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BACHELOR THESIS

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Construction of Dendroids and Their Properties

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In Prague on July 31, 2015 ....

Název práce: Konstrukce dendroidů a jejich vlastnosti

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Abstrakt: Tato práce se zabývá dendroidy jejich shore množinami. Je zkonstruován další příklad dendroidu, ve kterém sjednocení dvou protínajících se shore kontinuí není shore kontinuum. Dále se podařilo zjednodušit důkaz, že sjednocení konečně mnoha po dvou disjunktních shore kontinuí je opět shore kontinuum. Hlavním výsledekem je ale kladná odpověď na otázku, zda je sjednocení konečně mnoha uzavřených shore množin opět uzavřená shore množina pro případ dendroidů s jen konečně mnoha větvícími body.

Klíčová slova: teorie kontinuí, dendroid, fan, shore množina

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Abstract: This thesis is about dendroids and their properties. Another example of a dendroid with two intersecting shore continua whose union is not a shore continuum is constructed. Moreover, a simplification of a proof that the union of finitely many pairwise disjoint shore continua is again a shore continuum has been made. But the main result is an affirmative answer to the question whether the union of finitely many closed shore sets is again a closed shore set in the case dendroids with only finitely many branch points.

Keywords: continuum theory, dendroid, fan, shore set

Contents

Preface

Continua are compact, connected, and second countable topological spaces. They can be thought of as metric spaces and have many interesting properties. The topic of this thesis are particular kinds of continua called dendroids, which are arcwise connected and hereditarily unicoherent continua.

Continuum is unicoherent if it cannot be written as a union of two subcontinua whose intersection is not connected. Circle is a typical nonexample of such continuum, but even continua such as the topological sine curve with its end connected to the beginning (see [8, 1.6]) which resemble a circle, but do not actually contain a circle, can fail to have this property. If all subcontinua are unicoherent, the continuum is called hereditarily unicoherent; this property can be thought of as a generalization of not containing a circle.

A subset S of a continuum can have the property that the whole continuum is approached by subcontinua lying the complement of S. Such subsets are called shore sets, and are part of an active area of reseach. It is known (see [5]) that a union of disjoint shore continua in a dendroid is a shore set, and that disjointness is necessary. Here, the key theorem of the proof is generalized and its proof is simplifed. Additionally, another example of a dendroid with intersecting shore continua which fail to form a shore set is provided.

It has been recently shown (see [0]) that in dendroids, the union of disjoint closed shore sets can fail to be a shore set. There is a question whether it can hold for some particular classes of dendroids, see [0, Table 1, p. 213] for a nice summary. In this thesis, an affirmative answer is given in the case of dendroids with finitely many branch points (meeting points of three otherwise disjoint arcs), even for intersecting shore sets. In particular, this answers the case of fans, which are dendroids with just one branch point.

0 Preliminaries

The mathematical framework is the ZFC set theory. The natural numbers include zero.

In general, established terms are given in **bold**, and made-up terms are given in ***bold italic***.

Definition. A nonempty topological space is **connected** if it cannot be decomposed into two nonempty separated subsets.

Note that this means the empty space is neither connected, nor disconnected.[0]

A homeomorphism $h:[a,b]\to[a',b']$, $a<b$ and $a'<b'$, is called **orientation preserving** if $h(a)=a'$ and $h(b)=b'$.

Definition. Let X be a topological space. Continuous functions $p:[0,1]\to X$ up to a homeomorphism of the interval $[0,1]$ are called **unoriented paths**. If the homeomorphism is orientation preserving, they are called **oriented paths**. An **arc** is a path which is also an embedding.

Paths and arcs are assumed to be oriented, unless otherwise stated. A **subpath** of a path p is $p\restriction_{[a,b]}$ composed with an orientation preserving homeomorphism of $[0,1]$ and $[a,b]$ where $0\le a<b\le 1$. It is called a **left subpath** if $a=0$, and a **right subpath** if $b=1$. This also gives the definition of a subarc.

Note that an unoriented path represented by p has exactly two orientations, one is given by p itself and the other, called the **opposite** of p, is given by $p(1-t)$.

This also makes it possible to say that a path p **starts** in $p(0)$ and **ends** in $p(1)$. These are called the **endpoints** and the path **connects** them. Points of the form $p(t)$, where $0<t<1$, are called the **inner points** of p. Arcs use the same terminology.

Observation. Unoriented arcs are completely determined by their image.

So, in a topological space, subspaces homeomorphic to $[0,1]$ can be identified with arcs, provided the start and the end are specified. This identification is implicitly assumed throughout the text.

Definition. A nonempty topological space is **pathwise** or **arcwise connected** if any two distinct points can by connected by a path or an arc, respectively.

Fact.[8, 8.18 + 8.23] Pathwise connected Hausdorff spaces are arcwise connected.

[0]Some authors consider it to be connected.

Let X be a metric space, $x \in X$, $A, B \subseteq X$. Define the distance between x and A as $d(x, A) = \inf\{d(x, a) \mid a \in A\}$, and also the distance between A and B as $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$; the convention is $\inf \varnothing = \infty$ and $\sup \varnothing = 0$.

For a real $\varepsilon > 0$, $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ and $\overline{B}_\varepsilon(x) = \{y \in X \mid d(x, y) \leq \varepsilon\}$ are called the **open** and **closed balls** of radius ε (or **ε-balls**) with center x, respectively. Also, $B_\varepsilon(A) = \{x \in X \mid d(x, A) < \varepsilon\}$ and $\overline{B}_\varepsilon(A) = \{x \in X \mid d(x, A) \leq \varepsilon\}$ are called the **open** or **closed ε-neighborhoods** of A, respectively. A is **ε-dense** if $X \subseteq B_\varepsilon(A)$.

Open balls are open sets and closed balls are closed sets, but note the closure of an open ball is not necessarily the corresponding closed ball. The same applies to ε-neighborhoods.

Definition. A **continuum** is a connected second countable compact topological space. A continuum is **trivial** or **degenerate** if it is a one point set.

Second countability can be replaced by metrizability (see [4, 4.28]). In this text, it is implicitly assumed that continua are metric spaces with some unspecified metric d.

1 Dendroids and shore sets

1.0 Introduction

Definition.[7, 1.7.19] A continuum is **decomposable** if it can be covered by two of its proper subcontinua. It is **hereditarily decomposable** if all nondegenerate subcontinua are decomposable.

Not all nondegenerate continua are decomposable, see [8, 1.10] and [8, 1.23] for some examples.

Definition.[7, 1.7.30] A continuum is **unicoherent** if whenever it is covered by two of its subcontinua A and B, it holds that $A \cap B$ is a continuum. It is **hereditarily unicoherent** if all subcontinua are unicoherent.

Note that $A \cap B$ is always nonempty, otherwise the nonempty closed sets A, B separate the space, which cannot happen.

Not every unicoherent space is hereditarily unicoherent, see [6, Figure 14, p. 51] for a simple example.

Definition.[7, 6.9.1] A **dendroid** is an arcwise connected hereditarily unicoherent continuum.

Fact.[8, 11.58] Every subcontinuum of a dendroid is arcwise connected.

Corollary. Subcontinuum of a dendroid is a dendroid.

