EXHAUSTIVE STRUCTURES ON BOOLEAN ALGEBRAS

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m2: GEOMETRY, TOPOLOGY, GLOBAL ANALYSIS AND GENERAL STRUCTURES

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PREFACE

The main motivation of this work was a long years open problem III.1.11 of D. Maharam [Mah47] inspired by J. von Neumann question [Mau81]. When finishing this thesis the problem was solved by M. Talagrand [Tal06] in a negative way. Even that the solution of Maharam Problem is not included here it is an excellent ending of the long story and this thesis maps several areas of mathematics that were discovered on the way to the solution.

The notation used here is quite standard; as a main reference book we use the Handbook of Boolean algebras [Mon89]. The proofs are ended by $\square$ and examples by $\triangle$. When referring to some theorem or definition we always use the reference including the number of a chapter i.e. 'theorem V.7.4'; despite the fact we do not use the chapter number in labels.

The work is divided into five chapters. The first chapter serves as a quick guide to the basic techniques used further.

I. THE BASICS

There are almost no proofs in this chapter except those where we want to illustrate some techniques. One can find here basic definitions and known facts and nontrivial poset consisting of convergent sequences I.1.14, which we use later.

Somewhat new is the following observation I.3.20

> **Theorem.** *Let $\mathbb{B}$ be a subalgebra of $\mathbb{C}$. Then there is an ideal $\mathcal{I}$ on $\mathbb{C}$ such that canonical homomorphism*
>
> $$\begin{aligned} i:\mathbb{B} &\longrightarrow \mathbb{C}/\mathcal{I} \\ b &\longmapsto [b]_{\mathcal{I}}, \end{aligned}$$
>
> *is a regular embedding of $\mathbb{B}$ into $\mathbb{C}/\mathcal{I}$.*

with the direct description of the minimal ideal satisfying the theorem

$$\mathcal{I} = \{u \in \mathbb{C} : \exists \text{ max. disjoint family } X \subset \mathbb{B} \text{ such that } u \wedge x = \mathbf{0} \text{ for any } x \in X\}.$$

Also this result is used further in the text.

II. ALMOST DISJOINT REFINEMENT

First part of this chapter serves as a very brief introduction to, not only, generic extensions of ZFC models and to the method of forcing. As in the first chapter when we give proofs it is only to illustrate needed techniques.

The only reason why this material forms a separate chapter is the part about the Almost disjoint refinements. Where we answered the question raised by L. Soukup:

> Does the family $([\omega]^\omega)^V$ has an almost disjoint refinement in generic extension, which adds a new real?

in affirmative. Remind that a family $\mathcal{S} \subset [\omega]^\omega$ has an *almost disjoint refinement* if there is an almost disjoint family $\mathcal{A}$ such that for any $X \in \mathcal{S}$ there is $A \in \mathcal{A}$ such that $A \subset^* X$.

In fact our answer is not restricted to generic extensions only:

> **Theorem.** *In any ZFC extension* M *of* V *adding a new real there is an almost disjoint refinement for* $([\omega]^\omega)^V$.

III. EXHAUSTIVE FUNCTIONS

In this chapter we formulate basic properties of (uniformly) exhaustive functions on Boolean algebra, where

(i) A real function $f : P \to \mathbb{R}$ is called *exhaustive* if for each disjoint sequence $\langle a_n : n \in \omega\rangle \in P^\omega$, $\lim_{n\to\infty} f(a_n) = 0$.

(ii) $f : P \to \mathbb{R}$ is called *uniformly exhaustive* if for each positive $\varepsilon > 0$ there is a $k \in \omega$ such that for every disjoint sequence $\langle a_n : n \in \omega\rangle \in P^\omega$ $|\{n \in \omega : |f(a_n)| \geq \varepsilon\}| \leq k$.

Further we made the basic classification of Boolean algebras. First using the fragmentation properties:

A Boolean algebra $\mathbb{B}$ is called

(i) σ-*centered* if $\mathbb{B}^+ = \bigcup_{n\in\omega} P_n$, each P_n being centered;

(ii) σ-*linked* if $\mathbb{B}^+ = \bigcup_{n\in\omega} P_n$, each P_n being linked;

(iii) σ-*bounded cc* if $\mathbb{B}^+ = \bigcup_{n\in\omega} P_n$, each P_n being $(n+1)$-cc;

(iv) σ-*finite cc* if $\mathbb{B}^+ = \bigcup_{n\in\omega} P_n$, each P_n being ω-cc,

Second, using the existence of a function with special properties:

(v) XBA stands for the class of algebras carrying a strictly positive exhaustive functional,

(vi) UpmBA stands for the class of algebras carrying a strictly positive supermeasure,

(vii) EBA stands for the class of algebras carrying a strictly positive exhaustive submeasure,

(viii) MBA stands for the class of measure algebras (i.e. algebras carrying a strictly positive finitely additive measure),

We show the basic interrelationship among these classes:

Theorem. *Boolean algebra $\mathbb{B}$ carries a strictly positive exhaustive function if and only if $\mathbb{B}$ is a σ-finite* cc.

Theorem. *The following properties are equivalent.*

(i) *$\mathbb{B}$ carries a strictly positive supermeasure;*

(ii) *$\mathbb{B}$ carries a strictly positive uniformly exhaustive function;*

(iii) *$\mathbb{B}$ satisfies the σ-bounded* cc.

Theorem. *Any σ-centered algebra $\mathbb{B}$ carries a strictly positive measure.*

and describe the inclusion diagram concerning these classes, finally we give some examples which serves as an illustration that defined classes are generally distinct.

A separate part of this chapter concerns productivity of σ-finite and σ-bounded algebras. To our knowledge, these are new results:

Theorem. *Let I be an arbitrary index set and let $\{\mathbb{B}_i : i \in I\}$ be an arbitrary family of Boolean algebras satisfying the σ-finite* cc *(resp. σ-bounded* cc*). Then the free product*

$$\mathbb{B} = \bigotimes_{i \in I} \mathbb{B}_i$$

satisfies the σ-finite cc *(resp. σ-bounded* cc*) as well.*

We conclude the chapter with an introduction of the following relations on Boolean algebras: Let f be a monotone function on Boolean algebra $\mathbb{B}$ such that $f(\mathbf{0}) = 0$. Let $\varepsilon > 0$, for $a, b \in \mathbb{B}$ define

$$b <_{f,\varepsilon} a \quad \text{if and only if} \quad b \leq a \quad \& \quad f(a - b) \geq \varepsilon.$$

and show that one can decide the (uniform) exhaustivity of a function f by the (height, or) foundation of a relation $<_{f,\varepsilon}$.

Theorem. *Let f be a monotone function on Boolean algebra $\mathbb{B}$ and $f(\mathbf{0}) = 0$. Then*

(i) *the relation $<_{f,\varepsilon}$ is well founded on $\mathbb{B}$ for any $\varepsilon > 0$ if and only if f is an exhaustive function on $\mathbb{B}$,*

(ii) *the height of the relation $<_{f,\varepsilon}$ is finite for any $\varepsilon > 0$ if and only if f is a uniformly exhaustive function on $\mathbb{B}$.*

This contribute to the characteristics of (uniformly) exhaustive functions and touches the topics that are discussed in the next chapter.

IV. ALGEBRAIC PROPERTIES

This chapter is far from uniform, rather it is a collection of separate observations and facts with stress to the lattice structure of $\mathrm{Sub}(\mathbb{B})$; all submeasures on Boolean algebra $\mathbb{B}$.

We start with the description of a pavement construction of a submeasures

A *pavement* for a Boolean algebra $\mathbb{B}$ is a subset $D \subseteq \mathbb{B}$ together with a mapping $w : D \to [0, \infty)$ such that for some finite $D_0 \subseteq D$, $\mathbf{1} = \bigvee D_0$. Having pavement (D, w) on $\mathbb{B}$ one can define a submeasure μ_w on $\mathbb{B}$ as follows

$$\mu_w(a) = \inf\{\sum_{d \in F} w(d) : F \in [D]^{<\omega} \quad a \leq \vee F\}.$$

This construction is used in a modified Popov's example to construct exhaustive subpathological function and pathological submeasure, where:

Nonnegative bounded function on Boolean algebra is called *subpathological* if There is no nontrivial submeasure below f, i.e. μ_f is identical zero.

Submeasure φ on Boolean algebra $\mathbb{B}$ is *pathological* if $\varphi(\mathbf{1}) > 0$ and there is no nontrivial measure below φ, i.e.

$$\{\psi \in \mathrm{Meas}(\mathbb{B}) : \psi \leq \varphi\} = \{\bar{0}\}.$$

We continue with the structure description of a lattice $\mathrm{Sub}(\mathbb{B})$ and characterise minimal submeasures:

> A submeasure $\varphi \in \mathrm{Sub}(\mathbb{B})$ is minimal if and only if it is a $2-$additive function, i.e. a function $f : \mathbb{B} \to \mathbb{R}$ satisfying
>
> (i) $(\forall a \in \mathbb{B}) \quad f(a) + f(-a) = f(\mathbf{1})$, respectively
>
> (ii) for any partition a_1, a_2, a_3 of unity $f(a_1) + f(a_2) + f(a_3) = f(\mathbf{1})$.

We define *envelope* submeasures, i.e. submeasures that are supremum of measures in $\mathrm{Sub}(\mathbb{B})$ lattice and introduce the touching lemma:

> **Theorem.** *Let φ be an envelope on $\mathbb{B}$ then for any $a \in \mathbb{B}$ there is a measure $m \leq \varphi$ on $\mathbb{B}$ such that $\varphi(a) = m(a)$.*

and distinguish minimal and envelope submeasures:

> **Theorem.** *Every minimal submeasure is either a measure or is not even a supremum of measures below; i.e: is not an envelope.*

In the second part of this chapter we introduce the intersection number technique and give proofs to the famous J. L. Kelly and N. J. Kalton - J. W. Roberts theorems. This technique is used to Fremlin - Kupka operator to generalise and extend the results of D. H. Fremlin and J. Kupka.

This chapter is concluded with the notion of algebraic homogenisation of submeasures.

V. TOPOLOGICAL PROPERTIES

In the last chapter we focus on topological characterisation of Maharam algebras. This chapter is based on [BGJ98], [BFH99] and [BJP05] and brings among others a consistent solution to the von Neumann's problem.

VON NEUMANN'S PROBLEM

First let us remind the notion of Measure algebra: Consider the quotient algebra $\mathcal{M}$ of Borel sets in the interval $[0,1]$ modulo null sets, i.e. sets of Lebesgue measure 0. $\mathcal{M}$ is an atomless σ-algebra, and carries a (σ-additive strictly positive) measure, a numerical function m with the following properties

$$m(\mathbf{0}) = 0,\ m(a) > 0 \text{ for } a \neq \mathbf{0}, \text{ and } m(\mathbf{1}) = 1$$
$$m(\bigvee_{n\in\omega} a_n) = \sum_{n\in\omega} m(a_n) \text{ whenever } a_n \text{ are pairwise disjoint.}$$

An atomless σ-algebra that carries a measure is called Measure algebra.

The main motivation of the following is combinatorial characterisation of a Measure algebra. It is an easy observation that Measure algebra satisfies the countable chain condition (ccc); i.e. all the antichains are countable. But of course there are ccc algebras that do not carry a measure. Hence we have to look for some other property.

To characterise measure algebras, von Neumann formulated the following weak distributivity law:

$$\text{if } a_0^n \leq a_1^n \leq \ldots \text{ for } n = 1, 2, \ldots, \text{ then}$$
$$\bigwedge_n \bigvee_k a_k^n = \bigvee_{f:\omega\to\omega} \bigwedge_n a_{f(n)}^n.$$

The Measure algebra satisfies this form of distributivity.

The problem of von Neumann from The Scottish Book [Mau81] (Problem 163) asks whether every weakly distributive complete ccc Boolean algebra carries a countably additive measure.

That was an observation of Maharam that in order to prove ccc and weak distributivity one does not need a measure on $\mathbb{B}$, but an ostensibly weaker property:

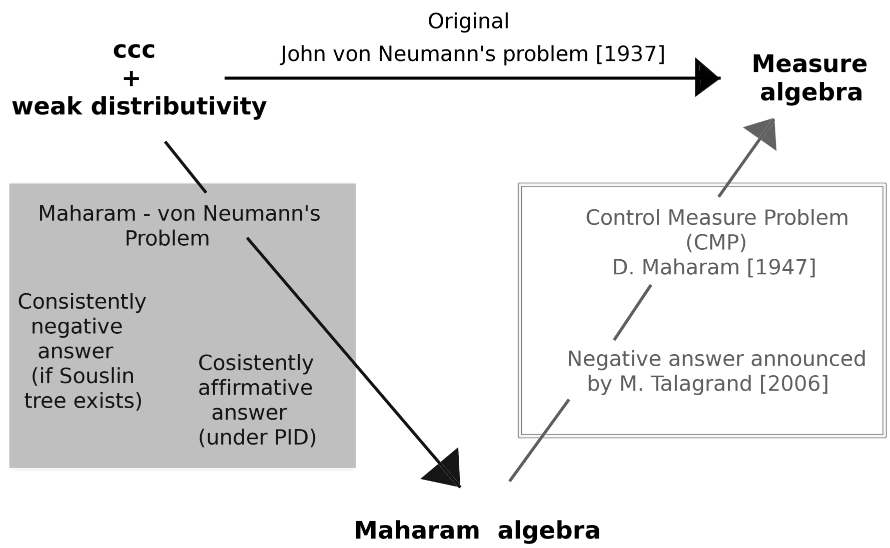
A function m on $\mathbb{B}$ is a *continuous submeasure* if

$$m(\mathbf{0}) = 0,\ m(a) > 0 \text{ for } a \neq \mathbf{0},\ \text{and } m(\mathbf{1}) = 1,$$
$$m(a) \leq m(b) \text{ if } a \leq b,$$
$$m(a \vee b) \leq m(a) + m(b),$$
$$\lim_n m(a_n) = 0 \text{ for every decreasing sequence } a_n \text{ with } \bigwedge_n a_n = 0.$$

We call a continuous submeasure a Maharam submeasure, and complete Boolean algebra $\mathbb{B}$ a Maharam algebra if it carries a Maharam submeasure. Every measure is Maharam submeasure.

Theorem. (D. Maharam) *A Maharam algebra satisfies ccc and is weakly distributive.*

Von Neumann's problem can be divided into two distinctly different questions. Weak distributivity is a consequence of a property possibly weaker than measurability, namely the existence of a continuous strictly positive submeasure (a Maharam submeasure); the Control Measure Problem of [Mah47] asks whether every complete Boolean algebra that carries a continuous submeasure must also carry a measure.



The second question is the following modified von Neumann problem: does every weakly distributive ccc complete Boolean algebra carry a strictly positive Maharam submeasure? This statement is not provable in ZFC, as the algebra associated with a Souslin tree is a counterexample. In chapter V, or in[BJP05] it is shown that it is consistent that the modified von Neumann problem holds.

Michel Talagrand announced the solution of the Control Measure Problem. In [Tal06] he constructs a submeasure on the Cantor algebra that is exhaustive but not uniformly exhaustive (and therefore not equivalent to a measure).

The interested reader can found a lot about the history of von Neumann problem in the B. Balcar and T. Jech survey article [BJ06].

ORDER SEQUENTIAL TOPOLOGY

We start with the description of algebraic convergence on Boolean algebra.

Let $\mathbb{B}$ be a Boolean σ-algebra. An infinite sequence $\{a_n\}_n$ *converges* to a, $\lim_n a_n = a$, if $\limsup_n a_n = \liminf_n a_n = a$, where $\limsup_n a_n = \bigwedge_n \bigvee_{k\geq n} a_k$, $\liminf_n a_n = \bigvee_n \bigwedge_{k\geq n} a_k$. Equivalently, we define $\lim_n a_n = \mathbf{0}$ whenever there exists a decreasing sequence b_n with $\bigwedge_n b_n = \mathbf{0}$ such that $a_n \leq b_n$ for all n. Then we let $\lim_n a_n = a$ if $\lim_n(a_n \bigtriangleup a) = \mathbf{0}$.

Algebraic convergence satisfies basic properties of convergence and is compatible with Boolean operations:

(a) If $a_n = a$ for all n then $\lim_n a_n = a$.

(b) If $\{a_n\}_n$ converges to a and π is a permutation of ω then $\{a_{\pi(n)}\}_n$ also converges to a.

(c) $\lim_n a_n = \mathbf{0}$ if and only if $\limsup_n a_n = \mathbf{0}$,

(d) if the a_n are pairwise disjoint then $\lim_n a_n = \mathbf{0}$,

(e) $\limsup_n (a_n \vee b_n) = \limsup_n a_n \vee \limsup_n b_n$,

(f) if $\lim_n a_n = a$ and $\lim_n b_n = b$ then $\lim_n -a_n = -a$, $\lim_n (a_n \vee b_n) = a \vee b$ and $\lim_n (a_n \wedge b_n) = a \wedge b$.

Having algebraic convergence one can simply extend it to a sequential topology on Boolean algebra:

A set $F \subseteq \mathbb{B}$ is *closed* if $\lim a_n \in F$ whenever $\{a_n\}_n$ is a sequence in F. Let τ_s denote the topology on $\mathbb{B}$ so obtained; it is the sequential topology on $\mathbb{B}$. The space $(\mathbb{B}, \tau_s)$ is T_1 (every singleton is closed). The closure $\mathrm{cl}(A)$ of a set $A \subseteq \mathbb{B}$ is generally obtained by taking limits of convergent sequences and iterating this ω_1 times. Maharam pointed out that the iteration is not necessary if $\mathbb{B}$ is ccc and weakly distributive: in this case $\mathrm{cl}(A)$ is the set of all limits of sequences of A.

THE DECOMPOSITION THEOREM

D. Maharam [Mah47] characterised algebras that carry a continuous submeasure as those on which the sequential topology τ_s is metrizable. In [BGJ98] this is improved to the condition that $\mathbb{B}$ is ccc and $(\mathbb{B}, \tau_s)$ is a Hausdorff space.

The Decomposition theorem shows that there are basically only two possibilities. First, the Boolean algebra with order sequential topology is Hausdorff hence Metrizable space. Second, if not Hausdorff then the order sequential topology goes wild, since every nonempty open set is dense.

Theorem. *Let $\mathbb{B}$ be a complete* ccc *Boolean algebra. Then there are disjoint elements $d, m \in \mathbb{B}$ such that $d \vee m = \mathbf{1}$ and*

(i) In the space $(\mathbb{B} \restriction d, \tau_s)$ the closure of every nonempty open set is the whole space.

(ii) The Boolean algebra $\mathbb{B} \restriction m$ carries a strictly positive Maharam submeasure, i.e. the τ_s topology is metrizable.

The Decomposition Theorem is a useful tool when translating combinatorial properties into topological ones. It appeared that the well known topological property characterises Maharam algebras.

Let us say that $\mathbb{B}$ has the G_δ-property if a singleton $\{\mathbf{0}\}$ is a G_δ-set in $(\mathbb{B}, \tau_s)$.

Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra. Then $\mathbb{B}$ carries a strictly positive Maharam submeasure if and only if*

(i) *$\mathbb{B}$ is weakly distributive, and*

(ii) *$\mathbb{B}$ has the G_δ property.*

This theorem is core for consistency proof of the von Neumann - Maharam problem. In [BJP05] it is shown than under the P-Ideal Dichotomy (PID) in every ccc weakly distributive Boolean algebra the singleton $\{\mathbf{0}\}$ is a G_δ set in order sequential topology, hence every ccc weakly distributive Boolean algebra is Maharam.

In June 2004 Stevo Todorcevic improved those results to the following purely combinatorial characterisation of Maharam algebras.

Theorem. (S. Todorcevic [Tod04]) *A complete Boolean algebra $\mathbb{B}$ is a Maharam algebra if and only if*

(i) *$\mathbb{B}$ is weakly distributive, and*

(ii) *$\mathbb{B}$ satisfies the σ-finite chain condition.*

WEAK DISTRIBUTIVITY

In [BGJ98] there is the following topological equivalent to weak distributivity.

Theorem. *Let $\mathbb{B}$ be a ccc Boolean algebra. $\mathbb{B}$ is weakly distributive if and only if the topological space $(\mathbb{B}, \tau_s)$ is Fréchet.*

Remind that a topological space X is Fréchet if for every set $A \subset X$, every point in the closure of A is the limit of some sequence in A.

The Decomposition theorem allows us to formulate another topological equivalent to weak distributivity in the form of the following Baire-like theorem.

Theorem. *Let $\mathbb{B}$ be a ccc Boolean algebra. $\mathbb{B}$ is weakly distributive if and only if for each $\{U_n\}$, where U_n is downward closed and cl$(U_n) = \mathbb{B}$ for any $n \in \omega$ the intersection $\bigcap_n U_n$ is algebraically dense, i.e. for each $b \in \mathbb{B}^+$ there is $u \in \bigcap_n U_n$ such that $u \leq b$.*

INDEPENDENT REALS

Another well known topological property, a sequential compactness (every sequence has a convergent subsequence) has also an interesting translation to algebraic properties.

Clearly when we have a sequence $\{a_n\} \subset \mathbb{B}$ such that

$$\bigvee_{i \in I} a_i = \mathbf{1} \text{ and } \bigwedge_{i \in I} a_i = \mathbf{0}. \tag{1}$$

for each infinite $I \subset \omega$, then by the definition of sequential topology $(\mathbb{B}, \tau_s)$ cannot by sequentially compact.

On the other hand once having such a sequence then forcing with $\mathbb{B}$ adds an independent (splitting) real. The sequence $\{a_n\}$ is a name for independent real.

Remind that $\mathbb{B}$ adds an independent real if there exists some $X \subset \omega$ in $V[G]$, the generic extension by $\mathbb{B}$, such that neither X nor its complement has an infinite subset Y such that $Y \in V$.

If algebra $\mathbb{B}$ adds an independent real f, then $\{\dot{f}(n)\}$ is the desired sequence satisfying (1), where $\dot{f} : \omega \to \mathbb{B}$ is a Boolean name for f.

Theorem. *Let $\mathbb{B}$ be a complete* ccc *Boolean algebra. $\mathbb{B}$ does not add independent reals if and only if $(\mathbb{B}, \tau_s)$ is sequentially compact.*

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In Prague,

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I. Basic Notions and Known Results

We start with a summary of basic notions and known results concerning preorders, partial orders, Boolean algebras, and submeasures. We also give a characterisation of complete ccc weakly distributive Boolean algebras.

1. Quasiorders and partial orders

Let $(P, \leq)$, $P \neq \emptyset$, be a preorder, $x, y \in P$. Elements x,y are *disjoint* or *incompatible*, $x \perp y$, if there is no $z \in P$ so that $z \leq x$ and $z \leq y$. Otherwise x and y are *compatible*, denoted by $x \parallel y$. Let us now summarise some basic notions.

1.1 Definition. A set $X \subseteq P$ is called

(i) *disjoint* or a *disjoint family* if for distinct x,$y \in X$, $x \perp y$;

(ii) *dense* if $(\forall x \in P)(\exists y \in X)(y \leq x)$;

(iii) *open dense* if it is dense and downward closed , i.e., $(\forall y \in X)((\leftarrow, y] \subseteq X)$, where $(\leftarrow, y] = \{x \in P : x \leq y\}$;

(iv) *predense* if $(\forall x \in P)(\exists y \in X)(x \parallel y)$;

(v) *linked* if $(\forall x, y \in X)(x \parallel y)$;

(vi) *centered* if for any finite subfamily $\{x_1, \dots, x_n\} \subseteq X$ there is a $z \in P$ such that $z \leq x_i$ for $i = 1, \dots, n$;

(vii) κ-*cc*, κ a cardinal number, if there is no disjoint family $Y \subseteq X$ of cardinality κ; in particular, ω-cc means that any disjoint subfamily of X is finite;

(viii) κ-*closed* if for any $\alpha < \kappa$ and any descending sequence $\{x_\xi : \xi < \alpha\}$ of elements of X, there is a $z \in X$ below all of the x_ξ's.

1.2 Definition. A preorder $(P, \leq)$ is called

(i) *atomless* if $(\forall x \in P)(\exists y_1, y_2 \in P)(y_1, y_2 \leq x \;\&\; y_1 \perp y_2)$;

(ii) *separative* if $x \leq y$ if and only if $\perp(x) \supseteq \perp(y)$, for any $x, y \in P$, where

$$\perp(x) = \{z \in P \ : \ z \perp x\};$$

(iii) ccc if P is ω_1-cc;

(iv) σ-*closed* if P is ω_1-closed.

1.3 Fact. P *is separative if and only if for any* $x, y \in P$, $x \not\leq y$, *there is a* $z \in P$ *so that* $z \leq x$ *and* $z \perp y$.

For any preorder $(P, \leq)$, one can define a relation $\preceq$ on P by

$$x \preceq y \Leftrightarrow \perp(x) \supseteq \perp(y).$$

Then $\preceq$ is a preordering of P which

(a) extends the original preorder $\leq$,

(b) is separative,

(c) preserves disjointness, i.e. $x \perp y$ in $\leq$ implies $x \perp y$ in $\preceq$.

Moreover, conditions (a),(b), and (c) uniquely determine the relation $\preceq$. Taking the quotient $(P/\sim, \preceq)$, where $x \sim y$ means $x \preceq y$ & $y \preceq x$ (i.e. $\perp(x) = \perp(y)$), one obtains a separative partial order called the *separative quotient* of $(P, \leq)$.

1.4 Definition. Let $(P, \leq)$ be a preorder and $\mathcal{D}$ a family of dense subsets of P. A filter F on P is called *$\mathcal{D}$-generic* if $(\forall D \in \mathcal{D})\, D \cap F \neq \emptyset$. *Martin's number* of P is $m(P) = \min\{|\mathcal{D}| : \mathcal{D}$ is a family of dense subsets of P such that there is no $\mathcal{D}-$ generic filter on $P\}$. If P is not atomless, we set $m(P) = \infty$. *Point density* of P is $\mathrm{pd}(P) = \min\{|\mathcal{F}| : \mathcal{F}$ is a family of filters on P such that $P = \bigcup \mathcal{F}\}$.

Notice that for a preorder P, its Martin's number and the point density are the same as for its separative quotient, respectively.

1.5 Definition. A partial order P has

(i) *Knaster property* (in short *property K*) if any uncountable set $X \subseteq P$ has a linked uncountable subset;

(ii) *precaliber* $\aleph_1$ if any uncountable set $X \subseteq P$ has a centered uncountable subset.

Notice, that every partial order with property K satisfies ccc. In general, ccc is not *productive*, i.e. the product of two ccc posets need not itself be ccc. Nevertheless, any product of P with property K and a ccc poset Q, $P \times Q$, is ccc. Moreover, one well-known equivalent of Martin's axiom MA_{ω_1} states that

1.6 Theorem. (K. Kunen F. Rowbottom) *Under MA_{ω_1} every* ccc *poset has the precaliber $\aleph_1$. Moreover if the poset P has size at most $\aleph_1$ then it is σ-centered.*

The proof of this theorem can be found in [Wei84]. There is an open problem concerning this topic due to S. Todorcevic.

1.7 Problem. [Tod00a] *Is MA_{ω_1} equivalent with 'every* ccc *poset has property K'?*

1.8 FRAGMENTATIONS

By a *fragmentation* of a structure we mean an expression of the structure as a union of parts that have some designated property. The following are examples of well-used fragmentations.

1.9 Definition. A poset P is called

(i) *σ-centered* if $P = \bigcup_{n \in \omega} P_n$, each P_n being centered;

(ii) *σ-linked* if $P = \bigcup_{n \in \omega} P_n$, each P_n being linked;

(iii) *σ-bounded cc* if $P = \bigcup_{n \in \omega} P_n$, each P_n being $(n+1)$-cc;

(iv) *σ-finite cc* if $P = \bigcup_{n \in \omega} P_n$, each P_n being ω-cc.

1.10 Problem. (Horn, Tarski problem [HT48]) *It is still an open problem whether classes of posets satisfying (iii) and (iv) respectively, are distinct; i.e. whether every σ-finite cc poset is σ-bounded cc.*

1.11 EXAMPLE. Let I be a non-empty set. Consider the partial order

$$\mathrm{Fn}(I, \omega) = \{p \in {}^K\omega : \ K \in [I]^{<\omega}\},$$

ordered by the inverse inclusion. Let $X = \prod_{i \in I} \omega$ be a topological product of countable discrete spaces. For $p \in \mathrm{Fn}(I, \omega)$, denote by $[p]$ the basic open subset of the product, $[p] = \{f \in X : p \subseteq f\}$. Clearly, this gives an embedding of $\mathrm{Fn}(I, \omega)$ into $\mathrm{Open}^+(X)$; that is all nonempty open subsets of X ordered by inclusion. Note that $\{[p] : p \in \mathrm{Fn}(I, \omega)\}$ is a base of the topology on X. △

1.12 Fact. *For partial orders* $Fn(I, \omega)$ *or* $(Open^+(X), \subseteq)$*, the following conditions are equivalent:*

(i) to be σ-centered,

(ii) to be σ-linked,

(iii) $|I| \leq 2^\omega$.

This follows immediately from a special case of Hewitt, Marczewski, Pondiczéry theorem, which says that a product of at most 2^ω separable spaces is separable and from the fact that any separative σ-linked poset has size at most 2^ω, for more details see (III.2.6).

1.13 Fact. $Fn(I, \omega)$ *and therefore* $(Open^+(X), \subseteq)$*, is σ-bounded cc for arbitrary index set* I.

Proof. Proof For each n, m put

$$P_{m,n} = \{p \in \mathrm{Fn}(I, \omega) : \ |p| = m \ \& \ \mathrm{rng}\,(p) \subseteq n\}.$$

Since any maximal disjoint family of members of $P_{m,n}$ is of size at most n^m, $\{P_{m,n} : m, n \in \omega\}$ is a countable fragmentation witnessing σ-boundedness cc. □

1.14 TODORCEVIC'S PARTIAL ORDER - (PART 1)

Here we describe a class of separative posets that satisfy ccc, the construction is inspired by the interesting example of Borel partial order introduced by S. Todorcevic [Tod84].

Let X be a metrizable space without isolated points. We define partial order $\mathbb{T}(X) = (P, \le)$ by the following. Let P be the family of all infinite compact subsets of X, which have only finitely many accumulation points, i.e.

$$P = \{A \,:\, A \in [X]^\omega, A \text{ compact and the derivation } A' \text{ is finite}\},$$

with ordering given by

$$A \le B \quad \text{if and only if} \quad A \supseteq B \;\&\; A' \cap B = B'.$$

Note that elements of $\mathbb{T}(X)$ are in fact disjoint unions of convergent sequences together with their limit points.

1.15 Lemma. $\mathbb{T}(X)$ *is separative partial order.*

Proof. Assume $A, B \in P$ and $A \not\le B$. i.e., either $B \setminus A \neq \emptyset$ or ($B \subseteq A$ and $A' \cap B \supsetneq B'$). In the first case, there is a $y \in B \setminus A$ such that $y \notin B'$. Since X has no isolated point one can take some one-to-one sequence $\langle y_n : n \in \omega\rangle$ with $\lim y_n = y$ and set $C = A \cup \{y_n : n \in \omega\} \cup \{y\}$. Then $C \in P$, $C \le A$ and $C \perp B$. In the second case, there is an $x \in A' \cap (B \setminus B')$ and this means $A \perp B$. □

1.16 EXAMPLE. For rational numbers $\mathbb{Q}$, Todorcevic's partial order $\mathbb{T}(\mathbb{Q})$ is σ-centered ordering. For arbitrary nonempty finite subset F of $\mathbb{Q}$, the subset $X = \{A \in \mathbb{T}(\mathbb{Q}) \,:\, A' = F\}$ is a centered family. △

1.17 Theorem. *Let X be a separable metric space without isolated points. $\mathbb{T}(X)$ is a separative partial order satisfying* ccc.

Moreover, if the space X is Polish without isolated points, then the partial order $\mathbb{T}(X)$ is Borel.

Proof. (i) $\mathbb{T}(X)$ satisfies ccc. Consider an uncountable family $\langle A_\alpha : \alpha \in \omega_1\rangle$ of elements from P. Shrinking, if necessary, we can assume that all derivatives A'_α have the same size, say $n \in \omega$. Applying the Δ-system lemma we can assume that the family $\langle A'_\alpha : \alpha \in \omega_1\rangle$ is a Δ-system with a kernel K. When $K = A'_\alpha$, then $\langle A_\alpha : \alpha \in \omega_1\rangle$ is a centered family and we are done.

So assume that $A^*_\alpha = A'_\alpha \setminus K \neq \emptyset$. Then the A^*_α's are pairwise disjoint sets of the same size $k = n - |K|$. Since the A_α's are countable sets, we can assume (after another shrinking), that for any $\beta \in \omega_1$, A^*_β is disjoint with $\bigcup_{\alpha<\beta} A_\alpha$. We have $A_\alpha \cap A'_\beta = K \subseteq A'_\alpha$ for any $\alpha < \beta < \omega_1$.

We are looking for $\alpha, \beta \in \omega_1$ such that $\alpha < \beta$ and $A_\beta \cap A'_\alpha \subseteq A'_\beta$, for this is equivalent to compatibility of A_α and A_β.

Assume that such a pair does not exist. It means that for any $\alpha < \beta$, $A^*_\alpha \cap (A_\beta \setminus A'_\beta) \neq \emptyset$. We fix arbitrary linear ordering of X. The set A^*_α with the inherited ordering may be viewed as a point in X^k. The space X^k is hereditary separable, hence the space $\{A^*_\alpha : \alpha \in \omega_1\}$ is separable and has a countable base, so there is some $\xi \in \omega_1$ and a point $\langle p_1, p_2, \dots, p_k\rangle = A^*_\xi$ such that for each open neighbourhood U of it, the set $\{\alpha \in \omega_1 : A^*_\alpha \in U\}$ is uncountable. Choose a decreasing sequence of open balls $B_n \subseteq X^k$ such that $\bigcap_{n\in\omega} B_n = \{\langle p_1, p_2, \dots, p_n\rangle\}$ and denote $J_n = \{\alpha \in \omega_1 : A^*_\alpha \in B_n\}$.

Pick an increasing sequence of ordinals $\alpha_n \in J_n$. Consider A_γ for some $\gamma \geq \sup_n\{\alpha_n, \xi\}$. Since $A_\gamma \cap A^*_{\alpha_n} \neq \emptyset$ for each $n \in \omega$, there is some i, $1 \leq i \leq k$ such that $p_i \in A'_\gamma$, therefore $p_i \in A^*_\gamma$. But $p_i \in A^*_\xi$ and A^*_γ are pairwise disjoint, a contradiction.

We proved that for some $\alpha < \beta$, A_α and A_β must be compatible.

(ii) Let us verify now that $\mathbb{T}(X)$ is Borel. It suffices to show that the underlying set P is Borel and that the relation $\leq$ and the induced relation $\perp$ are Borel sets in some Polish space.

Consider a space $K(X)$ of all compact subsets of X with the Vietoris topology. A basis for this topology consists of all sets

$$B(V, U_1, \dots, U_n) = \{A \in K(X) : A \subseteq V \,\&\, A \cap U_1 \neq \emptyset \,\&\, \dots \,\&\, A \cap U_n \neq \emptyset\}$$

for $V, U_1, \dots, U_n$ open sets in X. The hyperspace $K(X)$ is separable and completely metrizable (e.g. by Hausdorff metric), so it is a Polish space [Kec95].

Let the symbol $B(q, \varepsilon)$ denote an open ball in X with the centre q and radius ε. We shall proceed in four small steps.

(a) For any $m \in \omega$, the set

$$\{A \in K(X) : |A| \geq m\}$$

is open in $K(X)$. Indeed, it equals

$$\bigcup\{B(X, U_1, \dots, U_m) : U_1, \dots, U_m \text{ pairwise disjoint nonempty open sets}\}.$$

Therefore $K_{\mathrm{inf}}(X) = \{A \in K(X) : A \text{ is infinite}\}$ is a G_δ-set in $K(X)$, i.e. a Π^0_2 set.

(b) Analogously, the set $\{A \in K(X) : |A \cap U| \geq m\}$ is open in $K(X)$, whenever U is a nonempty open set in X and m is a non-negative integer.

(c) Using the fact that every infinite compact set has a nonempty derivative, we can verify that the set $D_k = \{A \in K(X) : |A'| \geq k\}$ equals

$$\bigcup_{n>0} \bigcup_{\substack{\langle q_1,\dots,q_k\rangle \in Y^k \\ \rho(q_i - q_j) > 2/n}} \bigcap_{m\in\omega} \{A : |B(q_1, 1/n) \cap A| \geq m \,\&\, \dots \,\&\, |B(q_k, 1/n) \cap A| \geq m\},$$

where Y is a countable dense subset of X. Thus it is a Σ^0_3 set. Consequently, the set

$$P = K_{\mathrm{inf}}(X) \cap \bigcup_{k\in\omega} (K(X) \setminus D_k)$$

is a Σ^0_4 set in $K(X)$.

(d) The relation $\leq$ is Borel. First, the inclusion is a closed relation, i.e., the set $\mathrm{Inc} = \{\langle A, B\rangle : A, B \in K(X) \,\&\, A \supseteq B\}$ is closed in $K(X) \times K(X)$.

Second, the inequality $A' \cap B \supsetneq B'$ means that there is a point $x \in A'$ which is isolated in B, $x \in A' \cap (B \setminus B')$, so we need to consider the set $H = \{\langle A, B\rangle \in P \times P : A' \cap (B \setminus B') \neq \emptyset\}$. Observe that a set

$$S = \bigcup_{n>0} \bigcup_{q\in Y} \bigcap_{m\in\omega} \{\langle A, B\rangle \in K(X)^2 : \, |B(q, 1/n) \cap A| \geq m \,\&\, |B(q, 1/n) \cap B| = 1\}$$

is a Π^0_4 set in $K(X)^2$.

We can verify that $H = (P\times P)\cap S$ and $\leq = \mathrm{Inc}\cap((P\times P)\setminus S)$ and $\perp = H\cup H^{-1}$. From that we see that the relations $\leq$ and $\perp$ are Borel sets in $K(X)^2$. □

1.18 Remark. What we do not know is whether for X, Y Polish spaces without isolated points are Boolean algebras $\mathrm{RO}(T(X))$ and $\mathrm{RO}(T(Y))$ isomorphic?

2. TOPOLOGICAL SPACES

A topological space (X, τ) is for us a non-empty set X endowed with some topology. We a priori do not assume any separation axioms. Recall that a sequence $\langle x_n : n \in \omega\rangle$ in a topological space converges to a point x (i.e. $\lim x_n = x$) if for any open neighbourhood U of x almost all x_n's belongs to U, i.e. $\{n \in \omega : x_n \notin U\}$ is finite. If the space is Hausdorff then every sequence converges to at most one point.

2.1 EXAMPLE. Let X be an infinite set with the topology τ consisting of cofinite subsets of X. Then (X, τ) is T_1, compact topological space and any one-to-one sequence has all points of the space as its limit. △

If $A \subset X$ is a *closed subset* of a space X then every limit of a sequence of points from A belongs to A.

2.2 Definition. A space (X, τ) is called

(i) *sequential* if a subset A is closed whenever it contains all its limit points, i.e. limits of all the τ convergent sequences of elements of A, and

(ii) *Fréchet* if for every set $A \subset X$, every point in its closure is the limit of some sequence of points from A.

It is clear that any Fréchet space is sequential and any metrizable space is Fréchet.

2.3 Lemma. *Let X be a sequential and let Y be an arbitrary topological space. A mapping $f : X \to Y$ is continuous if and only if for any sequence $\langle x_n : n \in \omega\rangle$ converging to x in the space X the sequence $\langle f(x_n) : n \in \omega\rangle$ converges to $f(x)$ in Y.*

So for sequential topological spaces, continuous mappings and sequentially continuous mappings coincide.

2.4 Fact. *(i) The class of sequential spaces is closed under disjoint sums and quotient; it is not generally closed under subspaces neither products.*

(ii) The class of Fréchet spaces is closed under disjoint sums, quotients and also to subspaces, it is not closed under the products.

The category of sequential topological spaces is correflexive in the category of all topological spaces.

2.5 SEQUENTIAL MODIFICATION

For any topological space (X, τ) there is a sequential space (X, τ_s) with the same underlying set and with the sequential topology $\tau_s \supset \tau$ called *sequential modification*.

The topology τ_s is the strongest among topologies τ' on X for which the identity $\mathrm{id}_X : (X, \tau) \to (X, \tau')$ is sequentially continuous.

A subset $A \subset X$ is closed in (X, τ_s) if and only if the limit $a = \lim a_n \in A$ of every τ-convergent sequence $\langle a_n \in A : n \in \omega\rangle$ belongs to A.

It should be clear that $\tau_s \supset \tau$ and (X, τ_s) is a sequential space. When starting space (X, τ) is sequential then $\tau = \tau_s$.

2.6 POWER SET $\mathcal{P}(X)$ WITH COMPACT AND SEQUENTIAL TOPOLOGY

Let X be a non-empty set. On the power set $\mathcal{P}(X)$ we have a compact Hausdorff topology when we identify subsets of X with their characteristic functions, i.e. elements of $\prod_{x \in X}\{0, 1\}^X$ endowed with the product topology of discrete space $\{0, 1\}$. We call such topology *the compact topology* and denote it τ_c. A sequence of functions $\langle f_n : n \in \omega\rangle \subset 2^X$ converges to $f \in 2^X$ in τ_c topology if and only if it converges pointwise. Equivalently in set-theoretical setting a sequence of subsets $\langle Y_n : n \in \omega\rangle$ of X converges to a set Y if and only if

$$Y = \bigcap_{k \in \omega} \bigcup_{n \geq k} Y_n = \bigcup_{k \in \omega} \bigcap_{n \geq k} Y_n.$$

Now look at the sequential modification τ_s of the compact topology τ_c on 2^κ, where $\kappa = |X|$. If $\kappa \leq \omega$, then the space $(2^\kappa, \tau_c)$ is metrizable and thus $\tau_s = \tau_c$.

2.7 Fact. *For any κ the subspace $[\kappa]^{\leq\omega}$ of $(\mathcal{P}(X), \tau_c)$ is sequential, even Fréchet.*

Proof. Let $A \subset 2^\kappa$ be a family of functions with at most countable support, i.e. $(\forall g \in A)\ |g^{-1}(\{1\})| \leq \omega$. Let $f \in 2^\kappa$ be of the same type and $f \in \mathrm{cl}_{\tau_c}(A) \setminus A$. We look for a sequence $\langle g_n : n \in \omega\rangle \subset A$ such that g_n's converges to f coordinatewise. Take a non-empty at most countable set $a_0 \subset \kappa$ such that $f^{-1}[\{1\}] \subset a_0$. Let $\{a_0(n) : n \in \omega\}$ be non-decreasing cover of a_0 consisting of finite subsets. Consider a neighbourhood $[f \restriction a_0(0)]$ of f and pick an element $g_0 \in A$ from this neighbourhood. Put $a_1 = g^{-1}[\{1\}] \setminus a_0$ and let $\{a_1(n) : n \geq 1\}$ be a non-decreasing cover of a_1 consisting of finite subsets. Now consider a neighbourhood $[f \restriction a_0(1) \cup a_1(1)]$ of f and pick $g_1 \in A$. Then $a_2 = g^{-1}[\{1\}] \setminus (a_0 \cup a_1)$ with some cover $\{a_2(n) : n \geq 2\}$ determines $[f \restriction a_0(2) \cup a_1(2) \cup a_2(2)]$ and one can obtain $g_3 \in A$ and continue in similar fashion. The sequence $\langle g_n : n \in \omega\rangle$ is as desired. $\square$

2.8 Corollary. *For any infinite κ topology τ_c and τ_s coincides on the subset $[\kappa]^{\leq\omega}$.*

2.9 Theorem. *For any cardinal number $\kappa > 0$ the space $(2^\kappa, \tau_c)$ is the Čech-Stone compactification of $([\kappa]^{\leq\omega}, \tau_c)$.*

Proof. The set $I = [\kappa]^{\leq\omega} = \{f \in 2^\kappa : f^{-1}[\{1\}]| \leq \omega\}$ is a dense subset of 2^κ, because $[\kappa]^{<\omega}$ is dense. It is sufficient to show that any continuous mapping $f : I \to [0, 1]$ can be continuously extended to the whole 2^κ. This follows from the following special case of the Mazur theorem. □

2.10 Theorem. (S. Mazur) *Assume $f : [\kappa]^{\leq\omega} \to [0, 1]$ is continuous. Then there is at most countable set $M \subset \kappa$ such that $(\forall x, y \in [\kappa]^{\leq\omega})$ $(x \cap M = y \cap M \to f(x) = f(y))$.*

Proof. Assume $\kappa > \omega$, otherwise the assertion is trivial, and consider the opposite.

By recursion for $\alpha < \omega_1$ we can choose $M_\alpha \in [\kappa]^{\leq\omega}$ and $x_\alpha, y_\alpha \in [\kappa]^{\leq\omega}$ such that

(i) if $\alpha < \beta$ then $M_\alpha \subset M_\beta$ and $x_\alpha, y_\alpha \subset M_{\alpha+1}$ and

(ii) $x_\alpha \cap M_\alpha = y_\alpha \cap M_\alpha$ and $f(x_\alpha) \neq f(y_\alpha)$.

The interval $[0, 1]$ has a countable base so there are disjoint closed intervals $I_1 = [a, b]$ and $I_2 = [c, d]$ and uncountable set $I \subset \omega_1$ such that $f(x_\alpha) \in I_1$, $f(y_\alpha) \in I_2$ for each $\alpha \in I$.

Let $\alpha_0, \alpha_1, \alpha_2, \dots$ be first ω many ordinal numbers from I. Put $\beta = \sup_n \alpha_n$ and $M = \bigcup_{n\in\omega} M_{\alpha_n}$. $\mathcal{P}(M)$ is closed subspace of $[\kappa]^{\leq\omega}$, thus $f \restriction \mathcal{P}(M)$ is continuous. $(\mathcal{P}(M), \tau_c)$ is Cantor discontinuum, the sequence $\langle x_{\alpha_n} : n \in \omega\rangle$ has a convergent subsequence, say $x_{\alpha_i} \to x$. But necessarily the subsequences $\langle y_{\alpha_i} : i \in \omega\rangle$ also converges to x. From sequential continuity we obtain that $f(x_{\alpha_i}) \to f(x) \in I_1$ and $f(y_{\alpha_i}) \to f(x) \in I_2$, a contradiction.

To finish the proof of Theorem I.2.9, fix $M \subset [\kappa]^{\leq\omega}$ satisfying Theorem I.2.10. It is sufficient to put $g(Y) = f(Y \cap M)$ for any $Y \subset \kappa$. The function $g : \mathcal{P}(\kappa) \to [0, 1]$ extends f and is continuous, as follows from the definition of g and continuity of f. □

The next assertion is due to V. Trnková [Trn69] and W. Główczyński [Gło91].

2.11 Theorem. *For κ uncountable, the space $(2^\kappa, \tau_s)$ is not regular, therefore $\tau_s \supsetneq \tau_c$.*

Proof. It is sufficient to prove theorem for $\kappa = \omega_1$, for $(\mathcal{P}(\omega_1), \tau_s)$ is closed subspace of $(\mathcal{P}(\kappa), \tau_s)$, for $\kappa \geq \omega_1$. Let U be the set of all $X \subset \omega_1$ whose complement is uncountable. U is an open neighbourhood of $\emptyset$ in topology τ_s, we shortly denote this fact by $U \in \mathcal{N}_0$. This is because any τ_c limit of a sequence of subsets of ω_1 with at most countable complement has at most countable complement.

Consider Ulam matrix $\{A_{\alpha,n} : n \in \omega, \alpha \in \omega_1\}$ [BŠ00], i.e.

$A_{\alpha,n} \cap A_{\alpha,m} = \emptyset$, for $m \neq n$,

$A_{\alpha,n} \cap A_{\beta,n} = \emptyset$, for $\alpha \neq \beta$,

$\bigcup_{n\in\omega} A_{\alpha,n} = \omega_1 \setminus \alpha$, for all $\alpha < \omega_1$.

Let $V \in \mathcal{N}_0$. As for every $\alpha < \omega_1$ the sequence $\langle \bigcup_{n\geq k} A_{\alpha,n} : k \in \omega\rangle$ converges to $\emptyset$ in τ_c topology, it converges to $\emptyset$ also in τ_s topology. For each α there exists some k_α such that $X_\alpha = \bigcup_{n\geq k_\alpha} A_{\alpha,n}$ is in V. There exists some $k \in \omega$ and uncountable set $W \subset \omega_1$ such that $k_\alpha = k$ for all $\alpha \in W$.

Put $C = \{\alpha \in \omega_1 : \alpha$ limit ordinal and $\bigcup(\alpha \cap W) = \alpha\}$. C is closed unbounded subset of ω_1. We claim that $\omega_1 \setminus \alpha \in \mathrm{cl}(V)$ for every $\alpha \in C$.

Let $\alpha_0 < \alpha_1 < \ldots$ be a sequence in W such that $\alpha = \lim_n \alpha_n$. Then $\lim_n X_{\alpha_n} = \omega_1 \setminus \alpha$ in τ_c topology, therefor $X \in \mathrm{cl}(V) = \omega_1 - \alpha$. We proved that there is no $V \in \mathcal{N}_0$ such that $\mathrm{cl}(V) \subset U$. $\square$

2.12 EXAMPLE. Let F be a σ-subfield of $\mathcal{P}(\omega_1)$ generated by all countable subsets of ω_1. Then (F, τ_s) is regular space and $(F, \tau_s) \neq (F, \tau_c)$.

Proof. Put $F_0 = [\omega_1]^{\leq\omega}$ and $F_1 = \{A \subset \omega_1 : |\omega_1 - A| \leq \omega\}$. Then F_0 and F_1 are disjoint and $F = F_0 \cup F_1$. (F_0, τ_c) as a subspace of compact Hausdorff space $(\mathcal{P}(\omega_1), \tau_c)$ is a regular space. The operation 'complement' is homeomorphism of $\mathcal{P}(\omega_1)$ in both topologies τ_c, τ_s. By Corollary I.2.8, τ_c, τ_s coincides on F_0 and F_1. The space (F, τ_s) is regular. The set F_0 is clopen proper subset in (F, τ_s), but is dense in (F, τ_c), therefore $\tau_s \supsetneq \tau_c$ on F. $\square$

$\triangle$

2.13 Definition. A topological space X is *sequentially compact* if any sequence contains a convergent subsequence.

2.14 Lemma. [Trn69] *A space* $(2^\kappa, \tau_c)$ *is sequentially compact if and only if* $\kappa < \mathfrak{s}$, *where* $\mathfrak{s}$ *is the splitting number.*

The proof of the following is straightforward.

2.15 Proposition. *Let* X *be a sequential* T_1 *space. Then* X *is sequentially compact if and only if it is* countably compact*; i.e. any countable open cover of* X *has a finite subcover.*

2.16 EXTREMAL PROPERTY OF COMPACT HAUSDORFF TOPOLOGY

Let (X, τ) be a compact Hausdorff space. If τ' is a stronger topology on X, i.e. $\tau' \supsetneq \tau$, then τ' is not compact.

If τ' is weaker then τ, i.e. $\tau' \subsetneq \tau$, then τ' is not Hausdorff.

2.17 Corollary. *Suppose* (X, τ) *is a Hausdorff space,* $\tau' \supseteq \tau$ *stronger topology and* $A \subseteq X$ *is a compact set is the space* (X, τ'). *Then* A *is also compact in* (X, τ) *and topologies* τ' *and* τ *coincides on* A.

2.18 POLISH SPACES

A topological space (X, τ) is called *Polish* if it is separable and completely metrizable. Hence Polish space has a countable base of open sets and moreover its size is either 2^ω or at most countable.

Baire space $\mathcal{N} = \prod_{n\in\omega} \omega$ consisting from infinite sequences of natural numbers and Cantor discontinuum $\mathcal{C} = \prod_{n\in\omega}\{0, 1\}$ are basic examples of zero-dimensional Polish spaces. The following characterisation of a Polish space without isolated point can be found in Kechris book [Kec95].

2.19 Theorem. *Polish space X is continuous one-to-one image of a Baire space $\mathcal{N}$ if and only if X has no isolated point.*

Borel(τ) denotes σ-field of Borel subsets of the space (X, τ). It is the least σ-field containing all open sets.

A mapping $f : X \to Y$ is called *Borel measurable* or simply Borel if for any open set $U \subset Y$ the set $f^{-1}[U]$ is Borel in X.

2.20 ISOMORPHISM THEOREM

Polish spaces X and Y are Borel isomorphic if and only if $|X| = |Y|$.

It means that for two uncountable Polish spaces X, Y there is a bijection $f : X \to Y$ such that for any $A \subset X$ $f[A] \in \mathrm{Borel}(Y)$ if and only if $A \in \mathrm{Borel}(X)$.

If X is Polish space and $|X| \leq \omega$, then $\mathrm{Borel}(X) = \mathcal{P}(X)$. Hence any bijection among countable spaces is Borel isomorphism. The proof of 2.19 is in [Kec95].

Isomorphism theorem says that different uncountable spaces determine unique σ-field of sets, namely $\mathrm{Borel}(2^\omega)$. This field plays an important rôle in mathematics and we will meet it in the future text.

2.21 EXTENDING POLISH TOPOLOGY

Let (X, τ) be a Polish space. Fix at most countably many Borel sets $A_n \in \mathrm{Borel}(\tau)$. Then there is a topology τ_1 on X stronger then τ such that

(i) (X, τ_1) is Polish space,

(ii) $\mathrm{Borel}(\tau_1) = \mathrm{Borel}(\tau)$,

(iii) all A_n's are clopen sets in topology τ_1.

Proof. We will proceed in three steps:

1) If $\emptyset \neq U \subset X$ is open, then U and $X \setminus U$ are G_δ sets. Therefore both with subspace topology are Polish. Then $\tau_1 = \{A \subset X : A \cap U \text{ open } \& \ A \cap (X \setminus U) \text{ open in } X \setminus U\}$ is Polish topology extending τ with U clopen.

