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MASTER'S THESIS

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Adaptive space-time discontinuous Galerkin method for the solution of non-stationary problems

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Study programme:	Mathematics
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I would like to thank my supervisor Vít Dolejší for his help in completing my master's thesis. I thank my friend, Jan Vlachý, for helping me with the language. My thanks also go to my family for their long term support.

I declare that I completed this master's thesis independently, and only with the cited sources, literature, and other professional sources.

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Název práce: Adaptivní časoprostorová nespojitá Galerkinova metoda pro řešení nestacionárních úloh

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Abstrakt: Tato práce se zabývá numerickým řešením nelineárních konvekčně-difuzních úloh s pomocí časovo-prostorové nespojité Galerkinové metody, která je vhodná pro časovou i prostorovou lokální adaptaci. Naším cílem je vyvinout aposteriorní odhady chyby, které odraží prostorové, časové a algebraické chyby. Tyto odhady jsou založeny na residuu v duálních normách. Odvodíme tyto odhady a numericky ověříme jejich vlastnosti. Na konci práce navrhneme adaptivní algoritmus a aplikujeme ho při simulaci nestacionáního vazkého stlačitelného proudění.

Klíčová slova: časovo-prostorová nespojitá Galerkinova metoda, reziduum v duální normě, nestacionární nelineární konvekčně-difuzní rovnice.

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Abstract: This thesis studies the numerical solution of non-linear convection-diffusion problems using the space-time discontinuous Galerkin method, which perfectly suits the space as well as time local adaptation. We aim to develop a posteriori error estimates reflecting the spatial, temporal, and algebraic errors. These estimates are based on the measurement of the residuals in dual norms. We derive these estimates and numerically verify their properties. Finally, we derive an adaptive algorithm and apply it to the numerical simulation of non-stationary viscous compressible flows.

Keywords: space-time discontinuous Galerkin method, residuum in the dual norm, order of convergence.

Contents

Introduction

The space-time discontinuous Galerkin method (STDGM) is a numerical method to solve the compressible flow problem. The problem is described by Navier-Stokes equations, and its solution is a vector-valued function describing the physical phenomenon. The STDGM uses a numerical discretization in the space coordinates as well as in the time coordinate, and uses a space of piecewise polynomial functions as the discrete functional space that can represent interelement discontinuity. Therefore, the method is suitable for problems with discontinuous solutions or a steep gradient, where continuous methods yield a result with oscillations. The time discretization allows the numerical discretization to be conservative, and if we choose the polynomial degree with respect to the time appropriately, we can achieve sufficient accuracy even for nonstationary problems with a time-dependent mesh. The STDGM allows the polynomial degrees to vary for each element and each time step, and so it enables local adaptivity. The STDGM then yields high order of convergence, but also requires longer computational time.

To solve the compressible flow problem, we would like to use STDGM to construct an algorithm that can be adaptive with respect to both the time step and the spatial step. We demonstrate the STDGM in Chapter 2 for a continuous problem where the solution is a scalar function. Then, we introduce the concept of the residuum in the dual norm in Chapter 3. Using this concept, we postulate certain criteria to optimize calculations during the computation. We want our algorithm to satisfy following properties:

- the approximate solution satisfies a given condition,
- the computational time is as small as possible,

In this thesis, we explore how to construct the algorithm, we explain the choices made, and we verify certain properties related to the residuum in the dual norm. In Chapter 4 the algorithm is constructed using previous knowledge.

This thesis extends the results published in [[1] : V. Dolejší, F. Roskovec, and M. Vlasák. Residual based error estimates for the space-time discontinuous Galerkin method applied to the compressible flows. Comput. Fluids, 117:304–324, 2015.]

The numerical experiments were carried out using software ADGFEM. The software was written in Fortran by Vít Dolejší and his students in the Departmant of Numerical Mathematics.

1. Continuous problem

1.1 Formulation of continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $\partial\Omega$ its boundary, and $T > 0$. We put $Q_T = \Omega \times (0,T) \subset \mathbb{R}^3$, $x = (x_1, x_2) \in \Omega$. Let us consider scalar equations of a non-stationary problem with a convection and diffusion term

$$\frac{\partial u}{\partial t} + \frac{\partial f_1(u)}{\partial x_1} + \frac{\partial f_2(u)}{\partial x_2} - \operatorname{div}(\mathbb{K}(u)\nabla u) = g \quad \forall (x,t) \in Q_T, \tag{1.1a}$$

$$u(x,0) = u^0(x) \quad \forall x \in \Omega, \tag{1.1b}$$

$$u(x,t) = u_D(x,t) \quad \forall (x,t) \in \partial\Omega \times (0,T), \tag{1.1c}$$

the goal is to find $u : Q_T \to \mathbb{R}$ from given functions $\boldsymbol{f} = (f_1,f_2)$, $\mathbb{K} = \{K_{ij}\}_{i,j=1}^2$, g, a given function u^0 for the initial value (at $t = 0$), and a given function u_D on the boundary.
The functions f_s represent the convective terms, in other words, f_s is the flux of quantity u in the directions x_s when we omit the viscous term (for example perfect gas has only inviscid term). The functions $K_{ij} \in \mathbb{R}$ represent the diffusion terms. Both f_s and K_{ij} can be non-linear functions. In Section 3.4, we put $\mathbb{K}(u) \equiv \varepsilon\mathbb{I}$, where $\varepsilon > 0$ is small number, and $\mathbb{I}$ is the identity matrix. We can write the term $\operatorname{div}(\mathbb{K}(u)\nabla u)$ component-wise as below

$$\begin{aligned}\nabla \cdot (\mathbb{K}(u)\nabla u) =& \nabla \cdot \left(\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \end{pmatrix} \right) \\ =& \frac{\partial}{\partial x_1}\left(K_{11}\frac{\partial u}{\partial x_1} + K_{12}\frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_2}\left(K_{21}\frac{\partial u}{\partial x_1} + K_{22}\frac{\partial u}{\partial x_2} \right).\end{aligned}$$

The continuous problem (1.1) is a model problem for Navier-Stokes equations described in Section 5.1.

We assume that the data satisfies the following assumptions:

(D1) $\boldsymbol{f} = (f_1, f_2),\ f_s \in C^1(\mathbb{R}),\ f_s(0) = 0,\ s = 1, 2,$

(D2) $\mathbb{K}$ is a bounded and positive definite matrix, that is
$\exists k_2 > 0 : K_{ij}(u) \leqslant k_2 < \infty \quad \forall u \in \mathbb{R},\ i, j = 1, 2$,
$\exists k_1 > 0 : x^T\mathbb{K}(u)x \geqslant k_1 x^T x > 0 \quad \forall u \in \mathbb{R},$ for a.e. $x \in \mathbb{R}^2 \setminus \{0\}$,

(D3) $g \in C([0,T]; L^2(\Omega))$,

(D4) u_D is the trace of some $u^* \in C([0,T]; H^1(\Omega)) \cap L^\infty(Q_T)$ on $\partial\Omega \times (0,T)$,

(D5) $u^0 \in L^2(\Omega)$.

1.2 Functional spaces

We use the Sobolev space for the spatial space and the Bochner space for the space-time space. We review their definitions below.

Let $\Omega \in \mathbb{R}^2$ be a bounded domain. The Lebesgue spaces are defined as follows:

$$L^p(\Omega) = \{\varphi \text{ measurable functions; } \|\varphi\|_{L^p(\Omega)} = \left(\int_\Omega |\varphi(x)|^p \mathrm{d}x\right)^{1/p} < \infty\},$$
$$L^\infty(\Omega) = \{\varphi \text{ measurable functions; } \|\varphi\|_{L^\infty} = \mathrm{essup}_{x\in\Omega}|\varphi(x)| < \infty\},$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$

$$(\varphi,\psi) = \int_\Omega \varphi(x)\psi(x)\mathrm{d}x, \quad \varphi,\psi \in L^2(\Omega).$$

The Sobolev spaces are defined

$$W^{k,p}(\Omega) = \{\varphi; \quad \forall\alpha\ |\alpha| \leqslant k,\ D^\alpha\varphi \text{ exists in the sense of distributions,}$$
$$D^\alpha\varphi \in L^p(\Omega),\ \|\varphi\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leqslant k} \|D^\alpha\varphi\|^p_{L^p(\Omega)}\right)^{1/p} < \infty\},$$

where $D^\alpha\varphi$ is the weak α-th derivative of φ. We use the notation $H^k(\Omega) \equiv W^{k,2}(\Omega)$, because for $p = 2$, the Sobolev space forms a Hilbert space as in the following:

$$H^k(\Omega) = \{\varphi : \Omega \to \mathbb{R};\ D^\alpha\varphi \in L^2(\Omega) \text{ in the sense of distributions } \forall\alpha\ |\alpha| \leqslant k,$$
$$\|\varphi\|_{H^k(\Omega)} = \left(\sum_{|\alpha|\leqslant k} \|D^\alpha\varphi\|^2_{L^2(\Omega)}\right)^{1/2} < \infty\},$$

with $|\cdot|_{H^k(\Omega)}$ being a seminorm defined in $H^k(\Omega)$ by the following:

$$|\varphi|_{H^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha\varphi\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Remark 1.1. *The broken Sobolev space is used instead of the Sobolev space for the discontinuous Galerkin method. We define the broken Sobolev space later.*

Let $(X, \|\cdot\|_X)$ be a Banach space with a seminorm $|\cdot|_X$. The Bochner spaces are defined as below:

$$L^p(0,T;X) = \{\varphi : (0,T) \to X, \text{ strongly measurable,}$$
$$\|\varphi\|_{L^p(0,T;X)} = \left(\int_0^T \|\varphi\|^p_X \mathrm{d}t\right)^{1/p} < \infty\},$$
$$L^\infty(0,T;X) = \{\varphi : (0,T) \to X, \text{ strongly measurable,}$$
$$\|\varphi\|_{L^\infty(0,T;X)} = \mathrm{essup}_{t\in(0,T)}\|\varphi(t)\|_X < \infty\},$$
$$C([0,T];X) = \{\varphi : (0,T) \mapsto X\}, \text{ continuous}, \|\varphi\|_{C([0,T];X)} = \sup_{t\in[0,T]} \|\varphi\|_X \leq \infty\},$$

$$W^{k,p}(0,T;X) = \{\varphi; \quad \frac{\partial^s\varphi}{\partial t^s} \in L^p(0,T;X) \quad \forall s = 0,..,k\ ,$$
$$\|\varphi\|_{W^{k,p}(0,T;X)} = \left(\int_0^T \sum_{s=0}^k \|\frac{\partial^s\varphi}{\partial t^s}(t)\|^p_X \mathrm{d}t\right)^{1/p} < \infty\},$$

where $\frac{\partial^s \varphi}{\partial t^s}$ is the weak s-th time derivative of φ.

$$\begin{aligned} H^1(0,T;H^2(\Omega)) &= W^{1,2}(0,T;W^{2,2}(\Omega)), \\ H_0^2(\Omega) &= \{\varphi \in H^2(\Omega); \varphi|_{\partial\Omega} = 0 \text{ in the sense of traces}\}. \end{aligned} \tag{1.2}$$

Moreover, we set seminorms

$$\begin{aligned} |\varphi|^2_{L^2(0,T;X)} &= \left(\int_0^T |\varphi(t)|_X^2 \mathrm{d}t \right)^{1/2}, \quad |\varphi|_{C([0,T];X)} = \sup_{t\in[0,T]} |\varphi(t)|_X, \\ |\varphi|^2_{H^k(0,T;X)} &= \left(\int_0^T |\frac{\partial^k \varphi}{\partial t^k}(t)|_X^2 \mathrm{d}t \right)^{1/2}. \end{aligned} \tag{1.3}$$

1.3 Weak formulation

Using the above-described functional spaces, we introduce a weak formulation of scalar problem (1.1). We define

$$\begin{aligned} a(u,v) &= \int_\Omega \mathbb{K}(u)\nabla u \cdot \nabla v \,\mathrm{d}x, \quad u \in H^1(\Omega) \cap L^\infty(\Omega), v \in H^1(\Omega), \\ b(u,v) &= \int_\Omega \nabla \cdot \boldsymbol{f}(u) v \,\mathrm{d}x, \quad u \in H^1(\Omega) \cap L^\infty(\Omega), v \in L^2(\Omega). \end{aligned}$$

Definition 1.1. *We say that function u is the weak solution of* (1.1)*, if the following conditions are satisfied*

$$u - u^* \in L^2(0,T;H_0^1(\Omega)),\ u \in L^\infty(Q_T), \tag{1.4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v) + b(u(t),v) + a(u(t),v) = (g(t),v) \quad \forall v \in H_0^1(\Omega) \text{ and for a.e. } t \in (0,T), \tag{1.4b}$$

$$u(x,0) = u^0(x), \quad \forall x \in \Omega. \tag{1.4c}$$

If $u' \in L^2(0,T;L^2(\Omega))$ then (1.4b) can be written as

$$\int_0^T \left((\frac{\partial}{\partial t} u(t), v(t)) + b(u(t),v(t)) + a(u(t),v(t)) \right) \mathrm{d}t = \int_0^T (g(t),v(t)) \mathrm{d}t \quad \forall v \in L^2(0,T;H_0^1(\Omega)).$$

The proof of existence and uniqueness of a weak solution in the sense of Definition 1.1 for the continuous problem (1.1) is presented in [3].

2. Discontinuous Galerkin method

The weak formulation uses infinite-dimensional Bochner spaces on whole domain Ω. To compute a solution, we introduce a discretization of the domain as well as a discretization of the functional space. The discretized functional space is chosen to be finite dimensional. The domain Ω is discretized as a mesh into small pieces, called elements. On every element in the finite-dimensional space, we compute a partial solution, and these then compose the complete solution on the entire mesh. We call this the approximate solution as it approximates the weak solution (1.4). The preparation to define the approximate solution is presented in Sections 2.1 – 2.4 after which the evaluation of the approximate solution is presented in Sections 2.5 – 2.6.

2.1 Triangulations - mesh discretization

Let I_τ be a partition of the time interval $(0, T)$. We divide $(0, T)$ by time levels $0 = t_0 < t_1 < \ldots < t_r = T$, and put $I_m = (t_{m-1}, t_m)$, $m = 1, \ldots, r$ time intervals, $\tau_m = t_m - t_{m-1}$ the length of a time interval and $\tau = \max_{m=1,\ldots,r} \tau_m$. Then time partition is defined as $\mathcal{I}_\tau := \{I_m, m = 1, \ldots, r\}$. We have $[0, T] = \cup_{m=1}^{r} \overline{I_m}$, and I_m are disjoint intervals.

The domain $\Omega \subset \mathbb{R}^d, d = 2$ can in general assume various shapes, but it has to satisfy the condition $\Omega \in C^{k,\mu}$ (see [4] for the definition). We often require that Ω is a Lipschitz domain or has a Lipshitz boundary. In this thesis, we will for simplification assume that Ω is a *bounded polygonal domain*. If, for example, $\partial\Omega$ is in fact piecewise polynomial (of degree more than two), or of a more complicated shape, we approximate the boundary by piecewise linear or quadratic functions.

We divide Ω into small disjoint triangles or quadrilaterals. At every time level t_m, $m = 0, \ldots, r$, we consider generally different space partition $\mathcal{T}_{h,m}$ of the closure $\overline{\Omega}$ into a finite number of closed triangles with mutually disjoint interiors. Numbering all the elements, we write $\mathcal{T}_{h,m} = \{K_i\}_{i \in E_m}$, where $E_m = \{1, \ldots, N_m\}$ is the index set, and $N_m = \#\mathcal{T}_{h,m}$ is the number of elements in partition $\mathcal{T}_{h,m}$. When we do not need to distinguish elements, we use K instead of K_i. We call $\mathcal{T}_{h,m}$ a triangulation of Ω at time level t_m.

Let $K, K' \in \mathcal{T}_{h,m}$. We say that K and K' are neighbours, if the set $\partial K \cap \partial K'$ has positive Lebesgue $(d-1)$-dimensional measure. Next, Γ is a face of K if $\Gamma = \partial K \cap \partial K'$, where K' is a neighbour of K or if $\Gamma = \partial K \cap \partial\Omega$.

We denote by $\mathcal{F}_{h,m}$ the system of all faces of all elements $K \in \mathcal{T}_{h,m}$. We define the set of all inner faces $\mathcal{F}_{h,m}^I = \{\Gamma \in \mathcal{F}_{h,m}; \Gamma \subset \Omega\}$ and the set of all boundary faces $\mathcal{F}_{h,m}^B = \{\Gamma \in \mathcal{F}_{h,m}; \Gamma \subset \partial\Omega\}$. Then $\mathcal{F}_{h,m} = \mathcal{F}_{h,m}^I \cup \mathcal{F}_{h,m}^B$.

Let $h_K = \mathrm{diam}(K)$, $h_m = \max_{K \in \mathcal{T}_{h,m}} h_K$, $h = \max\{h_m, m = 1 \ldots, r\}$, $\mathbf{n}_\Gamma$ be a unit normal vector belonging to face Γ and let ρ_K denote the radius of the largest 2-dimensional ball inscribed in K. For boundary faces $\Gamma \in \mathcal{F}_{h,m}^B$, the normal vector $\boldsymbol{n}_\Gamma = ((\boldsymbol{n}_\Gamma)_1, (\boldsymbol{n}_\Gamma)_2)$ has the same orientation as the outer normal to $\partial\Omega$, and for inner faces $\Gamma \in \mathcal{F}_{h,m}^I$, the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary but fixed.

For any inner face $\Gamma \in \mathcal{F}^I_{h,m}$, there exist two neighbouring elements K_i, K_j, $i \neq j$ such that $\Gamma = K_i \cap K_j$. If $\boldsymbol{n}_\Gamma$ is the outer normal to the element K_i and the inner normal to the element K_j, we put $K_\Gamma^{(L)} = K_i, K_\Gamma^{(R)} = K_j$. For boundary faces $\Gamma \in \mathcal{F}^B_{h,m}$, there exists an element $K_i \in \mathcal{T}_{h,m}$ such that $\Gamma = K_i \cap \partial\Omega$. We put $K_\Gamma^{(L)} = K_i$.

Let us recall the definition of a conforming mesh:

Definition. *Let $K, K' \in \mathcal{T}_{h,m}$ be two different elements. The triangulation $\mathcal{T}_{h,m}$ is a conforming mesh, if $K \cap K'$ is one of the following objects: an empty set, a common vertex, or a common face of K and K'. Otherwise, $\mathcal{T}_{h,m}$ is a nonconforming mesh.*

We consider a generally nonconforming mesh $\mathcal{T}_{h,m}$ and require the following assumptions:

(T1) the triangulations $\mathcal{T}_{h,m}, m = 0, \dots, r$, are shape-regular, i.e. there exists a real number $C_r > 0$ such that $h_K \leqslant C_r \rho_K \; \forall K \in \mathcal{T}_{h,m}, \; \forall m \in \{0, \dots, r\}$,

(T2) the triangulations $\mathcal{T}_{h,m}, m = 0, \dots, r$, are locally quasi-uniform, i.e. there exists a real number $C_q > 0$ such that $h_K \leqslant C_q h_{K'} \; \forall K, K' \in \mathcal{T}_{h,m}$, K, K' are neighbours, $\forall m \in \{0, \dots, r\}$.

The requirement of locally quasi-uniform mesh ensures that triangles change gradually, while the condition of shape-regularity ensures that the shape of triangles is non-deformed. We remark that if the triangulation $\mathcal{T}_{h,m}$ is conforming and shape-regular, then it is locally quasi-uniform. The complete discussion about the mesh assumptions can be found in [7].

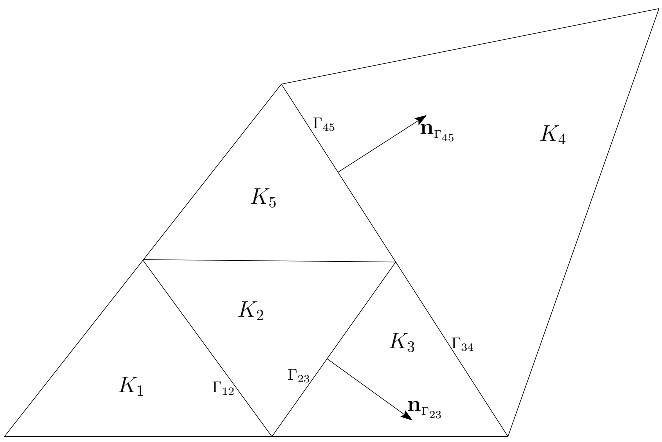

Figure 2.1: Spatial elements K_i in a non-conforming triangulation $\mathcal{T}_{hm}$

Note that in the stationary problem, we can consider the same space partition $\mathcal{T}_h$ over all time levels. In the case of a non-stationary problem, re-meshing the space partition at each time level can refine the mesh where it changes and enlarge it as needed. Therefore, we use a generally different space partition $\mathcal{T}_{h,m}$.

Obviously, $\overline{\Omega} = \cup_{K_i \in \mathcal{T}_{h,m}} K_i$, $m = 0, \dots, r$. We denote by $\mathcal{T}_h := \{\mathcal{T}_{h,m}, m = 1, \dots, r\}$ the set of triangulations on all time levels.

