tag{1.6}$$

More importantly the probability that we will eventually reach something positive is

$$\mathsf{P}(M \geq 1) = 1 - \mathsf{P}(M = 0) = \frac{18}{19} \doteq 0.947368.$$

Finally, median is 12 and 95% quantile is 55 ($\mathsf{P}(M \leq 55) \geq 0.95$)

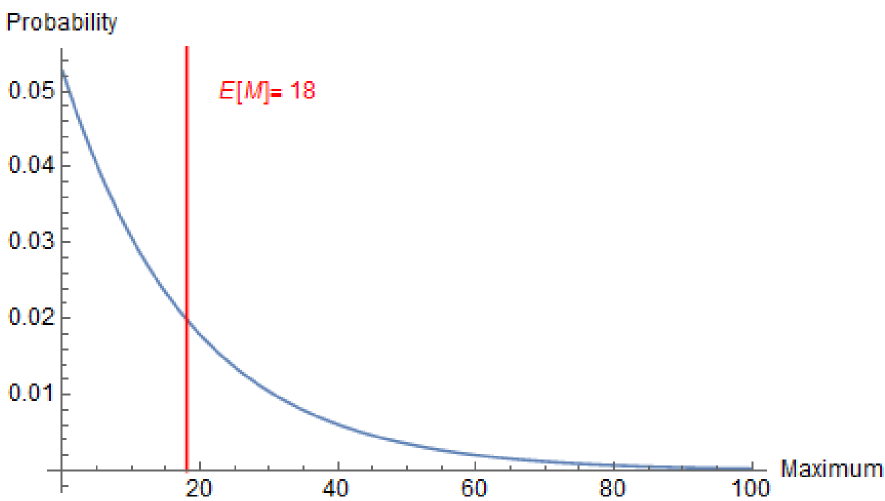

Figure 1.1: Distribution of the Maximum with the Expected value

1.2 Last Exit Time

Knowing that our game almost surely ends in the negative numbers, it is natural to ask question when is the last time τ we reach zero. The Last Exit Time is defined as

$$\tau = \max\{n \geq 0 | S(n) = 0\}.$$

The random walk $S(n)$ can be at zero only at even times. As in the previous section, the main goal is to find the distribution of τ. Therefore,

$$\mathsf{P}(\tau = 2n)$$

needs to be found for all $n \in \mathbb{N}_0$.

The evolution of the random walk before the time $2n$ is not relevant. The last exit time τ is at $2n$ only in the following situation. $S(2n) = 0$, next step must be down to -1 ($S(2n+1) = -1$). After that, the random walk must stay under -1, so it does not reach zero again. This means that $\max_{k \geq 2n+1} S(k) = -1$. An illustration of that is in Figure 1.2.

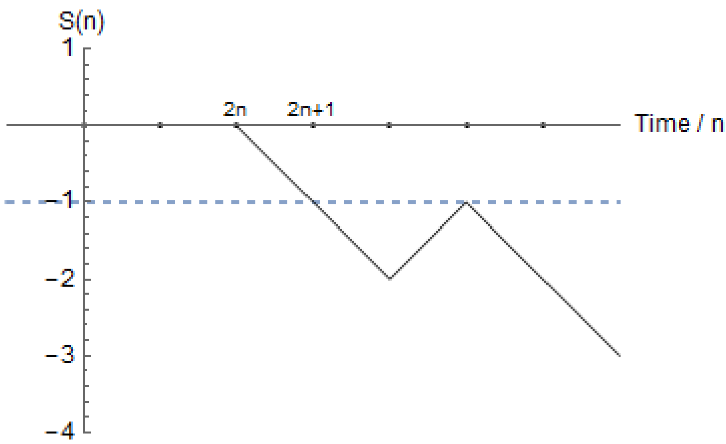

Figure 1.2: Part of a random walk with $\tau = 2n$

In the terms of the probabilities:

$$\mathsf{P}(\tau = 2n) = \mathsf{P}\left(S(2n) = 0 \;\&\; S(2n+1) = -1\right) \cdot \mathsf{P}(\max_{k \geq 2n+1} S(k) = -1). \tag{1.7}$$

The definition of the conditional probability can be used for computing the first factor of the equation above.

$$\begin{aligned}
\mathsf{P}[S(2n) = 0 \;\&\; S(2n+1) = -1] &= \\
&= \mathsf{P}[S(2n) = 0] \cdot \mathsf{P}[S(2n+1) = -1 | S(2n) = 0] = \\
&= \binom{2n}{n} p^n (1-p)^n \cdot p = \\
&= \binom{2n}{n} p^{n+1} (1-p)^n.
\end{aligned} \tag{1.8}$$

The second factor of Equation 1.7 can be solved the following way. The probability that we are at -1 and will not go above it is the same as the probability that we are at 0 and will not go above it. We only changed the starting point of $S(n)$ to -1. This probability was computed in Section 1.1, so Equation 1.6 can be used.

$$\mathsf{P}\left(\max_{k \geq 2n+1} S(k) = -1\right) = \mathsf{P}\left(\max_{k \geq 0} S(k) = 0\right) = \mathsf{P}\left(M = 0\right) = \frac{1}{19}. \tag{1.9}$$

Putting Equation 1.8 and Equation 1.9 together into Equation 1.7, we get our initial probability

$$\mathsf{P}(\tau = 2n) = \frac{1}{19} \cdot \binom{2n}{n} p^{n+1}(1-p)^n.$$

Using Wolfram Mathematica,

$$\mathsf{E}[\tau] = \sum_{n=0}^{\infty} 2n \cdot \mathsf{P}(\tau = 2n) = \sum_{n=0}^{\infty} \frac{2n}{19} \binom{2n}{n} p^{n+1}(1-p)^n = 1,368, \tag{1.10}$$

$$\text{var}[\tau] = \mathsf{E}[\tau^2] - (E[\tau])^2 = 5,617,008 - (1,368)^2 = 3,745,584, \tag{1.11}$$

$$\sigma[\tau] = \sqrt{\text{var}[\tau]} \doteq 1935.35. \tag{1.12}$$

The basic quantiles (also using Wolfram Mathematica) are: median is 622 and 95% quantile is 5,256.

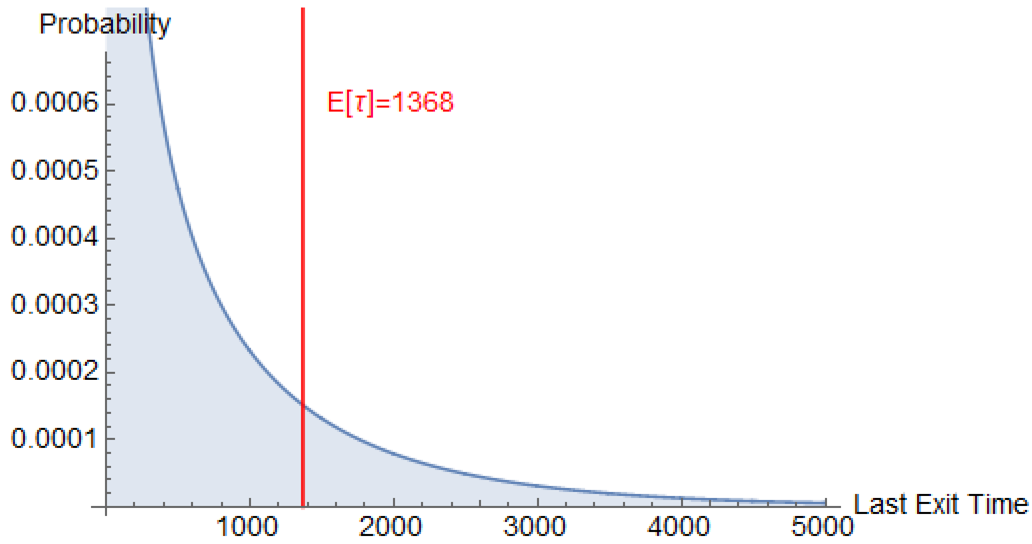

Figure 1.3: Distribution of the Last Exit Time with the Expected value

1.3 Visits of Zero

Let us assume that we would like to know how many times the process $S(n)$ returns to zero. This number tells us how many times the random walk crosses the zero which is the moment when our profit and loss are balanced. The random variable Z that counts the number of returns to zero is defined as

$$Z = \sum_{n=0}^{\infty} \chi_{\{S(n)=0\}},$$

where χ is characteristic function.

Z reaches only values from $\mathbb{N}$ ($\mathsf{P}(Z \geq 1) = 1$). Again, our goal is to find the distribution of Z

$$\mathsf{P}(Z = k) = \mathsf{P}\left(\sum_{n=0}^{\infty} \chi_{\{S(n)=0\}} = k\right)$$

for positive integers k.

If $Z = k$, we can divide the random walk on k sections as shown in Figure 1.4 (the red dashed lines).