Fact.[8, 11.54] Dendroids are hereditarily decomposable.

Observation. A topological space is hereditarily unicohent, if and only if the intersection of every two of its intersecting subcontinua is connected.

Proof. Let X be the space and let A, B be two of its intersecting subcontinua. Then $A \cup B$ is a continuum covered by A and B, so if X is hereditarily unicoherent, $A \cap B$ is a continuum and must be connected.

For the other direction, let C be a nondegenerate subcontinuum of X, and suppose C is covered by two of its subcontinua A and B. It has been already noted that $A \cap B$ must be nonempty, hence it is connected by the assumption. Clearly, $A \cap B$ is compact, so $A \cap B$ is indeed a continuum. ■

Observation. Let a,b be distinct points in a dendroid. Then there is a unique arc from a to b.

Proof. By arcwise connectedness, there is at least one such arc f, so let g be another arc from a to b. $g \cap f$ contains a and b, so it must be connected by the hereditary unicoherence. But the only connected subset of g containing its endpoints is the whole g and similarly for f. Therefore, $g = g \cap f = f$. ■

The arc from the above observation is denoted ab. In addition to this, let aa be the set $\{a\}$.

Corollary. Let C and D be disjoint subcontinua of a dendroid. Then there is a unique arc starting in C and ending in D such that none of its inner points are in C or D.

Proof. Let $a \in C$ and $b \in D$ be arbitrary. From the compactness of C, there is the last point c on ab such that $c \in C$. Similarly, there is the first point $d \in D$ on ab such that $d \in D$. The continua C and D are arcwise connected and disjoint, so $ac \subseteq C$, $db \subseteq D$, and c lies before d on the arc ab. Hence, cd is an arc starting in C, ending in D, and with no inner points in C or D.

Suppose there is another arc $c'd'$ with such properties. Clearly, $c'c \cap cd = \{c\}$ and $cd \cap dd' = \{d\}$ since $c'c \subseteq C$ and $dd' \subseteq D$. This means $c'd'$ contains cd because C and D are disjoint. So, cc' and dd' are degenerate since $c'd'$ does not contain points from C or D as its inner points. Therefore, $c'd' = cd$. ■

The arc from the above observation is denoted CD. If $C = \{p\}$ or $D = \{q\}$, the arc can be written as pD or Cq, respectively.

1.1 Unions of Shore Continua

Definition.[5] A subset S of a dendroid X is a **shore set** if for all $\varepsilon > 0$, there is a continuum $C \subseteq X \setminus S$ which is ε-dense in X.

The following lemma is a slight modification of [5, Lemma 1] which is easily seen to be equivalent.

Lemma 0. Let X be a dendroid and let ab be an arc with points $x_0, \ldots, x_{n-1}$ in this order. Then for all $\varepsilon > 0$, there is $\delta > 0$ such that all arcs pq, where $p \in B_\delta(a)$ and $q \in B_\delta(b)$, have points $y_0, \ldots, y_{n-1}$ in this order so that $y_k \in B_\varepsilon(x_k)$ for all k.

Definition.[9, p. 178] Let X be a dendroid, $p \in X$, and $A \subseteq X$. Define $Q_p(A)$ to be the set $\{x \in X \mid px \cap A \neq \varnothing\}$. If $A = \{a\}$, it can be written as $Q_p(a)$.

This means $Q_p(A)$ are the points unreachable from p when passing through A is forbidden.

Theorem 1. Let X be a dendroid, $p \in X$, and $A, B \subseteq X$ closed. It holds that $\big(\overline{Q_p(A)} \cap \overline{Q_p(B)}\big)^\circ = \big(\overline{Q_p(A) \cap Q_p(B)}\big)^\circ$.

Proof. The inclusion $\big(\overline{Q_p(A) \cap Q_p(B)}\big)^\circ \subseteq \big(\overline{Q_p(A)} \cap \overline{Q_p(B)}\big)^\circ$ is trivial, so only the other inclusion is discussed. Let $U \subseteq \big(\overline{Q_p(A)} \cap \overline{Q_p(B)}\big)^\circ$ be open.

Construct a sequence (q_n) of points of U, a sequence (c_n) of points of $A \cup B$, and a sequence (ε_n) of positive reals such that:

(0) $q_0 \in U \cap Q_p(A)$ and $c_0 \in pq_0 \cap A$,

(1) $B_{\varepsilon_0}(q_0) \subseteq U$, $\overline{B}_{\varepsilon_{n+1}}(q_{n+1}) \subseteq B_{\varepsilon_n}(q_n)$, and $\varepsilon_n \to 0$,

(2) for all $x \in B_{\varepsilon_n}(q_n)$, $B_{\varepsilon_n}(c_n) \cap px \neq \varnothing$,

(3) $q_{n+1} \in B_{\varepsilon_n}(q_n) \cap Q_p(B)$ and $c_{n+1} \in pq_{n+1} \cap B$ when n is even,
$q_{n+1} \in B_{\varepsilon_n}(q_n) \cap Q_p(A)$ and $c_{n+1} \in pq_{n+1} \cap A$ when n is odd.

(1) can be satisfied by choosing ε_n sufficiently small in each step, and it implies that $\overline{B}_{\varepsilon_m}(q_m) \subseteq B_{\varepsilon_n}(q_n)$ for all $m > n$. Particularly, $\overline{B}_{\varepsilon_{n+1}}(q_{n+1}) \subseteq B_{\varepsilon_0}(q_0) \subseteq U$ for all n. Additionally, $\varepsilon_n \to 0$ ensures that the sequence q_n is Cauchy, so it has a limit $q \in \bigcap_{n \in \mathbb{N}} B_{\varepsilon_n}(q_n) \subseteq U$.

(0) and (3) can be satisfied because $Q_p(A)$ and $Q_p(B)$ are dense in U. Lemma 0 ensures that (2) holds if ε_n is sufficiently small (and it holds trivially if $q_n = p$).

For all $n \in \mathbb{N}$, $q \in B_{\varepsilon_n}(q_n)$. So, (2) implies $B_{\varepsilon_n}(c_n) \cap pq \neq \varnothing$, hence $d(pq, c_n) < \varepsilon_n$. Even n give $d(pq, A) = 0$ and odd n give $d(pq, B) = 0$. Since A, B, and pq are compact, it must be the case that $pq \cap A \neq \varnothing$ and $pq \cap B \neq \varnothing$. This means that $q \in Q_p(A) \cap Q_p(B)$.