We define the space-time element

$$Q_{m.i} := K_i \times I_m, \quad K_i \in \mathcal{T}_{h,m}, \; m = 1, \dots, r.$$

The space-time mesh at time level t_m is $\mathcal{T}_{h,m} \times I_m$, $m = 1, \ldots, r$.
The space-time triangulation of the domain $Q_T = \Omega \times (0, T)$ is

$$\mathcal{M}_{h\tau} = \{\mathcal{T}_{h,m} \times I_m, m = 1, \ldots, r\}.$$

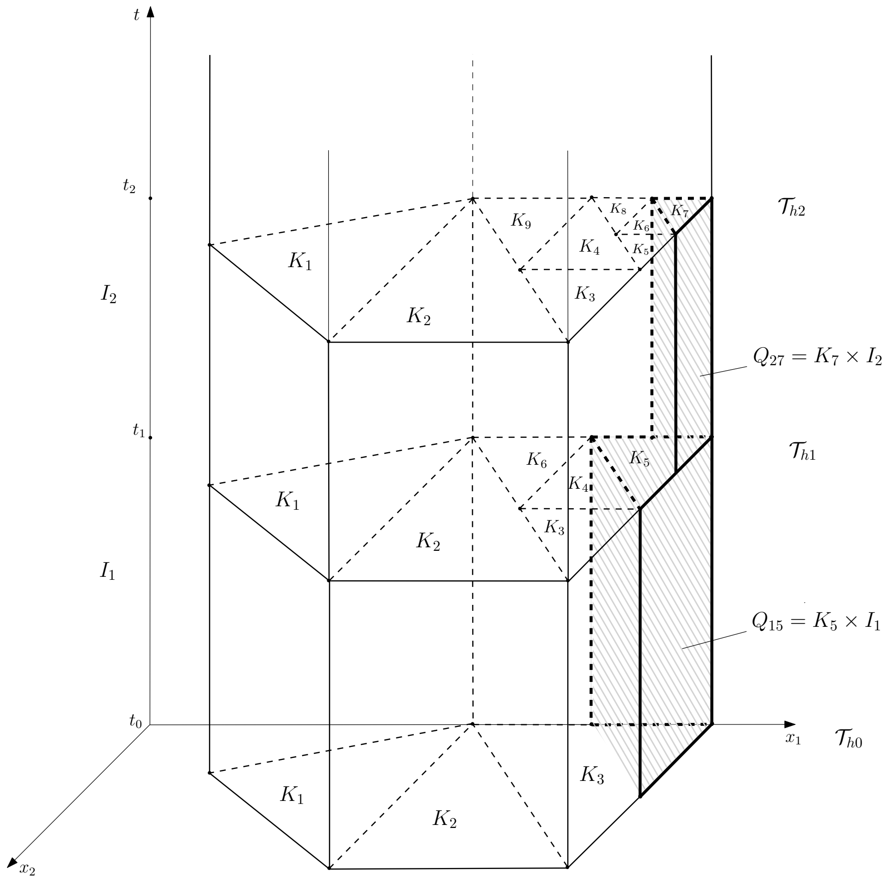

Figure 2.2: Space-time elements $Q_{m,i}$ in a mesh $\mathcal{M}_{h\tau}$

2.2 Discontinuous Galerkin functional spaces

We consider the broken Sobolev space over a triangulation $\mathcal{T}_{h,m}$

$$H^k(\mathcal{T}_{h,m}) := \{v : \Omega \to \mathbb{R};\ v|_K \in H^k(K)\ \forall K \in \mathcal{T}_{h,m}\}$$

equipped with the seminorm

$$|v|_{H^k(\mathcal{T}_{h,m})} = \left(\sum_{K \in \mathcal{T}_{hm}} |v|^2_{H^k(K)} \right)^{1/2}. \tag{2.1}$$

Further we define the broken Bochner space over a mesh $\mathcal{T}_{h,m} \times I_m$

$$H^1(I_m; H^2(\mathcal{T}_{h,m})) := \{\psi : \Omega \times I_m \to \mathbb{R};\ \psi|_{K \times I_m} \in H^1(I_m; H^2(K))\ \forall K \in \mathcal{T}_{h,m}\}$$

and over a space-time mesh $\mathcal{M}_{h\tau}$

$$H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h)) := \{\psi : Q_T \to \mathbb{R};\ \psi|_{K \times I_m} \in H^1(I_m; H^2(K)) \\ \forall K \in \mathcal{T}_{h,m}, I_m \in \mathcal{I}_\tau, m = 1, .., r\},$$

where Bochner space $H^1(t_{m-1}, t_m; H^2(K))$ is defined in (1.2).

For $v \in H^1(\mathcal{T}_{h,m})$ and $\Gamma \in \mathcal{F}^I_{h,m}$, we introduce the following notation

$$\begin{aligned} v|_\Gamma^{(L)} &= \text{ the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \\ v|_\Gamma^{(R)} &= \text{ the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma, \\ \langle v \rangle_\Gamma &= \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)} \right), \\ [v]_\Gamma &= v|_\Gamma^{(L)} - v|_\Gamma^{(R)}, \end{aligned} \tag{2.2}$$

where $\langle v \rangle_\Gamma$ and $[v]_\Gamma$ are the mean value and jump of the traces of v onf Γ, respectively. Remark that $[v]_\Gamma$ is dependent on the orientation of $\boldsymbol{n}_\Gamma$, but $[v]_\Gamma \boldsymbol{n}_\Gamma$ and $\langle v \rangle_\Gamma$ do not depend on the orientation of $\boldsymbol{n}_\Gamma$.

For $\Gamma \in \mathcal{F}^B_{h,m}$ boundary faces and $v \in H^1(\mathcal{T}_{h,m})$ we use the notation

$$\begin{aligned} v|_\Gamma^{(L)} &= \text{ the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \\ \langle v \rangle_\Gamma &= [v]_\Gamma = v|_\Gamma^{(L)}. \end{aligned} \tag{2.3}$$

Further, we use the convention that if $[\cdot]_\Gamma, \langle \cdot \rangle_\Gamma, \boldsymbol{n}_\Gamma$ are arguments of $\int_\Gamma \ldots$ we omit the subscript Γ and write simply $[\cdot], \langle \cdot \rangle, \boldsymbol{n}$. Also, we write $\int_K u(t)\, \mathrm{d}x$ instead of $\int_K u(t)(x)\, \mathrm{d}x$.

Next, let $\psi \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$. Then, we define a jump of function ψ with respect to the time on the time level t_m by

$$\psi|_m^\pm := \lim_{t \to t_m \pm} \psi(t), \quad \{\psi\}_m = \psi|_m^+ - \psi|_m^-$$

The broken Sobolev space and the broken Bochner space are infinite-dimensional spaces, but we need finite-dimensional spaces to compute a solution. Hence, we define several convenient spaces of discontinuous piecewise polynomial functions.

Let $p \in \mathbb{N}^+$ be a positive integer, $q \in \mathbb{N}^+ \cup \{0\}$ a non-negative integer. Then let $P_p(M)$ be a space of all polynomials defined on a spatial domain M with degree less than or equal to p, and $P^q(I_m)$ a space of all polynomials defined on a time interval I_m with degree less than or equal to q.
At a time level $t_m, m = 0, \ldots, r$ we define the space of piecewise polynomial functions with respect to spatial variables

$$S_{h,p,m} := \{\varphi : \Omega \to \mathbb{R};\ \varphi|_K \in P_p(K)\ \forall K \in \mathcal{T}_{h,m}\}.$$

On a time interval I_m, we define

$$S^{\tau,q}_{h,p,m} = S^{\tau,q}(I_m; S_{h,p,m}) := \{\psi : \Omega \times I_m \to \mathbb{R}; \quad \psi(x,t) = \sum_{s=0}^{q} t^s \varphi_s(x) \text{ with } \varphi_s \in S_{h,p,m}\}.$$

On a whole domain Q_T, we define

$$S^{\tau,q}_{h,p} = S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}) := \{\psi : \Omega \times (0,T) \to \mathbb{R};\ \psi|_{\Omega \times I_m} \in S^{\tau,q}_{h,p,m}, m = 1, \ldots, r\}.$$

Let $\psi \in S^{\tau,q}(I_m; S_{h,p,m})$, then $\psi(t) \in S_{h,p,m}\ \forall t \in I_m$. We have also that $\psi|_{K\times I_m} \in P_p(K) \times P^q(I_m)$, so ψ is a polynomial of degree $\leq q$ with respect to the time variable $t \in I_m$ and is a polynomial of degree $\leq p$ with respect to the spatial variable $x \in K$. The function ψ is continuous on a element $Q_{K,m} = K \times I_m$ but generally discontinuous on the mesh $\mathcal{T}_{h,m} \times I_m$.
Space $S^{\tau,q}(I_m; S_{h,p,m})$ is a space consisting of piecewise-polynomial functions on $\mathcal{T}_{h,m} \times I_m$.

2.3 Space semi-discretization

In this section, we recall the discretization with respect to spatial coordinates. The space semi-discretization of (1.4) is accomplished using the interior-penalty discontinuous Galerkin method (cf. [7], Chapter 2).
Let u be a sufficiently regular solution of problem (1.1) such that $u(t) \in H^2(\Omega) \subset H^2(\mathcal{T}_h)$, let $\psi(t) \in H^2(\mathcal{T}_{h,m})$.
For inner faces $\Gamma \in \mathcal{F}^I_{h,m}$ and for a.e. $t \in (0,T)$, we have

$$\begin{aligned}
&\int_\Gamma [u(t)]_\Gamma \,\mathrm{d}S = 0,\\
&\int_\Gamma \langle u(t)\rangle_\Gamma \,\mathrm{d}S = \int_\Gamma u(t)|_\Gamma^{(L)} \,\mathrm{d}S = \int_\Gamma u(t)|_\Gamma^{(R)} \,\mathrm{d}S = \int_\Gamma u(t)|_\Gamma \,\mathrm{d}S, \qquad (2.4)\\
&\int_\Gamma \langle \nabla u(t)\cdot \boldsymbol{n}_\Gamma\rangle_\Gamma \,\mathrm{d}S = \int_\Gamma (\nabla u(t))|_\Gamma \cdot \boldsymbol{n}_\Gamma \,\mathrm{d}S.
\end{aligned}$$

Then, for all faces $\Gamma \in \mathcal{F}_{h,m}$ and for a.e. $t \in (0,T)$, we have

$$(\mathbb{K}(u(t))\nabla u(t))|_\Gamma^{(L)} = (\mathbb{K}(u(t))\nabla u(t))|_\Gamma^{(R)} = \langle \mathbb{K}(u(t))\nabla u(t)\rangle_\Gamma. \qquad (2.5)$$

Note that for $\psi(t) \in H^2(\mathcal{T}_{h,m})$ the previous equivalents are not valid because, although $\psi(t) \in H^2(K)$, it is generally not in $H^2(\Omega)$.

Let m be arbitrary but fixed, $m \in \{1,\ldots,r\}$. We multiply equation (1.1) by an arbitrary function $\psi \in H^1(I_m; H^2(\mathcal{T}_{h,m}))$, integrate over $K \in \mathcal{T}_{h,m}$ and sum over all $K \in \mathcal{T}_{h,m}$. We have for a.e. $t \in I_m$

$$\begin{aligned}
\sum_{K\in\mathcal{T}_{h,m}} \int_K \frac{\partial u}{\partial t}(t)\psi(t)\,\mathrm{d}x + \sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^{2} \frac{\partial f_s(u(t))}{\partial x_s}\psi(t)\,\mathrm{d}x&\\
- \sum_{K\in\mathcal{T}_{h,m}} \int_K \operatorname{div}(\mathbb{K}(u(t))\nabla u(t))\psi(t)\,\mathrm{d}x = \sum_{K\in\mathcal{T}_{h,m}} \int_K g(t)\psi(t)\,\mathrm{d}x.& \qquad (2.6)
\end{aligned}$$

Let t be fixed, $u(t) \in H^2(\Omega), \psi(t) \in H^2(\mathcal{T}_{h,m})$. Applying Green's theorem, we obtain

$$\sum_{K\in\mathcal{T}_{h,m}}\int_K \frac{\partial u}{\partial t}(t)\psi(t)\,\mathrm{d}x - \sum_{K\in\mathcal{T}_{h,m}}\int_K \sum_{s=1}^{2} f_s(u(t))\frac{\partial\psi(t)}{\partial x_s}\,\mathrm{d}x$$

$$+\sum_{K\in\mathcal{T}_{h,m}}\int_{\partial K}\boldsymbol{f}(u(t))\cdot\boldsymbol{n}_{\partial K}\psi(t)\,\mathrm{d}S + \sum_{K\in\mathcal{T}_{h,m}}\int_K(\mathbb{K}(u(t))\nabla u(t))\cdot\nabla\psi(t)\,\mathrm{d}x$$

$$-\sum_{K\in\mathcal{T}_{h,m}}\int_{\partial K}(\mathbb{K}(u(t))\nabla u(t))\cdot\boldsymbol{n}_{\partial K}\psi(t)\,\mathrm{d}S = \sum_{K\in\mathcal{T}_{h,m}}\int_K g(t)\psi(t)\,\mathrm{d}x, \quad (2.7)$$

where $\boldsymbol{n}_{\partial K}$ is a unit outer normal to ∂K.

We first approximate the flux f_s through faces Γ by the numerical flux. This approximation serves to stabilize the discontinuous Galerkin method. Then, in Section 2.5, we will choose a numerical flux to use.
Let $w\in\mathbb{R}$. The numerical flux H is defined

$$H(w|_\Gamma^{(L)}, w|_\Gamma^{(R)}, \boldsymbol{n}|_\Gamma) \approx f_1(w)(\boldsymbol{n}_\Gamma)_1 + f_2(w)(\boldsymbol{n}_\Gamma)_2, \quad \Gamma\in\mathcal{F}_{h,m} \qquad (2.8)$$

and has to satisfy the following properties

(H1) $H(u,v,\boldsymbol{n})$ is defined in $\mathbb{R}\times\mathbb{R}\times B_1$, where $B_1=\{\boldsymbol{n}\in\mathbb{R}^2; |\boldsymbol{n}|=1\}$, and is locally Lipschitz-continuous with respect to u, v.

(H2) $H(u,v,\boldsymbol{n})$ is consistent , i.e. $H(u,u,\boldsymbol{n}) = \boldsymbol{f}(u)\cdot\boldsymbol{n}$

(H3) $H(u,v,\boldsymbol{n})$ is conservative, i.e. $H(u,v,\boldsymbol{n}) = -H(v,u,-\boldsymbol{n})$

The assumption (H3) ensures the conservation property, that is, the quantity of the flux from left to right is the same as the quantity from right to left with the opposite orientation.
Let us write $H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)$ instead of $H(u(t)|_\Gamma^{(L)}, u(t)|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)$ in the following expression.
Therefore,

$$\sum_{K\in\mathcal{T}_{h,m}}\int_{\partial K} f_s(u)(\boldsymbol{n}_\Gamma)_s\psi(t)\,\mathrm{d}S$$

$$= \sum_{\Gamma\in\mathcal{F}_{h,m}^I}\int_\Gamma\sum_{s=1}^{2}\left(f_s(u(t)|_\Gamma^{(L)})(\boldsymbol{n}_\Gamma)_s\psi(t)|_\Gamma^{(L)} + f_s(u(t)|_\Gamma^{(R)})(-\boldsymbol{n}_\Gamma)_s\psi(t)|_\Gamma^{(R)}\right)\,\mathrm{d}S$$

$$+\sum_{\Gamma\in\mathcal{F}_{h,m}^B}\int_\Gamma\sum_{s=1}^{2} f_s(u(t)|_\Gamma^{(L)})(\boldsymbol{n}_\Gamma)_s\psi(t)|_\Gamma^{(L)}\,\mathrm{d}S$$

$$\approx \sum_{\Gamma\in\mathcal{F}_{h,m}^I}\int_\Gamma\left(H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)\psi(t)|_\Gamma^{(L)} + H(u|_\Gamma^{(R)}, u|_\Gamma^{(L)}, -\boldsymbol{n}_\Gamma)\psi(t)|_\Gamma^{(R)}\right)\,\mathrm{d}S$$

$$+\sum_{\Gamma\in\mathcal{F}_{h,m}^B}\int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)\psi(t)|_\Gamma^{(L)}\,\mathrm{d}S$$

$$\overset{(H3)}{=} \sum_{\Gamma\in\mathcal{F}_{h,m}^I}\int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)[\psi(t)]_\Gamma\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_{h,m}^B}\int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)\psi(t)|_\Gamma^{(L)}\,\mathrm{d}S,$$

where $u|_\Gamma^{(R)}, \Gamma \in \mathcal{F}^B_{h,m}$, has to be specified. We can define $u|_\Gamma^{(R)}$ by the extrapolation of the function u^* from (D4) or by the extrapolation of $u|_\Gamma^{(L)}$. Here we put $u|_\Gamma^{(R)} = u|_\Gamma^{(L)}$.

We define the convective form $b_{h,m}(w_1, w_2)$ for $w_1, w_2 \in H^2(\mathcal{T}_{h,m})$ by

$$b_{h,m}(u(t), \psi(t)) := -\sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^{2} f_s(u(t)) \frac{\partial \psi(t)}{\partial x_s} \,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)[\psi(t)]_\Gamma \,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma H(u|_\Gamma^{(L)}, u|_\Gamma^{(L)}, \boldsymbol{n}_\Gamma)\psi(t)|_\Gamma^{(L)} \,\mathrm{d}S. \tag{2.9}$$

We continue with the viscous terms. After some manipulations, we get

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_{h,m}} \int_{\partial K} (\mathbb{K}(u(t))\nabla u(t)) \cdot \boldsymbol{n}_{\partial K} \psi(t) \,\mathrm{d}S \\
&= \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma (\mathbb{K}(u(t))\nabla u(t))|_\Gamma^{(L)} \cdot \boldsymbol{n}_\Gamma \psi(t)|_\Gamma^{(L)} + (\mathbb{K}(u(t))\nabla u(t))|_\Gamma^{(R)} \cdot (-\boldsymbol{n}_\Gamma)\psi(t)|_\Gamma^{(R)} \,\mathrm{d}S \\
&\qquad + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma (\mathbb{K}(u(t))\nabla u(t))|_\Gamma^{(L)} \cdot \boldsymbol{n}_\Gamma \psi(t)|_\Gamma^{(L)} \,\mathrm{d}S \\
&\overset{(2.5)}{=} \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \langle \mathbb{K}(u(t))\nabla u(t)\rangle_\Gamma \cdot \boldsymbol{n}_\Gamma [\psi(t)]_\Gamma \,\mathrm{d}S \\
&\qquad + \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma (\mathbb{K}(u(t))\nabla u(t))|_\Gamma^{(L)} \cdot \boldsymbol{n}_\Gamma \psi(t)|_\Gamma^{(L)} \,\mathrm{d}S.
\end{aligned}$$

We add the following artificial term to the viscous term

$$\begin{aligned}
&\theta \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \langle \mathbb{K}(u(t))^T \nabla\psi(t)\rangle_\Gamma \cdot \boldsymbol{n}_\Gamma [u(t)]_\Gamma \,\mathrm{d}S \\
&\qquad + \theta \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma (\mathbb{K}(u(t))^T \nabla\psi(t))|_\Gamma^{(L)} \cdot \boldsymbol{n}_\Gamma u(t)|_\Gamma^{(L)} \,\mathrm{d}S \\
&\left(\overset{(2.4)\&(1.1c)}{=} 0 + \theta \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma (\mathbb{K}(u(t))^T \nabla\psi(t))|_\Gamma^{(L)} \cdot \boldsymbol{n}_\Gamma u_D(t) \,\mathrm{d}S \right)
\end{aligned}$$

in order to stabilize the equation, where $\mathbb{K}(u(t))^T$ is the matrix transpose of $\mathbb{K}(u(t))$, θ is a chosen parameter.

Let us define diffusion form $a_{h,m}(w_1, w_2), w_1, w_2 \in H^2(\mathcal{T}_{h,m})$ by

$$\begin{aligned}
a_{h,m}(u(t), \psi(t)) :=& \sum_{K\in\mathcal{T}_{h,m}} \int_K (\mathbb{K}(u(t))\nabla u(t)) \cdot \nabla\psi(t) \,\mathrm{d}x \\
&- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \langle \mathbb{K}(u(t))\nabla u(t)\rangle_\Gamma \cdot \boldsymbol{n}_\Gamma [\psi(t)]_\Gamma + \theta \langle \mathbb{K}(u(t))^T \nabla\psi(t)\rangle_\Gamma \cdot \boldsymbol{n}_\Gamma [u(t)]_\Gamma \,\mathrm{d}S \\
&- \sum_{\Gamma\in\mathcal{F}^B_{h,m}} \int_\Gamma (\mathbb{K}(u(t))\nabla u(t))|_\Gamma^{(L)} \cdot \boldsymbol{n}_\Gamma \psi(t)|_\Gamma^{(L)} + \theta (\mathbb{K}(u(t))^T \nabla\psi(t))|_\Gamma^{(L)} \cdot \boldsymbol{n}_\Gamma u(t)|_\Gamma^{(L)} \,\mathrm{d}S.
\end{aligned} \tag{2.10}$$

We can choose the parameter θ in set $\{-1, 0, 1\}$. If $\theta = -1$, we speak about the symmetric interior penalty Galerkin (SIPG) variant of the interior and boundary penalty approximation of the diffusion term. If $\theta = 0$, artificial terms are not added, we call it the incomplete (IIPG) variant. For $\theta = 1$, we obtain the nonsymetric (NIPG) formulation.

Our discretization space consists of discontinuous functions. In order to achieve inter-element continuity in a weaker way, we add the interior and boundary penalty form

$$J_{h,m}(u(t), \psi(t)) := k_1 \left(\sum_{\Gamma \in \mathcal{F}^I_{h,m}} \int_\Gamma \frac{c_W}{h_\Gamma}[u(t)][\psi(t)]\, dS + \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma \frac{c_W}{h_\Gamma} u(t)|_\Gamma \psi(t)|_\Gamma\, dS \right), \tag{2.11}$$

where parameter k_1 is from positive definite property of the matrix $\mathbb{K}$ in (D2). We define h_Γ such that the mesh $\mathcal{T}_{h,m}$ satisfies assumptions (T1) and (T2). For example, we can choose $h_\Gamma = \begin{cases} \max(h_{K^{(L)}_\Gamma}, h_{K^{(R)}_\Gamma}) & \text{for } \Gamma \in \mathcal{F}^I_{h,m} \\ h_{K^{(L)}_\Gamma} & \text{for } \Gamma \in \mathcal{F}^B_{h,m} \end{cases}$. Other choices of h_Γ can be found in [7]. The choice of c_W is in each variant NIPG, IIPG and SIPG different, and it can vary from a small number to large. For more details about the choice of c_W see [8].