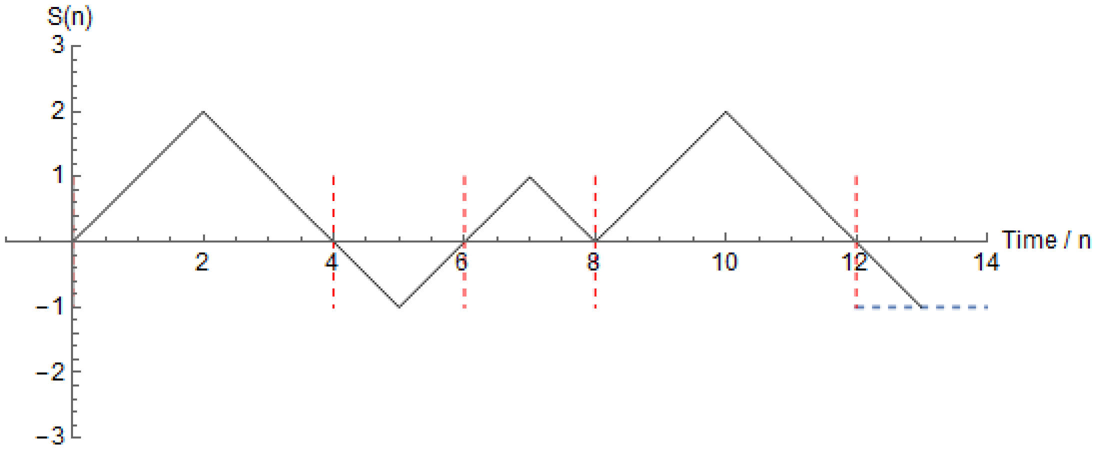

Figure 1.4: Example of a part of a random walk with $Z = 5$

The sections $1, \cdots, k-1$ are same in the following sense: they begin in zero, they end in zero and there is no intermediate value at zero. The last section is different - it starts at zero, goes to -1 in the next step and after that never reaches zero again. All the sections are independent of each other; consequently, the result $\mathsf{P}(Z = k)$ is just a product of probabilities of each part. Let the probability of section i be p_i for $i \in \{1, \cdots, k\}$. Then

$$\mathsf{P}(Z = k) = \prod_{i=1}^{k} p_i.$$

First of all, let us take a look on the last section and its probability p_k. We touch zero at some point and it is the last time we achieve it. Thus the following progress of the random walk must be down to -1 and then continue under the -1, so it never reaches 0 again. This is actually almost probability that we have already derived in Section 1.2. We go to -1 (probability $\frac{19}{37}$) and stay there (with probability $\frac{1}{19}$ from Equation 1.9). As a result,

$$p_k = \frac{19}{37} \cdot \frac{1}{19} = \frac{1}{37}.$$

Probabilities p_i are different, but they are computed the same way for all $i \in \{1, \cdots, k-1\}$. Let us fix i. The random walk $S(n)$ can be seen as a Markov chain with 2 states - we reach zero again or we do not. The probability that we do not is p_k, the probability that we reach zero again is $1 - p_k$. Therefore, this Markov Chain has a geometric distribution. Thus,

$$\mathsf{P}(Z = k) = \frac{1}{37} \cdot \left(\frac{36}{37}\right)^{k-1}$$

The expected value, the variation, and the standard deviation are

$$\mathsf{E}[Z] = \sum_{k=1}^{\infty} k \cdot \frac{1}{37} \cdot \left(\frac{36}{37}\right)^{k-1} = 36, \tag{1.13}$$

$$\text{var}[Z] = 1,332, \tag{1.14}$$

$$\sigma[Z] \doteq 36.5. \tag{1.15}$$

The important quantiles are: median is 25 and 95% quantile is 109.

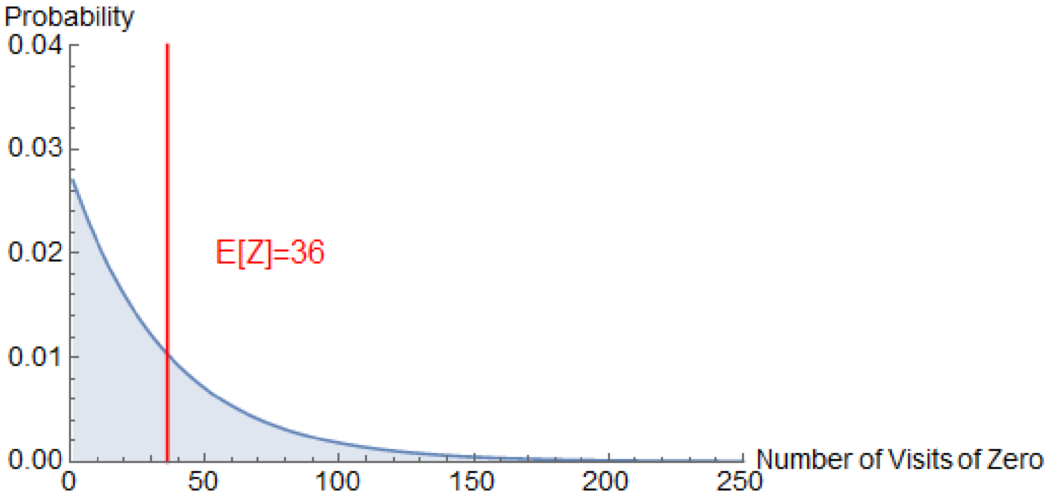

Figure 1.5: Distribution of the Number of Visits of Zero with the Expected value

1.4 Monte Carlo

In this section I will use data I got from Monte Carlo simulations. Results in the previous sections have proper proofs. Therefore, the Monte Carlo simulations are not necessary. This chapter only demonstrates the results. I used Monte Carlo method to simulate $N = 1,000,000$ random walks. For $i = \{1, \cdots, N\}$ let R_i be representation of random walk $S(n)$. From each R_i I remembered the value of the Maximum M_i, the Last Exit Time τ_i, and the Number of Visits of Zero Z_i. One stopping rule was used during the simulations - when the random walk reached value -100, the simulation stopped.

Terms such as the sample mean, the sample variation etc. will be used. First of all, their definitions will be given because they vary from author to author.

Definition 3. *Let $X_1, \cdots, X_n$ be a random sample of any distribution.*
(a) Sample mean $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$.
(b) Sample variance $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X_n})^2$.
(c) Sample standard deviation $s_n = \sqrt{S_n^2}$.

Theorem 3. *Let $X_1, \cdots, X_n$ be a random sample of any distribution with the expected value μ and with finite the variation σ^2. Then*

$$\sqrt{n}\frac{\overline{X_n} - \mu}{S_n} \xrightarrow{D} N(0,1).$$

Proof. From central limit theorem (CLT) we know that

$$\sqrt{n}\frac{\overline{X_n} - \mu}{\sigma} \xrightarrow{D} N(0,1).$$

We also know that $S_n^2 \xrightarrow{P} \sigma^2$. Thus,

$$\frac{S_n}{\sigma} \xrightarrow{P} 1.$$

Using Slutsky's theorem

$$\sqrt{n}\frac{\overline{X_n} - \mu}{S_n} = \frac{\sigma}{S_n}\sqrt{n}\frac{\overline{X_n} - \mu}{\sigma} \xrightarrow{D} N(0,1).$$

□

1.4.1 Maximum

Let M_i be the Maximum that was reached during the i-th walk. The mean of M_i observed in this simulation was

$$\overline{M_N} \doteq 17.9886.$$

Note that $E[M] = 18$ (from Equation 1.3). Graph of relative frequency with the mean $\overline{M_N}$ is shown in Figure 1.6.

The sample variation of the Maximum was $S_N^2 \doteq 343.14$, $\text{var}[M] = 342$ from Equation 1.4.

The sample standard deviation was $s_N \doteq 18.52$, $\sigma[M] \doteq 18.49$ from Equation 1.5.