2) Let $\tau_n \supset \tau$, $n \in \omega$ be Polish topologies with $\mathrm{Borel}(\tau_n) = \mathrm{Borel}(\tau)$ for every $n \in \omega$. Then topology τ_∞ generated by $\bigcup\{\tau_n : n \in \omega\}$ is also Polish topology on X and $\tau_\infty \subset \mathrm{Borel}(\tau)$.

$\prod_{n\in\omega}(X,\tau_n)$ is Polish as a product of Polish spaces. A mapping

$$\mathrm{id} : X \to \prod_{n\in\omega}(X,\tau_n),$$

where $\mathrm{id}(x) = \langle x, x, x, \dots \rangle$ embeds the set X onto closed subset. Topology on X given by this embedding is Polish and moreover it is just topology generated by $\bigcup\{\tau_n : n \in \omega\} = \tau_\infty$. So (X, τ_∞) is Polish.

3) Put $\mathcal{S} = \{A \subset X : \ \exists\tau' \text{ Polish}, \ \tau' \supset \tau, \ A \text{ is clopen in } (X,\tau'), \ \tau' \subset \mathrm{Borel}(\tau)\}$. The family $\mathcal{S}$ contains open sets by 1). Trivially $\mathcal{S}$ is closed to a complements. Consider a sequence $A_n \in \mathcal{S}$, $n \in \omega$. Let $\tau_n \supset \tau$ be Polish such that A_n is clopen in τ_n, $\tau_n \subset \mathrm{Borel}(\tau)$. By 2) $\bigcup A_n$ is open in τ_∞. Now apply 1) to τ_∞ and we get that $\bigcup A_n \in \mathcal{S}$. Hence we proved that $\mathrm{Borel}(\tau) \subset \mathcal{S}$ and this also completes the proof of the theorem. $\square$

2.22 Corollary. *Let (X,τ) be a Polish space.*

(i) There is a Polish topology $\tau_1 \supset \tau$, such that $\tau_1 \subset$ Borel(τ) and (X,τ_1) is zero-dimensional space.

(ii) If $f : (X,\tau) \to Y$ is Borel mapping of X into second countable space Y; i.e. Y has a countable base. Then there is a Polish topology $\tau_2 \supset \tau$ such that $\tau_2 \subset$ Borel(τ) and the mapping $f : (X,\tau_2) \to Y$ is continuous.

Proof. (i) Let $\mathcal{B}$ be a countable base of τ-topology. First note that if U is open in τ then by the step 1) of the previous proof there is a topology τ_1 such that U is clopen in τ_1 and $\mathcal{B} \cup \{U\} \cup \{X \setminus U\}$ is a subbase of τ_1. Hence applying I.2.21 to the base $\mathcal{B}$ we obtain the desired topology τ_1 with clopen base.

(ii) Simply apply I.2.21 to the set $\{f^{-1}[B] : \ B \in \mathcal{B}(Y)\}$, where $\mathcal{B}(Y)$ is a countable base of Y. $\square$

2.23 SEMICONTINUOUS FUNCTIONS

Let X be a topological space. Natural and useful generalisation of continuous functions are lower and upper semicontinuous functions. A function $f : X \to \mathbb{R}$ is *lower semicontinuous* if $f^{-1}(r, \to)$ is an open set, for each $r \in \mathbb{R}$. It is *upper semicontinuous* if $f^{-1}(\leftarrow, r)$ is an open set for each $r \in \mathbb{R}$.

A function f is lower (upper) semicontinuous in the point $x \in X$ if for every real number r satisfying $f(x) > r$ ($f(x) < r$) there is an open neighbourhood $U \subset X$ of x such that $f(y) > r$ ($f(y) < r$) for every $y \in U$.

So f is lower (upper) semicontinuous if and only if for every real number $r \in \mathbb{R}$ the set $\{x \in X : \ f(x) \le r\}$ (the set $\{x \in X : \ f(x) \ge r\}$) is closed and f is continuous if and only if it is both, lower and upper semicontinuous.

2.24 EXAMPLE. A canonical example of lower (upper) semicontinuous function is the characteristic function χ_U of an open (closed) set $U \subseteq X$. $\triangle$

2.25 Fact. *If* f *and* g *are lower (upper) semicontinuous then* $\min\{f, g\}$, $\max\{f, g\}$, $f + g$ *are lower (upper) semicontinuous. The same holds for* $f \cdot g$ *provided that* f *and* g *are nonnegative.*

2.26 Theorem. *Let* X *be metrizable space and let* $f : X \to (-\infty, \infty]$ *be bounded from below. Then* f *is lower semicontinuous if and only if there is an increasing sequence* $f_0 \leq f_1 \leq f_2 \leq \dots$ *of continuous functions from* X *to* $\mathbb{R}$ *such that*

$$f(x) = \sup_{n \in \omega} f_n(x).$$

Proof. If f is the supremum of an increasing sequence of continuous functions, then $\forall a \in \mathbb{R}$ the set $f^{-1}(a, \infty] = \bigcup_{n \in \omega} f_n^{-1}(a, \infty)$ is open.

For the converse, when f is not identically ∞, consider compatible metric d on X and put

$$f_n(x) = \inf \{f(y) + n \cdot d(x, y) : y \in X\}.$$

This sequence is as required.

When f is identically ∞, put $f_n \equiv n$. □

2.27 EXAMPLE. A subset $A \subseteq X$ is F_σ ($\sum_2^0$) set if and only if there is lower semicontinuous function $f : X \to [0, +\infty]$ such that $A = \{x \in X : f(x) < \infty\}$.

Proof. Since $A = \bigcup_{n \in \omega} f^{-1}[0, n]$ it is clearly a F_σ set.

Conversely, let $A = \bigcup_{n \in \omega} F_n$, where each F_n is closed set and $F_n \subseteq F_{n+1}$. Consider the function $f : X \to [0, \infty]$ given by $f(x) = 0$ on F_0, $f(x) = n$ on $F_n \setminus F_{n-1}$ and $f(x) = \infty$ on $X \setminus A$. The function f is lower semicontinuous and $x \in A$ if and only if $f(x) < \infty$. □

△

3. BOOLEAN ALGEBRAS

Let $\mathbb{B} = \langle B, \wedge, \vee, -, \mathbf{0}, \mathbf{1} \rangle$ be a Boolean algebra. We always assume that $\mathbf{0} \neq \mathbf{1}$. The canonical ordering $\leq$ on $\mathbb{B}$ is related to the Boolean operations via

$$x \leq y \quad \text{if and only if} \quad x \wedge y = x.$$

Thus, the canonical ordering restricted to the non-zero elements of the algebra, denoted by $\mathbb{B}^+$, is a separative partial order.

The properties of elements, subsets, and the whole partial order defined in I.1.1, I.1.2, and I.1.9 are said to hold for the algebra $\mathbb{B}$ if they hold for the partial order $(\mathbb{B}^+, \leq)$. Of course, $\mathbf{0}$ is considered disjoint with all elements of $\mathbb{B}^+$, therefore for any $x, y \in \mathbb{B}$, $x \perp y$ means $x \wedge y = \mathbf{0}$ and $x \parallel y$ means $x \wedge y \neq \mathbf{0}$. We allow that a dense subset of $\mathbb{B}$ may contain $\mathbf{0}$ as its element.

For any $a \in \mathbb{B}^+$, $\mathbb{B} \upharpoonright a$ denotes the factor of algebra $\mathbb{B}$ determined by the element a, i.e. $\mathbb{B} \upharpoonright a = \{x \in \mathbb{B} : x \leq a\}$ with appropriate restriction of Boolean operations. $St(\mathbb{B})$ denotes the Stone space of an algebra $\mathbb{B}$.

Our primary reference for Boolean algebras is Volume 1 of Handbook of Boolean Algebras [Kop89].

3.1 Definition. *Symmetric difference,* defined by $a \triangle b = (a - b) \vee (b - a)$, is a derived operation on $\mathbb{B}$ that is associative and commutative. Moreover, the structure $(\mathbb{B}, \triangle)$ is an Abelian group with $\mathbf{0}$ as the neutral element. Since $a \triangle a = \mathbf{0}$, each $a \in \mathbb{B}$ is also the opposite element to itself. Notice that for any ideal I on $\mathbb{B}$, $(I, \triangle)$ is a subgroup of $(\mathbb{B}, \triangle)$.

3.2 DISTRIBUTIVITY

A *partition of unity*, or a *partition*, is a disjoint family $P \subseteq \mathbb{B}^+$ such that $\bigvee P = \mathbf{1}$. Let κ, τ, λ be cardinal numbers. We say that an algebra $\mathbb{B}$ is (κ, τ, λ) *distributive* if for any family $\{P_\alpha : \alpha \in \kappa\}$ of partitions of unity, where each P_α has the size at most τ, there is a dense set Q with the property that every $q \in Q$ is compatible with less than λ elements of each P_α.

$(\kappa, \cdot, \lambda)$ distributivity means that there are no restrictions to the size of partitions P_α's.

Note that an algebra $\mathbb{B}$ is $(\kappa, \cdot, 2)$ distributive if and only if any κ-many partitions of unity have a common refinement if and only if the intersection $\cap_{\alpha<\kappa} D_\alpha$ of κ-many open dense sets D_α is again dense set.

Cardinal invariants $\mathfrak{h}$ and $\mathfrak{s}$ concerning families of infinite subsets of natural numbers are characterised through distributivity properties of the algebra $\mathcal{P}(\omega)/\mathrm{fin}$ as follows:

3.3 Definition.

$$\mathfrak{h} = \min\{\kappa : \mathcal{P}(\omega)/\mathrm{fin} \text{ is not } (\kappa, \cdot, 2) \text{ distributive }\},$$

$$\mathfrak{s} = \min\{\kappa : \mathcal{P}(\omega)/\mathrm{fin} \text{ is not } (\kappa, 2, 2) \text{ distributive}\}.$$

Important notion for us is a *weak distributivity*. Boolean algebra $\mathbb{B}$ is weakly distributive if it is (ω, ω, ω) distributive. When $\mathbb{B}$ is σ-complete then $\mathbb{B}$ is weakly distributive if and only if for any matrix $\{a(n,k) \in \mathbb{B} : n, k \in \omega\}$ such that each row is nondecreasing sequence, we have

$$\bigwedge_{n\in\omega} \bigvee_{k\in\omega} a(n,k) = \bigvee_{f\in{}^\omega\omega} \bigwedge_{n\in\omega} a(n, f(n)).$$

The canonical example of a ccc, weakly distributive Boolean algebra is a measure algebra, i.e. a complete Boolean algebra that carries a strictly positive σ-additive measure. The weak distributivity has many equivalent formulations, let us here state one of the most frequent:

A complete Boolean algebra $\mathbb{B}$ is weakly distributive if and only if for each system of nondecreasing sequences $\langle a_n^m : n \in \omega\rangle \nearrow_m \mathbf{1}$

$$\bigvee_{f\in{}^\omega\omega} \bigwedge_{n\in\omega} a_n^{f(n)} = \mathbf{1}.$$

3.4 Remark. A *Souslin line* is a complete ccc dense linear order that does not have a countable dense subset (a counterexample to Souslin's problem).

A *Souslin tree* is an ω_1-tree with no uncountable chains or antichains. A *Souslin algebra* is an atomless complete ccc Boolean algebra that satisfies the $(\omega, \omega, 2)$ distributive law.

A Souslin line, a Souslin tree and a Souslin algebra can be constructed from each other (see [Kur35], [Mil43] or [Jec02] for details). Thus the existence of a ccc, atomless $(\omega, \omega, 2)$ distributive Boolean algebra is equivalent to the existence of a Souslin tree.

Notice that the Martin's number (see I.1.4) of an atomless algebra $\mathbb{B}$ is the minimal cardinality of a family of nowhere dense sets covering the Stone space of $\mathbb{B}$, while $m(\mathbb{B}) = \infty$ if $\mathbb{B}$ has an atom. A Boolean algebra $\mathbb{B}$ is $(\kappa, \cdot, \omega)$-distributive if and only if any union of κ nowhere dense subsets of the space $St(\mathbb{B})$ is nowhere dense.

3.5 Lemma. *Boolean algebra $\mathbb{B}$ is weakly distributive if and only if the ideal of nowhere dense sets in the Stone space $St(\mathbb{B})$ is a σ-ideal.*

Furthermore, the point density of $\mathbb{B}$, $\text{pd}(\mathbb{B})$, is exactly the topological density of the Stone space of $\mathbb{B}$, thus

$$\text{pd}(\mathbb{B}) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a family of ultrafilters on } \mathbb{B} \text{ with } \bigcup \mathcal{F} = \mathbb{B}^+\}.$$

It follows that $\mathbb{B}$ is σ-centered if and only if $\text{pd}(\mathbb{B}) \leq \omega$ if and only if $St(\mathbb{B})$ is a separable space.

3.6 CONSTRUCTION OF COMPLETE BOOLEAN ALGEBRAS

In the following we reveal two natural sources of Boolean algebras. One source are topological spaces.

Let X be a non-empty topological space. For $Y \subseteq X$, the *regularization* of Y is $r(Y) = \text{int}(\text{cl}(Y))$. A set $U \subseteq X$ is a *regular open* set if $r(U) = U$.

3.7 Fact. *For any non-empty topological space X, the family of all regular open sets of X, denoted by $RO(X)$, ordered by the inclusion, is a complete Boolean algebra. Boolean operations on $RO(X)$ are then determined as follows:*

(i) $\mathbf{0} = \emptyset$, $\mathbf{1} = X$,

(ii) $U \wedge V = U \cap V$,

(iii) $U \vee V = r(U \cup V)$,

(iv) $-U = r(X - U)$,

(v) $\bigwedge_{i \in I} U_i = r(\bigcap_{i \in I} U_i)$.

The second natural source for constructing Boolean algebras are partial orders or more generally preorders.

3.8 Fact. *Any preorder* $(P, \leq)$ *determines a unique (up to isomorphism) complete Boolean algebra denoted by* $\mathsf{RO}(P, \leq)$.

Proof. For $(P, \leq)$ we have the relation $\perp$ of disjointness. Denote

$$A^{\perp} = \{x \in P : (\forall\, a \in A) \quad x \perp a\}$$

for $A \subset P$. So $A^{\perp}$ is the set of all elements disjoint with every element from A. If $A \subset P$ then $A \subset (A^{\perp})^{\perp}$ and $((A^{\perp})^{\perp})^{\perp} = A^{\perp}$.

Put $\mathsf{RO}(P) = \{A \subset P : (A^{\perp})^{\perp} = A\}$. Then $\mathsf{RO}(P)$ ordered by inclusion is a complete Boolean algebra. Boolean operations are $A \wedge B = A \cap B$, $A \vee B = (A \cup B)^{\perp\perp}$ and $-A = A^{\perp}$. □

The preceding construction is closely related with the description from I.3.7. More precisely, let family $\{(\leftarrow, x] : x \in P\}$ form a subbase for a topology τ on the set P. Then $\mathsf{RO}(P, \leq)$ is exactly the complete Boolean algebra $\mathsf{RO}(P, \tau)$. In this setting the regular open sets $A \in \mathsf{RO}(P)$ are exactly those where $A^{\perp\perp} = A$.

The mapping $\phi : P \to \mathsf{RO}(P, \tau)$ defined by $\phi(x) = r_{\tau}((\leftarrow, x])$ for $x \in P$ is order-preserving, preserves disjointness, and the image $\phi[P]$ is a dense subset of $\mathsf{RO}(P, \tau)$. Notice that ϕ is an embedding if and only if for every $x \in P$ the set $(\leftarrow, x]$ is regular open in the topology τ, which is equivalent with the fact that $(P, \leq)$ is a separative partial order.

3.9 FUNDAMENTAL EXAMPLES OF BOOLEAN ALGEBRAS

(i) $\{\mathbf{0}, \mathbf{1}\}$ is the simplest algebra. The countable product $\prod_{n\in\omega}\{\mathbf{0}, \mathbf{1}\}$ is naturally isomorphic to $\mathcal{P}(\omega)$, the algebra of all subsets of the set of natural numbers.
The group $(\mathcal{P}(\omega), \triangle)$ is isomorphic to the product $\prod_{n\in\omega}\mathbb{Z}_2$, where the group $\mathbb{Z}_2$ is the usual $\mathbb{Z}$ with the operation addition modulo 2.
(ii) Let us consider $2^{\omega} = \prod_{n\in\omega}\{0, 1\}$ as a topological space that is a product of topological spaces $\{0, 1\}$ with the discrete topology. Then 2^{ω} is the well-known *Cantor space*.

By *Cantor algebra* we mean the Boolean algebra of all clopen subsets of the Cantor space. The Cantor algebra $\mathcal{A}$ is isomorphic to $\mathrm{Free}(\omega)$, the free algebra with countably many free generators. It is atomless and countable. In fact, these two properties uniquely characterise the Cantor algebra.
(iii) Let *Borel*(2^{ω}), or simply *Borel*, denote the σ-algebra of all Borel subsets of the Cantor space. Then $\mathcal{A}$, the Cantor algebra, is a subalgebra and σ-completely generates *Borel*. There are three fundamental σ-additive ideals on *Borel* corresponding to different notions of 'being small':

(a) *Topological smallness*, meager sets.
$\mathcal{M} = \{X \in \mathit{Borel} : X \text{ is meager}\}$ is a σ-ideal and the quotient $\mathbb{C} = \mathit{Borel}/\mathcal{M}$ is a complete Boolean algebra, well known as the *Cohen algebra*. It is isomorphic to the regular completion of Cantor algebra $\mathcal{A}$, so $\mathbb{C} \simeq \mathrm{Comp}(\mathcal{A})$.

(b) *Measure-theoretic smallness*, negligible sets.
$\mathcal{N} = \{x \in \mathit{Borel} : \mu(X) = 0\}$, where μ is a Haar measure on the space 2^ω, is a σ-ideal and the quotient $\mathbb{M} = \mathit{Borel}/\mathcal{N}$ is a complete Boolean algebra well known as the *Measure algebra*.

(c) *Set-theoretic smallness*, to be at most countable.
The quotient *Borel/countable sets* is just a σ-complete algebra and its completion, $\mathbb{S}$, is known as the *Sacks algebra*.

All algebras $\mathbb{C}$, $\mathbb{M}$, and $\mathbb{S}$ are atomless, include the Cantor algebra $\mathcal{A}$ as a subalgebra and are completely generated by $\mathcal{A}$. Therefore, in any generic extension via those algebras, there are new reals (i.e. new subsets of ω). $\mathbb{C}$ and $\mathbb{M}$ satisfy ccc, while $\mathbb{S}$ is $(2^\omega)^+$-cc.

Measure algebra is weakly distributive and Sacks algebra is $(\omega, \cdot, \omega)$ distributive $\triangle$

3.10 COMPLETE BOOLEAN ALGEBRAS ARE INJECTIVE

Here we point out some important properties of complete Boolean algebras. The following important theorem is due to R. Sikorski.

3.11 Theorem. (R. Sikorski [Sik64]) *Let $\mathbb{B}$ be a complete Boolean algebra. Let A_1 be a subalgebra of an algebra A and $f : A_1 \to \mathbb{B}$ a homomorphism. Then f can be extended to a homomorphism $\overline{f} : A \to \mathbb{B}$.*

For proof of the theorem, see [Kop89], Theorem 5.9, page 70.

3.12 Corollary. *Let A be a subalgebra of a complete Boolean algebra $\mathbb{B}$. Then there is a subalgebra C of $\mathbb{B}$ such that $A \subseteq C \subseteq \mathbb{B}$, A is dense in C and C is a regular completion of A.*

3.13 GENERATORS OF BOOLEAN ALGEBRAS AND INDEPENDENT SUBSETS

Let us now recall the notion of generators and of independent subsets of a Boolean algebras.

3.14 Definition. We say that family $\langle x_i : i \in I \rangle \subset \mathbb{B}$ is *independent* if for each finite $I_1, I_2 \in [I]^{<\omega}$, $I_1 \cap I_2 = \emptyset$

$$\bigwedge_{i \in I_1} x_i \wedge \bigwedge_{i \in I_2} -x_i \neq \emptyset.$$

Note that if $\langle u_i : i \in I \rangle$ is an independent family, then every $u_i \neq \mathbf{0}$ and for $i \neq j$, $u_i \neq u_j$. So we can speak of $\{u_i : i \in I\}$ as of an independent set.

3.15 Definition. For a complete Boolean algebra $\mathbb{B}$, $\mathfrak{g}(\mathbb{B}) = \min\{|X| : X \subseteq \mathbb{B}$ is a set of complete generators of $\mathbb{B}\}$.

For a σ-complete Boolean algebra $\mathbb{B}$,

$$\mathfrak{g}_\omega(\mathbb{B}) = \min\{|X| : X \subset \mathbb{B} \text{ is a set of } \sigma\text{-complete generators of } \mathbb{B}\}.$$

3.16 Theorem. (R. McKenzie) *Let $\mathbb{B}$ be an infinite complete Boolean algebra. Then there is an independent subset X of $\mathbb{B}$ of size $\mathfrak{g}(\mathbb{B})$ that completely generates $\mathbb{B}$.*

For proof, see [Kop89], page 205.

3.17 REGULAR SUBALGEBRAS

A subalgebra $\mathbb{B}$ of algebra $\mathbb{C}$ is called *regular* if for any $X \subset \mathbb{B}$ for which there is a supremum $\bigvee^{\mathbb{B}} X$ of X in $\mathbb{B}$, the same element is a supremum of X in $\mathbb{C}$, i.e. $\bigvee^{\mathbb{B}} X = \bigvee^{\mathbb{C}} X$.

An embedding $i : \mathbb{B} \to \mathbb{C}$ is *regular* if the image $i[\mathbb{B}]$ is a regular subalgebra of algebra $\mathbb{C}$.

3.18 Proposition. *For a subalgebra $\mathbb{B} \subset \mathbb{C}$ the following are equivalent*

(i) *$\mathbb{B}$ is a regular subalgebra of $\mathbb{C}$,*

(ii) *every maximal disjoint family in $\mathbb{B}$ is maximal in $\mathbb{C}$,*

(iii) *every predense set in $\mathbb{B}$ is predense in $\mathbb{C}$,*

(iv) *for each $c \in \mathbb{C}^+$ there is a* 'pseudoprojection' $b_c \in \mathbb{B}$; *i.e. for every $a \leq b_c$, $a \in \mathbb{B}$*
$$a \wedge c \neq \mathbf{0},$$

(v) *for every generic (cf. II.1.6) filter F on $\mathbb{C}$, $F \cap \mathbb{B}$ is a generic filter on $\mathbb{B}$.*

Proof. The proof of implications *(i)* $\leftrightarrow$ *(ii)* $\leftrightarrow$ *(iii)* $\leftrightarrow$ *(v)* and *(vi)*$\to$ *(ii)* are straight forward.

To show *(ii)*$\to$ *(vi)* let $c \in \mathbb{C}^+$. Take arbitrary maximal disjoint family B_c from the set $\{b \in \mathbb{B} \ : \ b \wedge c = \mathbf{0}\}$. From *(ii)* it follows that B_c is not maximal in $\mathbb{B}$, hence there is some b_c disjoint with B_c and we are done. □

3.19 EXAMPLE. Let $\mathbb{C}$ be a complete algebra, $\mathbb{B} \subset \mathbb{C}$ subalgebra. Then $\mathbb{B}$ is a regular subalgebra of $\mathbb{C}$ if and only if $\langle \mathbb{B} \rangle$, the complete subalgebra of $\mathbb{C}$ completely generated by the set $\mathbb{B}$, is completion of $\mathbb{B}$, equivalently $\mathbb{B}$ is dense in $\langle \mathbb{B} \rangle$. △

3.20 Theorem. *Let $\mathbb{B}$ be a subalgebra of $\mathbb{C}$. Then there is an ideal $\mathcal{I}$ on $\mathbb{C}$ such that canonical homomorphism*

$$\begin{aligned} i : \mathbb{B} &\longrightarrow \mathbb{C}/\mathcal{I} \\ b &\longmapsto [b]_{\mathcal{I}}, \end{aligned}$$

is a regular embedding of $\mathbb{B}$ into $\mathbb{C}/\mathcal{I}$.

Proof. Define a set $\mathcal{I} \subset \mathbb{C}$:

$$\mathcal{I} = \{u \in \mathbb{C} : \exists \text{ max. disjoint family } X \subset \mathbb{B} \text{ such that } u \wedge x = \mathbf{0} \text{ for any } x \in X\}.$$

A set $\mathcal{I}$ is downward closed. Let $u, v \in \mathcal{I}$. Take maximal disjoint families X and Y that guarantees that u respectively v belongs to $\mathcal{I}$. Then $z = \{x \wedge y \neq \mathbf{0} : x \in$

X & $y \in Y\}$ is a maximal disjoint family of elements from $\mathbb{B}$ and $u \vee v$ is disjoint with any element of z. Therefore $u \vee v \in \mathcal{I}$ hence $\mathcal{I}$ is an ideal.

No $b \in \mathbb{B}^+$ belongs to $\mathcal{I}$, so the mapping $i : \mathbb{B} \to \mathbb{C}/\mathcal{I}$ is an embedding. Let $\{c_i : i \in I\}$ be a maximal disjoint family in $i[\mathbb{B}]$. Then $\{[c_i] : i \in I\}$ is a maximal disjoint family in $\mathbb{C}/\mathcal{I}$. Assume that $[u]$ is disjoint with every $[c_i]$ in $\mathbb{C}/\mathcal{I}$, i.e. $c_i \wedge u \in \mathcal{I}$ and hence there is a maximal disjoint set $X_i \subset \mathbb{B} \restriction c_i$ such that u is disjoint with any element of X_i. Disjoint set $\bigcup\{X_i : i \in I\}$ is maximal in $\mathbb{B}$ and so $u \in \mathcal{I}$, i.e. $[u] = \mathbf{0} \in \mathbb{C}/\mathcal{I}$. □

It is clear, that the ideal $\mathcal{I}$ cannot intersect with subalgebra $\mathbb{B}$, i.e. $\mathcal{I} \cap \mathbb{B} = \emptyset$. In fact this simple property is the principal one.

3.21 Proposition. *Let $\mathbb{B}$ be a subalgebra of $\mathbb{C}$ and let $\mathcal{J}$ be a maximal ideal such that $\mathbb{B} \cap \mathcal{J} = \{\mathbf{0}\}$. Then canonical embedding*

$$i : \mathbb{B} \longrightarrow \mathbb{C}/\mathcal{J},$$

is a regular one. In this case $i[\mathbb{B}]$ is even dense in $\mathbb{C}/\mathcal{J}$.

Proof. Suppose that $i[\mathbb{B}]$ is not dense in $\mathbb{C}/\mathcal{J}$. Then there is some $c \in \mathbb{C}$, $c \notin \mathcal{J}$ such that for any $b \in \mathbb{B}^+$ $b \not\leq_{\mathcal{J}} c$. Since $\mathcal{J}$ is maximal and $c \notin \mathcal{J}$ there is some $j \in \mathcal{J}$ such that there is $b \in \mathbb{B}^+$ so that $b \leq c \vee j$ i.e. $b \leq_{\mathcal{J}} c$; a contradiction. □

3.22 Remark. The ideal $\mathcal{I}$ from lemma I.3.20 is the crucial one, because any maximal ideal $\mathcal{J}$ which do not intersect $\mathbb{B}$ has to contain $\mathcal{I}$. Moreover let

$$\mathcal{K} = \{\mathcal{J} \ : \ \mathcal{J} \text{ is an ideal on } \mathbb{C} \text{ maximal with respect to } \mathcal{J} \cap \mathbb{B}^+ = \emptyset\}$$

then

$$\bigcap \mathcal{K} \ = \ \mathcal{I},$$

and

$$\bigcup \mathcal{K} = \{c \in \mathbb{C} \ : \ \nexists b \in \mathbb{B}^+ \ b \leq c\}.$$

Proof. Suppose that $\mathcal{I} - \mathcal{J} \neq \emptyset$ and $a \in \mathcal{I} - \mathcal{J}$. Since $\mathcal{J}$ is maximal then there is some $j \in \mathcal{J}$ for which there is some $b \in \mathbb{B}^+$ so that $b \leq j \vee a$. Since $a \in \mathcal{I}$, there is a maximal antichain M in $\mathbb{B}$ such that for each $m \in M$ $m \wedge a = \mathbf{0}$.

Now $b \in \mathbb{B}$ have to intersect with some $m \in M$, so $\mathbf{0} \neq m \wedge b \leq j \vee a$, but m and a are disjoint hence $m \wedge b \leq j$, which is in contradiction with assumption that $\mathcal{J}$ does not intersect with $\mathbb{B}$.

Clearly $\bigcap \mathcal{K} \ \supset \ \mathcal{I}$. Take arbitrary $c \in \mathbb{C}^+ \setminus \mathcal{I}$, hence the set $\{b \in \mathbb{B} \ : \ b \leq -c\}$ is not dense in $\mathbb{B}$ because otherwise $c \in \mathcal{I}$. It means that there is some $b_0 \in \mathbb{B}$ so that

$$\forall b \in \mathbb{B}^+ \quad (b \leq -c) \ \to \ b - b_0 \neq \mathbf{0}.$$

That is $b_0 \wedge -c \notin \mathbb{B}$ and one can take a maximal ideal $\mathcal{J}$ containing this element, which shows that $c \notin \bigcap \mathcal{K}$; and we are done. □

3.23 Proposition. *If $\mathbb{B}$ is a complete algebra and $\mathbb{B}$ is a subalgebra of $\mathbb{C}$, then $\mathbb{B}$ is a retract of $\mathbb{C}$, i.e. there is a homomorphism $f : \mathbb{C} \to \mathbb{B}$ such that $f \circ f = id_{\mathbb{B}}$. Therefore $\mathbb{B} \approx \mathbb{C}/Ker(f)$.*

Proof. Using the Sikorski theorem I.3.11 the identity mapping $id_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ can be extended to a homomorphism $f : \mathbb{C} \to \mathbb{B}$. □

3.24 UNIVERSALITY OF THE σ-FIELD *Borel*(2^ω)

We have the Stone duality between the category of Boolean algebras and the category of compact Hausdorff spaces. Specially we know that every Boolean algebra is isomorphic to a field of sets.

It is well known that in the category of Boolean algebras there are free objects, free Boolean algebras that can be distinguished by the size of free generators.

3.25 Definition. Let κ be a cardinal number. Boolean algebra $\mathbb{B}$ is *free Boolean algebra* with κ (free) generators if there is a subset $X \subset \mathbb{B}$ of size κ, such that

(i) X is a set of generators of $\mathbb{B}$,

(ii) for any Boolean algebra $\mathbb{C}$ and any mapping $f : X \to \mathbb{C}$ there is an unique extension $\overline{f} \supset f$ to a Boolean homomorphism $\overline{f} : \mathbb{B} \to \mathbb{C}$.

3.26 Fact. *For every cardinal κ there is the unique free Boolean algebra with κ free generators, denoted by $Fr(\kappa)$.*

For an infinite cardinal κ the free algebra $\mathrm{Fr}(\kappa)$ is isomorphic to the $\mathcal{A}(\kappa)$ i.e. the algebra of clopen subsets of the generalised Cantor space 2^κ. If κ is finite then the free algebra is $\mathcal{P}(2^\kappa)$.

Now consider the category of σ-complete Boolean algebras together with σ-complete homomorphisms. Also in this category there are free objects, i.e. free σ-complete algebras of any number of (σ-free) generators. They can be easily described as a σ-fields of sets due to Rasiowa - Sikorski theorem (or equivalently to the Baire category theorem for Stone spaces).

3.27 Theorem. *For cardinal κ, consider the Cantor space 2^κ. The algebra of subsets of 2^κ which is σ-generated by clopen sets i.e: Baire (2^κ) is free σ-complete Boolean algebra with κ many σ-free generators.*

Immediate consequence of this theorem is the Loomis - Sikorski theorem on a σ-representability.

3.28 Theorem. *Every σ-complete Boolean algebra $\mathbb{B}$ is a σ-representable i.e: there is a σ-complete epimorphism from a σ-field of sets onto $\mathbb{B}$, i.e. $\mathbb{B}$ is isomorphic to the quotient of a σ-field of sets F and a σ-ideal $I \subset F$,*

$$\mathbb{B} \cong F/I.$$

Proof. Let X be the set of σ-generators of the algebra $\mathbb{B}$, $\kappa = |X|$. A one-to-one mapping $f : \kappa \to X$ can be extended to σ-homomorphism $h : \mathit{Baire}(2^\kappa) \to \mathbb{B}$, where $h(\langle \alpha, 0 \rangle) = f(\alpha)$. Homomorphism h is onto $\mathbb{B}$ and the kernel $h^{-1}\{\mathbf{0}\}$ is a σ-ideal in *Baire*(2^κ), hence

$$\mathbb{B} \cong \mathit{Baire}(2^\kappa)/h^{-1}\{\mathbf{0}\}.$$

□

3.29 Remark. Although for any uncountable κ there are free algebras in the category of κ-complete algebras, they are not explicitly nicely describable. They are not κ-fields of sets, since κ-representability implies some type of distributivity, see [Sik64]

We use the universality of the σ-field *Borel*(2^ω) in two directions.

(i) As the Cantor space 2^ω has a countable base, *Baire*$(2^\omega) = \text{Borel}(2^\omega)$. Therefore for any σ-complete algebra $\mathbb{B}$ with countably many σ-complete generators, i.e. $g_\omega(\mathbb{B}) = \omega$, there is a σ-ideal $\mathcal{I}$ on 2^ω with a base consisting of Borel sets such that $\mathbb{B} \cong \text{Borel}(2^\omega)/\mathcal{I}$. Many classical forcing notions are representable this way and there are more or less natural σ-ideals of Borel sets with interesting forcing properties of partial orders of the type $(\text{Borel} - \mathcal{I}, \subseteq)$ or equivalently $\text{Borel}/\mathcal{I}$ [Zap04].

(ii) Standard Borel spaces are isomorphic [Kec95]; i.e. any two uncountable Polish spaces are Borel isomorphic. So instead of the Cantor space one can consider any uncountable Polish space and σ-ideal with Borel base on it .

3.30 EXAMPLE. Let $\mathcal{N} = \omega^\omega$ be the Baire space, K_σ an σ-ideal on $\mathcal{N}$ generated by compact sets. Note that $A \subset \omega^\omega$ belongs to K_σ if and only if it is bounded with respect to the ordering $\leq^*$ of eventual domination. Since any closed subset $F \subset \mathcal{N}$ can be uniquely decomposed $F = S \cup P$, where $S \in K_\sigma$ and P is the set of all branches of some superperfect tree, $\text{Borel}(\mathcal{N})/K_\sigma$ ordered by inclusion is equivalent to the *Miller forcing*. Recall that superperfect tree is a tree consisting of some finite sequences of natural numbers $\emptyset \neq T \subset {}^{<\omega}\omega$, such that every $s \in T$ has an extension $t \supseteq s$ with infinitely many distinct immediate extensions in T. $\triangle$

One can characterise when the $\text{Borel}(2^\omega)/\mathcal{I}$ is weakly distributive via 'the continuous reading of names'; see also [Zak05].

3.31 Theorem. (J. Zapletal [Zap04]) *Let $\mathcal{I}$ be a σ-ideal on a Polish space X. The Boolean algebra Borel$(X)/\mathcal{I}$ is weakly distributive if and only if for each Borel set $B \in \mathcal{I}^+$ and for each Borel function $f : B \to \mathcal{N}$ there is a compact set $K \subset B$, $K \in \mathcal{I}^+$ such that $f \restriction K$ is continuous mapping.*

Proof. We will benefit from paragraph I.2.21. Note that by the isomorphism theorem I.2.20 we can consider $\text{Borel}(2^\omega)$ instead of $\text{Borel}(X)$.

Suppose now that the algebra $\text{Borel}(X)/\mathcal{I}$ is weakly distributive. Take arbitrary Borel set $B \in \mathcal{I}^+$ and Borel function $f : B \to \mathcal{N}$ and modify the starting topology (X, τ) so that the modification $\tau' \supset \tau$ is zero-dimensional and the sets B and $f^{-1}(U)$ are clopen for each U, a member of base of the topology on $\mathcal{N}$ (cf. I.2.21). Note that $\text{Borel}(X, \tau) = \text{Borel}(X, \tau')$. Consider the family $\mathcal{C}'_n = \{C \text{ clopen } : \text{ diam } (C) < \frac{1}{n}\}$, where we take diameters in complete metric that generates τ'. Polish spaces are saparable, hence one can pick $\mathcal{C}_n = \{C^n_i \in \mathcal{C}'_n : i \in \omega\}$ a disjoint covering of the Borel set B for each $n \in \omega$. Since $\text{Borel}(X)/\mathcal{I}$ is weakly distributive and $\mathcal{I}$ is σ-ideal there is a set $K \in \mathcal{I}^+$

such that $|\{i : K \cap C_i^n \neq \emptyset\}| < \omega$ for each $n \in \omega$. Hence the set $\mathrm{cl}K$ is closed and totally bounded, i.e. compact in the topology τ'. Note that $\mathrm{cl}K \subset B$ because B is clopen in τ' topology. Both τ, τ' are Hausdorff topologies, topology τ' extends the topology τ and so identity mapping $\mathrm{id} : (X, \tau') \to (X, \tau)$ is continuous. Continuous image of compact set is compact and so $\mathrm{cl}K$ is compact also in τ topology. By I.2.17 topologies τ and τ' have to coincide on a compact set $\mathrm{cl}K$ and by our assumption the function $f \restriction K$ is continuous.

For the opposite direction let $A_{nk} \in \mathrm{Borel}(X)/\mathcal{I}$ be a matrix with disjoint rows. Since $\mathcal{I}$ is a σ-ideal one can pick up Borel representatives $\{a_{nk} : n \in \omega, k \in \omega\}$ again with disjoint rows. Now pick up arbitrary Borel set $B \in \mathcal{I}^+$ and define a function $f : B \to \mathcal{N}$ so that $f(x)(n) = k$ if and only if $x \in a_{nk}$. Function f is clearly Borel. By the assumption of compact reading of names we get a compact set $K \in \mathcal{I}^+$ so that $K \subset B$ and $f \restriction K$ is continuous. Continuous image of a compact set is again compact and compact sets in $\mathcal{N}$ are bounded. Let $g : \omega \to \omega$ be such that $f[K] \leq g$, i.e.

$$K \subset \bigcap_{n\in\omega} \bigcup_{i \leq g(n)} a_{ni}$$

and so $\mathrm{Borel}(X)/\mathcal{I}$ is weakly distributive. □

3.32 Remark. Note that weak distributivity of $\mathrm{Borel}(2^\omega)/\mathcal{I}$ does not generally imply the weak distributivity of its completion. This holds if $\mathrm{Borel}(2^\omega)/\mathcal{I}$ is $(\omega, \cdot, \omega_1)$ distributive, which follows for example when the forcing $\mathrm{Borel}(2^\omega)/\mathcal{I}$ is proper.

3.33 SEQUENCES IN BOOLEAN ALGEBRAS

In order to describe a suitable topology on Boolean algebras which correspond with Boolean operations we start with the description of convergent sequences.

3.34 Definition. Let $\mathbb{B}$ be a σ-complete Boolean algebra, for a sequence $\langle x_n : n \in \omega\rangle$ in $\mathbb{B}$, we define limes superior and limes inferior as usual, i.e.

$$\overline{\lim}\, x_n = \bigwedge_{k\in\omega} \bigvee_{n\geq k} x_n \qquad \text{and} \qquad \underline{\lim}\, x_n = \bigvee_{k\in\omega} \bigwedge_{n\geq k} x_n.$$

We say that a sequence $\langle x_n : n \in \omega\rangle$ has $x \in \mathbb{B}$ as a limit if $\overline{\lim}\, x_n = \underline{\lim}\, x_n = x$, and write $\lim x_n = x$.

We start the investigations of basic properties with the Vladimirov's equality which puts together $\underline{\lim}$, $\overline{\lim}$ and the operation of symmetric difference.

3.35 Lemma. (D. A. Vladimirov [Vla69]) *Let $\mathbb{B}$ be a σ-complete Boolean algebra, then*

(i) For arbitrary sequence $\langle a_n : n \in \omega\rangle$

$$\overline{\lim}_n a_n - \underline{\lim}_n a_n = \overline{\lim}_n (a_n \vartriangle a_{n+1}).$$

(ii) Let $\langle a_n : n \in \omega \rangle$, $\langle b_n : n \in \omega \rangle$ *be sequences, then*

$$\overline{\lim}_n a_n \vee \overline{\lim}_n b_n \;=\; \overline{\lim}_n (a_n \vee b_n).$$

Proof. *(i)* In order to prove the desired equality

$$\bigwedge_k \bigvee_{n \geq k} a_n - \bigvee_k \bigwedge_{n \geq k} a_n = \bigwedge_k \bigvee_{n \geq k} a_n \vartriangle a_{n+1}$$

we show both inequalities.
($\geq$) This inequality is quite trivial. Clearly $\bigwedge_k \bigvee_{n \geq k} a_n \geq \bigwedge_k \bigvee_{n \geq k} a_n \vartriangle a_{n+1}$ and whenever $p \leq \bigwedge_{n \geq k} a_n$ then $p \perp \bigvee_{n \geq k} a_n \vartriangle a_{n+1}$. Hence

$$\bigwedge_k \bigvee_{n \geq k} a_n - \bigvee_k \bigwedge_{n \geq k} a_n \;\geq\; \bigwedge_k \bigvee_{n \geq k} a_n \vartriangle a_{n+1}.$$

($\leq$) We show that

$$\forall k \in \omega \qquad \bigvee_{n \geq k} a_n \;\leq\; \bigvee_{n \geq k} a_n \vartriangle a_{n+1} \quad \vee \quad \bigvee_l \bigwedge_{n \geq l} a_n,$$

using the ‘density argument’ and show that:

$$\forall k \in \omega \quad \forall p \leq \bigvee_{n \geq k} a_n \quad \exists \mathbf{0} \neq r \leq p \quad \Big(r \leq \bigvee_{n \geq k} a_n \vartriangle a_{n+1} \quad \text{or} \quad r \leq \bigvee_l \bigwedge_{n \geq l} a_n\Big).$$

Let $p \neq \mathbf{0}$ then either there exists $m \geq k$ such that $p \wedge (a_m \vartriangle a_{m+1}) \neq \mathbf{0}$, which proves the density argument, or there is no such $m \geq k$. Then because $p \leq \bigvee_{n \geq k} a_n$ there is some n_0 such that $p \wedge a_{n_0} \neq \mathbf{0}$. Let $r = p \wedge a_{n_0}$. We know that $r \wedge a_{n_0} \vartriangle a_{n_0+1} = \mathbf{0}$ and so $r \leq a_{n_0+1}$, similarly $r \leq a_{n_0+k}$ for each $k \in \omega$. Finally we have $r \leq \bigwedge_{n \geq n_0} a_n$ and so $r \leq \bigvee_k \bigwedge_{n \geq k} a_n$.
(ii) The inequality ($\leq$) is clear. To show the remaining inequality we start with

$$\bigvee_{n \geq k} a_n \vee b_n \;=\; \bigvee_{n \geq k} a_n \vee \bigvee_{n \geq k} b_n.$$

We denote $c_k = \bigvee_{n \geq k} a_n$ and $d_k = \bigvee_{n \geq k} b_n$. Now

$$\overline{\lim}_n (a_n \vee b_n) \;=\; \bigwedge_{k \in \omega} c_k \vee d_k \;=\; \bigwedge_{j \in \omega} \bigwedge_{k \in \omega} c_k \vee d_j.$$

The latter equality hold because sequences c_k and d_k are descending and we are done:

$$\bigwedge_{j \in \omega} \Big(\bigwedge_{k \in \omega} c_k \vee d_j\Big) \;=\; \bigwedge_{j \in \omega} \overline{\lim}_n a_n \vee d_j \;=\; \overline{\lim}_n a_n \vee \overline{\lim}_n b_n.$$

□

3.36 Corollary. *A sequence* $\langle a_n : n \in \omega \rangle$ *in Boolean algebra converges if and only if* $\overline{\lim}_n (a_n \vartriangle a_{n+1}) \;=\; \mathbf{0}$.

The following assertion seems to be folklore.

3.37 Theorem. *Let $\mathbb{B}$ be a complete weakly distributive* ccc *Boolean algebra and let $A \subseteq \mathbb{B}$ be a subalgebra which completely generates $\mathbb{B}$. Then $\{\bigwedge_B X : X \in [A]^\omega\}$ is a dense subset of $\mathbb{B}$.*

Proof. Let us consider limits of all convergent sequences of elements from A, i.e.,

$$C = \{x \in \mathbb{B} : (\exists \langle a_n : n \in \omega\rangle \in A^\omega) \lim a_n = x\}.$$

Since $\mathbb{B}$ is weakly distributive and satisfies ccc, the set C is already closed under taking limits of convergent sequences, see for example [BGJ98]. This means that $C \supseteq A$ and C is closed under countable joins and meets, therefore $C = \mathbb{B}$.

Take $x \in \mathbb{B}^+$. There is a sequence $\langle a_n : n \in \omega\rangle \in A^\omega$ such that

$$\overline{\lim} a_n = \bigwedge_{k\in\omega} \bigvee_{n\geq k} a_n = x.$$

For given k, consider a disjoint sequence $\langle \overline{a}_{k,n} : k \leq n < \omega\rangle$, where $\overline{a}_{k,n} = a_n - \bigvee\{a_i : k \leq i < n\}$. Then $\overline{a}_{k,n} \in A$ and $\bigvee_{n\geq k} a_n = \bigvee_{n\geq k} \overline{a}_{k,n}$. Using weak distributivity of $\mathbb{B}$, there is a finite set $I_k \subseteq \omega \setminus k$ for every k, such that $y = \bigwedge_{k\in\omega} \bigvee_{n\in I_k} \overline{a}_{k,n} \neq \mathbf{0}$. Since $y \leq x$ and $\bigvee\{\overline{a}_{k,n} : n \in I_k\} \in A$, we are done. $\square$

4. SUBMEASURES ON BOOLEAN ALGEBRA

We consider only real-valued functions on an algebra $\mathbb{B}$. By a measure we usually mean a finitely additive measure.

4.1 Definition. A *submeasure* on a Boolean algebra $\mathbb{B}$ is a function $\mu : \mathbb{B} \to \mathbb{R}^+$ with the properties

(i) $\mu(\mathbf{0}) = 0$,

(ii) $\mu(a) \leq \mu(b)$ whenever $a \leq b$ (monotone),

(iii) $\mu(a \vee b) \leq \mu(a) + \mu(b)$ (subadditive).

A submeasure μ on $\mathbb{B}$ is

(iv) *exhaustive* if $\lim \mu(a_n) = 0$ for every sequence $\{a_n : n \in \omega\}$ of disjoint elements,

(v) *strictly positive* if $\mu(a) = 0$ only if $a = \mathbf{0}$,

(vi) a (finitely additive) *measure* if for any disjoint a and b, $\mu(a \vee b) = \mu(a) + \mu(b)$.

A submeasure μ on a σ-complete $\mathbb{B}$ is called

(vii) *Maharam* or *continuous* if for any descending sequence $\langle a_n : n \in \omega \rangle$ with $\bigwedge_{n\in\omega} a_n = \mathbf{0}$, $\lim_{n\to\infty} \mu(a_n) = 0$.

(viii) σ - *subadditive* if for any $\langle a_n : n \in \omega \rangle \subset \mathbb{B}$

$$\mu(\bigvee_{n\in\omega} a_n) \leq \sum_{n\in\omega} \mu(a_n).$$

The *norm* of μ is the number $\mu(\mathbf{1})$.

For any submeasure μ on $\mathbb{B}$, $Null(\mu) = \{a \in \mathbb{B} : \mu(a) = 0\}$ is an ideal on $\mathbb{B}$. If μ has a positive norm, i.e. $\mu \not\equiv 0$, then the quotient $\mathbb{B}/Null(\mu)$ carries a strictly positive submeasure $\overline{\mu}$, where $\overline{\mu}[a] = \mu(a)$ for each $a \in \mathbb{B}$.

We will see that a submeasure μ on a σ-complete Boolean algebra $\mathbb{B}$ is Maharam if and only if μ as a function from $\mathbb{B}$ to $\mathbb{R}$ is continuous with respect to the sequential topology on $\mathbb{B}$. Moreover, a measure is continuous if and only if it is σ-additive. We supply this definitions by following immediate conclusions.

At first it is clear that exhaustivity does not depend on zero elements.

4.2 Fact. *Let μ be a non-zero submeasure. Then $\overline{\mu}$ on $\mathbb{B}/Null(\mu)$ is an exhaustive submeasure if and only if μ is exhaustive.*

When focusing on strictly positive submeasures, exhaustivity limit the chain condition property.

4.3 Fact. *If an algebra $\mathbb{B}$ carries a strictly positive exhaustive submeasure, then $\mathbb{B}$ satisfy* ccc.

Submeasure is a generalisation of a measure. The stand-alone definition of a submeasure is way too much trivial. Notice that every Boolean algebra carries a submeasure: Simply put $\mu(\mathbf{0}) = 0$ and $\mu(a) = 1$ for any $a \neq \mathbf{0}$. Such submeasure is not exhaustive. Measure satisfies even more restrictive exhaustivity condition.

4.4 Definition. A submeasure $\mu : \mathbb{B} \to \mathbb{R}^+$ is called *uniformly exhaustive* if for each positive $\varepsilon > 0$ there is a $k \in \omega$ such that for every disjoint sequence $\langle a_n : n \in \omega \rangle \in \mathbb{B}^\omega$ $|\{n \in \omega : |\mu(a_n)| \geq \varepsilon\}| \leq k$.

4.5 Fact. *Any measure is a (uniformly) exhaustive submeasure.*

The definition of continuous submeasure was clearly motivated by the σ-additivity of a measure. Hence one cannot be surprised by the following fact.

4.6 Lemma. *Every Maharam submeasure is σ - subadditive.*

If a Boolean algebra $\mathbb{B}$ carries a strictly positive σ-additive measure, then $\mathbb{B}$ is weakly distributive and ccc. The same hold for Maharam algebras.

4.7 Lemma. *If a Boolean algebra $\mathbb{B}$ carries a strictly positive Maharam submeasure, then $\mathbb{B}$ satisfies* ccc *and is weakly distributive.*

We focus on different properties of submeasures further in this text. For more details on submeasures see also [Fre03].

4.8 Definition. *Pseudometric and metric induced by a submeasures.* Assume that μ is a submeasure on a Boolean algebra $\mathbb{B}$. Set $\rho_\mu(a, b) = \mu(a \triangle b)$ for any $a, b \in \mathbb{B}$. Then ρ_μ is generally a pseudometric on the set $\mathbb{B}$. Moreover, the Boolean operations $\wedge$, $\vee$, $-$ are uniformly continuous. We say that the pseudometric ρ_μ is *induced* by μ.

If μ is a strictly positive, then ρ_μ is a metric. In this case one can apply the well-known construction to obtain a metric completion of the space $(\mathbb{B}, \rho_\mu)$.

Denote by $(C, \overline{\rho}_\mu)$ the metric completion of $(\mathbb{B}, \rho_\mu)$. Then $\mathbb{B}$ is a topologically dense subspace of C. Since the Boolean operations on $\mathbb{B}$ are uniformly continuous, they can be uniquely extended to operations on C. The axioms of Boolean algebra must therefore hold for these operations and thus C is a Boolean algebra with $\mathbb{B}$ being its subalgebra. Similarly, a submeasure μ on $\mathbb{B}$ can be uniquely extended to a submeasure $\overline{\mu}$ on C. Moreover $\overline{\rho}_\mu = \rho_{\overline{\mu}}$, i.e. the metric on C is induced by the submeasure $\overline{\mu}$. This method is used to construct some useful super algebras of $\mathbb{B}$.

In general, an algebra constructed in such way does not have to be necessarily a strictly larger super algebra. To see that, consider a submeasure ν on $\mathbb{B}$ identically equal 1 on $\mathbb{B}^+$. The induced metric ρ_ν is a $\{0, 1\}$-discrete metric on $\mathbb{B}$ and thus $(\mathbb{B}, \rho_\mu)$ itself is a complete metric space.

The exhaustivity of submeasures plays an important role, as indicates the following fact.

4.9 Fact. *In the metric space $(\mathbb{B}, \rho_\mu)$, every decreasing sequence $\langle a_n : n \in \omega\rangle \in \mathbb{B}^\omega$ is a Cauchy sequence if and only if μ is an exhaustive submeasure.*

For the proof see (III.1.2).

4.10 Theorem. *Let $\mathbb{B}$ be a Boolean algebra and μ a strictly positive exhaustive submeasure on $\mathbb{B}$ with the induced metric ρ. The metric completion $\mathbb{C}$ of the metric space $(\mathbb{B}, \rho)$ has a structure of a Boolean algebra such that $\mathbb{B}$ is a subalgebra of $\mathbb{C}$ and there is a unique extension of μ to a strictly positive Maharam submeasure $\overline{\mu}$ on $\mathbb{C}$ moreover $\mathbb{C}$ is a complete Boolean algebra.*

For a detailed proof of the Theorem I.4.10, see [Fre03].

When μ is a strictly positive measure, then $\overline{\mu}$ is a strictly positive σ-additive measure.

4.11 Theorem. (W. Orlicz) *The space $(\mathbb{B}, \rho_\mu)$ is complete pseudometric space if and only if μ is σ-subadditive submeasure.*

We will deal further with (uniformly) exhaustive and Maharam submeasures in the following chapters. One of the crucial motivation for studying various properties of a submeasures is to find a border line between a submeasure and a measure. Further in the text we show that an algebra carrying uniformly exhaustive Maharam submeasure has to carry a measure.

Let us conclude this chapter with a similar result, which puts together fragmentation properties with measurability of a Boolean algebra

4.12 KELLEY'S FRAGMENTATION

4.13 Definition. *Kelley's fragmentation* of a partial ordered set $(P, \leq)$ is a fragmentation $P = \bigcup\{P_n : n \in \omega\}$, where each P_n has the following property:

for any positive integer k and $\langle x_1, x_2, \ldots, x_k \rangle \in P_n$ there is a subset $I \subseteq \{1, 2, \ldots, k\}$, $|I| \geq k/(n+1)$, such that $\langle x_i : i \in I \rangle$ is a centered family.