Since u is the regular solution, the first term of $J_{h,m}$ vanishes on any faces $\Gamma \in \mathcal{F}^I_{h,m}$. So

$$J_{h,m}(u(t), \psi(t)) \overset{(2.4)\&(1.1c)}{=} 0 + k_1 \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma \frac{c_W}{h_\Gamma} u_D(t)\psi(t)|_\Gamma\, \mathrm{d}S.$$

Finally, we define the form $l_{h,m}(w_1, w_2), w_1, w_2 \in H^2(\mathcal{T}_{h,m})$ representing the right-hand side of equation (2.6) after discretization, adding the artificial term and a penalty

$$\begin{aligned} l_{h,m}(u(t), \psi(t)) := \sum_{K \in \mathcal{T}_{h,m}} \int_K g(t)\psi(t)\, \mathrm{d}x + k_1 \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma \frac{c_W}{h_\Gamma} u_D(t)\psi(t)|_\Gamma\, \mathrm{d}S \\ + \theta \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma (\mathbb{K}(u(t))^T \nabla\psi(t))|^{(L)}_\Gamma \cdot \boldsymbol{n}_\Gamma u_D(t)\, \mathrm{d}S. \end{aligned} \tag{2.12}$$

Furthermore, we define

$$c_{h,m}(u, v) := a_{h,m}(u, v) + b_{h,m}(u, v) + J_{h,m}(u, v) - l_{h,m}(u, v), \quad u, v \in H^2(\mathcal{T}_{h,m}). \tag{2.13}$$

The form $c_{h,m}$ is non-linear with respect to its first argument and linear with respect to its second argument.

Discontinuous space semi-discretization is constructed in finite-dimensional space $S_{h,p,m} \subset H^2(\mathcal{T}_{h,m})$, and because of the approximation by numerical fluxes, the following holds:

$$(\frac{\partial u}{\partial t}(t), \psi(t)) + c_{h,m}(u(t), \psi(t)) = 0, \tag{2.14}$$

for a.e. $t \in I_m$, $u(t), \psi(t) \in S_{h,p,m}$, with u being a sufficiently regular solution.

2.4 Full space-time discretization

In what follows, we review the full space-time discontinuous Galerkin discretization of (1.4) (cf. [7], Chapter 4).
We will use $L^2(\Omega)$-projection on $S_{h,p,m}$, denoted by $\Pi_{h,m}$. In other words, if $\psi \in L^2(0,T;L^2(\Omega))$ then

$$\Pi_{h,m}\psi(t) \in S_{h,p,m} \text{ and } (\Pi_{h,m}\psi(t) - \psi(t), \varphi) = 0 \quad \forall\varphi \in S_{h,p,m},\ t \in I_m. \tag{2.15}$$

Let u be a sufficiently regular solution of problem (1.1) satisfying the conditions

$$u \in L^2(0,T;H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega)). \tag{2.16}$$

Then $u \in C(0,T;L^2(\Omega))$, so it is continuous with respect to the time coordinate t. We integrate (2.14) over the time slab I_m

$$\int_{I_m} (\frac{\partial u}{\partial t}(t), \psi(t)) + c_{h,m}(u(t),\psi(t))\,\mathrm{d}t = 0, \quad m = 1,\dots,r. \tag{2.17}$$

Using integration by parts with respect to the time and properties $u^+_{m-1} = u^-_{m-1}$, we have

$$\begin{aligned}\int_{I_m} (\frac{\partial u}{\partial t}(t), \psi(t))\,\mathrm{d}t &= -\int_{t_{m-1}}^{t_m} (u(t), \frac{\partial \psi}{\partial t}(t))\,\mathrm{d}t + (u^-_m, \psi^-_m) - (u^+_{m-1}, \psi^+_{m-1}) \\ &= -\int_{I_m} (u(t), \frac{\partial \psi}{\partial t}(t))\,\mathrm{d}t + (u^-_m, \psi^-_m) - (u^-_{m-1}, \psi^+_{m-1}) \\ &= \int_{I_m} (\frac{\partial u(t)}{\partial t}, \psi(t))\,\mathrm{d}t - (u^-_m, \psi^-_m) + (u^+_{m-1}, \psi^+_{m-1}) + (u^-_m, \psi^-_m) - (u^-_{m-1}, \psi^+_{m-1}) \\ &= \int_{I_m} (\frac{\partial u}{\partial t}(t), \psi(t))\,\mathrm{d}t + (\{u\}_{m-1}, \psi^+_{m-1}).\end{aligned}$$

Definition 2.1. *We say that function $u_{h\tau} \in S^{\tau,q}_{h,p}$ is the approximate (or space-time discrete) solution of problem* (1.1)*, if it satisfies*

$$\int_{I_m} \left((\frac{\partial u_{h\tau}}{\partial t}(t), \psi(t)) + c_{h,m}(u_{h\tau}(t), \psi(t)) \right) dt + (\{u_{h\tau}\}_{m-1}, \psi|^+_{m-1}) = 0, \quad \forall \psi \in S^{\tau,q}_{h,p},\ m = 1,\dots,r \tag{2.18a}$$

$$(u_{h\tau}|^-_0, \varphi) = (\Pi_{h,0}u^0, \varphi), \quad \forall\varphi \in S_{h,p,0} \tag{2.18b}$$

where $\Pi_{h,0}$ is the $L^2(\Omega)$-projection given by (2.15)*.*

Let $e_{h\tau} := u_{h\tau} - u$, where u is the sufficient regular solution and $u_{h\tau}$ the approximate solution of problem (1.1). Let semi-norms be given by (1.3) for Bochner spaces. In the broken Bochner space $L^2(\mathcal{I}_\tau; H^1(\mathcal{T}_h))$, we consider the following seminorm:

$$|e_{h\tau}|_{L^2(\mathcal{I}_\tau;H^1(\mathcal{T}_h))} = \left(\sum_{m=1}^{r} \int_{I_m} \sum_{K\in\mathcal{T}_{hm}} \int_K |\nabla u_{h\tau}(x,t) - \nabla u(x,t)|^2 \mathrm{d}x\mathrm{d}t \right)^{1/2}. \tag{2.19}$$

In [2], the following a priori error estimate was derived.

Theorem. *Let u be the exact solution of problem* (1.1) *satisfying the regularity conditions* $\|\nabla u(t)\|_{L^\infty(\Omega)} \leqslant C_R$ *for a.e.* $t \in (0,T)$, *a constant* $C_R > 0$ *and* $u \in H^{q+1}(0,T;H^1(\Omega)) \cap C([0,T];H^{p+1}(\Omega))$. *Let the mesh satisfy conditions (T1) and (T2), the time steps satisfy condition* $\forall m \in \{1,\dots,r\}$ $C_* h_m^2 \leqslant \tau_m \leqslant C^* k_1$ *for some constants* $C_*, C^* > 0$. *Then there exists a constant* $C_1 > 0$ *independent of* $h, \tau, m, r, u, u_{h\tau}$ *such that*

$$|e_{h\tau}|^2_{L^2(\mathcal{I}_\tau;H^1(\mathcal{T}_h))} \leqslant C_1(h^{2p}|u|^2_{C([0,T];H^{p+1}(\Omega))} + \tau^{2q+\gamma}|u|^2_{H^{q+1}(0,T;H^1(\Omega))}), \tag{2.20}$$

where

$$\gamma = \begin{cases} 0 & \textit{if } \tau_m \leqslant C_B h_{K_\Gamma^{(L)}} \textit{ for all } \Gamma \in \mathcal{F}^B_{hm} \textit{ and for a constant } C_B > 0, \\ 2 & \textit{if } u_D(x,t) = \sum_{i=0}^q t^i \psi_i(x) \textit{ for } \psi_i \in H^{p+1}(\partial\Omega), i = 0,\dots,q. \end{cases} \tag{2.21}$$

2.5 Linearization of the forms in the continuous problem

The equation (2.18a) represents a non-linear algebraic system because of the non-linearity of the forms $a_{h,m}, b_{h,m}, l_{h,m}$ with respect to their first argument. In order to solve this system, a Newton-like method was developed from the damped Newton method, where we replace Jacobi matrix by a suitable linearization. The linearization is presented as follows:

We define the *linearized diffusion form* $\tilde{a}_{h,m}(\overline{u}, u, \varphi)$, where $\overline{u}, u, \varphi \in H^2(\mathcal{T}_{h,m})$, based on (2.10) by

$$\begin{aligned} \tilde{a}_{h,m}(\overline{u}, u, \varphi) := & \sum_{K \in \mathcal{T}_{h,m}} \int_K (\mathbb{K}(\overline{u})\nabla u) \cdot \nabla\varphi \, \mathrm{d}x \\ & - \sum_{\Gamma \in \mathcal{F}^I_{h,m}} \int_\Gamma \langle \mathbb{K}(\overline{u})\nabla u\rangle \cdot \boldsymbol{n}[\varphi] + \theta\langle \mathbb{K}(\overline{u})^T \nabla\varphi\rangle \cdot \boldsymbol{n}[u] \, \mathrm{d}S \\ & - \sum_{\Gamma \in \mathcal{F}^B_{h,m}} \int_\Gamma (\mathbb{K}(\overline{u})\nabla u)|^{(L)} \cdot \boldsymbol{n}\varphi|^{(L)} + \theta(\mathbb{K}(\overline{u})^T \nabla\varphi|^{(L)} \cdot \boldsymbol{n}u|^{(L)} \, \mathrm{d}S. \end{aligned} \tag{2.22}$$

Based on (2.9) there are two parts to linearize: 1) function f_s inside a space element K and 2) the numerical flux.

In element $K \in \mathcal{T}_{h,m}$, convective term $f_s, s = 1,2$ will be approximated as the Euler flux, i.e.

$$f_s(u) \approx A_s(u)u, \text{ where } A_s(u) = \frac{\partial f_s}{\partial u}(u). \tag{2.23}$$

The numerical flux H is used as follows:

$$H(u^{(L)}, u^{(R)}, \boldsymbol{n}) = \begin{cases} \sum_{s=1}^2 f_s(u^{(L)})n_s & \text{if } \sum_{s=1}^2 A_s(\langle u\rangle)n_s > 0, \\ \sum_{s=1}^2 f_s(u^{(R)})n_s & \text{if } \sum_{s=1}^2 A_s(\langle u\rangle)n_s \leqslant 0. \end{cases} \tag{2.24}$$

Let denote Γ^+ the set of faces $\Gamma \in \mathcal{F}^I_{hm}$ satisfying the condition $\sum_{s=1}^2 A_s(\langle u\rangle)n_s > 0$ and Γ^- the set of faces satisfying the second condition in (2.24). Therefore we define the *linearized convective form* for $\overline{u}, u, \varphi \in H^2(\mathcal{T}_{h,m})$ as follows:

$$\tilde{b}_{h,m}(\overline{u}, u, \varphi) := -\sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^{2} A_s(\overline{u})u\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}^B_{hm}} \int_\Gamma \sum_{s=1}^{2} A_s(\overline{u})u^{(L)}n_s\varphi^{(L)}\,\mathrm{d}S$$
$$+ \sum_{\Gamma\in\Gamma^+} \int_\Gamma \sum_{s=1}^{2} A_s(\overline{u})u^{(L)}n_s[\varphi]\,\mathrm{d}S + \sum_{\Gamma\in\Gamma^-} \int_\Gamma \sum_{s=1}^{2} A_s(\overline{u})u^{(R)}n_s[\varphi]\,\mathrm{d}S. \quad (2.25)$$

and

$$\tilde{c}_{h,m}(\overline{u}, u, \varphi) := \tilde{a}_{h,m}(\overline{u}, u, \varphi) + \tilde{b}_{h,m}(\overline{u}, u, \varphi) + J_{h,m}(u, \varphi),\ \overline{u}, u, \varphi \in H^2(\mathcal{T}_{h,m}). \quad (2.26)$$

The semi-linearized form $\tilde{c}_{h,m}$ is non-linear with respect to its first argument and linear with respect to its second and third arguments. Due to approximations (2.23) and (2.24) it holds

$$\tilde{c}_{h,m}(u, u, \varphi) - l_{h,m}(u, \varphi) \approx c_{h,m}(u, \varphi)$$

We use the convention that for $u_{h\tau}, \psi \in S^{\tau,q}_{h,p}$, we write $c_{h,m}(u_{h\tau}, u_{h\tau}, \psi)$ instead of $c_{h,m}(u_{h\tau}(t), u_{h\tau}(t), \psi(t))$. Equation (2.18a) is now in the form

$$\int_{I_m} \left((\frac{\partial u_{h\tau}}{\partial t}, \psi) + \tilde{c}_{h,m}(u_{h\tau}, u_{h\tau}, \psi) - l_{h,m}(u_{h\tau}, \psi) \right) \mathrm{d}t + (\{u_{h\tau}\}_{m-1}, \psi|^+_{m-1}) = 0,$$
$$\forall\psi \in S^{\tau,q}_{h,p},\ m = 1, \ldots, r. \quad (2.27)$$

2.6 Solution of the continuous problem

In this section, we review basis functions in the finite dimensional space $S^{\tau,q}_{h,p,m}$, $m = 1, \ldots, r$, and the Newton-like method, which solves (2.18a).

2.6.1 Basis functions

Let $m \in \{1, \ldots, r\}$ be arbitrary but fixed. Let $K_\mu \in \mathcal{T}_{h,m}, \mu \in E_m$, be arbitrary but fixed. We consider a space-time element $Q_{m,\mu} = K_\mu \times I_m$.

Let p denote the degree of the space polynomial on the element K_μ, q denote the degree of the time polynomial on $Q_{m,\mu}$. In this thesis, we use the same degree p for all elements in $\mathcal{T}_{h,m}$ and the same degree q for all the time slabs $I_m, m = 1, \ldots, r$.

Let $\overline{p} =$ dim $P_p(K_\mu)$, $\{\varphi^{m,\mu}_j\}^{\overline{p}}_{j=1}$ be the basis of the space $P_p(K_\mu)$. We note that the basis is constructed on the reference element $\hat{K}$ and then transformed to a basis on the element K_μ. In our case of polygonal Ω, this transformation is affine. Basis functions $\varphi^{m,\mu}_j$ used here are after the transformation. Let $\{\phi^{m,\mu}_j\}^q_{j=0}$ be the basis of the space $P^q(I_m)$.

Let us define basis functions on space $P_p(K_\mu) \times P^q(I_m)$. For $l = 0, \ldots, q,\ k = 1, \ldots, \overline{p}$, we put

$$\psi^{m,\mu}_{kl}(x, t) := \begin{cases} \varphi^{m,\mu}_k(x)\phi^{m,\mu}_l(t) & \text{for } (x, t) \in Q_{m,\mu} \\ 0 & \text{otherwise} \end{cases} \quad (2.28)$$

and $\psi_s^{m,\mu} := \psi_{kl}^{m,\mu}$, $s = l\overline{p} + k$. This ordering gives us the basis in the form

$$\begin{array}{c}\phi_0^{m,\mu}\varphi_1^{m,\mu}, \dots, \phi_0^{m,\mu}\varphi_{\overline{p}}^{m,\mu},\\ \phi_1^{m,\mu}\varphi_1^{m,\mu}, \dots, \phi_1^{m,\mu}\varphi_{\overline{p}}^{m,\mu},\\ \vdots \\ \phi_q^{m,\mu}\varphi_1^{m,\mu}, \dots, \phi_q^{m,\mu}\varphi_{\overline{p}}^{m,\mu}.\end{array}$$

The set of basis functions

$$B_{m,\mu} = \{\psi_s^{m,\mu},\ s = 1, \dots, (q+1)\overline{p}\} \quad \mu \in E_m \tag{2.29}$$

generates the space $P_p(K_\mu) \times P^q(I_m)$ on one space-time element $Q_{m,\mu}$ with dimension $\dim_{m,\mu}$. Let

$$B_m = \cup_{\mu \in E_m} B_{m,\mu}.$$

Then B_m forms the basis of $S_{h,p,m}^{\tau,q}$.

$$\begin{aligned}\dim_{m,\mu} &= (q+1)\overline{p},\\ \dim_m := \dim S_{h,p,m}^{\tau,q} &= \sum_{\mu \in E_m} \dim_{m,\mu},\\ \dim S_{h,p}^{\tau,q} &= \sum_{m=1}^{r} \dim_m.\end{aligned} \tag{2.30}$$

Consider $B_m = \{\psi_i^m, i = 1, \dots, \dim_m\}$. For $\psi_i^m \in B_m$ we use the convention that If $(i \bmod \overline{p}(q+1)) = 0$ then

$$\mu = \frac{i}{\overline{p}(q+1)},\ s = \overline{p}(q+1),\ k = \overline{p},\ l = q$$

else

$$\mu = \lfloor \frac{i}{\overline{p}(q+1)} \rfloor + 1,\ s = i \bmod \overline{p}(q+1),\ k = s \bmod \overline{p},\ l = \frac{s-k}{\overline{p}},$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Using this convention we have

$$\psi_i^m(x,t) = \psi_s^{m,\mu}(x,t) = \varphi_k^{m,\mu}(x)\phi_l^{m,\mu}(t), \quad (x,t) \in K_\mu \times I_m. \tag{2.32}$$

This ordering gives us the basis B_m in a form

$$\begin{array}{c}\phi_0^{m,1}\varphi_1^{m,1}, \dots, \phi_q^{m,1}\varphi_{\overline{p}}^{m,1},\\ \phi_0^{m,2}\varphi_1^{m,2}, \dots, \phi_q^{m,2}\varphi_{\overline{p}}^{m,2},\\ \vdots \\ \phi_0^{m,N_m}\varphi_1^{m,N_m}, \dots, \phi_q^{m,N_m}\varphi_{\overline{p}}^{m,N_m}.\end{array}$$

For example, with $\langle M \rangle$ denoting the linear span of set M, $K_\mu \subset \mathbb{R}^2$, we have

$$\begin{array}{ll} q = 2 & \{\phi_l\}_{l=0}^2 = \{1, t, t^2\},\\ p = 2 & \langle \{\varphi_k\}_{k=1}^6 \rangle = \langle \{1, x_1, x_2, x_1x_2, x_1^2, x_2^2\} \rangle.\end{array}$$

2.6.2 Newton-like method

Let $u_{h\tau} \in S_{h,p}^{\tau,q}$ have the form

$$u_{h\tau}(x,t) = \sum_{m=1}^{r} u_{h\tau}^m(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{2.33}$$

where

$$u_{h\tau}^m = \begin{cases} u_{h\tau}|_{\Omega\times I_m} \in S_{h,p,m}^{\tau,q} & \text{for } (x,t) \in \Omega \times I_m, \\ 0 & \text{for } (x,t) \notin \Omega \times I_m. \end{cases} \tag{2.34}$$

We can express $u_{h\tau}^m$ in the basis B_m by

$$u_{h\tau}^m(x,t) = \sum_{i=1}^{\dim_m} u_i^m \psi_i^m(x,t) = \sum_{\mu=1}^{N_m}\sum_{k=1}^{\overline{p}}\sum_{l=0}^{q} u_{kl}^{m\mu}\varphi_k^{m,\mu}(x)\phi_l^{m,\mu}(t). \tag{2.35}$$

There is an isomorphism between $u_{h\tau}^m \in S_{hp}^{mq} \longleftrightarrow \{u_i^m\}_{i=1}^{\dim_m} =: U^m \in \mathbb{R}^{\dim_m}$.