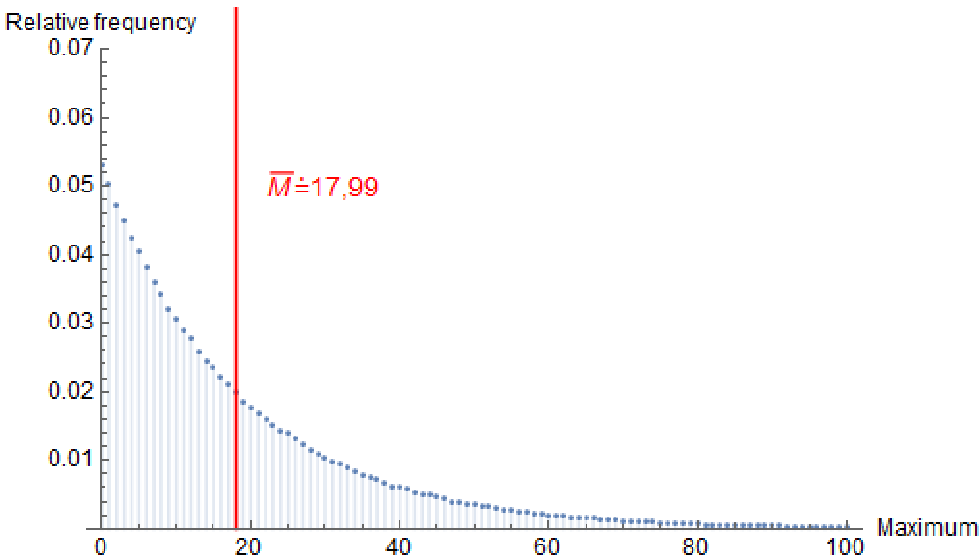

Figure 1.6: Relative frequency of the Maximum with sample mean

From these results and Theorem 3 we can construct a confidence interval for the expected value μ_M. For $N \to \infty$

$$\mathsf{P}\left[u_{\frac{\alpha}{2}} < \sqrt{N}(\overline{M_N} - \mu_M)/S_N < u_{1-\frac{\alpha}{2}}\right] \to 1 - \alpha.$$

The empirical 95% confidence interval for μ_M is

$$(17.9582, 18.0191).$$

Note that $E[M] = 18$ (from Equation 1.3). In Figure 1.7 there are both - the cumulative frequency of the Maximum in the sample from Monte Carlo and the real distribution of the Maximum. The maximal value of the Maximum in the sample was 267, $\mathsf{P}(M \geq 267) = \frac{1}{1{,}859{,}741}$.

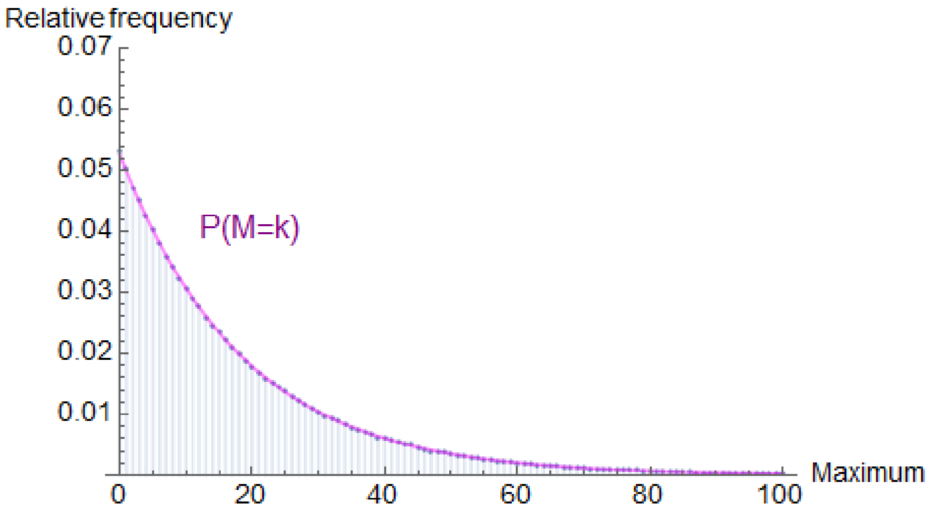

Figure 1.7: Relative frequency of the Maximum and the distribution of the Maximum

1.4.2 Last Exit Time

I will proceed the same way as in the previous subsection. τ_i is the Last Exit Time of the i-th walk.

The sample mean of the Last Exit Time was

$$\overline{\tau_N} \doteq 1,366.23,$$

the $E[\tau] = 1,368$ from Equation 1.10. There is a graph of the relative frequency with the sample mean in Figure 1.8.

The sample variation was $S_N^2 \doteq 3.8085 \cdot 10^6$, $\text{var}[\tau] = 3,745,584$ from Equation 1.11.

The sample standard deviation was $s_N \doteq 1,951.53$, $\sigma[\tau] \doteq 1935.35$ from Equation 1.12.

From these results and Theorem 3 we can construct a confidence interval for the expected value μ_τ. For $N \to \infty$

$$\mathsf{P}\left[u_{\frac{\alpha}{2}} < \sqrt{N}(\overline{\tau_N} - \mu_\tau)/S_N < u_{1-\frac{\alpha}{2}}\right] \to 1 - \alpha.$$

The empirical 95% confidence interval for μ_τ is

$$\mu_\tau \in (1363.02, 1369.44).$$

In Figure 1.9 there are both - the cumulative frequency of the sample from Monte Carlo and the real distribution of the Last Exit Time. The maximal Last Exit Time of the sample was 34,433.

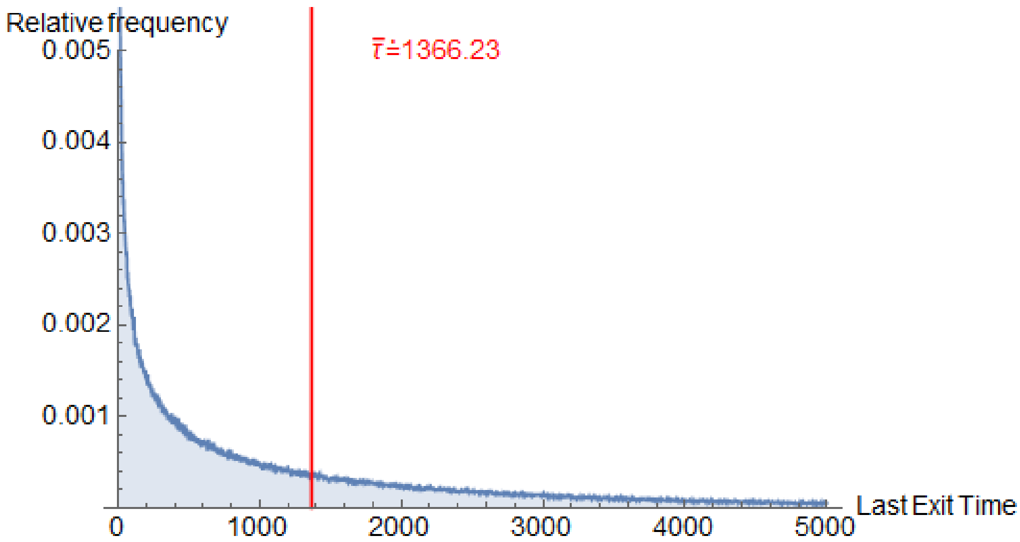

Figure 1.8: Relative frequency of the Last Exit Time with sample mean

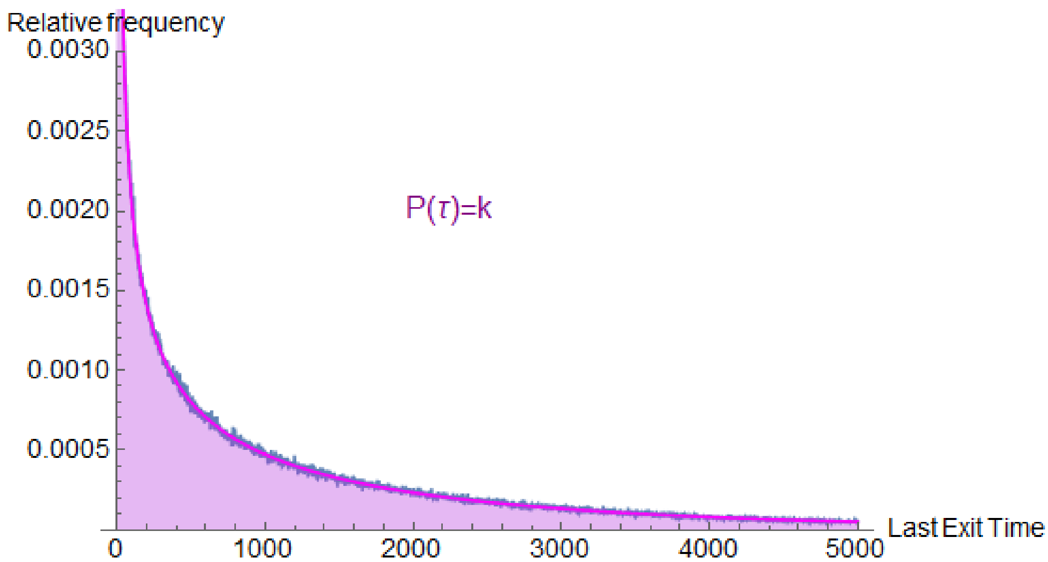

Figure 1.9: Relative frequency of the Last Exit Time and the distribution of the Last Exit Time

1.4.3 Visits of Zero

Let Z_i be the number of return times to zero of the i-th walk.

The sample mean of the Numer of Visits of Zero was

$$\overline{Z_N} \doteq 36.935,$$

$E[Z] = 36$ from Equation 1.13. There is graph of the relative frequency with sample mean in Figure 1.10.

The sample variation was $S_N^2 \doteq 1.7022 * 10^6$, $\text{var}[Z] = 1,332$ from Equation 1.14.

The sample standard deviation was $s_N \doteq 1,304.7$, $\sigma[Z] \doteq 36.5$ from Equation 1.15.