Notice that when $\langle P_n : n \in \omega \rangle$ is a Kelley's fragmentation of $(P, \leq)$, then putting $Q_{n^2+2n} = \bigcup_{i \leq n} \widehat{P_n}$, where $\widehat{P_i}$ is an upward closure of P_i, we obtain again a Kelley's fragmentation $\{Q_{n^2+2n} : n \in \omega\}$ of $(P, \leq)$.

4.14 Theorem. (J. L. Kelley, [Kel59]) *A Boolean algebra $\mathbb{B}$ carries a strictly positive measure if and only if there is a dense subset $P \subseteq \mathbb{B}^+$ having a Kelley's fragmentation.*

Quite recently, F. Galvin and K. Prikry [GP01] showed that Kelley's theorem holds true with a fragmentation where repetitions are avoided, i.e., $\langle x_1, x_2, \ldots, x_k \rangle$ can be replaced by $\{x_1, x_2, \ldots, x_k\}$. We will further investigate this topic in chapter IV.

II. Generic extensions

In this chapter we touch on extensions of transitive models of ZFC presented in the spirit of the Prague Set-theoretical seminar. We sketch generic extensions and we introduce adding new reals. We conclude the chapter with the fact that whenever a new real is added then all infinite groundmodel reals have almost disjoint refinement.

1. Basic concept

We deal with transitive models of set theory ZFC and their extensions. We admit models to be proper classes. The presentation is in the spirit of Vopěnka's Prague set-theoretical seminar [VH72].

1.1 Definition. A model N is called an *extension* of a model M if M and N have the same class of ordinal numbers and $M \subseteq N$. In this case, M is called a ground model and N its extension.

Note that if N is an extension of M, it means that N is a transitive class and a model of ZFC with the same ordinals as in M and the extension N 'knows' about M i.e. $X \cap M \in N$ for any $X \in N$.

1.2 Theorem. (P. Vopěnka, B. Balcar) [VB67] *Any extension is uniquely determined by new sets of ordinals. More precisely, when N_1 and N_2 are extensions of M, and*

$$\{x \in N_1 : x \subseteq On\} = \{y \in N_2 : y \subseteq On\},$$

then $N_1 = N_2$. Also, the inclusion

$$\{x \in N_1 : x \subseteq On\} \subseteq \{y \in N_2 : y \subseteq On\}$$

is equivalent to $N_1 \subseteq N_2$ for extensions N_1, N_2 of M.

For proof of the Theorem see [Jec02, Theorem 13.28]. Notice that the Axiom of choice is crucial for the Theorem.

1.3 Definition. An extension W of a ground model V is called *generic*, if there is a set $\sigma \in W$ such that

(i) $\sigma \subseteq V$, or equivalently, σ is a part of a set b from V, i.e., $\sigma \subseteq b \in V$ and

(ii) for any $\rho \in W$, $\rho \subseteq V$, there is a relation $r \in V$ (depending on ρ) such that $\rho = r''\sigma$.

Such a set σ is called a *generic set* for an extension W over V. This set determines all parts of the ground model, which belong as a set to the extension, through the relations from the ground model.

Note that for any $\rho \in W$, $\rho \subseteq V$, there is a set $b \in V$ with $\rho \subseteq b$. Using AC, there is a one-to-one mapping into some ordinal number $f : b \longrightarrow \alpha$, $f \in V$. Therefore $\rho_1 = f''\rho \subseteq \alpha$ and $\rho = f^{-1}{}''\rho_1$.

Trivially, a composition of relations from V is again a relation from V. Hence, if W is a generic extension, then there is a generic set σ for W which is a subset of some ordinal number. Also, any set $\sigma \subseteq V$, $\sigma \in W$, is a generic set for W over V if and only if every part of ordinals, which is a set in W, is determined by σ through a relation from V.

From this we have transitivity of generic extensions.

1.4 Proposition. *Let $V \subseteq W \subseteq Z$ be transitive models of ZFC. If W is a generic extension of V and Z is a generic extension of W, then Z is a generic extension of V.*

Proof. There is an ordinal α and $\sigma_1 \subseteq \alpha$, σ_1 generic for W over V and an ordinal β and $\sigma_2 \subseteq \beta$, σ_2 generic for Z over W. Let γ be an ordinal and $\rho \subseteq \gamma$, $\rho \in Z$ be arbitrary. Then there is some $\tau \in W$, $\tau \subseteq \beta \times \gamma$ such that $\tau''\sigma_2 = \rho$. We have $\tau \in W$, $\tau \subseteq V$, so there must be $r \in V$, $r \subseteq \alpha \times (\beta \times \gamma)$ with $\tau = r''\sigma_1$. Put $\bar{r} = \{\langle\langle x, y\rangle, z\rangle : \langle x, \langle y, z\rangle\rangle \in r\}$.

Then $\bar{r}''(\sigma_1 \times \sigma_2) = \rho$ and $\bar{r} \in V$. Also, $\sigma_1 \times \sigma_2 \subseteq V$, $\sigma_1 \times \sigma_2 \in Z$. Thus $\sigma_1 \times \sigma_2$ is a generic set for Z over V. $\square$

1.5 Fact. *Whenever we have a generic set $\sigma \subset a \in V$ then there is a uniquely determined extension $M \supset V$ containing σ; M is a model of ZFC and it is a minimal model extending V and containing σ.*

1.6 Definition. *Canonical form of generic set.* Let $(a, \leq)$ be a preorder in V. We say that a set $G \subset a$ is a *generic filter* over V if for any $f, g \in G$ there is $b \in G$ such that $b \leq f$ and $b \leq g$ and if $h \geq g$ then $h \in G$ (i.e. G is a filter) and for an arbitrary dense set $D \subseteq a$ from the groundmodel V the intersection $G \cap D$ is not empty.

Note that if preorder is atomless then no generic filter belongs to the ground model.

1.7 Definition. We call sets $\sigma, \rho \subseteq V$ *similar* if there are relations $r, s \in V$ such that $r''\sigma = \rho$ and $s''\rho = \sigma$.

The following theorem gives us a powerful tool: whenever we work with a generic set over a transitive model V of ZFC we can suppose that the generic set is in *a canonical form*, namely in the form of a generic ultrafilter over V on some complete atomless Boolean algebra $\mathbb{B} \in V$.

1.8 Theorem. (B. Balcar, P. Vopěnka [BŠ00]) *Let V be a transitive model of* ZF *and let $\sigma \subset a \in V$ be a generic over V for some generic extension. Then there is a preorder $\leq$ on a in V such that σ is a generic filter on $(a, \leq)$ over V.*

A *forcing notion* is a synonym for a partially ordered set $(P, \leq)$, a preordering is also acceptable. From the paragraph I.3.6 we know that algebra $RO(P)$ is the canonical complete Boolean algebra associated with P. $RO(P)$ together with a mapping $\phi : P \to RO(P)$ is a completition of P, i.e. $\phi[P]$ is dense in $RO(P)$. Now we can formulate the addition to theorem II.1.8 as follows:

$$G = \{a \in RO(P) : \exists x \in \sigma \quad \phi(x) \le a\}$$

is a generic filter on Boolean algebra $RO(P)$ over V which is similar to σ. Therefore G and σ determine the same generic extension.

Generic extension of a groundmodel V is hence given by some generic filter G over a Boolean algebra, we denote the extension as $V[G]$.

1.9 Remark. Note that a generic filter G on a complete Boolean algebra $\mathbb{B}$ over groundmodel V is exactly a maximal filter on $\mathbb{B}$, which is *set complete*, i.e. $\bigwedge a \in G$, for any $a \subset G$ such that $a \in V$.

When describing generic extension using Boolean algebras one can use 'Boolean valued names' instead of relations. Consider G a generic filter on a complete Boolean algebra $\mathbb{B}$. We know that sets from the extension $V[G]$ are reachable via relations from groundmodel V. Namely let $r''G \in V[G]$, where $r \in V$ is a relation. One can describe each such set via a groundmodel function $f : \mathrm{rng}(r) \to \mathbb{B}$, called a *Boolean name*:

$$\begin{aligned} f : \mathrm{rng}(r) &\longrightarrow \mathbb{B} \\ x &\longmapsto \bigvee\{b \in \mathbb{B} : \langle b, x\rangle \in r\}. \end{aligned}$$

It is now clear, that $f^{-1}{''}G = r''G$. The main advantage is that whenever we want to deal with a 'new' subset a of a groundmodel set A; i.e. $A \in V,\ a \in V[G]$ and $a \subset A$ in, we will use its Boolean name usually denoted by a dot:

$$\dot{a} : A \to \mathbb{B}, \quad \dot{a} \in V.$$

Let $\dot{a}$ be such a Boolean name, then its meaning is given by a generic object G as follows

$$\dot{a}_G = \{x \in A : \dot{a}(x) \in G\} \quad \subset A.$$

We saw that generic extensions are somewhat easier to describe than extensions in general and moreover have some nice properties. One of important properties of generic extensions is the fact that they are closed under 'intermodels'.

1.10 Theorem. *Let $V[G]$ be a generic extension of the groundmodel V, where G is a generic filter on a complete Boolean algebra $\mathbb{B}$. Whenever N is an extension of V, such that $V \subset N \subset V[G]$, then N is a generic extension of V determined by some complete subalgebra $\mathbb{C} \subset \mathbb{B}$, $\mathbb{C} \in V$ and $N = V[\mathbb{C} \cap G]$.*

Proof. The class N is a model of ZFC. Therefore $\mathcal{P}^N(\mathbb{B}) \in N$ and by the Axiom of Choice there is an ordinal α and a relation $\rho \subset \alpha \times \mathbb{B},\ \rho \in N$ such that for any $\tau \subset \mathbb{B}, \tau \in N,\ \tau \neq \emptyset$ there is $\xi < \alpha$ such that $\rho''(\xi) = \tau$. Now $\rho \subset V,\ \rho \in V[G]$ hence there is a mapping $f : \alpha \times \mathbb{B} \to \mathbb{B},\ f \in V$ such that $f^{-1}[G] = \rho$.

Put $a = \mathrm{rng}(f)$. Then $a \cap G$ and ρ are similar. Let $\mathbb{C}$ be a complete subalgebra of $\mathbb{B}$ completely generated in V by the set a.

Now working in N we are able by recursion to extend the set $a \cap G \in N$ uniquely to get a set complete ultrafilter on $\mathbb{C}$, and verify that $G_1 = \mathbb{C} \cap G$.

$V[G_1] = N$ because any set $\sigma \in N$, $\sigma \subset V$ is similar to $\rho(\xi)$ for some $\xi \subset \alpha$. □

On the other hand every generic extension can be decomposed into 'two step' extension:

Let $\mathbb{C}$ be a complete subalgebra of $\mathbb{B}$, $\mathbb{C}, \mathbb{B} \in V$. Let G be a generic filter on $\mathbb{C}$ over V. Then

$$\hat{G} = \{b \in \mathbb{B} : (\exists c \in G)\ c \le b\}$$

is filter on $\mathbb{B}$ in $V[G]$.

$\hat{I}$ is a dual ideal, i.e.

$$\hat{I} = \{b \in \mathbb{B} : (\exists c \in \mathbb{C})\ b \le c\ \&\ -c \in G\}$$

Note that $\hat{G}$, $\hat{I}$, respectively are closed under meets, joins, respectively over arbitrary set from V:

To prove this consider projection $p : \mathbb{B} \to \mathbb{C}$ given by $p(b) = \bigvee\{c \in \mathbb{C} : c \le b\}$. Now if $X \subset \hat{G}$, $X \in V$, then $p[X] \in V$ and $p[X] \subset G$ and it follows from genericity of G that $\bigwedge p[X] \in G$, hence $\bigwedge X \in \hat{G}$.

We can show that $\mathbb{B}/\hat{I}$ is a complete Boolean algebra in $V[G]$. This is equivalent to the fact that preordering $\le_G$ on $\mathbb{B}$ defined by

$$b_1 \le_G b_2 \quad \text{if and only if} \quad b_1 - b_2 \in \hat{I}$$

is complete in $V[G]$.

1.11 Claim. $\mathbb{B}/\hat{I}$ *is a complete Boolean algebra in* $V[G]$.

Proof. Let $f : \mathbb{B} \to \mathbb{C}$ be a Boolean $\mathbb{C}$-name for a subset of $\mathbb{B}$, $f \in V$. Put

$$c = \bigvee_{b \in \mathbb{B}} f(b) \wedge b, \quad \text{then } c \in \mathbb{B}.$$

We have to check that c is the supremum in $(\mathbb{B}, \le_G)$ of the set $\sigma = \{b : f(b) \in G\}$, or equivalently $[c] = \bigvee\{[b] : b \in \sigma\}$ in $\mathbb{B}/\hat{I}$.

At first, for $b \in \sigma$, i.e. $f(b) \in G$ we have that $b \le_G f(b) \wedge b$ and $f(b) \wedge b \le_G b$, so $[b] = [f(b) \wedge b]$ for all $b \in \sigma$. Since $f(b) \wedge b \le c$, c is an upper bound for σ.

Now assume $d \in \mathbb{B}$ is an upper bound for σ, then whenever $f(b) \in G$ then $f(b) \wedge b \le_G d$ and so $(f(b) \wedge b) - d \in \hat{I}$. On the other hand if $f(b) \notin G$ then $f(b) \wedge b \in \hat{I}$ and so $(f(b) \wedge b) - d \in \hat{I}$.

Finally, $\bigvee_{b \in \mathbb{B}}(f(b) \wedge b - d) \in \hat{I}$ and so $c = \bigvee_{b \in \mathbb{B}}(f(b) \wedge b \le_G d$. □

Let $\mathbb{C}$ be a complete subalgebra of $\mathbb{B}$. Now assume that $F \subset \mathbb{B}$ is a generic filter on $\mathbb{B}$ over V.

Consider $G = F \cap \mathbb{C}$, G is a generic filter on $\mathbb{C}$ over V.

1.12 Claim. F *is a generic filter on* $(\mathbb{B}, \leq_G)$ *over* $V[G]$*, equivalently* $F/G = \{[b] : b \in F\}$ *is a generic filter on* $\mathbb{B}/\hat{I}$ *over* $V[G]$*, i.e.*

$$V[F] = V[G][F/G].$$

Proof. Certainly F is a filter on $(\mathbb{B}, \leq_G)$. For a genericity of F suffices to show that for any Boolean $\mathbb{C}$-name $f : \mathbb{B} \to \mathbb{C}$ such that if $\sigma = \{b : f(b) \in G\}$ and $\sigma \cap F = \emptyset$, then $\sup(\sigma) = \bigvee_{b \in \mathbb{B}} f(b) \wedge b = c \notin F$.

But if $b \notin \sigma$ then $f(b) \notin F$ and if $b \in \sigma$ then $b \notin F$. So $c \notin F$ because F is a set complete filter. □

1.13 ADDING NEW REALS

Now we describe what we mean by a name for a *real* (here we identify reals with subsets of ω). Name $\dot{f}$ for a real is a Boolean valued function $\dot{f} : \omega \to \mathbb{B}$. One can see it as a generalisation of characteristic functions. The set of all names i.e. $\mathbb{B}^\omega$ is also a Boolean algebra with Boolean operations defined coordinatewise.

Let $G \subset \mathbb{B}$ be a generic filter. Then the meaning of a name $\dot{f}$ is the following subset of ω

$$\dot{f}_G = \{n \in \omega \ : \ \dot{f}(n) \in G\}.$$

If $\mathbb{B}$ is complete, then $\mathbb{B}^\omega$ is also complete Boolean algebra and these are the names for all reals in the generic extension, namely

$$\big(\mathcal{P}(\omega)\big)^{V[G]} = \{f_G \ : \ f \in \mathbb{B}^\omega\}.$$

We say that a forcing notion $(P, \leq)$ adds a new real if for any generic filter on $(P, \leq)$

$$(\mathcal{P}(\omega))^V \neq (\mathcal{P}(\omega))^{V[G]}.$$

Equivalently, $(P, \leq)$ adds a new real if there is a Boolean name $r : \omega \to RO(P)$ such that for each $p \in P$ there is $n \in \omega$ such that

$$p \parallel r(n) \quad \& \quad p \parallel -r(n).$$

We summarise here a classification of a newly added reals, at first we show an example of Cohen real, i.e. the real added via the Cohen forcing.

1.14 EXAMPLE. We say that a forcing P adds a *Cohen real* if the Cantor algebra $\mathcal{A}$ can be regularly embedded into $RO(P)$.

It is equivalent with existence of a Boolean name $c : \omega \to RO(P)$ with the property that for every set D dense in the partial order $(\bigcup_{n \in \omega} {}^n\{0,1\}, \supseteq)$

$$\bigvee_{f \in D} \bigwedge_{i \in dom(f)} f(i)c(i) = \mathbf{1},$$

where Boolean operations are in $RO(P)$ and $0c(i) = c(i)$, $1c(i) = -c(i)$. △

Let us remind some well-known notions concerning the interrelationship of functions and subsets of ω in the extension and the ground model.

1.15 Definition. Let M denote an extension of V.

(i) $X \subseteq \omega$ in the extension is said to be an *independent* (or *splitting*) *real* over V if for all $Y \in [\omega]^\omega \cap V$ both $X \cap Y$ and $Y - X$ are infinite.

(ii) A function $f \in M$, $f \in \omega^\omega$, is a *dominating real* over V if for all $g \in \omega^\omega \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$.

(iii) M is an ${}^\omega\omega$*-bounding extension* of V if every $f \in M$, $f \in \omega^\omega$ is bounded by a $g \in \omega^\omega \cap V$, i.e. $f(n) \leq g(n)$ for any n.

(iv) A function $h \in \omega^\omega$ in the extension is said to be an *unbounded real* over V if for all $f \in \omega^\omega \cap V$ the set $\{n \in \omega : h(n) > f(n)\}$ is infinite.

1.16 Definition. We say that the Cantor algebra $\mathcal{A}$ is *almost regularly embedded* into a Boolean algebra $\mathbb{B}$ if there is $\mathcal{A}'$, a subalgebra of $\mathbb{B}$, so that

(i) $\mathcal{A}'$ is isomorphic to the Cantor algebra $\mathcal{A}$, and

(ii) there is a set $\{x_n : n \in \omega\}$ of generators of $\mathcal{A}'$ such that for any infinite subset X of ω, $\bigvee_{\mathbb{B}}\{x_n : n \in X\} = \mathbf{1}$ and $\bigwedge_{\mathbb{B}}\{x_n : n \in X\} = \mathbf{0}$.

1.17 Definition. We call a sequence $\langle x_n : n \in \omega\rangle \in \mathbb{B}^\omega$ *splitting* if for every $b \in \mathbb{B}^+$ $x_n \wedge b \neq \mathbf{0}$ and $b - x_n \neq \mathbf{0}$ for all but finitely many n.

The following theorem summarize several conditions for adding independent reals.

1.18 Theorem. *For any Boolean algebra $\mathbb{B}$ the following are equivalent.*

(i) Cantor algebra is almost regularly embedded in algebra $\mathbb{B}$,

(ii) there is a sequence $\{x_n : n \in \omega\}$ in $\mathbb{B}$ such that for any infinite subset X of ω, $\bigvee_B\{x_n : n \in X\} = \mathbf{1}$ and $\bigwedge_B\{x_n : n \in X\} = \mathbf{0}$,

(iii) there is a splitting sequence in $\mathbb{B}$,

(iv) there is a splitting sequence in $\mathbb{B}$ which form an independent family cf. I.3.14,

(v) there is $f \in \mathbb{B}^\omega$ such that for every generic G on $\mathbb{B}$ f_G is an splitting real in $V[G]$.

Proof. *(i)* $\to$ *(ii)* is clear from the definitions.
(ii) $\to$ *(iii)* Arguing by contradiction, we establish that the sequence $\langle x_n : n \in \omega\rangle$ satisfying (ii) is splitting: let $b \in \mathbb{B}^+$ be an element not split by the sequence, then one of the sets $\{n \in \omega : x_n \leq -b\}$ and $\{n \in \omega : x_n \geq b\}$ is infinite, contradicting *(ii)*.
(iii) $\to$ *(iv)* Let $\{x_n : n \in \omega\}$ be a splitting sequence. We define $y_0 = x_0$. Since $\{x_n : n \in \omega\}$ is splitting, there exists $n_1 \in \omega$ such that $\forall m \geq n_1$ x_m splits ϵx_0, for $\epsilon \in \{-1, 1\}$ and we put $y_1 = x_{n_1}$. By induction we construct a splitting sequence $\langle y_n : n \in \omega\rangle$ since it is subsequence of $\langle x_n : n \in \omega\rangle$. By the construction $\bigwedge_{n \in K} \epsilon y_n \neq \mathbf{0}$ for any finite K so the sequence $\langle y_n : n \in \omega\rangle$ is independent.

(iv) $\rightarrow$ *(i)* We define $\mathcal{A}$ as a Boolean algebra generated by the splitting sequence $\langle y_n : n \in \omega \rangle$. This algebra is countable and atomless, since atoms can not be split so is isomorphic to Cantor algebra.
(iii) $\rightarrow$ *(v)* The function $f(n) = x_n$, where $\langle x_n : n \in \omega \rangle$ is a splitting sequence, is an independent real for any generic G on $\mathbb{B}$. Fix some G and assume that there is an infinite set $a \subseteq \omega$ in V that is not split by f_G. We can assume that the set $a \cap f_G$ is finite and so $a \setminus f_g$ is cofinite, hence both belong to groundmodel V. Now the element $v = \bigwedge_{n \in a \cap f_G} x_n \wedge \bigwedge_{n \in a \setminus f_G} -x_n$ belongs to G and so $v \in \mathbb{B}^+$ but is not split by any element of $\langle x_n : n \in \omega \rangle$.
(v) $\rightarrow$ *(iii)* We show that the sequence $\langle f(n) : n \in \omega \rangle$ is splitting, where $f \in \mathbb{B}^\omega$ is an independent real for any generic G on $\mathbb{B}$. Let $b \in \mathbb{B}^+$ be an element that is not split. It means that one of the following sets $b_0 = \{n \in \omega : -b \geq f(n)\}$, $b_1 = \{n \in \omega : b \leq f(n)\}$ is infinite. We chose G so that it contains b. Then $b_0 \cap f_G = \emptyset$ or $b_1 \cap f_G = b_1$ contradicting the fact that f is an independent real, since both b_0 and b_1 belong to V. $\square$

The easy reformulation of the condition *(ii)* of the theorem II.1.18 allows us to formulate combinatorial condition to forcing or Boolean algebra to not to add independent reals.

1.19 Corollary. *Complete Boolean algebra $\mathbb{B}$ as a forcing notion does not add independent reals if*

$$\forall \langle a_n : n \in \omega \rangle \in \mathbb{B}^\omega \quad \exists X \in [\omega]^\omega \bigwedge_{Y \in [X]^\omega} \left(\overline{\lim}_{n \in Y} a_n - \underline{\lim}_{n \in Y} a_n \right) = \mathbf{0}.$$

In chapter V we show the following topological condition for not adding independent reals.

1.20 Theorem. *Let $\mathbb{B}$ be a complete* ccc *Boolean algebra. Then $\mathbb{B}$ does not add independent reals if and only if the sequential topology on $\mathbb{B}$ is countably (sequentially) compact* (I.2.15).

1.21 EXAMPLE. (i) Cohen forcing adds Cohen real, which is a splitting real,

(ii) Random forcing is ${}^\omega\omega$-bounding forcing which adds independent reals,

(iii) Sacks forcing is ${}^\omega\omega$-bounding which is not adding splitting reals and is $(2^\omega)^+ -$ cc,

(iv) Miller forcing is $(2^\omega)^+ -$ cc and adds unbounded reals and no splitting real hence it adds no dominating real [Mil84],

(v) Laver forcing adds dominating real and no Cohen real,

(vi) Hechler forcing (σ-centered) adds a dominating real and also Cohen real,

(vii) Mathias forcing adds a dominating real and no Cohen real.

$\triangle$

2. ALMOST DISJOINT REFINEMENT

At first we should mention that we will mainly focus on the almost disjoint families on ω rather then on an arbitrary set. Let us remind definitions.

2.1 Definition. The family $\mathcal{A} \subset [\omega]^\omega$ is an *almost disjoint family* or simply *AD* family if any two $A, B \in \mathcal{A}$, $A \neq B$ are almost disjoint, i.e $A \cap B$ is finite. We will indicate this fact by $A \cap B =^* \emptyset$.

If the family $\mathcal{A}$ is maximal with respect to this property, we will talk about a *maximal almost disjoint family* or *MAD* family.

In the following $A \subset^* B$ will denote the fact that $A \setminus B$ is finite.

Please note that in fact AD family is any disjoint family in the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ and MAD family is nothing else than a decomposition of unity in the same algebra.

We say that a family $\mathcal{S} \subset [\omega]^\omega$ has an *almost disjoint refinement (ADR)* if there is an almost disjoint family $\{A_X : X \in \mathcal{S}\}$ such that $A_X \in [X]^\omega$ for every $X \in \mathcal{S}$. Instead of this 'indexed' refinement we use the following simpler definition and we show that these two are equivalent. We will use any of this equivalents without further mentioning.

2.2 Definition. Family $\mathcal{S} \subset [\omega]^\omega$ has an *ADR* if there is an almost disjoint family $\mathcal{A}$ such that for any $X \in \mathcal{S}$ there is $A \in \mathcal{A}$ such that $A \subset^* X$

2.3 Fact. *For a family $\mathcal{S} \subset [\omega]^\omega$ the following are equivalent:*

(i) *There is an almost disjoint family $\{A_X : X \in \mathcal{S}\}$ such that $A_X \in [X]^\omega$ for every $X \in \mathcal{S}$.*

(ii) *a family S has ADR, i.e. there is an almost disjoint family $\mathcal{A}$ such that for any $X \in \mathcal{S}$ there is $A \in \mathcal{A}$ such that $A \subset^* X$,*

(iii) *there is an almost disjoint family $\mathcal{A}$ such that for any $X \in \mathcal{S}$*

$$|\{A \in \mathcal{A} : |X \cap A| = \omega\}| = 2^\omega.$$

Proof. *(i)* $\to$ *(ii)* This implication is trivial since the disjoint family from *(i)* satisfies also *(ii)*.

(ii) $\to$ *(iii)* Let $\mathcal{A}$ be an almost disjoint family as in *(ii)*. In $[\omega]^\omega$ there is an maximal almost disjoint family $\langle B_i^A : i \in 2^\omega\rangle$ of a size 2^ω below any $A \in \mathcal{A}$. Hence $\langle B_i^A : i \in 2^\omega,\ A \in \mathcal{A}\rangle$ satisfies *(iii)*.

(iii) $\to$ *(i)* First enumerate $\mathcal{S} = \{X_\alpha : \alpha \in 2^\omega\}$ and for any $X \in \mathcal{S}$ denote $\mathcal{A}_X = \{A \in \mathcal{A} : |X \cap A| = \omega\}$, $|\mathcal{A}_X| = 2^\omega$. Now proceed by induction and for each $X_\alpha \in \mathcal{S}$ choose some $A_\alpha \in \mathcal{A}_{X_\alpha} - \bigcup\{A_\beta : \beta < \alpha\}$. Clearly the family $\{A_\alpha \cap X_\alpha : \alpha \in 2^\omega\}$ gives an almost disjoint refinement for $\mathcal{S}$. $\square$

The strongest property concerning the existence of an almost disjoint refinement for some family of subsets of ω is the following refinement principle by countable set RPC (see [BS89a] for details). The RPC is formulated using the tall ideal, we will see later that AD families and tall ideals are closely related.

2.4 Definition. An ideal $\mathcal{I}$ on ω is a *tall ideal* if

$$\forall X \in [\omega]^\omega \quad \exists Y \in [X]^\omega \quad Y \in \mathcal{I}.$$

2.5 Definition. *Refinement principle by countable sets* (RPC):

For every tall ideal $\mathcal{I}$, the set $\mathcal{P}(\omega) - \mathcal{I}$ has an almost disjoint refinement.

2.6 Remark. This principle follows from Martin's Axiom and holds true in many generic extensions. Till now it is not known whether RPC is a ZFC statement.

Let us recall an useful cardinal invariant $\mathfrak{a}$:

2.7 Definition.

$$\mathfrak{a} = \min\{|\mathcal{A}| \ : \ \mathcal{A} \text{ is an infinite MAD family }\}.$$

2.8 Proposition. *If* $\mathfrak{a} = 2^\omega$ *then RPC holds true.*

Proof. Let $\mathcal{I}$ be a tall ideal. Enumerate $\mathcal{I}^+ = \{X_\alpha : \alpha < 2^\omega\}$. We pick up an almost disjoint refinement for $\mathcal{I}^+$. Since $\mathcal{I}$ is tall ideal, one can find some $I_0 \subset X_0$, $I_0 \in \mathcal{I}$. Suppose we already picked $\{I_\alpha \ : \ \alpha < \gamma\}$ for some $\gamma < 2^\omega$. The family

$$\mathcal{A} \ = \ \{X_\gamma \cap I_\alpha \ : \ \alpha < \gamma\}$$

is not a MAD family, since $\mathfrak{a} = 2^\omega$ hence there is some infinite $X \subset \omega$ almost disjoint with $\mathcal{A}$. The ideal $\mathcal{I}$ is tall, hence there is some $I_\gamma \subset X$. The family $\{I_\alpha \ : \ \alpha < 2^\omega\}$ is a desired almost disjoint refinement for $\mathcal{I}^+$. □

As we already mentioned, there is a close relation between an AD family and a tall ideal. Let $\mathcal{A}$ be an AD family. Then one can define an ideal

$$\mathcal{I}_\mathcal{A} \ = \ \{X \subset \omega \ : \ |A \in \mathcal{A} \ : \ |X \cap A| = \omega| < \omega\}$$

2.9 Fact. *Let* $\mathcal{A}$ *be an AD family. Then* $\mathcal{I}_\mathcal{A}$ *is a tall ideal.* $\mathcal{I}_\mathcal{A}$ *is nontrivial ideal whenever* $\mathcal{A}$ *is an infinite AD family.* □

We denote $\mathcal{I}^+$ all *positive sets* with respect to an ideal $\mathcal{I}$, i.e. $\mathcal{I}^+ = \mathcal{P}(\omega) - \mathcal{I}$.

In the following we will use the main techniques that are used in [BS89a], let us restate them here. We start with the introduction of the *Base Tree* of $\mathcal{P}(\omega)/\mathrm{fin}$. Base Tree is a special kind of a dense subset of $\mathcal{P}(\omega)/\mathrm{fin}$:

2.10 Theorem. (B. Balcar, J. Pelant and P. Simon [BPS80]) *There is a* Base Tree $(T, \supset^*)$ *for* $[\omega]^\omega$*, i.e.*

(i) $(T, \supseteq^*) \subset [\omega]^\omega$ *is a tree,*

(ii) let $B \in T$ *then the family of immediate successors of* B *in* T *is a maximal almost disjoint family below* B *of a full size; i.e.* 2^ω.

(iii) for each $A \in [\omega]^\omega$ *there is* $B \in T$ *such that* $B \subset^* A$,

(iv) the hight of T *is* $\mathfrak{h}$*; (for definition of* $\mathfrak{h}$ *see I.3.3).*

One reason for which we are interested in dense subfamilies of $\mathcal{P}(\omega)$ is because one can reconstruct the whole $\mathcal{P}(\omega)$ from its dense part.

2.11 Proposition. *Let* V *and* M *be models of ZFC and let* $H \subset V$ *be a dense set in* $[\omega]^\omega \cap V$. *If* $H \subset M$ *then*

$$\mathcal{P}^V(\omega) \subset \mathcal{P}^M(\omega),$$

moreover if H *is dense in* $[\omega]^\omega \cap M$ *in* M*, then* $\mathcal{P}^V(\omega) = \mathcal{P}^M(\omega)$.

Proof. We identify ω with the countable set $A = \{\, ^n\{0,1\} : n \in \omega\}$. One can also identify $\mathcal{P}(\omega)$ with the set of characteristic functions 2^ω. Let H be a dense set of $([A]^\omega)^V$.

Now let $X \in \mathcal{P}^V(\omega)$, we take its characteristic function $\chi : \omega \to \{0,1\}$. χ determines infinite subset $X_\chi = \{\chi \restriction n : n \in \omega\} \subset A$. H is dense, hence there is infinite $h \in H$ such that $h \subset X_\chi$. Since $h \in M$, one can easily reconstruct $\chi \in M$ as

$$\chi = \bigcup h.$$

So in M there is a characteristic function of the set X and we are done. The moreover part is now obvious. $\square$

We will also use following general combinatorial facts, see [BV80]. For proofs of theorems II.2.10 and II.2.12 see also [BS89a], or [BŠ00].

2.12 Theorem. (B. Balcar, P. Vojtáš [BV80])

(i) *Let* $\langle I_n : n \in \omega\rangle$ *be a decomposition of* ω *into finite sets. Then the family* $\mathcal{S} = \{X \in [\omega]^\omega : \limsup |X \cap I_n| = \infty\}$ *has ADR.*

(ii) *Let* $\langle S_n \in [\omega]^\omega\rangle$ *be a disjoint family. Then the family* $\mathcal{S} = \{X \in [\omega]^\omega : \{n : |X \cap S_n| = \omega\}$ is infinite$\}$ *has ADR.*

2.13 Definition. AD family $\mathcal{A}$ is called *completely separable* if for each $X \in \mathcal{I}^+_{\mathcal{A}}$ there is some $A \in \mathcal{A}$ such that $A \subset^* X$.

The existence of completely separable AD in ZFC can be found in [BS89a]. The existence of completely separable MAD in ZFC is still open. One reformulation of RPC (which appeared in [BS89a]) uses refinement by completely separable AD and a special kind of Base Tree:

2.14 Theorem. *The following are equivalent:*

(i) *RPC*

(ii) *for each infinite MAD family* $\mathcal{A}$, $\mathcal{I}^+_{\mathcal{A}}$ *has ADR by a completely separable AD family,*

(iii) *there is a Base Tree* T *such that any MAD family* $\mathcal{A} \subset T$ *is completely separable.*

2.15 ALMOST DISJOINT REFINEMENT OF GROUNDMODEL REALS

Lajos Soukup pose on the Set Theory Workshop (November 17 - November 19, 2005, Vienna) the following question:

Does the family $([\omega]^\omega)^V$ has an almost disjoint refinement in generic extension, which adds a new real?

We shall consider a little more general situation, when we take into account arbitrary ZFC extension M of V. Clearly, to have a chance for the refinement, the extension M have to add a new real, i.e.

$$(\mathcal{P}(\omega))^V \subsetneq (\mathcal{P}(\omega))^M.$$

So from now on we will consider, that the extension M adds new reals and we ask about the existence (of course in M) of the mapping

$$\varphi : ([\omega]^\omega)^V \to [\omega]^\omega$$

such that for each $x \neq y,\ x, y \in ([\omega]^\omega)^V$

(i) $\varphi(x) \subset x$ and

(ii) $\varphi(x) \cap \varphi(y) =^* \emptyset$.

The following theorem gives an affirmative answer to L. Soukup's question.

2.16 Theorem. *In any ZFC extension M of V adding a new real there is an almost disjoint refinement for $([\omega]^\omega)^V$.*

Proof. The proof of the theorem uses Base Tree techniques and the following observation:

2.17 Lemma. *There is an infinite $\sigma \subset \omega$, $\sigma \in M$ such that for each $X \in [\omega]^\omega \cap V$ there is $Y \in [X]^\omega \cap V$ which miss σ, i.e. $Y \cap \sigma = \emptyset$.*

Proof. Instead of ω one can consider a countable set

$$A = \bigcup \{ {}^n\{0, 1\} \ : \ n \in \omega\}.$$

Let χ be a characteristic function of a new real. Define $\sigma = \{\chi \restriction n : n \in \omega\}$, note that σ is set of compatible functions. Then σ has desired properties:

Let $X \subset A$, $X \in V$ be infinite. From Ramsey theorem it follows that X contains either infinite subset Y of compatible functions or it contains infinite subset Y of pairwise disjoint functions. In the latter case clearly $|Y \cap \sigma| \leq 1$. Now suppose that Y is set of compatible functions and $Y \cap \sigma$ is infinite. Then $\bigcup Y = \chi$, but $\bigcup Y \in V$ and $\chi \notin V$, a contradiction. Hence $Y \cap \sigma =^* \emptyset$ and we are done. □

2.18 Corollary. *In any ZFC extension M of V adding a new real there is $\sigma \subset \omega$, $\sigma \in M$ such that σ does not contain infinite groundmodel set.*

2.19 Corollary. *In any ZFC extension M of V adding a new real $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is not a regular subalgebra of $\mathcal{P}(\omega)/\mathrm{fin}$, i.e. there is a MAD family in $(\mathcal{P}(\omega)/\mathrm{fin})^V$ which is no longer MAD in M; cf.* (I.3.18)*(iv)*.

The following lemma concludes the PROOF OF THE THEOREM II.2.16.

2.20 Lemma. *Whenever an extension* M *of* V *destroys some MAD, then the family* $([\omega]^{\omega})^V$ *has an ADR in* M.

Proof. Let $(T, \supseteq^*) \subset [\omega]^{\omega}$ be a Base Tree for $[\omega]^{\omega} \cap V$, $T \in V$; see (II.2.10) and let $\mathcal{A} \in V$ be a destructible MAD family with its 'destructor' $\sigma \in [\omega]^{\omega}$, $\sigma \in M$. We denote T_{α} the α-level of the tree T.

By recursion we construct a Base Tree $T^* \in V$ for $[\omega]^{\omega} \cap V$. We start with the root $t \in T_0$ of the tree T an let it untouched. The set t is an infinite subset of ω so there is a bijection $b : t \to \omega$ in V. So $b^{-1}[\mathcal{A}]$ is a destructible MAD family on t with the destructor $b^{-1}(\sigma) \in M$. There is a common refinement of the MAD families $b^{-1}[\mathcal{A}]$ and T_1. This common refinement will be next level T_1^* of the tree T^*.

Let T_{α}^* level be constructed. For every $t \in T_{\alpha}^*$ we pick some bijection $b_t : t \to \omega$. The $T_{\alpha+1}^*$ level will be the common refinement of $T_{\alpha+1}$ and maximal almost disjoint family

$$\{b_t^{-1}[A] \;:\; t \in T_{\alpha}^*,\; A \in \mathcal{A}\}.$$

On the limit stages $\gamma < \mathfrak{h}$. We take T_{γ}^* common refinement of the T_{α}^* for each $\alpha \leq \gamma$. Such a refinement exists; cf. definition of $\mathfrak{h}$ (I.3.3).

The just constructed tree $T^* \in V$ is clearly a Base Tree for $([\omega]^{\omega})^V$. Moreover for each $t \in T^*$ we have a subset $b_t^{-1}(\sigma) \in M$. Note that each $b_t^{-1}(\sigma)$ is almost disjoint with every $s \in T_{\beta}^*$ for each $\beta > \alpha$. Hence for for each $t \neq s$, $b_s^{-1}(\sigma)$ is almost disjoint with $b_t^{-1}(\sigma)$ and

$$\{b_t^{-1}(\sigma) \;:\; t \in T^*\}$$

is an almost disjoint refinement of $([\omega]^{\omega})^V$, and we are done. □

□

Now we define a family, that seems to be the crucial one, when talking about adding new reals and almost disjoint refinement. In a special case when there are no independent reals in the extension this family form an ideal.

2.21 Definition. Let H be the family of subsets of ω which do not contain infinite sets from groundmodel

$$H = \{\sigma \in M : \sigma \subset \omega \quad \& \quad \neg \exists\, a \in ([\omega]^{\omega})^V \; a \subset \sigma\}.$$

2.22 Theorem. *The following hold in* M.

(i) H *is an open dense subset of* $([\omega]^{\omega}, \subseteq)$.

(ii) H *is an ideal if and only if* M *does not add independent reals.*

Proof. First note that if M adds a new real $\chi \subset \omega$, $\chi \notin V$, then H contains infinite set. It is easy to see, that σ given by lemma II.2.17 is an infinite set belonging to H.

To prove *(i)*, let $A \in ([\omega]^{\omega})^V$, then there is a bijection f in V between ω and A and there is a subset $\sigma \subset \omega$ in M which does not contain infinite groundmodel set so $f[\sigma] \in H$ is subset of A. Generally if $A \in [\omega]^\omega$, then $A \in H$ or there is some $A' \in ([\omega]^\omega)^V$, $A' \subset A$ and we can use the same reasoning.

(ii) Suppose that M adds an independent real σ. Clearly $\sigma \in H$ and $-\sigma \in H$, hence H is not an ideal.

On the other hand if H is not an ideal, then there are $a, b \in H$ such that there is $X \in ([\omega]^\omega)^V$ and $X \subset a \cup b$. Again we can identify X and ω in groundmodel and then $X \cap a$ is an independent real in M. □

2.23 Remark. We know that $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is not a regular subalgebra of the algebra $\mathcal{P}(\omega)/\mathrm{fin}$. From the paragraph I.3.17 we know that there is some maximal ideal $\mathcal{J}$ so that the canonical homomorphism

$$i : (\mathcal{P}(\omega)/\mathrm{fin})^V \longrightarrow (\mathcal{P}(\omega)/\mathrm{fin})/\mathcal{J}$$

is a regular embedding; cf. I.3.21. In case that H is an ideal, then it is the greatest ideal with this property and vice versa: If there exist the greatest ideal not intersecting with $(\mathcal{P}(\omega)/\mathrm{fin})^V$ then M does not add independent reals; cf. I.3.22.

EXAMPLES AND SPECIAL CASES

We know that when adding new reals, there is an almost disjoint refinement for groundmodel reals. In the following we will discuss some examples and possibly show a direct proof for this fact for the sake of clarity. Please, keep in mind that in this section extension M of the groundmodel V is always supposed to add new reals.

We start with a trivial one, when continuum is collapsed.

2.24 Proposition. *If $|(\mathcal{P}(\omega)) \cap V| < 2^\omega$ in M then the family $([\omega]^\omega)^V$ has ADR.*

Proof. This is direct application of well known Balcar-Vojtáš Theorem, see [Kop89, Theorem 3.14]. □

When M adds an unbounded real than the existence of ADR can be seen via the following shortcut.

2.25 Proposition. *If there is an unbounded real in M then the family $([\omega]^\omega)^V$ has ADR.*

Proof. Let $\sigma : \omega \to \omega$ be an unbounded real in M. We can assume that σ is strictly increasing and $\sigma(0) = 0$.

Put $I_n = [\sigma(n), \sigma(n+1))$. $\{I_n : n \in \omega\}$ is a partition of ω into intervals. It is sufficient to show that $[\omega]^\omega \cap V$ satisfies condition *(i)* from Theorem II.2.12.

Suppose not, i.e. there is $X \in ([\omega]^\omega)^V$ and $0 < k \in \omega$ such that $|X \cap [\sigma(n), \sigma(n+1))| < k$ for all $n \in \omega$. Let $\langle x_n : n \in \omega\rangle$ be an enumeration of X. Then a function $h : \omega \to \omega$ defined by $h(n) = x_{n \cdot k}$ belongs to V and $\sigma \leq h$, a contradiction. □

2.26 Remark. Note that if M adds a Cohen real then assumptions of the theorem are satisfied. Especially, the fact that ADR holds when adding a Cohen real is well known, with a different proof cf. S. H. Hechler [Hec78].

III. Exhaustive functions

In this chapter we describe the inclusion diagram for several natural classes of ccc Boolean algebras. These classes are determined by the existence of suitable real functions. We introduce here a notion of exhaustive and uniformly exhaustive function. We also show that classes determined by the existence of a function are closely related with the fragmentation properties; defined in the previous section (I.1.9). Namely we show that a given fragmentation property of a Boolean algebra $\mathbb{B}$ or a partial order P is in fact equivalent to the existence of a real function of a certain type over $\mathbb{B}$ or P. With such an equivalence at hand we engage the productivity of fragmentation properties.

We conclude this chapter by several examples illustrating that some implications from inclusion diagram are irreversible.

1. Definitions and basic facts

Let $(P, \leq)$ be a partial order and $f, g : P \to \mathbb{R}$ then we set $f \leq g$ if for each $p \in P$ $f(p) \leq g(p)$.

1.1 Definition. Exhaustive and uniformly exhaustive functions.

(i) A real function $f : P \to \mathbb{R}$ is called *exhaustive* if for each disjoint sequence $\langle a_n : n \in \omega\rangle \in P^\omega$, $\lim_{n\to\infty} f(a_n) = 0$.

(ii) $f : P \to \mathbb{R}$ is called *uniformly exhaustive* if for each positive $\varepsilon > 0$ there is a $k \in \omega$ such that for every disjoint sequence $\langle a_n : n \in \omega\rangle \in P^\omega$ $|\{n \in \omega : |f(a_n)| \geq \varepsilon\}| \leq k$.

Note that f is exhaustive if and only if for each positive $\varepsilon > 0$ and for each disjoint sequence $\langle a_n : n \in \omega\rangle \in P^\omega$ there is a $k \in \omega$ such that $|\{n \in \omega : |f(a_n)| \geq \varepsilon\}| \leq k$; i.e. uniformly exhaustive functions are exhaustive.

For a Boolean algebra $\mathbb{B}$ and $f : \mathbb{B} \to \mathbb{R}$ we define (uniform) exhaustivity in the same way. Note that for an exhaustive function $f : \mathbb{B} \to \mathbb{R}$ the value $f(\mathbf{0})$ is equal to 0 since the sequence $\langle \mathbf{0} : n \in \omega\rangle$ is disjoint.

1.2 Fact. *(i) A function $f : P \to \mathbb{R}$ is exhaustive if and only if $|f|$ is exhaustive if and only if the function $\min(1, |f|)$ is exhaustive.*

(ii) The family of all exhaustive functions forms a linear space over $\mathbb{R}$.

(iii) If $f, g : P \to \mathbb{R}$, $|f| \leq |g|$ and g is an exhaustive function then f is also an exhaustive function.

(iv) A monotone function $f : \mathbb{B} \to \mathbb{R}$ is exhaustive if and only if for any sequence $\langle a_n : n \in \omega\rangle$ and for any $\varepsilon > 0$ there is natural number k such that

$$(\forall\, p > k) \quad f\Big(\bigvee_{n\leq p} a_n - \bigvee_{n\leq k} a_n\Big) < \varepsilon$$

Proof. (i), (ii), (iii) is clear from the definition.

(iv) Suppose that f is a monotone exhaustive function and there is a sequence $\langle a_n : n \in \omega \rangle$ such that for any $k \in \omega$ there is $k^* > k$ such that $f(\bigvee_{n \le k^*} a_n - \bigvee_{n \le k} a_n) \ge \varepsilon$. Put $b_0 = \bigvee_{i \le 0^*} a_i$, $b_1 = (\bigvee_{i \le (0^*)^*} a_i) - b_0$, generally $b_n = \bigvee_{i \le 0^{(n+1)*}} a_i - \bigvee_{i<n} b_i$. Clearly the sequence $\langle b_n : n \in \omega \rangle$ is disjoint and for each $n \in \omega$, $f(b_n) \ge \varepsilon$, which contradicts the exhaustivity.

On the other hand if $\langle a_n : n \in \omega \rangle$ is a disjoint sequence and $\varepsilon > 0$ than there is by the assumption $k \in \omega$ such that $f(a_p) < \varepsilon$ for each $p > k$; since f is monotone. Hence $\lim_n f(a_n) = 0$. □

In *(i)*, *(ii)* and *(iii)* the exhaustivity can by replaced by uniform exhaustivity. So, we can restrict ourselves to non-negative, bounded, (uniformly) exhaustive functions. Next lemma shows that even more is possible.

1.3 Lemma. *Let $f : P \to \mathbb{R}$ be a bounded, non-negative exhaustive (resp. uniformly exhaustive) function. Then there is a function $h : P \to \mathbb{R}$ such that $h \ge f$ and h is monotone and exhaustive (resp. uniformly exhaustive).*

Proof. For each $p \in P$, let $h(p) = \sup\{f(a) : a \le p\}$. Clearly, $h \ge f$ and h is monotone. Suppose that h is not exhaustive. Then, for some $\varepsilon > 0$, there is a disjoint sequence $\langle a_n : n \in \omega \rangle$ such that $(\forall n)h(a_n) \ge \varepsilon$. For each n, take $b_n \le a_n$ such that $f(b_n) \ge \varepsilon/2$. Then $\langle b_n : n < \omega \rangle$ contradicts the exhaustivity of f. In case of uniform exhaustivity one can use the same argument. □

1.4 Lemma. *(i) Let $\mathbb{B}$ be a Boolean algebra, P dense in $\mathbb{B}$ and let $f : P \to \mathbb{R}$ be a monotone, bounded, and (uniformly) exhaustive function. Then there is $g : \mathbb{B} \to \mathbb{R}$ such that $g \supset f$ and g is a monotone and exhaustive function.*

(ii) Let $\mathbb{B}$ be a subalgebra of Boolean algebra $\mathbb{C}$, then every monotone (uniformly) exhaustive function f on $\mathbb{B}$ can be extended to monotone (uniformly) exhaustive function on $\mathbb{C}$.

Proof. (i) For $a \in B$ put $g(a) = \sup\{f(b) : b \le a \ \&\ b \in P\}$.

(ii) We proceed in the same way as in *(i)*, for $a \in \mathbb{C}$ put $g(a) = \sup\{f(b) : b \le a \ \&\ b \in \mathbb{B}\}$, with $g(a) = 0$, in case that there are no $b \le a$ such that $b \in \mathbb{B}$. Note that monotone functions on subalgebra are automatically bounded by $f(\mathbf{1})$. □

Next definition summarises additional natural properties of functions on Boolean algebras. These properties are motivated by properties of measures and submeasures.

1.5 Definition. A function $f : \mathbb{B} \to \mathbb{R}$ is called

(i) *non-negative* if $f(x) \ge 0$ for all $x \in \mathbb{B}$,

(ii) *strictly positive* if $f(x) > 0$ for all $x \ne \mathbf{0}$,

(iii) *monotone* if $f(x) \le f(y)$ for $x \le y$,

(iv) *subadditive* if $a \perp b$ then $f(a \vee b) \leq f(a) + f(b)$,

(v) *superadditive* if $a \perp b$ then $f(a \vee b) \geq f(a) + f(b)$,

(vi) *additive* if $a \perp b \to f(a \vee b) = f(a) + f(b)$,

(vii) *submeasure* if f is monotone, subadditive and $f(\mathbf{0}) = 0$,

(viii) *supermeasure* if f is non-negative and superadditive,

(ix) *measure* if f is non-negative and additive.

(x) *normalised* if $f(\mathbf{1}) = 1$.

We denote the collection of *null sets* of a function f by

$$\mathit{Null}(f) = \{a \in \mathbb{B} : f(a) = 0\}.$$

Note that a function f is a measure if it is a submeasure and supermeasure simultaneously. If f is a supermeasure then $f(\mathbf{0}) = 0$ since $f(\mathbf{0}) \geq 2f(\mathbf{0})$. Every supermeasure is monotone and every submeasure is non-negative.

For a submeasure f, the null sets $\mathit{Null}(f)$ form an ideal on $\mathbb{B}$. For a supermeasure f, $\mathit{Null}(f)$ is, in general, only downward closed set.

To denote measures, submeasures and supermeasures we use Greek letters λ, μ, ν and so on. Note that an identically zero function is a measure (a trivial measure).

1.6 EXAMPLE. For any Boolean algebra $\mathbb{B}$ there are simple examples of submeasure, supermeasure and measure on $\mathbb{B}$.

(i) *Submeasure*. Let $\mu(\mathbf{0}) = 0$ and $\mu(a) = 1$ for all $a \in \mathbb{B}^+$.

(ii) *Supermeasure*. Let $\nu(a) = 0$ for $a < \mathbf{1}$ and $\nu(\mathbf{1}) = 1$.

(iii) *Measure*. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{B}$ and put $\vartheta(a) = 0$ for $a \notin \mathcal{U}$ and $\vartheta(a) = 1$ for $a \in \mathcal{U}$ (ϑ is a characteristic function of $\mathcal{U}$).

(iv) *Exhaustive submeasure*. It follows from the definition I.4.1, that every Maharam submeasure is exhaustive. Also note that every measure is uniformly exhaustive. △

1.7 Fact. *Every supermeasure ν on $\mathbb{B}$ is uniformly exhaustive.*

Proof. For $\varepsilon > 0$ consider a set $U = \{x \in \mathbb{B} : \nu(x) \geq \varepsilon\}$. Then any disjoint set $X \subseteq U$ has size at most $\nu(\mathbf{1})/\varepsilon$ since $\nu(\mathbf{1}) \geq \nu(\bigvee X) \geq \sum_{x \in X} \nu(x) \geq \varepsilon |X|$. □

Let us recall the definition of a cone in a real vector space.

1.8 Definition. Let V be a vector space over $\mathbb{R}$, a subset $K \subset V$ is called a *cone* if

(i) $(\forall x \in K)\ (\forall 0 \leq r \in \mathbb{R}) \quad rx \in K$,

(ii) $(\forall x, y \in K) \quad x + y \in K$.

Monotone functions on Boolean algebra $\mathbb{B}$ are bounded ($f(\mathbf{0}) \leq f(x) \leq f(\mathbf{1})$), specially note that any submeasure (supermeasure) is a bounded function. In the following we deal only with bounded functions on $\mathbb{B}$ and we shall denote them $\mathrm{Fn}(\mathbb{B})$, usually this vector space is denoted by $l^\infty(\mathbb{B})$.

1.9 Definition. Let $\mathbb{B}$ be a Boolean algebra, we denote $\mathrm{Fn}(\mathbb{B})$ the set of all bounded functions on $\mathbb{B}$ and $\mathrm{Fn}^+(\mathbb{B})$ the subset of all nonnegative bounded functions. Similarly we denote $\mathrm{Sub}(\mathbb{B})$ the set of all submeasures on $\mathbb{B}$ and $\mathrm{Upm}(\mathbb{B})$ the set of all supermeasures on $\mathbb{B}$.