Equation (2.27) is then equivalent with

$$\int_{t_{m-1}}^{t_m} \Big((\frac{\partial u_{h\tau}^m}{\partial t}, \psi) + \tilde{a}_{h,m}(u_{h\tau}^m, u_{h\tau}^m, \psi) + \tilde{b}_{h,m}(u_{h\tau}^m, u_{h\tau}^m, \psi)$$
$$+ J_{h,m}(u_{h\tau}^m, \psi) - l_{h,m}(u_{h\tau}^m, \psi) \Big) \,\mathrm{d}t + (u_{h\tau}^m(t_{m-1}^+), \psi(t_{m-1}^+)) - (u_{h\tau}^m(t_{m-1}^-), \psi(t_{m-1}^+)) = 0,$$
$$\forall \psi \in S_{h,p,m}^{\tau,q}, \quad m = 1, \dots, r \tag{2.36}$$

Now we describe how to obtain a system of linear equations
The terms $(\frac{\partial u_{h\tau}^m}{\partial t}, \psi)$, $J_{h,m}$ and $(u_{h\tau}^m(t_{m-1}^+), \psi(t_{m-1}^+))$ are linear with respect to $u_{h\tau}^m$, so we treat $u_{h\tau}^m$ implicitly as an unknown function.
The terms $\tilde{a}_{h,m}, \tilde{b}_{h,m}$ are non-linear with respect to the first argument and linear with respect to second and third arguments, so we use a known value in first argument, and treat second implicitly.
The term $l_{h,m}$ is non-linear with respect to the first argument, so $u_{h\tau}^m$ will be explicitly treated.
In the last term, we use the value of the previous result and put

$$u_{h\tau}^m(t_{m-1}^-) := u_{h\tau}^{m-1}(t_{m-1}^-),$$
$$u_{h\tau}^1(t_0^-) := \Pi_{h,0} u^0.$$

Equation (2.36) is equivalent to a system of equations where we replace arbitrary testing function $\psi \in S_{h,p,m}^{\tau,q}$ in (2.36) with all basis functions $\psi_j^m \in B_m$. Thus

$$\sum_{i=1}^{\dim_m} u_i^m c_{ij}^m(U^m) = q_j^m(U^m), \quad j = 1, \dots, \dim_m, \tag{2.37}$$

where

$$c_{ij}^m(\{u_k^m\}_k) := \int_{t_{m-1}}^{t_m} \Big((\psi_i', \psi_j) + \tilde{c}_{h,m}(\sum_k u_k^m \psi_k^m, \psi_i^m, \psi_j^m) \Big) \,\mathrm{d}t + (\psi_i^m(t_{m-1}^+), \psi_j^m(t_{m-1}^+)),$$
$$q_j^m(\{u_k^m\}_k) := \int_{t_{m-1}}^{t_m} l_{h,m}(\sum_k u_k^m \psi_k^m, \psi_j^m) \,\mathrm{d}t + (u_{h\tau}^{m-1}(t_{m-1}^-), \psi_j^m(t_{m-1}^+)).$$

We define functions

$$F_j^m(U^m) := \sum_{i=1}^{\dim_m} u_i^m c_{ij}^m(U^m) - q_j^m(U^m), \quad j = 1, \dots, \dim_m. \tag{2.38}$$

For simplicity, we write

$$\mathbb{C}^m(U^m) = \{c_{ij}(U^m)\}_{i,j=1}^{\dim_m},\ Q^m(U^m) = \{q_j^m(U^m)\}_{j=1}^{\dim_m},\ F^m(U^m) = \{F_j^m(U^m)\}_{j=1}^{\dim_m}. \tag{2.39}$$

Then

$$F^m(U^m) = \mathbb{C}^m(U^m)U^m - Q^m(U^m), \quad m = 1, \dots, r \tag{2.40}$$

and we obtain the system of non-linear algebraic equations

$$F^m(U^m) = 0. \tag{2.41}$$

In virtue of (2.27), (2.36), (2.41), problem (2.18) reads

$$\begin{aligned} &(1)\ u_{h\tau}^m(t_{m-1}^-) = u_{h\tau}^{m-1}(t_{m-1}^-), \quad m = 1, \dots, r \\ &(2)\ \text{find } U^m \in \mathbb{R}^{\dim_m} : F^m(U^m) = 0, \quad m = 1, \dots, r \end{aligned} \tag{2.42}$$

The idea is to solve instead of the system of non-linear equations $\mathbb{C}^m(U^m)U^m = Q^m(U^m)$, the system of linear equations $\mathbb{C}^m(\overline{U})U^m = Q^m(\overline{U})$ with $\overline{U}$ is known. In particular, we proceed as follows:
To solve (2.41), we use a damped Newton-like method which generates a sequence of approximations $U^{(n)}, n = 0, 1, 2, \dots$ to the actual numerical solution U^m using the following algorithm:

$$\text{find } d^n \in \mathbb{R}^{\dim_m} \text{ such that } \mathbb{C}^m(U^{(n)})d^n = -F^m(U^{(n)}) \tag{2.43a}$$

$$U^{(n+1)} := U^{(n)} + \lambda^n d^n \tag{2.43b}$$

where $\mathbb{C}^m$ is defined by (2.39) and we use the matrix $\mathbb{C}^m(U)$ to approximate the Jacobi matrix of $F^m(U)$ in the damped Newton method, where $\lambda^n \in (0, 1]$ is the selected damping parameter. Let $\tilde{U}^m$ be the result of the iteration process (2.43). At the beginning of the iteration process, we put $\lambda^n = 1$ and evaluate a monitoring function $\delta^n := \|F(U^{(n+1)})\|/\|F(U^{(n)})\|$. If $\delta^n < 1$, we proceed to the next Newton iteration. Otherwise, we set $\lambda^n := \lambda^n/2$ and repeat the current Newton iteration. This setting is for the case when an initial value $U^{(0)}$ is far from the solution U^m. When $U^{(n)}$ is close to U^m, then $\lambda^n = 1$.

We have several approximations (1.1) ≈ (2.18) ≈ (2.43). We obtain the solution

$$\tilde{u}_{h\tau}(x, t) = \sum_{m=1}^{r} \tilde{u}_{h\tau}^m(x, t), \text{ where } \tilde{u}_{h\tau}^m \longleftrightarrow \tilde{U}^m,\ m = 1, \dots, r$$

which approximates the approximate solution $u_{h\tau}$ of (2.18).

2.7 Discussion

We have so far presented the theory of STDG method, but many questions remain for the numerical implementation. In the numerical implementation, a space $S_{hp}^{\tau q}$, in which the approximate solution is sought, has to be specified. The polynomial degrees p, q, time step τ_m and spatial mesh $\mathcal{T}_{hm}$ can be fixed or they can be adaptively changed.
To create an algorithm we need to make choices, and we want to propose an algorithm such that satisfies the properties outlined in Introduction. And so we ask:

Should the polynomial degrees p, q be constant across all elements as described previously or should they be adaptive based on a setting? The p-adaptation can be seen at work in [9].

Should the time step τ_m be fixed at the beginning or selected adaptively? If adaptively, how should we change it?

When to re-mesh the time-dependent spatial mesh $\mathcal{T}_{hm}$? The h-adaptation can be seen in [9] and [11].

How should the iteration process (2.43) terminate? Once it falls under a set tolerance or based on some other criteria?

To answer these questions, the next chapter defines a posteriori error estimates, and Chapter 4 proposes an algorithm that uses these estimates to accomplish the following:

- Decide if the time step τ_m is accepted or if not, how it will be selected,
- Decide if the mesh $\mathcal{T}_{h,m}$ is accepted or needs to be refined,
- Stop the iteration process.

3. Residual estimation

A common approach to stopping an algorithm is to stop once the difference Dif of two approximate solutions in a suitable norm falls under a certain tolerance ω, where the second approximate solution is computed on a more refined spatial mesh. If Dif $> \omega$, then we have to decide whether to change the mesh discretization or the time step. Maybe, we can also compute the difference of two approximate solutions, where the second is computed under a more refined spatial mesh to decide for spatial refinement, or the second is computed with a higher degree of q with respect to the time to decide for the time refinement. The difference of two approximate solutions requires the evaluation, namely a Newton-like method, to repeat many times. Here, we follow this common approach to develop a posteriori estimates based on the measurement of the residuum in the dual norms.

We review the definition of residual measures and residual estimators in Section 3.1, 3.2 (cf. [10]). We verify certain desirable properties of residual estimators in Section 3.4. The residual estimators together with certain criteria are then used as estimates to guide the changes in the time step and the spatial step.

We start by introducing spaces that we will use throughout:

$$H^1(\mathcal{I}_\tau; S_{h,p}) := \{\psi \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h));\ \psi(.,t) \in S_{h,p,m} \text{ for a.e. } t \in I_m, m = 1, \dots, r\}$$
$$S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h)) := \{\psi \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h));\ \psi(x,.) \in P^q(\mathcal{I}_\tau) \text{ for a.e. } x \in \Omega\}$$

The space $H^1(\mathcal{I}_\tau; S_{h,p})$ consists of piecewise-polynomial functions of degree $\leqslant p$ with respect to the spatial coordinates while $S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ consists of piecewise-polynomial functions of degree $\leqslant q$ with respect to the time coordinate.
Finally, let $S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$ consist of piecewise-polynomial functions with respect to the both spatial and time coordinates.
Let $A_{h,m}$ denote the form obtained from (2.18):

$$A_{hm}(u,\psi) := \int_{I_m} \left((\frac{\partial u}{\partial t}, \psi) + c_{h,m}(u,\psi) \right) \mathrm{d}t + (\{u\}_{m-1}, \psi|^{+}_{m-1})$$
$$\forall u, \psi \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h)),\ m = 1, \dots, r, \quad (3.1a)$$

$$u_0^- := u^0. \quad (3.1b)$$

and

$$A_{h\tau}(u,\psi) := \sum_{m=1}^{r} A_{hm}(u,\psi). \quad (3.2)$$

In Figure 3.1, the approximate solution $u_{h\tau}$ can be obtained by space-discretization and then full space-time discretization, as presented in Chapter 2. Alternatively, we can derive it by first doing the time-discretization and then the full time-space discretization. We recall the definition of solutions $u, u_h, u_\tau, u_{h\tau}, \tilde{u}_{h\tau}$ as follows:
First, let $u \in H^1(0,T; H^2(\Omega))$ formally denote the sufficiently regular solution of (1.1), then equations (2.14), (3.1), and (3.2) yield

$$A_{h\tau}(u,\psi) = 0 \quad \forall \psi \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h)). \quad (3.3)$$

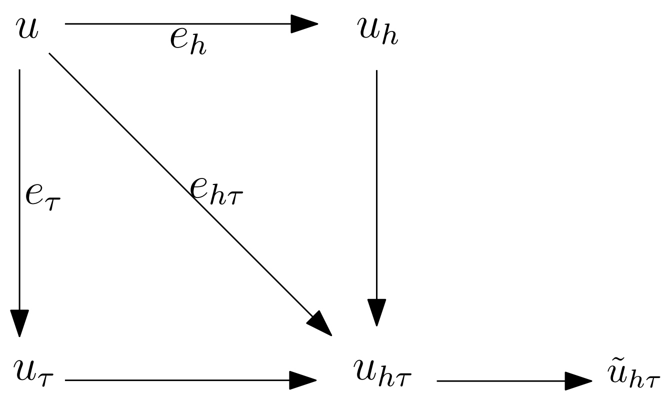

Figure 3.1: Two ways of the discretization for the approximate solution $u_{h\tau}$.

Second, let $u_h \in H^1(\mathcal{I}_\tau; S_{h,p})$ be the solution obtained from (2.14), i.e.

$$
\begin{aligned}
&(\frac{\partial u_h}{\partial t}(t), \psi(t)) + c_{h,m}(u_h(t), \psi(t)) = 0 \; \forall \psi \in H^1(I_m; S_{h,p,m}) \; \forall t \in I_m, \; m = 1, \ldots, r, \\
&(u_h|^+_{m-1}, \varphi) = (u_h|^-_{m-1}, \varphi) \quad \forall \varphi \in S_{h,p,m}.
\end{aligned}
\tag{3.4}
$$

In other words, we solve equation (1.1) in $H^1(I_m)$ with respect to the time variable t and in $S_{h,p,m}$ with respect to the spatial variable x, and so u_h formally gives the exact solution with respect to the time coordinate and approximate solution with respect to the spatial coordinates. We call u_h the *space semi-discrete* solution of (1.1); this solution has the property

$$A_{h\tau}(u_h, \psi) = 0 \quad \forall \psi \in H^1(\mathcal{I}_\tau; S_{h,p}). \tag{3.5}$$

Third, we denote by $u_\tau \in S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ the solution of

$$A_{hm}(u_\tau, \psi) = 0 \quad \forall \psi \in S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h)). \tag{3.6}$$

In this case, we solve equation (1.1) in $S^{\tau,q}(\mathcal{I}_\tau)$ with respect to the time variable and in $H^2(\mathcal{T}_h)$ with respect to the spatial variable. Formally, u_τ gives the solution discretely in the time coordinate and exactly in the spatial coordinates. We call u_τ the *time semi-discrete* solution of (1.1); it has the property

$$A_{h\tau}(u_\tau, \psi) = 0 \quad \forall \psi \in S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h)). \tag{3.7}$$

Fourth, let $u_{h\tau} \in S^{\tau,q}_{h,p} = S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$ be the approximate solution of (1.1) given by (2.18). Then

$$A_{h\tau}(u_{h\tau}, \psi) = 0 \quad \forall \psi \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}). \tag{3.8}$$

Last, let $\tilde{u}_{h\tau}$ be the approximate solution computed by the iterative method (2.43). Then in general for $\psi \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}) \quad A_{h\tau}(\tilde{u}_{h\tau}, \psi) \approx 0$.

We summarise the above-defined solutions of the equation (1.1):

- exact solution $u \in H^1(0, T; H^2(\Omega)) \subset H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ satisfying (3.3),
- space semi-discrete solution $u_h \in H^1(\mathcal{I}_\tau; S_{h,p})$ satisfying (3.5),
- time semi-discrete solution $u_\tau \in S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ satisfying (3.7),

- approximate (or space-time discrete) solution $u_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$ satisfying (3.8),
- computed approximate solution $\tilde{u}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$.

3.1 Residual measures

Let $V := H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$, V' be its dual space, and $\langle \cdot, \cdot \rangle$ denote the duality between V and V'. Let $A : V \to V'$ such that $\langle Aw, \cdot \rangle = A_{h\tau}(w, \cdot)$, $w \in V$. The residual measure in the dual norm of space V is given by

$$\|Aw - Av\|_{V'} := \sup_{\psi \in V, \psi \neq 0} \frac{\langle Aw - Av, \psi \rangle}{\|\psi\|_V} = \sup_{\psi \in V, \psi \neq 0} \frac{A_{h\tau}(w, \psi) - A_{h\tau}(v, \psi)}{\|\psi\|_V} \quad \forall w, v \in V. \tag{3.9}$$

We define residual measures as follows:
The *space-time algebraic residual measure* is defined as the difference in the dual norm of the space $H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ between $\tilde{u}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$ (the computed approximate solution) and $u \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ (the regular strong solution):

$$\begin{aligned} \mathcal{E}_{STA}(\tilde{u}_{h\tau}) &:= \sup_{\psi \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h)), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi) - A_{h\tau}(u, \psi)}{\|\psi\|_{H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))}} \\ &\overset{(3.3)}{=} \sup_{\psi \in H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h)), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))}}. \end{aligned} \tag{3.10}$$

The *time algebraic residual measure* is defined as the difference between $\tilde{u}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$ (which is discrete in time) and $u_h \in H^1(\mathcal{I}_\tau; S_{hp})$ (which is exact in time) in dual norm of $H^1(\mathcal{I}_\tau; S_{hp})$

$$\begin{aligned} \mathcal{E}_{TA}(\tilde{u}_{h\tau}) &:= \sup_{\psi \in H^1(\mathcal{I}_\tau; S_{hp}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi) - A_{h\tau}(u_h, \psi)}{\|\psi\|_{H^1(\mathcal{I}_\tau; S_{hp})}} \\ &\overset{(3.5)}{=} \sup_{\psi \in H^1(\mathcal{I}_\tau; S_{hp}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{H^1(\mathcal{I}_\tau; S_{hp})}}. \end{aligned} \tag{3.11}$$

The *space algebraic residual measure* is defined as the difference between the approximate solution $\tilde{u}_{h\tau}$ (which is discrete in space) and the time semi-discrete solution $u_\tau \in S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ (which is exact in space) in the dual norm of $S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$

$$\begin{aligned} \mathcal{E}_{SA}(\tilde{u}_{h\tau}) &:= \sup_{\psi \in S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h)), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi) - A_{h\tau}(u_\tau, \psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))}} \\ &\overset{(3.7)}{=} \sup_{\psi \in S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h)), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))}}. \end{aligned} \tag{3.12}$$

The *algebraic residual measure* is defined as the difference between $\tilde{u}_{h\tau}$ and $u_{h\tau}$ in the dual norm of $S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$

$$\begin{aligned} \mathcal{E}_{A}(\tilde{u}_{h\tau}) &:= \sup_{\psi \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi) - A_{h\tau}(u_{h\tau}, \psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})}} \\ &\overset{(3.8)}{=} \sup_{\psi \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})}}. \end{aligned} \tag{3.13}$$

For the residual measures, the following properties hold: For u being the exact solution, we have

$$\mathcal{E}_A(u) = \mathcal{E}_{TA}(u) = \mathcal{E}_{SA}(u) = \mathcal{E}_{STA}(u) = 0. \tag{3.14}$$

Since $S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}) \subset H^1(\mathcal{I}_\tau; S_{h,p}) \subset H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$ and $S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}) \subset S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h)) \subset H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$, it holds for $\vartheta_{h\tau} \in S_{hp}^{\tau q}$

$$\mathcal{E}_A(\vartheta_{h\tau}) \leqslant \mathcal{E}_{TA}(\vartheta_{h\tau}) \leqslant \mathcal{E}_{STA}(\vartheta_{h\tau}), \tag{3.15}$$

$$\mathcal{E}_A(\vartheta_{h\tau}) \leqslant \mathcal{E}_{SA}(\vartheta_{h\tau}) \leqslant \mathcal{E}_{STA}(\vartheta_{h\tau}). \tag{3.16}$$

In what follows, we present the relation between the estimate $e_{h\tau}$ and the residual measure $\mathcal{E}_{AST}$. For simplicity, we consider the relation for a linear problem.

We consider the following problem:

$$\begin{aligned} u' - \Delta u &= f \quad \text{on } \Omega \times (0,T), \\ u(x,t) &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u^0 \quad x \in \Omega. \end{aligned}$$

The solution is sought in the continuous Bochner space, defined as $Y := \{\psi \in L^2(0,T; H_0^1(\Omega));\ \psi' \in L^2(0,T; H^{-1});\ \psi(T) = 0\}$, where H^{-1} is the dual space of $H_0^1(\Omega)$. Then the residual measure for $u_{h\tau}, u \in L^2(0,T; H_0^1(\Omega))$ is the following:

$$\mathcal{E}_{STA}(u_{h\tau}) = \sup_{0 \neq \psi \in Y} \frac{\int_0^T (-e_{h\tau}, \psi') + (\nabla e_{h\tau}, \nabla\psi)\,\mathrm{d}t}{\|\psi\|_Y}. \tag{3.17}$$

Lemma 3.1. *Let* $\|\psi\|_Y := \|\psi\|_{L^2(0,T;H_0^1(\Omega))} + \|\psi'\|_{L^2(0,T;H^{-1})}$. *Then*

$$\mathcal{E}_{STA} \leqslant \|e_{h\tau}\|_{L^2(0,T;H_0^1(\Omega))} \leqslant 3\mathcal{E}_{STA}. \tag{3.18}$$

Proof. See [1]. □

We extend the previous example to the problem

$$\begin{aligned} u' - \Delta u &= f \quad \text{on } \Omega \times (0,T), \\ u(x,t) &= u_D \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u^0 \quad \forall x \in \Omega. \end{aligned} \tag{3.19}$$

with the boundary function u_D as (D4) and the solution sought in the broken Bochner space $V = H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$, with the space-time discretization of the IIPG variant. Applying the space-time discretization as presented in Chapter 2, for $u \in H^1(0,T; H^2(\Omega))$ and $\psi \in H^1(\mathcal{I}_\tau; H^1(\mathcal{T}_h))$, the general forms now look like:

$$\begin{aligned}
a_{hm}(u,\psi) &:= \sum_i \int_{K_i} \nabla u \cdot \nabla\psi \,\mathrm{d}x - \sum_{\Gamma\in\mathcal{F}^I} \int_\Gamma \langle\nabla u\rangle \cdot \boldsymbol{n}[\psi]\,\mathrm{d}S - \sum_{\Gamma\in\mathcal{F}^B} \int_\Gamma \nabla u \cdot \boldsymbol{n}\psi\,\mathrm{d}S, \\
J_{hm}(u,\psi) &:= \sum_{\Gamma\in\mathcal{F}^I} \sigma_\Gamma \int_\Gamma [u][\psi]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B} \sigma_\Gamma \int_\Gamma u\psi\,\mathrm{d}S, \quad \text{where } \sigma_\Gamma = \frac{c_W}{h_\Gamma}, \\
l_{hm}(\psi) &:= \int_\Omega f\psi\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}^B} \sigma_\Gamma \int_\Gamma u_D\psi\,\mathrm{d}S, \\
c_{hm}(u,\psi) &:= a_{hm}(u,\psi) + J_{hm}(u,\psi) - l_{hm}(\psi), \\
A_{hm}(u,\psi) &:= \int_{I_m} ((u',\psi) + c_{hm}(u,\psi))\,\mathrm{d}t + (\{u\}_{m-1}, \psi_{m-1}^+).
\end{aligned}$$

We use the following manipulations

$$
\begin{aligned}
&\int_{t_{m-1}}^{t_m} (u', \psi)\,\mathrm{d}t + (\{u\}_{m-1}, \psi_{m-1}^+) \\
&\quad = -\int_{t_{m-1}}^{t_m} (u, \psi')\,\mathrm{d}t + (u_m^-, \psi_m^-) - (u_{m-1}^+, \psi_{m-1}^+) + (u_{m-1}^+ - u_{m-1}^-, \psi_{m-1}^+) \\
&\qquad\qquad = \int_{t_{m-1}}^{t_m} (-u, \psi')\,\mathrm{d}t + (u_m^-, \psi_m^-) - (u_{m-1}^-, \psi_{m-1}^+). \qquad (3.20)
\end{aligned}
$$

For the exact solution $u \in H^1(0,T;H^2(\Omega))$ and the approximate solution $u_{h\tau} \in S_{hp}^{\tau q}$, we have

$$
\begin{aligned}
A_{hm}(u_{h\tau}, \psi) - A_{hm}(u, \psi) &= \int_{I_m} (-u_{h\tau} + u, \psi')\,\mathrm{d}t + ((u_{h\tau} - u)|_m^-, \psi_m^-) \\
&\quad - ((u_{h\tau} - u)|_{m-1}^-, \psi_{m-1}^+) + \int_{I_m} (a_{hm}(u_{h\tau} - u, \psi) + J_{hm}(u_{h\tau} - u, \psi))\,\mathrm{d}t \\
= \int_{I_m} ((-e_{h\tau}, \psi') + a_{hm}(e_{h\tau}, \psi) &+ J_{hm}(e_{h\tau}, \psi))\,\mathrm{d}t + (e_{h\tau}|_m^-, \psi_m^-) - (e_{h\tau}|_{m-1}^-, \psi_{m-1}^+).
\end{aligned} \tag{3.21}
$$