From these results and Theorem 3 we can construct a confidence interval for

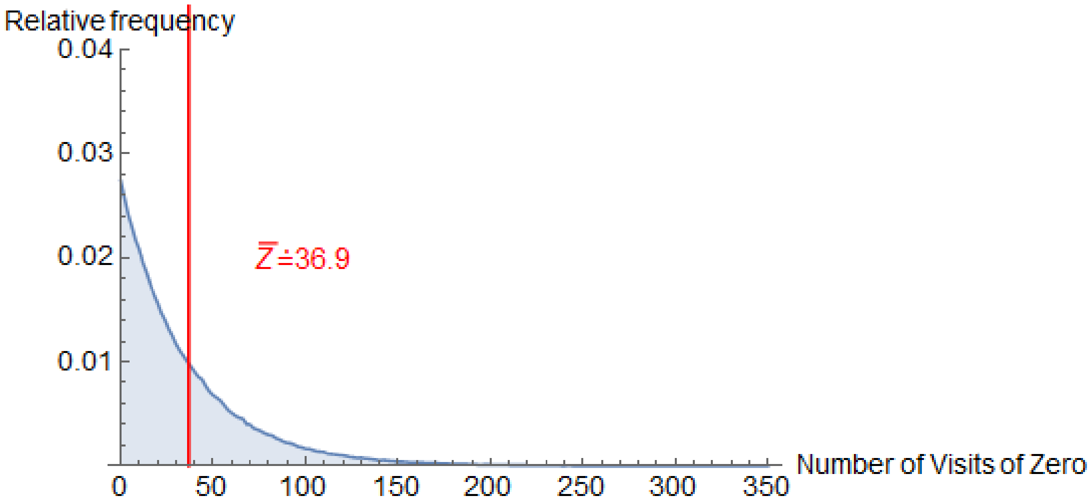

Figure 1.10: Relative frequency of the Number of Visits of Zero with sample mean

the expected value μ_Z. For $N \to \infty$

$$\mathsf{P}\left[u_{\frac{\alpha}{2}} < \sqrt{N}(\overline{Z_N} - \mu_Z)/S_N < u_{1-\frac{\alpha}{2}}\right] \to 1 - \alpha.$$

The empirical 95% confidence interval for μ_Z is

$$\mu_Z \in (34.7889, 39.0809).$$

In Figure 1.11 there are both - the cumulative frequency from the data from Monte Carlo and the real distribution of the Number of Visits of Zero.

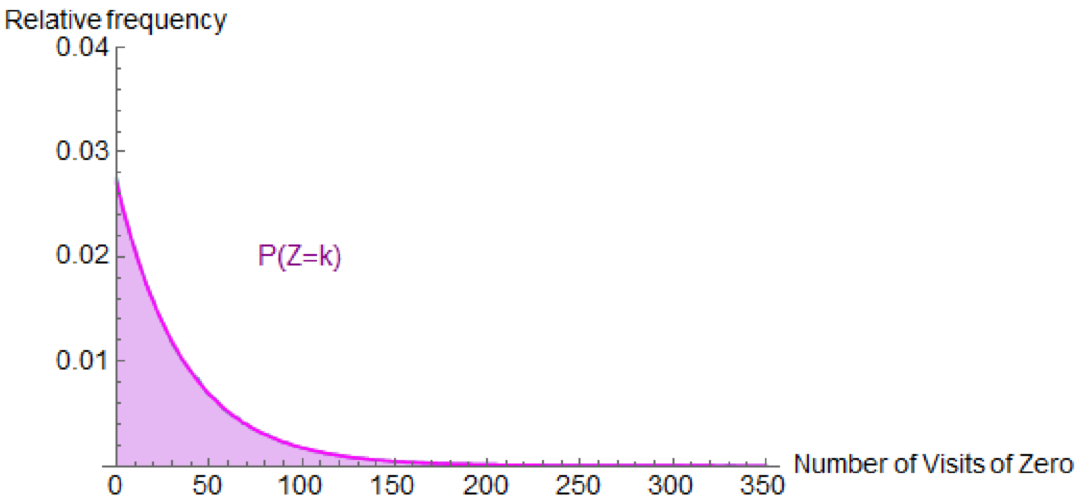

Figure 1.11: Relative frequency of the Number of Visits of Zero and the the distribution of Number of Visits of Zero

2. Single Number Strategy

The same statistics as in Chapter 1 will be described in this chapter. The difference is that we consider betting on a single number. In European roulette that means we have one number we bet on and 36 numbers we do not. The player bets on a single number and probabilities of winning and losing are $\frac{1}{37}$ and $\frac{36}{37}$. Comparing it to the probabilities in the Red-Black strategy, it appears that we have smaller chances to win something. But that is true only if we consider one single bet. Considering a process of single bets leads us to a different conclusions as explained in the following sections.

Let us define *a discrete stochastic process* $Y = \{Y(n), n \in \mathbb{N}\}$, where n represents time and $Y(n)$ represents single bet at time n (profit/loss). As in the previous chapter we will consider only simple \$1. Therefore, $\{Y(1), Y(2), Y(3), ...\}$ is a sequence of independent and identically distributed random variables with the following distribution

$$Y(n) = \begin{cases} +35, & 1-p = \frac{1}{37} \\ -1, & p = \frac{36}{37}. \end{cases}$$

Let us define $S(n)$ as a cumulative profit/loss,

$$S(n) = \sum_{i=1}^{n} Y(i),$$
$$S(0) = 0.$$

It is clear that for all $n \in \mathbb{N}$

$$S(n) = S(n-1) + Y(n).$$

The expected value is $\mathsf{E}[Y_n] = 35 \cdot \frac{1}{37} + (-1) \cdot \frac{36}{37} = -\frac{1}{37}$. The stochastic process $S(n)$ forms *a one-dimensional random walk with negative drift.* The expected value is the same as in the Red-Black strategy. Due to that we can use some results from Chapter 1 such as

$$\lim_{n \to \infty} S(n) = -\infty \ a.s.$$

2.1 Maximum

As in Section 1.1 we determine the distribution of the Maximum. The random variable Maximum M that can be reached in the single number strategy is defined as

$$M = \max_n S(n), \ n \in \mathbb{N}_0.$$

It can reach values from $\mathbb{N}_0$ and for finding its distribution we need to know probability $\mathsf{P}(M = k)$ for all $k \in \mathbb{N}_0$,

$$\mathsf{P}(M = k) = \mathsf{P}(\max_n S(n) = k).$$

In the Single Number strategy $Y(n) = +35$ or -1; consequently, $Y(n)$ skips numbers when going up but does not skip them when going down. Since we know that the random walk will almost surely end up in $-\infty$, every step up must be eventually at some point followed by 35 steps down (not necessarily one after another). So again, as in the Red-Black strategy, no value k can be skipped. This leads to

$$\begin{aligned}\mathsf{P}(M = k) &= \mathsf{P}(M \geq k) - \mathsf{P}(M \geq k+1) = \\ &= \mathsf{P}(S(n) \text{ hits } k \text{ eventually}) - \mathsf{P}(S(n) \text{ hits } k+1 \text{ eventually}).\end{aligned}$$

We will use the following transformation to map the wealth process to a simple random walk:

$$W(n) = k - S(n).$$

The process W has a fixed starting point k and drifts to $+\infty$. The process $W(n)$ hitting zero at some point is equivalent to $S(n)$ hitting k. Therefore, for solving of our problem we only need to compute probability

$$\begin{aligned}\alpha(k) &= \mathsf{P}(W(n) \text{ hits } 0 \text{ eventually} | W(0) = k) = \\ &= \mathsf{P}(S(n) \text{ hits } k \text{ eventually} | S(0) = 0).\end{aligned}$$

The random variable $S(n)$, $n > 0$, depends only on $S(n-1)$ and the new value of $Y(n)$, so it satisfies the Markov property. This gives us method how to compute $\alpha(k)$,

$$\alpha(k) = \sum_{j=1}^{\infty} \alpha(j) p(k,j),$$

where $p(k,j)$ is probability of moving from a position k to j. Playing the Single Number strategy, $S(n)$, $n \geq 1$, can be reached only from $S(n)+1$ with probability $\frac{36}{37}$ and from $S(n)-35$ with probability $\frac{1}{37}$. This fact reduces the previous equation to:

$$\alpha(k) = p \cdot \alpha(k+1) + (1-p) \cdot \alpha(k-35). \tag{2.1}$$

For solving this equation we can use the same method as in Section 1.1. Substituting $\alpha(k) = x^k$ $(x \neq 0)$ in Equation 2.1 we get

$$\begin{aligned}x^k &= p \cdot x^{k+1} + (1-p) \cdot x^{k-35}, \\ x^{35} &= p \cdot x^{36} + 1 - p, \\ 0 &= p \cdot x^{36} - x^{35} + 1 - p.\end{aligned}$$