1.10 Fact. *Let $\mathbb{B}$ be a Boolean algebra, then*

(i) *$Fn(\mathbb{B})$ is a real vector space and $Fn^+(\mathbb{B}) \subset Fn(\mathbb{B})$ forms a cone,*

(ii) *the set of all exhaustive submeasures on $\mathbb{B}$ is a cone in real vector space $Fn(\mathbb{B})$,*

(iii) *$Upm(\mathbb{B})$ is a cone in $Fn(\mathbb{B})$.*

As we already seen (III.1.7) every supermeasure is an uniformly exhaustive function. On the other hand later we show (IV.2.9) that on every infinite atomless Boolean algebra $\mathbb{B}$ there is an exhaustive function which is not uniformly exhaustive. The question whether any exhaustive submeasure is uniformly exhaustive is one of the equivalent of so called *Maharam's problem*, which was solved quite recently by M. Talagrand [Tal06].

1.11 Maharam's problem. *Is every exhaustive submeasure uniformly exhaustive?*

The problem was solved in negative; i.e M. Talagrand found a Boolean algebra which carries strictly positive exhaustive submeasure but no uniformly strictly positive exhaustive submeasure.

2. CLASSES OF BOOLEAN ALGEBRAS

Now we give a classification of Boolean algebras based on the existence of a strictly positive functions of a special kind. We start with the definition of used symbols:

2.1 Definition. Classes of Boolean algebras.

(i) ccc stands for the class of ccc Boolean algebras, similarly *σ-centered* stands for the class of σ-centered algebras and *σ-linked* stands for the class of σ-linked algebras,

(ii) K stands for the class of algebras with the Knaster property (for a definition see I.1.5),

(iii) XBA stands for the class of algebras carrying a strictly positive exhaustive functional,

(iv) UpmBA stands for the class of algebras carrying a strictly positive supermeasure,

(v) EBA stands for the class of algebras carrying a strictly positive exhaustive submeasure,

(vi) MBA stands for the class of measure algebras (i.e. algebras carrying a strictly positive finitely additive measure),

Those classes of Boolean algebras are closed under taking subalgebras, countable products and, mainly, under taking regular completions.

Later (III.2.14, III.2.15) we will focus on closedness of these classes under the free product. It is well known that the class ccc for example is not generally closed under the free product.

2.2 Proposition. *Each of the classes defined above is closed under*

(i) subalgebras,

(ii) countable products and

(iii) taking regular completion.

Proof. Let $\mathcal{C}$ denote an arbitrary class from the diagram.

(i) $\mathcal{C}$ is defined using an existence of a fragmentation or a functional. Both cases can be restricted to any subalgebra $\mathbb{A} \subseteq \mathbb{B}$. Hence $\mathcal{C}$ is closed under taking subalgebras.

(ii) Let $\langle \mathbb{B}_n : n \in \omega \rangle$ be a countable sequence of algebras $\mathbb{B}_n \in \mathcal{C}$. We show that a product $\mathbb{B} = \prod \mathbb{B}_n \in \mathcal{C}$. Elements $\overline{x}$ of the product $\mathbb{B}$ are sequences $\langle x_n : n \in \omega \rangle$ with $x_n \in \mathbb{B}_n$. We distinguish three cases: $\mathcal{C}$ defined by a functional, $\mathcal{C}$ defined by a fragmentation and $\mathcal{C}$ defined by a ccc like property.

Suppose $\mathcal{C}$ is defined by a functional. For $n \in \omega$ take a functional f_n on $\mathbb{B}_n$ witnessing $\mathbb{B}_n \in \mathcal{C}$ such that $f_n(\mathbf{1}_{\mathbb{B}_n}) \leq 1/2^{n+1}$. For $\overline{x} \in \mathbb{B}$ put $f(\overline{x}) = \sum_{n\in\omega} f_n(x_n)$. Then $f : \mathbb{B} \to \mathbb{R}$ is a strictly positive function on $\mathbb{B}$ witnessing $\mathbb{B} \in \mathcal{C}$.

Suppose $\mathcal{C}$ is defined using fragmentation (the case of σ-linked and σ-centered Boolean algebras). For each $\mathbb{B}_n$ take a fragmentation $\{V_n^m : m \in \omega\}$. Put $\overline{V_n^m} = \{\overline{x} \in \mathbb{B} : \overline{x}(n) \in V_n^m\}$. Then $\{\overline{V_n^m} : n \in \omega, m \in \omega\}$ is a desired fragmentation of $\mathbb{B}$.

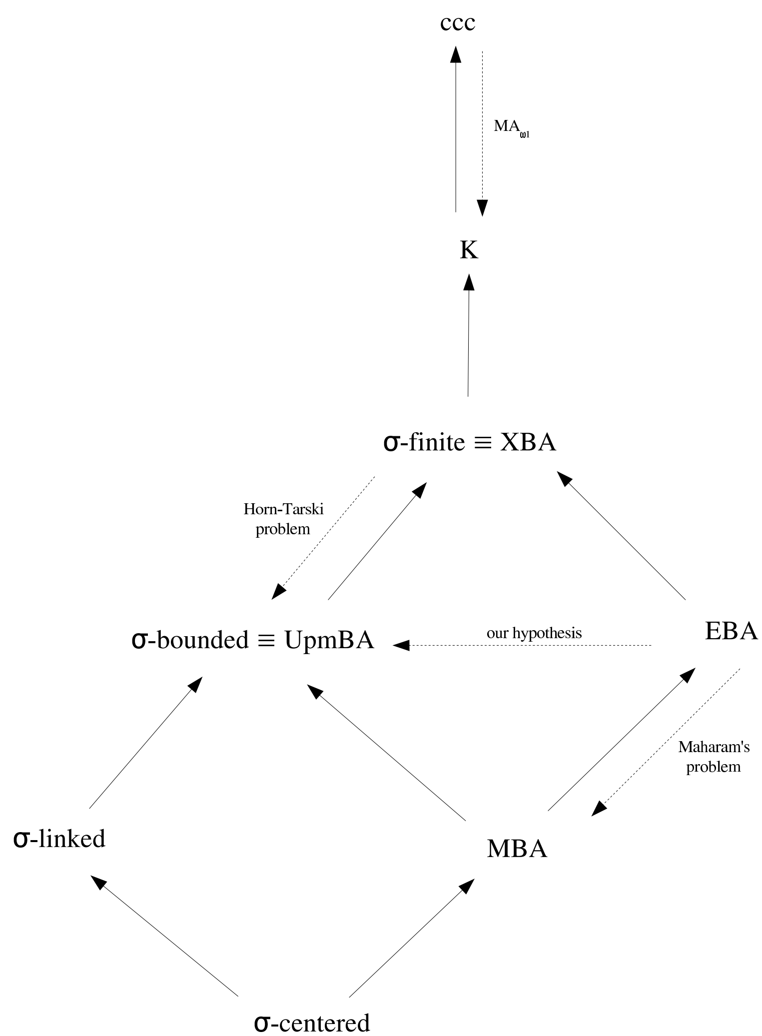
For the last case, suppose $\mathcal{C} = \mathsf{K}$ (the case of $\mathcal{C} = \mathsf{ccc}$ is easy). Let $X \subseteq \mathbb{B}$ be uncountable. There is $n \in \omega$ and an uncountable $Y \subseteq X$ such that $y(n) \neq \mathbf{0}_n$ for all $y \in Y$. Hence there is a linked uncountable $Y' \subseteq Y$ ($\{y(n) : y \in Y'\}$ is linked in $\mathbb{B}_n$).

(iii) We complete the proof by showing that $\mathcal{C}$ is closed under taking regular completion. Let $\mathbb{B} \in \mathcal{C}$ and $\overline{\mathbb{B}}$ be the regular completion of $\mathbb{B}$. The case of the ccc, K, σ-centered and σ-linked classes is clear. For $\mathcal{C} = \mathsf{XBA}$ use III.1.4. The same argument works for UpmBA.

Let $\mathcal{C} = \mathsf{EBA}$ or $\mathcal{C} = \mathsf{MBA}$. Let μ be a strictly positive exhaustive submeasure, resp. measure on $\mathbb{B}$. By Theorem I.4.10 there is a complete Boolean algebra $\mathbb{A}$ containing $\mathbb{B}$ as a subalgebra and a function $\overline{\mu}$ on $\mathbb{A}$ extending μ. Hence $\mathbb{A} \in \mathcal{C}$. Sikorski extension theorem (see I.3.11) gives an injective homomorphism $\varphi : \overline{\mathbb{B}} \to \mathbb{A}$ such that $\varphi \restriction \mathbb{B}$ is the identity. Since $\mathcal{C}$ is closed under subalgebras then $\overline{\mathbb{B}} \in \mathcal{C}$. □

2.3 INCLUSION DIAGRAM

For the illustration of the interplay of various properties, we put them in the following diagram. The diagram shows inclusions between classes. An arrow $\rightarrow$ stands for $\subset$. In the rest of this paragraph we prove the indicated inclusions and in the paragraph 4. we give some examples of Boolean algebras which illustrates the irreversibility of arrows.



2.4 Remark. (i) Under MA_{ω_1} the class of ccc algebras coincides with the class of Boolean algebras satisfying the Knaster property (I.1.6).

(ii) Later in this chapter we show that the question of whether classes XBA and UpmBA differ is equivalent to the Horn, Tarski problem; see proposition I.1.10.

(iii) We conjecture that EBA $\subset$ UpmBA (our conjecture is denoted by a dotted line in the diagram). See more arguments in III.3.1.

2.5 Remark. The negative solution of Maharam problem III.1.11 shows us that the classes EBA and MBA differs.

In our approach to prove the inclusions from the diagram we start with the most obvious ones.

The σ-centered and σ-linked Boolean algebras are limited in size. This is not the case for the remaining considered classes since the measure algebra can be of arbitrary size (e.g. algebra for adding $\kappa \geq \aleph_0$ Random reals).

2.6 Proposition. *(i) An algebra $\mathbb{B}$ is $\sigma-$centered if and only if it can be embedded into power set algebra $\mathcal{P}(\omega)$.*

(ii) If an algebra $\mathbb{B}$ is $\sigma-$linked then $(\mathbb{B}, \leq, \mathbf{0}, \mathbf{1})$ can be embedded into the algebra $(\mathcal{P}(\omega), \subset, \emptyset, \omega)$ as an ordering; hence its size is at most 2^ω.

Proof. (i) Let $\mathbb{B}$ be a $\sigma-$centered Boolean algebra. Let $\{P_n : n \in \omega\}$ be the appropriate fragmentation; i.e: $\mathbb{B} = \bigcup_{n\in\omega} P_n$. Without loss of generality we may assume, that P_n's are ultrafilters. Then the mapping

$$\begin{aligned} \varphi : \mathbb{B} &\longrightarrow \mathcal{P}(\omega) \\ b &\longmapsto \{n \in \omega : b \in P_n\} \end{aligned}$$

is clearly the desired homomorphic embedding. On the other hand if Boolean algebra $\mathbb{B}$ is embedded in $\mathcal{P}(\omega)$, then it is clearly $\sigma-$centered.

(ii) For $\sigma-$linked Boolean algebra $\mathbb{B}$ one can use the same embedding; assuming maximal linked sets $\bigcup_{n\in\omega} P_n = \mathbb{B}^+$. The embedding is no longer a homomorphism of algebras, but it is easy to check, that it preserves the ordering. □

The classes defined using the existence of a particular function are closely related with the fragmentation properties of Boolean algebras.

2.7 Theorem. *Boolean algebra $\mathbb{B}$ carries a strictly positive exhaustive function if and only if $\mathbb{B}$ is a σ-finite* cc*.*

Proof. Let f be a strictly positive exhaustive functional on $\mathbb{B}$. Put $V_n = \{x \in \mathbb{B} : f(x) \geq 1/n\}$, for $n \in \omega$. Then $\{V_n : n \in \omega\}$ is a fragmentation of $\mathbb{B}$ and each V_n satisfies ω-cc, and therefore $\mathbb{B}$ satisfies σ-finite cc.

In the opposite direction, let $\{V_n : n \in \omega\}$ be a fragmentation of $\mathbb{B}$ witnessing that $\mathbb{B}$ is σ-finite cc. We can assume that each V_n is upward closed and each $V_{n+1} \supset V_n$. Put $f(a) = \sup\{1/(n+1) : a \in V_n\}$ for $a \in \mathbb{B}^+$ and set $f(\mathbf{0}) = 0$. Then f is a strictly positive exhaustive functional on $\mathbb{B}$. □

The following observation is a simple reformulation of the previous theorem and we will meet a similar structure later, when dealing with topology on Boolean algebras.

2.8 Corollary. *Boolean algebra $\mathbb{B}$ carries a strictly positive exhaustive function if and only if there is a sequence $\langle U_n \subset \mathbb{B} : n \in \omega\rangle$ such that*

(i) $\mathbb{B} = U_0 \supset U_1 \supset \ldots U_n \ldots$

(ii) $\bigcap U_n = \{\mathbf{0}\}$,

(iii) for any disjoint sequence $d \in \mathbb{B}^\omega$ and for any $k \in \omega$ $\quad d \subset^* U_k$.

As the existence of strictly positive exhaustive function is bound with the existence of σ-finite cc fragmentation the uniformly exhaustive function is bound with the existence of σ-bounded cc fragmentation.

2.9 Theorem. *The following properties are equivalent.*

(i) $\mathbb{B}$ *carries a strictly positive supermeasure;*

(ii) $\mathbb{B}$ *carries a strictly positive uniformly exhaustive function;*

(iii) $\mathbb{B}$ *satisfies the σ-bounded* cc.

Proof. The implication (i)→(ii) follows from the fact that every supermeasure is an uniformly exhaustive function.

(ii)↔(iii). A uniformly exhaustive strictly positive function f on $\mathbb{B}$ gives a fragmentation $V_n = \{x \in \mathbb{B} : f(x) \geq 1/(n+1)\}$, for $n \in \omega$. Each V_n is m_n-cc for some $m_n \in \omega$.

Conversely, if we have a fragmentation $\{V_n : n \in \omega\}$ witnessing that $\mathbb{B}$ is the σ-bounded cc, we can assume that $V_n \subseteq V_{n+1}$, V_n is upward closed, and $\bigcup_{n\in\omega} V_n = \mathbb{B}^+$. Then f defined by $f(\mathbf{0}) = 0$ and $f(a) = \sup\{1/(n+1) : a \in V_n\}$ for $a \neq \mathbf{0}$, is a strictly positive monotone uniformly exhaustive function.

(ii)→(i). We can assume by the previous that there is a strictly positive monotone uniformly exhaustive normalised functional f on $\mathbb{B}$. In the following we shall describe a construction of a supermeasure on $\mathbb{B}$.

For positive n put $X_n = \{x \in \mathbb{B} : 1/n \leq f(x) < 1/(n-1)\}$ $(1/0 = \infty)$. Denote by k_n the maximal size of disjoint subsets of X_n. $k_n \in \omega$, since f is uniformly exhaustive. For $x \in X_n$, put $\phi(x) = 1/k_n{\cdot}2^n$ and $\phi(\mathbf{0}) = 0$. Then a functional μ on $\mathbb{B}$, where $\mu(a) = \sup\{\sum_{i=1}^{m} \phi(x_i) : \{x_1, \ldots, x_m\}$ disjoint finite set of elements $x_i \leq a\}$, is a strictly positive supermeasure and $\phi \leq \mu$, $\mu(\mathbf{1}) \leq 1$. □

2.10 Remark. It is easily seen that if Boolean algebra $\mathbb{B}$ carries a strictly positive measure, then it is σ-bounded cc. It was shown by H. Gaifman [Gai64] that the converse is false; for stronger example see III.4.12.

2.11 Remark. From the preceding theorems we see that the Horn - Tarski problem is equivalent to the question whether the classes UpmBA and XBA differ.

To conclude the proof of the arrows in the diagram it remains to show the following:

2.12 Proposition. *(i) Every algebra carrying a strictly positive exhaustive functional has the Knaster property.*

(ii) Any σ-centered algebra $\mathbb{B}$ *carries a strictly positive measure.*

Proof. (i) Let $X \subseteq \mathbb{B}$, $|X| = \omega_1$. There is an $n \in \omega$ such that $Y_n = \{x \in X : f(x) \geq 1/(n+1)\}$ is uncountable. Apply the partition relation $\omega_1 \to (\omega_1, \omega)^2$ (Special case of Erdős, Dushnik and Miller theorem, see [BŠ00].) to the disjointness relation restricted to the set Y_n, i.e. $\{x, y\} \in [Y_n]^2$ is coloured blue if $x \perp y$, otherwise it is coloured red. There is no blue infinite homogeneous subset. Thus there must exist an uncountable part of Y_n that is homogeneous in colour red, that is, it must be linked. This proves that $\mathbb{B}$ has the Knaster property.

(ii) Let $\mathbb{B}^+ = \bigcup\{F_n : n \in \omega\}$, each F_n an ultrafilter on $\mathbb{B}$. Take a sequence of positive reals $\langle a_n : n \in \omega\rangle$ such that the series $\sum a_n$ converges. Set $\mu(x) = \sum\{a_n : x \in F_n\}$ for each $x \in \mathbb{B}$. Then μ is a strictly positive measure on $\mathbb{B}$. □

2.13 PRODUCTIVITY

Now we turn our attention back to the free product. It is known (see [Gal80]) that ccc is not *productive*, i.e. it is not necessarily preserved by a free product. However, with an additional assumption that one of the algebras satisfy the so-called Knaster property (I.1.5), the ccc is preserved by the product of two algebras. To our knowledge, the productiveness of the σ-finite cc and σ-bounded cc have not been shown. Using the interplay between exhaustive functionals on a Boolean algebra, we prove that both, the σ-finite cc and the σ-bounded cc are in fact productive.

2.14 Theorem. *Let* I *be an arbitrary index set and let* $\{\mathbb{B}_i : i \in I\}$ *be an arbitrary family of Boolean algebras satisfying the σ-finite* cc*. Then the free product*

$$\mathbb{B} = \bigotimes_{i \in I} \mathbb{B}_i$$

satisfies the σ-finite cc *as well.*

Proof. Let $P = \{\prod_{i \in J}(\mathbb{B}_i - \{\mathbf{0}, \mathbf{1}\}) : \emptyset \neq J \subseteq I,\ J \text{ finite}\}$. We introduce a partial order on P by defining $p \leq q$ if and only if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and for every $i \in \mathrm{dom}(q)$, $p(i) \leq q(i)$ in the algebra $\mathbb{B}_i$. The set $\hat{P} = \{\hat{p} : p \in P\}$, where $\hat{p} = \bigwedge_{i \in \mathrm{dom}(p)} p(i)$ is dense in $\mathbb{B}$ and $\hat{p} < \hat{q}$ if and only if $p < q$.

Since each $\mathbb{B}_i$ satisfies the σ-finite cc, according to Theorem III.2.7 it carries a strictly positive exhaustive functional f_i. Thus, we define a real functional f on P by setting $f(p) = \min\left\{ \min\{f_i(p(i)) : i \in \mathrm{dom}(p)\}, \frac{1}{|\mathrm{dom}(p)|} \right\}$ for each $p \in P$. f is thus a strictly positive functional.

f is an exhaustive functional: if not, then there must be $\varepsilon > 0$ and a disjoint family $D = \{p_\alpha \in P : f(p) \geq \varepsilon\}$, $\alpha \in A$ of an infinite size. For each $\alpha \in A$, $|\mathrm{dom}(p_\alpha)| \leq \frac{1}{\varepsilon}$, and so there exists a Δ-system $D_1 \subseteq \{\mathrm{dom}(p_\alpha) : \alpha \in A\}$ of an infinite size with a non-empty kernel J, i.e. for each $p, q \in D_2 = \{r \in D : \mathrm{dom}(r) \in D_1\}$ the intersection of domains is the kernel $\mathrm{dom}(p) \cap \mathrm{dom}(q) = J$. Now each pair of elements p, q from D_2 is assigned as a colour the least $i \in J$ so that $p(i)$ and $q(i)$ are disjoint in $\mathbb{B}_i$. Applying the infinite Ramsey theorem $\omega \to \omega^2_{|J|}$, there exists an infinite $D_3 \subseteq D_2$ and a colour $j \in J$ so that D_3 is homogeneous in j, i.e. for any $p, q \in D_3$, $p(j) \wedge q(j) = \mathbf{0}$. Since $f_j(p(j)) \geq f(p) \geq \varepsilon$ for every $p \in D_3$, we obtained an infinite disjoint family of elements of $\mathbb{B}_j$ where f_j exceeds ε, contradicting the fact that f_j is exhaustive.

Extending f from P to $\hat{f}$ from $\hat{P}$ by defining $\hat{f}(\hat{p}) = f(p)$ we obtain a strictly positive exhaustive functional that can be then extended to a strictly positive exhaustive functional on $\mathbb{B}$. By Theorem III.2.7 we conclude that $\mathbb{B}$ satisfies the σ-finite cc. □

2.15 Theorem. *Let* I *be an arbitrary index set and let* $\{\mathbb{B}_i : i \in I\}$ *be an arbitrary family of Boolean algebras satisfying the* σ*-bounded* cc*. Then the free product*

$$\mathbb{B} = \bigotimes_{i \in I} \mathbb{B}_i$$

satisfies the σ*-bounded* cc *as well.*

Proof. Since each $\mathbb{B}_i$ is σ-bounded, it carries a strictly positive uniformly exhaustive functional f_i. Moreover each $\mathbb{B}_i$ admits a $\subseteq$-increasing fragmentation $B_{i,k}$, $k \in \omega$, so that every disjoint family $\{x \in B_{i,k} : f_i(x) \geq \frac{1}{k}\}$ has a size $< k+1$.

We define P and f, as in the proof of the previous theorem. We define a $\subseteq$-increasing fragmentation of P: $q \in P_k$ if and only if $q \in P$ and $f(q) \geq 1/k$. Note that if $p \in P_k$, then $|\mathrm{dom}(p)| \leq k$ and that $p(i) \in B_{i,k}$ for any $i \in \mathrm{dom}(p)$.

We are trying to find a bounding function g for the fragmentation $\{P_k : k \in \omega\}$. Define

$$g(k) = \min\{c : r(\frac{r(c,k)-1}{\binom{j}{k}} + 1, k) \geq k+1\}$$

where $r(m,n) =$ the minimal c so that $c \to (m)^2_n$; we are using the finite Ramsey theorem here, cf. [BŠ00]. We will show that g is a bounding function.

Fix a k and let us begin with assuming that we have a disjoint family $D \subseteq P_k$ of size $d \geq g(k)$. Without loss of generality we can assume that $|\mathrm{dom}(p)| = k$ for any $p \in D$ (by appropriately defining the values of p outside of $\mathrm{dom}(p)$, if necessary). From the definition of P_k it follows that for any $p \in D$, $f(p) \geq \frac{1}{k}$.

There exists a family $D_1 \subseteq D$ of size $d_1 \geq r(d,k) \geq r(g(k),k)$ and $1 \leq j \leq k$ so that for any $p, q \in D_1$, $|\mathrm{dom}(p) \cap \mathrm{dom}(q)| = j$.

To see that colour each pair $\{p,q\} \subseteq D$ by a colour $i = |\mathrm{dom}(p) \cap \mathrm{dom}(q)|$. Since we are using k colours and since $d \geq g(k)$, we can obtain a $D_1 \subseteq D$, homogeneous in colour j of size $d_1 \geq r(d,k)$.

There exists a Δ-system $D_2 \subseteq D_1$ of size $d_2 = \frac{d_1-1}{\binom{j}{k}}+1$ with a kernel J of size j.

To see that fix a $t \in D_1$. Let $\langle A_i : 1 \leq i \leq \binom{j}{k}\rangle$ be an enumeration of all subsets of $\mathrm{dom}(t)$ of size j. Assign each $p \in D_1 - \{t\}$ a colour i if $\mathrm{dom}(p) \cap \mathrm{dom}(t) = A_i$. By pigeon-hole principle there is a family $E \subseteq D_1 - \{t\}$ of size $d_2 = \frac{d_1-1}{\binom{j}{k}}$ and a colour i so that for any $p, q \in E$, $\mathrm{dom}(p) \cap \mathrm{dom}(q) = A_i$. Set $D_2 = E \cup \{t\}$. Now D_2 forms a Δ-system of size $d_2 = \frac{d_1-1}{\binom{j}{k}}+1$ with the non-empty kernel $J = A_i$.

There exists a family $D_3 \subseteq D_2$ of size $d_3 \geq r(d_2,k)$ and $i \in J$ so that for any $p, q \in D_3$, $p(i) \cap q(i) = \mathbf{0}$ in $\mathbb{B}_i$.

To see that colour each pair $p, q \in D_2$ by a colour i the least integer such that $p(i) \cap q(i) = \mathbf{0}$. D_3 is a subset of D_2 homogeneous in some colour i.

Let us estimate the size of d_3:

$$d_3 \geq r(d_2,k) \geq r(\frac{d_1-1}{\binom{j}{k}}+1,k) \geq r(\frac{r(d,k)-1}{\binom{j}{k}}+1,k) \geq$$
$$\geq r(\frac{r(g(k)d,k)-1}{\binom{j}{k}}+1,k) \geq k+1.$$

Thus, we obtained $\{p(i) : p \in D_3\}$, a disjoint family of elements of $\mathbb{B}_i$ of size $\geq k+1$ so that $f_i(p(i)) \geq f(p) \geq \frac{1}{k}$, therefore a disjoint family of elements of $B_{i,k}$ of size $\geq k+1$, a contradiction. Therefore, for any disjoint family D of elements of P_k, $|D| < g(k)$. It follows that P is σ-bounded, and hence $\mathbb{B}$ is σ-bounded as well. □

3. ADDITIONAL FACTS

The question whether $\mathrm{EBA} \subset \mathrm{UpmBA}$ or equivalently whether every Boolean algebra carrying exhaustive strictly positive submeasure is σ-bounded cc is still open. The following lemma gives us a partial answer to this question in very special case of countably completely generated Boolean algebra and Maharam submeasure.

3.1 Lemma. *Let μ be a strictly positive Maharam submeasure on a complete Boolean algebra $\mathbb{B}$ with a countable set of complete generators. Then $\mathbb{B}$ is σ−linked, hence σ-bounded* cc *algebra.*

Proof. Denote $D \subset \mathbb{B}$ a countable subalgebra that completely generates $\mathbb{B}$. Since $\mathbb{B}$ is ccc and weakly distributive, the set

$$S = \{\bigwedge\langle d_n : n \in \omega\rangle : \langle d_n\rangle \in D^\omega \text{ and } d_n \text{ is decreasing}\},$$

is dense in $\mathbb{B}$.

For $d \in D^+$ and rational $q \in \mathbb{Q}$, $0 < q < 1/2\mu(d)$ is the set $V(d, q) = \{b \in \mathbb{B} : \mu(d - b) < q\}$ linked. It remains to show that $\mathbb{B}^+ = \bigcup\{V(d, q) : d \in D^+ \text{ and } 0 < q < 1/2\mu(d)\}$. Let $b \in \mathbb{B}^+$ be arbitrary; there is $c \in S^+$ such that $c \leq b$ and $c = \bigwedge d_n$ for some $\langle d_n\rangle \in D^\omega$. From continuity of μ we can pick $n_0 \in \omega$ such that $\mu(d_{n_0} - c) < r < 1/2\mu(c) \leq 1/2\mu(d_{n_0})$. Clearly $b \in V(d_{n_0}, r)$. □

σ-boundedness of Boolean algebra is not enough to guarantee the existence of a strictly positive Maharam submeasure; cf. III.4.12. One of the greatest achievements concerning this topic is the famous Kalton, Roberts theorem [KR83]. We state the theorem here without proof; we provide the proof later on (IV.6.11).

3.2 Theorem. (N. J. Kalton, J. W. Roberts [KR83]) *Let $\mathbb{B}$ be a Boolean algebra and ν be a strictly positive, uniformly exhaustive submeasure on $\mathbb{B}$. Then $\mathbb{B}$ carries a strictly positive measure $\mu \leq \nu$.*

A very natural question is whether any Boolean algebra that carries simultaneously a strictly positive exhaustive submeasure and a supermeasure has to be measurable. The solution of Maharam problem answer the question generally in the negative. Algebra constructed by M. Talagrand [Tal06] does not carry a measure, but has a countable set of complete generators, hence it is σ-linked. In a special case when a supermeasure dominates a submeasure the answer is affirmative.

3.3 Proposition. *Let $\mu \leq \nu$ be a submeasure and a supermeasure on a Boolean algebra $\mathbb{B}$, respectively. Then there is $m : \mathbb{B} \to \mathbb{R}$ a measure on $\mathbb{B}$ such that $\mu \leq m \leq \nu$.*

Proof. The proof is based on compactness argument. Denote

$$S = \prod_{a \in \mathbb{B}} [\mu(a), \nu(a)]$$

topological product of closed non-empty intervals. S is non-empty compact space. Consider finite subalgebra $C \subset \mathbb{B}$.

Claim: A set

$$M_C = \{f \in S : f \restriction C \text{ is a measure on } C\}$$

is a non-empty closed subset of S.

C is finite, denote $\{a_i : i \leq n\}$ the set of its atoms and choose $f(a_i) \in [\mu(a_i), \nu(a_i)]$ at will. Now one can extend f to a measure on C with $\mu \leq f \leq \nu$ on C because

$$\mu(\bigvee_{i \in I} a_i) \leq \sum_{i \in I} \mu(a_i) \leq \sum_{i \in I} f(a_i) = f(\bigvee_{i \in I} a_i) \leq \sum_{i \in I} \nu(a_i) \leq \nu(\bigvee_{i \in I} a_i).$$

Measure f on C can be now extended to a function $f \in S$ arbitrary, hence M_C is non-empty.

Assume $f \in S \setminus M_C$ then there is an open neighbourhood in S disjoint with M_C, so M_C is closed which proves the Claim.

A family of all finite subalgebras is upward directed. Therefore

$$\{M_C : C \subset \mathbb{B},\ C \text{ is finite subalgebra of } \mathbb{B}\}$$

is a centered family of non-empty closed sets in a compact space and therefore have a non-empty intersection. Any function $\lambda \in \bigcap M_C$ is a desired measure. $\square$

It follows from the definition I.4.1 that every Maharam submeasure is exhaustive. On the other hand the existence of a strictly positive exhaustive submeasure implies the existence of a Maharam submeasure whenever the underlying Boolean algebra is weakly distributive, more precisely

3.4 Theorem. *Let $\mathbb{B}$ be a complete,* ccc*, weakly distributive Boolean algebra and let $f : \mathbb{B} \to \mathbb{R}$ be a strictly positive monotone function. Then there is strictly positive monotone lower semicontinuous function $g \leq f$.*

Moreover

(i) if f is an exhaustive submeasure, then g is Maharam submeasure,

(ii) if f is finitely additive measure, then g is σ-additive (i.e. continuous) measure.

Proof. We define function $g : \mathbb{B} \to \mathbb{R}$ as $g(a) = \inf\{\lim_n f(a \wedge c_n) : c_n \nearrow \mathbf{1}\}$. Thus g is monotone and we show that g is a strictly positive function. Assume on contrary, that there is $a \in \mathbb{B}^+$ such that $g(a) = 0$ i.e. for any $m \in \omega$ there is an increasing $\langle c_n^m : n \in \omega \rangle$ such that $\lim_n (a \wedge c_n^m) \leq 1/m$. We use weak distributivity of $\mathbb{B}$ and strict positivity of f to find $b \in \mathbb{B}$ such that $f(b) > 0$ and for any m there is some $n_m \in \omega$ so that $a \wedge c_n^m \geq b$ for $n > n_m$. Hence $\lim f(a \wedge c_n^m) \geq f(b) > 0$; which is a contradiction.

To show that g is lower semicontinuous (I.2.23) it is enough to prove that $\lim g(a_n) = g(a)$ for arbitrary increasing sequence $\langle a_n : n \in \omega \rangle \nearrow a$. Suppose for contradiction that $g(a_n) < g(a) - \varepsilon$. For each a_n there is some $c_m^n \nearrow \mathbf{1}$ such that

$$\lim_m f(a_n \wedge c_m^n) \; < \; g(a) \; - \; \varepsilon. \tag{III.1}$$

Now we use the 'diagonal property'; the equivalent (V.8.4) of weak distributivity for ccc Boolean algebras and find the diagonal $\langle m_n : n \in \omega \rangle$ such that $\lim_n c_{m_n}^n = \mathbf{1}$. The sequence

$$c'_k \; = \; \bigwedge_{n \geq k} c_{m_n}^n$$

is increasing with $\lim_k c'_k = \mathbf{1}$. We modify the sequence as follows

$$c_k \; = \; \left(a_k \; \wedge \; c'_k \right) \; \vee \; (-a).$$

Claim: $c_k \nearrow \mathbf{1}$.

Clearly c_k is increasing. Let $b \in \mathbb{B}^+$ be arbitrary. If $b \wedge (-a) \neq \mathbf{0}$ we are done, if $b \leq a$ then since $a_k \nearrow a$ there is some $n \in \omega$ such that $a_n \wedge b \neq \mathbf{0}$. The sequence $c'_k \nearrow \mathbf{1}$ hence there is some $m \in \omega$ such that $a_n \wedge b \wedge c'_m \neq \mathbf{0}$.

Now take $k = \max\{m, n\}$ and

$$b \wedge c_k \; = \; b \wedge \; (a_k \wedge c'_k) \vee (-a) \neq \mathbf{0},$$

which completes the claim.

By the definition of function g we have the following

$$g(a) \leq \lim_k f(a \wedge c_k), \text{ but}$$

$$f(a \wedge c_k) \; = \; f(a_k \wedge c_k) \; \leq \; f(c_{m_k}^k \wedge a_k) \overset{\text{eq.(III.1)}}{\leq} g(a) - \varepsilon.$$

Hence $g(a) \leq g(a) - \varepsilon$ which is a contradiction.

(i) Now we show that if f is a submeasure then g is a submeasure. Suppose that g is not monotone i.e. there are $a \leq b$ and some positive $\varepsilon > 0$ such that $g(a) > g(b) + \varepsilon$. Then one can find an increasing sequence $c_n \nearrow \mathbf{1}$ such that $g(b) \leq \lim_n f(b \wedge c_n) \leq g(b) + \varepsilon$. Clearly, since f is monotone $f(a \wedge c_n) \leq f(b \wedge c_n)$ hence $\lim_n f(a \wedge c_n) \leq g(b) + \varepsilon < g(a)$, a contradiction. Once we know g is monotone, to prove subadditivity it is sufficient to consider disjoint $a, b \in \mathbb{B}$. It is also clear that $g(a) = \inf\{\lim_n f(a \wedge c_n) : c_n \nearrow a\}$. Now choose arbitrary $\varepsilon > 0$ and find increasing $a_n \nearrow a$ and $b_n \nearrow b$ such that $\lim_n f(a \wedge a_n) \leq g(a) + \varepsilon/2$ and

$\lim_n f(b \wedge b_n) \leq g(b) + \varepsilon/2$. We get that $f(a \wedge a_n) + f(b \wedge b_n) \geq f((a \vee b) \wedge (a_n \vee b_n))$ so $g(a) + g(b) \geq g(a \vee b)$ and hence g is subadditive.

We already showed that g is strictly positive, hence a nontrivial submeasure. What remains, is to show that g is continuous. Let $a_n \searrow \mathbf{0}$ be a decreasing sequence, our aim is to show that $\lim_n g(a_n) = 0$. Suppose on contrary that $g(a_n) > \delta > 0$ for each $n \in \omega$. Fix n_0, the sequence $\langle (a_{n_0} - a_m) \rangle$ is increasing with limit a_{n_0}. Since g is lower semicontinuous $\lim g(a_{n_0} - a_m) = g(a_{n_0}) > \delta$ and there is n_1 such that

$$g((a_{n_0} - a_{n_1}) = d_0) > \frac{\delta}{2}.$$

We proceed by induction and construct an infinite disjoint sequence d_n such that $f(d_n) \geq g(d_n) > \delta/2$ which contradicts the fact that f is an exhaustive submeasure and we are done.

(ii) To show that g is a measure when f is a measure is similar to the argumentation for submeasures. Since g is continuous if f is submeasure, it is also continuous when f is measure. Which in fact means that g is σ-additive measure. $\square$

Recently (June 2004), the following theorem was shown by S. Todorcevic [Tod04]. Note that every Maharam submeasure is exhaustive, hence one implication is obvious. We give the different proof of this combinatorial characterisation of Maharam algebras in V.9.14.

3.5 Theorem. *Let $\mathbb{B}$ be a complete, weakly distributive Boolean algebra, then $\mathbb{B}$ is σ-finite* cc *if and only if $\mathbb{B}$ carries a strictly positive Maharam submeasure.*

4. EXAMPLES

In this part we show two examples to illustrate that the remaining arrows in the inclusion diagram cannot be reversed. Namely, we show that there is a ccc Boolean algebra which is not σ-finite cc and moreover under the assumption $\mathfrak{b} = \aleph_1$ it does not have even the Knaster property.

To cope with lower part of diagram we introduce the Localisation forcing. The completion of this partial order stands as an example of a σ-linked Boolean algebra not carrying a strictly positive exhaustive submeasure. This illustrates the irreversibility of arrows in the lower part of the inclusion diagram.

4.1 TODORCEVIC'S PARTIAL ORDER - (PART 2)

In I.1.14 we defined a Todorcevic's partial order $\mathbb{T}(X)$ of countable, compact subsets of X, where X is a Polish space (i.e. separable, completely metrizable space) without isolated points. We show that the ccc Boolean algebra $\mathrm{RO}(\mathbb{T}(X))$ determined by Todorcevic's partial order carries no strictly positive exhaustive function; i.e: having no σ-finite cc fragmentation. It was already shown by Todorcevic [Tod84] that $\mathbb{T}(\mathbb{R})$ is not σ-linked.

On the other hand it is easy to check that the partial order $\mathbb{T}(\mathbb{Q})$ is σ-centered.

4.2 Theorem. *Let X be a Polish space without isolated points. Then there is no strictly positive exhaustive functional on $\mathbb{T}(X)$, thus there exists a* ccc *Boolean algebra having no σ-finite* cc *fragmentation.*

Proof. Consider the separative ordering $\mathbb{T}(X) = (P, \leq)$. We know that it satisfies ccc from I.1.17. Our aim is to show that it is not σ-finite cc, thus the algebra $\mathrm{RO}(\mathbb{T}(X))$ does not carry a strictly positive exhaustive functional by III.2.7.

Assume the opposite, let $\{P_n : n \in \omega\}$ be a σ-finite cc fragmentation of $\mathbb{T}(X)$. Heading toward a contradiction, we look for an $A \in P$, in fact, it will be a convergent sequence, such that $A \notin P_n$ for any $n \in \omega$. We will construct a sequence of open balls B_n with $\mathrm{diam}(B_n) \leq 1/n$ and finite sets A_n in X such that $B_{n+1} \subset B_n$, $A_n \subset B_n \setminus B_{n+1}$ for all $n \in \omega$ and a Cauchy sequence $\langle c_n : n \in \omega \rangle$. Choose B_1 arbitrarily. Induction step: Assume that an interval B_n is known. Denote by D_n a maximal pairwise incompatible family of elements $A \in P_n$ with the property that the derived set of A meets the ball B_n, i.e., $A' \cap B_n \neq \emptyset$. The family D_n is finite, since it is a subset of P_n, so the set $A_n = B_n \cap \bigcup\{A' : A \in D_n\}$ is also finite. Choose B_{n+1} such that $\mathrm{diam}(B_{n+1}) \leq 1/(n+1)$, $\mathrm{cl}(B_{n+1}) \subset B_n$ and $A_n \cap B_{n+1} = \emptyset$ and some $c_n \in B_n \setminus (B_{n+1} \cup A_n)$.

Put $c = \lim c_n$ and $A = \bigcup_{n\in\omega} A_n \cup \{c_n : n \in \omega\} \cup \{c\}$. Since $\bigcup_{n\in\omega} P_n = P$ and $A \in P$, there must be some $n \in \omega$ with $A \in P_n$. Since $c \in B_n$ and $c \in A'$, the set A is incompatible with all elements from D_n, which contradicts the maximality of D_n. $\square$

We know that under MA_{ω_1} every ccc forcing has the Knaster property. It was also shown by Todorcevic [Tod84] that $\mathbb{T}$ does not have the Knaster property under the assumption that $\mathfrak{b} = \aleph_1$.

4.3 Theorem. (S. Todorcevic [Tod84]) $[\mathfrak{b} = \aleph_1]$ $\mathbb{T}(\mathbb{R})$ *does not have the Knaster property.*

Proof. We start with a system of functions $A = \{a_\alpha \in \omega^\omega : \alpha \in \omega_1\}$ witnessing that $\mathfrak{b} = \aleph_1$. Suppose that each a_α is increasing and let A be ordered by $\leq^*$ i.e: if $\alpha < \beta$ then $a_\alpha \leq^* a_\beta$.

Let

$$\{e_\alpha : \alpha \longrightarrow \omega : \alpha \in \omega_1\},$$

be such that each e_α is one-to-one function and for $\alpha < \beta$ we get $e_\alpha =^* e_\beta \restriction \alpha$; c.f: construction of Aronszajn tree [BŠ00].

In this setting we can interpret a_α's as irrational numbers and our aim is to find and fix an appropriate convergent sequence to $\aleph_1$-many a_α's.

Define for $b \in A$ the set

$$H(b) = \{a \leq^* b : e_b(a) < b(\triangle\,(a, b))\},$$

where $\triangle\,(a, b)$ is the least natural number in which the functions a and b are distinct.

Clearly, whenever the set $H(b)$ is infinite then it is a convergent sequence with the limit point b. If $H(b)$ is infinite then the set $\{e_b(a) : a \in H(b)\}$ is also infinite since $e_b : b \longrightarrow \omega$ is one-to-one function. Hence for arbitrary $t \in \omega$ there is $a \in H(b)$ such that $e_b(a) > b(t)$, that is by the definition of $H(b)$, that $\triangle\,(a, b) > t$; i.e: a approximates b up to length t. It is obvious that the set $H(b)$ has no other cluster point. In the following sequence of facts we show that there is $\aleph_1$ many b's in A with infinite $H(b)$.

Let $F \subset A$ be an arbitrary cofinal subset. Since $F \subset \omega^\omega$, there is $D \subset F$ countable and dense in F. Having D countable and F cofinal, there is an upper bound $c \in F$ of the set D, i.e: $d \leq^* c$ for every $d \in D$.

4.4 Fact. *There is* $m \in \omega$ *and* $s \in {}^m\omega$ *and cofinal* $F_0 \subset F \cap [s]$ *such that for every* $f \in F_0$ $c(n) < f(n)$ *for any* $n > m$.

Proof. For any $g \in F$ such that $g \geq^* c$ there is $m_g \in \omega$ such that $g(n) > c(n)$ for any $n > m_g$. There are only countably many distinct short functions $g \restriction m_g$. Hence for some $\overline{g} \in F$ the set $[\overline{g} \restriction m_{\overline{g}}] \cap F$ is cofinal. Put $s = \overline{g} \restriction m_{\overline{g}}$, $m = m_{\overline{g}}$ and $F_0 = [\overline{g} \restriction m_{\overline{g}}] \cap F$. □

What we get is the fact that for any $f \in F_0$ the function $e_f \restriction D =^* e_c \restriction D$ (recall that $D \leq^* c$). Since there are only countably many of finite modifications of $e_c \restriction D$, there is a cofinal $F_1 \subset F_0$ such that for any $f, g \in F_1$ $e_f \restriction D = e_g \restriction D$. That is the value $e_f(d)$ does not depend on the choice of $f \in F_1$, but depends only on $d \in D$.

4.5 Fact. *There is a finite function* t *extending* s *and a cofinal* $F_2 \subset F_1 \cap [t]$ *such that the set* $\{f(|t|) : f \in F_2\}$ *is infinite.*

Proof. Suppose otherwise that for any t extending s is the set $\{f(|t|) : f \in F_1\}$ finite. Then for an arbitrary $n > |s|$, the set $\{f(n) : f \in F_1\}$ is only finite; which is in contradiction to the fact that F_1 is cofinal. □

4.6 Fact. *For any natural* k *there is a function* $f \in F_2$ *such that* $|H(f)| > k$.

Proof. The set D is dense in F_2, we can find k many distinct $d_i \in D$ $(i < k)$ such that each d_i extends t. Recall that values $e_f(d_i)$ does not depend on the particular choice of $f \in F_2$. Since the set $\{f(|t|) : f \in F_2\}$ is infinite, there is some $f \in F_2$ such that

$$f(|t|) > \max_{i<k}\ e_f(d_i).$$

Clearly, $d_i \in H(f)$:

(a) $d_i \leq^* c_i \leq^* f$ for any $i < k$.

(b) $\triangle\,(f, d_i) \geq |t|$ since each d_i extends t, hence $e_f(d_i) < f(|t|) \leq f(\triangle\,(f, d_i))$; recall that f is an increasing function. □

We just proved that in any cofinal $F \subset A$ we are able to find a function $f \in F$ such that $H(f)$ is of an arbitrary size. Suppose that there are no functions with infinite $H(f)$ then for some $n \in \omega$ the set $\{f \in F_2 : |H(f)| \leq n\}$ has to be cofinal, a contradiction. Moreover, since $F \setminus \{f\}$ is cofinal we get uncountably many f's with infinite $H(f)$ and from the construction we know that $H(f) \cap D \neq \emptyset$.

Proof. END OF PROOF OF THEOREM III.4.3

By the previous discussion there is a cofinal $F \subset A$ such that $H(f)$ is infinite for any $f \in F$. We claim that the set $\{H(f) \cup \{f\} \in \mathbb{T} : f \in F\}$ is uncountable without uncountable linked subset.

Let $\overline{F}$ be an uncountable subset of F, then $\overline{F}$ is cofinal. There is some countable dense $D \subset \overline{F}$ and an upper bound $c \in \overline{F}$ for D. Choose arbitrary $a \in \overline{F}$ such that $a \geq^* c$ and distinct from c. By the assumption the set $H(a)$ is infinite and it follows that it meets the set D. Let $b \in H(a) \cap D$ then $H(a) \cup \{a\}$ and $H(b) \cup \{b\}$ are two disjoint sequences. Hence the set $\{H(f) \cup \{f\} \in \mathbb{T} : f \in F\}$ has no uncountable linked subset. □

4.7 LOCALISATION FORCING

Here we describe an example of a Boolean algebra $\mathbb{B}$ that carries a strictly positive supermeasure, in fact $\mathbb{B}$ is σ-linked, but carries no strictly positive exhaustive submeasure. This complete Boolean algebra is, in the context of the forcing notion, known as the localisation forcing.

4.8 Definition. The *localisation forcing* $(Loc, \leq)$ is the partial order

$$Loc = \{f : \omega \to [\omega]^{<\omega} : (\exists k \in \omega)(\forall n \in \omega)|f(n)| \leq \min(n, k)\}$$

ordered by the reverse inclusion, i.e. $f \leq g$ if and only if $f(n) \supseteq g(n)$ for all $n \in \omega$.

4.9 Fact. *$(Loc, \leq)$ is separative.*

Proof. Assume that $f \not\leq g$, i.e. there are natural numbers $n_0 > 0$ and $x \in g(n_0) - f(n_0)$. Take h such that $h(n) = f(n)$ for $n \neq n_0$ and $h(n_0) = f(n_0) \cup X$ for some $X \subseteq \omega$ where $x \notin X$ and $|h(n_0)| = n_0$. Then $h \leq f$ and is disjoint with g. □

4.10 Lemma. *$(Loc, \leq)$ is σ-linked.*

Proof. For $k \in \omega$, put $L_k = \{f \in Loc : k$ is minimal such that $(\forall n \in \omega)|f(n)| \leq \min(n, k)\}$. Let $\mathcal{S}_k = \{h : 2k \to [\omega]^{<\omega} : (\forall i < 2k)|h(i)| \leq \min(i, k)\}$.

For $h \in \mathcal{S}_k$, put $L_{k,h} = \{f \in L_k : f \restriction 2k = h\}$. We get $L_k = \bigcup_{h \in \mathcal{S}_k} L_{k,h}$ and $L_{k,h}$ is 2-linked for every $h \in \mathcal{S}_k$. Hence $Loc = \bigcup_{k \in \omega} L_k$ is σ-linked. □

4.11 Remark. One can easily modify the above proof to show that $(Loc, \leq)$ is σ-k-linked for any $k \geq 2$.

4.12 Theorem. *The Boolean algebra $\mathbb{B} = \mathrm{RO}(\mathrm{Loc}, \leq)$ carries a strictly positive supermeasure and no strictly positive exhaustive submeasure.*

Clearly, since Loc is σ-linked, it is σ-bounded hence by the previous $\mathbb{B}$ carries a strictly positive supermeasure. The remaining part of the proof will be given in a few propositions using the auxiliary notion of the localisation property.

4.13 Definition. We say that a Boolean algebra $\mathbb{B}$ satisfies the *localisation property* if there is a matrix $\{a_{n,k} : n, k \in \omega\} \subseteq \mathbb{B}$ such that

(1) $(\forall n > 0)\{a_{n,k} : k \in \omega\}$ is $(n+1)$-disjoint, i.e.

$$(\forall X \in [\omega]^{\geq n+1}) \bigwedge_{k \in X} a_{n,k} = \mathbf{0}, \text{ and}$$

(2) $(\forall f : \omega \to \omega) \liminf a_{n,f(n)} = \bigvee_{k\in\omega} \bigwedge_{n\geq k} a_{n,f(n)} = \mathbf{1}.$

4.14 Proposition. *Let* $\mathbb{B}$ *be a Boolean algebra that carries a strictly positive exhaustive submeasure. Then* $\mathbb{B}$ *does not satisfy the localisation property. Moreover, for any matrix* $\{a_{n,k} : n, k \in \omega\}$ *satisfying (1) of the above definition, there is an* $f : \omega \to \omega$ *such that* $\liminf a_{n,f(n)} = \mathbf{0}$.

4.15 Lemma. *Let* μ *be an exhaustive submeasure on* $\mathbb{B}$. *For a given* $n > 0$ *let* $d = \langle d_k : k \in \omega\rangle \in \mathbb{B}^\omega$ *be an* $(n+1)$*-disjoint sequence. Then* $\lim\limits_{k\to\infty} \mu(d_k) = 0$.

Proof. Since μ is the exhaustive submeasure on $\mathbb{B}$ it is sufficient to show that $d \leq f_1 \vee ... \vee f_n$ in $\mathbb{B}^\omega$ for some disjoint sequences $f_1,...,f_n \in \mathbb{B}^\omega$. We proceed by the induction: if $n = 1$ then d is disjoint sequence and we are done. Let us assume that $n > 1$. We put $f(m) = d(m) - \bigvee_{i<m} d(i)$, for all $m \in \omega$. Clearly, f is a disjoint sequence and $g = d - f$ is an n-disjoint sequence: By the definition, $g(m) = d(m) \wedge \bigvee_{i<m} d(i)$, in particular $g(0) = \mathbf{0}$. Let $x_1 < x_2 < \cdots < x_n$, we show that $\bigwedge_{j=1}^n g(x_j) = \mathbf{0}$. If $x_1 = 0$ we are done, assume that $x_1 \geq 1$ then $\bigwedge_{j=1}^n g(x_j) = \bigwedge_{j=1}^n d(x_j) \wedge \bigvee_{i<x_j} d(i) = \bigvee_{i<x_1} d(i) \wedge \bigwedge_{j=1}^n d(x_j)$. The last element is equal to $\mathbf{0}$ since by our assumption the sequence d is $(n+1)$-disjoint. □

Proof. OF PROPOSITION III.4.14

Let μ be a strictly positive exhaustive submeasure on $\mathbb{B}$. Let $\{a_{n,k} : n, k \in \omega\}$ be a matrix satisfying (1) of definition III.4.13. We shall define a function $f : \omega \to \omega$.

For $n > 0$, using the lemma, we can find $k \in \omega$ with $\mu(a_{n,k}) < \frac{1}{n}$ and put $f(n) = k$.

Hence, for every n, $\mu(a_{n,f(n)}) < \frac{1}{n}$, so $\liminf a_{n,f(n)} = \mathbf{0}$. □

4.16 Lemma. *The Boolean algebra* $\mathbb{B} = \mathrm{RO}(Loc, \leq)$ *satisfies the localisation property.*

Proof. Put $a_{0,k} = \mathbf{1}$ for every $k \in \omega$. For $n > 0$, $k \in \omega$ put $a_{n,k} = \bigvee_{\mathbb{B}}\{f \in Loc : k \in f(n)\}$. We show that the matrix $\{a_{n,k} : n, k \in \omega\}$ witnesses the localisation property of $\mathbb{B}$.

Suppose that (1) fails for $\{a_{n,k} : n, k \in \omega\}$, i.e. there is $n \in \omega$ and mutually distinct $k_1, \ldots, k_{n+1} \in \omega$ such that $a_{n,k_1} \wedge \cdots \wedge a_{n,k_{n+1}} \neq \mathbf{0}$. It follows that there are $f, f_1, \ldots, f_{n+1} \in Loc$ such that $k_i \in f_i(n)$ and $f \leq f_i$, for $i = 1, \ldots, n+1$. Hence $\{k_1, \ldots, k_{n+1}\} \subseteq f(n)$. But $|f(n)| \leq n$ - a contradiction.

For (2), let $\varphi : \omega \to \omega$ be an arbitrary function. We show that $\liminf a_{n,\varphi(n)} = \bigvee_k \bigwedge_{n\geq k} a_{n,\varphi(n)} = \mathbf{1}$. Let $f \in Loc$ be arbitrary. We find $g \in Loc$ and $k \in \omega$ such that $g \leq f$ and $g \leq \bigwedge_{n\geq k+1} a_{n,\varphi(n)}$.

Indeed, there must be $k \in \omega$ such that $f \in L_k$. Define $g : \omega \to [\omega]^{<\omega}$ by putting $g(n) = f(n)$ for $n \le k$ and $g(n) = f(n) \cup \{\varphi(n)\}$ for $n > k$. Clearly, $g \in Loc$, $g \le f$ and $g \le \bigwedge_{n \ge k+1} a_{n,\varphi(n)}$. □

THIS COMPLETES THE PROOF OF THEOREM III.4.12 □

4.17 Remark. Whether or not a Boolean algebra carries a strictly positive exhaustive submeasure is not significant with respect to adding a dominating real in generic extensions. The localisation forcing adds a dominating real. On the other hand, there are Boolean algebras, e.g. Hechler's forcing, that are σ-centered and therefore carry a strictly positive measure, also adding a dominating real.