Therefore, $U \cap (Q_p(A) \cap Q_p(B)) \neq \varnothing$ because it contains the point q. From the arbitrary choice of U, it follows that $\overline{Q_p(A) \cap Q_p(B)} \supseteq \big(\overline{Q_p(A)} \cap \overline{Q_p(B)}\big)^\circ$. Hence, $\big(\overline{Q_p(A) \cap Q_p(B)}\big)^\circ \supseteq \big(\overline{Q_p(A)} \cap \overline{Q_p(B)}\big)^\circ$. ∎

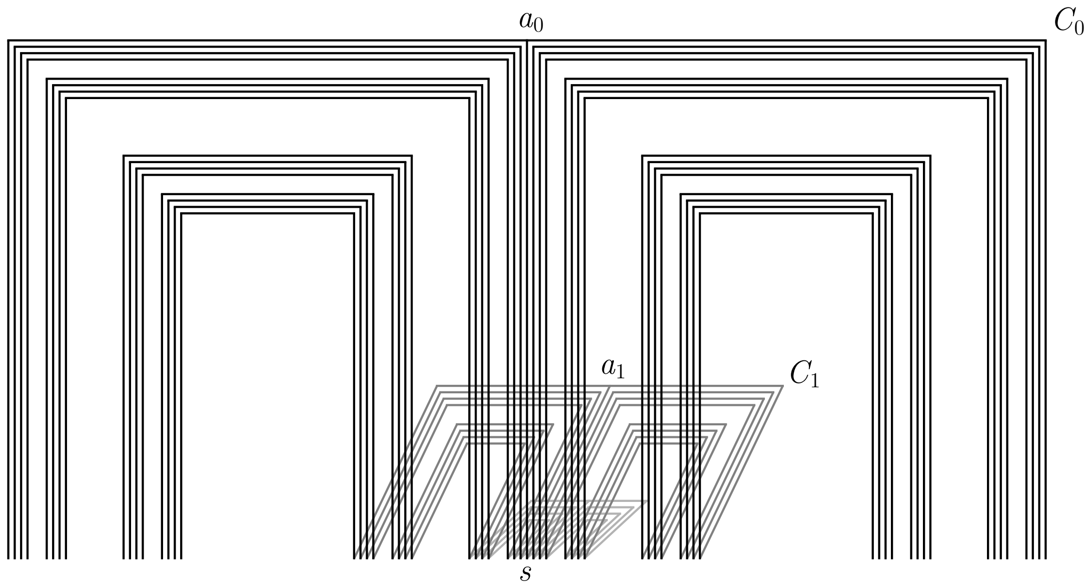
This theorem is more general and with a simpler proof than [5, Theorem 2] because $\big(\overline{Q_p(a_{-1})} \cap \overline{Q_p(a_1)}\big)^\circ = \big(\overline{Q_p(a_{-1}) \cap Q_p(a_1)}\big)^\circ = \big(\overline{\varnothing}\big)^\circ = \varnothing$ for any choice of $p \in a_{-1}a_1 \setminus \{a_{-1}, a_1\}$, and $Q_p(a_{-1})$, $Q_p(a_1)$ correspond to $L(a_{-1}, a_1)$, $R(a_{-1}, a_1)$ in the paper, respectively. A consequence is that for dendroids, a finite union of pairwise disjoint shore subcontinua is a shore set, as shown in [5, Theorem 3].

Observation. Let X be a dendroid with a subset A. If $X \setminus A$ has an arc component which is dense in X, then A is a shore set.

Proof. Let D be the dense component and let $\varepsilon > 0$. By compactness, there are $\frac{\varepsilon}{2}$-balls $B_0, \ldots, B_{n-1}$, $n \geq 1$ which cover X. For all i, there is a point $p_i \in D \cap B_i$, then $\bigcup_{i=0}^{n-1} p_0 p_i \subseteq D \subseteq X \setminus A$ is the required ε-dense continuum. ∎

Compare the following example with [5, Example 5], which also contains shore continua with the specified property.

Example 2. A dendroid X with a shore point s such that no arc component of $X \setminus \{s\}$ is dense in X. Moreover, there are two shore continua whose union is not a shore set.



The points at the bottom form two Cantor sets joined by the point s in the middle. $X = \bigcup_{n\in\mathbb{N}} C_n$.

For any $n \in \mathbb{N}$, each half of C_n (including the arc sa_n) is homeomorphic to the product $C \times I$ of the Cantor set and the unit interval, which is a compact. So, if X is embedded in $\mathbb{R}^3$ like in the picture, it is a union of closed sets tending to $\{s\}$ in the Hausdorff metric. Hence, X is a closed subset of $\mathbb{R}^3$, and it is compact because it is also bounded.

It is clear that any point $p_n \in C_n \setminus C_{n+1}$ is connected to C_{n+1} by an arc. Let p_{n+1} be its end and continue in the same manner. Concatenating all such arcs yields an arc ending in $\{s\}$ since C_n tend to $\{s\}$ in the Hausdorff metric. That means X is arcwise connected, so it is a continuum. It is also easy to see that C_n contracts to $C_n \cap C_{n+1}$, and concatenating them all yields a contraction of X to $\{s\}$.

For metrizable compacts, the inductive and the covering dimension coincide (see [4, 7.3.3]). Note that $\dim C \times I \leq^{[4,\ 7.4.10]} \dim C + \dim I = 0 + 1 = 1$. So the closed halves mentioned above have dimension 1, and since X is their countable union, it is the case that $\dim X \leq 1$ by [4, 7.2.1]. It is clear that $\operatorname{ind} X \geq 1$, therefore $\dim X = 1$.

This means X is a 1-dimensional contractible continuum, so it is a dendroid by [2, Proposition 1].

Let A_n be the arc component of a_n in $X \setminus \{s\}$. From the picture, it is easy to see that $(A_n \setminus sa_n) \cup \{a_n\})$ is a continuum in $X \setminus \{s\}$, which becomes increasingly dense as n increases. Hence, s is a shore point. But none of the components A_n is dense in X, and $\bigcup_{n\in\mathbb{N}} sa_n$ divides X into two open halves, therefore no arc component of $X \setminus \{s\}$ is dense in X. It is also clear that $\bigcup_{n\in\mathbb{N}} sa_{2n}$ and $\bigcup_{n\in\mathbb{N}} sa_{2n+1}$ are shore continua, but $\bigcup_{n\in\mathbb{N}} sa_n$ cannot be.

2 Main Result

2.0 Introduction

Let X be a topological space, and let $p \in X$. The **arc order** of p is the supremum of the cardinalities of all collections $\mathscr{A}$ of arcs starting in p such that $f \cap g = \{p\}$ for all $f, g \in \mathscr{A}$, $f \neq g$. Points of order 1 are called **endpoints**, points of order at least 3 are called **branch points**. See [1, p. 230].

Definition.[7, 6.9.1] A **fan** is a dendroid with exactly one branch point.

Let $\mathscr{U}$ be an open cover of a topological space X. The **nerve** of $\mathscr{U}$, denoted by $N(\mathscr{U})$, is the abstract simplicial complex on $\mathscr{U}$, whose faces are all $\{U_0, \ldots, U_{n-1}\} \subseteq \mathscr{U}$ satisfying $\bigcap_{i=0}^{n-1} U_i \neq \varnothing$. If X is a metric space, define **mesh** of $\mathscr{U}$, denoted by $\operatorname{mesh} \mathscr{U}$, to be $\sup_{U \in \mathscr{U}} \operatorname{diam} U$.

Every simple undirected graph can be viewed as an abstract simplicial complex. If such graph is a tree, the corresponding abstract simplicial complex is also called a tree. If $N(\mathscr{U})$ is a tree, $\mathscr{U}$ is said to be a **tree cover**.

Definition.[6, 55.1]+[7, 2.5.13] A continuum X is **treelike** if for all $\varepsilon > 0$, there is a tree cover $\mathscr{U}$ of X satisfying $\operatorname{mesh} \mathscr{U} < \varepsilon$.