This is a polynomial equation of 36-th degree and can be computed using

Wolfram Mathematica. This gives us the roots:

$$\begin{aligned}
x_1 &= -0.882643 - 0.078397i,\\
x_2 &= -0.882643 + 0.078397i,\\
x_3 &= -0.855197 - 0.232788i,\\
x_4 &= -0.855197 + 0.232788i,\\
x_5 &= -0.801135 - 0.380045i,\\
x_6 &= -0.801135 + 0.380045i,\\
x_7 &= -0.722094 - 0.515652i,\\
x_8 &= -0.722094 + 0.515652i,\\
x_9 &= -0.620465 - 0.635444i,\\
x_{10} &= -0.620465 + 0.635444i,\\
x_{11} &= -0.499321 - 0.735735i,\\
x_{12} &= -0.499321 + 0.735735i,\\
x_{13} &= -0.362323 - 0.813426i,\\
x_{14} &= -0.362323 + 0.813426i,\\
x_{15} &= -0.213606 - 0.866101i,\\
x_{16} &= -0.213606 + 0.866101i,\\
x_{17} &= -0.057652 - 0.892093i,\\
x_{18} &= -0.057652 + 0.892093i,\\
x_{19} &= +0.100855 - 0.890536i,\\
x_{20} &= +0.100855 + 0.890536i,\\
x_{21} &= +0.257172 - 0.861384i,\\
x_{22} &= +0.257172 + 0.861384i,\\
x_{23} &= +0.406657 - 0.805403i,\\
x_{24} &= +0.406657 + 0.805403i,\\
x_{25} &= +0.544935 - 0.724134i,\\
x_{26} &= +0.544935 + 0.724134i,\\
x_{27} &= +0.668081 - 0.619809i,\\
x_{28} &= +0.668081 + 0.619809i,\\
x_{29} &= +0.772853 - 0.495218i,\\
x_{30} &= +0.772853 + 0.495218i,\\
x_{31} &= +0.857135 - 0.353398i,\\
x_{32} &= +0.857135 + 0.353398i,\\
x_{33} &= +0.921415 - 0.196615i,\\
x_{34} &= +0.921415 + 0.196615i,\\
x_{35} &= +0.998443\\
x_{36} &= +1.
\end{aligned}$$

This is the particular solution. The general solution is in the form

$$\alpha(k) = \sum_{i=1}^{36} c_i x_i^k.$$

We need boundary conditions in order to determine c_i. From $S(0) = 0$, we have $\alpha(0) = 1$. We get another conditions from the fact that $S(n)$ converges to $-\infty$ almost surely for $n \to \infty$. Thus $\lim_{k\to\infty} \alpha(k) = 0$. For all $k \in \{-34, ..., -1\}$ we know that $S(n)$ can not skip them but it almost surely ends in $-\infty$, so it must reach them (a.s.) at some point. That gives us $\alpha(k) = 1$ for $k \in \{-34, ..., -1\}$. Now we have 36 conditions and 36 unknown variables. Using Wolfram Mathematica we get the solution:

$$
\begin{aligned}
c_1 &= 0.000409310 - 0.000017051i, \\
c_2 &= 0.000409310 + 0.000017051i, \\
c_3 &= 0.000409633 - 0.000051422i, \\
c_4 &= 0.000409633 + 0.000051422i, \\
c_5 &= 0.000410300 - 0.000086612i, \\
c_6 &= 0.000410300 + 0.000086612i, \\
c_7 &= 0.000411348 - 0.000123227i, \\
c_8 &= 0.000411348 + 0.000123227i, \\
c_9 &= 0.000412845 - 0.000161969i, \\
c_{10} &= 0.000412845 + 0.000161969i, \\
c_{11} &= 0.000414893 - 0.000203698i, \\
c_{12} &= 0.000414893 + 0.000203698i, \\
c_{13} &= 0.000417645 - 0.000249512i, \\
c_{14} &= 0.000417645 + 0.000249512i, \\
c_{15} &= 0.000421336 - 0.000300882i, \\
c_{16} &= 0.000421336 + 0.000300882i, \\
c_{17} &= 0.000426330 - 0.000359857i, \\
c_{18} &= 0.000426330 + 0.000359857i, \\
c_{19} &= 0.000433208 - 0.000429429i, \\
c_{20} &= 0.000433208 + 0.000429429i, \\
c_{21} &= 0.000442953 - 0.000514183i, \\
c_{22} &= 0.000442953 + 0.000514183i, \\
c_{23} &= 0.000457316 - 0.000621568i, \\
c_{24} &= 0.000457316 + 0.000621568i, \\
c_{25} &= 0.000479691 - 0.000764617i, \\
c_{26} &= 0.000479691 + 0.000764617i, \\
c_{27} &= 0.000517423 - 0.000968462i, \\
c_{28} &= 0.000517423 + 0.000968462i, \\
c_{29} &= 0.000589141 - 0.001288803i, \\
c_{30} &= 0.000589141 + 0.001288803i, \\
c_{31} &= 0.000755717 - 0.001879529i,
\end{aligned}
$$

$$
\begin{aligned}
c_{32} &= 0.000755717 + 0.001879529i, \\
c_{33} &= 0.001342830 - 0.003392991i, \\
c_{34} &= 0.001342830 + 0.003392991i, \\
c_{35} &= 0.982496, \\
c_{36} &= 0.
\end{aligned}
$$

Now we have all x_i and c_i. The solution for $\alpha(k)$ is

$$\alpha(k) = \sum_{i=1}^{35} c_i x_i^k.$$

Our initial goal was to find $\mathsf{P}(M = k)$,

$$\mathsf{P}(M = k) = \alpha(k) - \alpha(k+1) = \sum_{i=1}^{35} c_i x_i^k - \sum_{i=1}^{35} c_i x_i^{k+1} = \sum_{i=1}^{35} c_i x_i^k (1 - x_i).$$

The probability of $M = 0$ is

$$\mathsf{P}(M = 0) = \alpha(0) - \alpha(1) = 1 - \sum_{i=1}^{35} c_i x_i.$$

But since we have computed the roots only numerically, we do not know the proper distribution. Mathematica finds the results only with finite quantity of numbers after the decimal point. Our conjecture is that

$$\mathsf{P}(M = 0) = \frac{1}{36}.$$

Note that the following results are based only on numerical computations without theoretical proof:

$$
\begin{aligned}
\mathsf{P}(M = 0) &= \frac{1}{36} \doteq 0.02778, && (2.2) \\
\mathsf{P}(M > 0) &= \frac{35}{36} \doteq 0.97222, && (2.3) \\
\mathsf{E}[M] &= 630, && (2.4) \\
\mathrm{var}[M] &= 411,793, && (2.5) \\
\sigma[M] &= 641, && (2.6)
\end{aligned}
$$

the median is 434 and the 95% quantile is 1912.

The expected value of the Maximum playing Red-Black strategy is 18 and that happens to be 35 times smaller than the expected value playing the Single Number strategy. The variance is 1,200 times smaller.

2.2 Last Exit Time

Our next goal is to find the Last Exit Time during one simple walk. Let us start with the definition of the Last Exit Time τ.

$$\tau = \max\{n \geq 0 | S(n) = 0\}.$$

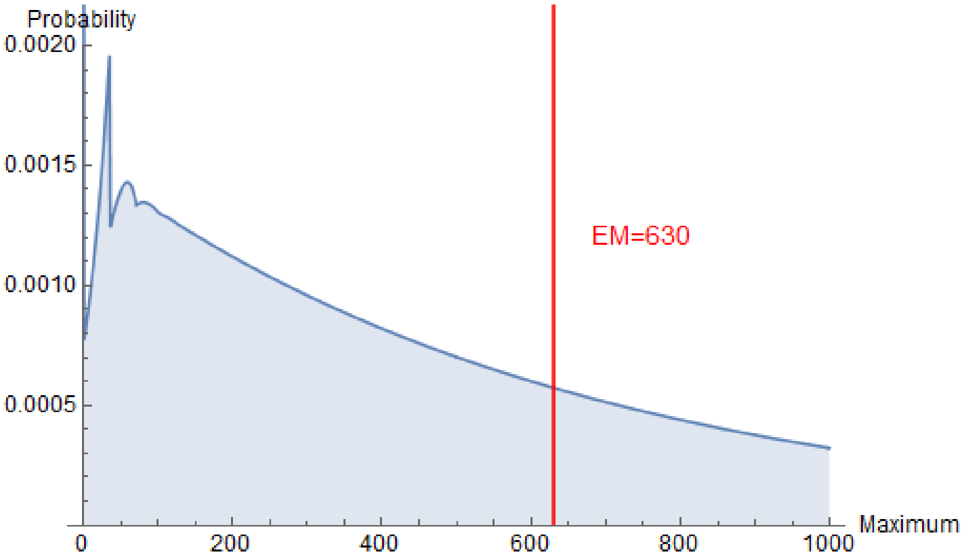

Figure 2.1: Theoretical distribution of the Maximum with the Expected value

τ is a discrete random variable, it can reach only zero or positive integers. But not all positive integers but only those that are multiple of 36 (the random walk can reach zero only in time $36n$, $n \in \mathbb{N}_0$). We would like to know the distribution of it and that means to find the probability

$$\mathsf{P}(\tau = 36n) = \mathsf{P}\left(\max\{k \geq 0 | S(k) = 0\} = 36n\right)$$

for all $k \in \mathbb{N}_0$. The progress of the random walk before the time $36n$ is not relevant. We only need that at the time $36n$ we reach 0 and never reach it again in the future. In other words at the time $36n$ we are at zero, at the time $36n+1$ we go down to -1 and from that moment our maximum is -1. In the mean of probabilities:

$$\mathsf{P}(\tau = 36n) = \mathsf{P}(S(36n) = 0 \ \& \ S(36n+1) = -1) \cdot \mathsf{P}(\max_{k \geq 36n+1} S(k) = -1). \quad (2.7)$$