5. RELATIONS GIVEN BY MONOTONE FUNCTIONS

We conclude this chapter with the introduction of well founded relation. One can distinguish uniformly exhaustive function by the height of this relation.

5.1 Definition. Let f be a monotone real function on Boolean algebra $\mathbb{B}$ such that $f(\mathbf{0}) = 0$. Let $\varepsilon > 0$, for $a, b \in \mathbb{B}$ define

$$b <_{f,\varepsilon} a \quad \text{if and only if} \quad b \le a \quad \& \quad f(a - b) \ge \varepsilon.$$

For any positive $\varepsilon > 0$ there is the largest downward closed subset of $\mathbb{B}$ on which the relation $<_{f,\varepsilon}$ is well founded.

5.2 Definition. Denote $K_0 = \{a \in \mathbb{B} : f(a) < \varepsilon\}$ the set of $<_{f,\varepsilon}$ minimal elements and $K_\alpha = \{a \in \mathbb{B} : \{b < a : f(a - b) \ge \varepsilon\} \subset \bigcup\{K_\beta : \beta < \alpha\}\}$.

There is $\bar{\alpha} \in On$ such that $K_{\bar{\alpha}} = K_{\bar{\alpha}+1}$. Denote $\mathrm{Ker}(f, \varepsilon) = K_{\bar{\alpha}}$, well founded kernel of the relation $<_{f,\varepsilon}$ and let $\tau_{f,\varepsilon} : \mathrm{Ker}_{f,\varepsilon} \to On$ be the type function. Whenever $<_{f,\varepsilon}$ is well founded on $\mathbb{B}$ we define its height as $\tau_{f,\varepsilon}(\mathbf{1})$.

5.3 EXAMPLE. Let $\mathbb{B} = \mathcal{P}(\omega)$ and define function f by the following:

$$f(x) = \begin{cases} 0 & \text{if } x = \mathbf{0} = \emptyset \\ \frac{1}{2} & \text{otherwise} \\ 1 & \text{if } x = \mathbf{1} = \omega \end{cases}$$

If $\varepsilon \le \frac{1}{2}$ then $K_0 = \{\mathbf{0}\}$, $K_1 = \{x \in \mathbb{B} : x \text{ is an atom of } \mathbb{B}\}$, generally $K_n = \{x \subset \omega : |x| \le n\}$. Clearly the height of the well founded kernel is ω i.e. $\mathrm{Ker}_{f,\varepsilon} = \bigcup\{K_n : n \in \omega\}$. Hence there is no infinite set in the well founded kernel.

On the other hand if $\frac{1}{2} < \varepsilon < 1$ then $K_0 = \mathcal{P}(\omega) \setminus \{\omega\}$ and $K_1 = \mathcal{P}(\omega)$. Hence the relation $<_{f,\varepsilon}$ is well founded on $\mathbb{B}$ and the height of the relation is 1. △

5.4 Lemma. *Let* $f \in Mon_0(\mathbb{B})$. *Then*

(i) the relation $<_{f,\varepsilon}$ *is well founded on* $\mathbb{B}$ *for any* $\varepsilon > 0$ *if and only if* f *is an exhaustive function on* $\mathbb{B}$,

(ii) the height of the relation $<_{f,\varepsilon}$ is finite for any $\varepsilon > 0$ if and only if f is a uniformly exhaustive function on $\mathbb{B}$.

Proof. (i) If the function f is not exhaustive, then there is a positive $\varepsilon > 0$ and a disjoint sequence $\langle a_n : n \in \omega\rangle$ such that $f(a_n) > \varepsilon$. Hence the sequence $b_n = \mathbf{1} - \bigvee\{a_i : i < n\}$ forms an infinite $<_{f,\varepsilon}$-descending chain and vice versa.

(ii) Let us assume that f is uniformly exhaustive, then for any $\varepsilon > 0$ the maximal length of $<_{f,\varepsilon}$-descending chain is equal to the constant given by uniform exhaustivity. Hence every $<_{f,\varepsilon}$-increasing chains in $\mathbb{B}$ is bounded, i.e. the height of the relation $<_{f,\varepsilon}$ is finite.

Now let the height be finite for any $\varepsilon > 0$ and assume that f is not uniformly exhaustive. Then there is a witness $\bar{\varepsilon} > 0$ such that there is a disjoint sequence $\langle a_n : n \in \omega\rangle$ such that for any $k \in \omega$ $|\{n \in \omega : f(a_n) > \bar{\varepsilon}\}| \geq k$ i.e. $\tau_{f,\varepsilon}(\mathbf{1}) > k$ for any $k \in \omega$, hence infinite. $\square$

5.5 EXAMPLE. If f is an uniformly exhaustive monotone function then the height of $<_{f,\varepsilon}$ relation is finite. One can ask a natural question if the height for exhaustive function is ω or if it can be larger. The answer is that the height of the $<_{f,\varepsilon}$ relation can be arbitrary for exhaustive function f. This example is due to E. Thümmel.

We proceed in two steps:

(i) If for some Boolean algebra $\mathbb{B}$ and some monotone exhaustive function f the height $\tau_{\varepsilon,f}(\mathbf{1}) = \alpha$ then there is Boolean algebra $\mathbb{B}'$ and exhaustive function f' such that $\tau_{\varepsilon,f'}(\mathbf{1}) > \alpha$.

Put $\mathbb{B}' = \mathbb{B} \times \{\mathbf{0}, \mathbf{1}\}$ and

$$f'(b,i) = \begin{cases} f(b) & \text{if } i = \mathbf{0} \\ \max\{f(b), \varepsilon\} & \text{if } i = \mathbf{1}. \end{cases}$$

Now $\tau_{\varepsilon,f'}(\mathbf{1_B}, \mathbf{0}) = \tau_{\varepsilon,f}(\mathbf{1}) = \alpha$ but $(\mathbf{1_B}, \mathbf{1}) >_\varepsilon (\mathbf{1_B}, \mathbf{0})$. So $\tau_{\varepsilon,f'}(\mathbf{1}) \geq \alpha + 1$, and we are done. Note that f' has the same norm as f.

(ii) If there is $\{(\mathbb{B}_\alpha, f_\alpha) : \alpha \in A\}$, f_α is exhaustive normalized function on $\mathbb{B}_\alpha$ such that $\tau_{\varepsilon,f_\alpha}(\mathbf{1}) = \alpha$ then there is Boolean algebra $\mathbb{B}$ and exhaustive normalized function f such that

$$\tau_{\varepsilon,f}(\mathbf{1}) \geq \sup_A \alpha.$$

Let $\mathbb{B}$ be a free product of $\mathbb{B}_\alpha$'s; $\mathbb{B} = \oplus_{\alpha\in A}\mathbb{B}_\alpha$. We have to define suitable f on $\mathbb{B}$. We start with the dense subset $P = \{b^+ \in \mathbb{B} : (\exists F \in [A]^{<\omega})\ (\forall \alpha \in F)\ (\exists b_\alpha \in \mathbb{B}_\alpha \setminus \{\mathbf{0}_\alpha, \mathbf{1}_\alpha\})\ b = \bigwedge_{\alpha\in F} b_\alpha\}$ and put

$$f^*(b) \ = \ \min_{\alpha\in F}(f_\alpha(b_\alpha), \frac{1}{|F|}), \ \text{ and } f^*(\mathbf{1}_\alpha) \ = \ 1, \text{ for each } \alpha \in A.$$

f^* is monotone function on P. Finally we define f for $b \in \mathbb{B}$

$$f(b) \ = \ \sup\{f(c) \ : \ c \in P \text{ and } c \leq b\}.$$

Such defined f is exhaustive (easy use of Δ-system lemma), $f \restriction \mathbb{B}_\alpha = f_\alpha$ and $\mathbb{B}_\alpha \subset \mathbb{B}$, it follows that $\tau_{\varepsilon,f_\alpha}(\mathbf{1}) \leq \tau_{\varepsilon,f}(\mathbf{1})$ for each $\alpha \in A$. Hence

$$\tau_{\varepsilon,f}(\mathbf{1}) \geq \sup_A \alpha.$$

$\triangle$

Following easy fact will lead us to the topic we will focus on in the next chapter.

5.6 Fact. *Let f be a monotone exhaustive function on Boolean algebra $\mathbb{B}$. If $\xi < \zeta$ then $\tau_{\xi,f}(\mathbf{1}) \geq \tau_{\zeta,f}(\mathbf{1})$ for each $u \in \mathbb{B}$.*

Now the following definition makes sense: Let f be an exhaustive function on Boolean algebra $\mathbb{B}$, then for any $u \in \mathbb{B}$ let

$$\nu_f(u) = \begin{cases} \sup\{\ \varepsilon \in (0,1]\ :\ \tau_{\varepsilon,f}(u) + 1 > \omega\ \} \\ 0 \quad \text{if there is no such } \varepsilon. \end{cases} \tag{III.2}$$

Thus defined ν_f is an exhaustive function on $\mathbb{B}$ (see IV.6.8, IV.6.4) and hence ν itself can be viewed as an operator on monotone exhaustive functions on Boolean algebra

$$\nu : f \longmapsto \nu_f.$$

From the previous (III.5.4) it follows that $\nu_f \equiv 0$ if and only if f is uniformly exhaustive. In the following chapter we will discus this (Fremlin - Kupka) operator (see IV.6.1) in more general setting.

IV. Lattices of Submeasures and Supermeasures

In the previous we introduced $\mathrm{Sub}(\mathbb{B})$, a set of all submeasures (III.1.9) on a given Boolean algebra $\mathbb{B}$, as a subset of a vector space. We know that $\mathrm{Sub}(\mathbb{B})$ forms a cone in $\mathrm{Fn}(\mathbb{B})$, a real vector space of all bounded functions on $\mathbb{B}$ (III.1.10).

1. General setting

Here we focus on its lattice structure. We will mention here the topology on the set $\mathrm{Sub}(\mathbb{B})$, which is naturally given as the topology of pointwise convergence or equivalently as a subspace of product $\prod_{a\in\mathbb{B}}\mathbb{R}$. We start with the definition of the ordering $\leq$ on $\mathrm{Fn}(\mathbb{B})$

$$f \leq g \text{ if and only if } f(a) \leq g(a) \text{ for all } a \in \mathbb{B}.$$

The structure $\mathrm{Fn}(\mathbb{B})$ with ordering $\leq$ forms a conditionally complete lattice.

1.1 Theorem. *$\mathrm{Sub}(\mathbb{B})$, partially ordered by $\leq$, is a conditionally complete lattice, i.e. every bounded family of submeasures has the least upper bound in $\mathrm{Sub}(\mathbb{B})$.*

Proof. For $\mathcal{S} \subseteq \mathrm{Sub}(\mathbb{B})$ put

$$\psi(a) = \sup\{\varphi(a) : \varphi \in \mathcal{S}\}, \text{ for every } a \in \mathbb{B}.$$

Since the family $\mathcal{S}$ is bounded, ψ is a real function. Obviously, ψ is a submeasure. $\square$

For an arbitrary non-empty family of submeasures $\mathcal{S} \subseteq \mathrm{Sub}(\mathbb{B})$, we get the greatest lower bound by putting

$$\psi = \sup\{\chi \in \mathrm{Sub}(\mathbb{B}) : (\forall\varphi \in \mathcal{S})\ \chi \leq \varphi\}.$$

It is easy to see that for two submeasures $\varphi, \psi \in \mathrm{Sub}(\mathbb{B})$ the infimum $\varphi \wedge \psi$ is given by the formula $(\varphi \wedge \psi)(a) = \inf\{\varphi(x \wedge a) + \psi(a - x) : x \in \mathbb{B}\}$, for each $a \in \mathbb{B}$.

If both φ and ψ are measures on $\mathbb{B}$ then $\varphi \wedge \psi$ is a measure, too. Hence $(\mathrm{Meas}(\mathbb{B}), \leq)$ is a lower sublattice of $(\mathrm{Sub}(\mathbb{B}), \leq)$.

Note that generally $\mathrm{Sub}(\mathbb{B})$ is not a sublattice of $\mathrm{Fn}(\mathbb{B})$.

2. Pavement construction of submeasures

Pavement construction of submeasures is a general method for constructing submeasures. Specially any function $f \in \mathrm{Fn}(\mathbb{B})$ determines some submeasure on Boolean algebra $\mathbb{B}$, namely

$$\mu_f = \sup\{\nu \ : \ \nu \in \mathrm{Sub}(\mathbb{B}) \ \& \ \nu \leq f\}.$$

This construction of submeasures can be viewed as an operator. Since every submeasure is a nonnegative function, we can restrict ourselves to $\mathrm{Fn}^+(\mathbb{B}) \subset \mathrm{Fn}(\mathbb{B})$, a set of all nonnegative bounded functions on a Boolean algebra $\mathbb{B}$.

$$\begin{array}{rcl} \mathcal{D} : \mathrm{Fn}^+(\mathbb{B}) & \longrightarrow & \mathrm{Sub}(\mathbb{B}) \\ f & \longmapsto & \mu_f. \end{array} \tag{IV.1}$$

2.1 Proposition. *Let $\mathbb{B}$ be a Boolean algebra and let $f \in Fn^+(\mathbb{B})$. Then for each $a \in \mathbb{B}$*

$$\mu_f(a) \;=\; \inf\{\sum_{b\in K} f(b) \;:\; K \in [\mathbb{B}]^{<\omega} \;\&\; a \leq \bigvee K\}.$$

Proof. Let us denote the function on the right side of the equation by ν, i.e for each $a \in \mathbb{B}$

$$\nu(a) = \;\inf\{\sum_{b\in K} f(b) \;:\; K \in [\mathbb{B}]^{<\omega} \;\&\; a \leq \bigvee K\}.$$

Our aim is now to show, that $\nu = \mu_f$. Clearly ν is a submeasure and $\nu \leq f$. Suppose, that $\varphi \leq \mu_f$ is a submeasure on $\mathbb{B}$. We claim that $\varphi \leq \nu$.

Suppose opposite, i.e. that there is some $a \in \mathbb{B}$ for which $\varphi(a) > \nu(a)$. For arbitrary small $\varepsilon > 0$ there is a finite $K \subset \mathbb{B}$ cover $\bigvee K \geq a$ for which

$$\sum_{k\in K} f(k) < \nu(a) + \varepsilon.$$

Since we suppose that a submeasure $\varphi \leq f$, we get a contradiction

$$\sum_{k\in K} \varphi(k) \leq \sum_{k\in K} f(k) < \varphi(a).$$

$\square$

Note that in the preceding lemma there were no need for global function on Boolean algebra $\mathbb{B}$, the only need was to cover the unite element $\mathbf{1} \in \mathbb{B}$. We can generalise this construction in the following way. This method gives the name to the construction.

2.2 Definition. A *pavement* for a Boolean algebra $\mathbb{B}$ is a subset $D \subseteq \mathbb{B}$ together with a mapping $w : D \to [0, \infty)$ such that for some finite $D_0 \subseteq D$, $\mathbf{1} = \bigvee D_0$. (Usually it is considered that $\mathbf{1} \in D$.)

Having pavement (D, w) on $\mathbb{B}$ one can define μ_w similarly as previously

$$\mu_w(a) = \inf\{\sum_{d\in F} w(d) : F \in [D]^{<\omega} \quad a \leq \vee F\}.$$

2.3 EXAMPLE. Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$ and ν be a submeasure on $\mathbb{B}$. Then ν is a pavement for algebra $\mathbb{C}$ and μ_ν is an extension of ν to a submeasure on $\mathbb{C}$. Moreover the norm of μ_ν equals to the norm of ν. $\triangle$

Let us remark, that not necessarily the submeasure μ_f is different from identical zero even if $f \in Fn^+(\mathbb{B})$ is strictly positive. This leads us to the following definition.

2.4 Definition. Nonnegative bounded function on Boolean algebra is called *subpathological* if μ_f is identical zero. In another words, there is no nontrivial submeasure below f.

The subpathological functions on a Boolean algebra $\mathbb{B}$ can be simply characterised.

2.5 Lemma. *A function $f \in Fn^+(\mathbb{B})$ is subpathological if and only if for any $\varepsilon > 0$, there is a finite $\langle a_i : i \in I\rangle$ covering of unity; i.e: $\bigvee_I a_i = \mathbf{1}$, for which*

$$\sum_{i \in I} f(a_i) < \varepsilon.$$

Proof. This characterisation of subpathological functions is an immediate corollary of the proposition IV.2.1. □

Now we are ready to show, that from a topological point of view the set of subpathological functions is very large.

2.6 Theorem. *The set of all subpathological functions on an atomless Boolean algebra $\mathbb{B}$ is G_δ dense set in the space (Fn^+, τ_p), where τ_p is inherited from the product topology.*

Proof. For any $\varepsilon > 0$ and for any finite partition $\mathcal{P}$ we put

$$\mathcal{O}(\mathcal{P}, \varepsilon) = \{f \in Fn^+(\mathbb{B}) : \sum_{p \in \mathcal{P}} f(p) < \varepsilon\}.$$

Clearly each $\mathcal{O}(\mathcal{P}, \varepsilon)$ is an open set. Since one can express all subpathological functions on $\mathbb{B}$ as

$$\bigcap_{k \in \omega} \bigcup_{\mathcal{P}} \mathcal{O}(\mathcal{P}, \frac{1}{k}),$$

where $\mathcal{P}$ is finite partition of unity, it remains to show that the set

$$U_k = \bigcup_{\mathcal{P}} \mathcal{O}(\mathcal{P}, \frac{1}{k})$$

is dense for each $k \in \omega$.

Let $f \in Fn^+(\mathbb{B})$, fix $a_0, \dots, a_{n-1} \in \mathbb{B}$ Since $\mathbb{B}$ is atomless there is a finite partition of unity $\mathcal{P}$ such that for each $p \in \mathcal{P}$, $p \neq a_i$, for each $i \in n$. Now define function

$$g(a) = \begin{cases} f(a) & \text{if } a \in \{a_0, \dots, a_{n-1}\} \\ \frac{1}{k\ |\mathcal{P}|+1} & \text{if } a \in \mathcal{P} \\ f(\mathbf{1}) & \text{otherwise.} \end{cases}$$

Clearly $g \in U_k$ and g lies in a chosen neighbourhood of f, given by $a_0, \dots, a_{n-1}$. This, together with Baire category theorem completes the proof. □

Thus functions which determine some nontrivial submeasure form F_σ meager set.

2.7 VARIATIONS OF POPOV EXAMPLE

In this example we use similar method to construct an exhaustive subpathological function and a pathological submeasure on a Cantor algebra, both strictly positive; the construction was motivated by work of V. A. Popov [Pop76].

(I) EXHAUSTIVE SUBPATHOLOGICAL FUNCTION

For a natural number $n \geq 2$ consider Boolean algebra $\mathcal{A}_n = \mathcal{P}(n)$ and a normalised submeasure μ_n on $\mathcal{A}_n$ such that

$$\mu_n(a) = \begin{cases} 0 \text{ if } a = \mathbf{0} \\ 1 \text{ if } a = \mathbf{1} \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

Let

$$X = \prod_{n \geq 2} n \tag{IV.2}$$

be a topological product of discrete spaces. The algebra of clopen subsets of the space X is the Cantor algebra $\mathcal{A}$ i.e. countable atomless Boolean algebra.

Moreover $\mathcal{A}$ is isomorphic to the free product $\otimes\langle \mathcal{A}_n : n \geq 2 \rangle$. We have a natural embedding $\mathcal{A}_n \hookrightarrow \mathcal{A}$.

We look for a function φ on $\mathcal{A}$ that extends μ_n for all $n \geq 2$, i.e.

$$\varphi \restriction \mathcal{A}_n = \mu_n.$$

Put $D = \{R : R \text{ is a relation, with } \mathrm{dom}(R) \subset \omega - \{0,1\} \text{ finite and such that } (\forall i \in \mathrm{dom}(R))\ R(i) \subsetneq i\}$. Each $R \in D$ determines an element of $\mathcal{A}$, namely $\bar{R} = \{f \in X : (\forall i \in \mathrm{dom}(R))\ f(i) \in R(i)\}$. Together with mapping w,

$$w(\bar{R}) = \frac{1}{|\mathrm{dom}(R)| + 1},$$

we have a pavement. Note that $\emptyset \in D$, $\bar{\emptyset} = X$ and $w(\bar{\emptyset}) = 1$.

Define a function φ on $\mathcal{A}$ by

$$\varphi(A) = \sup\{w(\bar{R}) : A \supset \bar{R},\ R \in D\}.$$

2.8 Claim. *The function φ is exhaustive, normalised, monotone such that*

$$(\forall R \in D)\ \varphi(\bar{R}) = w(\bar{R}), \quad (\forall n \geq 2)\ \varphi \restriction \mathcal{A}_n = \mu_n$$

and moreover, it is subpathological and not uniformly exhaustive.

Proof. Clearly φ is normalised and monotone. Whenever $S, R \in D$ $\bar{S} \subset \bar{R}$, then $|\mathrm{dom}(S)| \geq |\mathrm{dom}(R)|$, hence $(\forall R \in D)\ \varphi(\bar{R}) = w(\bar{R})$.

Now let $x \subsetneq n$ then $\varphi(\bar{R}_x) = 1/2$, where the relation $R_x = \{\langle n, i \rangle : i \in x\}$, $R_x \in D$. Hence $(\forall n \geq 2)\ \varphi \restriction \mathcal{A}_n = \mu_n$.

(i) φ is not uniformly exhaustive, because for given $n \geq 2$ the set $\{\langle i \rangle : i < n\}$, where $\langle i \rangle = \{f \in X : f(n) = i\}$, is partition of unity into n pieces and each of them has the φ value equal to $1/2$.

(ii) φ is an exhaustive function. Assume not. Then for some $\varepsilon > 0$ there is a sequence $\langle R_i : i \in \omega \rangle$, $R_i \in D$, with $\bar{R}_i \cap \bar{R}_j = \emptyset$, whenever $i \neq J$, such that $w(\bar{R}_i) \geq \varepsilon$ for any $i \in \omega$. The latter condition says that size of $\mathrm{dom}(R_i)$ is bounded, so we can assume, that all domains are of the same size, say k. Now apply a Δ-system technology to the family $\langle \mathrm{dom}(R_i) : i \in \omega \rangle$ and we obtain an infinite set $I \subset \omega$, and a nonempty set $a \subset \omega$, a kernel of a Δ-system $\langle \mathrm{dom}(R_i) : i \in I \rangle$. Since the sets $\bar{R}_i$ are pairwise disjoint we obtain infinitely many different subsets of the finite set $\bigcup\{\{j\} \times j : j \in a\}$, which is impossible.

(iii) φ is subpathological. Take $k \geq 2$ and $I \subset \omega$, $|I| = k$. Consider a clopen partition $\{X_0, X_1\}$

$$X_0 = \{f \in X : \sum_{i \in I} f(i) \text{ is odd}\},$$

$$X_1 = \{f \in X : \sum_{i \in I} f(i) \text{ is even}\}$$

Then $\varphi(X_j) \leq 1/(k+1)$. For any $R \in D$ with $|\mathrm{dom}(R)| < k$, $\bar{R} \cap X_j \neq \emptyset$ for each $j \in \{0, 1\}$. By the lemma IV.2.5 φ is subpathological. □

2.9 Remark. On every infinite atomless Boolean algebra $\mathbb{B}$ there is a normalised function $\bar{\varphi} \in \mathrm{Fn}(\mathbb{B})$, which is exhaustive but not uniformly exhaustive:

One can embed Cantor algebra $\mathcal{A} \hookrightarrow \mathbb{B}$ and extend φ to $\bar{\varphi}$ on $\mathbb{B}$ in the following way

$$\bar{\varphi}(b) = \sup\{\varphi(a) : a \in \mathcal{A},\ a \leq b\}, \text{ for each } b \in \mathbb{B}.$$

(II) PATHOLOGICAL SUBMEASURE

In the previous construction we used pavement D and weight function w to obtain an exhaustive subpathological function. In the following we will use this pavement together with the weight function in the pavement construction to obtain a pathological submeasure.

2.10 Definition. The submeasure given by pavement construction from the pavement D and weight w defined in the previous example is called *The Popov submeasure*.

2.11 Claim. *The Popov submeasure μ extends the weight function w, so it is normalised and μ is strictly positive.*

Proof. Fix $R \in D$ and put $d = |\mathrm{dom}(R)|$, hence $w(\bar{R}) = \frac{1}{d+1}$. Consider a covering of $\bar{R}$:

$$\bar{R} \subset \bigcup\{\bar{R}_i : 1 \leq i \leq p\}, \quad R_i \in D.$$

In order to prove that the Popov submeasure extends the weight function it is enough to show that $w(\bar{R}) \leq \sum w(\bar{R}_i)$.

Because we are looking for the smallest possible covering of $\bar{R}$, one can assume that $\mathrm{dom}(R_i) \supset \mathrm{dom}(R)$, $|\mathrm{dom}(R_i)| \geq d+1$ and that the ordering satisfies $|\mathrm{dom}(R_i)| \leq |\mathrm{dom}(R_j)|$, for every $1 \leq i \leq j \leq p$.

(i) Assume that for each $1 \leq i \leq p$

$$|\mathrm{dom}(R_i)| - d \geq i.$$

If this holds one can choose distinct numbers $\{n_i : 1 \leq i \leq p\}$ such that $n_i \in \mathrm{dom}(R_i) - \mathrm{dom}(R)$ and $x_i \in n_i$ such that $\langle n_i, x_i \rangle \notin R_i$. Then the finite function $f = \{\langle n_i, x_i \rangle : 1 \leq i \leq p\}$ determines a nonempty clopen set $[f]$ in the space X (cf. IV.2 on page 64) which is not covered by $\bigcup\{\bar{R}_i : 1 \leq i \leq p\}$; a contradiction.

(ii) Assume that there is a i_0, for which $\mathrm{dom}(R_{i_0}) - d < i_0$. Then $|\mathrm{dom}(R_i)| < i_0 + d$ for every $1 \leq i \leq i_0$ and

$$\sum_{i=1}^{i_0} w(\bar{R}_i) = \sum_{i=1}^{i_0} \frac{1}{|\mathrm{dom}(R_{i_0})| + 1} \geq \frac{i_0}{|\mathrm{dom}(R_{i_0})| + 1} > \frac{i_0}{i_0 + d} \geq \frac{1}{d+1}.$$

Which completes the proof. □

2.12 Definition. We say that submeasure φ on Boolean algebra $\mathbb{B}$ is *pathological* if $\| \varphi \| > 0$ and there is no nontrivial measure below φ, i.e.

$$\{\psi \in \mathrm{Meas}(\mathbb{B}) : \psi \leq \varphi\} = \{\bar{0}\}.$$

2.13 Lemma. *The Popov submeasure μ is pathological but non exhaustive submeasure.*

Proof. (i) Suppose that $\rho \leq \mu$ is a measure. Notice that for every $n \geq 2$ the set $\{\overline{\langle n, i \rangle} : i \in n\}$ is a disjoint partition of X into n pieces. Note that in our setting $\overline{\langle n, i \rangle} = \{f \in X : f(n) = i\}$, usually denoted by $[\langle n, i \rangle]$. Since ρ is a measure one can find a sequence $\langle x_n \in n : n \geq 2 \rangle$ such that

$$\lim_{n\to\infty} \rho(\overline{\langle n, x_n \rangle}) = 0.$$

Now choose arbitrary $\varepsilon > 0$; we show that $\|\rho\| < \varepsilon$. Fix some $k > 2/\varepsilon$. One can find a subsequence $\langle x_{n_i} : i \in k \rangle \subset \langle x_n \in n : n \geq 2 \rangle$ such that

$$\sum_{i=1}^{k} \rho(\overline{\langle n_i, x_{n_i} \rangle}) < \frac{\varepsilon}{2}.$$

Having this we define a relation R such that $\mathrm{dom}(R) = \{n_i : 1 \leq i \leq k\}$ and $R(n_i) = n_i \setminus \{x_i\}$. Clearly $\rho(\bar{R}) \leq \mu(\bar{R}) = w(\bar{R}) = \frac{1}{k+1} < \frac{\varepsilon}{2}$. Hence

$$\|\rho\| = \rho(X) = \rho(\bar{R} \cup \overline{\langle n_1, x_1 \rangle}) \cup \cdots \cup \overline{\langle n_k, x_k \rangle}) \leq \rho(\bar{R}) + \sum_{i=1}^{k} \rho(\overline{\langle n_i, x_{n_i} \rangle}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Which shows that the Popov submeasure is pathological.

(ii) It remains to show that the Popov submeasure is not exhaustive. We show that there is an atomless subalgebra $\mathcal{B} \subset \mathcal{A} = \mathit{Clop}(X)$ such that $\mu(u) \geq 1/2$ for each $u \in \mathcal{B}$.

Fix a sequence $\langle I_j : j \in \omega \rangle$ of finite subsets of ω such that $|I_j| \geq 2$ and $\max(I_j) < \min(I_{j+1})$, for each $j \in \omega$.

For any $n \in \omega$ and for any $\varphi \in {}^n2$ define

$$B_\varphi = \{f \in X : \ \forall j < n \sum_{i \in I_j} f(i) \equiv_{\mathrm{mod}(2)} \varphi(j)\}.$$

Clearly $B_\emptyset = X$. It is sufficient to show that for any $n \in \omega$ and for any $\varphi \in {}^n2$

$$\mu(B_\varphi) \geq \frac{1}{2},$$

announced subalgebra $\mathcal{B}$ is then generated by $\{B_\varphi : \varphi \in {}^{<\omega}2\}$.

Fix arbitrary $\emptyset \neq \varphi \in {}^{<\omega}2$. Let B_φ be covered by $\bigcup\{\bar{R}_i : 1 \leq i \leq p\}$. As in previous lemma we will assume that $|\mathrm{dom}(R_i)| \leq |\mathrm{dom}(R_{i+1})|$ and we distinguish two cases.

(a) Assume that there is some $i_0 \leq p$ such that $|\mathrm{dom}(R_{i_0})| < 2i_0$. Then

$$\sum_{i=1}^{p} \frac{1}{|\mathrm{dom}(R_i)| + 1} \geq \sum_{i=1}^{i_0} \frac{1}{|\mathrm{dom}(R_i)| + 1} \geq \frac{i_0}{2i_0} = \frac{1}{2}.$$

Hence $\mu(B_\varphi) \geq \frac{1}{2}$.

(b) Assume that $2i \leq |\mathrm{dom}(R_i)|$ for every $1 \leq i \leq p$. In this case we show that $\bar{R}_i$'s do not cover the set B_φ. Choose $D_i \in [\mathrm{dom}(R_i)]^2$ pairwise disjoint and take a function $g : \bigcup_{i=1}^{p} D_i \to \omega$ such that $g(k) \in k \setminus R_i(k)$ for each $1 \leq i \leq p$ and for any $k \in D_i$. Then any $f \in X$ such that for each i there is some $k \in D_i$ such that $f(k) = g(k)$, is not covered by $\bar{R}_i$'s. Our aim is now to find such f within the set B_φ.

Since the function g is given on D_i's only one can simply extend g to attain appropriate sum on I_j if the whole I_j is not covered by D_i's. Even if I_j is covered but there is some $D_i = \{d_i^1, d_i^2\} \subset I_j$ one can modify one of the value $(f(d_i^1))$ to attain appropriate sum and keep the other $(f(d_i^2) = g(d_i^2))$ to satisfy the condition for f to be not covered by $\bar{R}_i$'s.

It appears for a set I_j to be in a critical situation if it is covered by D_i's and $|D_i \cap I_j| \leq 1$, for some $1 \leq i \leq p$. In this case we choose one such i and for $d_i^1 \in D_i \cap I_j$ modify the value $g(d_i^1)$ to attain the desirable sum. Now we have to keep the value $g(d_i^2)$ which is no longer in I_j, say it belongs to I_k. Assuming that in the worst possible case the set I_k is also in a critical situation we have to modify some other value in I_k. By our assumption that $|I_i| \geq 2$ we have a plenty of room to do that. One can continue in this fashion to take care of all I_j's and obtain a partial function f. Every extension of f belongs to B_φ and it follows from the construction that it is not covered by any $\bar{R}_i$; a contradiction with the fact that the collection of $\bar{R}_i$'s was covering the set B_φ. $\square$

$\triangle$

3. MINIMAL SUBMEASURES

3.1 Definition. A submeasure $\varphi \in \mathrm{Sub}(\mathbb{B})$ is called *minimal* if there is no submeasure $\psi \leq \varphi$ with $\| \psi \| = \| \varphi \|$ and $\psi(a) < \varphi(a)$ for some $a \in \mathbb{B}$.

We show that the minimality of φ on every factor of $\mathbb{B}$ characterises φ to be a measure on $\mathbb{B}$.

3.2 Proposition. *Let $\varphi \in Sub(\mathbb{B})$. Then φ is a measure on $\mathbb{B}$ if and only if the submeasure $\varphi \restriction (\mathbb{B} \restriction a)$ is minimal for all $a \in \mathbb{B}^+$. In particular, every measure is a minimal submeasure.*

3.3 Definition. *Two*, respectively, *three - additive function* on a Boolean algebra $\mathbb{B}$ is a function $f : \mathbb{B} \to \mathbb{R}$ satisfying

(i) $(\forall a \in \mathbb{B}) \quad f(a) + f(-a) = f(\mathbf{1})$, respectively

(ii) for any partition a_1, a_2, a_3 of unity $f(a_1) + f(a_2) + f(a_3) = f(\mathbf{1})$.

3.4 Fact. *$3-$additive functions on $\mathbb{B}$ are exactly additive functions, i.e: for any disjoint elements a, b of $\mathbb{B}$ $f(a) + f(b) = f(a \vee b)$.*

The proof of Proposition IV.3.2 follows immediately from the next lemma from which it is clear, that $2-$additivity and minimality of submeasures are synonymous.

3.5 Lemma. *Let $\mathbb{B}$ be a Boolean algebra*

(i) A submeasure $\varphi \in Sub(\mathbb{B})$ is minimal if and only if it is a $2-$additive.

(ii) A function $f : \mathbb{B} \to \mathbb{R}$ is minimal submeasure if and only if it is $2-$additive, monotone function on Boolean algebra $\mathbb{B}$, for which

$f(a_1)+f(a_2)+f(a_3) \geq f(\mathbf{1})$, *for each* $a_1, a_2, a_3 \in \mathbb{B}$ *covering of unity* $\bigvee_{i \in \{1,2,3\}} a_i = \mathbf{1}$.

Proof. Clearly, the $2-$additive submeasure is minimal.
(i) Suppose now that φ is minimal. Without loss of generality, we can assume that φ is normalised.

To get a contradiction, suppose there is an $a \in \mathbb{B}$ such that $\varphi(a) + \varphi(-a) > 1$. Let ψ be the submeasure given by the formula

$$\psi(x) = \begin{cases} 0 & x = \mathbf{0}, \\ \varphi(a) & x \leq a, \\ 1 - \varphi(a) & x \leq -a \text{ and } x \neq \mathbf{0}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $1 - \varphi(a) = \psi(-a) < \varphi(-a)$ and $\| \psi \| = \| \varphi \| = 1$. Our aim is now to show that $\| \varphi \wedge \psi \| = 1$ which contradicts minimality of φ. Since the submeasure ψ has only four possible values it is easy to check that

$$(\varphi \wedge \psi)(\mathbf{1}) = \inf \{\varphi(x) + \psi(-x) : \ x \in \mathbb{B}\} = 1.$$

(ii) It remains to show that the monotone, $2-$additive function $f \in \mathrm{Fn}(\mathbb{B})$ satisfying the condition

$$f(a_1)+f(a_2)+f(a_3) \geq f(\mathbf{1}), \text{ for each } a_1, a_2, a_3 \in \mathbb{B} \text{ covering of unity } \bigvee_{i\in\{1,2,3\}} a_i = \mathbf{1}$$

is a submeasure.

Since f is monotone and $2-additive$

$$f(\mathbf{0}) + f(\mathbf{1}) = f(\mathbf{1}) \text{ and so } f(\mathbf{0}) = 0,$$

hence f is nonnegative. It remains to show that for each $a \perp b$

$$f(a) + f(b) \geq f(a \vee b).$$

Assumption on f yields that $f(a) + f(b) + f(-(a \vee b)) \geq f(\mathbf{1})$ and since f is $2-$additive we get $f(a\vee b)+f(-(a\vee b)) = f(\mathbf{1})$. Which completes the proof. $\square$

3.6 EXAMPLE. Let I be an ideal on $\mathbb{B}$. Put

$$\varphi_I^*(a) = \begin{cases} 0 & a \in I \\ \dfrac{1}{2} & a \notin I \quad \& \quad -a \notin I \\ 1 & -a \in I \end{cases} \quad \text{and} \quad \varphi_I^+(a) = \begin{cases} 0 & a \in I \\ 1 & a \notin I. \end{cases}$$

Then both φ_I^* and φ_I^+ are normalised submeasures on $\mathbb{B}$. Submeasure φ_I^* is minimal for any ideal I on $\mathbb{B}$, whereas φ_I^+ is minimal if and only if I is a prime ideal on $\mathbb{B}$ (and hence $\varphi_I^+ = \varphi_I^*$).

For two ideals $I, J \subseteq \mathbb{B}$, $I \neq J$, the submeasures φ_I^* and φ_J^* are not comparable in the lattice ordering $\leq$ of $\mathrm{Sub}(\mathbb{B})$.

We will show that for $I \neq J$

$$\varphi_I^* \wedge \varphi_J^* = \frac{1}{2}\varphi^+_{\langle I\cup J\rangle}.$$

Recall that $\varphi_I^* \wedge \varphi_J^*(a) = \inf\{\varphi_I^*(a \wedge x) + \varphi_J^*(a - x) : x \in \mathbb{B}\}$. We distinguish two cases. First, let $a \in \langle I \cup J\rangle$. Then there is $x \in \mathbb{B}$ such that $a \wedge x \in I$ and $a - x \in J$. Hence $\varphi_I^* \wedge \varphi_J^*(a) = 0 = \varphi^+_{\langle I\cup J\rangle}(a)$.

Second, let $a \notin \langle I \cup J\rangle$. Since $I \neq J$ there is $x \in I \setminus J$ (or, symmetrically, $x \in J \setminus I$). Hence $0 < \varphi_I^* \wedge \varphi_J^*(a) \leq \frac{1}{2} = \frac{1}{2}\varphi^+_{\langle I\cup J\rangle}(a)$. Note that $\varphi_I^* \wedge \varphi_J^*$ admits only values $0, \frac{1}{2}, 1$. We are done.

This example also shows that the infimum in the lattice of submeasures of $2-$additive submeasures is not necessarily a $2-$additive submeasure. In another words the infimum of two minimal submeasures is not necessarily minimal. This is different from the case of $3-$additive submeasures, i.e. measures. $\triangle$

We proved that $\mathrm{Sub}(\mathbb{B})$ is a conditionally complete lattice, i.e. any bounded $\mathcal{S} \subseteq \mathrm{Sub}(\mathbb{B})$ has the least upper bound and any non-empty $\mathcal{T} \subseteq \mathrm{Sub}(\mathbb{B})$ has the greatest lower bound. One can find useful the fact that for a chain $\mathcal{T} \subseteq \mathrm{Sub}(\mathbb{B})$, i.e. $\mathcal{T}$ is linearly ordered by $\leq$, we get the infimum ψ by putting, for all $a \in \mathbb{B}$,

$$\psi(a) = \inf\{\varphi(a) : \varphi \in \mathcal{T}\}.$$

We check that ψ is a submeasure. Let $a, b \in \mathbb{B}$. Our aim is to prove that $\psi(a \vee b) \leq \psi(a) + \psi(b)$. Since $\mathcal{T} \subseteq \mathrm{Sub}(\mathbb{B})$ we have for all $\varphi \in \mathcal{T}$

$$\varphi(a \vee b) \leq \varphi(a) + \varphi(b).$$

Thus $\psi(a \vee b) \leq \varphi(a) + \varphi(b)$. Fix now an arbitrary $\varphi_0 \in \mathcal{T}$. For all $\varphi \in \mathcal{T}$ such that $\varphi \leq \varphi_0$ we get

$$\psi(a \vee b) \leq \varphi_0(a) + \varphi(b).$$

Since $\mathcal{T}$ is linearly ordered we have for all $\varphi \in \mathcal{T}$ that $\varphi(b) \leq \varphi_0(b) \implies \varphi \leq \varphi_0$ and therefore

$$\psi(a \vee b) \leq \varphi_0(a) + \psi(b).$$

Since φ_0 was chosen arbitrary we get that ψ is subadditive.

Using the fact that every decreasing chain of submeasures has a lower bound in $\mathrm{Sub}(\mathbb{B})$, namely its infimum in the lattice $\mathrm{Fn}(\mathbb{B})$ is a submeasure again, we can apply Zorn's principle of minimality to prove the following fact.

3.7 Fact. *For an arbitrary submeasure $\varphi \in Sub(\mathbb{B})$ there is $\psi \in Sub(\mathbb{B})$ such that*

(i) $\psi \leq \varphi$,

(ii) $\| \psi \| = \| \varphi \|$ *and*

(iii) ψ *is minimal ($\equiv$ 2–additive).*

3.8 EXAMPLE. Let $\varphi \in \mathrm{Sub}(\mathbb{B})$ be a normalised submeasure. Then there is a minimal submeasure $\psi \in \mathrm{Sub}(\mathbb{B})$ below φ with $\varphi(\mathbf{1}) = \psi(\mathbf{1})$ and moreover

$$\psi(a) = \varphi(a) \text{ for all } a \in \mathbb{B} \text{ with } \varphi(a) \leq \tfrac{1}{2}, \text{ especially } \mathrm{Null}(\varphi) = \mathrm{Null}(\psi).$$

Simply put

$$\psi(a) = \begin{cases} \varphi(a) & \varphi(a) \leq \frac{1}{2}, \\ 1 - \varphi(-a) & \varphi(-a) < \frac{1}{2}, \\ \frac{1}{2} & \varphi(a) \geq \frac{1}{2} \ \& \ \varphi(-a) \geq \frac{1}{2}. \end{cases}$$

△

3.9 Theorem. *An additive function is exhaustive if and only if it is uniformly exhaustive.*

Proof. Let $f : \mathbb{B} \to \mathbb{R}$ be an additive and exhaustive function. First we claim, that f is bounded.

Assume not. We construct by the induction a sequence $b_0 = \mathbf{1} > b_1 > \dots$, such that

(i) $\sup_{a \leq b_n} |f(a)| = \infty$, and

(ii) $|f(b_{n-1} - b_n)| > n$.

Having b_n for $n \in \omega$, the sequence $\langle a_n = b_{n-1} - b_n : n \in \omega \rangle$ contradicts the exhaustivity of f.

Suppose that we have b_i, for $i \leq n$. Since $\sup_{a \leq b_n} |f(a)| = \infty$, there is some $c_n < b_n$ such that
$$|f(c_n)| > |f(b_n)| + n.$$

The assumption of additivity of f yields that

$\sup_{a \leq c_n} \ |f(a)| = \infty$, or

$\sup_{a \leq (b_n - c_n)} \ |f(a)| = \infty$.

If the latter supremum is infinite, we put $b_{n+1} = b_n - c_n$ and we are done. If the former supremum is infinite we put $b_{n+1} = c_n$ and it remains to show, that $|f(b_n - c_n)| > n$. But it is clear since by additivity of f we have $f(b_n - c_n) = f(b_n) - f(c_n)$ and so $|f(b_n - c_n)| = |f(b_n) - f(c_n)| \geq |f(c_n)| - |f(b_n)| > n$.

Now we are ready to show, that f is uniformly exhaustive. We know $-K \leq f(x) \leq K$, for each $x \in \mathbb{B}$. Assume that f is not uniformly exhaustive, than there is some positive $\varepsilon > 0$ such that for any $k \in \omega$ one can find k many disjoint elements $\langle a_i : i \leq k \rangle$ with $|f(a_i)| > \varepsilon$.

Take $k > 2K/\varepsilon$. One of the set

$\{i < k : f(a_i) > \varepsilon\}$, or

$\{i < k : -f(a_i) > \varepsilon\}$

have to be of size at least K/ε. Since f is an additive function, we get a contradiction with its boundedness. □

The previous theorem was motivated by the following Jordan decomposition theorem.

3.10 Proposition. *An additive function f on $\mathbb{B}$ is bounded if and only if $f = \mu - \nu$ where μ and ν are measures on $\mathbb{B}$.*

Proof. We put
$$\mu(a) = \sup\{f(b) : b \leq a\}, \text{ for each } a \in \mathbb{B}.$$

Let us note, that μ is nonnegative since the additivity of f yields that $f(\mathbf{0}) = 0$. We check that μ is a measure. Let $a \perp b$. Our aim is to show that $\mu(a \vee b) = \mu(a) + \mu(b)$. By the definition of $\mu(a \vee b)$ for any $\varepsilon > 0$ there is some $c \leq a \vee b$ such that $f(c) \geq 0$ and $f(c) \geq \mu(a \vee b) - \varepsilon$. The additivity of f yields that $\mu(a \vee b) - \varepsilon \leq f(c) = f(a \wedge c) + f(b \wedge c) \leq \mu(a) + \mu(b)$, so $\mu(a \vee b) \leq \mu(a) + \mu(b)$. The opposite inequality can be shown in the same way.

To complete the proof we put $\nu = \mu - f$. The function ν is clearly nonnegative and as a difference of two additive functions it is also additive, hence ν is a measure on $\mathbb{B}$ and we are done. □

3.11 EXAMPLE. In this example we show additive function that is not exhaustive.

Consider an infinite field S of subsets of a set X. The set $W = \{h : X \to \mathbb{Q} : \mathrm{rng}(h)$ is finite and $(\forall q \in \mathbb{Q})\ h^{-1}\{q\} \in S\}$ is a vector space over $\mathbb{Q}$. Let $D = \{d_n : n \in \omega\}$ be an infinite disjoint family of non empty sets from S. Characteristic functions χ_n of d_n are independent vectors, thus there is a base $\mathcal{B}$ of the vector space W which extends $\{\chi_n : n \in \omega\}$. Any mapping $\psi : \mathcal{B} \to \mathbb{R}$ is uniquely extendable to a linear function L_ψ on W. If $\lim \psi(\chi_n)$ is not equal $\mathbf{0}$, then functional f on S defined by $f(b) = L_\psi(\chi_b) = \sum_{i\in I} \alpha_i \psi(e_i)$, where $\chi_b = \sum_{i\in I} \alpha_i e_i$ and $e_i \in \mathcal{B}$ is an additive non exhaustive functional. △

Let mSub($\mathbb{B}$) denote the class of all minimal submeasures on a Boolean algebra $\mathbb{B}$. We shall deal now with three classes — submeasures, minimal submeasures and measures. One can easily see that

$$\mathrm{Sub}(\mathbb{B}) \supset \mathrm{mSub}(\mathbb{B}) \supset \mathrm{Meas}(\mathbb{B}).$$

In this section we show that Sub($\mathbb{B}$) is the complete closure of mSub($\mathbb{B}$) in terms of lattice, i.e. we can get every submeasure as a supremum of a certain set of minimal submeasures. We show that there are minimal submeasures 'unreachable' by measures.

3.12 Proposition. *Every submeasure μ on a Boolean algebra $\mathbb{B}$ is the supremum of a set of minimal submeasures. Moreover, for any $a \in \mathbb{B}$ there is $\nu \in$ mSub($\mathbb{B}$) such that $\nu \le \mu$ and $\mu(a) = \nu(a)$.*

Proof. Let $a \neq \mathbf{0}$, we can find for a submeasure $\mu_1 = \mu \restriction (\mathbb{B} \restriction a)$ a minimal submeasure ν_1 on $\mathbb{B} \restriction a$ such that $\nu_1 \le \mu_1$ and $\nu_1(a) = \mu(a)$ by fact IV.3.7. Extend ν_1 to a desired submeasure ν by putting

$$\nu(x) = \nu_1(x \wedge a).$$

Clearly, $\nu(a) = \mu(a)$ and $\nu \le \mu$. We check the 2-additivity of ν on $\mathbb{B}$. Let $x \in \mathbb{B}$. Then $\nu(\mathbf{1}) = \nu_1(a) = \nu_1(a \wedge x) + \nu_1(a - x) = \nu(x) + \nu(-x)$. □

4. PATHOLOGICAL SUBMEASURES

We showed that, in terms of lattices, minimal submeasures are dense in Sub($\mathbb{B}$). This is not the case of measures (even in mSub($\mathbb{B}$)). Nevertheless, the closure of measures in Sub($\mathbb{B}$) defines a subclass of 'nice', nonpathological submeasures.

4.1 Definition. A submeasure $\varphi \in$ Sub($\mathbb{B}$) is called an *envelope* if one can gain φ as a supremum of measures, i.e.

$$\varphi = \sup \{\mu \in \mathrm{Sub}(\mathbb{B}) : \mu \le \varphi \text{ and } \mu \text{ is a measure }\}.$$

4.2 EXAMPLE. Let φ be the largest normalised submeasure on $\mathbb{B}$, i.e. $\varphi(\mathbf{0}) = 0$ and $\varphi(a) = 1$ for $a \in \mathbb{B}^+$. Then $\varphi = \sup\{\varphi_I^+ : I$ is a maximal ideal on $\mathbb{B}\}$ is an envelope submeasure. △

4.3 Proposition. *Let φ be an envelope on $\mathbb{B}$ then for any $a \in \mathbb{B}$ there is a measure $m \leq \varphi$ on $\mathbb{B}$ such that $\varphi(a) = m(a)$.*

Proof. Let

$$\varphi = \sup\{\mu \in \mathrm{Sub}(\mathbb{B}) : \mu \leq \varphi \text{ and } \mu \text{ is a measure }\}$$

be an envelope submeasure. It suffices to show the proposition for $a = \mathbf{1}$.

First note that the space

$$X = \{f : \mathbb{B} \to \mathbb{R}^+ \ : \ 0 \leq f \leq \varphi\}$$

with pointwise topology is compact. Similarly the subspace $M \subset X$ of measures is compact. As we know that $\varphi(\mathbf{1}) = \sup\{\mu(\mathbf{1}) : \ \mu \in M\}$, the set $M_n = \{\mu \in M : \mu(\mathbf{1}) \geq \varphi(\mathbf{1}) - 1/n\}$ is nonempty. The set M_n are all closed and form centered family in a compact space, hence the intersection $\bigcap_{n\in\omega} M_n$ is nonempty. Clearly $\mu(\mathbf{1}) = \varphi(\mathbf{1})$ for any $\mu \in \bigcap_{n\in\omega} M_n$, which completes the proof. □

4.4 Definition. Submeasure is *strongly subadditive* if

$$\mu(A \cup B) \leq \mu(A) + \mu(B) - \mu(A \cap B).$$

Note that for measures this inequality is obvious. Later in this chapter we show the following theorem.

4.5 Theorem. *Each strongly subadditive submeasure is an envelope submeasure.*

Proof. Our aim is to use proposition IV.4.3 together with Theorem IV.5.5. It follows that it is enough to show that for each strongly subadditive submeasure μ the following inequality holds

$$\forall \bar{a} = \langle a_i : i \in n\rangle \in \mathbb{B}^n \quad \sum_{i\in n} \mu(a_i) \geq \mathrm{cal}(\bar{a}).$$

First we generalize the notion of caliber

4.6 Definition. Let $\bar{a} = \langle a_i : i \in n\rangle \in \mathbb{B}^n$ and $b \in \mathbb{B}$, then

$$\mathrm{cal}(\bar{a}, b) \ = \ \min_{\mathbb{B}\restriction b} \chi_{\bar{a}}.$$

□

4.7 EXAMPLE. Consider the Boolean algebra with four atoms; i.e. $\mathbb{B} = \mathcal{P}(At)$, where $|At| = 4$. We define the submeasure φ on this algebra so that φ will be an envelop but not strongly subadditive submeasure.

$$\begin{aligned}
\varphi(a) &= 1/2 && \text{for } a \in At,\\
\varphi(a \vee b) &= 1/2 && \text{for } a \neq b \in At\\
\varphi(a \vee b \vee c) &= 3/4 && \text{for different } a, b, c \in At\\
\varphi(\mathbf{1}) &= 1.
\end{aligned}$$

It is obvious that φ is an envelope submeasure and for $A = \{a, b\}$ and $B = \{b, c\}$ the strong submeasure inequality is not fulfilled. △

4.8 Fact. *Every minimal submeasure is either a measure or is not even a supremum of measures below; i.e: is not an envelope.*

Proof. Let φ be a minimal submeasure. If φ is an envelope then by (IV.4.3) there is a measure $\psi \leq \varphi$ such that $\varphi(\mathbf{1}) = \psi(\mathbf{1})$. Since φ is minimal we get $\varphi = \psi$. Hence φ is a measure.

□

4.9 EXAMPLE. There is a minimal submeasure which is not supremum of measures on $\mathbb{B}$ provided $|\mathbb{B}| \geq 5$. Take a submeasure φ_I^* for the ideal $I = \{\mathbf{0}\}$; from the example IV.3.6. Take any three pairwise disjoint elements $a_1, a_2, a_3 \in \mathbb{B}^+$. Then $\varphi_I^*(a_1) = \varphi_I^*(a_2) = \varphi_I^*(a_3) = \frac{1}{2}$ and so φ_I^* is not a measure. Since φ_I^* is minimal and satisfies criterion from above it is not an envelope.

Now we deal with pathological submeasures. We will show that on an atomless Boolean algebra there are (in topological sense) many pathological submeasures. On the other hand, there are no pathological submeasures on a finite Boolean algebra.

Indeed, let $\mathbb{B}$ be finite and ν be a submeasure on $\mathbb{B}$ with $\nu(\mathbf{1}) > 0$. Take the weight measure μ given by

$$\mu(y) = \frac{1}{|At(\mathbb{B})|} \min\{\nu(x) > 0 : x \in At(\mathbb{B})\},$$

for $y \in At(\mathbb{B})$ such that $\nu(y) > 0$ and $\mu(y) = 0$ for atoms with $\nu(y) = 0$. Thus μ is defined on $At(\mathbb{B})$ and can be uniquely extended to the whole algebra and $\mu \leq \nu$. △

The next lemma shows that there are still finite Boolean algebras with 'almost pathological' submeasures.