Fact.[3] Dendroids are treelike.

2.1 Key Proposition

A **walk** in an abstract simplicial complex is a sequence of vertices $v_0, \ldots, v_{n-1}$, where $n \geq 1$, such that $\{v_i, v_{i+1}\}$ is a face for all $0 \leq i < n-1$. The walk **starts** in v_0, **ends** in v_{n-1}, and the walk $v_i, \ldots, v_{j-1}$, where $0 \leq i < j \leq n$, is called a **subwalk**. A **path** is a walk with all vertices distinct.

Lemma 3. Let X be a topological space with an open cover $\mathscr{U}$ satisfying $\dim N(\mathscr{U}) \leq 1$, and let p be an path starting in $U \in \mathscr{U}$, ending in $V \in \mathscr{U}$. Then there is a walk $U = U_0, U_1, \ldots, U_{n-1} = V$ in $N(\mathscr{U})$ such that for any subwalk $U_i, \ldots, U_{j-1}$, there is a subpath p' of p starting in U_i, ending in U_{j-1}, and contained in $\bigcup_{k=i}^{j-1} U_k$, which is a left subpath if $i = 0$, and a right subpath if $j = n$. Moreover, p' actually visits each of $U_i, \ldots, U_{j-1}$.

Proof. Set $t_0 = 0$ and $U_0 = U$, then $p(t_0) \in U_0$. Whenever $p([t_i, 1]) \nsubseteq U_i$, it is possible to choose the smallest $t_{i+1} \in (t_i, 1]$ such that $p(t_{i+1}) \notin U_i$, because $X \setminus U_i$ is closed. In that case, choose U_{i+1} so that $p(t_{i+1}) \in U_{i+1}$.

This inductively defines two sequences $t_0, t_1, \ldots$ and $U_0, U_1, \ldots$. From the continuity, $t_{i+1} \in \overline{[t_i, t_{i+1})}$ implies $p(t_{i+1}) \in \overline{p([t_i, t_{i+1}))} \subseteq \overline{U}_i$. Moreover, $p(t_{i+1}) \in U_{i+1} \setminus U_i$, hence $U_i \neq U_{i+1}$ and $U_i \cap U_{i+1} \neq \varnothing$.

Suppose the sequences are infinite. Set $t = \sup_{i \in \mathbb{N}} t_i$, then there is $W \in \mathscr{U}$ such that $p(t) \in W$. Since p is continuous, there is $s \in [0, t)$ such that $p([s, t]) \subseteq W$. Choose $i \in \mathbb{N}$ so large that $t_i \geq s$, then $U_i \neq W$ because $t_{i+1} \notin U_i$, and similarly, $U_{i+1} \neq W$. But $U_{i+1} \cap W$ is a neighborhood of $p(t_{i+1}) \in \overline{U}_i$, so $U_i \cap U_{i+1} \cap W \neq \varnothing$ and $\dim N(\mathscr{U}) \geq 2$, which is a contradiction.

So, the sequences are finite. Let $n' \in \mathbb{N}$ be their length, then $U_0, \ldots, U_{n'-1}$ form a walk in $N(\mathscr{U})$. Also, $p([t_i, t_{i+1})) \subseteq U_i$ for all $0 \leq i < n' - 1$ and $p([t_{n'-1}, 1]) \subseteq U_{n'-1}$.

Let $0 \leq i' < j' \leq n'$. First, if $i' = 0$ and $j' = n'$, the path p itself starts in U_0, ends in $U_{n'-1}$ and is contained in $\bigcup_{k=0}^{n'-1} U_k$. Else, if $j' - i' = 1$, then $p \restriction_{[t_{i'}, t_{i'}+\varepsilon]}$ (or $p \restriction_{[1-\varepsilon, 1]}$ if $j' = n'$) gives a path contained in $U_{i'}$ for some sufficiently small $\varepsilon > 0$. And if $j' - i' > 1$, then $p \restriction_{[t_{i'}, t_{j'-1}]}$ (or $p \restriction_{[t_{i'}, 1]}$ if $j' = n'$) gives a path starting in $U_{i'}$, ending in $U_{j'-1}$, and contained in $\bigcup_{k=i}^{j'-1} U_k$. All the given paths are left subpaths of p when $i' = 0$, right subpaths when $j' = n'$, and visit each of $U_{i'}, \ldots, U_{j'-1}$.

If $U_{n'-1} = V$, set $n = n'$ and the proof is done. Otherwise, set $n = n' + 1$ and $U_{n'} = V$; this extends the walk since $p(1) \in U_{n'-1} \cap V$. Additional subwalks of the form $U_i, \ldots, U_{n'}$, $0 \leq i \leq n'$, need to be checked. If $i < n'$, then the path previously constructed for $i' = i$ and $j' = n'$ has all the required properties. If $i = n'$ ($\neq 0$), then $p \restriction_{[1-\varepsilon, 1]}$ gives a left subpath contained in $U_{n'}$ for some sufficiently small $\varepsilon > 0$. ■

Let X be a metric space, and $\varepsilon > 0$. A sequence of points $c_0, \ldots, c_{n-1}$, where $n \geq 1$, is called an ***ε*-chain** if $d(c_i, c_{i+1}) < \varepsilon$ for all $0 \leq i < n - 1$. A set $A \subseteq X$ is called ***ε*-chained** if for all $a, b \in A$, there is an ε-chain $a = c_0, c_1, \ldots, c_{m-1} = b$. Clearly, if $c_0, \ldots, c_{n-1}$ is a chain, then $\{c_0, \ldots, c_{n-1}\}$ is ε-chained.

Fact.[8, 4.13 + 4.16] Let X be a compact metric space, and let (C_n) be a sequence of compacts such that C_n is ε_n-chained. If $\varepsilon_n \to 0$, then (C_n) has a subsequence converging to a subcontinuum of X (in the Hausdorff metric).

Proposition 4. Let X be a dendroid, and let ab be an arc such that none of its inner points are branch points. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that all arcs pq, where $p \in B_\delta(a)$ and $q \in B_\delta(b)$, have a subarc $p'q'$ so that $p' \in B_\varepsilon(a)$, $q' \in B_\varepsilon(b)$, and $p'q' \subseteq B_\varepsilon(ab)$.

Proof. Suppose it is not the case. Then there is $\varepsilon > 0$ such that for all $\delta > 0$, there is an arc pq, $p \in B_\delta(a)$, $q \in B_\delta(b)$, whose every subarc $p'q'$, $p' \in B_\varepsilon(a)$, $q' \in B_\varepsilon(b)$, fails to satisfy $p'q' \subseteq B_\varepsilon(ab)$. Without loss of generality, suppose that $\varepsilon < \frac{1}{2} d(a, b)$.

Suppose $\mathscr{U}$ is a tree cover of X with $\operatorname{mesh}\mathscr{U} < \frac{\varepsilon}{2}$. Choose $U, V \in \mathscr{U}$ such that $a \in U$ and $b \in V$, then Lemma 3 gives a walk $U = U_0, U_1, \dots, U_{n-1} = V$. Pick the largest i such that $U_i \subseteq B_\varepsilon(a)$, and then the least $j > i$ such that $U_{j-1} \subseteq B_\varepsilon(b)$. Now, $U_i, \dots, U_{j-1}$ is a subwalk, so the lemma says there is a subarc cd of ab such that $c \in U_i$, $d \in U_{j-1}$, $cd \subseteq \bigcup_{k=i}^{j-1} U_k$, and $cd \cap U_k \neq \varnothing$ for each $i \leq k < j$.