The first factor of the product can be computed using the definition of the conditional probability,

$$\begin{aligned}
&\mathsf{P}[S(36n) = 0 \ \& \ S(36n+1) = -1] = \\
&\qquad = \mathsf{P}[S(36n) = 0] \cdot \mathsf{P}[S(36n+1) = -1 | S(36n) = 0] = \\
&\qquad = \binom{36n}{n} p^{35n} (1-p)^n \cdot p = \qquad (2.8) \\
&\qquad = \binom{36n}{n} p^{35n+1} (1-p)^n.
\end{aligned}$$

The second factor of Equation 2.7 can be solved the same way as in Section 1.2. The approach is simple – the probability that we are at -1 and will not go above is exactly the same probability that we are at 0 in the beginning and will never go above zero. Thus, Equation 2.2 can be used,

$$\mathsf{P}\left(\max_{k \geq 36n+1} S(k) = -1\right) = \mathsf{P}\left(\max_{k \geq 0} S(k) = 0\right) = \mathsf{P}(M = 0) = \frac{1}{36} \quad (2.9)$$

Now we have everything we needed and using Equation 2.7, Equation 2.8, and Equation 2.9 we get for all $n \in \mathbb{N}_0$

$$\mathsf{P}(\tau = 36n) = \frac{1}{36} \cdot \binom{36n}{n} p^{35n+1}(1-p)^n.$$

We would like to know something more about the distribution (such as the expected value etc.) but the $\binom{36n}{n}$ makes it impossible to compute it exactly even using software. Let us use the Stirling's approximation of factorial

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

where e is the Euler's number. Using it in our distribution

$$\binom{36n}{n} = \frac{(36n)!}{n!(35n)!} \approx \frac{6}{\sqrt{70 \cdot \pi n}} \cdot \frac{36^{36n}}{35^{35n}} = \frac{6}{\sqrt{70 \cdot \pi n}} \cdot \left(35 \left(\frac{36}{35}\right)^{36}\right)^n.$$

Thus,

$$\mathsf{P}(\tau = 36n) \approx \frac{p}{6} \cdot \frac{1}{\sqrt{70 \cdot \pi n}} \cdot \left(35 \left(\frac{36}{35}\right)^{36} \cdot p^{35}(1-p)\right)^n.$$

The approximation changes the real values of the probability. Note that if we use the approximation, $\mathsf{P}(\tau = 0)$ is not defined and $\mathsf{P}(\tau \to 0) = \infty$. There is comparison of the exact values and the approximation in the following table.

n	Exact	Approximation
0	0.027027	–
1	0.0100792	0.010931
2	0.00741327	0.00772642
3	0.0061339	0.00630619
4	0.00534681	0.00545923
5	0.00480038	0.0048102
10	0.00341621	0.00344481
20	0.00241645	0.00242655
50	0.00151465	0.00151718
100	0.00105162	0.00105249
1000	0.000235898	0.000235918

Using Wolfram Mathematica,

$$\mathsf{E}[\tau] \approx 46,658.9, \tag{2.10}$$

$$\mathrm{var}[\tau] \approx 4.41222 \cdot 10^9, \tag{2.11}$$

$$\sigma[\tau] \approx 66,424.5. \tag{2.12}$$

Median is 583, 95% quantile is approximately 128,135.

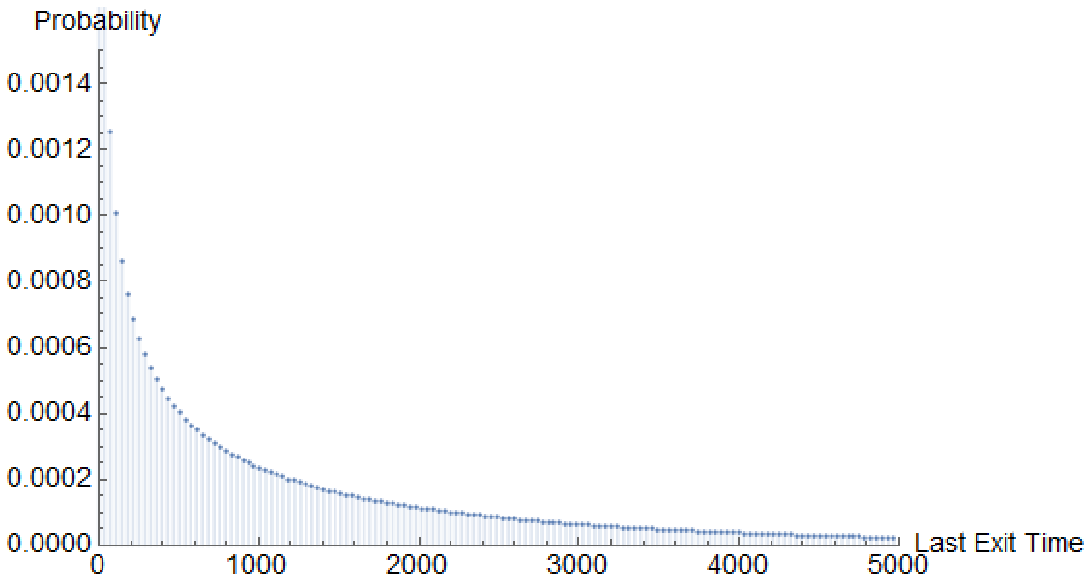

Figure 2.2: Theoretical distribution of the Last Exit Time

2.3 Visits of Zero

Now we would like to know how many times we visit zero. Let us define random variable Z that has a value of Visits of Zero. It is defined as:

$$Z = \sum_{n=0}^{\infty} \chi_{\{S(n)=0\}},$$

where χ is characteristic function. Random variable Z reaches only values from $\mathbb{N}$, which means $\mathsf{P}(Z \geq 1) = 1$. The aim of this section is to find the distribution of Z and that means we need to find

$$\mathsf{P}(Z = k) = \mathsf{P}\left(\sum_{n=0}^{\infty} \chi_{\{S(n)=0\}} = k\right)$$

for all positive k.

We can proceed the same way as in Section 1.3. Let us fix $k \in \mathbb{N}$. If $Z = k$, we can divide the random walk on k sections. The sections $1, \cdots, k-1$ are all the same because they begin in zero, end in zero but they never reach zero between the beginning and the end. The k-th section is different, we go to -1 and stay under it. Let p_i, $i \in \{1, \cdots, k\}$, be the probabilities of each section. The probabilities $p_1, \cdots, p_{k-1}$ are equal. As we can see, all the sections are independent of each other. Therefore,

$$\mathsf{P}(Z = k) = \prod_{i=1}^{k} p_i = p_k \prod_{i=1}^{k-1} p_i = p_k (p_1)^k. \tag{2.13}$$

First of all, let us find p_k. It can be found the similar way as in Section 1.3. Describing it in words: we are at zero at some time $36n$. Our next step must be down to -1 ($q = \frac{36}{37}$) and from now on we must stay under -1 (our Maximum is -1) and it is the same situations as this: we are at 0 and we must stay under it (and that is probability $\mathsf{P}(M=0)$). Note that we computed the probability $\mathsf{P}(M=0)$ only numerically. From this we get

$$p_k = \frac{36}{37} \cdot \mathsf{P}(M=0) = \frac{36}{37} \cdot \frac{1}{36} = \frac{1}{37}. \tag{2.14}$$

Secondly, we would like to know p_1. We know that distribution of random variable must sum up to 1 and using Equation 2.13 and Equation 2.14 we get

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} \mathsf{P}(Z=k) = \sum_{k=1}^{\infty} p_k (p_1)^{k-1} = \frac{1}{37} \sum_{k=1}^{\infty} (p_1)^{k-1} \\ 37 &= \sum_{k=1}^{\infty} (p_1)^{k-1} = \frac{1}{1-p_1} \\ p_1 &= \frac{36}{37}. \end{aligned}$$

That is all we needed to find the distribution of Z. Thus,

$$\mathsf{P}(Z=k) = \frac{1}{37} \cdot \left(\frac{36}{37}\right)^k.$$

It is the same distribution as in betting on color. The expected value, the variation etc. are:

$$\mathsf{E}[Z] = 36, \tag{2.15}$$
$$\mathrm{var}[Z] = 1,332, \tag{2.16}$$
$$\sigma[Z] = 36.5. \tag{2.17}$$

The important quantiles are: median is 25 and 95% quantile is 109.

