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Report of Lenka Slavíková PhD thesis

To the Thesis Committee:

This is a strong endorsement of Lenka Slavíková PhD thesis, submitted to the Faculty of Mathematics and Physics at Charles University.

The general subject of the thesis is applications of the Banach function spaces theory to the theory of Sobolev spaces and to the area of Harmonic Analysis. This are prominent areas of analysis that have featured work from outstanding mathematicians from the early 20th century to the present day. Not only is this topic of intrinsic interest, it also has applications in the theory of Partial Differential Equations but also in the area of Geometric Meaure Theory.

Lenka's thesis begins with a well written introduction where many primary developments of the theory is very nicely motivated. This summary indicates that she is well versed in the literature on the subject.

The thesis consists of four papers. I will briefly describe the original results of this thesis that I find to be of the most interest and that lie most closely to my personal area of research expertise.

The first and main paper is devoted to optimal embeddings of Sobolev type spaces. Indeed, Edmunds, Kerman and Pick [18] and later Kerman and Pick [27] solved the problem of optimality of Sobolev type embeddings on regular Euclidean domains in the context of rearrangement invariant Banach function spaces. The question that remained open concerned Sobolev type embeddings on irregular Euclidean domains and on domains in $\mathbb{R}^n$ equipped with measures different from the Lebesgue one. The domain of particular importance was the Gauss space, namely, $\mathbb{R}^n$ equipped with the Gauss measure $d\gamma_n(x) = (2\pi)^{-n/2}e^{-|x|^2/2}\,dx$. Indeed, the classical well known theorem by Gross [22] shows that whenever u is a weakly differentiable function belonging to $L^2(\mathbb{R}^n, \gamma_n)$ together with all its weak derivatives, then u does actually belong to the (smaller) Orlicz space $L^2 \log L(\mathbb{R}^n, \gamma_n)$. More recently, Cianchi and Pick gave in [14] a complete characterization of optimal first order Gaussian Sobolev embeddings in the general context of rearrangement invariant

Banach function spaces. The natural question that arised from this paper was how to generalize this result to higher order embeddings. The answer to this question was not straightforward since none of the previously known techniques could be applied here. The solution was obtained in paper I of this thesis which appeared in the Journal Advances of Mathematics which is a top journal.

The second result I want to point out involves the boundedness of the Hardy- Littlewood maximal operator between weighted Lebesgue spaces with different weights. The two weight problem was solved by Sawyer in [52] by means of a sort of testing condition. The drawback of Sawyer's condition is that it involves the operator M itself. It would be very desirable to avoid the operator on the condition. For instance, conditions closer in form to the two weight A_p condition, which is necessary but not to sufficient for the problem, would be of interest in applications. A very interesting step towards this direction was obtained by Neugebauer in [39] who proved a two-weight "power bump". This result was improved in [43] in different directions with the help of the theory of Banach function spaces. In particular a "bump" condition, "bumping" properly only one weight within the scale of appropriate Banach function spaces, is sufficient for the two weight problem. Furthermore the condition is completely optimal if you fix the scale of Banach function spaces like the Orlicz spaces. These conditions turned out to be relevant in Harmonic Analysis in connection with questions concerning singular integrals, commutators, potential operators etc. However, still a natural question persisted, namely whether this sufficient bump condition, at least in the scale of Orlicz spaces, was a necessary condition. This is the content of paper IV where Lenka shows that this not the case by building some really interesting examples. I am sure that these examples will be used in different situations. I find this result result very nice and people in the area will be interested.

In conclusion, I find this to be an excellent thesis, containing substantial original results. Lenka Slavíková shows considerable promise as a mathematician, and I look forward to hearing more of her results in the future!

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