$j - i > 2$ because $B_\varepsilon(a) \cap B_\varepsilon(b) = \varnothing$. Moreover, $U_i, U_{j-1} \notin \{U_{i+1}, \dots, U_{j-2}\}$ since U_k is not contained in $B_\varepsilon(a)$ or $B_\varepsilon(b)$ for all $i < k < j-1$. So, let $\mathscr{C}$ be the component of $\{U_{i+1}, \dots, U_{j-2}\}$ in $N(\mathscr{U}) \setminus \{U_i, U_{j-1}\}$ and $\mathscr{C}'$ the component in $N(\mathscr{U}) \setminus \{W \in \mathscr{U} \mid W \subseteq B_\varepsilon(a) \vee W \subseteq B_\varepsilon(b)\}$. Clearly, $\mathscr{C}' \subseteq \mathscr{C}$.

By the construction of $\mathscr{C}'$, all $W \in \mathscr{U} \setminus \mathscr{C}'$ neighboring $\mathscr{C}'$ in $N(\mathscr{U})$ (i.e. intersecting a member of $\mathscr{C}'$) are contained either in $B_\varepsilon(a)$, or in $B_\varepsilon(b)$. Let $\mathscr{E}$ be the set of such W for $B_\varepsilon(a)$, and $\mathscr{F}$ for $B_\varepsilon(b)$. Clearly, $U_i \in \mathscr{E}$ and $U_{j-1} \in \mathscr{F}$. Applying the initial assumption for $\delta > 0$ so small that $B_\delta(a) \subseteq U$ and $B_\delta(b) \subseteq V$ yields an arc pq, $p \in U$, $q \in V$, whose every subarc $p'q'$, $p' \in B_\varepsilon(a)$, $q' \in B_\varepsilon(b)$, fails to satisfy $p'q' \subseteq B_\varepsilon(ab)$.

Since $N(\mathscr{U})$ is a tree, $\mathscr{C}$ divides $\mathscr{U}$ into 2 components, one contains U and connects to $\mathscr{C}$ through U_i and the other contains V and connects through U_{j-1}. So, any walk from U to V must go through $U_i \in \mathscr{E}$, and then through $U_{j-1} \in \mathscr{F}$. For pq, Lemma 3 gives a walk $U = U'_0, U'_1, \dots, U'_{n'-1} = V$. Pick the largest i' such that $U'_{i'} \in \mathscr{E}$, and then the least $j' > i'$ such that $U_{j'-1} \in \mathscr{F}$. Now, $U'_{i'}, \dots, U'_{j'-1}$ is a subwalk, so the lemma says there is a subarc $p'q'$ of pq such that $p' \in U'_{i'}$, $q' \in U'_{j'-1}$, and $p'q' \subseteq \bigcup_{k=i'}^{j'-1} U'_k$. Also, $j' - i' > 2$ because $B_\varepsilon(a) \cap B_\varepsilon(b) = \varnothing$.

$U'_{i'} \subseteq B_\varepsilon(a)$ and $U'_{j'-1} \subseteq B_\varepsilon(b)$, hence, $p'q' \setminus B_\varepsilon(ab) \neq \varnothing$ by the properties of pq. This means there is a point $c_0 \in W \setminus B_\varepsilon(ab)$ for some $W \in \{U'_{i'}, \dots, U'_{j'-1}\}$, but $U'_{i'}, U'_{j'-1} \subseteq B_\varepsilon(a,b) \subseteq B_\varepsilon(ab)$, so $W \in \{U_{i'+1}, \dots, U_{j'-2}\} \subseteq \mathscr{C}'$. Note that the walk $U'_{i'+1}, \dots, U'_{j'-2} \in \mathscr{U} \setminus (\mathscr{E} \cup \mathscr{F})$ connects $U'_{i'} \in \mathscr{E}$ with $U'_{j'-1} \in \mathscr{F}$ in $N(\mathscr{U})$ and so does $\mathscr{C}'$. But $\mathscr{C}'$ is a maximal connected subset of $\mathscr{U} \setminus (\mathscr{E} \cup \mathscr{F})$, so, since $N(\mathscr{U})$ is a tree, it must be the case that $U'_{i'+1}, \dots, U'_{j'-2} \in \mathscr{C}'$.

$U_{i+1}, W \in \mathscr{C}'$, therefore, there is a path $W = W_0, W_1, \dots, W_{l-1} = U_{i+1}$ contained in $\mathscr{C}'$. Choosing $c_i \in W_i$ for all $1 \leq k < l$ yields a $(2 \operatorname{mesh}\mathscr{U})$-chain $c_0, \dots, c_{l-1}$. Set $C = \{c_0, \dots, c_{l-1}\}$, then $C \subseteq \bigcup \mathscr{C}' \subseteq X \setminus B_{\varepsilon - \operatorname{mesh}\mathscr{U}}(a,b) \subseteq X \setminus B_{\frac{\varepsilon}{2}}(a,b)$. Note that C is compact for it is finite, $C \setminus B_\varepsilon(ab) \neq \varnothing$ because of c_0, and $d(C, ab) < \operatorname{mesh}\mathscr{U}$ since $c_{l-1} \in U_{i+1}$ and $U_{i+1} \cap cd \neq \varnothing$ (a property of cd).

X is treelike, so there is a sequence $(\mathscr{U}_n)$ of tree covers such that $\operatorname{mesh}\mathscr{U}_n \to 0$. Let (C_n) be a sequence of the corresponding C's constructed above. (C_n) is a sequence of $(2 \operatorname{mesh}\mathscr{U})$-chained compacts in a compact space $X \setminus B_{\frac{\varepsilon}{2}}(a,b)$. Hence, by the fact above this proposition, there is a subsequence (C'_n) converging to a continuum $C' \subseteq X \setminus B_{\frac{\varepsilon}{2}}(a,b)$. Now, $C'_n \setminus B_\varepsilon(ab) \neq \varnothing$ by the properties of C'_n, therefore, $d(C', X \setminus B_\varepsilon(ab)) = 0$, which implies that $C' \setminus B_\varepsilon(ab) \neq \varnothing$. Similarly, $d(C'_n, ab) < \operatorname{mesh} \mathscr{U}_n$, hence, $d(C', ab) = 0$, and $C' \cap ab \neq \varnothing$.

Let $x \in C' \setminus B_\varepsilon(ab)$. C' is arcwise connected and intersects ab, so there is an arc $xy \subseteq C'$, $y \in ab$. From the closedness of $ab \cap C'$, it may be assumed that $xy \cap ab = \{y\}$. Then $y \neq a, b$ since $y \in C' \subseteq X \setminus B_{\frac{\varepsilon}{2}}(a,b)$, so y is an inner branch point of ab, which is a contradiction. ■

Corollary 5. Let X be a dendroid with a subcontinuum C and let aC be an arc such that none of its inner points are branch points. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that all arcs pC, where $p \in B_\delta(a)$, have a subarc $p'q'$ so that $p' \in B_\varepsilon(a)$, $q' \in B_\varepsilon(c)$, and $p'q' \subseteq B_\varepsilon(aC)$, where c is the end of aC.