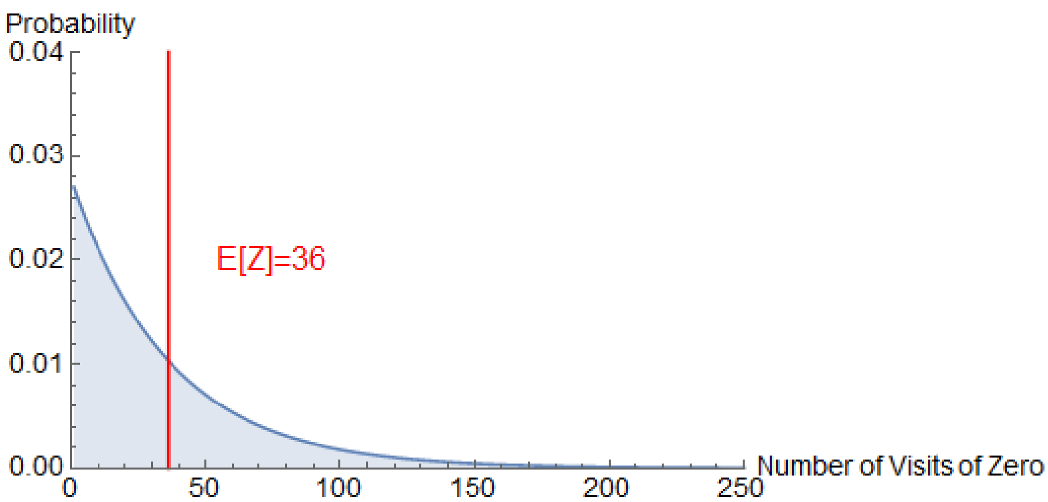

Figure 2.3: Theoretical distribution of the Number of Visits of Zero with the Expected value

2.4 Monte Carlo

Purpose of this section is to use data that I got from Monte Carlo method. Since the probability $\mathsf{P}(M = 0)$ was found only numerically, this section is important to support our conjecture that $\mathsf{P}(M = 0) = \frac{1}{36}$. I used pseudorandom numbers to simulate $1,000,000$ random walks. Let R_i be each simulation of $S(n)$. For all R_i I remembered the value of the Maximum M_i, the Last Exit Time τ_i, and the Number of Visits of Zero Z_i. These data can help to verify my numerical results in the previous sections. Again, 1 stopping rule was used. It is a boundary of the random walk – if it reached value -2,000, the simulation stopped.

2.4.1 Maximum

M_i is the Maximum of the i-th simulation. The mean of M_i was

$$\overline{M_N} \doteq 639.175.$$

The graph of relative frequency of the Maximum and its sample mean is in Figure 2.4. Interesting to confirm oscillatory behavior for small values of M predicted from the theoretical distribution of the Maximum.

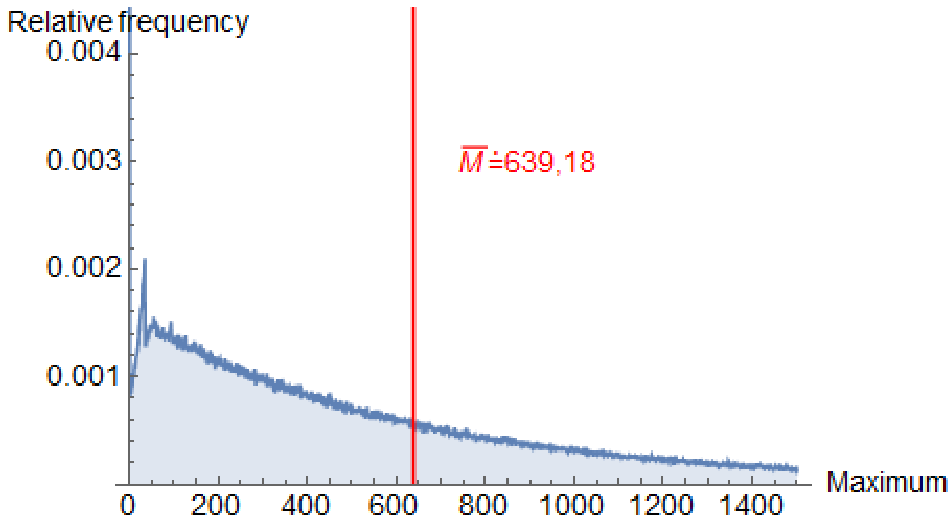

Figure 2.4: Relative frequency of the Maximum with sample mean

The sample variation of the Maximum was $S_N \doteq 46.1 \cdot 10^6$ and the sample standard deviation was $s_N = 6,788.2$. The numerical value of the expected value, the variation, and the standard deviation are in Equation 2.4, Equation 2.5, Equation 2.6.

From these results and Theorem 3 we can construct a confidence interval for the expected value μ_M. For $N \to \infty$

$$\mathsf{P}\left[u_{\frac{\alpha}{2}} < \sqrt{N}(\overline{M_N} - \mu_M)/S_N < u_{1-\frac{\alpha}{2}}\right] \to 1 - \alpha.$$

The empirical 95% confidence interval for μ_M is

$$\mu_M \in (628.00, 650.34).$$

In Figure 2.5 there are both - the relative frequency of the Maximum and the real distribution of it..

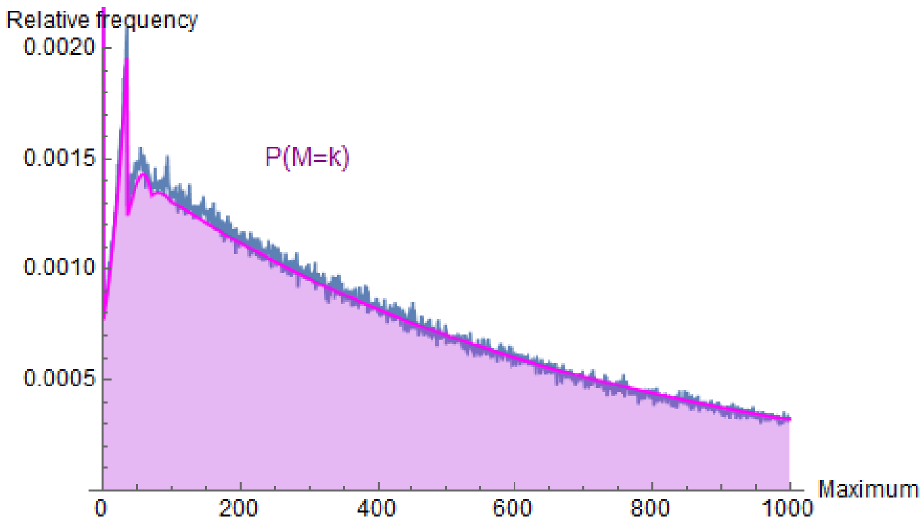

Figure 2.5: Relative frequency of the Maximum and the theoretical distribution of the Maximum

2.4.2 Last Exit Time

Let us assume τ_i the Last Exit Time of i-th simulation.The sample mean was

$$\overline{\tau_N} \doteq 46,432.8.$$

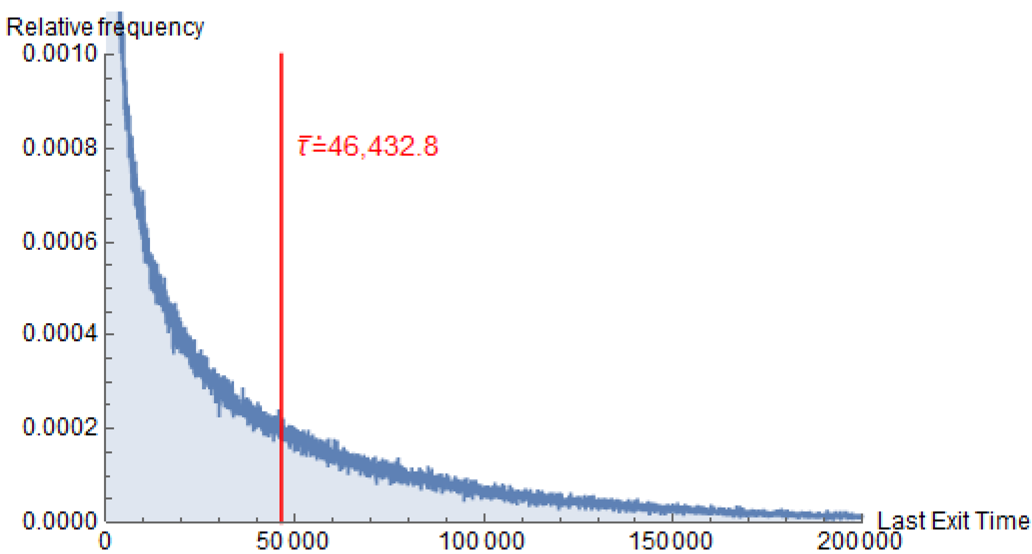

Figure 2.6: Relative frequency of the Last Exit Time with sample mean

The sample variation $S_N \doteq 3.9851 \cdot 10^{10}$.
The sample standard deviation $s_N \doteq 199,273.18$.
In Section 2.2 we derived approximations of the expected value etc. There are the results to compare:

$$\mathsf{E}\,[\tau] \approx 46,658.9,$$
$$\mathrm{var}[\tau] \approx 4.41222 \cdot 10^9,$$
$$\sigma[\tau] \approx 66,424.5.$$