4.10 Lemma. *For any $\varepsilon > 0$ there is a finite set $X \neq \emptyset$ and normalised submeasure ν on $\mathcal{P}(X)$ such that for any measure $m \leq \nu$ on $\mathcal{P}(X)$, we have $m(X) \leq \varepsilon$.*

Proof. Fix an integer $n > 0$ and let $X = [2n]^n$, the set of all n-element subsets of $2n$. For $i < 2n$ let $A_i = \{x \in X : i \in x\}$ and put $\nu(A_i) = \frac{1}{n+1}$. Extend ν to a normalised submeasure on a powerset algebra $\mathcal{P}(X)$ by a pavement construction, using the family $\{A_i : i < 2n\}$ together with ν. In fact ν is extended to the whole $\mathcal{P}(X)$ in the following way. For nonempty family $\mathcal{C} \subset X$ take a set $a \subset 2n$ of minimal size such that $x \cap a \neq \emptyset$ for each $x \in \mathcal{C}$. Note that $|a| \leq n+1$. Then let $\nu(\mathcal{C}) = \frac{|a|}{n+1}$, especially note that $\nu(X) = 1$.

Let $m \leq \nu$ be a measure on $\mathcal{P}(X)$. We show that $m(X) < \frac{2}{n+1}$.

$$\sum_{i<2n} m(A_i) = \sum_{x \in X} n \cdot m(\{x\}) = n \sum_{x \in X} m(\{x\}) = n \cdot m(X).$$

Since $m \leq \nu$ we get

$$\sum_{i<2n} m(A_i) \leq \frac{2n}{n+1}$$

and hence $m(X) \leq \frac{2}{n+1}$. □

We are about to show that every atomless Boolean algebra carries many pathological submeasures. Namely, the set of all pathological submeasures on an atomless algebra $\mathbb{B}$ is a residual subset of the compact space $\mathrm{Sub}_1(\mathbb{B})$ of all normalised submeasures on $\mathbb{B}$; compare with IV.2.6

4.11 Theorem. (J. P. R. Christensen [Chr76]) *Let $\mathbb{B}$ be an atomless Boolean algebra. Then the set of all pathological submeasures on $\mathbb{B}$ is dense G_δ in $Sub_1(\mathbb{B})$.*

Proof. Let k be a positive integer. Put

$$F_k = \{\nu \in \mathrm{Sub}_1(\mathbb{B}) : (\exists \mu \in \mathrm{Meas}(\mathbb{B})) \quad \mu \leq \nu \text{ and } \mu(\mathbf{1}) \geq \frac{1}{k}\}.$$

Clearly, $\bigcup F_k$ is the set of all normalised submeasures on $\mathbb{B}$ which are not pathological. We show that F_k is a closed nowhere dense set in $\mathrm{Sub}_1(\mathbb{B})$ for every $k > 0$.

We start with checking the closedness of F_k. Let $(I, \leq)$ be an upward directed index set and $\langle \nu_i : i \in I\rangle$ be a net in F_k converging to a submeasure ν. Our aim is to show that $\nu \in F_k$.

For each $i \in I$ we can take a measure $\mu_i \leq \nu_i$ with $\mu(\mathbf{1}) \geq \frac{1}{k}$ witnessing ν_i to be in F_k. Since $M_k = \{\mu \in \mathrm{Meas}(\mathbb{B}) : \frac{1}{k} \leq \mu(\mathbf{1}) \leq 1\}$ is a compact set there is a cluster point μ of $\langle \mu_i : i \in I\rangle$ such that for each neighbourhood U of measure μ and for each $i \in I$ there is $j \geq i$ such that $\mu_j \in U$

We show that $\mu \leq \nu$. Suppose otherwise that there is $a \in \mathbb{B}$ such that $\mu(a) > \nu(a)$. Choose $\varepsilon > 0$ such that $\mu(a) - \varepsilon > \nu(a) + \varepsilon$. Since $\langle \nu_i : i \in I\rangle$ is convergent net, there is a $j \in I$ such that $|\nu_i(a) - \nu(a)| < \varepsilon$ for each $i \geq j$. μ is a cluster point, hence it follows that there is $i \geq j$ such that $|\mu_i(a) - \mu(a)| < \varepsilon$, hence $\nu_i(a) < \mu_i(a) - \varepsilon$, a contradiction.

Hence $\mu \leq \nu$ and μ witnesses ν to be in F_k. F_k is then closed and its complement $U_k = \{\nu \in \mathrm{Sub}_1(\mathbb{B}) : (\forall \mu \in \mathrm{Meas}(\mathbb{B})) \quad \mu \leq \nu \implies \mu(\mathbf{1}) < \frac{1}{k}\}$ is open. We are done if we show that U_k is dense in $\mathrm{Sub}_1(\mathbb{B})$ for every $k > 0$.

Let $\nu \in \mathrm{Sub}_1(\mathbb{B})$ and $a_0, \ldots, a_{n-1} \in \mathbb{B}$. Take a finite partition $\mathcal{P}$ of $\mathbf{1}$ refining $\{a_0, \ldots, a_{n-1}\}$. For every $p \in \mathcal{P}$ pick up a finite subalgebra A_p of $\mathbb{B} \restriction p$ and submeasure ν_p on A_p with $\nu_p(p) = \nu(p)$ such that for every measure $\varphi \leq \nu_p$ on A_p

$$\varphi(p) \;<\; \frac{1}{k} \cdot \frac{1}{|\mathcal{P}|} \quad \text{(use lemma IV.4.10).}$$

We now consider $D = \bigcup_{p \in \mathcal{P}} A_p \;\cup \{a_0\} \cup \cdots \cup \{a_{n-1}\} \cup \{\mathbf{1}\}$ as a pavement together with weight w defined by the following

$$w(a) = \begin{cases} \nu_p(a) & \text{if } a \in A_p \\ \nu(a) & \text{if } a \in \{a_0, \ldots, a_{n-1}\} \\ 1 & \text{if } a = \mathbf{1}. \end{cases}$$

Since ν and ν_p, for each $p \in \mathcal{P}$ are submeasures and moreover $\nu_p(p) = \nu(p)$, it is clear that for any finite cover $D_0 \in [D]^{<\omega}$ of unity

$$\sum_{d \in D_0} w(d) \geq \nu(\mathbf{1}) = 1.$$

Hence ν_w is normalised submeasure, for which $\nu_w \restriction A_p \leq \nu_p$, for each $p \in P$ and so $\nu_w \in U_k$. It is also easy to see, that $\nu_w(a_i) = \nu(a_i)$, because for each finite cover $D_0 \in [D]^{<\omega}$ of a_i

$$\sum_{d \in D_0} w(d) \geq w(a_i) = \nu(a_i).$$

Which completes the proof. □

We have already shown that we can find a minimal submeasure with the same norm below any submeasure on any algebra $\mathbb{B}$. We now consider a more general setting. Let $S \subseteq \mathbb{B}$ be an arbitrary subset of $\mathbb{B}$ and $f : S \to [0, 1]$ be a function. We ask whether there is a minimal normalised submeasure $\mu \in \mathrm{mSub}_1(\mathbb{B})$, or measure, such that $\mu \restriction S \leq f$. We can also ask a "complementary" question: whether μ can be found above f.

The question is clearly motivated by pathological submeasures, finding measures below a function appears for example in [Fre04]. Conditions of that type also appear in paper of A. Horn and A. Tarski [HT48] as a criteria for a function to be extendable to a measure. We will give a necessary and sufficient condition for a function to majorize a minimal submeasure and slightly modify the proof to get a version for measures. In the sequel, we deal with normalised submeasures only for the sake of simplification.

4.12 Definition. Let $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle$ be a non-empty finite sequence of elements of a Boolean algebra $\mathbb{B}$ (multiple occurrence is allowed). Let $\mathcal{P}$ denote the set of all atoms of the subalgebra generated by $\{a_0, \dots, a_{n-1}\}$. Hence $\mathcal{P}$ is a partition of unity and for all $p \in \mathcal{P}$ and $i < n$ we have either $p \perp a_i$ or $p \leq a_i$. Note that for the sake of the subsequent definition one can take any partition of unity with the latter property.

A characteristic function of a family $\bar{a}$ is a function $\chi_{\bar{a}} : \mathcal{P} \to n$ defined by

$$\chi_{\bar{a}}(p) = |\{i < n : p \leq a_i\}|.$$

We call a *caliber* of $\bar{a}$ a maximum of $\chi_{\bar{a}}$, $\mathrm{cal}(\bar{a}) = \max \chi_{\bar{a}}$.

We call a *covering* of $\bar{a}$ a minimum of $\chi_{\bar{a}}$, $\mathrm{cov}(\bar{a}) = \min \chi_{\bar{a}}$.

4.13 Fact. *For any* $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle \in \mathbb{B}^n$, $n > 0$ *we have*

(i) $\mathit{cal}(\bar{a}) = \max\{|I| : I \subseteq n$ *and* $\bigwedge_{i \in I} a \neq \mathbf{0}\}$,

(ii) $\mathit{cov}(\bar{a}) = n - \mathit{cal}(\langle -a_0, \dots, -a_{n-1} \rangle)$.

5. DUALITY OF 2-ADDITIVE FUNCTIONS

We denote by $\neg f$ the function derived from f by the following formula $\neg f(a) = f(\mathbf{1}) - f(-a)$, we will mainly use it in case of normalised function $\neg f(a) = 1 - f(-a)$, which looks more natural. First we note, that 2-additive functions are in this sense self dual.

5.1 Fact. *A function* $f : \mathbb{B} \to \mathbb{R}$ *on Boolean algebra* $\mathbb{B}$ *is 2-additive if and only if* $f = \neg f$.

There is a duality in finding normalised 2–additive nonnegative functional above or below a given function $f : \mathbb{B} \to [0, 1]$. In this setting we can formulate this duality in general.

5.2 Fact. *Let* $f : \mathbb{B} \to [0, 1]$ *be arbitrary function. For any 2–additive normalised nonnegative functional* φ *on* $\mathbb{B}$ *the following conditions are equivalent*

(i) $\varphi \leq f$,

(ii) $\varphi \geq \neg f$.

Note that whenever φ satisfies one of the equivalent conditions from the preceding Fact it has to be non-negative.

5.3 Theorem. *Let* $\mathbb{B}$ *be a Boolean algebra,* $S \subseteq \mathbb{B}$ *an arbitrary subset of* $\mathbb{B}$ *and* $f : S \to [0, 1]$ *a function. Then*

(i) there is a minimal normalised submeasure ν *such that* $\nu \upharpoonright S \leq f$ *if and only if for every set* $\{a_i \in S : i < n\}$ *the following holds*

$$\bigvee_{i<n} a_i = \mathbf{1} \longrightarrow \sum_{i<n} f(a_i) \geq 1,$$

(ii) there is a minimal normalised submeasure μ *such that* $f \leq \mu \upharpoonright S$ *if and only if for every set* $\{a_i \in S : i < n\}$ *the following holds*

$$\bigwedge_{i<n} a_i = \mathbf{0} \longrightarrow \sum_{i<n} f(a_i) \leq n - 1.$$

Proof. The condition in *(i)* is clearly necessary for ν being a normalised submeasure. Without loss of generality we can consider $S = \mathbb{B}$ since we can define $f(b) = \mathbf{1}$ for all $b \in \mathbb{B} \setminus S$. We can also redefine if necessary $f(\mathbf{1}) = 1$. Now we use function $f : \mathbb{B} \to [0, 1]$ as a pavement and we get submeasure φ_f with the declared properties except minimality. So if φ_f is not minimal we simply put ν as a minimal submeasure below φ_f and we are done.

Item *(ii)* is just the dualization of *(i)*. Let μ be a minimal submeasure below $\neg f$. This submeasure is as desired since $f(a) = 1 - \neg f(-a) \leq 1 - \mu(-a) = \mu(a)$, the last equality holds because μ is minimal, hence 2–additive. □

5.4 Remark. Whenever S is centered, we can extend S to an ultrafilter on $\mathbb{B}$ and find even a measure (2-valued) with required properties.

In the previous we use submeasures for approximations, now we focus on approximations by measures. In this case we use characterisation over finite families of elements instead of sets.

5.5 Theorem. (D. H. Fremlin [Fre03]) *Let* $w : D \to [0, 1]$ *be a function defined on a subset* $D \subseteq \mathbb{B}$ *of Boolean algebra* $\mathbb{B}$. *Then*

(i) *there is a normalised measure* m *on* $\mathbb{B}$ *such that* $m \restriction D \leq w$ *if and only if for every family* $\bar{a} = \langle a_0, \ldots, a_{n-1} \rangle \in D^n$

$$\sum_{i<n} w(a_i) \geq \mathit{cov}(\bar{a}),$$

(ii) *there is a normalised measure* m *on* $\mathbb{B}$ *such that* $m \restriction D \geq w$ *if and only if for every family* $\bar{a} = \langle a_0, \ldots, a_{n-1} \rangle \in D^n$

$$\sum_{i<n} w(a_i) \leq \mathit{cal}(\bar{a}).$$

Proof. First note, that as in previous the conditions *(i)* and *(ii)* are mutually dual. A normalised measure m as a 2-additive functional satisfies $m \restriction D \leq w$ if and only if $m \restriction D \geq \neg w$. So if $\sum_{i<n} w(a_i) \geq \mathrm{cov}(\bar{a})$, then

$$\begin{aligned} n - \sum_{i<n} w(a_i) &\leq n - \mathrm{cov}(\bar{a}) \\ \sum_{i<n} 1 - w(a_i) &\leq \mathrm{cal}(-\bar{a}) \\ \sum_{i<n} \neg w(-a_i) &\leq \mathrm{cal}(-\bar{a}). \end{aligned}$$

Since clearly all the inequalities above are equivalent, the proof of duality between *(i)* and *(ii)* is complete. □

In this context the standard approach uses the Hahn - Banach theorem. We state here a version of this theorem which we are going to use in the rest of the proof.

5.6 Theorem. **HAHN - BANACH** *Let* V *be a real vector space and let* $p : V \longrightarrow [0, \infty)$ *be a real functional satisfying following conditions:*

(i) $\forall u, v \in V \; p(u + v) \leq p(u) + p(v)$,

(ii) $\forall v \in V \; \forall \alpha \geq 0 \; p(\alpha v) = \alpha p(v)$.

Then for any $v_0 \in V$ *there is a linear functional* $L : V \longrightarrow \mathbb{R}$ *such that* $L(v_0) = p(v_0)$ *and* $L(v) \leq p(v)$ *for every* $v \in V$.

Proof. CONTINUATION OF PROOF OF THE THEOREM IV.5.5: Since we proved the duality of *(i)* and *(ii)* it suffices to prove *(i)*. One can always assume that $\mathbb{B}$ is a field of subsets of a set X, for example X is the Stone space of $\mathbb{B}$. In order to use Hahn - Banach theorem we define a real vector space V. The vector space V consist of all finite values functions $f : X \longrightarrow \mathbb{R}$ such that $\forall \alpha \in \mathrm{rng}(f) \; f^{-1}(\alpha) \in \mathbb{B}$. Now we define real functional $p : V \longrightarrow [0, \infty)$ as

$$p(h) = \inf\{\sum_{i<n} \alpha_i w(a_i) : n > 0,\ \alpha_i \geq 0,\ \langle a_i : i < n \rangle \in D^n \text{ and } \sum_{i<n} \alpha_i \chi_{a_i} \geq h\},$$

where χ_a is the characteristic function of a set $a \subseteq X$. Note that we assumed that $\mathbb{B}$ is a field of subsets of X and that for any $d \in D$ $p(\chi_d) \leq w(d) \leq 1$ and since we can assume that $\mathbf{1} \in D$ we get that for any $b \in \mathbb{B}$ $p(\chi_b) \leq 1$.

Clearly functional p satisfies conditions *(i)* and *(ii)* from Hahn - Banach theorem. We noted that $p(\chi_X) = p(\chi_{\mathbf{1}}) \leq 1$. Now we show that $p(\chi_{\mathbf{1}}) = 1$. Suppose in contrary that $p(\chi_{\mathbf{1}}) < 1$, then there are $n > 0$, $\langle a_i : i < n\rangle \in D^n$ and $\langle \alpha_i > 0 : i < n\rangle$ such that $\sum \alpha_i w(a_i) < 1$ and $\sum \alpha_i \chi_{a_i} \geq \chi_{\mathbf{1}}$. We can increase a little coefficients α_i so that the both inequalities remains the same and each $\alpha_i = q_i/r_i$ becomes rational number. There are natural numbers s_i such that $\sum_{i<n} s_i w(a_i) < t$, where t is the least common multiple of r_i's. Applying to second inequality we obtain that $\sum_{i<n} s_i \chi_{a_i} \geq t\chi_{\mathbf{1}}$, therefore the family

$$\bar{a} = \langle \overbrace{a_0, \ldots, a_0}^{s_0 \text{ times}}, \overbrace{a_1, \ldots, a_1}^{s_1 \text{ times}}, \ldots, \overbrace{a_{n-1}, \ldots, a_{n-1}}^{s_{n-1} \text{ times}} \rangle$$

has the covering $\mathrm{cov}(\bar{a}) \geq t$, but this is in contrary with our assumption since we know that $\sum_{i<n} s_i w(a_i) < t$.

Let L be a linear functional given by Hahn - Banach theorem for chosen $v_0 = \chi_{\mathbf{1}}$ i.e. $p(\chi_{\mathbf{1}}) = L(\chi_{\mathbf{1}}) = 1$. Since L is a linear functional it determines a normalised measure m on $\mathbb{B}$, $m(b) = L(\chi_b)$. It remains to show that m is non-negative for any $b \in \mathbb{B}$. Suppose in contrary that for some $b \in \mathbb{B}$ $m(b) < 0$, then $m(-b) = 1 - m(b)$ is greater then 1, but this is impossible since $L(-b) \leq p(-b) \leq 1$.

For the proof of the opposite direction let $\bar{a} = \langle a_i : i < n\rangle \in D^n$. As in the previous we will think about a_i's as about subsets of X. We have a measure m on $\mathbb{B}$ such that $m \restriction D \leq w$. So $\sum_{i<n} w(a_i) \geq \sum_{i<n} m(a_i)$. Evaluation of last sum is the same as an integral over X of a function $f = \sum_{i<n} \chi_{a_i} m(a_i)$ with respect to the measure m. Since f is piecewise constant function the integral is equal to $\sum_{i<k} f(p_i)m(p_i)$, where $\langle p_i : i < k\rangle$ is disjoint refinement of $\langle a_i : i < n\rangle$. From the definition of covering we get that $\mathrm{cov}(\bar{a}) \leq f(p_i)$ for any $i < k$ and without loss of generality we can suppose that $\bar{a}$ covers X since otherwise $\mathrm{cov}(\bar{a}) = 0$, hence $\sum_{i<k} m(p_i) = m(X) = 1$, and we are done since we've proved that $\sum_{i<n} w(a_i) \geq \sum_{i<n} m(a_i) = \sum_{i<k} f(p_i)m(p_i) \geq \mathrm{cov}(\bar{a}) \sum_{i<k} m(p_i) = \mathrm{cov}(\bar{a})$. □

Now we mention some application of Theorem IV.5.5. Consider a subset $D \subset \mathbb{B}$ of algebra $\mathbb{B}$. Fix a real number $0 < \alpha \leq 1$. The following assertion is a core of Kelley's theorem.

5.7 Proposition. *There is a normalised measure* m *on* $\mathbb{B}$ *such that* $m(d) \geq \alpha$ *for every* $d \in D$ *if and only if for every positive natural number* $n \in \omega$ *and for each sequence* $\bar{a} \in D^n$

$$\mathrm{cal}(\bar{a}) \geq n \cdot \alpha.$$

Proof. It suffices to take a constant function $w : D \to [0, 1]$ with value α and apply Theorem IV.5.5 *(ii)*. □

5.8 Definition. Let $\emptyset \neq P \subset \mathbb{B}$. The *intersection number* of P is

$$\alpha(P) = \inf\{\frac{\text{cal}(\bar{a})}{n} \ : \ n > 0 \ \& \ \bar{a} \in P^n\}.$$

The *covering number* of P is

$$\beta(P) = \sup\{\frac{\text{cov}(\bar{a})}{n} \ : \ n > 0 \ \& \ \bar{a} \in P^n\}.$$

5.9 Corollary. *Let $D \subset \mathbb{B}$, then the intersection number of D is*

$$\alpha(D) = \sup_{m \in Meas_1(\mathbb{B})} \inf\{m(d) \ : \ d \in D\}.$$

It follows that for every normalised measure on Boolean algebra $\mathbb{B}$

$\alpha(\{b \in \mathbb{B} : \ m(b) \geq \frac{1}{k}\}) \geq \frac{1}{k}$, *and*

$\beta(\{b \in \mathbb{B} : \ m(b) < \frac{1}{k}\}) \leq \frac{1}{k}$.

5.10 Theorem. (J. L. Kelley [Kel59]) *A Boolean algebra $\mathbb{B}$ carries a strictly positive measure if and only if there is a fragmentation $\mathbb{B}^+ = \bigcup_{n \in \omega} P_n$ such that each P_n has a positive intersection number.*

Proof. By the proposition IV.5.5 we have for each P_n a measure m_n on $\mathbb{B}$ such that $\forall p \in P_n \ m_n(p) > 0$.

It is enough to put

$$m = \sum_{n \in \omega} \frac{m_n}{2^{n+1}}$$

to obtain strictly positive measure on $\mathbb{B}$. □

Note that it is sufficient to have such fragmentation on some dense subset of a Boolean algebra $\mathbb{B}$.

5.11 Theorem. [Chr76] *A submeasure μ on a Boolean algebra $\mathbb{B}$ is pathological if and only if $\mu \neq \mathbf{0}$ and*

$$\begin{aligned} \beta(\{b \in \mathbb{B} : \ \mu(b) < \varepsilon\}) \quad &= 1 \ \textit{ for every } \varepsilon > 0, \textit{ or equivalently} \\ \alpha(\{b \in \mathbb{B} : \ \mu(-b) < \varepsilon\}) \quad &= 0. \end{aligned}$$

Proof. Let us denote

$$P = \{a \in \mathbb{B} : \mu(-a) < \varepsilon\}.$$

We show that if μ is a pathological submeasure, then the set P cannot have the positive intersection number.

Let us suppose in contrary that the set P has the positive intersection number $\alpha > 0$. Then there is a normalised measure m such that for each $a \in P$ the measure $m(a) \geq \alpha$. We proceed by the induction. Suppose we have $b_0 \dots b_{n-1}$ disjoint elements of $\mathbb{B}$ such that $\mu(b_i) \leq \varepsilon m(b_i)$ and $s_n = \bigvee_{i<n} b_i$ such that

$-s_n \in P$, hence $m(-s_n) \geq \alpha$. Now put for any $x \in \mathbb{B}$ $\nu_n(x) = m(x - s_n)$. Since $\nu_n(\mathbf{1}) = m(-s_n) \geq \alpha$, the measure ν_n is nontrivial and put

$$\delta_n = \sup\{\ \nu_n(b)\ :\ \mu(b) \leq \varepsilon\nu_n(b)\ \}.$$

The value δ_n has to be positive since μ is pathological and therefor there is an element b_n disjoint with s_n such that

$$\nu_n(b_n) = m(b_n) > \frac{1}{2}\delta_n.$$

Clearly $\mu(b_n) \leq \varepsilon\nu_n(b_n) = \varepsilon m(b_n)$. The following easy computation shows that $-s_{n+1} = -(\ \bigvee_{n+1} b_i\ )$ is in the set P:

$$\mu(s_{n+1}) \leq \sum_{n+1} \mu(b_i) \leq \varepsilon \sum_{n+1} m(b_i) = \varepsilon m(s_{n+1}) \leq \varepsilon,$$

which completes the induction step.

Now let $\nu(x) = \lim_n m(x - s_n)$, for $x \in \mathbb{B}$. The measure ν is nontrivial since $\nu(\mathbf{1}) = \lim_n m(-s_n) \geq \alpha$. By the assumption the submeasure μ is pathological and hence there is $a \in \mathbb{B}$ such that $\mu(a) < \varepsilon\nu(a)$. Let us denote $a_n = a - s_n$, it follows immediately that $\mu(a_n) \leq \mu(a) < \varepsilon\nu(a) \leq \varepsilon m(a_n)$. The last inequality yields that a_n takes part in the definition of δ_n and hence $m(a_n) \leq \delta_n$; note that $a_n \perp s_n$. Now we can estimate $m(b_n)$:

$$m(b_n)\ >\ \frac{1}{2}\,\delta_n\ \geq\ \frac{1}{2}\,m(a_n)\ \geq\ \frac{1}{2}\,\nu(a)\ >\ 0.$$

This is a contradiction since b_n, $n \in \omega$ are disjoint and all have the measure $m(b_n)$ greater or equal to a positive constant.

On the other hand if the set P has zero intersection number $\alpha(P) = 0$, then a submeasure μ is pathological. Suppose on contrary that there is a nontrivial measure $m \leq \mu$. Then $P \subset Q = \{a \in \mathbb{B} :\ m(-a) < \varepsilon\}$. Since m is a measure

$$Q = \{a \in \mathbb{B} :\ m(a) \geq \varepsilon\},$$

hence by corollary IV.5.9 it has positive intersection number. But $\alpha(P) \geq \alpha(Q)$ since $P \subseteq Q$ (cf. corollary IV.5.9); a contradiction. $\square$

6. FREMLIN - KUPKA OPERATOR

Remind, that the pavement construction of submeasures described in the previous can be viewed as an operator; cf. equation (IV.1) on page 62

$$\begin{aligned} \mathcal{D} : \mathrm{Fn}^+(\mathbb{B}) &\longrightarrow \mathrm{Sub}(\mathbb{B}) \\ f &\longmapsto \sup\{\nu \in \mathrm{Sub}(\mathbb{B}) : \nu \leq f\}. \end{aligned}$$

In the following we will deal with another kind of operator which can also produce some interesting examples of submeasures. One of the main advantage of this operator is that it preserves submeasures, similarly to $\mathcal{D}$. Let us remark, that in [FK90], D. Fremlin and J. Kupka dealt with this operator only in the realm of submeasures.

6.1 Definition. *The Fremlin - Kupka operator* ϕ is an operator on Fn_0^+ on a Boolean algebra $\mathbb{B}$. For a nonnegative bounded function $f \in \mathrm{Fn}_0^+$, it is defined as a limit:

$$\begin{aligned} \phi : \mathrm{Fn}_0^+(\mathbb{B}) &\rightarrow \mathrm{Fn}_0^+(\mathbb{B}) \\ \phi(f)(a) &= \lim_{n\to\infty} \varphi_n(f)(a), \end{aligned}$$

where

$$\varphi_n(f)(a) = \sup \{\min \{f(a_i) : i \leq n\} : \langle a_i : i \leq n\rangle \text{is disjoint, and } (\forall i)\ a_i \leq a\}.$$

Note that the sequence $\langle \varphi_n(f)(a) : n \in \omega\rangle$ is nonincreasing because every sequence $\langle a_i : i \leq n+1\rangle$ taking part in the definition of φ_{n+1} also participate in the definition of φ_n as the same sequence with one element omitted. Note that we do not require the sequence to be a decomposition of a and also we allow **0**'s as elements of a disjoint sequence.

6.2 Proposition. *(i)* $\phi(f) \leq f$*, for any monotone function* $f \in Fn_0^+$*,*

(ii) for any $f \in Fn_0^+$ *the function* $\phi(f)$ *is monotone,*

(iii) if $f \in Fn_0^+$ *is subadditive then* $\phi(f)$ *is a submeasure.*

Proof. Case *(i)* is clear from the definition. Case *(ii)* is true even for all φ_n, $n \in \omega$; that is for all $a \leq b$ the inequality $\varphi_n(f)(a) \leq \varphi_n(f)(b)$ holds.

Let us now suppose the opposite in case *(iii)*, that is that there are some disjoint elements a and b such that $\phi(a \vee b) - \phi(a) - \phi(b) > 0$. Then there has to be some $n \in \omega$ and some positive δ such that $\varphi_{2n}(a \vee b) - \varphi_n(a) - \varphi_n(b) > \delta$. Since $\varphi_{2n}(a \vee b)$ is defined as a supremum over disjoint families, one can pick some disjoint $\langle c_i : i \leq 2n\rangle$ such that for any $i \in 2n$ $f(c_i) \geq \varphi_{2n}(a \vee b) - \vartheta$, where ϑ is some positive real significantly smaller than δ. We define sequences $\langle a_i = c_i \wedge a : i \leq 2n\rangle$ and $\langle b_i = c_i \wedge b : i \leq 2n\rangle$. Since f is subadditive, the following inequality holds $f(a_i) + f(b_i) \geq f(c_i)$ and since ϑ is smaller then δ we get that $f(a_i) + f(b_i) \geq f(c_i) \geq \varphi_{2n}(a \vee b) - \vartheta > \varphi_n(a) + \varphi_n(b)$. From the last inequality it is clear, that at least one of this cases: (a) $f(a_i) > \varphi_n(a)$, or (b) $f(b_i) > \varphi_n(b)$ must come up for any $i \in 2n$. Since the size of partition $\langle c_i : i \leq 2n\rangle$ is $2n+1$, the case (a) or the case (b) must apply at least $n+1$ times. Without loss on generality we can suppose that the case (a) happens $n+1$ times. In this case we obtain $n+1$ disjoint elements $\langle a_k : k \in n\rangle$ such that $a_k \leq a$ and $f(a_k) > \varphi_n(a)$ for any $k \in n$, which is a contradiction with the definition of $\varphi_n(a)$. $\square$

6.3 Corollary. (D. Fremlin, J. Kupka [FK90]) *The Fremlin - Kupka operator preserves submeasures. That is*

$$\phi : Sub(\mathbb{B}) \longrightarrow Sub(\mathbb{B}).$$

6.4 Proposition. *(i) If* $f \in Fn_0^+$ *is an exhaustive function, then* $\phi(f)$ *is also exhaustive,*

(ii) A function $f \in Fn_0^+$ *is uniformly exhaustive if and only if* $\phi(f) = 0$.

Proof. *(i)* If $\phi(f)$ is not exhaustive, then there is disjoint sequence $\langle a_i : i \in \omega \rangle$ and some $\varepsilon > 0$ such that $\phi(f)(a_i) > \varepsilon$ for each $i \in \omega$. One can easily find refinement $\langle b_i : i \in \omega \rangle$ of $\langle a_i : i \in \omega \rangle$ such that $f(b_i) > \varepsilon/2$, which contradicts exhaustivity of f.

(ii) Suppose that f is uniformly exhaustive, then for any $a \in \mathbb{B}$ the sequence $\langle \varphi_n(f)(a) : n \in \omega \rangle$ converges to 0. On the other hand if f is not uniformly exhaustive one can find some $\varepsilon > 0$ such that for arbitrary length $n \in \omega$ there is some finite disjoint sequence $\langle a_i : i \leq n \rangle$ such that $f(a_i) > \varepsilon$. Hence $\phi(f)(\mathbf{1}) \geq \varepsilon > 0$. □

In the preceding we saw how the Fremlin - Kupka operator treats uniformly exhaustive functions and now we show that the Fremlin - Kupka operator in fact annihilates any uniformly exhaustive contributions of a functions on which it is applied.

6.5 Proposition. *Let* $f, g \in Fn_0^+(\mathbb{B})$ *and let* g *be uniformly exhaustive. Then* $\phi(f) = \phi(max(\bar{0}, f - g))$.

Proof. Let $a \in \mathbb{B}$, if $\phi(f)(a) = 0$, then by IV.6.4 we know that $f \restriction a$ is uniformly exhaustive and then $\max(\bar{0}, f - g)$ is also uniformly exhaustive so $\phi(\max(\bar{0}, f - g))(a) = 0$.

Suppose that $\phi(f)(a) > 0$ then for some small positive $\eta > 0$ there is $n_0 \in \omega$ such that for all $n > n_0$ $\phi(f)(a) - \eta < \varphi_n(f)(a)$. For arbitrary small $\varepsilon > 0$ there is $k \in \omega$ given by uniform exhaustivity of g such that for any disjoint sequence $\langle a_i : i \in \omega \rangle$ there is at most k a_i's such that $g(a_i) > \varepsilon$. Since $\phi(f)(a) - \eta < \varphi_{n+k}(f)(a)$ we pick some disjoint family $\langle a_i : i \in n + k \rangle$ for which the inequality $\phi(f)(a) - \eta \leq \min\{f(a_i) : i \in n + k\} \leq \varphi_{n+k}(f)(a)$ holds true. Clearly there are at least n elements of $\langle a_i : i \in n + k \rangle$ such that $|g(a_i)| \leq \varepsilon$, let them denote by $\langle a_i : i \in n \rangle$. Now we get the following

$$\phi(f)(a) - \eta - \varepsilon \leq \min\{f(a_i) : i \in n\} - \varepsilon \leq \min\{(f - g)(a_i) : i \in n\} \leq \varphi_n(f - g)(a).$$

This yields that $\phi(f)(a) \leq \phi(f - g)(a)$.

On the other hand we can again assume, that $\phi(f - g)(a) > 0$ since otherwise we know that $f - g$ is uniformly exhaustive. It is just a routine check that from uniform exhaustivity of g and $f - g$ we get uniform exhaustivity of f.

Again we fix some small $\eta > 0$, $\varepsilon > 0$ and find appropriate $k \in \omega$. In this case we are even able to proof that $0 < \varphi_{n+k}(f - g) \leq \varphi_n(f)$. We pick some disjoint family $\langle a_i : i \in n + k \rangle$ such that $0 < \varphi_{n+k}(f - g)(a) - \eta \leq \min\{(f - g)(a_i) : i \in n + k\}$. Employing the uniform exhaustivity of g we can find at least n a_i's such that $|g(a_i)| \leq \varepsilon$ and denote them $\langle a_i : i \in n \rangle$. Clearly the following chain of inequalities $\min\{(f - g)(a_i) : i \in n + k\} \leq \min\{(f - g)(a_i) : i \in n\} \leq \min\{f(a_i) : i \in n\} + \varepsilon \leq \varphi_n(f)(a) + \varepsilon$ holds true. This completes our proof. □

6.6 Theorem. *Let $f \in Fn_0^+(\mathbb{B})$ be an exhaustive function. Then there is no nonzero uniformly exhaustive submeasure below $\phi(f)$.*

Proof. Assume that there is some nonzero uniformly exhaustive submeasure $\nu \leq \phi(f)$ and put $\varepsilon = \nu(\mathbf{1})$. We pick some convergent series $\sum \alpha_i = \varepsilon/2$, $\alpha_i > 0$. Since ν is uniformly exhaustive there is some constant k_0 such that any disjoint sequence $\langle a_i : i \in \omega \rangle$ has at most k_0 elements such that $\nu(a_i) \geq \alpha_0$. Because $\phi(f)(\mathbf{1}) \geq \varepsilon$, there is a disjoint sequence of arbitrary length $n > k_0$ such that for any of its members $f(a) > \varepsilon/2$. Since we choose $n > k_0$ we can pick some element a_0 such that $\nu(a_0) < \alpha_0$ and $f(a_0) > \varepsilon/2$. Now we know that $\varepsilon/2 < \nu(1 - a_0) \leq \phi(f)(1 - a_0)$ so we can continue in similar fashion and find some a_1 disjoint with a_0 such that $\nu(a_1) < \alpha_1$ and $f(a_1) > \varepsilon/2$. In this way one can find infinite disjoint sequence $\langle a_i : i \in \omega \rangle$ such that $\nu(a_i) < \alpha_i$ and $f(a_i) > \varepsilon/2$ which contradicts exhaustivity of f. □

6.7 Corollary. (D. Fremlin, J. Kupka [FK90]) *If μ is exhaustive, non uniformly exhaustive submeasure, then $\phi(\mu)$ is nonzero exhaustive pathological submeasure.*

6.8 Remark. As we mention at the end of chapter III when $f \in \mathrm{Fn}_0$ is a monotone exhaustive function on Boolean algebra $\mathbb{B}$ then

$$\nu_f \;\equiv\; \phi(f),$$

where ν is an operator defined by equation III.2 on page 60.

Proof. Fix $a \in \mathbb{B}^+$. For each $n \in \omega$ and $\varepsilon > 0$ there is a disjoint $\langle a_i \leq a : i \in n \rangle$ such that

$$f(a_i) > \phi(f)(a) - \varepsilon.$$

For each $\varepsilon > 0$ and each $n \in \omega$ we have an increasing $a'_i = \bigvee_{j \leq i} a_j$, $i \in \omega$ such that $f(a'_j - a'_i) > \phi(f)(a) - \varepsilon$ for every $i < j < n$. Hence $\tau_{\phi(f)(a),f}(a) > n$ for each $n \in \omega$ and so

$$\nu_f(a) \;\geq\; \phi(f)(a).$$

From the definition of ν_f it follows

$$(\,\forall \varepsilon < \nu_f(a)\,) \quad \tau_{f,\varepsilon}(a) + 1 \;>\; \omega.$$

It means that for each $n \in \omega$ there is an increasing $a_i \nearrow a$, $i \in n$ such that $a_i \leq_{f,\varepsilon} a_j$, for $i < j < n$. Let $a'_i = a_{i+1} - a_i$ and we get

$$\sup_{a_i \nearrow a} \{\, \min \{f(a_i) : i \in n\} \,\} \geq \varepsilon.$$

Hence $\phi(f)(a) \geq \varepsilon$ for each $\varepsilon \leq \nu_f(a)$, which completes the proof of the remaining inequality

$$\nu_f(a) \;\leq\; \phi(f)(a).$$

□

6.9 Theorem. (D. Fremlin, J. Kupka [FK90]) *Let μ be a pathological submeasure, then $\phi(\mu) \geq 1/3\mu$.*

The following combinatorial proposition is the key tool for evaluating intersection numbers and also plays important role in the proof of Fremlin - Kupka and Kalton - Roberts [KR83] theorem. The proof can by found in [Fre89], or [KR83].

6.10 Proposition. *Suppose that* q, p, m *are natural numbers such that* $1 \leq q \leq m$ *and* $18mq \leq p^2$. *Then for a sequence* $\langle a_1, \dots, a_m \rangle$ *with caliber* $\mathrm{cal}(a_1, \dots, a_m) = q$ *of elements of Boolean algebra* $\mathbb{B}$, *there is an* (m, p)*-matrix* $(b_{i,j} \in \mathbb{B} : 1 \leq i \leq m, 1 \leq j \leq p)$ *such that*

(i) each column consist of disjoint elements,

(ii) each row contains at most three nonzero elements.

and $\bigvee\{b_{i,j} : 1 \leq j \leq p\} = a_i$ *for every* $i \leq m$.

In the following we will consider μ to be a normalised submeasure and for arbitrary $\varepsilon > 0$ we define the set

$$P = \{a \in \mathbb{B} : \mu(-a) < \varepsilon\}.$$

Since μ is pathological submeasure, then the set P cannot have the positive intersection number by the theorem IV.5.11.

Proof. OF FREMLIN - KUPKA THEOREM

Since for any $a \in \mathbb{B}^+$ $\phi(\mu \restriction a) = \phi(\mu) \restriction a$ it suffices to show that $\phi(\mu)(\mathbf{1}) \geq 1/3$.

From the previous fact we know that the set P cannot have positive intersection number, hence for any $\delta > 0$ there are finitely many elements $a_1, \dots, a_m \in P$ such that for their caliber $q = \mathrm{cal}(a_1, \dots, a_m)$ the inequality $q/m < \delta$ holds.

Now we choose a natural number p such that $p^2 \geq 18mq \geq (p-1)^2$ and apply the previous combinatorial proposition. So we have a matrix $(b_{i,j} \in \mathbb{B} : 1 \leq i \leq m, 1 \leq j \leq p)$ with disjoint columns and with rows containing at most three non zero elements. Since $\mu(a_i) > 1 - \varepsilon$ for every $i \leq m$, there exists $g(i) \leq p$ such that $\mu(b_{i,g(i)}) > 1/3 - \varepsilon$. We can pick a $1 \leq j^* \leq p$ such that the set $E = \{i : g(i) = j^*\}$ has the maximal size. Clearly $m/p \leq |E|$, and it follows that $1/(\sqrt{18\delta}) \leq m/(p-1)$ and so the fraction m/p can be arbitrary large. Hence we are able to find arbitrarily large disjoint family of elements and each of them has size at least $1/3 - \varepsilon$. Since ε was arbitrary we get $\phi(\mu)(\mathbf{1}) \geq 1/3$ which completes the proof of Fremlin - Kupka theorem. □

6.11 Theorem. (N. J. Kalton, J. W. Roberts [KR83]) *If the Boolean algebra* $\mathbb{B}$ *carries a strictly positive uniformly exhaustive submeasure* μ, *then it carries a strictly positive measure* m.

Proof. Put

$$C_n = \{ a \in \mathbb{B} : \mu(a) > \frac{1}{n+1} \},$$

then $\mathbb{B}^+ = \bigcup_n C_n$. We claim that C_n has a positive intersection number.

Since μ is uniformly exhaustive there is $r \in \omega$ such that for each disjoint $a_0, a_1, \ldots, a_r$ there is $i \leq r$ such that

$$\mu(a_i) \leq \frac{1}{4(n+1)}.$$

Toward a contradiction suppose that $\alpha(C_n) = 0$; i.e. $\forall \delta > 0$ there are $a_1, a_2, \ldots, a_m$ inC_n so that $q = \mathrm{cal}(a_1, a_2, \ldots, a_m)$ and

$$\frac{q}{m} < \delta.$$

We use the combinatorial proposition IV.6.10 for $p \in \omega$ such that $p^2 \geq 18mq \geq (p-1)^2$ and obtain a matrix $(b_{ij} \in \mathbb{B} : 1 \leq i \leq m,\ 1 \leq j \leq p)$. In each row $i \leq m$ there is some $g(i) \leq p$ such that

$$\mu(b_{i\ g(i)}) > \frac{1}{4(n+1)}.$$

Now pick $j^* \leq p$ so that $E = \{i : g(i) = j^*\}$ has a maximal size. Clearly $\frac{m}{p} \leq |E|$. Following the easy computation

$$\begin{aligned}
p^2 \geq 18qm &\geq (p-1)^2 \\
\frac{18q}{m} &\geq \left(\frac{p-1}{m}\right)^2 \\
\sqrt{18\delta} &\geq \frac{p-1}{m} \\
\frac{1}{\sqrt{18\delta}} &\leq \frac{m}{p-1}
\end{aligned}$$

yields that $\frac{m}{p-1}$ can be arbitrary large, provided $\delta \to 0$. Hence $\frac{m}{p}$ and so $|E|$ can be arbitrary large, contradicting the uniform exhaustivity of μ.

C_n has a positive intersection number $\alpha_n > 0$ and so (IV.5.9) there is a measure m_n on $\mathbb{B}$ so that $m_n \upharpoonright C_n \geq \alpha$. Define

$$m = \sum \frac{1}{2^n} m_n;$$

m is the strictly positive measure on algebra $\mathbb{B}$. □

7. HOMOGENISATION OF SUBMEASURES

First of all we have to clarify what we mean by homogeneous submeasure. We mean that the submeasure is invariant under some kind of shift or generally under some set of automorphisms. Hence we will generally formulate it as a G-homogeneity, where G will be some group acting on Boolean algebra.

Let us recall what it means that group G has an action over set X.

7.1 Definition. Let G be a group and X be a set. By *acting* of group G over set X we mean function:

$$h : G \times X \longrightarrow X.$$

with following properties. We will denote $h_p : X \to X$ the function given by $h_p(x) = h(p, x)$, $p \in G$.

(i) $h_{p \circ q}(x) = h_p(h_q(x))$, where $\circ$ is the group operation,

(ii) $h_e \equiv \mathrm{id}_X$, where $e \in G$ is unit element.

Moreover let $\mathbb{B}$ be a Boolean algebra and let G be a group having action on $\mathbb{B}$. We say that G is acting on $\mathbb{B}$ via automorphisms if

$$(\forall p \in G) \quad h_p : \mathbb{B} \longrightarrow \mathbb{B} \quad \text{is an automorphism of Boolean algebras.}$$

Now we are ready to specify what we will understand by G-homogeneous submeasure.

7.2 Definition. Let G be a group having action h on a Boolean algebra $\mathbb{B}$ via homomorphisms and let μ be a submeasure on $\mathbb{B}$. We call μ G-*homogeneous submeasure* if

$$(\forall p \in G)(\forall a \in \mathbb{B}) \quad \mu(a) = \mu(h_p(a)).$$

For illustration of our definitions we introduce here some finitary example of homogenisation of submeasure.

7.3 EXAMPLE. Let the group G be finite $|G| = n$ and G is acting on Boolean algebra $\mathbb{B}$ via automorphisms. Let mu be submeasure on $\mathbb{B}$. Then the arithmetic mean

$$\mu^G(a) \; = \; \frac{1}{n} \sum_{p \in G} \mu(h_p(a))$$

is a G-homogeneous submeasure on $\mathbb{B}$ and $\|\mu\| = \|\mu^G\|$. $\triangle$

7.4 Theorem. *Let $\mathbb{B}$ be a Boolean algebra with submeasure μ and let G be a compact group having action on $\mathbb{B}$ via homomorphisms and moreover for any $a \in \mathbb{B}$ the mapping*

$$\begin{array}{ccc} G & \longrightarrow & [0, \|\mu\|] \\ p & \longmapsto & \mu(h_p(a)) \end{array}$$

is Λ-measurable, where Λ is uniquely determined normalised Haar measure on G. Then

(i) $\mu^G(a) = \int_G \mu(h_p(a)) d\Lambda$ *is a submeasure and* $\|\mu^G\| = \|\mu\|$,

(ii) μ^G *is a* G*-homogeneous submeasure.*

Proof. Due to our assumptions μ^G is well defined, since whenever we integrate, we integrate measurable function bounded by a constant $\mu(\mathbf{1})$.

(i) Clearly $\mu^G(\mathbf{0}) = 0$, since for any $p \in G$ the mapping h_p is homomorphism. Now assume, that $a \leq b$ we have to prove that $\mu^G(a) \leq \mu^G(b)$. For any $p \in G$ we get $h_p(a) \leq h_p(b)$ and since μ is a submeasure we get $\mu(h_p(a)) \leq \mu(h_p(b))$ and this inequality remain intact by integration, hence μ^G is monotone. To prove subadditivity pick some disjoint elements $a, b \in \mathbb{B}$ and as in previous we get that $\mu(h_p(a) \vee h_p(b)) = \mu(h_p(a \vee b) \leq \mu(h_p(a)) + \mu(h_p(b))$ for any $p \in G$ and again the inequality remain intact. It remains to evaluate $\mu^G(\mathbf{1}) = \int_G \mu(h_p(\mathbf{1}))d\Lambda = \int_G \mu(\mathbf{1})d\Lambda = \mu(\mathbf{1})$, since Λ is normalised.

(ii) We prove G-homogeneity by the following chains of equalities, let $q \in G$, then

$$\mu^G(h_q(a)) = \int_G \mu(h_p(h_q(a)))d\Lambda = \int_G \mu(h_{p\circ q}(a))d\Lambda =$$

$$= \int_{G\circ q} \mu(h_p(a))d\Lambda = \int_G \mu(h_p(a))d\Lambda = \mu^G(a).$$

□

One can also ask quite natural question, what properties of a submeasure are preserved by its homogenisation.

7.5 Proposition. *Let μ be a submeasure on Boolean algebra $\mathbb{B}$ and let G be a compact group satisfying assumptions of the theorem. Then*

(i) if μ is 2-additive submeasure, then μ^G is also 2-additive,

(ii) if μ is exhaustive submeasure, then μ^G is also exhaustive,

(iii) if μ is measure, then μ^G is a measure.

Proof. *(i)* Let us suppose that μ is 2-additive, then for any $p \in G$ and any $a \in \mathbb{B}$ $\mu(h_p(a)) + \mu(-h_p(a)) = \mu(\mathbf{1})$. Employing this fact we get

$$\mu^G(a) + \mu^G(-a) = \int_G \Big((\mu(h_p(a)) + \mu(h_p(-a))\Big)d\Lambda =$$

$$= \int_G \Big((\mu(h_p(a)) + \mu(-h_p(a))\Big)d\Lambda = \int_G \mu(\mathbf{1}) = \mu^G(\mathbf{1})$$

(ii) In this part we use known Lebesgue's bounded convergence theorem (the proof can be found for example in [HR79]):

7.6 Theorem. (Lebesgue's bounded convergence theorem) *Let $\{f_n : n \in \omega\}$ be a family of integrable functions such that $f_n \xrightarrow{a.e.} f$ and there is g integrable function such that $\forall n \in \omega \quad |f_n(x)| \leq |g(x)|$ a.e. (almost everywhere). Then f is integrable function and*

$$\int f_n \xrightarrow{n\to\infty} \int f.$$

Let $\langle a_i : i \in \omega\rangle$ be a disjoint sequence. Clearly for any $p \in G$ the sequence $\langle h_p(a_i) : i \in \omega\rangle$ is also disjoint. We define functions $f_n : G \longrightarrow \mathbb{R}$ as $f_n(p) = \mu(h_p(a_n))$. It follows from our assumption that for any $n \in \omega$ the function f_n is measurable and since it is bounded by a constant integrable function $g : G \longrightarrow \mathbb{R}$ $g(p) = \mu(\mathbf{1})$ all assumptions of Lebesgue's bounded convergence theorem are satisfied. Moreover since μ is exhaustive and the sequence $\langle h_p(a_i) : i \in \omega\rangle$ is disjoint for any $p \in G$ the functions f_n converges to $f \equiv 0$. Hence by the Lebesgue's bounded convergence theorem we get

$$\mu^G(a_n) = \int_G \mu(h_p(a_n))d\Lambda = \int_G f_n(p)d\Lambda \xrightarrow{n\to\infty} 0.$$

$\square$

7.7 EXAMPLE. Consider the group G be $(\mathcal{P}(\omega), \triangle)$ as a compact topological group, or equivalently $(2^\omega, \mathrm{mod}(2)) \simeq \Pi_n \mathbb{Z}_2$.

As usual $\mathcal{A} = \mathrm{Clop}(2^\omega)$ is Cantor algebra. Choose finite, or countable set $\{f_i : i \in I\}$ and evaluation by reals $\langle \alpha_i > 0 : i \in I\rangle$, such that $\sum_I \alpha_i = 1$.

Define measure m on $\mathcal{A}$ by

$$m(A) = \sum \{ \alpha_i : f_i \in A \}.$$

Then homogenisation of m by the group G is exactly standard product measure on $\mathcal{A}$. $\triangle$

V. Continuous submeasures

1. Introduction

At the beginning of this chapter we deal with sequential topology on Boolean algebras. This chapter is based on and extends [BFH99], [BJP05]. Of such topologies we require at least homogeneity and one-sided continuity of the Boolean operations. It is natural to describe and investigate such a topology on a Boolean algebra $\mathbb{B}$ in terms of zero-convergence structures, i.e. special ideals on $\mathbb{B}^\omega$, the usual countable product of $\mathbb{B}$.

$\mathbb{B}^\omega$ with coordinate-wise definitions of the operations is again a Boolean algebra. Some motivation for our interest in such products follows from viewing the elements of $\mathbb{B}^\omega$ as $\mathbb{B}$-names for reals (i.e. subsets of ω) in generic extensions of V via forcing with $\mathbb{B}$. It is clear that for $f \in \mathbb{B}^\omega$ and $n \in \omega$, $f(n)$ is a $\mathbb{B}$-value of the formula $n \in f$, i.e. $f(n) = \|n \in f\|_{\mathbb{B}}$.

For a given Boolean algebra $\mathbb{B}$ and a generic filter G on $\mathbb{B}$ over V,

$$R(\mathbb{B}, G) = \{\, \{n \in \omega : \|n \in f\|_{\mathbb{B}} \in G\} : f \in \mathbb{B}^\omega\}$$

represents a set of reals in $V[G]$ coded by $\mathbb{B}$. If the algebra $\mathbb{B}$ is complete, then $R(\mathbb{B}, G) = \mathcal{P}(\omega)^{V[G]}$. For investigations of topologies on $\mathbb{B}$ or the Abelian group $(\mathbb{B}, \triangle)$ ($\triangle$ denotes symmetric difference), the structure

$$\mathcal{Z}(\mathbb{B}) = \{f : \omega \to \mathbb{B} : (\forall I \in [\omega]^\omega) \ \bigwedge\{f(n) : n \in I\} = \mathbf{0}\}$$

is quite relevant. In forcing terms $\mathcal{Z}(\mathbb{B})$ represents names of reals that do not contain any infinite subset from the ground model in any generic extension, cf. II.1.9. Formally

$$f \in \mathcal{Z}(\mathbb{B}) \quad \text{if and only if}$$
$$(\forall G\ \mathbb{B}\text{-generic over } V)\ (\forall X \in \mathcal{P}(\omega) \cap V) \text{ if } X \subset f_G \text{ then } X \text{ is finite.}$$

In the previous we already meet this structure when investigating almost disjoint refinement cf. II.2.21. If the extension M is generic, i.e. $M = V^{\mathbb{B}}$ for some complete Boolean algebra $\mathbb{B}$, then

$$H = \{f_G : f \in \mathcal{Z}(\mathbb{B})\}.$$

It is well known that once forcing with $\mathbb{B}$ adds a new real, it also adds a new real that does not include any infinite subset from the ground model. Therefore, if forcing with $\mathbb{B}$ adds a new real, it also adds a new real with a name from $\mathcal{Z}(\mathbb{B})$, provided that $\mathbb{B}$ is complete algebra.

The main interest is investigation of zero-convergence structures on $\mathbb{B}$ that are special ideals on $\mathbb{B}^\omega$. We shall denote by $\varphi \in \omega^{\omega}\!\nearrow$ the fact that φ is a strictly increasing sequence of non-negative integers.

Our motivation comes from famous Control Measure problem posed by D. Maharam in 1947 [Mah47] and solved by M. Talagrand in January 2006 [Tal06]. M. Talagrand found an algebra that carries a strictly positive continuous submeasure does not admit a σ-additive measure.

It is natural to consider the following four classes of Boolean algebras.

MBA: the class of all Boolean algebras carrying strictly positive finitely additive measure.

McBA: the class of measurable algebras, i.e. complete algebras carrying strictly positive probability measure.

EBA: the class of algebras carrying strictly positive exhaustive submeasure.