Proof. Choose any $b \in (aC \setminus \{c\}) \cap B_{\frac{\varepsilon}{2}}(c)$. Then $ab \cap C = \varnothing$, so there is $\varepsilon' \in (0, \frac{\varepsilon}{2})$ such that $B_{\varepsilon'}(ab) \cap C = \varnothing$. By the proposition, there is $\delta' > 0$ such that for all arcs pq, $p \in B_{\delta'}(a)$, $q \in B_{\delta'}(b)$, there is a subarc $p'q'$, $p' \in B_{\varepsilon'}(a)$, $q' \in B_{\varepsilon'}(b)$, such that $p'q' \subseteq B_{\varepsilon'}(ab)$. Without loss of generality, $\delta' \leq \varepsilon'$.

Using Lemma 0 for the point b on ac yields $\delta > 0$ such that all arcs pc, $p \in B_\delta(a)$, have a point $q \in B_{\delta'}(b) \subseteq B_{\varepsilon'}(ab)$. Without loss of generality, $\delta \leq \delta'$. Note that $q \notin C$, which implies $pq \cap C = \varnothing$ since $q \in pc$ and C is arcwise connected. So, pq is a subarc of pC such that $p \in B_\delta(a) \subseteq B_{\delta'}(a)$, $q \in B_{\delta'}(b)$. Then, by the above usage of the proposition, there is a subarc $p'q'$ of pC such that $p' \in B_{\varepsilon'}(a) \subseteq B_\varepsilon(a)$, $q' \in B_{\varepsilon'}(b) \subseteq B_{\frac{\varepsilon}{2}}(b) \subseteq B_\varepsilon(c)$, and $p'q' \subseteq B_{\varepsilon'}(ab) \subseteq B_\varepsilon(ac)$. ■

2.2 Unions of Closed Shore Sets

Definition. Let X be a dendroid with a subcontinuum C, and let $A \subseteq X \setminus C$. Define $R_A(C)$ to be the set of all $p \in X \setminus C$ such that $pC \cap A = \varnothing$.

Let X and C be as above. Additionally, let $S(C)$ be the set of all $p \in X \setminus C$ such that no inner point of pC is a branch point. C is a ***core*** if no point of C is a shore point. A core C is ***simple core*** if $S(C)^\circ$ is dense in $X \setminus C$.

Lemma 6. Let X be a dendroid with a core C, and let $A \subseteq X \setminus C$. Then A is a shore set, if and only if $R_A(C)$ is dense in $X \setminus C$.

Proof. Suppose A is a shore set. Let U be a nonempty open subset of $X \setminus C$, and pick any $c \in C$. Since c is not a shore point, any sufficiently dense subcontinuum of X must contain it. Therefore, because A is a shore set, there is a continuum $D \subseteq X \setminus A$ such that $c \in D$ and $D \cap U \neq \varnothing$. Then for any $p \in D \cap U$, it holds that $pC \subseteq D \subseteq X \setminus A$, so $p \in R_A(C) \cap U$.

For the other direction, suppose $R_A(C)$ is dense in $X \setminus C$, and let $\varepsilon > 0$. By compactness, there are finitely many $\frac{\varepsilon}{2}$-balls $(B_i)_{i \in I}$ which cover X. Let J be the set of all $i \in I$ such that $B_i \subseteq X \setminus C$. Then for each $j \in J$, by the density of $R_A(C)$, B_j contains a point p_j such that $p_j C \cap A = \varnothing$. Let $D = C \cup \bigcup_{j \in J} p_j C$, that is a continuum disjoint from A, and $X \subseteq \bigcup_{i \in I} B_i \subseteq B_\varepsilon(C) \cup \bigcup_{j \in J} B_\varepsilon(p_j) \subseteq B_\varepsilon(D)$. ■

Let X be a metric space. A path p is said to be ***n*-wiggly** around (A, ε), where $A \subseteq X$ and $\varepsilon > 0$, if there exist $0 \le t_0 \le \ldots \le t_{2n-1} \le 1$ such that $p(t_k) \in B_\varepsilon(A)$ for even k's, and $p(t_k) \notin \overline{B}_{2\varepsilon}(A)$ for odd k's. All paths are 0-wiggly around any (A, ε) for there are no points to be checked. If $A = \{p\}$, the path is said to be n-wiggly around (p, ε).

Lemma 7. Let X be metric space, p be a path, $A \subseteq X$, and $\varepsilon > 0$. Then there is a maximum n such that p is n-wiggly around (A, ε).

Proof. If it is not the case, then for all $n \ge 1$, there are $0 \le t_0 \le \ldots \le t_{2n-1} \le 1$ such that $p(t_k) \in B_\varepsilon(A)$ for even k's, and $p(t_k) \notin \overline{B}_{2\varepsilon}(A)$ for odd k's. Pick an even k such that $t_{k+1} - t_k$ is the least possible. Set $u_n = t_k$ and $v_n = t_{k+1}$, then $p(u_n) \in B_\varepsilon(A)$, $p(v_n) \notin \overline{B}_{2\varepsilon}(A)$, and $|u_n - v_n| \le \frac{1}{n}$.

The sequence (u_n) has a convergent subsequence (u_{n_k}) with limit $u \in [0, 1]$. And since $|u_n - v_n| \to 0$, the sequence (v_{n_k}) has the same limit. But $p(u) \in \overline{B}_\varepsilon(A)$ because p is continuous and $p(u_n) \in B_\varepsilon(A)$ for all $n \ge 1$. Similarly, $p(u) \notin B_{2\varepsilon}(A)$ because $p(v_n) \notin \overline{B}_{2\varepsilon}(A)$. That is a contradiction. ■

The number n from the above lemma is called the ***wiggliness*** around (A, ε) of the path p.

Lemma 8. Let X be a dendroid, ab an arc, $A \subseteq X$, and $\varepsilon > 0$. Then there is $\delta > 0$ such that around (A, ε), the wiggliness of all arcs pq, where $p \in B_\delta(a)$ and $q \in B_\delta(b)$, is at least the wiggliness of ab.

Proof. Let n be the wiggliness of ab around (A, ε). Then there are points $x_0, \ldots, x_{2n-1}$ in this order on ab such that $x_k \in B_\varepsilon(A)$ for even k's, and $x_k \notin \overline{B}_{2\varepsilon}(A)$ for odd k's.

Let $\varepsilon' > 0$ be so small that $B_{\varepsilon'}(x_k) \subseteq B_\varepsilon(A)$ for even k's, and $B_{\varepsilon'}(x_k) \subseteq X \setminus \overline{B}_{2\varepsilon}(A)$ for odd k's. Then, by Lemma 0, there is $\delta > 0$ such all arcs pq, $p \in B_\delta(a)$, $q \in B_\delta(b)$, have points $y_0, \ldots, y_{2n-1}$ in this order so that $y_k \in B_{\varepsilon'}(x_k) \subseteq B_\varepsilon(A)$ for even k's, and $y_k \in B_{\varepsilon'}(x_k) \subseteq X \setminus \overline{B}_{2\varepsilon}(A)$ for odd k's. ■

Theorem 9. Let X be a dendroid with a simple core C. Then the union of finitely many closed shore sets is a shore set.