Then, we obtain the residual measure

$$
\begin{aligned}
&\mathcal{E}_{STA}(u_{h\tau}) \\
&\quad = \sup \left\{ \frac{1}{\|\psi\|_{H^1(\mathcal{I}_\tau;H^1(\mathcal{T}_h))}} \sum_m \left(\int_{I_m} (-e_{h\tau}, \psi') + a_{hm}(e_{h\tau}, \psi) + J_{hm}(e_{h\tau}, \psi)\ \mathrm{d}t \right.\right. \\
&\qquad\qquad \left.\left. + (e_{h\tau}|_m^-, \psi_m^-) - (e_{h\tau}|_{m-1}^-, \psi_{m-1}^+) \right), 0 \neq \psi \in H^1(\mathcal{I}_\tau; H^1(\mathcal{T}_h)) \right\}. \qquad (3.22)
\end{aligned}
$$

In the following, we employ:
Discrete Cauchy inequality Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be two sequences of real numbers. Then

$$
\left| \sum_{i=1}^n a_i b_i \right| \leqslant \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{i=1}^n b_i^2 \right)^{1/2}. \tag{3.23}
$$

Cauchy–Schwarz inequality Let $M \subset R^2$ be a Lebesgue-measurable set, $f, g \in L^2(M)$. Then

$$
\left| \int_M fg\,\mathrm{d}x \right| \leqslant \left(\int_M f^2\,\mathrm{d}x \right)^{1/2} \left(\int_M g^2\,\mathrm{d}x \right)^{1/2}. \tag{3.24}
$$

We also define

$$
\|\nabla_h \psi\|_{L^2(I_m;L^2(\Omega))} := \left(\int_{I_m} \sum_i \int_{K_i} |\nabla \psi|^2\,\mathrm{d}x\mathrm{d}t \right)^{1/2} \qquad \forall \psi \in V. \tag{3.25}
$$

Now, consider the following:

Lemma 3.2. *For the problem* (3.19), *we have*

$$\mathcal{E}_{STA}(u_{h\tau}) \leqslant \|e_{h\tau}\|_V, \tag{3.26}$$

where

$$\|\psi\|^2_{H^1(\mathcal{I}_\tau;H^1(\mathcal{T}_h))} := \sum_m \Bigg(\|\psi'\|^2_{L^2(I_m;L^2(\Omega))} + \|\nabla_h \psi\|^2_{L^2(I_m;L^2(\Omega))} + \int_{I_m} J_{hm}(\psi,\psi)\ dt \\ + \|\psi^-_m\|^2_{L^2(\Omega)} + \|\psi^+_{m-1}\|^2_{L^2(\Omega)} \Bigg), \tag{3.27}$$

$$\|e_{h\tau}\|^2_V := \sum_m \Bigg(\|e_{h\tau}\|^2_{L^2(I_m;L^2(\Omega))} + \|\nabla_h e_{h\tau}\|^2_{L^2(I_m;L^2(\Omega))} + \int_{I_m} J_{hm}(e_{h\tau}, e_{h\tau})\,\mathrm{d}t \\ + \int_{I_m} \sum_{\Gamma\in\mathcal{F}^{IB}_{hm}} \int_\Gamma \sigma_\Gamma^{-1}(\langle \nabla e_{h\tau}\rangle \cdot \boldsymbol{n})^2\,\mathrm{d}S\mathrm{d}t + \|e_{h\tau}|^-_m\|^2_{L^2(\Omega)} + \|e_{h\tau}|^-_{m-1}\|^2_{L^2(\Omega)} \Bigg). \tag{3.28}$$

Proof. For the first term of (3.21) we have

$$\int_{I_m} (-e_{h\tau}, \psi')\,\mathrm{d}t = \int_{I_m} \sum_i \int_{K_i} -e_{h\tau}\psi'\,\mathrm{d}x\mathrm{d}t = \sum_i \int_{I_m\times K_i} -e_{h\tau}\psi'\,\mathrm{d}x\mathrm{d}t \\ \overset{(3.24)}{\leqslant} \sum_i \left(\int_{I_m\times K_i} e^2_{h\tau}\mathrm{d}x\mathrm{d}t\right)^{1/2} \left(\int_{I_m\times K_i} \psi^2\mathrm{d}x\mathrm{d}t\right)^{1/2} \\ \overset{(3.23)}{\leqslant} \sqrt{\sum_i \|e_{h\tau}\|^2_{L^2(I_m;L^2(K_i))}}\sqrt{\sum_i \|\psi'\|^2_{L^2(I_m;L^2(K_i))}} = \|e_{h\tau}\|_{L^2(I_m;L^2(\Omega))}\|\psi'\|_{L^2(I_m;L^2(\Omega))}. \tag{3.29}$$

We manipulate the terms in the bilinear form a_{hm} in the same way as above which results in

$$\int_{I_m} \sum_i \int_{K_i} \nabla e_{h\tau} \cdot \nabla\psi\,\mathrm{d}x\mathrm{d}t \\ \leqslant \left(\int_{I_m} \sum_i \int_{K_i} |\nabla e_{h\tau}|^2\,\mathrm{d}x\mathrm{d}t\right)^{1/2} \left(\int_{I_m} \sum_i \int_{K_i} |\nabla\psi|^2\,\mathrm{d}x\mathrm{d}t\right)^{1/2} \\ = \|\nabla_h e_{h\tau}\|_{L^2(I_m;L^2(\Omega))}\|\nabla_h\psi\|_{L^2(I_m;L^2(\Omega))}. \tag{3.30}$$

Using the properties (2.3) for the third term in a_{hm}, we get

$$-\int_{I_m} \sum_{\Gamma\in\mathcal{F}^{IB}} \int_\Gamma \sqrt{\sigma_\Gamma^{-1}}\langle\nabla e_{h\tau}\rangle \cdot \boldsymbol{n}\sqrt{\sigma_\Gamma}[\psi]\,\mathrm{d}S\mathrm{d}t \\ \leqslant \left(\int_{I_m} \sum_{\Gamma\in\mathcal{F}^{IB}_{hm}} \int_\Gamma \sigma_\Gamma^{-1}(\langle\nabla e_{h\tau}\rangle\cdot\boldsymbol{n})^2\,\mathrm{d}S\mathrm{d}t\right)^{1/2} \left(\int_{I_m} \overbrace{\sum_{\Gamma\in\mathcal{F}^{IB}_{hm}} \int_\Gamma \sigma_\Gamma[\psi]^2\,\mathrm{d}S\,\mathrm{d}t}^{J_{hm}(\psi,\psi)}\right)^{1/2}. \tag{3.31}$$

We manipulate the third term in (3.21) to obtain

$$\int_{I_m} J_{hm}(e_{h\tau}, \psi)\,\mathrm{d}t = \int_{I_m} \sum_{\Gamma\in\mathcal{F}_{hm}^{IB}} \sigma_\Gamma \int_\Gamma [e_{h\tau}][\psi]\,\mathrm{d}S\mathrm{d}t$$
$$\leqslant \left(\int_{I_m} J_{hm}(e_{h\tau}, e_{h\tau})\,\mathrm{d}t\right)^{1/2} \left(\int_{I_m} J_{hm}(\psi,\psi)\,\mathrm{d}t\right)^{1/2}. \quad (3.32)$$

For the last two terms, we have

$$(e_{h\tau}|_m^-, \psi_m^-) = \sum_i \int_{K_i} e_{h\tau}|_m^- \psi_m^-\,\mathrm{d}x$$
$$\leqslant \left(\sum_i \int_{K_i} (e_{h\tau}|_m^-)^2\,\mathrm{d}x\right)^{1/2} \left(\sum_i \int_{K_i} (\psi_m^-)^2\,\mathrm{d}x\right)^{1/2} = \|e_{h\tau}|_m^-\|_{L^2(\Omega)} \|\psi_m^-\|_{L^2(\Omega)}, \quad (3.33)$$

$$(-e_{h\tau}|_{m-1}^-, \psi_{m-1}^+) \leqslant \|e_{h\tau}|_{m-1}^-\|_{L^2(\Omega)} \|\psi_{m-1}^+\|_{L^2(\Omega)}. \quad (3.34)$$

Then from the definition of the norms (3.27) and (3.28), we obtain the inequality (3.26). □

We have relation (3.18) between the error $e_{h\tau}$ and the residual measure $\mathcal{E}_{STA}$. Although the spatial discretization is done in space $H^1(\Omega)$ and not in broken space $H^2(\mathcal{T}_{hm})$, we use this relation as follows:

The algebraic residual measure $\mathcal{E}_A(\tilde{u}_{h\tau})$ gives information about the computed approximate solution $\tilde{u}_{h\tau}$ in the space $S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$. In particular, if $\mathcal{E}_A(\tilde{u}_{h\tau})$ is very small, then $\tilde{u}_{h\tau}$ is close to $u_{h\tau}$.

The space algebraic residual measure $\mathcal{E}_{SA}(\tilde{u}_{h\tau})$ gives information about the approximate solution $\tilde{u}_{h\tau} \in S^{\tau q}(\mathcal{I}_\tau; S_{hp})$ and the space $S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$, which is piecewise polynomial with respect to the time variable and is infinite-dimensional with respect to the spatial variables.
If $\mathcal{E}_{SA}(\tilde{u}_{h\tau})$ is very small, we can assume that the approximate solution $\tilde{u}_{h\tau}$ approximates well the spatial one ($\tilde{u}_{h\tau} \approx u_\tau$).
A larger value $\mathcal{E}_{SA}$ and a very small value of $\mathcal{E}_A$ suggest a problem in defining the space $S_{hp}^{\tau q}$. In this case, if we want a better approximate solution, we can increase p or change the mesh $\mathcal{T}_{hm}$.

The time algebraic residual measure $\mathcal{E}_{TA}(\tilde{u}_{h\tau})$ gives information about the approximate solution $\tilde{u}_{h\tau} \in S^{\tau q}(\mathcal{I}_\tau; S_{hp})$ and the space $H^1(\mathcal{I}_\tau; S_{hp})$. If $\mathcal{E}_{TA}$ is very small, we can assume that the result $\tilde{u}_{h\tau}$ is a good approximation with respect to time ($\tilde{u}_{h\tau} \approx u_h$). A larger value of $\mathcal{E}_{TA}$ and a very small value of $\mathcal{E}_A$ suggest that if we want a better approximate solution $u_{h\tau}$, we can increase q or reduce the time step τ_m.

The space-time algebraic residual measure $\mathcal{E}_{STA}(\tilde{u}_{h\tau})$ gives information about the approximate solution $\tilde{u}_{h\tau}$ and the space $H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$. If $\mathcal{E}_{STA}(u_{h\tau})$ is very small, then the approximate solution $\tilde{u}_{h\tau}$ approximates u well ($\tilde{u}_{h\tau} \approx u$).

3.2 Residual estimators

The spaces $H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$, $S^{\tau,q}(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$, $H^1(\mathcal{I}_\tau; S_{hp})$ are infinite-dimensional, so the evaluation of residual measures $\mathcal{E}_*, * \in \{SA, TA, STA\}$ is practically impossible. Hence, we introduce the residual estimators that apply to finite-dimensional spaces as follows:

$$\eta_{STA}(\tilde{u}_{h\tau}) := \sup_{\psi \in S^{\tau,q+1}(\mathcal{I}_\tau; S_{h,p+1}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{S^{\tau,q+1}(\mathcal{I}_\tau; S_{h,p+1})}}, \tag{3.35}$$

$$\eta_{TA}(\tilde{u}_{h\tau}) := \sup_{\psi \in S^{\tau,q+1}(\mathcal{I}_\tau; S_{h,p}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{S^{\tau,q+1}(\mathcal{I}_\tau; S_{h,p})}}, \tag{3.36}$$

$$\eta_{SA}(\tilde{u}_{h\tau}) := \sup_{\psi \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p+1}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau; S_{h,p+1})}}, \tag{3.37}$$

$$\eta_{A}(\tilde{u}_{h\tau}) := \sup_{\psi \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}), \psi \neq 0} \frac{A_{h\tau}(\tilde{u}_{h\tau}, \psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})}} = \mathcal{E}_A(\tilde{u}_{h\tau}). \tag{3.38}$$

We call $\eta_{STA}, \eta_{TA}, \eta_{SA}, \eta_A$ *space-time algebraic*, *time algebraic*, *space algebraic* and *algebraic* residual estimators, respectively.

Remark 3.1. *It is possible to define η_{TA} in the spaces $S_{h,p}^{\tau,q+2}$ or $S_{h,p}^{\tau,q+3}, \dots$ so that η_{TA} can be more accurate with respect to $\mathcal{E}_{TA}$. However, numerical experiments show that the choice of the space $S_{h,p}^{\tau,q+1}$ is already sufficient.*
The same holds for η_{SA}, we can define it in the space $S_{h,p+2}^{\tau,q}$ or $S_{h,p+3}^{\tau,q}$ and so on, but experiments show that the choice $p+1$ is sufficient.

For the residual estimators, the following properties hold:
For the exact solution $u \in H^1(0,T; H^2(\Omega)) \subset H^1(\mathcal{I}_\tau; H^2(\mathcal{T}_h))$, we obtain

$$\eta_A(u) = \eta_{TA}(u) = \eta_S(u) = \eta_{ST}(u) = 0. \tag{3.39}$$

We have $S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}) \subset S^{\tau,q}(\mathcal{I}_\tau; S_{h,p+1}) \subset S^{\tau,q+1}(\mathcal{I}_\tau; S_{h,p+1})$ and $S^{\tau,q}(\mathcal{I}_\tau; S_{h,p}) \subset S^{\tau,q+1}(\mathcal{I}_\tau; S_{h,p}) \subset S^{\tau,q+1}(\mathcal{I}_\tau; S_{h,p+1})$, then it holds

$$\eta_A(\vartheta_{h\tau}) \leqslant \eta_{TA}(\vartheta_{h\tau}) \leqslant \eta_{STA}(\vartheta_{h\tau}), \quad \eta_A(\vartheta_{h\tau}) \leqslant \eta_{SA}(\vartheta_{h\tau}) \leqslant \eta_{STA}(\vartheta_{h\tau}) \quad \forall \vartheta_{h\tau} \in S_{hp}^{\tau q}. \tag{3.40}$$

The suprema in the estimates η_* are taken over smaller spaces than the suprema in the estimates $\mathcal{E}_*$, and so it holds

$$\eta_A(\vartheta_{h\tau}) = \mathcal{E}_A(\vartheta_{h\tau}) \quad \forall \vartheta_{h\tau} \in S_{hp}^{\tau q}, \tag{3.41}$$

$$\eta_*(\vartheta_{h\tau}) \leqslant \mathcal{E}_*(\vartheta_{h\tau}) \quad \forall \vartheta_{h\tau} \in S_{hp}^{\tau q}, \quad * \in \{SA, TA, STA\}. \tag{3.42}$$

The a priori error estimate satisfies inequality (2.20). We expect that our residual estimators will behave as the a priori error estimate and so we expect the following:

$$\eta_{TA} = O(\tau^{q+1}), \tag{3.43a}$$

$$\eta_{SA} = O(h^p), \tag{3.43b}$$

$$\eta_{TA} \text{ is independent from } h \text{ and } \eta_{SA} \text{ is independent from } \tau. \tag{3.43c}$$

3.2.1 Evaluation of the residual estimators

Now, we want to compute the residual estimators to use in our algorithm. Let us recall (2.33) and (2.34). For $u_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau; S_{h,p})$ then

$$u_{h\tau}(x,t) = \sum_{m=1}^{r} u_{h\tau}^m(x,t), \quad (x,t) \in \Omega \times (0,T),$$

where

$$u_{h\tau}^m = \begin{cases} u_{h\tau}|_{\Omega\times I_m} \in S^{\tau,q}(I_m; S_{h,p,m}) & \text{for } (x,t) \in \Omega \times I_m, \\ 0 & \text{for } (x,t) \notin \Omega \times I_m. \end{cases}$$

We define the residual estimator on the time interval I_m and on the space-time element $K_\mu \times I_m$ by

$$\eta_*^m(\tilde{u}_{h\tau}) := \sup_{\substack{0\neq\psi\in X, \\ \mathrm{supp}(\psi)\subset\Omega\times I_m}} \frac{A_{h\tau}(\tilde{u}_{h\tau},\psi)}{\|\psi\|_X} = \sup_{\substack{0\neq\psi\in X, \\ \mathrm{supp}(\psi)\subset\Omega\times I_m}} \frac{A_{hm}(\tilde{u}_{h\tau},\psi)}{\|\psi\|_X}, \tag{3.44}$$

$$\eta_*^{m,\mu}(\tilde{u}_{h\tau}) := \sup_{\substack{0\neq\psi\in X, \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} \frac{A_{h\tau}(\tilde{u}_{h\tau},\psi)}{\|\psi\|_X} = \sup_{\substack{0\neq\psi\in X, \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} \frac{A_{hm}(\tilde{u}_{h\tau},\psi)}{\|\psi\|_X}, \tag{3.45}$$

where $* \in \{A, TA, SA, STA\}$ and $X \in \{S_{hp}^{\tau q}, S_{h,p}^{\tau,q+1}, S_{h,p+1}^{\tau,q}, S_{h,p+1}^{\tau,q+1}\}$ is their corresponding space, respectively.
Instead of writing $\eta_*^m(\tilde{u}_{h\tau})$, we can write $\eta_*^m(\tilde{u}_{h\tau}^m)$ with a convention that

$$\tilde{u}_{h\tau}^m(t_{m-1}^-) := \tilde{u}_{h\tau}^{m-1}(t_{m-1}^-). \tag{3.46}$$

For example, the algebraic residual estimator at time level m and on a space-time element is

$$\eta_A^m(\tilde{u}_{h\tau}^m) = \sup_{\substack{0\neq\psi\in S^{\tau,q}(\mathcal{I}_\tau;S_{h,p}), \\ \mathrm{supp}(\psi)\subset\Omega\times I_m}} \frac{A_{hm}(\tilde{u}_{h\tau}^m,\psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau;S_{h,p})}}, \tag{3.47}$$

$$\eta_A^{m,\mu}(\tilde{u}_{h\tau}^m) = \sup_{\substack{0\neq\psi\in S^{\tau,q}(\mathcal{I}_\tau;S_{h,p}), \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} \frac{A_{hm}(\tilde{u}_{h\tau}^m,\psi)}{\|\psi\|_{S^{\tau,q}(\mathcal{I}_\tau;S_{h,p})}}. \tag{3.48}$$

The evaluation of $\eta_*^{m,\mu}$ was presented originally in [9] for steady state problems. We extend it to our non-stationary case. At time level m and on spatial element K_μ, we have defined the basis of the space $P_p(K_\mu) \times P^q(I_m)$ as (2.29)

$$B_{m,\mu} = \{\psi_i^{m,\mu},\ i = 1, \ldots, N\}, \quad N = \dim_{m,\mu}. \tag{3.49}$$

We define the basis similarly for spaces $P_{p+1}(K_\mu) \times P^q(I_m)$, $P_p(K_\mu) \times P^{q+1}(I_m)$, $P_{p+1}(K_\mu) \times P^{q+1}(I_m)$. We want to evaluate the residual estimator $\eta_*^{m,\mu}(\tilde{u}_{h\tau}^m)$, which is defined as

$$\eta_*^{m,\mu}(\tilde{u}_{h\tau}^m) = \sup_{\substack{0\neq\psi\in X, \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} \frac{A_{hm}(\tilde{u}_{h\tau}^m,\psi)}{\|\psi\|_X} = \sup_{\substack{\psi\in X, \|\psi\|_X=1, \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} A_{hm}(\tilde{u}_{h\tau}^m,\psi), \tag{3.50}$$

where $* \in \{A, SA, TA, STA\}$, X is their corresponding space and $B_{m,\mu}$ is their corresponding basis on one space-time element $K_\mu \times I_m$. Let $(\cdot,\cdot)_X$ be the scalar product in X generating the norm $\|\cdot\|_X$. We put

$$\begin{aligned} \mathbb{S} = \{s_{ij}\}_{j,j=1}^N, \quad & s_{ij} = (\psi_i^{m,\mu}, \psi_j^{m,\mu})_X, \\ \boldsymbol{d} = \{d_i\}_{i=1}^N, \quad & d_i = A_{hm}(\tilde{u}_{h\tau}^m, \psi_i^{m,\mu}). \end{aligned} \tag{3.51}$$

Let $\psi \in X, \mathrm{supp}(\psi) \subset K_\mu \times I_m$ then

$$\psi = \sum_{i=1}^{N} \alpha_i \psi_i^{m,\mu}, \alpha_i \in \mathbb{R}, i = 1, \dots, N \qquad \|\psi\|_X^2 = \sum_{i,j=1}^{N} \alpha_i \alpha_j s_{ij}. \tag{3.52}$$

Hence, there is an isomorphism between $\psi \in X \longleftrightarrow \alpha = \{\alpha_i\}_{i=1}^N \in \mathbb{R}^N$.
Let us define a linear functional Θ and a function Φ such as

$$\begin{aligned} \Theta(\alpha_1, \dots, \alpha_N) &:= A_{hm}(\tilde{u}_{h\tau}^m, \psi) = \sum_{i=1}^{N} \alpha_i A_{hm}(\tilde{u}_{h\tau}^m, \psi_i^{m,\mu}) = \sum_{i=1}^{N} \alpha_i d_i \,, \\ \Phi(\xi) &:= \sum_{j,i=1}^{N} \xi_i \xi_j s_{ij} - 1, \qquad \xi = \{\xi_i\}_{i=1}^N \in \mathbb{R}^N. \end{aligned} \tag{3.53}$$

We have the following equivalence

$$\begin{array}{ccc} A_{hm}(\tilde{u}_{h\tau}^m, \cdot) \text{ has a maximum at } \psi & \Longleftrightarrow & \Theta \text{ has a maximum at } \alpha \\ \|\psi\|_X = 1 & & \Phi(\alpha) = 0 \end{array}$$