CcBA: the class of all complete algebras carrying strictly positive continuous submeasure; i.e. Maharam submeasures.

The diagram below shows the obvious inclusion relations:

$$\begin{tikzcd}
 & \mathrm{EBA} & \\
\mathrm{CcBA} \arrow[ur] & & \mathrm{MBA} \arrow[ul] \\
 & \mathrm{McBA} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The following theorem gives additional information. Note that the relations between the classes with measure are the same as those with submeasure. The proof of the theorem is scattered throughout Fremlin's work [Fre89].

1.1 Theorem. *(i) The class MBA consists exactly of all subalgebras of algebras belonging to the class McBA.*

(ii) The class EBA consists exactly of all subalgebras of algebras in CcBA.

(iii) The class McBA consists of all algebras from the class MBA which are complete and weakly distributive.

(iv) The class CcBA consists of all algebras in EBA that are complete and weakly distributive.

We start with the description of Zero-convergence structure on Boolean algebra and the topology given by the Zero-convergence structures.

2. CONVERGENT STRUCTURES ON BOOLEAN ALGEBRAS

Let us first introduce the largest possible zero convergence structure $\mathcal{Z}(\mathbb{B})$ on a Boolean algebra $\mathbb{B}$.

$$\mathcal{Z}(\mathbb{B}) = \{f : \omega \to \mathbb{B} : (\forall I \in [\omega]^{\omega}) \ \bigwedge\{f(n) : n \in I\} = \mathbf{0}\}.$$

Clearly, $\emptyset \neq \mathcal{Z}(\mathbb{B}) \subseteq \mathbb{B}^{\omega}$.

When clear from the context, we drop the reference to $\mathbb{B}$ and use just $\mathcal{Z}$, and similarly for other structures defined on Boolean algebras that will be introduced later.

2.1 Definition. Let $\mathbb{B}$ be a Boolean algebra and let $\mathcal{I}$ be an ideal on $\mathbb{B}^{\omega}$. $\mathcal{I}$ is *zero-convergence structure on* $\mathbb{B}$ if

(i) $\mathcal{I} \subseteq \mathcal{Z}$,

(ii) $\mathcal{I}$ is closed under subsequences, i.e. whenever $f \in \mathcal{I}$ and $\varphi \in \omega^{\omega}\!\nearrow$, then $f \circ \varphi \in \mathcal{I}$.

Note that $\mathcal{Z}$ itself need not be (an ideal) a zero-convergence structure on $\mathbb{B}$.

2.2 EXAMPLE. Consider a Cantor algebra $\mathcal{A}$, i.e. $\mathcal{A} \approx \mathtt{Clop}(2^{\omega})$, the algebra of clopen subsets of the Cantor space 2^{ω}. Equivalently, $\mathcal{A}$ is a free algebra with countably many independent generators, say $\langle x_n : n \in \omega\rangle$. Then f defined by $f(n) = x_n$ belongs to $\mathcal{Z}(\mathcal{A})$, and so does $-f = \langle -x_n : n \in \omega\rangle$. Since $f \vee -f = \mathbf{1}_{\mathcal{A}^{\omega}}$, $\mathcal{Z}(\mathcal{A})$ cannot be an ideal. △

Therefore, the largest possible zero-convergence structure with respect to $\subseteq$ need not be a zero-convergence structure at all, nevertheless the maximality principle is applicable, hence each zero-convergence structure on $\mathbb{B}$ can be extended to a maximal one.

Conditions under which $\mathcal{Z}(\mathbb{B})$ itself is a zero-convergence structure are discussed later (paragraph 6.) in this chapter.

2.3 Definition. Let $\mathbb{B}$ be a Boolean algebra and let $A \subseteq \mathbb{B}^{\omega}$. *The Urysohn closure* of A, $\mathcal{U}(A)$, is a subset of $\mathbb{B}^{\omega}$ with the property that every subsequence of a sequence from $\mathcal{U}(A)$ has a subsequence that belongs to A, i.e.

$$\mathcal{U}(A) = \{f \in \mathbb{B}^{\omega} : (\forall \varphi \in \omega^{\omega}\!\nearrow)(\exists \psi \in \omega^{\omega}\!\nearrow)(f \circ \varphi \circ \psi) \in A\}.$$

The following are easy observations.

2.4 Fact. *(i) for any $A \subseteq \mathbb{B}^{\omega}$, $\mathcal{U}(\mathcal{U}(A)) = \mathcal{U}(A)$,*

(ii) $\mathcal{Z}$ is Urysohn closed,

(iii) if $\mathcal{I}$ is a zero-convergence structure, then $\mathcal{U}(\mathcal{I})$ is also a zero-convergence structure,

(iv) if $\mathcal{I}$ is a zero-convergence structure, then $\mathcal{U}(\mathcal{I})$ is closed under permutations.

The set of zero-convergences on $\mathbb{B}^\omega$ is partially ordered by set inclusion. In general they do not form a lattice, however every upward directed family of zero-convergences has an upper bound.

If $\mathcal{I}$ is a zero-convergence structure on $\mathbb{B}$, we consider the elements of $\mathcal{I}$ as sequences converging to $\mathbf{0}$. We can extend this to a notion of convergence $s(\mathcal{I})$ of sequences on $\mathbb{B}$ by defining $x_n \xrightarrow[s(\mathcal{I})]{} x$ whenever $\langle x_n \vartriangle x : n \in \omega\rangle \in \mathcal{I}$, where $\vartriangle$ denotes the Boolean operation of symmetric difference. It is easy to verify that the following hold.

2.5 Fact. *(i) every sequence has at most one limit, i.e. if* $x_n \xrightarrow[s(\mathcal{I})]{} x$ *and* $x_n \xrightarrow[s(\mathcal{I})]{} y$*, then* $x = y$*,*

(ii) if $x \in \mathbb{B}$*, then the constant sequence* $\langle x : n \in \omega\rangle$ *has* x *as its limit,*

(iii) if $x_n \xrightarrow[s(\mathcal{I})]{} x$ *and* $\langle y_n : n \in \omega\rangle$ *is a subsequence of* $\langle x_n : n \in \omega\rangle$*, then* $y_n \xrightarrow[s(\mathcal{I})]{} x$*,*

(iv) if $x_n \leq y_n \leq z_n$ *for every* n *and* $x_n \xrightarrow[s(\mathcal{I})]{} x$ *and* $z_n \xrightarrow[s(\mathcal{I})]{} x$*, then* $y_n \xrightarrow[s(\mathcal{I})]{} x$*,*

(v) the convergence respects Boolean operations, i.e. if $x_n \xrightarrow[s(\mathcal{I})]{} x$ *and* $y_n \xrightarrow[s(\mathcal{I})]{} y$*, then* $x_n \vee y_n \xrightarrow[s(\mathcal{I})]{} x \vee y$ *and* $-x_n \xrightarrow[s(\mathcal{I})]{} -x$*.*

The notions of zero convergence and convergence are really identical in the sense that a convergence structure $s(\mathcal{I})$ induced by a zero-convergence structure $\mathcal{I}$ is a convergence structure on $\mathbb{B}$, i.e. a structure satisfying V.2.5 *(i)* - *(v)*, while for a convergence structure s on $\mathbb{B}$, $s_0 = \{f \in \mathbb{B}^\omega : f(n) \xrightarrow[s]{} \mathbf{0}\} \subseteq \mathcal{Z}$ and is a zero-convergence structure on $\mathbb{B}$.

3. EXAMPLES OF ZERO-CONVERGENCE

In this section we discuss some basic examples of zero-convergent structures on an arbitrary Boolean algebra $\mathbb{B}$. We define exhaustive convergence structures motivated by exhaustive submeasures and their continuity. We introduce and characterise a zero-convergence structure $\mathcal{E}$ that is the intersection of all maximal zero-convergence structures and introduce the most important 'order sequential zero-convergence structure'.

3.1 Definition. For $f \in \mathbb{B}^\omega$, the set $\{n : f(n) \neq \mathbf{0}\}$ is the *support of* f. We shall call f a *finite* element of $\mathbb{B}^\omega$ if its support is finite. Set

$$Fin(\mathbb{B}) = \{f \in \mathbb{B}^\omega : f \text{ has finite support}\}.$$

It is natural in this context that some notions for ideals on $\mathcal{P}(\omega)$ can be translated to ideals on $\mathbb{B}^\omega$, for instance the notion of P-ideal.

3.2 Definition. An ideal I on $\mathbb{B}^\omega$ is a P-ideal if for any countable family $\{f_k : k \in \omega\} \subset I$ $(\exists g \in I)$ $(\forall k \in \omega)$ $f_k - g \in$ *Fin*, where *Fin* is an ideal on $\mathbb{B}^\omega$ of sequences with finite support.

Let us remark that *Fin* is the Urysohn closure of $\{\mathbf{0}\}$. Since $\mathcal{P}(\omega)$ can be naturally embedded to $\mathbb{B}^\omega$, *Fin* plays the same role on $\mathbb{B}^\omega$ as does *fin* $= \{X \subseteq \omega : X \text{ is finite}\}$ on $\mathcal{P}(\omega)$; one can be generated from the other.

3.3 Fact. *Fin is the least Urysohn closed zero-convergence structure and the topology* $\tau(\mathit{Fin})$ *which it determines is discrete.*

Although the ideal *Fin* is not very interesting from the convergence point of view, it becomes more interesting in the context of quotient algebras. For any $\mathbb{B}$, the quotient algebra $\mathbb{B}^\omega/\mathit{Fin}$ is σ-closed, i.e. any descending sequence of non-zero elements has a non-zero lower bound. If $\mathbb{B}^\omega/\mathit{Fin}$ has a dense subset of size $\leq 2^\omega$, then $\mathbb{B}^\omega/\mathit{Fin}$ has a base tree (not necessarily homogeneous in height). For the basic case when $\mathbb{B} = \{\mathbf{0}, \mathbf{1}\}$ and hence $\mathbb{B}^\omega/\mathit{Fin} = \mathcal{P}(\omega)/\mathit{fin}$, see [BPS80]. Let $\mathcal{A}$ be the Cantor algebra. Under the CH, algebras $\mathcal{A}^\omega/\mathit{Fin}$ and $\mathcal{P}(\omega)/\mathit{fin}$ are the same. The reason is that both algebras are saturated structures as models of Boolean algebra. Recently A. Dow solved the long-standing problem and showed that consistently the completions of those two algebras may be different. Moreover, the height of $\mathcal{A}^\omega/\mathit{Fin}$ can be smaller than that of $\mathcal{P}(\omega)/\mathit{fin}$. Alternative, simpler proof of Dow's result can be found in [BH01].

3.4 Definition. Let k be a positive integer, $d \in \mathbb{B}^\omega$ is called a k-*disjoint sequence*, if for any $X \subset \omega$ of size k, $\bigwedge\{d(n) : n \in X\} = \mathbf{0}$. We use the term *disjoint sequence* for a 2-disjoint sequence. Let

$$D(\mathbb{B}) = \{\, f \in \mathbb{B}^\omega : (\exists m \in \omega)(\forall X \in [\omega]^m)\ \bigwedge\{f(i) : i \in X\} = \mathbf{0}\,\}.$$

It is clear that $\mathit{Fin} \subseteq D \subseteq \mathcal{Z}$.

3.5 Proposition. D *is the zero-convergence structure generated by all disjoint sequences, i.e. for any* $f \in D$ *there are disjoint sequences* $d_1, \ldots, d_k$ *so that* $f \leq d_1 \vee \ldots \vee d_k$.

Proof. When $d_1, \ldots, d_k$ are disjoint sequences, then for any $X \subseteq \omega$, $|X| = k+1$, using the usual distributivity and the pigeon hole principle,

$$\bigwedge_{i \in X} d_1(i) \vee \ldots \vee d_k(i) = \mathbf{0}.$$

We shall argue the opposite direction using induction. Let $f \in D$ be an m-disjoint sequence. If $m = 2$, f is disjoint, and so we assume that $m > 2$. Put $d(n) = f(n) - \bigvee_{i<n} f(i)$ for every $n \in \omega$. Then d is a disjoint sequence. We show that $g = f - d$ is $(m-1)$-disjoint. For every $n \in \omega$ we have $g(n) = f(n) \wedge \bigvee_{i<n} f(i)$, in particular, $g(0) = \mathbf{0}$. Let us check that for arbitrary $x_1 < x_2 < \cdots < x_{m-1}$ we get $\bigwedge_{j=1}^{m-1} g(x_j) = \mathbf{0}$.

In case of $x_1 = 0$ we are done, so assume that $x_1 > 0$. We have

$$\bigwedge_{j=1}^{m-1} g(x_j) = \bigwedge_{j=1}^{m-1} \Big(f(x_j) \wedge \bigvee_{i<x_j} f(i)\Big) = \bigvee_{i<x_1} \Big(f(i) \wedge \bigwedge_{j=1}^{m-1} f(x_j)\Big).$$

Since f is m-disjoint each member of the latter join is $\mathbf{0}$, hence g is a $(m-1)$-disjoint sequence. □

3.6 Definition. Let $\mathcal{I}$ be a zero-convergence structure. $\mathcal{I}$ is called *exhaustive* if $D \subseteq \mathcal{I}$. In such case, the induced convergence structure $s(\mathcal{I})$ is referred to as *exhaustive*.

We can ask about the description of the largest zero-convergence structure or equivalently the weakest topology in which all exhaustive submeasures are continuous.

3.7 Definition. Put

$$L(\mathbb{B}) = \{f \in \mathbb{B}^\omega : (\forall X \subseteq \omega, \text{infinite})(\exists Y \subseteq X, \text{finite}) \ \bigwedge\{f(i) : i \in Y\} = \mathbf{0}\},$$

these are all sequences without infinite centered subsequences.

3.8 Proposition. $L(\mathbb{B})$ *is a Urysohn closed zero-convergence structure.*

Of course $D \subset L$, moreover $U(D) \subseteq L$. We show the situation where $U(D)$ is strictly smaller then L.

3.9 EXAMPLE. Let us consider the algebra $\mathbb{B} = \mathcal{P}([\omega]^{<\omega})$ and the set $S = \{s \in [\omega]^{<\omega} : \min(s) \geq |s|\}$. Now we define a sequence $S_n = \{s \in S : n \in s\}$, which belongs to $L(\mathbb{B})$ and not to $U(D(\mathbb{B}))$.

Clearly the sequence $\langle S_n : n \in \omega\rangle$ belongs to $L(\mathbb{B})$, because whenever we fix arbitrary $n_1 < n_2 < \cdots < n_{n_1+1}$ then by the property $\min(s) \geq |s|$ the intersection $\bigcap_{i=1}^{n_1+1} S_{n_i}$ is empty.

The sequence $\langle S_n : n \in \omega\rangle$ does not belong to $U(D(\mathbb{B}))$. It suffices to show that $\forall X \in [\omega]^\omega \ \forall m \in \omega \ \exists Y \in [X]^m$ such that $\bigcap_{i=1}^m S_{y_i} \neq \emptyset$. As Y we can fix arbitrary m members from X with the property $m \leq x_1 < x_2 < \cdots < x_m$, and we are done. △

3.10 EXAMPLE. Recall that zero-convergence structures are ordered by inclusion and that there are maximal ones. For an arbitrary algebra $\mathbb{B}$, set

$$\mathcal{E}(\mathbb{B}) = \bigcap\{\mathcal{I} : \mathcal{I} \text{ is a maximal zero-convergence structure on } \mathbb{B}\}.$$

$\mathcal{E}$ is again a zero-convergence structure, which is Urysohn closed, for it is an intersection of Urysohn closed structures. △

3.11 ORDER SEQUENTIAL ZERO-CONVERGENCE

In what follows we will focus on order sequential zero-convergence and order sequential topology.

3.12 Definition. Let C denote a σ-completion of $\mathbb{B}$. Then *order zero-convergence* structure on $\mathbb{B}$ is defined by

$$Os(\mathbb{B}) = \{f \in \mathbb{B}^\omega : (\exists g \in C^\omega, \ g \searrow \mathbf{0}_{\mathbb{B}}) \ f \leq g\}.$$

Therefore $Os(\mathbb{B}) = \mathbb{B}^\omega \cap Os(C)$.

The order zero-convergence structure on Boolean algebra $\mathbb{B}$ can be degenerated; for instance, if $\mathbb{B} = \mathcal{P}(\omega)/\mathit{fin}$, then $\mathrm{Os} = \mathcal{Z} = \mathrm{L}$.

The convergence structure induced by the order zero-convergence structure on $\mathbb{B}$ is the most frequently studied one in the context of σ-fields of sets or σ-complete Boolean algebras.

3.13 Fact. *For any Boolean algebra $\mathbb{B}$, $\mathrm{Os}(\mathbb{B})$ is an exhaustive zero-convergence, not necessarily Urysohn closed, however $\mathrm{L} \subseteq \mathcal{U}(\mathrm{Os})$.*

Instead of a proof let us recall a few notions. Let $\mathbb{C}$ be a σ-completion of $\mathbb{B}$. For any sequence $\langle x_n : n \in \omega \rangle \in \mathbb{C}^\omega$ we have defined notion of limes superior, limes inferior and notion of limit; cf. I.3.34.

Let us remark that Os zero-convergence is exactly the convergence given by the sequences that converges to **0**, cf. I.3.33

$$\mathrm{Os}(\mathbb{C}) \ = \{ \, f \in \mathbb{C}^\omega \ : \ \overline{\lim}\, f(n) = \mathbf{0} \, \}.$$

Since $\overline{\lim}\,(x_n \vee y_n) = \overline{\lim}\, x_n \vee \overline{\lim}\, y_n$ for arbitrary $\langle x_n \rangle, \langle y_n \rangle$ (see I.3.35), then $\mathrm{Os}(\mathbb{C})$ is a zero-convergence. Since for any disjoint sequence $\langle x_n \rangle$, $\lim x_n = \mathbf{0}$, $\mathrm{Os}(\mathbb{C})$ is exhaustive. $\mathrm{Os}(\mathbb{B})$ has the same property; it follows from the fact that $\mathbb{B}$ is dense in $\mathbb{C}$.

4. SEQUENTIAL TOPOLOGY ON BOOLEAN ALGEBRAS

A convergence structure s on a Boolean algebra $\mathbb{B}$ gives rise to a sequential topology on $\mathbb{B}$ in the following way: consider all topologies τ on $\mathbb{B}$ so that whenever $x_n \xrightarrow[s]{} x$, then $x_n \xrightarrow[\tau]{} x$. There is a largest topology with respect to inclusion among all such topologies, and we denote it by $\tau(s)$ and call it *the sequential topology determined by* s.

Alternatively, the topology $\tau(s)$ can be described through the closure operation: for $A \subseteq \mathbb{B}$, let $u(A) = \{x : x \text{ is the } \textit{s-limit} \text{ of a sequence } \langle x_n \rangle \text{ of elements of} A\}$. Then

$$\mathrm{cl}_{\tau(s)}(A) = \bigcup_{\alpha<\omega_1} u^{(\alpha)}(A),$$

where $u^{(\alpha+1)}(A) = u(u^{(\alpha)}(A))$ and $u^{(\alpha)}(A) = \bigcup\{u^{(\beta)}(A) : \beta < \alpha\}$ for a limit α.

It follows from V.2.5*(ii)* that every singleton is a closed set, i.e. $\tau(s)$ is a T_1 topology. Moreover, $(\mathbb{B}, \tau(s))$ is a sequential topological space, and it is Fréchet if and only if $\mathrm{cl}_{\tau(s)}(A) = u(A)$ for every $A \subseteq \mathbb{B}$.

4.1 Fact. *A sequence $\langle x_n \rangle$ converges to x in the topology $\tau(s)$, $x_n \xrightarrow[\tau(s)]{} x$, if and only if any subsequence of $\langle x_n \rangle$ has a subsequence that converges to x in s.*

Let $\mathcal{I}$ be a zero-convergence structure on a Boolean algebra $\mathbb{B}$. In the way described above, the convergence structure $s(\mathcal{I})$ determines a sequential topology, which we denote by $\tau(\mathcal{I})$. It follows that the Urysohn closure of $\mathcal{I}$, $\mathrm{U}(\mathcal{I})$, is the set of all sequences of elements of $\mathbb{B}$ that converge to **0** in the topology

$\tau(\mathcal{I})$. Moreover, if $\mathcal{I}$ is Urysohn closed, then $x_n \xrightarrow[s(\mathcal{I})]{} x$ if and only if $x_n \xrightarrow[\tau(\mathcal{I})]{} x$. Note that from larger zero-convergent structure we obtain a weaker sequential topology.

Let us state here the basic obvious fact which puts together zero-convergence structure and continuity of submeasure.

4.2 Fact. *(i) Let $\mathcal{I}$ be a zero-convergence structure on $\mathbb{B}$. Then for any submeasure μ on $\mathbb{B}$, μ is continuous in $\tau(\mathcal{I})$ if and only if* $(\forall f \in \mathcal{I})\ \mu(f(n)) \longrightarrow 0$.

(ii) Let S be a non-empty set of submeasures on $\mathbb{B}$ such that for any $a \in \mathbb{B}^+$ there is some $\mu \in S$ with $\mu(a) > 0$. Then $\{f \in \mathbb{B}^\omega : (\forall \mu \in S) \lim_n \mu(f(n)) = 0\}$ is a Urysohn closed zero-convergence structure.

The following fact gives the motivation and justification of the term 'exhaustive zero-convergence' introduced in V.3.6. It follows immediately from the definition of exhaustivity and Fact V.4.2*(i)*.

4.3 Fact. *For any submeasure μ on $\mathbb{B}$, μ is exhaustive if and only if it is continuous in the $\tau(D)$ topology.*

4.4 Fact. *Let $\mathbb{B}$ be a Boolean algebra with topology τ_L given by zero-convergence structure* L. *The sequence $\langle a_n : n \in \omega \rangle$ converges to $a \in \mathbb{B}$ in a topology τ_L if and only if*

$$\forall X \in [\omega]^{<\omega}\ \exists Y \in [X]^\omega \textit{ such that } \bigvee_{n \in Y} a_n \geq a \ \&\ \bigwedge_{n \in Y} a_n \leq a.$$

The following theorem is a modification of the result of R. Frič [Fri01], who proved the theorem for measures. This is one of the situations when the global properties of exhaustive submeasures and measures on Boolean algebras are the same.

4.5 Theorem. L *is a Urysohn closed zero-convergence structure. Moreover,*

$$L = \{f \in \mathbb{B}^\omega : \textit{for every exhaustive submeasure } \mu \text{ on } \mathbb{B},\ \mu(f(n)) \longrightarrow 0\}.$$

In the proof of the theorem in addition to Frič's methods we are going to use the following equivalence which we already proved III.1.2*(iv)*.

4.6 Proposition. *Let μ be a submeasure on $\mathbb{B}$. Then μ is exhaustive if and only if for any $\langle x_n : n \in \omega \rangle \in \mathbb{B}^\omega$ and any $\varepsilon > 0$, there is a $k \in \omega$ such that*

$$(\forall p \geq k)\ \mu(\bigvee_{i \leq p} x_i - \bigvee_{i \leq k} x_i) < \varepsilon.$$

Proof. PROOF OF THEOREM V.4.5

An ultrafilter F on $\mathbb{B}$ corresponds uniquely to a $\{0,1\}$-valued measure μ_F defined by $\mu_F(x) = 1$ if and only if $x \in F$, and 0 otherwise.

Given the simple observation that $L = \{f \in \mathbb{B}^\omega : (\forall \mu\ \{0,1\}\text{-valued measure})\ \mu(f(n)) \longrightarrow 0\}$ and V.4.2*(ii)*, it follows that L is a zero-convergence structure, Urysohn closed, and moreover it is exhaustive.

In the following we will show that any exhaustive submeasure μ on $\mathbb{B}$ is continuous in the topology determined by L.
Let $\langle x_n \rangle \in L$ and let $\varepsilon > 0$. We want to show that for some n_0, $\mu(x_n) \leq \varepsilon$ whenever $n \geq n_0$. Using V.4.6, by induction we can construct a $\varphi \in \omega^{\omega}\!\nearrow$ with the property that

$$\mu\Big(\bigvee_{k \leq i \leq p} x_i - \bigvee_{k \leq i \leq \varphi(k)} x_i \Big) < \varepsilon/2^k.$$

Set $a_k = \bigvee\limits_{k \leq i \leq \varphi(k)} x_i$. It follows that $x_k \leq a_k$ and $x_k \leq \bigwedge\limits_{i \leq k} a_i \vee \bigvee\limits_{i \leq k} (a_k - a_i)$.
Since for $i \leq k$, $a_k - a_i = \bigvee\limits_{k \leq j \leq \varphi(k)} x_j - \bigvee\limits_{i \leq j \leq \varphi(i)} x_j$, $\mu(a_k - a_i) < \varepsilon/2^i$ and thus $\mu\big(\bigvee\limits_{i \leq k} (a_k - a_i) \big) < 2\varepsilon$.
Set $b_k = \bigwedge_{i \leq k} a_i$. $\langle b_k \rangle$ is a descending sequence. If $\langle b_k \rangle$ is not in L, then $\langle b_k \rangle$ has the finite intersection property and hence can be extended to an ultrafilter F on $\mathbb{B}$. Then for any k there is an $i \geq k$ so that $x_i \in F$, and so there is $\langle y_n \rangle$, a subsequence of $\langle x_n \rangle$, with $\mu_F(y_n) \longrightarrow 1$, a contradiction with the definition of L. Thus $\langle b_k \rangle \in L$, and, consequently, for some k_0, $b_k = \mathbf{0}$ for any $k \geq k_0$, and therefore $\mu(b_k) \longrightarrow 0$.
Since $x_k \leq \bigwedge\limits_{i \leq k} a_i \vee \bigvee\limits_{i \leq k} (a_k - a_i)$, for sufficiently large k,

$$\mu(x_k) \leq \mu\Big(\bigvee_{i \leq k} (a_k - a_i) \Big) < 2\varepsilon.$$

$\square$

From now on whenever we mention sequential topology on Boolean algebra we will mean the following order sequential topology.

4.7 Definition. The sequential topology on Boolean algebra $\mathbb{B}$ given by $\mathrm{Os}(\mathbb{B})$ zero-convergence is called *the order sequential topology* and in what follows we denote it simply τ_s instead of $\tau(\mathrm{Os})$.

4.8 EXAMPLE. *(The order sequential convergence on a powerset algebra)* Let $\mathbb{B} = \mathcal{P}(X)$ for an infinite set X. A sequence $\langle X_n : n \in \omega \rangle$, $X_n \subseteq X$, belongs to $\mathcal{Z}$ if and only if $\{X_n : n \in \omega\}$ is a point-finite family of sets. Moreover, $\mathcal{Z}$ is a zero-convergence structure. For this example let s denote the convergence structure induced by $\mathcal{Z}$. When we identify $\mathcal{P}(X)$ with 2^X via characteristic functions, then the convergence in the topology $\tau(s)$ is exactly the pointwise convergence of sequences on 2^X. It is well known that the corresponding sequential topology $\tau(s)$ on 2^X is a product topology if and only if X is at most countable, see I.2.11. For an uncountable X, the sequential space $(2^X, \tau(s))$ is a Hausdorff, but not a regular, topological space, see [Głó91] or [BGJ98]. Moreover, the topology $\tau(s)$ is stronger than the usual product topology τ_c on 2^X and $\tau(s)$ coincides with the sequential modification τ_s of compact topology τ_c see I.2.14. If we consider the spaces of continuous real-valued functions on 2^X with respect to those two topologies, it is shown in [BH01] that $C(2^X, \tau)) \subsetneq C(2^X, \tau(s))$ if and only if size of X is at least as large as the first submeasurable cardinal. $\triangle$

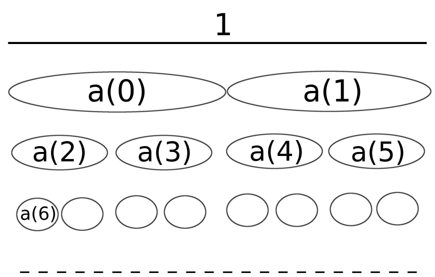
Order zero-convergence structures for the Cantor algebra $\mathcal{A}$ and its completions are not Urysohn closed.

4.9 Definition. A sequence $\langle a_n : n \in \omega \rangle \in \mathbb{B}^\omega$ is *independent sequence* if for every finite $F \subset \omega$ and $\varepsilon : F \to \{0, 1\}$ the intersection

$$\bigwedge_{n \in F} a_n^{\varepsilon(n)} \neq \mathbf{0}$$

is nonempty, where $a_n^1 = a_n$ and $a_n^0 = -a_n$.

4.10 EXAMPLE. Here we show the sequence that converges in the sense of topology, but do not converges in the algebraic sense. Such a sequence can be found in both Cohen and random algebra. In fact this simple construction is based on the existence of an appropriate independent sequence. It is well known fact that both Cohen and random contains such a sequence.



Having an independent sequence one can easily construct a sequence as in the picture. For such a sequence

$$\bigvee_{n \geq k} a(n) = \mathbf{1}, \text{ and } \bigwedge_{n \geq k} a(n) = \mathbf{0},$$

for any $k \in \omega$. Hence such a sequence cannot converge in the algebraic sense.

On the other hand it follows from Ramsey Theorem that the sequence $\langle a(n) : n \in \omega \rangle$ converges to $\mathbf{0}$ in the τ_s topological sense, since $\langle a(n) : n \in \omega \rangle \in U(Os)$. $\triangle$

4.11 SEMICONTINUITY OF MONOTONE FUNCTIONS

We consider σ-complete Boolean algebras as a topological space endowed with order sequential topology; i.e. topology given by order sequential zero-convergence see V.3.12. We reformulate conditions for real valued functions on Boolean algebra B to be semicontinuous, respective continuous in the topological space $(\mathbb{B}, \tau_s)$. Main interest concerns monotone functions and submeasures.

In the sequel, let B be a σ-complete Boolean algebra. Since $(\mathbb{B}, \tau_s)$ is a sequential space, continuity of a mapping is equivalent to sequential continuity I.2.3. If a sequence $\langle x_n : n \in \omega \rangle$ converges to x in τ_s topology $x_n \to_{\tau_s} x$, it does not imply that the sequence converges to x algebraically; cf. example V.4.10.

We check that for continuity it is sufficient to consider only algebraically convergent sequences; cf. V.4.2.

4.12 Fact. *A function $f : \mathbb{B} \to \mathbb{R}$ is continuous in τ_s if and only if for every sequence $\langle x_n : n \in \omega \rangle$ algebraically converging to x, $f(x) = \lim_n f(x_n)$.*

We start with continuity of measure and submeasure.

4.13 Theorem. *Let $\mathbb{B}$ be a σ-complete Boolean algebra. Then*

(i) *submeasure μ on $\mathbb{B}$ is continuous with respect to τ_s topology if and only if μ is Maharam submeasure,*

(ii) *finitely additive measure m on $\mathbb{B}$ is continuous with respect to τ_s topology if and only if m is σ-additive measure.*

Proof. *(i)* is a direct consequence of V.4.2*(i)*
(ii) Let $\langle a_n : n \in \omega\rangle$ be disjoint sequence. Then by continuity of m

$$m(\bigvee_n a_n) \;=\; m(\lim_n \bigvee_{i\le n} a_i) \;=\; \lim_n m(\bigvee_{i\le n} a_i) \;=\; \lim_n \sum_0^n m(a_i) = \sum_0^\infty m(a_i).$$

If m is σ-additive, then m is Maharam, hence by *(i)* continuous: Let $a_n \searrow \mathbf{0}$, we can suppose that $a_0 = \mathbf{1}$. Then $b_n = b_n - b_{n+1}$ is disjoint sequence with $\bigvee b_n = \mathbf{1}$. Clearly $m(\bigwedge_n a_n) = m(\mathbf{1}) - \sum_n m(b_n) = 0$; and we are done. □

In fact in the proof of previous theorem when checking continuity we checked only nonincreasing sequences, hence in fact only semicontinuity; cf. I.2.23, let us restate the theorem here in terms of semicontinuity.

4.14 Corollary. *Let μ be a submeasure on σ-complete Boolean algebra then the following are equivalent*

(i) $\mu : \mathbb{B} \to \mathbb{R}$ *is τ_s-continuous function,*

(ii) μ *is Maharam submeasure,*

(iii) $\mu : \mathbb{B} \to \mathbb{R}$ *is τ_s-upper semicontinuous function.*

When investigating Measure algebra one can use complements, hence lower and upper semicontinuity coincides.

4.15 Corollary. *Let m be a measure on σ-complete Boolean algebra then the following are equivalent*

(i) $m : \mathbb{B} \to \mathbb{R}$ *is τ_s-continuous function,*

(ii) $m : \mathbb{B} \to \mathbb{R}$ *is τ_s-upper semicontinuous function,*

(iii) $m : \mathbb{B} \to \mathbb{R}$ *is τ_s-lower semicontinuous function,*

(iv) m *is σ-additive measure.*

5. DISCRETE VERSUS CONTINUOUS EXHAUSTIVE SUBMEASURES

Exhaustive submeasures share many properties with measures. We show that there is one-to-one correspondence between exhaustive submeasures on the algebra of clopen sets, $\mathcal{A} = \mathrm{Clop}(2^\omega)$ - *the discrete version* and continuous submeasures on $\mathrm{Borel}(2^\omega)$ - *the continuous version*.

Let μ be a continuous submeasure on Borel(2^ω). Since μ is exhaustive functional $\hat{\mu} = \mu \restriction \mathcal{A}$ is also exhaustive submeasure on $\mathcal{A}$. We know that algebra $\mathcal{A} \subset$ Borel(2^ω) is dense subset of the space $(\text{Borel}(2^\omega), \tau_s)$ and μ is continuous function, therefore μ is by $\hat{\mu}$ determined uniquely.

On the opposite direction we have the following well known

5.1 Theorem. *Every exhaustive submeasure μ on $\mathcal{A}$ can be uniquely extended to continuous submeasure μ defined on Borel(2^ω).*

Proof. First, we deal with an extension of μ to open and closed subsets of the Cantor space. Let U be an open set. There is a nondecreasing sequence of $V_n \in \mathcal{A}$ such that $U = \bigcup V_n$. Put

$$\mu(U) = \lim_{n\to\infty} \mu(V_n).$$

Similarly, for closed set F, there is nonincreasing sequence $V_n \in \mathcal{A}$ such that $F = \bigcap V_n$ and we put $\mu(F) = \lim \mu(V_n)$. Value μ does not depend on the representation of U or V by clopen sets, it follows from compactness. In fact

$$\mu(U) = \sup\{\mu(V) : V \in \mathcal{A} \;\&\; V \subset U\},$$

$$\mu(F) = \inf\{\mu(V) : V \in \mathcal{A} \;\&\; V \supset F\}.$$

Till now, we used only monotonicity of μ on $\mathcal{A}$ and certainly we obtained an extension of the starting submeasure μ.

Claim.

(a) μ is monotone and subadditive on open sets,

(b) for every open U and $\varepsilon > 0$ there is a clopen $V \subset U$ such that $\mu(U - V) < \varepsilon$, the dual version holds for closed sets.

(c) If F is closed, U is open and $F \subset U$, then $\mu(U) - \mu(F) \le \mu(U - F)$.

(a) We use subadditivity of μ on $\mathcal{A}$. Let U_1, U_2 be an open sets and $U_1 = \bigcup\{U_1(n) : n \in \omega\}$ and $U_2 = \bigcup\{U_2(n) : n \in \omega\}$ be arbitrary approximating sequences consisting of clopen sets. Then for $U(n) = U_1(n) \cup U_2 n$, we have $U_1 \cup U_2 = \bigcup\{U(n) : n \in \omega\}$ and $\mu(U_1 \cup U_2) = \lim \mu(U(n)) \le \lim \mu(U_1(n)) + \lim \mu(U_2(n)) \le \mu(U_1) + \mu(U_2)$.

(b) We use exhaustivity and monotonity of μ and apply III.1.2*(iv)*.

(c) Assume $\langle V_n : n \in \omega\rangle$ and $\langle F_n : n \in \omega\rangle$ sequences of clopen sets such that $U = \bigcup U_n$ and $F = \bigcap F_n$. Since μ is subadditive on clopen sets and $U_n \setminus F_n \subset U \setminus F$ for each $n \in \omega$ we get

$$\begin{aligned}\mu(U) - \mu(F) \;=\; \lim\ \mu(U_n) - \lim\ \mu(F_n) \;&=\; \lim\ (\mu(U_n) - \mu(F_n)) \;\le \\ &\le\; \lim\ \mu(U_n \setminus F_n) \;\le\; \mu(U \setminus F),\end{aligned}$$

which completes the proof of the Claim.

We have just proved that we are able to extend the starting μ from clopen to both open and closed sets. For second, consider a family $S \subset \mathcal{P}(2^\omega)$ defined by $A \in S$ if and only if for every $\varepsilon > 0$ there are closed F and open U such that

$$F \subset A \subset U \quad \& \quad \mu(U - F) < \varepsilon.$$

By claim (b) the family S contains all open sets. Moreover, if $A \in S$ then also clearly $-A \in S$. To show that S is σ-field of sets, consider the countable family $\{A_n : n \in \omega\} \subset S$. Fix $\varepsilon > 0$. Take F_n's closed, U_n's open such that $F_n \subset A_n \subset U_n$ and $\mu(U_n - F_n) \leq \varepsilon 2^{-n}$ for all $n \in \omega$.

Put $U = \bigcup_{i\in\omega} U_i$, U is open and by claim (b) there is a clopen $V \subset U$ such that $\mu(U \setminus V) < \varepsilon$. It follows from the compactness that there is $n_0 \in \omega$ such that $V \subset \bigcup_{i\leq n_0} U_i$. By the claim (a)

$$\mu(U \setminus \bigcup_{i\leq n_0} F_i) \;\leq\; \mu(U \setminus V) + \sum_0^{n_0} \mu(U_i \setminus F_i) < 3\varepsilon,$$

moreover $\bigcup_{i\leq n_0} F_i \subset \bigcup_{i\in\omega} A_i \subset U$.

Thus we proved that $\bigcup_{i\in\omega} A_i \in S$ and therefore $\mathrm{Borel}(2^\omega) \subset S$.

For $A \in S$ put

$$\hat{\mu}(A) = \inf\{\mu(U) : A \subset U \;\&\; U \text{ is open}\}.$$

By claim (c) one can see that $\hat{\mu}(A) = \sup\{\mu(F) : F \subset A \;\&\; F \text{ is closed}\}$, and that $\hat{\mu}$ extends the starting submeasure and that $\hat{\mu}$ is defined on $S \supset \mathrm{Borel}(2^\omega)$. We need to verify that $\hat{\mu}$ is a continuous submeasure. Let A_1, A_2 be from S. For $\varepsilon > 0$ choose U_1, U_2 open such that $\mu(U_i) \leq \hat{\mu}(A_i) + \varepsilon$, $A_i \subset U_i$, $i = 1, 2$. By the claim (a) we have $\hat{\mu}(A_1 \cup A_2) \leq \mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2) \leq \hat{\mu}(A_1) + \hat{\mu}(A_2) + 2\varepsilon$, hence $\hat{\mu}$ is a submeasure.

Now suppose that $\langle A_n : n \in \omega\rangle$ is decreasing sequence of sets from S such that $\bigcap_{n\in\omega} A_n = \emptyset$. The sequence $\langle \hat{\mu}(A_n) : n \in \omega\rangle$ is nonincreasing, suppose for the contradiction that $\lim \hat{\mu}(A_n) = \varepsilon > 0$. There is a sequence $\langle F_n : n \in \omega\rangle$ of closed sets such that $F_n \subset A_n$ and $\hat{\mu}(A_n \setminus F_n) < \varepsilon 2^{-(n+2)}$.

For any n

$$\bigcap_{i\leq n} F_i \cup \bigcup_{k\leq n} A_k \setminus F_k \supset A_n.$$

Thus

$$\mu(\bigcap_{i\leq n} F_i) > \hat{\mu}(A_n) - \sum_{k\leq n} \mu(A_k \setminus F_k) > \frac{\varepsilon}{2},$$

so $\bigcap_{i\leq n} F_i \neq \emptyset$. That yields $\emptyset \neq \bigcap_{i\in\omega} F_i \subset \bigcap_{i\in\omega} A_i = \emptyset$, a contradiction. Therefore $\lim \hat{\mu}(A_n) = 0$ and hence $\hat{\mu}$ is continuous. □

6. When $\mathcal{Z}$ is a zero-convergence structure

In this section we focus on the question when the $\mathcal{Z}$ itself is a zero-convergence structure and give here several characterisations. This part is based on and extends the work of B. Balcar, F. Franek and J. Hruška [BFH99].

The main theorem of this section characterises Boolean algebras that do not add independent reals in the generic extension.

6.1 Definition. We say that the Cantor algebra $\mathcal{A}$ is *almost regularly embedded* into a Boolean algebra $\mathbb{B}$ if there is $\mathcal{A}'$, a subalgebra of $\mathbb{B}$, so that

(i) $\mathcal{A}'$ is isomorphic to $\mathcal{A}$, and

(ii) there is a set $\{x_n : n \in \omega\}$ of generators of $\mathcal{A}'$ such that for any infinite subset X of ω, $\bigvee_B\{x_n : n \in X\} = \mathbf{1}$ and $\bigwedge_B\{x_n : n \in X\} = \mathbf{0}$.

6.2 Theorem. *For any $\mathbb{B}$ the following are equivalent.*

(i) $\mathcal{Z}$ is a zero-convergence structure; i.e $\mathcal{Z}$ is an ideal on $\mathbb{B}^\omega$,

(ii) for any $a \in \mathbb{B}^+$, the Cantor algebra $\mathcal{A}$ cannot be almost regularly embedded into $\mathbb{B}{\restriction}a$.

6.3 Remark. Recall, that $\mathbb{B}$ is $(\omega, 2)$-distributive if for any sequence $\langle a_n : n \in \omega\rangle \in \mathbb{B}^\omega$ and for any $b \in \mathbb{B}^+$, there is a $c \leq b$, $c \neq \mathbf{0}$, such that for any $n \in \omega$, either $c \leq a_n$ or $c \wedge a_n = \mathbf{0}$. Thus an $(\omega, 2)$-distributive algebra satisfies *(ii)*, and so Jakubík's result [BFH99] that for an $(\omega, 2)$-distributive Boolean algebra, $\mathcal{Z}$ is a zero-convergence structure, follows as a direct consequence of the theorem.

Proof. Proving $\neg$*(i)* $\to$ $\neg$*(ii)*. $\mathcal{Z}$ is not a zero-convergence structure if and only if $\mathcal{Z}$ is not an ideal if and only if there are $a \in \mathbb{B}^+$ and $f, g \in \mathcal{Z}$ such that $f \vee g = k_a$. Then $\{f(n) : n \in \omega\} \subseteq \mathbb{B}{\restriction}a$. Let $\mathcal{A}'$ be a subalgebra of $\mathbb{B}{\restriction}a$ generated by $\{f(n); n \in \omega\}$. Then $\mathcal{A}'$ is countable. Moreover, it is atomless, otherwise there is an atom $c \neq \mathbf{0}$ and so for any n, either $c \leq f(n)$ or $c \wedge f(n) = \mathbf{0}$. One of those cases must happen infinitely many times. The former contradicts the fact that $f \in \mathcal{Z}$, while the latter contradicts the fact that $g \in \mathcal{Z}$. Thus $\mathcal{A}'$ is isomorphic to the Cantor algebra.

For proving $\neg$*(ii)* $\to$ $\neg$*(i)*, set $f(n) = x_n$ and $g(n) = a - x_n$. It follows that $f \vee g = k_a$, hence $\mathcal{Z}$ is not an ideal. □

In the following we will characterise some Boolean algebras that satisfy the theorem using their forcing properties. We shall explain how some of the notions discussed previously can be reinterpreted in terms of properties of reals in generic extensions and restated in the forcing language.

Let us repeat here well known basic relations concerning the interrelationship of functions and subsets of ω in a generic extension and the ground model.

6.4 Definition. Let M denote a generic extension of V.

(i) $X \subseteq \omega$ in the extension is said to be an *independent* (or *splitting*) *real* over V if for all $Y \in [\omega]^\omega \cap V$ both $X \cap Y$ and $Y - X$ are infinite.

(ii) A function $f \in M$, $f \in \omega^\omega$, is a *dominating real* over V if and only if for all $g \in \omega^\omega \cap V$ for all but finitely many $n \in \omega$, $g(n) \leq f(n)$.

(iii) Function $h \in \omega^\omega$ in the extension is said to be an *unbounded real* over V if for all $f \in \omega^\omega \cap V$ is the set $\{n \in \omega : h(n) > f(n)\}$ infinite.

(iv) M is an *ω^ω-bounding extension* of V if every $f \in M$, $f \in \omega^\omega$ is bounded by a $g \in \omega^\omega \cap V$, i.e. $f(n) \leq g(n)$ for any n.

If $\mathbb{B}$ is a Boolean algebra and $\mathbb{C}$ its completion, sequences from $\mathbb{C}^\omega$ can be viewed as canonical names for all reals in a generic extension when forcing with $(\mathbb{B}^+, \leq)$ or $(\mathbb{C}^+, \leq)$. Sequences from $\mathbb{B}^\omega$ can be viewed as names for elements of a subfield of all reals in the generic extension. If G is a generic filter on $\mathbb{C}$ over V, then a real (= subset of ω) in $V[G]$ named by $f \in \mathbb{C}^\omega$ is $f_G = \{n : f(n) \in G\}$.

6.5 EXAMPLE. Assume that a complete Boolean algebra adds an unbounded real $h \in V[G]$. We can suppose that h is strictly increasing, so h is an enumeration of some subset $\rho \subseteq \omega$. Then there is a name f for ρ such that $f \in \mathcal{Z}(\mathbb{B})$: let $g \in \mathbb{B}^\omega$ be a name for ρ, then for every infinite subset $X \in [\omega]^\omega \cap V$,

$$\bigwedge_{n \in X} g(n) \notin G.$$

Suppose the opposite; then clearly $X \subseteq \rho$ and so for an enumeration $e(X) \geq e(\rho) = h$, which contradicts the fact that h is unbounded.

Now we put

$$c = \bigvee_{X \in [\omega]^\omega \cap V} \bigwedge_{n \in X} g(n).$$

Then $c \notin G$ and put $f(n) = g(n) - c$. Clearly $f_G = g_G$ and f belongs to $\mathcal{Z}(\mathbb{B})$. $\triangle$

6.6 Definition. We call a sequence $\langle x_n : n \in \omega \rangle \in \mathbb{B}^\omega$ *splitting* if for every $b \in \mathbb{B}^+$ $x_n \wedge b \neq \mathbf{0}$ and $b - x_n \neq \mathbf{0}$ for all but finitely many n.

6.7 Lemma. *If there exists a sequence $\langle x_n : n \in \omega \rangle \in \mathbb{B}^\omega$ such that for every $\varphi \in \omega^{\omega}\!\nearrow$ the sequence $\langle c_n^\varphi := \bigvee_{n \leq i \leq \varphi(n)} x_i : n \in \omega \rangle$ is splitting, then $\mathbb{B}$ adds an unbounded real.*

Proof. Define function $f(n) = x_n$ and let G be a generic on $\mathbb{B}$ over V. We put $h = e(f_G)$ enumeration of $f_G \subseteq \omega$. Now pick arbitrary $\varphi \in \omega^{\omega}\!\nearrow \cap V$ and assume that $h \leq \varphi$. Then for all $n \in \omega$ c_n^φ belongs to G since $h \leq \varphi$ and so for arbitrary infinite set $X \subset \omega$ from the groundmodel V we get $\bigwedge_{n \in X} c_n^\varphi \in G$. This is a contradiction, since every infinite meet of a splitting sequence should be zero. $\square$

The following theorem describes the equivalences of almost regular embedding of Cantor algebra.

6.8 Theorem. *For any Boolean algebra $\mathbb{B}$ the following are equivalent.*

(i) Cantor algebra is almost regularly embedded in algebra $\mathbb{B}$,

(ii) there is a sequence $\{x_n : n \in \omega\}$ in $\mathbb{B}$ such that for any infinite subset X of ω, $\bigvee_B \{x_n : n \in X\} = \mathbf{1}$ and $\bigwedge_B \{x_n : n \in X\} = \mathbf{0}$,

(iii) *there is a splitting sequence in* $\mathbb{B}$,

(iv) *there is a splitting independent sequence in* $\mathbb{B}$,

(v) *there is* $f \in \mathbb{B}^\omega$ *such that for every generic* G *on* $\mathbb{B}$ f_G *is an independent real in* $V[G]$.

Proof. *(i)* $\to$ *(ii)* is clear from the definitions.

(ii) $\to$ *(iii)* Arguing by contradiction, we establish that the sequence $\langle x_n : n \in \omega \rangle$ satisfying *(ii)* is splitting: let $b \in \mathbb{B}^+$ be element not split by the sequence, then one of the sets $\{n \in \omega : x_n \leq -b\}$ and $\{n \in \omega : x_n \geq b\}$ is infinite, contradicting *(ii)*.

(iii) $\to$ *(iv)* Let $\{x_n : n \in \omega\}$ be a splitting sequence. We define $y_0 = x_0$. Since $\{x_n : n \in \omega\}$ is splitting, there exists $n_1 \in \omega$ such that $\forall m > n_1$ x_m splits ϵx_0, where $\epsilon \in \{-1, 1\}$ and we put $y_1 = x_{n_1}$. By induction we construct a splitting sequence $\langle y_n : n \in \omega \rangle$ since it is subsequence of $\langle x_n : n \in \omega \rangle$. By the construction $\bigwedge_{n \in K} \epsilon y_n \neq \mathbf{0}$ for any finite K so the sequence $\langle y_n : n \in \omega \rangle$ is independent.

(iv) $\to$ *(i)* We define $\mathcal{A}$ as a Boolean algebra generated by the sequence $\langle y_n : n \in \omega \rangle$. This algebra is countable and atomless, since atoms can not be split so is isomorphic to Cantor algebra.

(iii) $\to$ *(v)* Function $f(n) = x_n$, where $\langle x_n : n \in \omega \rangle$ is a splitting sequence, is an independent real for any generic G on $\mathbb{B}$. Fix some G and assume that there is an infinite set $a \subseteq \omega$ in V that is not split by f_G. We can assume that the set $a \cap f_G$ is finite and so $a \setminus f_g$ is cofinite, hence both belong to groundmodel V. Now the element $v = \bigwedge_{n \in a \cap f_G} x_n \wedge \bigwedge_{n \in a \setminus f_G} -x_n$ belongs to G and so $v \in \mathbb{B}^+$ but is not split by any element of $\langle x_n : n \in \omega \rangle$.

(v) $\to$*(iii)* We show that the sequence $\langle f(n) : n \in \omega \rangle$ is splitting, where $f \in \mathbb{B}^\omega$ is an independent real for any generic G on $\mathbb{B}$. Let $b \in \mathbb{B}^+$ be an element that is not split. It means that one of the following sets $b_0 = \{n \in \omega : -b \geq f(n)\}$, $b_1 = \{n \in \omega : b \leq f(n)\}$ is infinite. We chose G so that it contains b. Then $b_0 \cap f_G = \emptyset$ or $b_1 \cap f_G = b_1$ contradicting the fact that f is an independent real, since both b_0 and b_1 belong to V. $\square$

Hence any forcing notion that does not add an independent real gives an example of a complete Boolean algebra $\mathbb{B}$ for which $\mathcal{Z}(\mathbb{B})$ is a zero-convergence structure. Among them the ones that add a real, but not an independent real, are the non-trivial and interesting ones. There are several examples of such forcing notions. The most familiar are Sacks forcing [Sac71], Miller forcing [Mil84], Blass-Shelah forcing [BS89b], and Matet forcing [Bla89]. Therefore, Boolean algebras of regular open sets of these partial orders and all their dense subalgebras are examples of non $(\omega, 2)$-distributive Boolean algebras for which $\mathcal{Z}$ is a zero-convergence structure.

It is well known that among the forcing notions mentioned above, only Sacks forcing is ω^ω-bounding. On the other hand, any forcing notion adding a dominating real adds also an independent real and so it is not an example of an algebra where $\mathcal{Z}$ is a zero-convergence structure.

Recall that a topological space X is sequentially compact if every sequence $\langle a_n : n \in \omega \rangle$, $a_n \in X$, has a convergent subsequence.

Since we are dealing with sequences, the most interesting cases concern ccc Boolean algebras. For instance, whenever $\mathbb{B}$ is a regular subalgebra of a $\mathbb{C}$, then $\mathcal{Z}(\mathbb{B}) = \mathcal{Z}(\mathbb{C}) \cap \mathbb{B}^\omega$. If $\mathbb{C}$ satisfies ccc, then $\mathcal{Z}(\mathbb{B}) = \mathcal{Z}(\mathbb{C}) \cap \mathbb{B}^\omega$ implies that $\mathbb{B}$ is a regular subalgebra. In the preceding we introduced several equivalences for the fact that $\mathcal{Z}(\mathbb{B})$ itself is an ideal. In the following theorem we show that this is equivalent with sequential compactness of the topology given by the Os ideal.

6.9 Theorem. *For a* ccc *complete Boolean algebra $\mathbb{B}$, the following statements are equivalent.*

(i) *$\mathcal{Z}(\mathbb{B})$ is zero-convergence.*

(ii) *$\mathcal{Z}(\mathbb{B})$ is Urysohn closure of $\mathrm{Os}(\mathbb{B})$.*

(iii) *The topological space $(\mathbb{B}, \tau_s)$, where τ_s is the order sequential topology, is sequentially compact.*

Proof. *(i) $\to$ (iii)* Since $\mathcal{Z}(\mathbb{B})$ is zero-convergence there is no sequence $\langle a_n : n \in \omega \rangle \subset \mathbb{B}^\omega$ and no constant $a \in \mathbb{B}^+$ such that $\bigwedge a_{\varphi(n)} = \mathbf{0}$ and $\bigvee a_{\varphi(n)} = a$ for any $\varphi \in \omega^{\omega}\!\nearrow$.

Given arbitrary sequence $\langle a_n : n \in \omega \rangle$ we have to show that there is a τ-convergent subsequence $\langle b_n : n \in \omega \rangle$. It is sufficient to show that sequence $\langle b_n \rangle$ converges in $s(\mathrm{Os})$.