Proof. It is sufficient to prove it for two closed shore sets A, B. Suppose that $A \cup B$ is not a shore set. $A, B \subseteq X \setminus C$, so by Lemma 6, $R_{A\cup B}(C)$ is not dense in $X \setminus C$. The set $S(C)^\circ$ is dense in $X \setminus C$, so there is an open set $U \subseteq S(C)$ disjoint from $R_{A\cup B}(C)$.

There is $\lambda > 0$ such that $\overline{B}_{2\lambda}(C) \cap (A \cup B) = \varnothing$ because C and $A \cup B$ are disjoint compact sets. Construct a sequence (p_n) of points of U, and sequences (δ_n), (ε_n) of positive reals such that:

(0) $p_0 \in U \cap R_A(C)$, $\overline{B}_{2\delta_0}(p_0) \cap (A \cup B) = \varnothing$, and $B_{\varepsilon_0}(p_0C) \cap A = \varnothing$,

(1) $\overline{B}_{\delta_0}(p_0) \subseteq U$, $\overline{B}_{\delta_{n+1}}(p_{n+1}) \subseteq B_{\delta_n}(p_n)$, and $\delta_n \to 0$,

(2) $p_{n+1} \in B_{\delta_n}(p_n) \cap R_B(C) \setminus R_A(C)$ and $B_{\varepsilon_{n+1}}(p_{n+1}C) \cap B = \varnothing$ when n is even, $p_{n+1} \in B_{\delta_n}(p_n) \cap R_A(C) \setminus R_B(C)$ and $B_{\varepsilon_{n+1}}(p_{n+1}C) \cap A = \varnothing$ when n is odd,

(3) around both of (p_0, δ_0) and (C, λ), the wiggliness of all arcs pq, $p \in B_{\varepsilon_n}(p_n)$, $q \in B_{\varepsilon_n}(c_n)$, is at least the wiggliness of p_nC,

(4) any arc pC, where $p \in B_{\delta_n}(p_n)$, has a subarc $p'q'$ such that $p' \in B_{\varepsilon_n}(p_n)$, $q' \in B_{\varepsilon_n}(c_n)$, and $p'q' \subseteq B_{\varepsilon_n}(p_nc_n)$;

where c_n denotes the end p_nC.

A, B are closed shore sets, so they are nowhere dense. Hence, $A \cup B$ is also nowhere dense. Moreover, $R_A(C)$ is dense in U by Lemma 6, so there is a point p_0 such that the first two conditions of (0) are satisfied. The last condition is fulfilled when ε_0 is sufficiently small since A and p_0C are disjoint compact sets.

(1) can be satisfied by choosing δ_n sufficiently small in each step, and it implies that $\overline{B}_{\delta_m}(p_m) \subseteq B_{\delta_n}(p_n)$ for all $m > n$. Particularly, $\overline{B}_{\delta_{n+1}}(p_{n+1}) \subseteq B_{\delta_0}(p_0) \subseteq U$ for all n. Additionally, $\delta_n \to 0$ ensures that the sequence (p_n) is Cauchy, so it has a limit $p_\omega \in \bigcap_{n\in\mathbb{N}} B_{\delta_n}(p_n) \subseteq U$.

Let n be even. Note that $U \cap R_A(C) = U \cap R_A(C) \setminus R_B(C)$ because U is disjoint from $R_{A\cup B}(C) = R_A(C) \cap R_B(C)$. And since $R_A(C)$ is dense in U by Lemma 6, there is a point $p_{n+1} \in B_{\delta_n}(p_n) \cap R_B(C) \setminus R_A(C)$. The sets $p_{n+1}C$ and B are disjoint and compact, so if ε_{n+1} is sufficiently small, $B_{\varepsilon_{n+1}}(p_{n+1}C) \cap B = \varnothing$. The case of n being odd is similar, hence (2) can be satisfied.

Clearly, (3) can be satisfied by Lemma 8, assuming ε_n are sufficiently small. For each n, $p_n \in U \subseteq S(C)$, so p_nC contains no inner branch points. Hence, if δ_n is sufficiently small, Corollary 5 says that (4) holds. This finishes the construction.

Let n be even. The arc $p_{n+1}C$ has a subarc $p'q'$ satisfying the conditions in (4) because $p_{n+1} \in B_{\delta_n}(p_n)$. It follows from (2) (and (0) for $n = 0$) that $p_{n+1}c_{n+1}$ contains a point $a \in A$, while $B_{\varepsilon_n}(p_nc_n)$ is disjoint from A. The point a lies either on $p_{n+1}p'$, or on $q'c_{n+1}$ since $p'q' \subseteq B_{\varepsilon_n}(p_nc_n)$.

It follows from (3) that the wiggliness of $p'q'$ around (p_0, δ_0) is at least the wiggliness of $p_n c_n$. This means that if $a \in p_{n+1}p'$, the wiggliness of $p_{n+1}c_{n+1}$ around (p_0, δ_0) is strictly greater than the wiggliness of $p_n c_n$ because $p_{n+1} \in B_{\delta_0}(p_0)$ and $a \notin \overline{B}_{2\delta_0}(p_0)$. If $a \in q'c_{n+1}$, then the wiggliness of $p_{n+1}c_{n+1}$ around (C, λ) is strictly greater than the wiggliness of $p_n c_n$ because $c_{n+1} \in B_\lambda(C)$ and $a \notin \overline{B}_{2\lambda}(C)$.

Similar reasoning applies when n is odd. Therefore, around either (p_0, δ_0) or (C, λ), the wiggliness of the arcs $p_n C$ increases without bound. But $p_\omega \in B_{\varepsilon_n}(p_n)$ for all n, so (3) applies, and the wiggliness is bounded by the wiggliness of $p_\omega C$. That is a contradiction. ■

Corollary 10. Let X be a dendroid with finitely many branch points. Then the union of finitely many closed shore sets is a shore set.

Proof. If X is degenerate, then X itself is a simple core. If X is an arc, then any of its inner points are a simple core. Otherwise, X has at least one branch point because it contains a maximal arc by [10, Lemma 3].

Hence, let $b_0, \dots, b_{n-1}$, $n \geq 1$, be the branch points, T the continuum $\bigcup_{i,j=0}^{n-1} b_i b_j$, A_i the union of all arcs $b_i p$, $p \in X$, such that $b_i p \cap T = \{b_i\}$, and B the set of all b_i such that $(A_i)^\circ \neq \varnothing$.

Suppose that $T^\circ = \varnothing$. Then B is nonempty because $X = T \cup \bigcup_{i=0}^{n-1} A_i$, therefore $C = \bigcup_{pq \in B} pq$ is a continuum. Let D be the union of all A_i which intersect C. By the construction, $\big(X \setminus (C \cup D)\big)^\circ = \varnothing$, which implies that D is dense in X since $C^\circ \subseteq T^\circ = \varnothing$.