To find the maximum value of the function Θ with respect to constraint $\Phi(\xi) = 0$ we use the method of Lagrange multipliers, that is, we introduce a Lagrange multiplier $\lambda \in \mathbb{R}$ and find the critical values of Lagrange function

$$\begin{aligned} L(\xi, \lambda) &:= \Theta(\xi) + \lambda \Phi(\xi) \\ &= \sum_{i=1}^{N} \xi_i d_i + \lambda \sum_{j,i=1}^{N} \xi_i \xi_j s_{ij} - \lambda \\ &= \xi^T \boldsymbol{d} + \lambda \xi^T \mathbb{S} \xi - \lambda. \end{aligned} \tag{3.54}$$

For the critical points of L, the following relations hold:

$$\begin{aligned} 0 &= \frac{\partial L}{\partial \xi_i} = d_i + 2\lambda \sum_{j+1}^{N} \xi_j s_{ij}, \quad i = 1, \dots, N, \\ 0 &= \frac{\partial L}{\partial \lambda} = \sum_{j,i=1}^{N} \xi_i \xi_j s_{ij} - 1. \end{aligned} \tag{3.55}$$

Equivalently:

$$\boldsymbol{d} + \mathbb{S}(2\lambda\xi) = 0, \tag{3.56}$$

$$\xi^T \mathbb{S} \xi = 1. \tag{3.57}$$

Let $\overline{\xi} := 2\lambda\xi$. We solve (3.56) as a system of linear equations $\mathbb{S}\overline{\xi} = -\boldsymbol{d}$ and we obtain a vector $\overline{\xi}$. We obtain λ through the following manipulations:

$$2\lambda \mathbb{S}\xi = -\boldsymbol{d} \;\Leftrightarrow\; 4\lambda^2 \xi^T \mathbb{S} \xi = -2\lambda \xi^T \boldsymbol{d} \;\Leftrightarrow 4\lambda^2 = -\overline{\xi}^T \boldsymbol{d} \;\Leftrightarrow\; 2\lambda = \pm\sqrt{-\overline{\xi}^T \boldsymbol{d}}.$$

Then

$$\xi^{1,2} = \pm \frac{\overline{\xi}}{\sqrt{-\overline{\xi}^T \boldsymbol{d}}}.$$

Obviously $\Theta(\xi^1) = -\Theta(\xi^2)$. Let

$$\alpha := \xi^i, \quad \text{where } \Theta(\xi^i) \geq 0, i \in \{1, 2\}. \tag{3.58}$$

Finally

$$\eta_*^{m,\mu}(\tilde{u}_{h\tau}^m) = \sum_{i=1}^{N} \alpha_i d_i. \tag{3.59}$$

3.3 Properties of a norm

Definition 3.1. *Let X be a broken Bochner space over a triangulation $\mathcal{M}_{h\tau}$. Scalar product $(\cdot,\cdot)_X$ in the space X satisfies the element-orthogonality condition if*

$$(\psi_1, \psi_2 \in X, \text{supp}(\psi_i) \subset K_i \times I_{m_i}, i = 1, 2,\ (K_1, m_1) \neq (K_2, m_2)) \Rightarrow (\psi_1, \psi_2)_X = 0. \tag{3.60}$$

For example, the scalar product

$$(u, \psi)_X = \sum_{m=1}^{r} \int_{I_m} \left(\sum_{K \in \mathcal{T}_{hm}} \int_K \nabla u \cdot \nabla \psi \, \mathrm{d}x + J_{hm}(u, \psi) \right) \mathrm{d}t \quad u, \psi \in X. \tag{3.61}$$

does not satisfy the element-orthogonality condition (3.60). Also, the scalar product, which induces the norm (3.27), does not satisfy (3.60). However, if we put

$$(u, \psi)_X = \sum_{m=1}^{r} \int_{I_m} \sum_{K \in \mathcal{T}_{hm}} \int_K (u\psi + \nabla u \cdot \nabla \psi) \, \mathrm{d}x\mathrm{d}t, \tag{3.62}$$

then $(\cdot,\cdot)_X$ satisfies the condition orthogonality between elements (3.60). The property of element-orthogonality of the scalar product allows us to compute the residual estimator by parts, according to the following lemma.

Lemma 3.3. *Let $(\cdot,\cdot) : X \times X \to \mathbb{R}$ be a scalar product generating the norm $\|\cdot\|_X$ and $(\cdot,\cdot)$ satisfy the element-orthogonality condition. Then*

$$\eta_*(\psi_{h\tau})^2 = \sum_{m=1}^{r} \eta_*^m(\psi_{h\tau})^2 = \sum_{m=1}^{r} \sum_{\mu=1}^{N_m} \eta_*^{m,\mu}(\psi_{h\tau})^2 \quad \forall \psi_{h\tau} \in X. \tag{3.63}$$

Proof. The proof is presented originally in [9].
We define linear function F such that

$$F(\psi) := A_{hm}(\tilde{u}_{h\tau}^m, \psi), \psi \in X. \tag{3.64}$$

The relation (3.63) is, with the convention (3.46), equivalent with

$$\left(\sup_{\substack{\psi \in X \\ \|\psi\|_X = 1}} A_{h\tau}(\tilde{u}_{h\tau}, \psi) \right)^2 = \sum_{m=1}^{r} \left(\sup_{\substack{\psi \in X, \|\psi\|_X = 1 \\ \text{supp}(\psi) \subset \Omega \times I_m}} A_{hm}(\tilde{u}_{h\tau}^m, \psi) \right)^2$$

$$= \sum_{m=1}^{r} \sum_{\mu=1}^{N_m} \left(\sup_{\substack{0 \neq \psi \in X, \|\psi\|_X = 1 \\ \text{supp}(\psi) \subset K_\mu \times I_m}} A_{hm}(\tilde{u}_{h\tau}^m, \psi) \right)^2. \tag{3.65}$$

We will first prove the second equality of (3.65), that is

$$\left(\sup_{\substack{\psi\in X, \|\psi\|_X=1 \\ \mathrm{supp}(\psi)\subset\Omega\times I_m}} A_{hm}(\tilde{u}^m_{h\tau}, \psi) \right)^2 = \sum_{\mu=1}^{N_m} \left(\sup_{\substack{\psi\in X, \|\psi\|_X=1 \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} A_{hm}(\tilde{u}^m_{h\tau}, \psi) \right)^2. \tag{3.66}$$

Let $\psi^m_{K_\mu}$ denote the function in which $A_{hm}(\tilde{u}^m_{h\tau}, \cdot)$ has a maximum on one space-time element $K_\mu \times I_m$. We obtain $\psi^m_{K_\mu}$ from (3.52) and (3.58). So

$$\begin{aligned} \psi^m_{K_\mu} \in X, \quad \|\psi^m_{K_\mu}\|_X = 1, \quad \mathrm{supp}(\psi^m_{K_\mu}) \subset K_\mu \times I_m, \\ \eta^{m\mu}_*(\tilde{u}^m_{h\tau}) = A_{hm}(\tilde{u}^m_{h\tau}, \psi^m_{K_\mu}) = \sup_{\substack{\psi\in X, \|\psi\|_X=1 \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} A_{hm}(\tilde{u}^m_{h\tau}, \psi). \end{aligned} \tag{3.67}$$

Then (3.64) yields

$$F(\psi^m_{K_\mu}) = \sup_{\substack{\psi\in X, \|\psi\|_X=1 \\ \mathrm{supp}(\psi)\subset K_\mu\times I_m}} F(\psi) \tag{3.68}$$

The equality relation (3.66) is now in the form

$$\sup_{\substack{\psi\in X, \|\psi\|_X=1 \\ \mathrm{supp}(\psi)\subset\Omega\times I_m}} F(\psi) = \left(\sum_{\mu=1}^{N_m} \left(F(\psi^m_{K_\mu}) \right)^2 \right)^{1/2}. \tag{3.69}$$

To prove that this equality holds, we must show two things:

(i) Consider function ψ^m, with properties $\psi \in X$, $\|\psi^m\|_X = 1$, $\mathrm{supp}(\psi^m) \subset \Omega \times I_m$, such that the following equality is satisfied

$$F(\psi^m) = \left(\sum_{\mu=1}^{N_m} \left(F(\psi^m_{K_\mu}) \right)^2 \right)^{1/2}. \tag{3.70}$$

We put

$$\psi^m := \sum_{\mu=1}^{N_m} \xi_\mu \psi^m_{K_\mu} = \sum_{i=1}^{N_m} \xi_i \psi^m_{K_i}, \quad \xi_i \in \mathbb{R},\ i \in E_m, \tag{3.71}$$

$$\|\psi^m\|_X = (\sum_{i=1}^{N_m} \xi_i \psi^m_{K_i}, \sum_{j=1}^{N_m} \xi_j \psi^m_{K_j})_X = \sum_{i,j=1}^{N_m} \xi_i\xi_j (\psi^m_{K_i}, \psi^m_{K_j})_X \overset{(3.60)}{=} \sum_{i=1}^{N_m} \xi_i^2.$$

Let us define

$$\xi_i := \frac{F(\psi^m_{K_i})}{\sqrt{\sum_{i=1}^{N_m} F(\psi^m_{K_i})^2}}, \quad i \in E_m. \tag{3.72}$$

Then $\sum_{i=1}^{N_m} \xi_i^2 = 1$ and so

$$F(\psi^m) = F(\sum_{i=1}^{N_m} \xi_i \psi^m_{K_i}) = \sum_{i=1}^{N_m} \xi_i F(\psi^m_{K_i}) = \frac{\sum_i F(\psi^m_{K_i})^2}{\sqrt{\sum_{i=1}^{N_m} F(\psi^m_{K_i})^2}} = \sqrt{\sum_{i=1}^{N_m} F(\psi^m_{K_i})^2}, \tag{3.73}$$

as we wanted to show.

(ii) Now let ψ^m be a function in which F attains a maximum, i.e.

$$F(\psi^m) = \sup_{\substack{\psi\in X, \|\psi\|_X=1 \\ \mathrm{supp}(\psi)\subset\Omega\times I_m}} F(\psi). \tag{3.74}$$

Let Θ be an arbitrary function in X with the properties

$$\mathrm{supp}(\Theta) \subset \Omega \times I_m, \quad \|\Theta\|_X = 1. \tag{3.75}$$

We have $B_m = \{\psi_i^m, i = 1, \dots, \dim_m\}$ the basis of the broken Bochner space at time level m. We have also $B_{m\mu} = \{\psi_i^{m\mu}, i = 1, \dots, \dim_{m\mu}\}$ the basis on one space-time element $K_\mu \times I_m$ and the relation (2.29). Then

$$\Theta(x,t) = \sum_{i=1}^{\dim_m} \alpha_i^m \psi_i^m = \sum_{\mu=1}^{N_m} \sum_{s=1}^{\dim_{m\mu}} \alpha_s^{m\mu} \psi_s^{m\mu}$$

and

$$\begin{aligned}\|\Theta\|_X^2 &= (\sum_k \sum_i \alpha_i^{mk}\psi_i^{mk}, \sum_l \sum_j \alpha_j^{ml}\psi_j^{ml})_X \\ &= \sum_{k,l} \sum_{i,j} \alpha_i^{mk}\alpha_j^{ml}(\psi_i^{mk}, \psi_j^{ml})_X \overset{(3.60)+(3.51)}{=} \sum_\mu \sum_{i,j} \alpha_i^{m\mu}\alpha_j^{m\mu} s_{ij}^{m\mu} = 1.\end{aligned} \tag{3.76}$$

Thus

$$F(\Theta) = F(\sum_\mu \sum_s \alpha_s^{m\mu}\psi_s^{m\mu}) = \sum_\mu \xi'_\mu F(\sum_s \frac{\alpha_s^{m\mu}}{\xi'_\mu}\psi_s^{m\mu}). \tag{3.77}$$

We want coefficients ξ'_μ such that $\| \sum_s \frac{\alpha_s^{m\mu}}{\xi'_\mu}\psi_s^{m\mu}\|_X = 1$. This means

$$\begin{aligned}1 = \| \sum_s \frac{\alpha_s^{m\mu}}{\xi'_\mu}\psi_s^{m\mu}\|_X^2 &= (\sum_i \frac{\alpha_i^{m\mu}}{\xi'_\mu}\psi_i^{m\mu}, \sum_j \frac{\alpha_j^{m\mu}}{\xi'_\mu}\psi_j^{m\mu})_X = \frac{1}{{\xi'_\mu}^2}\sum_{i,j} \alpha_i^{m\mu}\alpha_j^{m\mu}s_{ij}^{m\mu} \\ \Leftrightarrow \sum_{i,j} \alpha_i^{m\mu}\alpha_j^{m\mu}s_{ij}^{m\mu} &= {\xi'_\mu}^2, \quad \mu \in E_m \\ \overset{(3.76)}{\Rightarrow} \sum_\mu {\xi'_\mu}^2 &= 1.\end{aligned}$$

So

$$F(\Theta) \overset{(3.68)}{\leqslant} \sum_\mu \xi'_\mu F(\psi_{K_\mu}^m) \leqslant \left(\sum_\mu {\xi'_\mu}^2\right)^{1/2} \left(\sum_\mu F(\psi_{K_\mu}^m)^2\right)^{1/2} = F(\psi^m), \tag{3.78}$$

which concludes the proof.

The first equality relation of (3.65) is proven by similar considerations. □

If a norm does not satisfy the element-orthogonality, then η_*^m on $\mathcal{T}_{h,m} \times I_m$ can be computed by the same method presented above, but the system of linear equations is as large as when solving one Newton iteration. In this case, we compute the residual estimators as in the case that the norm satisfies the element-orthogonality property.

Remark 3.2. *The numerical experiments were carried out in Section 3.4 and 5.3. In the case of linear problems, the evaluation of residual estimators consumes the same amount of time as the iteration process. However, in the case of non-linear problems and difficult physical phenomena, the evaluation of residual estimators represents just a very small percentage of the overall computational time.*

We would like to have a norm in space X such that

(a) the norm allows us to find a relation between the residual measure and the a priori error estimate, for example the relation in Lemma 3.1, so that using residual estimators has value,

(b) it satisfies the 'orthogonality between elements' condition (3.60), so that the evaluation of residual estimators is fast compared to the overall computational time.

We also consider a norm which does not satisfy (3.60) for a comparison. Let us review the norm in a broken Bochner space $L^2(\mathcal{I}_\tau; L^2(\Omega))$ and $L^2(\mathcal{I}_\tau; H^1(\mathcal{T}_h))$.

$$\begin{aligned}\|u\|^2_{L^2(\mathcal{I}_\tau;L^2(\Omega))} &= \sum_{m=1}^{r} \|u\|^2_{L^2(I_m;L^2(\Omega))}\\ &= \sum_{m=1}^{r} \int_{I_m} \|u(t)\|^2_{L^2(\Omega)}\mathrm{d}t = \sum_{m=1}^{r} \int_{I_m} \sum_{K\in\mathcal{T}_{hm}} \int_K |u(x,t)|^2 \mathrm{d}x\mathrm{d}t. \qquad (L^2(L^2))\end{aligned}$$

$$\begin{aligned}\|u\|^2_{L^2(\mathcal{I}_\tau;H^1(\mathcal{T}_h))} &= \sum_{m=1}^{r} \|u\|^2_{L^2(I_m;H^1(\mathcal{T}_{hm}))} = \sum_{m=1}^{r} \int_{I_m} \|u(t)\|_{H^1(\mathcal{T}_{hm})}\,\mathrm{d}t\\ &= \sum_{m=1}^{r} \int_{I_m} \sum_{K\in\mathcal{T}_{hm}} \int_K |u(x,t)|^2 + |\nabla u(x,t)|^2\,\mathrm{d}x\mathrm{d}t \qquad (L^2(H^1))\\ &= \|u\|^2_{L^2(\mathcal{I}_\tau;L^2(\Omega))} + \|\nabla_h u\|^2_{L^2(\mathcal{I}_\tau;L^2(\Omega))}.\end{aligned}$$

Let $u \in H^1(\mathcal{I}_\tau; H^1(\mathcal{T}_h))$. At the time level m, we consider the following norms $(\|\cdot\|_{H^1(I_m;H^1(\mathcal{T}_{hm}))})$

$$\|u\|_{EO,m} := \left(\|u'\|^2_{L^2(I_m;L^2(\Omega))} + \|u\|^2_{L^2(I_m;L^2(\Omega))} + \epsilon\|\nabla_h u\|^2_{L^2(I_m;L^2(\Omega))}\right)^{1/2}, \qquad (3.79)$$

$$\|u\|_{DG,m} := \sqrt{\epsilon}\left(\|u'\|^2_{L^2(I_m;L^2(\Omega))} + \|\nabla_h u\|^2_{L^2(I_m;L^2(\Omega))} + \int_{I_m} J_{hm}(u,u)\mathrm{d}t\right)^{1/2}. \qquad (3.80)$$

Then the corresponding norms in the $H^1(\mathcal{I}_\tau; H^1(\mathcal{T}_h))$ are given by

$$\|u\|_{EO} = \left(\sum_{m=1}^{r} \|u\|^2_{EOm}\right)^{1/2}, \qquad (\mathrm{EO})$$

$$\|u\|_{DG} = \left(\sum_{m=1}^{r} \|u\|^2_{DGm}\right)^{1/2}. \qquad (\mathrm{DG})$$

Remark 3.3. *The form A_{hm}, which was defined in* (3.1)*, includes the interior and boundary penalty J_{hm}, which represents a jump between elements. In the definition* (EO)*, the norm satisfies the element-orthogonality condition, but it does not satisfy the bounded condition of the form A_{hm}. In contrast, the norm* (DG) *may give a bounded form, but it does not satisfy the element-orthogonality condition.*

In the following, we will test four norms, $(L^2(L^2))$, $(L^2(H^1))$, (EO) and (DG), in order to test the properties (3.43). If the norm satisfies the properties, we will use it in our proposed algorithm in Chapter 4.

3.4 Verification and numerical experiments

In this section, we consider three examples and carry out computations to verify properties (3.43). We also carry out experiments to test the robustness with respect to the polynomial approximation degree and the input data.

3.4.1 Experimental order of convergence

Let us introduce the experimental order of convergence (EOC), for example for the time residual estimator, which approximates the order of convergence with respect to the time. Let η_{TA_1}, η_{TA_2} be the algebraic time residual estimators for the time step τ_1, τ_2, respectively. We assume that

$$\begin{aligned} \eta_{TA_1} &\approx c\tau_1^{q+1}, \\ \eta_{TA_2} &\approx c\tau_2^{q+1}, \end{aligned} \tag{3.81}$$

where c is a positive constant. Using the following manipulation, we obtain the EOC with respect to the time

$$\begin{aligned} \log \frac{\eta_{TA_1}}{\eta_{TA_2}} &\approx \log(\frac{\tau_1}{\tau_2})^{q+1} = (q+1)\log\frac{\tau_1}{\tau_2} \\ q+1 &\approx \frac{\log \eta_{TA_1} - \log \eta_{TA_2}}{\log \tau_1 - \log \tau_2}. \end{aligned}$$

For each pair of successive time steps τ_1, τ_2 and the corresponding errors η_{TA_1}, η_{TA_2}, or for each pair of successive spatial steps h_1, h_2 and the corresponding errors η_{SA_1}, η_{SA_2}, the experimental order of convergence is calculated using the formula:

$$\mathrm{EOC} = \frac{\log \eta_{TA_1} - \log \eta_{TA_2}}{\log \tau_1 - \log \tau_2} \quad (\mathrm{EOC}_\tau) \qquad \mathrm{EOC} = \frac{\log \eta_{SA_1} - \log \eta_{SA_2}}{\log h_1 - \log h_2} \quad (\mathrm{EOC}_h)$$

3.4.2 Examples

To carry out the verification, we choose different scalar problems. To verify $\eta_{TA} = O(\tau^{q+1})$, i.e. the EOC with respect to the time step τ, we select a function that changes essentially with respect to the time variable and slightly with respect to the spatial variable. In contrast, to verify (3.43b), i.e. the EOC with respect to the spatial h, we select a function that changes essentially in the spatial variable and slightly in the time variable.

Let $\Omega = (0,1)^2, T = 0.5$. We consider the scalar convection-diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} - \varepsilon \Delta u = g \quad & \forall (x,t) \in Q_T = \Omega \times (0,T), \\ u(x,t) = 0 \quad & \forall x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u^0(x) \quad & \forall x \in \Omega, \end{aligned} \tag{3.82}$$

where $\varepsilon = 0.1$ is the diffusion coefficient.

Example 1. For verification (3.43a), the data, i.e. function g and initial function u^0, are chosen such that the exact solution of (3.82) has the following form:

$$u(x_1, x_2, t) = (0.1 + \exp(10t))x_1(1-x_1)x_2(1-x_2). \tag{3.83}$$

Example 2. For verification (3.43b), the data are chosen such that the exact solution of (3.82) has the form:

$$u(x_1, x_2, t) = (1+t)x_1(1-x_1)x_2(1-x_2). \tag{3.84}$$

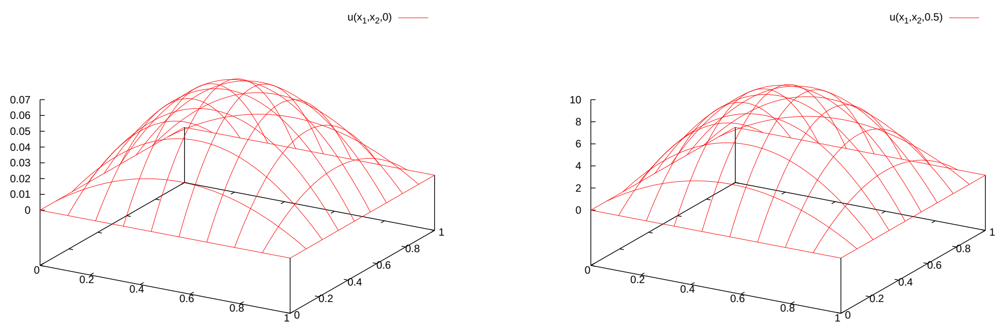

Figure 3.2: Function (3.83) at dimensionless times $t = 0$ and $t = 0.5$.