Let us assume that $\overline{\lim} a_n > \underline{\lim} a_n$ because otherwise we are done. There is a subsequence $\langle d_n \rangle$ of $\langle a_n \rangle$ such that its every subsequence has the same limes superior. We obtain it as follows: Suppose that there is a subsequence $\langle c_n^1 \rangle$ of $\langle a_n \rangle$ such that $\overline{\lim} a_n > \overline{\lim} c_n^1$ and also there is a subsequence $\langle c_n^2 \rangle$ of $\langle c_n^1 \rangle$ such that $\overline{\lim} c_n^1 > \overline{\lim} c_n^2$ and so on. Since $\mathbb{B}$ satisfy ccc and $\mathcal{P}(\omega)/\mathit{fin}$ is σ-closed, the process has to stop after countably many steps and the pseudointersection works. Similarly we can find subsequence $\langle b_n \rangle$ of $\langle d_n \rangle$ which is homogeneous in limes inferior. It suffice to show that $\overline{\lim} b_n = \underline{\lim} b_n$.

Assume in contrary that $\overline{\lim} b_n > \underline{\lim} b_n$ and define sequence $\langle e_n = b_n - \underline{\lim} b_n \ : n \in \omega \rangle$. From the properties of $\langle b_n \rangle$ it is clear that $\bigwedge e_{\varphi(n)} = \mathbf{0}$ and $\bigvee e_{\varphi(n)} = \overline{\lim} b_n - \underline{\lim} b_n \in \mathbb{B}^+$ for any $\varphi \in \omega^{\omega}\!\nearrow$ but such sequence cannot exists. Hence $\overline{\lim} b_n = \underline{\lim} b_n$.

The implication *(ii) $\to$ (i)* is trivial.

Now we prove the implication *(iii) $\to$ (ii)*. Let $\langle a_n \rangle$ be a sequence from $\mathcal{Z}(\mathbb{B})$ so from the definition of $\mathcal{Z}(\mathbb{B})$ its $\underline{\lim} a_n = \mathbf{0}$. By the sequential compactness there is a subsequence $\langle b_n \rangle$ which is convergent in $s(\mathrm{Os})$. Since $\mathcal{Z}(\mathbb{B})$ is closed under subsequences $\underline{\lim} b_n = \mathbf{0}$ so as the $\overline{\lim} b_n = \mathbf{0}$ since $\langle b_n \rangle$ is convergent. Now we proved that the sequence $\langle b_n \rangle$ belongs to Os hence $\mathcal{Z}(\mathbb{B}) \subset \mathrm{U}(\mathrm{Os})$ and the opposite inclusion follows directly from the definition. □

As a corollary we get the following theorem. The direct proof of this theorem can be found in [BJP05].

6.10 Theorem. *Let $\mathbb{B}$ be a complete* ccc *Boolean algebra. $\mathbb{B}$ does not add independent reals if and only if $(\mathbb{B}, \tau_s)$ is sequentially compact.*

7. THE DECOMPOSITION THEOREM

Let us first remind the definition of weak distributivity.

7.1 Definition. Let κ be an infinite cardinal. A Boolean algebra $\mathbb{B}$ is (ω, κ)-weakly distributive if for every sequence $\{P_n\}$ of maximal antichains, each of size at most κ, there exists a maximal antichain Q with the property that each $q \in Q$ meets only finitely many elements of each P_n. $\mathbb{B}$ is *weakly distributive* if it is (ω, ω)-weakly distributive.

If $\mathbb{B}$ is κ^+ complete Boolean algebra then $\mathbb{B}$ is (ω, κ)-weakly distributive if and only if it satisfies the following distributive low:

$$\bigwedge_n \bigvee_\alpha a_{n\alpha} = \bigvee_{f:\omega \to [\kappa]^{<\omega}} \bigwedge_n \bigvee_{\alpha \in f(n)} a_{n\alpha}.$$

Let us recall some known characterisations of weak distributivity.

7.2 Theorem. *For* ccc *Boolean algebra $\mathbb{B}$ the following are equivalent*

(i) *$\mathbb{B}$ is weakly distributive,*

(ii) *the ideal of nowhere dense sets in the Stone space $\mathrm{St}(\mathbb{B})$ is a σ-ideal, i.e. meager sets are nowhere dense,*

(iii) *topological space $(\mathbb{B}, \tau_s)$ is Fréchet* [BGJ98]*,*

(iv) *the extension $V^{Compl(\mathbb{B})}$ of groundmodel V is ${}^\omega\omega$-bounding.*

In the proof of the decomposition theorem we use the following characterisation of Maharam algebras.

7.3 Theorem. (B. Balcar, W. Głowcziński, T. Jech [BGJ98]) *The complete* ccc *Boolean algebra $\mathbb{B}$ is Maharam if and only if the order sequential topology τ_s on $\mathbb{B}$ is Hausdorff.*

7.4 Theorem. THE DECOMPOSITION THEOREM *Let $\mathbb{B}$ be a complete* ccc *Boolean algebra. Then there are disjoint elements $d, m \in \mathbb{B}$ such that $d \vee m = \mathbf{1}$ and*

(i) *In the space $(\mathbb{B} \restriction d, \tau_s)$ the closure of every nonempty open set is the whole space.*

(ii) *The Boolean algebra $\mathbb{B} \restriction m$ carries a strictly positive Maharam submeasure.*

The elements d and m are uniquely determined, and either can be $\mathbf{0}$. If $m \neq \mathbf{0}$ then $\mathbb{B}$ carries a nontrivial continuous submeasure while if $m = \mathbf{0}$ then every continuous real valued function on $\mathbb{B}$ is constant. For the reader convenience we state here lemmas from [BGJ98] which we employ in the proof of the theorem.

Let

$$\mathcal{N} = \{U : U \text{ is an open neighbourhood of } \mathbf{0} \text{ and is downward closed}\}$$

where *downward closed* means that $a < b \in U$ implies $a \in U$.

7.5 Lemma. *If* $(\mathbb{B}, \tau_s)$ *is Fréchet and* A *is downward closed then* $\mathrm{cl}(A) = \bigcap\{A \vee V : V \in \mathcal{N}\}$*, and* $\mathrm{cl}(A)$ *is downward closed.*

7.6 Lemma. *Let* $\mathbb{B}$ *be a* σ*-complete Boolean algebra such that* $(\mathbb{B}, \tau_s)$ *is Fréchet. Then for every* $U \in \mathcal{N}$ *there exists* $V \in \mathcal{N}$ *such that* $V \vee V \vee V \subseteq U \vee U$.

Proof. First we prove the theorem in the case when $\mathbb{B}$ is weakly distributive. Let $\mathbb{B}$ be a weakly distributive complete ccc Boolean algebra. By V.7.2*(iii)* the space $(\mathbb{B}, \tau_s)$ is Fréchet. By lemma V.7.5 if $(\mathbb{B}, \tau_s)$ is Fréchet then $\mathcal{N}$ is a neighbourhood base of $\mathbf{0}$ and the closure $\mathrm{cl}(A)$ of each downward closed set A is $\bigcap\{A \vee U : U \in \mathcal{N}\}$ and is also downward closed.

Now let

$$D = \bigcap\{\mathrm{cl}(U) : U \in \mathcal{N}\} \text{ and } d = \bigvee D.$$

D is both downward closed and topologically closed, and it follows from the remarks above that

$$D = \bigcap\{U \vee V : U, V \in \mathcal{N}\} = \bigcap\{U \vee U : U \in \mathcal{N}\}.$$

If $a \notin D$ then for some $U \in \mathcal{N}$, $a \notin U \vee U$ and hence U and $a \vartriangle U$ are disjoint; in other words, a is Hausdorff separated from $\mathbf{0}$. It follows that if we let $m = -d$, then the space $(\mathbb{B} \restriction m, \tau_s)$ is a Hausdorff space. By V.7.3, $\mathbb{B} \restriction m$ carries a strictly positive Maharam submeasure. It remains to show that in $(\mathbb{B} \restriction d, \tau_s)$, every nonempty open set is dense.

Claim. D is closed under $\vee$.

Let $U \in \mathcal{N}$. By lemma V.7.6, there exists a $U_1 \in \mathcal{N}$ such that $U_1 \vee U_1 \vee U_1 \subset U \vee U$. Similarly, there exists a $U_2 \in \mathcal{N}$ such that $U_2 \vee U_2 \vee U_2 \subset U_1 \vee U_1$, and letting $V = U_1 \cap U_2$ we get $V \vee V \vee V \vee V \subset U \vee U$. Hence $D = \bigcap\{V \vee V \vee V \vee V : V \in \mathcal{N}\}$. It follows that $D \vee D = D$.

Now let $\{a_n : n \in \omega\}$ be a maximal antichain in D. The sequence $\{\bigvee_{k=0}^{n} a_k : n \in \omega\}$ is in D and converges to d. Since D is closed, we have $d \in D$, and so $\mathbb{B} \restriction d = D$.

For every $U \in \mathcal{N}$, $\mathrm{cl}(U) \supset D$. Now let G be an arbitrary topologically open set in $\mathbb{B} \restriction d$. There exist $a \in D$ and $U \in \mathcal{N}$ such that $G \supset (a \vartriangle U) \cap D$. Since $\mathrm{cl}(a \vartriangle U) \supset a \vartriangle D = D$, we have $\mathrm{cl}(G) \supset D$ and the theorem follows for the weakly distributive case.

In the general case, there exists an element $d_1 \in \mathbb{B}$ such that $\mathbb{B} \restriction -d_1$ is weakly distributive, and such that $\mathbb{B} \restriction d_1$ is nowhere weakly distributive. There exists an infinite matrix $\{a_{kl}\}$ such that each row is a partition of d_1 and for every nonzero $x \leq d_1$ there is some $k \in \omega$ such that $x \wedge a_{kl} \neq \mathbf{0}$ for infinitely many l.

Let d_2 and m, with $d_2 \vee m = -d_1$, be the decomposition of the weakly distributive algebra $\mathbb{B} \restriction -d_1$, so that $\mathbb{B} \restriction m$ carries a strictly positive Maharam submeasure and $(\mathbb{B} \restriction d_2, \tau_s)$ has the property that every nonempty open set is dense in the space. Let $d = d_1 \vee d_2$, and let us prove that in $(\mathbb{B} \restriction d, \tau_s)$, every nonempty open set is dense.

Let U be an open neighbourhood of $\mathbf{0}$ in $\mathbb{B} \restriction d$. The space $(\mathbb{B} \restriction d_2, \tau_s)$ is a closed subspace of $(\mathbb{B} \restriction d, \tau_s)$, and $V = U \cap \mathbb{B} \restriction d_2$ is an open neighbourhood of $\mathbf{0}$ in $\mathbb{B} \restriction d_2$.

Let $c \leq d$ be arbitrary; we shall prove that c is in the closure of U. Let $c_1 = c \wedge d_1$ and $c_2 = c \wedge d_2$. Since c_2 is in the closure of V and $\mathbb{B} \restriction d_2$ is Fréchet, there exists a sequence $\langle z_n : n \in \omega \rangle$ in V that converges to c_2. We shall prove that $c_1 \vee z_n \in \mathrm{cl}(U)$ for each $n \in \omega$, and then it follows that $c = \lim_n (c_1 \vee z_n)$ is in $\mathrm{cl}(U)$.

Thus let $n \in \omega$ be fixed. For every k and every l let $y_{kl} = c_1 \wedge \bigvee_{i \geq l} a_{ki}$. Since the sequence $\langle y_{0l} : l \in \omega \rangle$ converges to $\mathbf{0}$, we have $\lim_l (y_{0l} \vee z_n) = z_n$, and since $z_n \in U$, there exists some l_0 such that $y_{0l_0} \vee z_n \in U$. Let $x_0 = y_{0l_0}$.

Next we consider the sequence $\langle y_{1l} \vee x_0 \vee z_n : l \in \omega \rangle$. This sequence converges to $x_0 \vee z_n \in U$ and so there exists some l_1 such that $x_1 \vee z_n \in U$ where $x_1 = y_{1l_1} \vee x_0$. We proceed by induction and obtain a sequence $\langle l_k : k \in \omega \rangle$ and an increasing sequence $\langle x_k : k \in \omega \rangle$ with $x_k \vee z_n \in U$ for each k. The sequence $\langle x_k : k \in \omega \rangle$ converges to c_1 because otherwise, if $b \neq \mathbf{0}$ is the complement of $\bigvee_k x_k$ in c, then $b \leq \bigwedge_k \bigvee_{i < l_k} a_{ki}$ and so b meets only finitely many elements in each row of the matrix. Hence $c_1 \vee z_n = \lim_k (x_k \vee z_n) \in \mathrm{cl}(U)$. $\square$

We use this theorem to different proof of the well known fact. Note that random real, i.e. the generic real added by random forcing is independent.

7.7 Lemma. *If $\mathbb{B}$ is atomless and carries a strictly positive Maharam submeasure then $\mathbb{B}$ adds an independent real.*

Proof. Let μ be a strictly positive Maharam submeasure on $\mathbb{B}$.

Case 1. The submeasure μ is uniformly exhaustive. By Kalton - Roberts theorem IV.6.11, $\mathbb{B}$ is a measure algebra which is known to add independent reals.

Case 2. There exists an $\varepsilon > 0$ and a sequence $\langle P_n : n \in \omega \rangle$ of finite antichains with $|P_n| \geq n$, and $\mu(a) \geq \varepsilon$ for each $a \in P_n$. We can find infinitely many functions f_k, $k \in \omega$, such that $f_k(n) \in P_n$ for every $n \in \omega$ and when $k \neq l$ then $f_k(n) \neq f_l(n)$ for eventually all n. Since $\mathbb{B}$ does not add independent reals, by Theorem V.6.10 τ_s is sequentially compact and so we can find convergent subsequences g_k of f_k, with $\mathrm{dom}(g_{k+1}) \subset \mathrm{dom}(g_k)$. If $a_k = \lim_n g_k(n)$ then the a_k's are mutually disjoint, and $\mu(a_k) \geq \varepsilon$ for each $k \in \omega$ (by continuity of μ). Thus μ is not exhaustive; a contradiction. $\square$

8. WEAKLY DISTRIBUTIVE BOOLEAN ALGEBRAS

In the following we proceed to another characterisations of weak distributivity; cf. V.7.2. To prevent the confusion let us remind the notion of P-ideal on $\mathbb{B}^\omega$ and P-ideal on the set $\mathbb{B}$.

8.1 Definition. (i) An ideal I on $\mathbb{B}^\omega$ is a *P-ideal* if for any countable family $\{f_k : k \in \omega\} \subset I$ there is $g \in I$ such that for each $k \in \omega)$ $f_k - g \in$ *Fin*, where *Fin* is an ideal on $\mathbb{B}^\omega$ of sequences with finite support.

(ii) An ideal $\mathcal{I}$ on the set $\mathbb{B}$ is called *P-ideal* if for any $\{I_n \in \mathcal{I} : n \in \omega\}$ there exists $I \in \mathcal{I}$ such that $I_n \subseteq^* I$ for every $n \in \omega$.

The next theorem shows that the behaviour of the ideal $\mathrm{Os}(\mathbb{B})$ is somewhat crucial for the weak distributivity of the Boolean algebra $\mathbb{B}$.

8.2 Theorem. *Complete Boolean algebra $\mathbb{B}$ satisfying* ccc *is weakly distributive if and only if* $\mathrm{Os}(\mathbb{B})$ *is a P-ideal on* $\mathbb{B}^\omega$.

Proof. As we already mention, Os is always an ideal. Let us suppose that Os is a P-ideal on $\mathbb{B}^\omega$. Let $\langle P_n : n \in \omega\rangle$ be a countable family of disjoint decompositions of the unity. Define a function $g_n(m) = -\bigvee_{i=0}^{m} P_n(i)$, $g_n \in \mathrm{Os}$ since every P_n is a decomposition of the unity. By our assumption there exists $g \in \mathrm{Os}$ such that for all $n \in \omega$, $g_n - g \in$ *Fin*. Now we define a decomposition of the unity Q : let $Q(0) = -g(0)$ and $Q(m) = -g(m) - \bigvee_{i=0}^{m-1} Q(i)$. Q is a decomposition of the unity since $g \in \mathrm{Os}$. For every $n \in \omega$ there exists $k_n \in \omega$ such that $\forall m > k_n\ g_n(m) \leq g(m)$; that means $\bigvee_{i=0}^{m} Q(i) \leq \bigvee_{i=0}^{m} P_n(i)$ and so Q witnessing the weak distributivity, since every $Q(m)$ meets at most finitely many members of each P_n.

On the other hand let $\langle f_n : n \in \omega\rangle \subseteq \mathrm{Os}$, since $\mathbb{B}$ is complete we can assume that $f_n \searrow \mathbf{0}$. Define $g_n = \bigvee_{i=0}^{n} f_i$, $g_n \in \mathrm{Os}$ since Os is an ideal. As in previous every function g_n determines a partition of the unity and by weak distributivity of algebra $\mathbb{B}$ we get a partition of the unity Q which determines a function $f(n) = \bigvee_{n\leq k} Q(k)$. Clearly f belongs to Os and for all $n \in \omega$ there exists $k(n) \in \omega$ such that $f(n) \geq g_n(k(n))$. Therefore $g_n - f \in$ *Fin*, which completes the proof. □

8.3 Definition. (i) Recall the well known *diagonal sequence property*. If $\langle x_{n,k} : n \in \omega, k \in \omega\rangle$ is double indexed sequence of elements of Boolean algebra $\mathbb{B}$ such that for every $n \in \omega$ $x_{n,k} \xrightarrow{k} x_n$ and $x_n \xrightarrow{n} x$ then for every n we can choose $k(n)$ such that $x_{n,k(n)} \xrightarrow{n} x$.

(ii) The similar notion is *weak diagonal sequence property* which says that for double indexed sequence $\langle x_{n,k} : n \in \omega, k \in \omega\rangle$ of elements of $\mathbb{B}$ such that for every $n \in \omega$ $x_{n,k} \xrightarrow{k} x_n$ and $x_n \xrightarrow{n} x$ there exists $\varphi \in \omega^{\omega}\!\nearrow$ and function $k : \mathrm{rng}(\varphi) \to \omega$ such that $x_{\varphi(n),k(\varphi(n))} \xrightarrow{n} x$.

8.4 Theorem. *For* ccc *complete Boolean algebra $\mathbb{B}$ the following conditions are equivalent.*

(i) *$\mathbb{B}$ has the diagonal sequence property,*

(ii) $\mathbb{B}$ *has the weak diagonal sequence property,*

(iii) $\mathbb{B}$ *is weakly distributive.*

Proof. Implication *(i)→(ii)* is trivial.

(ii)→(iii) It is enough to show that Os is P-ideal. Let $\langle f_n : n \in \omega \rangle \subseteq \mathrm{Os}$. Since $\mathbb{B}$ is complete Boolean algebra we can assume that $f_n \searrow \mathbf{0}$ and since Os is an ideal we can also assume that for $m \leq n$ the inequality $f_m \leq f_n$ holds. Now by weak diagonal sequence property there are $\varphi \in \omega^{\omega}\!\nearrow$ and $k : \mathrm{rng}(\varphi) \to \omega$ such that $f_{\varphi(n)}(k(\varphi(n))) \xrightarrow{n} \mathbf{0}$; f_n is nonincreasing hence we can chose k increasing. Now we define sequence $g \in \mathbb{B}^{\omega}$, $g(m) := f_{\varphi(n)}(k(\varphi(n)))$ for $m \in [k(\varphi(n)), k(\varphi(n+1)))$. Clearly $g \in \mathrm{Os}$ and $f_n - g \in \mathit{Fin}$ since for all $m \geq k(\varphi(n))$ $g(m) = f_{\varphi(n)}(k(\varphi(n))) \geq f_n(k(\varphi(n)) \geq f_n(m)$, for $m \in [k(\varphi(n)), k(\varphi(n+1)))$.

(iii) $\to$ *(i)* Assume that $\langle x_{n,k} : n \in \omega, k \in \omega \rangle$ is double indexed sequence of elements of $\mathbb{B}$ such that for every $n \in \omega$ $x_{n,k} \xrightarrow{k} x_n$ and $x_n \xrightarrow{n} x$. We define countable family of partitions of unity $\langle P_n : n \in \omega \rangle$. Let $P_n(0) := -(x_{n,0} \triangle x_n)$ and $P_n(m) := -\bigvee_{i=0}^{m-1} P_n(i) - (x_{n,m} \triangle x_n)$. By weak distributivity of algebra $\mathbb{B}$ we obtain partition unity Q such that for all $n \in \omega$ there exists $k(n) \in \omega$ for which the inequality $\bigvee_{i=0}^{n} Q(i) \leq \bigvee_{i=0}^{k(n)} P_n(i)$ holds true. From this inequality we derive that $x_{n,k(n)} \triangle x_n \xrightarrow{n} \mathbf{0}$ since the function $-\bigvee_{i=0}^{n} Q(i) \searrow \mathbf{0}$ is its majorant. It remains to show that $x_{n,k(n)} \xrightarrow{n} x$, it means that $\bigwedge_k \bigvee_{n \geq k} x_{n,k(n)} \triangle x = \mathbf{0}$, but it is clear since $x_{n,k(n)} \triangle x \leq (x_{n,k(n)} \triangle x_n) \vee (x_n \triangle x)$. □

The following theorem has a very similar structure with many ‘Baire-like’ theorems. Here one has to be extra careful. When we talk about dense set D in Boolean algebra $\mathbb{B}$, we mean that for each $a \in \mathbb{B}^+$ there is some $d \in D^+$ for which $d \leq a$.

When we talk about dense set X in topological space $(\mathbb{B}, \tau_s)$ we mean that each τ_s-open set U intersect with X.

8.5 Theorem. *Let* $\mathbb{B}$ *be a* ccc *complete Boolean algebra, then* $\mathbb{B}$ *is weakly distributive if and only if for every collection* $\langle U_n : n \in \omega \rangle$ *of downward closed dense sets in the topological space* $(\mathbb{B}, \tau_s)$ *the intersection*

$$\bigcap_{n \in \omega} U_n$$

is a dense set in the Boolean algebra $\mathbb{B}$*.*

Proof. Let $\mathbb{B}$ be a ccc, weakly distributive Boolean algebra and let $\langle U_n : n \in \omega \rangle$ be downward closed topologically dense sets. Fix $a \in \mathbb{B}^+$. Since the space $(\mathbb{B}, \tau_s)$ is Fréchet and each U_n is topologically dense, there is a sequence

$$\langle a_n^m : m \in \omega \rangle \subset U_n, \quad a_n^m \nearrow_m a, \text{ for each } n.$$

From the weak distributivity it follows that there is positive $c \in \bigcap_{n \in \omega} U_n$, $c \leq a$.

To show weak distributivity let $a_n^m \nearrow \mathbf{1}$ be nondecreasing sequences. Put

$$U_n = \{ b \in \mathbb{B}^+ : (\exists m \in \omega)\, b \leq a_n^m \}.$$

U_n's are downward closed and topologically dense, hence by the assumption $\bigcap U_n$ is dense in algebra **B**. It means that for each $c \in \mathbb{B}^+$ there is some positive $b \in \bigcap U_n$ such that $b \leq c$; which completes the proof. □

Let us conclude the weak distributivity part with the recent result of B. Balcar and T. Jech [BJ07]. Note that random algebra is preserved by this simple iteration.

8.6 Theorem. *If $\mathbb{B}$ is a Maharam algebra and if $\dot{\mathbb{C}}$ is a Maharam algebra in $V^{\mathbb{B}}$, then $\mathbb{B} \star \dot{\mathbb{C}}$ is a Maharam algebra.*

9. ALGEBRAIC CHARACTERISATION OF MAHARAM ALGEBRA

In this section we are going to characterise Maharam algebra using its algebraic properties. We already mention the topological characterisation of Maharam algebra; cf. V.7.3. We employ the decomposition theorem to produce another topological characterisation V.9.13 and to prove the algebraic one. Main theorem of this section is due to S. Todorcevic.

Theorem. (Todorcevic [Tod04]) *Let $\mathbb{B}$ be a complete Boolean algebra. Then $\mathbb{B}$ carries a strictly positive Maharam submeasure if and only if*

(i) *$\mathbb{B}$ is weakly distributive, and*

(ii) *$\mathbb{B}$ satisfies the σ-finite* cc.

We give here a different proof of this theorem using our techniques.

9.1 SET - STRUCTURES DERIVED FROM CONVERGENCE ONES

The general idea of this paragraph is simple: Given some set of sequences say $A \subseteq \mathbb{B}^\omega$ we can forget the sequence structure of A and define something we call a 'A-set' as a family of ranges of the sequences in A, formally:

$$A_{\text{set}} = \{X \in [\mathbb{B}]^{\leq \aleph_0} : \exists f \in A \quad \operatorname{rng}(f) = X\}$$

As an example of this notion we can look at the basic structure $\mathcal{Z}(\mathbb{B})$. It is easily seen, that

$$\mathcal{Z}_{\text{set}} = \{X \in [\mathbb{B}]^{\leq \aleph_0} : \forall a \in \mathbb{B}^+ \text{ the set } \{x \in X : x \geq a\} \text{ is finite }\}.$$

One can ask a natural question how such systems of sets of elements of the Boolean algebra $\mathbb{B}$ behaves in case that we start from some particular Zero - convergence structure especially from $\mathrm{Os}(\mathbb{B})$ and if it is possible to reconstruct the structure again. In case of $\mathcal{Z}_{\text{set}}$ the reconstruction of structure $\mathcal{Z}(\mathbb{B})$ is again easily seen: the set $\{f \in {}^\omega\mathbb{B} : f \text{ is finite to one and } \operatorname{rng}(f) \in \mathcal{Z}_{\text{set}}\}$ contains every sequence from $\mathcal{Z}(\mathbb{B})$ with an infinite support. Since $\mathcal{Z}(\mathbb{B})$ contains all sequences with finite support we have to add them

$$\mathcal{Z}(\mathbb{B}) = \{f \in {}^{\omega}\mathbb{B} : f \text{ is fin to one } \& \operatorname{rng}(f) \in \mathcal{Z}_{\text{set}}\} \cup \{f \in {}^{\omega}\mathbb{B} : \operatorname{supp}(f) \text{ is finite }\}.$$

In the following we focus on some properties of an ideal of Zero - convergence $\mathrm{Os}(\mathbb{B})$.

9.2 Definition. Let A be a subset of Boolean algebra $\mathbb{B}$. We define for every positive element $b \in \mathbb{B}^+$ set of b-compatible elements of A as the set

$$\{a \in A \ : \ a \parallel b\}$$

and we will denote it $\mathrm{Cp}(b, A)$.

9.3 Fact. *Let $\mathbb{B}$ be a ccc Boolean algebra, then $X \in \mathrm{Os}_{set}$ if and only if there exists a maximal antichain A in $\mathbb{B}$ such that for every $a \in A$ the set $\mathrm{Cp}(a, X)$ is finite.*

Proof. Let us have $X \in \mathrm{Os}_{\text{set}}$, then there is a witness $f \in \mathrm{Os}$ such that $\operatorname{rng}(f) = X$. By the definition of Os there must be a function $g : \omega \to \mathrm{Compl}(\mathbb{B})$ such that $g \searrow \mathbf{0}$ and $f \leq g$. By induction we define a countable antichain $A' = \{A'(i) : i \in \omega\}$ in $\mathrm{Compl}(\mathbb{B})$: Set $A'(0) = -g(0)$ and $A'(n) = -g(n) - \bigvee_{i=0}^{n-1} A'(i)$. It suffices to take a maximal antichain A in $\mathbb{B}$ as a refinement of the maximal antichain A'.

On the other hand let us have $X = \langle X_i : i \in \omega\rangle$ as stated in the fact, i.e. there is a maximal antichain $A = \{A(i) : i \in \omega\}$, such that $\forall a \in A$ the set $\mathrm{Cp}(a, X)$ is finite. Note that such X has to be finite, because $\mathbb{B}$ is ccc Boolean algebra. Define functions $g, f : \omega \to \mathbb{B}$ by induction: $A(0)$ is compatible with only finitely many, say k_0, elements of X. We set $f(i) = X_i$ and $g(i) = \mathbf{1}$ for $i \leq k_0$. We continue in a similar fashion, if $A(n)$ is not compatible with any X_i, $i > k_{n-1}$ then we set $f(k_{n-1} + 1) = X_{k_{n-1}+1}$ and $g(k_{n-1} + 1) = -\bigvee_{i=0}^{n} A(i)$. Otherwise $A(n)$ is compatible with finitely many, say m, elements of $\langle X_i : i > k_{n-1}\rangle$ and we set $k_n = k_{n-1} + m$ and $f(k_{n-1} + i) = X_{k_{n-1}+i}$ and $g(k_{n-1} + i) = -\bigvee_{i=0}^{n-1} A(n)$ for $i \leq m$. Thus defined functions f and g satisfy: $\operatorname{rng}(f) = X$ and $f \in \mathrm{Os}$ since $g \searrow \mathbf{0}$ and $f \leq g$. $\square$

Since an ideal $\mathrm{Os}(\mathbb{B})$ contains every sequence with finite support we can again easily describe its reconstruction from Os_{set} in the same fashion as we did for $\mathcal{Z}(\mathbb{B})$.

9.4 Fact. *Let $\mathbb{B}$ be a ccc Boolean algebra, then the sequence f belongs to the $\mathrm{Os}(\mathbb{B})$ if and only if it has a finite support or is finite to one and rng$(f) \in \mathrm{Os}_{set}$.*

The previous two proposition lead us to alternative definition of the Zero - convergence ideal $\mathrm{Os}(\mathbb{B})$ for a ccc Boolean algebra $\mathbb{B}$ in which we do not employ a σ-completion of $\mathbb{B}$.

9.5 Corollary. *Let $\mathbb{B}$ be a ccc Boolean algebra, then the sequence $f \in {}^{\omega}\mathbb{B}$ belongs to the Zero - convergence $\mathrm{Os}(\mathbb{B})$ if and only if there is a maximal antichain A in $\mathbb{B}$ such that the set $\{n \in \omega : f(n) \parallel a\}$ is finite for every $a \in A$.*

The following theorem characterises weakly distributive algebras among ccc Boolean algebras in terms of Os_{set}.

9.6 Theorem. *A* ccc *Boolean algebra* $\mathbb{B}$ *is weakly distributive if and only if* $\mathrm{Os}_{set}(\mathbb{B})$ *is a P-ideal on* $[\mathbb{B}]^{\leq\aleph_0}$.

Proof. *(i)* Let us have a sequence $\langle X_n : n \in \omega \rangle$, $X_n \in \mathrm{Os}_{\mathrm{set}}(\mathbb{B})$. By the definition of $\mathrm{Os}_{\mathrm{set}}(\mathbb{B})$ there is a collection of decompositions of unity P_n. By weak distributivity there is a decomposition of unity P such that $\forall p \in P \quad \forall n \in \omega$ the set $\mathrm{Cp}(p, P_n)$ is finite. We enumerate $P = \{p_i : i \in \omega\}$ and set

$$X = \bigcup_n \{ x \in X_n : x \perp \cup_{j \leq n} p_j \}.$$

Now a decomposition P is the witness of $X \in \mathrm{Os}_{\mathrm{set}}(\mathbb{B})$ and $\forall n \in \omega$ $X_n \setminus X$ is finite.
(ii) Let us have a collection $\{P_n : n \in \omega\}$ of maximal antichains. Clearly $P_n \in \mathrm{Os}_{\mathrm{set}}(\mathbb{B})$. Due to our assumptions there is a set $X \in \mathrm{Os}_{\mathrm{set}}(\mathbb{B})$ such that $P_n \setminus X$ is finite for all $n \in \omega$ and some decomposition P which witnesses this fact i.e: $\forall p \in P$ the set $\mathrm{Cp}(p, X)$ is finite, so the algebra $\mathbb{B}$ is weakly distributive. □

The following principle was formulated by S. Todorcevic

9.7 Definition. *P-Ideal Dichotomy (PID)* Let S be an infinite set. Then for every P-ideal $\mathcal{I} \subset [S]^{\leq\omega}$ either

(i) $\exists Y \subset S$ uncountable such that $[Y]^{\leq\omega} \subset \mathcal{I}$, or

(ii) $\exists \{S_n : n \in \omega\}$ such that $\bigcup_n S_n = S \quad$ and $\quad \forall n \in \omega \quad \forall I \in \mathcal{I} \quad |S_n \cap I| < \omega$.

The principle PID follows from the Proper Forcing Axiom and is also consistent with GCH [Tod00b]. For related principles with many interesting applications, see [AT97].

9.8 Lemma. *If* $\mathbb{B}$ *is* ccc *Boolean algebra then there exists no uncountable* $X \subset \mathbb{B}$ *such that* $[X]^\omega \subset \mathrm{Os}_{set}$.

Proof. Let $X \subset \mathbb{B}^+$ be uncountable, where $\mathbb{B}^+ = \mathbb{B} - \{\mathbf{0}\}$. First we claim that there exists some $b \in \mathbb{B}^+$ such that for every nonzero $a \leq b$ the set $X_a = \{x \in X : x \wedge a \neq \mathbf{0}\}$ is uncountable. To see this assume that for every $b \in \mathbb{B}^+$ there is some nonzero $a \leq b$ such that the set X_a is at most countable; thus the set $D = \{a \in \mathbb{B}^+ : X_a$ is at most countable$\}$ is dense in $\mathbb{B}$. By ccc D has a countable subset $W \subset D$ such that $\bigvee W = \mathbf{1}$. Now $X = \bigcup_{a \in W} X_a$; a contradiction.

Let $b \in \mathbb{B}^+$ be as in the claim. If $Y \subset X$ is countable than we can find a countable set $Z \subset (X - Y)$ such that $\bigvee Z \geq b$. This is because $\bigvee(X - Z) \geq b$ (by the claim), and so a countable $Z \subset (X - Y)$ with $\bigvee Z = \bigvee(X - Y)$ exists by ccc.

Now let $X_0 \in [X]^\omega$ be such that $\bigvee X_0 \geq b$ and by induction let $X_n \in [X - \bigcup_{i<n} X_i]^\omega$ such that $\bigvee X_n \geq b$. Clearly $\bigcup X_n \in [X]^\omega$; we claim that $\bigcup X_n \notin \mathrm{Os}_{\mathrm{set}}$. Otherwise by corollary V.9.5 there is an antichain W such that every $w \in W$ is incompatible with all but finitely many $x \in \bigcup X_n$. To obtain a contradiction it is enough to choose some $w \in W$ such that $w \wedge b \neq \mathbf{0}$. □

9.9 Corollary. (S. Quickert [Qui02]) *Under PID every* ccc*, weakly distributive Boolean algebra is σ-finite* cc.

9.10 Corollary. *Let $\mathbb{B}$ be weakly distributive,* ccc *complete Boolean algebra. PID implies that every singleton is a* G_δ *set in* $(\mathbb{B}, \tau_s)$.

Proof. It is enough to show that $\{\mathbf{0}\}$ is a G_δ set in $(\mathbb{B}, \tau_s)$. Assuming PID for I_s, it follows that $\mathbb{B} = \bigcup_{n=0}^{\infty} S_n$ with each S_n meeting only finitely many elements of each $A \in I_s$. It follows that $\mathbf{0}$ is not in the closure of $S_n - \{\mathbf{0}\}$ for any n. Let $U_n = \mathbb{B} - \mathrm{cl}(S_n - \{\mathbf{0}\})$. Each U_n is an open neighbourhood of $\mathbf{0}$ and $\bigcap_{n\in\omega} U_n = \{\mathbf{0}\}$, hence $\{\mathbf{0}\}$ is G_δ in $(\mathbb{B}, \tau_s)$. □

9.11 Theorem. *Assuming PID, every weakly distributive* ccc *complete Boolean algebra carries a strictly positive Maharam submeasure.*

Proof. Let $m, d \in \mathbb{B}$ be given by the decomposition theorem. If $m = \mathbf{1}$ the space $(\mathbb{B}, \tau_s)$ is completely metrizable. Suppose now that $d > \mathbf{0}$. By the corollary V.9.10 there is a family $\{U_n : n \in \omega\}$ of open neighbourhoods of $\mathbf{0}$ such that $\bigcap_{n\in\omega} U_n = \{\mathbf{0}\}$. We may assume that $U_{n+1} \subset U_n$, and since $\mathbb{B}$ is weakly distributive, the space is Fréchet and we may assume that each U_n is downward closed. By the decomposition theorem, d is in the closure of every nonempty open set, and since the space is Fréchet, there exists for each n a sequence $\{a^n_k\}_k$ in U_n that converges to d. By weak distributivity there exists a function $k(n)$ such that the sequence $\{b_n\}_n = \{a^n_{k(n)}\}_n$ converges to d. Since $d > \mathbf{0}$ there exists a $c > \mathbf{0}$ such that $b_n \geq c$ for eventually all n, say all $n \geq n_0$. Since each U_n is downward closed and $b_n \in U_n$, it follows that $c \in U_n$ for all $n \geq n_0$, a contradiction. □

9.12 Corollary. *Assuming PID, every weakly distributive* ccc *complete Boolean algebra adds independent reals.*

This result was obtained independently by B. Velickovic [Vel05].

Let us say that $\mathbb{B}$ has the G_δ-*property* if $\{\mathbf{0}\}$ is a G_δ-set in $(\mathbb{B}, \tau_s)$. In the proof of theorem V.9.11 we applied PID by using the G_δ-property. Thus we proved the following equivalence in ZFC.

9.13 Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra. Then $\mathbb{B}$ carries a strictly positive Maharam submeasure if and only if*

(i) $\mathbb{B}$ *is weakly distributive, and*

(ii) $\mathbb{B}$ *has the* G_δ *property.*

Note that the G_δ-property implies ccc, in fact it implies σ-finite cc. The preceding theorem was recently improved by S. Todorcevic to algebraic characterisation of Maharam algebras.

9.14 Theorem. (S. Todorcevic [Tod04]) *Let $\mathbb{B}$ be a complete Boolean algebra. Then $\mathbb{B}$ carries a strictly positive Maharam submeasure if and only if*

(i) $\mathbb{B}$ *is weakly distributive, and*

(ii) $\mathbb{B}$ *satisfies the σ-finite* cc.

Proof. Using the Decomposition theorem V.7.4 it is enough to show that $d < \mathbf{1}$. Suppose in contrary that $d = \mathbf{1}$.

Since $\mathbb{B}$ is σ-finite cc there are upward closed sets V_n such that

$$\mathbb{B}^+ = \bigcup_{n\in\omega} V_n.$$

So we have downward closed sets $-V_n = \{a \in \mathbb{B} : a \notin V_n\}$ such that $\bigcap -V_n = \{\mathbf{0}\}$.

From the definition of σ-finite cc it follows that V_n does not contain infinite antichain. But in such a case $-V_n$ are τ_s-topologically dense which is a contradiction with Theorem V.8.5.

To conclude the proof we have to show that $-V_n$ are τ_s-topologically dense. Let $U \neq \emptyset$ be τ_s-open. We show that U have to contain an infinite antichain. It is enough to show that arbitrary $a \in U$ can be split up to disjoint $a_1, a_2 \leq a$, $a_1, a_2 \in U$. Without loss on generality suppose $a = \mathbf{1}$. Since U is an open set, the set $V = \{-x : x \in U\}$ is also open. By the assumption $d = \mathbf{1}$ the intersection $U \cap V$ is nonempty. Now choose arbitrary $a_1 \in U \cap V$, then $a_2 = -a_1 \in U \cap V$. Clearly $a_1, a_2 \in U$ are disjoint which completes the proof. □

10. REGULAR IDEALS ON $\mathbb{B}^\omega$

In the previous we investigate the Os ideal on complete Boolean algebra. One is naturally interested in how the factor algebra

$$\mathbb{B}^\omega/\mathrm{Os}$$

behaves as a forcing notion. Generalisation to more general ideals instead of Os was motivated by M. Hrušák. First we characterise ideals $\mathcal{I}$ on $\mathbb{B}^\omega$ for which the mapping

$$\begin{aligned} e : \mathbb{B} &\longrightarrow \mathbb{B}^\omega \\ b &\longmapsto \langle b : n \in \omega\rangle \end{aligned}$$

determines a regular embedding

$$\hat{e} : \mathbb{B} \longrightarrow \mathbb{B}^\omega/\mathcal{I}.$$

Note that trivially $\mathbb{B}$ is a regular subalgebra of B^ω.

10.1 Definition. Let $\mathbb{B}$ be a Boolean algebra, $\mathcal{I}$ an ideal on $\mathbb{B}^\omega$ such that *Fin* $\subset \mathcal{I}$ and $e(b) \notin \mathcal{I}$ for any $b \in \mathbb{B}^+$, i.e. $\mathcal{I}$ does not contain nonzero constants. We say that $\mathcal{I}$ is *regular* if $f : \omega \to \mathbb{B} \ \in \mathcal{I}$ if and only if there is a maximal antichain of unity $\langle c_n \in \mathbb{B}^+ : n \in J\rangle$ such that $f \wedge e(c_n) \in \mathcal{I}$ for each $n \in J$.

Clearly the mapping $e : \mathbb{B} \to \mathbb{B}^\omega$ is an embedding of Boolean algebras.

10.2 Fact. *For ideal $\mathcal{I} \subset \mathbb{B}^\omega$, $\mathcal{I} \supset$ Fin, the embedding $e : \mathbb{B} \to \mathbb{B}^\omega/\mathcal{I}$ is regular if and only if $\mathcal{I}$ is the regular ideal.*

Proof. If $\mathcal{I}$ is not regular then there is some $f \notin \mathcal{I}$ such that there is a decomposition of unity $\langle c_n : n \in K\rangle$ such that $f \wedge e(c_n) \in \mathcal{I}$ for each $n \in K$. We claim that $\langle e(c_n) : n \in K\rangle$ is not a decomposition of unity in $\mathbb{B}^\omega/\mathcal{I}$, hence the embedding is not regular. Clearly the elements $e(c_n)$ are positive ($\mathcal{I}$ does not contains constants) and disjoint but they do not form a decomposition. Function $f \notin \mathcal{I}$ is positive and disjoint with each $e(c_n)$ since $f \wedge e(c_n) \in \mathcal{I}$.

On the other hand if the embedding $e : \mathbb{B} \to \mathbb{B}^\omega/\mathcal{I}$ is not regular, then there is a decomposition $\langle c_n : n \in K\rangle$ such that the $\langle e(c_n) : n \in K\rangle$ is not maximal i.e. there is a positive function $f : \omega \to \mathbb{B}$ such that $f \wedge e(c_n) \in \mathcal{I}$. □

10.3 Definition. Let $\mathbb{B}$ be a complete Boolean algebra, $\mathcal{I}$ be an ideal on $\mathbb{B}^\omega$ and let G be a generic ultrafilter on $\mathbb{B}$ over groundmodel V. We define

$$\mathcal{I}^G = \{f_G : f \in \mathcal{I}\},$$

where $f_G = \{n \in \omega : f(n) \in G\}$.

10.4 Lemma. *Let $\mathbb{B}$ be a complete Boolean algebra. Then for any generic G on $\mathbb{B}$*

$$\mathrm{Os}^G = \mathit{fin} = [\omega]^{<\omega}.$$

Proof. Let $f \in \mathrm{Os}$ and suppose on contrary that f_G is infinite set for some generic G. Since $f \in \mathrm{Os}$, there exists $g \searrow \mathbf{0}$ such that $f \leq g$. Clearly whenever $f(n) \in G$ then $g(n) \in G$. Since g is monotone and f_G is infinite we have $g(n) \in G$ for every $n \in \omega$. This is the contradiction since $\mathbf{0} = \bigwedge\{g(n) : n \in \omega\} \in G$.

On the other hand let us suppose that $f \notin \mathrm{Os}$ and set $d = \overline{\lim} f > \mathbf{0}$. We choose generic G such that $d \in G$. Clearly $\forall k \in \omega\ d \leq \bigvee\{f(n) : n > k\}$, that means that $\forall k \in \omega\ \exists m > k$ such that $f(m) \in G$; hence the set f_G is infinite. □

10.5 Fact. *The canonical embedding of Boolean algebra $\mathbb{B}$ into $\mathbb{B}^\omega/\mathrm{Os}(\mathbb{B})$ is regular whenever $\mathbb{B}$ is σ-complete.*

Proof. Let us suppose that $f \in \mathbb{B}^\omega - \mathrm{Os}$. It suffices to show that there exists some $d \in \mathbb{B}$ such that $e(d) \wedge f \notin \mathrm{Os}$. We set $d = \overline{\lim} f$, clearly $d > \mathbf{0}$ since $f \notin \mathrm{Os}$. Following the simple computation: $\overline{\lim}(e(d) \wedge f) = d \wedge \overline{\lim} f = d > \mathbf{0}$, yields that $e(d) \wedge f \notin \mathrm{Os}$. Hence $e(d)$ is the required pseudoprojection (cf. I.3.18); which completes the proof. □

10.6 Fact. *If an ideal $\mathcal{I}$ is regular, then $\mathrm{Os} \subset \mathcal{I}$.*

Proof. Suppose $f \in \mathrm{Os}$, by the definition there is a decreasing function $g : \omega \to \mathbb{B}$ such that $\bigwedge_{n\in\omega} g(n) = \mathbf{0}$ and $f \leq g$. Clearly $g - e(g(n)) \in \mathit{Fin}$ for each $n \in \omega$ and hence function g belongs to $\mathcal{I}$. Since $f \leq g$, f belongs also to $\mathcal{I}$. □

10.7 Corollary. *If a Boolean algebra $\mathbb{B}$ is σ-complete, then the ideal $\mathrm{Os}(\mathbb{B})$ is the least regular ideal containing Fin.*

When we slightly modify the convergence structure Os to its Urysohn closure $U(\mathrm{Os})$ we obtain, by a similar argument that the canonical embedding of $\mathbb{B}$ into $\mathbb{B}^\omega/U(\mathrm{Os})$ is also regular whenever the Boolean algebra $\mathbb{B}$ is ccc and complete.

10.8 Corollary. *Let $\mathbb{B}$ be a complete,* ccc *Boolean algebra. Then the canonical embedding of the Boolean algebra $\mathbb{B}$ into $\mathbb{B}^\omega/\mathrm{U}(\mathrm{Os}(\mathbb{B}))$ is regular.*

10.9 Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra. If an ideal $\mathcal{I} \subset \mathbb{B}^\omega$ is regular then*

$$\mathbb{B}^\omega/\mathcal{I} \quad \cong \quad \mathbb{B} \star (\mathcal{P}(\omega)/\mathcal{I}^G)^{V(\mathbb{B})}.$$

Proof. We define

$$\begin{aligned} \varphi : \mathbb{B} \star \mathcal{P}(\omega)/\mathcal{I}^G &\longrightarrow \mathbb{B}^\omega/\mathcal{I} \\ (b, f) &\longmapsto e(b) \wedge f, \end{aligned}$$

where f is a $\mathbb{B}$-name for a subset of ω. Let us remind the ordering i.e.

$$(b, f) \leq (c, g) \text{ if and only if } b \leq c \;\&\; b \Vdash \text{“}[f]_\mathcal{I} \leq [g]_\mathcal{I}\text{"},$$

where $b \Vdash$ “$[f]_\mathcal{I} \leq [g]_\mathcal{I}$" means that $e(b) \wedge f \leq_\mathcal{I} e(b) \wedge g$.

It is a routine check to verify that φ preserves ordering, disjoint relation and that $\varphi[\mathbb{B} \star \mathcal{P}(\omega)/\mathcal{I}^G]$ is dense in $\mathbb{B}^\omega/\mathcal{I}$. □

The following shows how the Boolean algebra $\mathbb{B}^\omega/\mathrm{Os}(\mathbb{B})$ behaves from the forcing point of view.

10.10 Corollary. (A. Kamburelis) *If $\mathbb{B}$ is a complete Boolean algebra, then the Boolean algebra $\mathbb{B}^\omega/\mathrm{Os}(\mathbb{B})$ is isomorphic with an iteration of $\mathbb{B}$ and $\mathrm{P}(\omega)/fin$, that means*

$$\mathbb{B}^\omega/\mathrm{Os}(\mathbb{B}) \;\cong\; \mathbb{B} \star (\mathcal{P}(\omega)/\mathit{fin})^{V(\mathbb{B})}.$$

10.11 THE BEHAVIOUR OF $\mathbb{B}^\omega/\mathcal{Z}(\mathbb{B})$.

In general $\mathcal{Z}(\mathbb{B})$ is not an ideal in $\mathbb{B}^\omega$. From preceding we know that $\mathcal{Z}(\mathbb{B})$ is an ideal if and only if $\mathbb{B}$ as a forcing notion does not add independent reals (cf. V.6.8). Equivalent property is:

$$\mathcal{Z}(\mathbb{B}) \text{ is the largest ideal in } \mathbb{B}^\omega \text{ such that } \mathcal{Z}(\mathbb{B}) \cap e[\mathbb{B}] \;=\; \{e(\mathbf{0})\}.$$

Under this condition $\mathcal{Z}(\mathbb{B})$ is a regular ideal.

It was mentioned that Sacks or Miller algebras are examples for which $\mathcal{Z}(\mathbb{B})$ is an ideal. Both, Miller and Sacks does not satisfies ccc.

In [BJP05] there are described examples of atomless ccc algebras for which $\mathcal{Z}(\mathbb{B})$ is an ideal.

In chapter I we discussed general situation $\mathbb{B} \subset \mathbb{C}$, $\mathbb{B}$ subalgebra of $\mathbb{C}$. We described the smallest ideal $\mathcal{I}$ such that $\mathbb{B}$ is a regular subalgebra of $\mathbb{C}/\mathcal{I}$; cf. I.3.20.

10.12 Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra such that $\mathcal{Z}(\mathbb{B})$ is an ideal. Then*

$$\mathbb{B}^\omega/\mathcal{Z}(\mathbb{B}) \;\cong\; \mathbb{B} \star (\mathcal{P}(\omega)/\mathit{fin})^V.$$

Proof. Let G be a generic on $\mathbb{B}$ over V. Then $e : \mathbb{B} \to \mathbb{B}^\omega/\mathcal{Z}(\mathbb{B})$ is a regular embedding.

$$\begin{aligned}\{f_G : f \in \mathbb{B}^\omega\} &= (\mathcal{P}(\omega))^{V[G]} \\ \{f_G : f \in \mathcal{Z}(\mathbb{B})\} &= (H)^{V[G]},\end{aligned}$$

where H is an ideal defined in II.2.21; note that $\mathbb{B}$ does not add independent reals here. From II.2.23 it follows that

$$(\mathcal{P}(\omega)/H)^{V[G]} \cong (\mathcal{P}(\omega)/fin)^V.$$

□

10.13 Corollary. *Let $\mathbb{B}$ be a complete* ccc *Boolean algebra such that $\mathcal{Z}(\mathbb{B})$ is an ideal. Then*

$$\mathbb{B}^\omega/U(Os) \cong \mathbb{B} \star (\mathcal{P}(\omega)/fin)^V.$$

Now let $\mathbb{B}$ be a complete Boolean algebra. We can ask how to describe in $V^{\mathbb{B}}$ an ideal $\mathcal{I}_{\min}$ determined (note that whenever adding new real the following embedding is not regular) by a couple

$$(\mathcal{P}(\omega)/fin)^V \subset (\mathcal{P}(\omega)/fin)^{V(\mathbb{B})}.$$

i.e. the minimal ideal $\mathcal{I}_{\min}$ such that the embedding

$$(\mathcal{P}(\omega)/fin)^V \hookrightarrow (\mathcal{P}(\omega)/fin)^{V(\mathbb{B})}/\mathcal{I}_{\min}$$

is regular.

10.14 Theorem. *Let $\mathbb{B}$ be a complete Boolean algebra and G be a generic in $\mathbb{B}$ over V. Then*

$$\mathcal{I}_{\min} = U(Os)^G.$$

Proof. Let $f \in V$ be such that $f_G = \rho \subset \omega$ destroys a MAD $\mathcal{A} \in V$. We find a name $g \in U(Os)$ for the set ρ. Suppose $f \notin U(Os)$; i.e. there is $X \subset \omega$ infinite such that $f \restriction Y \notin Os$ for each $Y \in [X]^\omega$. Let

$$\mathfrak{X} = \{ X \in [\omega]^\omega : \forall Y \in [X]^\omega \ f \restriction Y \notin Os \}.$$

For $X \in \mathfrak{X}$ there is $A \in \mathcal{A}$ such that $X \cap A$ is infinite; denote this infinite intersection $Y_X = X \cap A$. Since $X \in \mathfrak{X}$ $f \restriction Y_X \notin Os$; i.e. $\overline{\lim}_{Y_X} f \notin G$. Otherwise if

$$\bigwedge_{k\in\omega} \bigvee_{k\leq n\in Y_X} f(n) \in G,$$

then $\bigvee_{k\leq n\in Y_X} f(n) \in G$ for each $k \in \omega$ and we get that the set $f_G \cap (A \cap X)$ is infinite, which contradicts the fact that f_G destroys MAD $\mathcal{A}$. Now put

$$c = \bigvee_{X\in\mathfrak{X}} \overline{\lim}_{n\in Y_X} f(n) \notin G,$$

and $g(n) = f(n) - c$; clearly $g_G = f_G = \rho$ and $g \in U(Os)$.

Let $f \in U(Os) \setminus Os$ i.e. for each infinite X there is $Y_X \in [X]^\omega$ such that $f \restriction Y_X \in Os$. The family

$$\mathcal{F} = \{ Y_X : X \in [\omega]^\omega \}$$

is dense in $\mathcal{P}(\omega)/fin$. Now pick an arbitrary MAD $\mathcal{A} \subset \mathcal{F}$. Clearly f_G is infinite set ($f \notin Os$) and destroys the MAD $\mathcal{A}$. □

This result together with corollary II.2.19 yields the following equivalence. This equivalence was achieved independently by M. S. Kurilić and A. Pavlović in [KP07].

10.15 Corollary. *For complete Boolean algebra $\mathbb{B}$ the following are equivalent*

(i) $U(Os(\mathbb{B})) = Os(\mathbb{B})$,

(ii) there is no $V^{\mathbb{B}}$-destructible MAD in V,

(iii) algebra $\mathbb{B}$ as a forcing notion does not add new reals.

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