If $\big(\overline{X \setminus (C \cup D)}\big)^\circ \neq \varnothing$, then $\big(\overline{X \setminus (T \cup C \cup D)}\big)^\circ \neq \varnothing$ since T is nowhere dense. Note that $X \setminus (T \cup C \cup D)$ is contained in the union of all A_i such that $b_i \notin C$. Because the union of nowhere dense sets is nowhere dense, there is i, $b_i \notin C$, such that $\big(\overline{A_i}\big)^\circ \neq \varnothing$. D is dense in X, so $\big(\overline{A_i} \cap \overline{D}\big)^\circ \neq \varnothing$. Similarly, from the definition of D, there is j, $A_j \cap C \neq \varnothing$, such that $\big(\overline{A_i} \cap \overline{A_j}\big)^\circ \neq \varnothing$. Then $A_i \cap C = \varnothing$ since $b_i \notin C$, so $i \neq j$. But that contradicts Theorem 1 for any $p \in b_i b_j \setminus \{b_i, b_j\}$ because $A_i \subseteq Q_p(b_i)$, $A_j \subseteq Q_p(b_j)$, and $Q_p(b_i) \cap Q_p(b_j) = \varnothing$.

This means that $\big(\overline{X \setminus (C \cup D)}\big)^\circ = \varnothing$, which can be rewritten as $\overline{(C \cup D)^\circ} = X$. Therefore, $S(C)^\circ \supseteq D^\circ$ is dense in $X \setminus C$, and even in X since $C^\circ = \varnothing$. Any sufficiently dense subcontinuum of X intersects all A_i for which $(A_i)^\circ \neq \varnothing$. Hence, if $|B| \geq 2$, it must contain all arcs $b_i b_j$, and so the whole C. In that case no point of C can be a shore point, and C is a simple core.

So, suppose that $B = \{b\}$, then $C = \{b\}$. All dendroids are decomposable, which means there are proper subcontinua C_0, C_1 such that $X = C_0 \cup C_1$. If C_0 does not intersect T, it must be a subset of some arc starting in T. An arc is still a proper subcontinuum of X, so C_0 can by replaced by such arc, and similarly for C_1. Hence, it can be assumed that C_0 and C_1 intersect T.

Note that $C_0 \setminus C_1 = X \setminus C_1$ and $C_1 \setminus C_0 = X \setminus C_0$ are nonempty open sets, and D has dense interior, so any sufficiently dense subcontiuum C' of X must intersect $D \cap C_0 \setminus C_1$ and $D \cap C_1 \setminus C_0$ at some points c_0, c_1, respectively. Then $Tc_0 = bc_0$ because $c_0 \in D$ and $C = \{b\}$. Moreover, C_0 is arcwise connected and intersects T, so $c_1 \in bc_0 = Tc_0 \implies c_1 \in C_0$. Hence, $c_1 \notin bc_0$, and similarly, $c_0 \notin bc_1$. Therefore, $bc_0 \cap bc_1 = \{b\}$ since no point of bc_0 or bc_1 other than b is a branch point. So, $b \in c_0c_1 \subseteq C'$, which means b cannot be a shore point, and $C = \{b\}$ is a simple core.

Now look at the case $T^\circ \neq \varnothing$. Then is an arc b_ib_j with no inner branch points such that $(b_ib_j)^\circ \neq \varnothing$. Clearly, $n \geq 2$, so assume that all dendroids with less than n branch points have a simple core.

Choose a subarc pq of b_ib_j such that $pq \subseteq (b_ib_j)^\circ \setminus \{b_i, b_j\}$. Then $U = pq \setminus \{p, q\}$ is an open subset of X. Subcontinua of dendroids are arcwise connected, so $X \setminus U$ cannot be a continuum for there would be another arc between p and q. That means $X \setminus U$ breaks into 2 closed components X_0 and X_1 containing p and q, respectively. X_0 and X_1 are dendroids containing b_i and b_j, respectively. That means they have less than n branch points, and therefore, by the assumption, they contain simple cores C_0 and C_1.

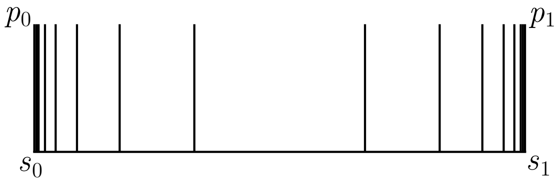
Pick any $c_0 \in C_0$ and $c_1 \in C_1$, and let C be any ε-dense subcontinuum of X. By unicoherence, $C \cap X_0$ is an ε-dense subcontinuum of X_0. Hence, $c_0 \in C \cap X_0 \subseteq C$ if $\varepsilon > 0$ is sufficiently small since c_0 is not a shore point in X_0, and similarly for c_1. Therefore, $c_0c_1 \subseteq C$ for a sufficiently small $\varepsilon > 0$, which means no point of c_0c_1 is a shore point in X. Additionally, no points of C_0 and C_1 are shore points from the arbitrary choice of c_0 and c_1.

That means the continuum $C = C_0 \cup C_1 \cup c_0c_1$ is a core. Let $U \subseteq X \setminus C$ be open, then $U \cap X_0 \neq \varnothing$ or $U \cap X_1 \neq \varnothing$ since $pq \subseteq c_0c_1 \subseteq C$. Therefore, without loss of generality, $U \subseteq X_0$. Then in X_0, $S(C_0)^\circ \cap U \neq \varnothing$ because C_0 is a simple core. Hence, $S(C)^\circ \cap U \neq \varnothing$ in X because all branch points of X in X_0 are already branch points of X_0. ■

In particular, this answers the questions posed by [0, Table 1, p. 213] for fans.

The following example shows some limitations of the simple core approach.

Example 11. A dendroid X with two different shore points s_0 and s_1 (whose union must be a shore set) containing no simple core.



It it easy to see that X is a dendroid. Any simple core must intersect $s_0 s_1$ in a nonempty closed subset disjoint from $\{s_0, s_1\}$, but that prevents it from reaching both p_0 and p_1 without crossing a branch point.

A ***hairy tree*** is a dendroid, whose all branch points are contained in a subcontinuum T which is a union of finitely many arcs. For instance, $T = s_0 s_1$ in the above example.

Question. Is a union of two closed shore sets again a closed shore set in hairy trees? If not, what about the disjoint case?

Conclusion

[0, Example 14] shows than even for planar dendroids with just countably many branch points, a union of disjoint closed shore sets can fail to be a shore set. Therefore, Corollary 10 is a nice complement of that result.

Example 11 shows some limitations of the simple core approach. An extension to hairy trees seems worth investigating. The problem is that one cannot simply omit the closedness of simple core.

While Theorem 1 nicely simplifies the original [5, Theorem 2], the generalization part is not used anywhere. It was originally devised for an attempt to prove that the union of disjoint closed shore sets is a shore set in a dendroid, which was doomed to fail. It may be worth investigating if it is of any help in some other special cases of dendroids.

Speaking of which, the table [0, Table 1, p. 213] still shows some gaps for smooth dendroids (there is a point p such that for all sequences $q_n \to q$, the sequence pq_n converges to pq in the Hausdorff metric). These results will probably not be of much help for λ-dendroids (where arcwise connectedness is replaced by hereditarily decomposability).

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