Example 3. Let $\Omega = (-1,1)^2, T = 1$. The scalar convection-diffusion equation has the following form:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} - \varepsilon \Delta u = 0 \quad & \forall (x,t) \in Q_T, \\ u(x,t) = u_D(x,t) \quad & \forall (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u^0(x) \quad & \forall x \in \Omega, \end{aligned} \tag{3.85}$$

where $\varepsilon = 0.01$ and the data u_D, u^0 are chosen in such a way that the exact solution has the form

$$u(x_1, x_2, t) = \frac{1}{1 + \exp(\frac{x_1+x_2+1-t}{2\varepsilon})}. \tag{3.86}$$

For problem (3.85) and its solution (3.86), we conduct numerical experiments for both cases, the EOC with respect to τ and the EOC with respect to h.

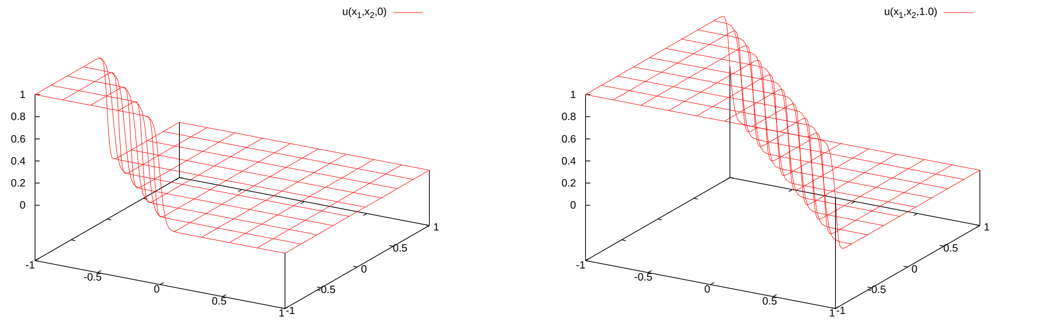

Figure 3.3: Function (3.86) with $\varepsilon = 0.01$ at dimensionless times $t = 0$ and $t = 1$.

The IIPG variant is used with $c_W = 20$. Linear iteration process (2.43) is solved by GMRES-ILU solver with stopping criterion (4.8). We do not adapt the time step nor the spatial mesh. We set a fixed spatial step and change time step manually for testing (EOC_τ) or we set a fixed time step and change the spatial step manually for testing (EOC_h).

Remark 3.4. *We obtain the approximate solution $u_{h\tau}$ for linear scalar equation* (3.82)*, while the approximate solution $\tilde{u}_{h\tau}$ is obtained for non-linear scalar equation* (3.85) *because of non-linear convective terms $f_s(u) = \frac{u^2}{2}, s = 1, 2$.*

Let us repeat the definition in (1.3). Let $e_{h\tau} = \tilde{u}_{h\tau} - u$, where $\tilde{u}_{h\tau} = u_{h\tau}$ for (3.82). The discretization error in the $L^2(\mathcal{I}_\tau; H^1(\mathcal{T}_h))$-seminorm is computed by

$$|e_{h\tau}|_{L^2(\mathcal{I}_\tau; H^1(\mathcal{T}_h))} = \left(\sum_{m=1}^{r} \int_{I_m} \sum_{K \in \mathcal{T}_{hm}} \int_K |\nabla \tilde{u}_{h\tau}(x,t) - \nabla u(x,t)|^2 \mathrm{d}x \mathrm{d}t \right)^{1/2}.$$

Let the ratio of the error in the dual norm and actual error in the $L^2(H^1)$-seminorm be given by

$$i_X := \frac{\eta_{STA}}{|e_{h\tau}|_{L^2(\mathcal{I}_\tau; H^1(\mathcal{T}_h))}}. \tag{3.87}$$

Note that i_X is not the standard effectivity index, where in place of η_{STA} is the difference between two approximate solutions in a suitable norm.

3.4.3 The order of convergence with respect to the time

The setting for variant (3.83) (equation (3.82) with solution (3.83)): We set fixed uniform mesh $\mathcal{T}_{hm}$ with number of elements $N_m = 512$, then $h = \frac{1}{16}$. The function (3.83) has order 4 with respect to the spatial variable, so we set $p = 4$. Computations were carried out with fixed time steps $\tau = \frac{0.1}{2^n}, n \in \{0, \dots, 5\}$ and with $q = 1, 2, 3$.

Setting for equation (3.85)-(3.86) with respect to (EOC_τ): Let mesh $\mathcal{T}_{hm}$ be fixed with $N_m = 1152$ elements, so $h = 1/12$, and spatial polynominal degree $p = 3$. We carried out computations with 6 time steps and time polynomial degree $q = 1, 2$.

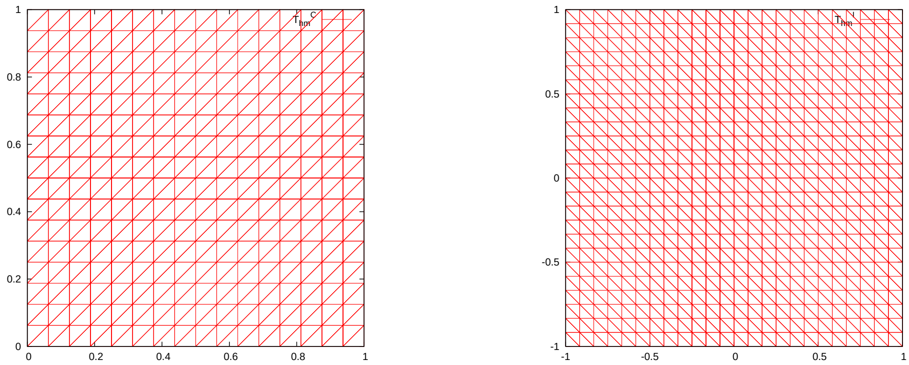

Figure 3.4: The uniform mesh $\mathcal{T}_{hm}$ for (3.82)-(3.83) (left) and (3.85)-(3.86) (right).

We present the results of computations for 4 norms $(L^2(L^2))$, $(L^2(H^1))$, (EO) and (DG) in tables 3.1, 3.2, 3.3, 3.4 for (3.83) and in tables 3.5, 3.6, 3.7, 3.8 for (3.86).

For problem (3.82) with solution (3.83), the result yields:

- norms (EO) and (DG) exhibit the desired order of convergence with respect to the time, EOC $\approx q+1$ for η_T,
- for very small time step η_T changes negligibly,
- for $q = 2, 3$, η_S changes negligibly when the time step decreased,
- for $\tau = 1/160$ or smaller, the error in seminorm and the residual estimator both change negligibly,

For problem (3.85) with solution (3.86) with respect to (EOC$_\tau$) the result yields:

- norms (EO) and (DG) satisfy the expectation (3.43a), i.e. EOC $\approx q+1$ for η_T,
- for $q = 2$ and τ very small, η_T does not change,
- the error in seminorm and η_S decrease negligibly with decreasing τ. They also change negligibly with increasing q.

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/16	1/10	4	1	2.632E-01	7.048E-07	1.227E-03	1.329E+01	1.329E+01	5.050E+01
1/16	1/20	4	1	6.427E-02	3.808E-07	5.543E-04	6.691E+00	6.691E+00	1.041E+02
	(EOC)			(2.03)	(0.89)	(1.15)	(0.99)	(0.99)	
1/16	1/40	4	1	1.570E-02	1.471E-07	2.577E-04	3.316E+00	3.316E+00	2.112E+02
	(EOC)			(2.03)	(1.37)	(1.11)	(1.01)	(1.01)	
1/16	1/80	4	1	3.871E-03	1.775E-08	1.122E-04	1.646E+00	1.646E+00	4.251E+02
	(EOC)			(2.02)	(3.05)	(1.20)	(1.01)	(1.01)	
1/16	1/160	4	1	9.604E-04	3.681E-09	4.285E-05	8.190E-01	8.190E-01	8.528E+02
	(EOC)			(2.01)	(2.27)	(1.39)	(1.01)	(1.01)	
1/16	1/319	4	1	2.361E-04	8.146E-09	1.372E-05	4.049E-01	4.049E-01	1.715E+03
	(EOC)			(1.42)	(-0.81)	(1.15)	(0.71)	(0.71)	
1/16	1/10	4	2	1.974E-02	1.626E-07	2.814E-04	4.866E-01	4.866E-01	2.465E+01
1/16	1/20	4	2	2.452E-03	2.019E-08	5.409E-05	1.238E-01	1.238E-01	5.050E+01
	(EOC)			(3.01)	(3.01)	(2.38)	(1.97)	(1.97)	
1/16	1/40	4	2	3.020E-04	6.961E-09	1.170E-05	3.086E-02	3.086E-02	1.022E+02
	(EOC)			(3.02)	(1.54)	(2.21)	(2.00)	(2.00)	
1/16	1/80	4	2	3.735E-05	3.872E-09	2.871E-06	7.676E-03	7.676E-03	2.055E+02
	(EOC)			(3.02)	(0.85)	(2.03)	(2.01)	(2.01)	
1/16	1/160	4	2	4.642E-06	1.175E-08	6.997E-07	1.913E-03	1.913E-03	4.121E+02
	(EOC)			(3.01)	(-1.60)	(2.04)	(2.00)	(2.00)	
1/16	1/319	4	2	5.701E-07	1.489E-08	1.366E-07	4.712E-04	4.712E-04	8.265E+02
	(EOC)			(2.13)	(-0.24)	(1.66)	(1.42)	(1.42)	
1/16	1/10	4	3	1.162E-03	2.670E-09	1.581E-05	3.882E-02	3.882E-02	3.340E+01
1/16	1/20	4	3	7.316E-05	8.815E-09	1.732E-06	4.979E-03	4.979E-03	6.806E+01
	(EOC)			(3.99)	(-1.72)	(3.19)	(2.96)	(2.96)	
1/16	1/40	4	3	4.532E-06	4.895E-09	1.851E-07	6.226E-04	6.226E-04	1.374E+02
	(EOC)			(4.01)	(0.85)	(3.23)	(3.00)	(3.00)	
1/16	1/80	4	3	2.811E-07	8.158E-09	2.054E-08	7.758E-05	7.758E-05	2.760E+02
	(EOC)			(4.01)	(-0.74)	(3.17)	(3.00)	(3.00)	
1/16	1/160	4	3	1.749E-08	1.112E-08	1.454E-08	9.674E-06	9.674E-06	5.531E+02
	(EOC)			(4.01)	(-0.45)	(0.50)	(3.00)	(3.00)	
1/16	1/319	4	3	1.105E-09	2.344E-08	2.609E-08	1.190E-06	1.191E-06	1.077E+03
	(EOC)			(2.80)	(-0.76)	(-0.59)	(2.12)	(2.12)	

Table 3.1: Variant (3.83) with norm $(L^2(L^2))$

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/16	1/10	4	1	2.632E-01	7.925E-10	3.553E-06	1.326E+01	1.326E+01	5.039E+01
1/16	1/20	4	1	6.427E-02	7.531E-10	1.613E-06	6.676E+00	6.676E+00	1.039E+02
	(EOC)			(2.03)	(0.07)	(1.14)	(0.99)	(0.99)	
1/16	1/40	4	1	1.570E-02	1.565E-10	7.540E-07	3.309E+00	3.309E+00	2.107E+02
	(EOC)			(2.03)	(2.27)	(1.10)	(1.01)	(1.01)	
1/16	1/80	4	1	3.871E-03	1.883E-10	3.304E-07	1.642E+00	1.642E+00	4.241E+02
	(EOC)			(2.02)	(-0.27)	(1.19)	(1.01)	(1.01)	
1/16	1/160	4	1	9.604E-04	2.751E-10	1.273E-07	8.172E-01	8.172E-01	8.509E+02
	(EOC)			(2.01)	(-0.55)	(1.38)	(1.01)	(1.01)	
1/16	1/319	4	1	2.361E-04	2.612E-10	4.130E-08	4.041E-01	4.041E-01	1.711E+03
	(EOC)			(1.42)	(0.05)	(1.14)	(0.71)	(0.71)	
1/16	1/10	4	2	1.974E-02	3.766E-10	8.154E-07	4.855E-01	4.855E-01	2.459E+01
1/16	1/20	4	2	2.452E-03	6.940E-10	1.574E-07	1.236E-01	1.236E-01	5.039E+01
	(EOC)			(3.01)	(-0.88)	(2.37)	(1.97)	(1.97)	
1/16	1/40	4	2	3.020E-04	9.465E-10	3.411E-08	3.079E-02	3.079E-02	1.020E+02
	(EOC)			(3.02)	(-0.45)	(2.21)	(2.00)	(2.00)	
1/16	1/80	4	2	3.735E-05	4.813E-10	8.367E-09	7.659E-03	7.659E-03	2.050E+02
	(EOC)			(3.02)	(0.98)	(2.03)	(2.01)	(2.01)	
1/16	1/160	4	2	4.642E-06	5.426E-10	2.112E-09	1.909E-03	1.909E-03	4.112E+02
	(EOC)			(3.01)	(-0.17)	(1.99)	(2.00)	(2.00)	
1/16	1/319	4	2	5.701E-07	1.656E-09	1.704E-09	4.702E-04	4.702E-04	8.248E+02
	(EOC)			(2.13)	(-1.13)	(0.22)	(1.42)	(1.42)	
1/16	1/10	4	3	1.162E-03	4.561E-10	4.623E-08	3.874E-02	3.874E-02	3.333E+01
1/16	1/20	4	3	7.316E-05	9.581E-10	5.167E-09	4.969E-03	4.969E-03	6.791E+01
	(EOC)			(3.99)	(-1.07)	(3.16)	(2.96)	(2.96)	
1/16	1/40	4	3	4.532E-06	8.843E-10	1.039E-09	6.213E-04	6.213E-04	1.371E+02
	(EOC)			(4.01)	(0.12)	(2.31)	(3.00)	(3.00)	
1/16	1/80	4	3	2.811E-07	1.287E-09	1.289E-09	7.741E-05	7.741E-05	2.754E+02
	(EOC)			(4.01)	(-0.54)	(-0.31)	(3.00)	(3.00)	
1/16	1/160	4	3	1.749E-08	6.406E-10	6.407E-10	9.653E-06	9.653E-06	5.519E+02
	(EOC)			(4.01)	(1.01)	(1.01)	(3.00)	(3.00)	
1/16	1/319	4	3	1.105E-09	2.895E-09	2.895E-09	1.188E-06	1.188E-06	1.075E+03
	(EOC)			(2.80)	(-1.53)	(-1.53)	(2.12)	(2.12)	

Table 3.2: Variant (3.83) with norm $(L^2(H^1))$

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/16	1/10	4	1	2.632E-01	7.110E-10	2.291E-06	5.414E-02	5.414E-02	2.057E-01
1/16	1/20	4	1	6.427E-02	2.904E-10	6.460E-07	1.363E-02	1.363E-02	2.121E-01
	(EOC)			(2.03)	(1.29)	(1.83)	(1.99)	(1.99)	
1/16	1/40	4	1	1.570E-02	8.500E-11	1.643E-07	3.377E-03	3.377E-03	2.150E-01
	(EOC)			(2.03)	(1.77)	(1.98)	(2.01)	(2.01)	
1/16	1/80	4	1	3.871E-03	8.788E-11	3.704E-08	8.379E-04	8.379E-04	2.164E-01
	(EOC)			(2.02)	(-0.05)	(2.15)	(2.01)	(2.01)	
1/16	1/160	4	1	9.604E-04	1.926E-10	7.207E-09	2.085E-04	2.085E-04	2.171E-01
	(EOC)			(2.01)	(-1.13)	(2.36)	(2.01)	(2.01)	
1/16	1/319	4	1	2.361E-04	1.233E-10	6.204E-09	5.134E-05	5.134E-05	2.174E-01
	(EOC)			(1.42)	(0.45)	(0.15)	(1.42)	(1.42)	
1/16	1/10	4	2	1.974E-02	1.518E-10	2.049E-07	4.103E-03	4.103E-03	2.078E-01
1/16	1/20	4	2	2.452E-03	1.315E-10	3.196E-08	5.222E-04	5.222E-04	2.129E-01
	(EOC)			(3.01)	(0.21)	(2.68)	(2.97)	(2.97)	
1/16	1/40	4	2	3.020E-04	1.133E-10	4.066E-09	6.505E-05	6.505E-05	2.154E-01
	(EOC)			(3.02)	(0.22)	(2.97)	(3.00)	(3.00)	
1/16	1/80	4	2	3.735E-05	1.207E-10	4.162E-10	8.092E-06	8.092E-06	2.166E-01
	(EOC)			(3.02)	(-0.09)	(3.29)	(3.01)	(3.01)	
1/16	1/160	4	2	4.642E-06	2.000E-10	2.019E-10	1.008E-06	1.008E-06	2.172E-01
	(EOC)			(3.01)	(-0.73)	(1.04)	(3.00)	(3.00)	
1/16	1/319	4	2	5.701E-07	2.461E-10	2.463E-10	1.240E-07	1.240E-07	2.175E-01
	(EOC)			(2.13)	(-0.21)	(-0.20)	(2.12)	(2.12)	
1/16	1/10	4	3	1.162E-03	1.246E-10	1.388E-08	2.440E-04	2.440E-04	2.099E-01
1/16	1/20	4	3	7.316E-05	1.461E-10	1.000E-09	1.565E-05	1.565E-05	2.139E-01
	(EOC)			(3.99)	(-0.23)	(3.80)	(3.96)	(3.96)	
1/16	1/40	4	3	4.532E-06	1.926E-10	1.995E-10	9.784E-07	9.784E-07	2.159E-01
	(EOC)			(4.01)	(-0.40)	(2.33)	(4.00)	(4.00)	
1/16	1/80	4	3	2.811E-07	7.785E-11	7.790E-11	6.096E-08	6.096E-08	2.168E-01
	(EOC)			(4.01)	(1.31)	(1.36)	(4.00)	(4.00)	
1/16	1/160	4	3	1.749E-08	2.600E-10	2.600E-10	3.809E-09	3.809E-09	2.178E-01
	(EOC)			(4.01)	(-1.74)	(-1.74)	(4.00)	(4.00)	
1/16	1/319	4	3	1.105E-09	3.697E-10	3.697E-10	4.374E-10	4.375E-10	3.959E-01
	(EOC)			(2.80)	(-0.36)	(-0.36)	(2.19)	(2.19)	

Table 3.3: Variant (3.83) with norm (EO)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/16	1/10	4	1	2.632E-01	7.517E-10	2.244E-06	4.669E-03	4.669E-03	1.774E-02
1/16	1/20	4	1	6.427E-02	2.662E-10	6.326E-07	1.175E-03	1.175E-03	1.829E-02
	(EOC)			(2.03)	(1.50)	(1.83)	(1.99)	(1.99)	
1/16	1/40	4	1	1.570E-02	6.632E-11	1.609E-07	2.913E-04	2.913E-04	1.855E-02
	(EOC)			(2.03)	(2.01)	(1.98)	(2.01)	(2.01)	
1/16	1/80	4	1	3.871E-03	1.862E-11	3.628E-08	7.226E-05	7.226E-05	1.867E-02
	(EOC)			(2.02)	(1.83)	(2.15)	(2.01)	(2.01)	
1/16	1/160	4	1	9.604E-04	2.653E-11	7.056E-09	1.798E-05	1.798E-05	1.872E-02
	(EOC)			(2.01)	(-0.51)	(2.36)	(2.01)	(2.01)	
1/16	1/319	4	1	2.361E-04	1.956E-11	6.074E-09	4.427E-06	4.427E-06	1.875E-02
	(EOC)			(1.42)	(0.31)	(0.15)	(1.42)	(1.42)	
1/16	1/10	4	2	1.974E-02	3.993E-11	2.007E-07	3.538E-04	3.538E-04	1.792E-02
1/16	1/20	4	2	2.452E-03	1.592E-11	3.130E-08	4.503E-05	4.503E-05	1.836E-02
	(EOC)			(3.01)	(1.33)	(2.68)	(2.97)	(2.97)	
1/16	1/40	4	2	3.020E-04	1.607E-11	3.981E-09	5.610E-06	5.610E-06	1.858E-02
	(EOC)			(3.02)	(-0.01)	(2.98)	(3.00)	(3.00)	
1/16	1/80	4	2	3.735E-05	1.883E-11	3.905E-10	6.978E-07	6.978E-07	1.868E-02
	(EOC)			(3.02)	(-0.23)	(3.35)	(3.01)	(3.01)	
1/16	1/160	4	2	4.642E-06	3.186E-11	4.191E-11	8.694E-08	8.694E-08	1.873E-02
	(EOC)			(3.01)	(-0.76)	(3.22)	(3.00)	(3.00)	
1/16	1/319	4	2	5.701E-07	3.408E-11	3.533E-11	1.069E-08	1.069E-08	1.875E-02
	(EOC)			(2.13)	(-0.07)	(0.17)	(2.12)	(2.12)	
1/16	1/10	4	3	1.162E-03	1.647E-11	1.360E-08	2.104E-05	2.104E-05	1.810E-02
1/16	1/20	4	3	7.316E-05	1.615E-11	9.691E-10	1.350E-06	1.350E-06	1.845E-02
	(EOC)			(3.99)	(0.03)	(3.81)	(3.96)	(3.96)	
1/16	1/40	4	3	4.532E-06	2.029E-11	5.478E-11	8.438E-08	8.438E-08	1.862E-02
	(EOC)			(4.01)	(-0.33)	(4.14)	(4.00)	(4.00)	
1/16	1/80	4	3	2.811E-07	1.802E-11	1.819E-11	5.257E-09	5.257E-09	1.870E-02
	(EOC)			(4.01)	(0.17)	(1.59)	(4.00)	(4.00)	
1/16	1/160	4	3	1.749E-08	4.231E-11	4.268E-11	3.305E-10	3.305E-10	1.890E-02
	(EOC)			(4.01)	(-1.23)	(-1.23)	(3.99)	(3.99)	
1/16	1/319	4	3	1.105E-09	5.442E-11	5.491E-11	5.804E-11	5.849E-11	5.293E-02
	(EOC)			(2.80)	(-0.26)	(-0.26)	(1.76)	(1.76)	

