Referee report on the dissertation

Hierarchical Problems with Evolutionary Equilibrium Constraints

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The dissertation at hand is a collection of, in large parts, independent results which are tied through the general theme, which is the *mathematical program with equilibrium constraints (MPEC)* in the most general form

$$\begin{aligned} \min_{(u,x)} \quad & J(u,x) \\ \text{s.t.} \quad & x \in S(u), \\ & u \in U_{ad}. \end{aligned} \tag{1}$$

An emphasis is put on the analysis of the (in general set-valued) *solution mapping* S, which ties the state variable x to the control variable u. Here the state variable is usually time-dependent, whereas the control variable can be finite-dimensional (Chapter 4) or also time-dependent (Chapter 5 and 6). At many places, the solution mapping is assumed to be (locally) single-valued.

The thesis is divided in 7 chapters: Chapter 1 introduces the problem (1) and discusses various instances of it, e.g., parameter identification and bilevel optimization problems, and gives references to the literature for these kinds of problems. Moreover it contains an outline of things to come.

Section 2 provides the necessary tools from variational analysis, tangent and normal cones, subdifferentials, coderivatives and Lipschitz properties for set-valued mappings etc., which are used throughout. Moreover, the *implicit programming approach* for (1), which applies in the case that the solution mapping S is single-valued, is outlined.

The main part of the thesis are the Chapters 3 to 6, each of which can be read independently.

Chapter 3 is devoted to the computation of normal cones to the union of polyhedral sets, which arises naturally in, e.g., generalized equation solving or disjunctive programming. The key concept for the analysis is the *normally* *admissible stratification* of a union of finitely many polyhedra (see Definition 3.2.2), which was coined in strata theory for general manifolds. The existence of a normally admissible stratification of an arbitrary union of finitely many polyhedra is proven in Lemma 3.2.3. Based on this, Theorem 3.2.7 provides formulas for the graph of the regular and limiting normal cone to a union of finitely many polyhedra, which as a consequence gives the desired normal cone formulas at specific points, see Corollary 3.2.8. The author also compares his results to existing ones. More precisely, he discusses the results by Dontchev and Rockafellar (SIOPT 6(4), 1996) as well as by Henrion and Outrata (Optimization 57(1), 2008) and shows that his results are more general than the existing ones. To conclude Section 3 and to display their impact, the results are applied to the solution set of a discretized (time-dependent) differential inclusion which, in a sense, also provides the link to the general topic of the thesis.

Chapter 4 is concerned with sensitivity analysis for solution functions of differential inclusions with smooth left-hand and closed-valued right-hand side, involving a finite-dimensional (time-independent) parameter (control variable) and a time-dependent state variable. The main result, Theorem 4.2.2, establishes a connection between Lipschitz properties of the solution mapping for the discretized problem and Lipschitz continuity of the original solution mapping.

The obtained results are then applied to ordinary differential equations (mainly for illustration's sake) and, later on, to a sweeping process, culminating in Theorem 4.3.4, which provides a Lipschitz continuity result for the solution mapping of the sweeping process under certain assumptions.

Chapter 5 deals with an optimal control problem governed by a an ordinary differential inclusion and equation. After establishing a set of standing assumptions, a general discretization scheme is proposed and some technical estimates for the solutions to the discretized problems are proven. This leads to the (theoretical) main result, Theorem 5.2.5, which shows that, under the standing assumptions, for any sequence of feasible discretized control variables, there exists a weakly convergent subsequence in L^2 such that there exist Lipschitz continuous discretized state variables converging uniformly such that the above limits give a feasible point of the original continuous problem. Moreover, some (semi-)continuity properties of the optimal value are established.

This result is then formulated in terms of optimal solutions (instead of merely feasible points) in Theorem 5.2.8.

The following Section 5.3 is devoted to the numerical solution of the discretized problems. To this end, the discretized problem is first reformulated only as a problem in the (discretized) state variable, using the implicit programming approach outlined in Section 2.2. As shown by Lemma 5.3.1, this is viable, since under the standing assumptions, the solution mapping is locally Lipschitz, in particular single valued. The rest of the section deals with variational properties of the solution mapping and subdifferential analysis of the nonsmooth part of the reformulated problem.

The Chapter is rounded off by some numerical experiments.

Chapter 6 is devoted to an optimal control problem governed by ordinary differential equations in Banach spaces, involving an abstract parameter.

First, a discretization of the original problem with respect to time and the abstract spaces in which the variables live is proposed. It is established that the solution mapping to the discretized problem, as a function of the abstract parameter, is single-valued, hence an implicit programming approach is applied and differentiability properties of the function that maps each abstract parameter to its objective value are studied. The main result in this regard, Theorem 6.3.1, shows that, in particular, under a set of standing assumptions, the latter mapping is continuously differentiable. Moreover, it contains some necessary optimality conditions for the discretized problem.

Afterwards, these results are applied to adhesive contact problems and some numerical experiments conclude the chapter.

Chapter 7 contains a synopsis of the analysis carried out in Chapter 3-6 as well as possible starting points for future research based on the exhibited material.

The research presented in this dissertation is, to my knowledge, original and, apart from the publications by the author and his collaborators on which the thesis is based, new. The dissertation is very well sourced, and related existing results are referenced whenever they occur. The material, which lies at the intersection of variational analysis and optimization (optimal control), is current, relevant, well motivated and also mathematically quite deep. The author makes masterful use of involved concepts from variational analysis, such as normal cone operators, generalized derivatives and Lipschitzian notions for set-valued mappings. This is already reflected in the very efficient presentation of the variatonal tools in Section 2, which he brings to bear with great ease; sometimes, for the uninitiated, even a little too casual, see below. On the other hand, this is, to a certain extent, compensated for by numerous insightful examples.

Moreover, I checked most of the results and their proofs fairly thoroughly and, in addition to that, most parts of the paper are already accepted for publication in peer reviewed journals, which is another certificate for the correctness of the mathematics contained in the dissertation.

In general, the use of the English language is good (despite of many missing definite and indefinite articles), but at places I had a hard time of understanding some of the more convoluted statements, which, on the other hand could also be a symptom of the author's more condensed, brief way of writing mathematically. For example, in the central Lemma 3.2.3, at first glance, it is a little hard to tell apart the assumptions from the statement, although, admittedly, it is understandable at second glance. Another example is the statement of Theorem 4.2.2.

As already foreshadowed above, I think that certain parts of the thesis could be made more accessible for uninitiated readers. In my opinion, a dissertation can (maybe should) be more elaborate and self-contained than a research paper. I would have liked the Chapters 3-6 to be *proper* supersets of the papers that they are based on. To give an example, in Section 3.2, only the statements for the regular normal cones are actually proven. For the unexperienced reader, however, it is not clear how those for the limiting normal cones follow from that. Another example is the (generalized MFCQ-type) constraint qualification used in Lemma 4.2.1 and how it implies the inequality (4.11). The author invokes the well-known result from Rockafellar and Wets, but no words are spend on how his constraint qualification is equivalent (or implies) the one used in the latter reference.

I attribute this nonchalance to the author's expertise in the field of variational analysis and a certain fear of being too obvious.

All in all, there is no doubt, however, that this thesis is a valuable piece of research and highly deserves a 'pass'.

(Dr. Tim Hoheisel)