The empirical 95% confidence interval for μ_τ is

$$\mu_\tau \in (46,105; 46,760.6).$$

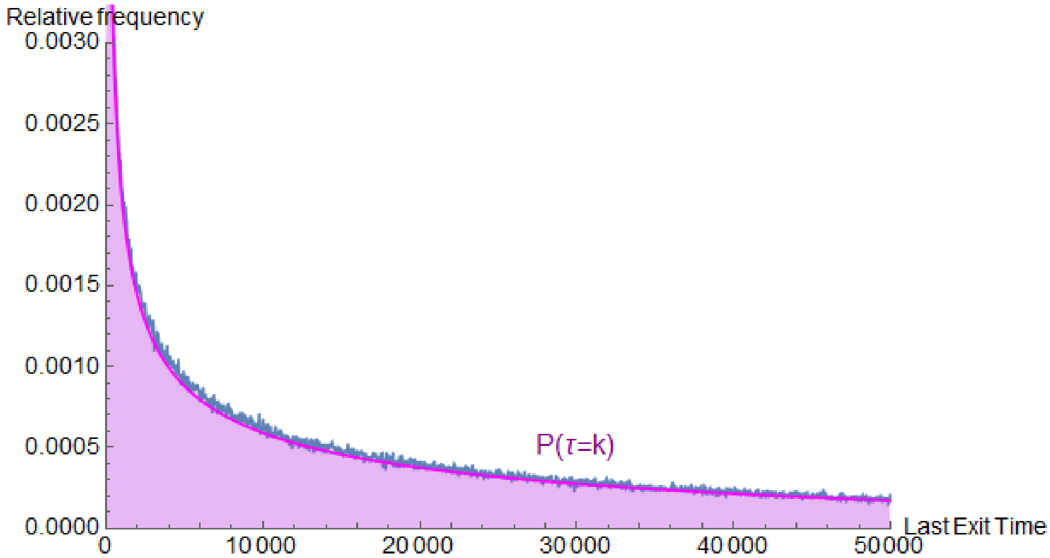

Figure 2.7: Relative frequency of the Last Exit Time and the theoretical distribution of the Last Exit Time

2.4.3 Visits of Zero

Let Z_i be the Number of Visits of Zero of i-th walk. The sample mean was

$$\overline{Z_N} \doteq 35.239.$$

$E[Z] = 36$ from Equation 2.15. There is graph of the relative frequency with sample mean in Figure 2.8.

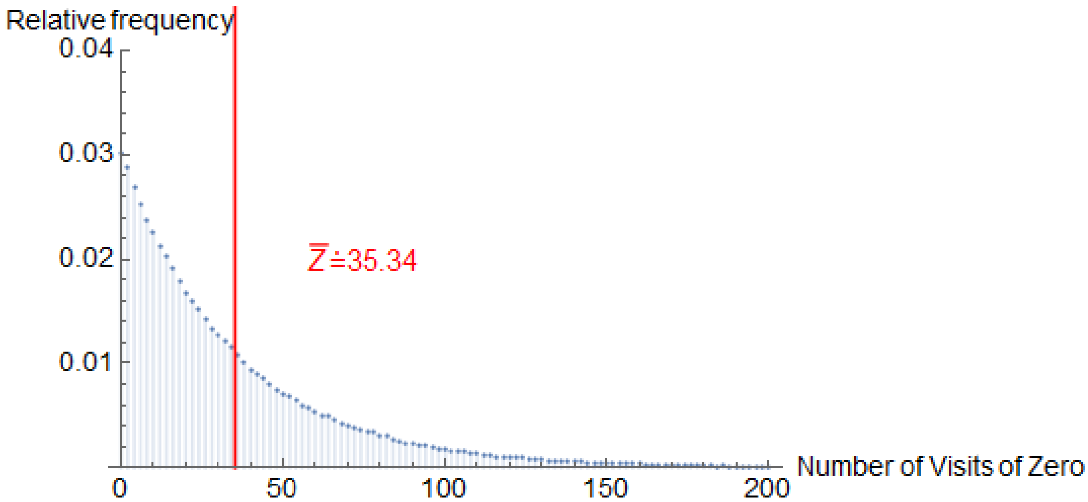

Figure 2.8: Relative frequency of the Number of Visits of Zero with sample mean

The sample variation was $S_N^2 \doteq 762,303.1$ and $\mathrm{var}[Z] = 1,332$ from Equation 2.16.

The sample standard deviation was $s_N \doteq 876.1$ and $\sigma[Z] = 36.5$ from Equation 2.17.

From these results and Theorem 3 we can construct a confidence interval for the expected value μ_Z. For $N \to \infty$

$$\mathsf{P}\left[u_{\frac{\alpha}{2}} < \sqrt{N}(\overline{Z_N} - \mu_Z)/S_N < u_{1-\frac{\alpha}{2}}\right] \to 1 - \alpha.$$

The empirical 95% confidence interval for μ_Z is

$$\mu_Z \in (33.8; 36.674).$$

In Figure 2.9 there are both – the relative frequency from the data from Monte Carlo and the real distribution of the Number of Visits of Zero.

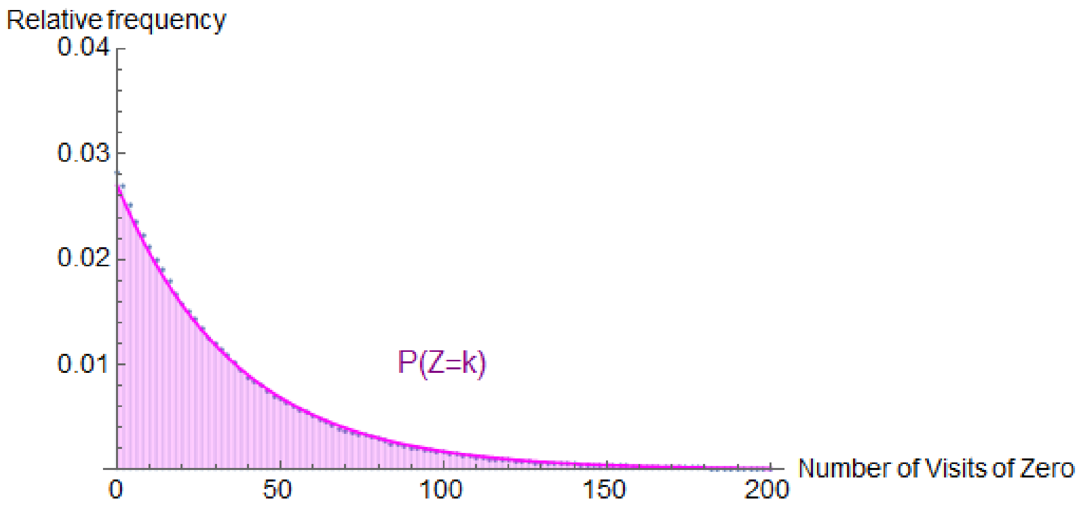

Figure 2.9: Relative frequency of the Number of Visits of Zero and the theoretical distribution of the Number of Visits of Zero

Conclusion

The Red-Black strategy and the Single Number strategy have the same expected value, but they differ in the variation. The variation is the only reason for the differences derived in the previous chapters. Note that all results in the Single Bet strategy were found only numerically without a proper theoretical proof. Therefore, they are only conjectures of the real values. The Monte Carlo simulations supported our approximate results.

The expected value of the Maximum is 18 in the Red-Black strategy and approximately 630 in the Single Bet strategy and that is 35 times higher. The probability that we will eventually win something is approximately 0.947 in the Red-Black strategy and approximately 0.9722 in the Single Bet strategy. As a result, the betting on number is a better option from this point of view.

The expected value of the Last Exit Time is 1,368 in the Red-Black strategy and approximately 46,659 in the Single Bet strategy. Thus, the Single Bet strategy can last longer around or even above zero and is a better option from this point of view.

The expected value of the Number of Visits of Zero is the same for both strategies (36). From this point of view the strategies are equal.

In the beginning we were talking about whether or how to diversify out asset \$1. From the previous chapters it seems that the diversification may look safer but the more risky Single Bet strategy might have better attributes owning to the higher variation. Therefore, the conclusion of this thesis is that the not-diversified strategy can bring us benefits and playing roulette is better with a more risky strategy.

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