Table 3.4: Variant (3.83) with norm (DG)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/12	1/10	3	1	1.107E+00	2.123E-04	4.316E-01	7.489E+00	7.932E+00	7.162E+00
1/12	1/20	3	1	4.512E-01	4.144E-04	7.596E-01	4.208E+00	4.340E+00	9.619E+00
	(EOC)			(1.30)	(-0.96)	(-0.82)	(0.83)	(0.87)	
1/12	1/40	3	1	3.056E-01	3.994E-04	8.915E-01	2.161E+00	2.346E+00	7.676E+00
	(EOC)			(0.56)	(0.05)	(-0.23)	(0.96)	(0.89)	
1/12	1/80	3	1	2.889E-01	4.990E-04	9.134E-01	1.100E+00	1.431E+00	4.954E+00
	(EOC)			(0.08)	(-0.32)	(-0.04)	(0.97)	(0.71)	
1/12	1/160	3	1	2.876E-01	4.229E-04	9.166E-01	5.570E-01	1.073E+00	3.730E+00
	(EOC)			(0.01)	(0.24)	(-0.01)	(0.98)	(0.42)	
1/12	1/319	3	1	2.875E-01	5.968E-04	9.170E-01	2.810E-01	9.591E-01	3.336E+00
	(EOC)			(0.00)	(-0.24)	(-0.00)	(0.49)	(0.08)	
1/12	1/10	3	2	4.933E-01	7.766E-04	1.550E+00	1.012E+00	1.869E+00	3.788E+00
1/12	1/20	3	2	2.985E-01	5.180E-04	1.198E+00	2.944E-01	1.235E+00	4.136E+00
	(EOC)			(0.72)	(0.58)	(0.37)	(1.78)	(0.60)	
1/12	1/40	3	2	2.878E-01	5.284E-04	9.489E-01	7.892E-02	9.522E-01	3.309E+00
	(EOC)			(0.05)	(-0.03)	(0.34)	(1.90)	(0.37)	
1/12	1/80	3	2	2.875E-01	5.260E-04	9.195E-01	2.043E-02	9.198E-01	3.199E+00
	(EOC)			(0.00)	(0.01)	(0.05)	(1.95)	(0.05)	
1/12	1/160	3	2	2.875E-01	4.054E-04	9.173E-01	5.397E-03	9.173E-01	3.191E+00
	(EOC)			(0.00)	(0.38)	(0.00)	(1.92)	(0.00)	
1/12	1/319	3	2	2.875E-01	6.018E-04	9.170E-01	1.727E-03	9.170E-01	3.190E+00
	(EOC)			(-0.00)	(-0.28)	(0.00)	(0.81)	(0.00)	

Table 3.5: Equation (3.85)-(3.86) for (EOC_τ) with norm ($L^2(L^2)$)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/12	1/10	3	1	1.107E+00	9.128E-07	1.571E-03	5.741E+00	5.741E+00	5.184E+00
1/12	1/20	3	1	4.512E-01	1.525E-06	2.742E-03	3.263E+00	3.263E+00	7.233E+00
	(EOC)			(1.30)	(-0.74)	(-0.80)	(0.81)	(0.81)	
1/12	1/40	3	1	3.056E-01	1.793E-06	3.214E-03	1.683E+00	1.683E+00	5.508E+00
	(EOC)			(0.56)	(-0.23)	(-0.23)	(0.96)	(0.96)	
1/12	1/80	3	1	2.889E-01	1.501E-06	3.292E-03	8.574E-01	8.574E-01	2.968E+00
	(EOC)			(0.08)	(0.26)	(-0.03)	(0.97)	(0.97)	
1/12	1/160	3	1	2.876E-01	1.358E-06	3.304E-03	4.334E-01	4.334E-01	1.507E+00
	(EOC)			(0.01)	(0.14)	(-0.01)	(0.98)	(0.98)	
1/12	1/319	3	1	2.875E-01	1.370E-06	3.306E-03	2.180E-01	2.180E-01	7.584E-01
	(EOC)			(0.00)	(-0.01)	(-0.00)	(0.49)	(0.49)	
1/12	1/10	3	2	4.933E-01	2.617E-06	5.596E-03	6.682E-01	6.682E-01	1.355E+00
1/12	1/20	3	2	2.985E-01	1.766E-06	4.321E-03	1.911E-01	1.912E-01	6.405E-01
	(EOC)			(0.72)	(0.57)	(0.37)	(1.81)	(1.81)	
1/12	1/40	3	2	2.878E-01	2.026E-06	3.421E-03	5.107E-02	5.118E-02	1.779E-01
	(EOC)			(0.05)	(-0.20)	(0.34)	(1.90)	(1.90)	
1/12	1/80	3	2	2.875E-01	1.694E-06	3.314E-03	1.311E-02	1.352E-02	4.704E-02
	(EOC)			(0.00)	(0.26)	(0.05)	(1.96)	(1.92)	
1/12	1/160	3	2	2.875E-01	1.658E-06	3.306E-03	3.317E-03	4.684E-03	1.629E-02
	(EOC)			(0.00)	(0.03)	(0.00)	(1.98)	(1.53)	
1/12	1/319	3	2	2.875E-01	1.333E-06	3.306E-03	8.389E-04	3.411E-03	1.186E-02
	(EOC)			(0.00)	(0.15)	(0.00)	(0.98)	(0.23)	

Table 3.6: Equation (3.85)-(3.86) for (EOC_τ) with norm ($L^2(H^1)$)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/12	1/10	3	1	1.107E+00	4.406E-06	9.260E-03	2.359E-02	2.535E-02	2.289E-02
1/12	1/20	3	1	4.512E-01	1.083E-05	2.093E-02	6.710E-03	2.198E-02	4.871E-02
	(EOC)			(1.30)	(-1.30)	(-1.18)	(1.81)	(0.21)	
1/12	1/40	3	1	3.056E-01	1.713E-05	2.946E-02	1.731E-03	2.951E-02	9.658E-02
	(EOC)			(0.56)	(-0.66)	(-0.49)	(1.96)	(-0.43)	
1/12	1/80	3	1	2.889E-01	1.197E-05	3.213E-02	4.410E-04	3.214E-02	1.112E-01
	(EOC)			(0.08)	(0.52)	(-0.13)	(1.97)	(-0.12)	
1/12	1/160	3	1	2.876E-01	1.447E-05	3.280E-02	1.124E-04	3.280E-02	1.141E-01
	(EOC)			(0.01)	(-0.27)	(-0.03)	(1.97)	(-0.03)	
1/12	1/319	3	1	2.875E-01	1.335E-05	3.296E-02	3.107E-05	3.296E-02	1.147E-01
	(EOC)			(0.00)	(0.06)	(-0.00)	(0.91)	(-0.00)	
1/12	1/10	3	2	4.933E-01	6.475E-06	1.032E-02	5.722E-03	1.181E-02	2.394E-02
1/12	1/20	3	2	2.985E-01	1.004E-05	2.300E-02	8.192E-04	2.301E-02	7.710E-02
	(EOC)			(0.72)	(-0.63)	(-1.16)	(2.80)	(-0.96)	
1/12	1/40	3	2	2.878E-01	1.272E-05	3.006E-02	1.102E-04	3.006E-02	1.045E-01
	(EOC)			(0.05)	(-0.34)	(-0.39)	(2.89)	(-0.39)	
1/12	1/80	3	2	2.875E-01	1.477E-05	3.224E-02	2.039E-05	3.224E-02	1.121E-01
	(EOC)			(0.00)	(-0.21)	(-0.10)	(2.43)	(-0.10)	
1/12	1/160	3	2	2.875E-01	1.505E-05	3.282E-02	1.516E-05	3.282E-02	1.142E-01
	(EOC)			(0.00)	(-0.03)	(-0.03)	(0.43)	(-0.03)	
1/12	1/319	3	2	2.875E-01	1.429E-05	3.296E-02	1.429E-05	3.296E-02	1.147E-01
	(EOC)			(-0.00)	(0.04)	(-0.00)	(0.04)	(-0.00)	

Table 3.7: Equation (3.85)-(3.86) for (EOC_τ) with norm (EO)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/12	1/10	3	1	1.107E+00	4.750E-06	9.101E-03	8.798E-03	1.267E-02	1.144E-02
1/12	1/20	3	1	4.512E-01	1.326E-05	2.059E-02	2.528E-03	2.074E-02	4.598E-02
	(EOC)			(1.30)	(-1.48)	(-1.18)	(1.80)	(-0.71)	
1/12	1/40	3	1	3.056E-01	2.318E-05	2.899E-02	6.537E-04	2.900E-02	9.490E-02
	(EOC)			(0.56)	(-0.81)	(-0.49)	(1.95)	(-0.48)	
1/12	1/80	3	1	2.889E-01	1.727E-05	3.162E-02	1.674E-04	3.162E-02	1.095E-01
	(EOC)			(0.08)	(0.43)	(-0.13)	(1.96)	(-0.12)	
1/12	1/160	3	1	2.876E-01	1.728E-05	3.228E-02	4.551E-05	3.228E-02	1.122E-01
	(EOC)			(0.01)	(-0.00)	(-0.03)	(1.88)	(-0.03)	
1/12	1/319	3	1	2.875E-01	9.242E-06	3.244E-02	1.407E-05	3.244E-02	1.128E-01
	(EOC)			(0.00)	(0.44)	(-0.00)	(0.83)	(-0.00)	
1/12	1/10	3	2	4.933E-01	3.701E-06	1.016E-02	2.211E-03	1.040E-02	2.108E-02
1/12	1/20	3	2	2.985E-01	9.935E-06	2.263E-02	3.188E-04	2.263E-02	7.582E-02
	(EOC)			(0.72)	(-1.42)	(-1.16)	(2.79)	(-1.12)	
1/12	1/40	3	2	2.878E-01	1.688E-05	2.958E-02	4.591E-05	2.958E-02	1.028E-01
	(EOC)			(0.05)	(-0.76)	(-0.39)	(2.80)	(-0.39)	
1/12	1/80	3	2	2.875E-01	1.554E-05	3.172E-02	1.647E-05	3.172E-02	1.103E-01
	(EOC)			(0.00)	(0.12)	(-0.10)	(1.48)	(-0.10)	
1/12	1/160	3	2	2.875E-01	1.550E-05	3.229E-02	1.551E-05	3.229E-02	1.123E-01
	(EOC)			(0.00)	(0.00)	(-0.03)	(0.09)	(-0.03)	
1/12	1/319	3	2	2.875E-01	1.893E-05	3.244E-02	1.893E-05	3.244E-02	1.128E-01
	(EOC)			(-0.00)	(-0.14)	(-0.00)	(-0.14)	(-0.00)	

Table 3.8: Equation (3.85)-(3.86) for (EOC_τ) with norm (DG)

3.4.4 The order of convergence with respect to the space

The setting for variant (3.84) (equation (3.82) with solution (3.84)): We put fixed $q = 2$ and $\tau = 0.05$. The mesh is uniformly refined. We carried out computations with spatial step $h = {}^1\!/_{2^n}, n \in \{3, \ldots, 7\}$ and with spatial polynomial degree $p = 1, 2, 3$.

The setting for equation (3.85)-(3.86) for testing (EOC_h): For $q = 2$ and $\tau = 0.01$, we carried out computations with spatial step $h = {}^1\!/_{6n}, n = 1, 2, 4, 8$ and with spatial polynomial degree $p = 1, 2, 3$. The mesh $\mathcal{T}_{hm}$ is refined uniformly.

The results are presented in tables 3.9, 3.10, 3.11, 3.12 for (3.84) and 3.13, 3.14, 3.15, 3.16 for (3.86).
For variant (3.84), the result yields:

- the norms $(L^2(H^1))$, (EO) and (DG) satisfy hypothesis (3.43b), that is, EOC $\approx p$ for η_S,
- η_T changes slightly for decreasing h.

For equation (3.85) equipped with solution (3.86) with respect to (EOC_h), the result yields:

- the property (3.43b) holds for 3 norms $(L^2(H^1))$, (EO) and (DG), where EOC $\approx p$ for η_S,
- η_T changes slightly for decreasing h,
- the error in seminorm changed slightly when $p = 1$, and it changed more when p is bigger (for example $p = 3$).

We have chosen norms (EO) and (DG). With one of these norms the expectations (3.43) hold. We will use a whole algorithm in Section 5.3 to solve a Navier-Stokes equation.

Remark 3.5. *We modified the EO-norm, so that we have considered the orthogonality between time basis functions $\phi_i^{m,\mu}$ on the space-time element $Q_{m,\mu}$. In this case, the matrix $\mathbb{S}$ in (3.51) is block diagonal. The previous problems were carried out with this modification EO-norm. We have shown that the results between EO-norm and the modified EO-norm vary only slightly. Therefore, in Section 5.3, the modified EO-norm is used.*

h	τ	p	q	$\lvert e_{h\tau}\rvert_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/8	1/20	1	2	2.409E-02	1.858E-04	2.530E-01	2.383E-03	2.530E-01	1.050E+01
1/16	1/20	1	2	1.217E-02	5.098E-05	2.526E-01	6.711E-04	2.526E-01	2.075E+01
(EOC)				(0.98)	(1.87)	(0.00)	(1.83)	(0.00)	
1/32	1/20	1	2	6.114E-03	4.145E-05	2.530E-01	1.737E-04	2.530E-01	4.139E+01
(EOC)				(0.99)	(0.30)	(-0.00)	(1.95)	(-0.00)	
1/64	1/20	1	2	3.063E-03	4.783E-05	2.534E-01	6.400E-05	2.534E-01	8.272E+01
(EOC)				(1.00)	(-0.21)	(-0.00)	(1.44)	(-0.00)	
1/128	1/20	1	2	1.533E-03	8.192E-05	2.536E-01	8.279E-05	2.536E-01	1.654E+02
(EOC)				(1.00)	(-0.78)	(-0.00)	(-0.37)	(-0.00)	
1/8	1/20	2	2	1.557E-03	1.110E-05	2.265E-02	1.327E-04	2.265E-02	1.455E+01
1/16	1/20	2	2	3.936E-04	1.813E-06	1.144E-02	1.961E-05	1.144E-02	2.908E+01
(EOC)				(1.98)	(2.61)	(0.98)	(2.76)	(0.98)	
1/32	1/20	2	2	9.891E-05	6.722E-07	5.760E-03	2.665E-06	5.760E-03	5.824E+01
(EOC)				(1.99)	(1.43)	(0.99)	(2.88)	(0.99)	
1/64	1/20	2	2	2.479E-05	6.700E-07	2.891E-03	7.464E-07	2.891E-03	1.166E+02
(EOC)				(2.00)	(0.00)	(0.99)	(1.84)	(0.99)	
1/128	1/20	2	2	6.205E-06	5.448E-07	1.448E-03	5.464E-07	1.448E-03	2.334E+02
(EOC)				(2.00)	(0.30)	(1.00)	(0.45)	(1.00)	
1/8	1/20	3	2	6.258E-05	6.969E-08	9.316E-04	4.702E-06	9.317E-04	1.489E+01
1/16	1/20	3	2	7.795E-06	1.184E-07	2.288E-04	3.548E-07	2.288E-04	2.935E+01
(EOC)				(3.01)	(-0.76)	(2.03)	(3.73)	(2.03)	
1/32	1/20	3	2	9.731E-07	3.321E-08	5.680E-05	3.968E-08	5.680E-05	5.837E+01
(EOC)				(3.00)	(1.83)	(2.01)	(3.16)	(2.01)	
1/64	1/20	3	2	1.216E-07	8.117E-08	1.416E-05	8.124E-08	1.416E-05	1.164E+02
(EOC)				(3.00)	(-1.29)	(2.00)	(-1.03)	(2.00)	
1/128	1/20	3	2	2.456E-08	9.075E-07	3.768E-06	9.113E-07	3.769E-06	1.535E+02
(EOC)				(2.31)	(-3.48)	(1.91)	(-3.49)	(1.91)	

Table 3.9: Variant (3.84) with norm $(L^2(L^2))$

h	τ	p	q	$\lvert e_{h\tau}\rvert_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/8	1/20	1	2	2.409E-02	2.733E-06	3.892E-03	1.078E-03	4.038E-03	1.676E-01
1/16	1/20	1	2	1.217E-02	7.131E-07	1.938E-03	3.211E-04	1.964E-03	1.613E-01
(EOC)				(0.98)	(1.94)	(1.01)	(1.75)	(1.04)	
1/32	1/20	1	2	6.114E-03	1.316E-07	9.695E-04	8.174E-05	9.730E-04	1.591E-01
(EOC)				(0.99)	(2.44)	(1.00)	(1.97)	(1.01)	
1/64	1/20	1	2	3.063E-03	1.613E-07	4.853E-04	2.060E-05	4.857E-04	1.586E-01
(EOC)				(1.00)	(-0.29)	(1.00)	(1.99)	(1.00)	
1/128	1/20	1	2	1.533E-03	1.460E-07	2.428E-04	5.191E-06	2.429E-04	1.584E-01
(EOC)				(1.00)	(0.14)	(1.00)	(1.99)	(1.00)	
1/8	1/20	2	2	1.557E-03	6.719E-08	2.118E-04	4.715E-05	2.170E-04	1.394E-01
1/16	1/20	2	2	3.936E-04	1.732E-08	5.360E-05	7.370E-06	5.410E-05	1.375E-01
(EOC)				(1.98)	(1.96)	(1.98)	(2.68)	(2.00)	
1/32	1/20	2	2	9.891E-05	9.335E-09	1.350E-05	9.908E-07	1.353E-05	1.368E-01
(EOC)				(1.99)	(0.89)	(1.99)	(2.90)	(2.00)	
1/64	1/20	2	2	2.479E-05	1.811E-08	3.387E-06	1.283E-07	3.390E-06	1.367E-01
(EOC)				(2.00)	(-0.96)	(1.99)	(2.95)	(2.00)	
1/128	1/20	2	2	6.205E-06	4.025E-08	8.495E-07	4.439E-08	8.497E-07	1.369E-01
(EOC)				(2.00)	(-1.15)	(2.00)	(1.53)	(2.00)	
1/8	1/20	3	2	6.258E-05	4.362E-09	6.644E-06	2.848E-07	6.650E-06	1.063E-01
1/16	1/20	3	2	7.795E-06	1.627E-10	8.178E-07	3.278E-08	8.185E-07	1.050E-01
(EOC)				(3.01)	(4.74)	(3.02)	(3.12)	(3.02)	
1/32	1/20	3	2	9.731E-07	8.184E-10	1.016E-07	2.662E-09	1.017E-07	1.045E-01
(EOC)				(3.00)	(-2.33)	(3.01)	(3.62)	(3.01)	
1/64	1/20	3	2	1.216E-07	2.121E-08	2.471E-08	2.142E-08	2.489E-08	2.047E-01
(EOC)				(3.00)	(-4.70)	(2.04)	(-3.01)	(2.03)	
1/128	1/20	3	2	2.456E-08	3.162E-07	3.162E-07	3.270E-07	3.270E-07	1.331E+01
(EOC)				(2.31)	(-3.90)	(-3.68)	(-3.93)	(-3.72)	

Table 3.10: Variant (3.84) with norm $(L^2(H^1))$

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/8	1/20	1	2	2.409E-02	7.823E-06	1.227E-02	9.076E-06	1.227E-02	5.094E-01
1/16	1/20	1	2	1.217E-02	4.431E-06	6.124E-03	4.635E-06	6.124E-03	5.030E-01
(EOC)				(0.98)	(0.82)	(1.00)	(0.97)	(1.00)	
1/32	1/20	1	2	6.114E-03	1.273E-06	3.065E-03	1.319E-06	3.065E-03	5.014E-01
(EOC)				(0.99)	(1.80)	(1.00)	(1.81)	(1.00)	
1/64	1/20	1	2	3.063E-03	5.901E-07	1.534E-03	5.965E-07	1.534E-03	5.009E-01
(EOC)				(1.00)	(1.11)	(1.00)	(1.15)	(1.00)	
1/128	1/20	1	2	1.533E-03	5.629E-07	7.678E-04	5.633E-07	7.678E-04	5.008E-01
(EOC)				(1.00)	(0.07)	(1.00)	(0.08)	(1.00)	
1/8	1/20	2	2	1.557E-03	4.068E-07	6.689E-04	4.564E-07	6.689E-04	4.297E-01
1/16	1/20	2	2	3.936E-04	5.991E-08	1.694E-04	6.765E-08	1.694E-04	4.305E-01
(EOC)				(1.98)	(2.76)	(1.98)	(2.75)	(1.98)	
1/32	1/20	2	2	9.891E-05	2.474E-08	4.267E-05	2.509E-08	4.267E-05	4.314E-01
(EOC)				(1.99)	(1.28)	(1.99)	(1.43)	(1.99)	
1/64	1/20	2	2	2.479E-05	3.463E-09	1.071E-05	3.504E-09	1.071E-05	4.320E-01
(EOC)				(2.00)	(2.84)	(1.99)	(2.84)	(1.99)	
1/128	1/20	2	2	6.205E-06	1.754E-08	2.683E-06	1.754E-08	2.683E-06	4.324E-01
(EOC)				(2.00)	(-2.34)	(2.00)	(-2.32)	(2.00)	
1/8	1/20	3	2	6.258E-05	2.706E-09	2.099E-05	3.147E-09	2.099E-05	3.354E-01
1/16	1/20	3	2	7.795E-06	4.407E-10	2.586E-06	4.634E-10	2.586E-06	3.317E-01
(EOC)				(3.01)	(2.62)	(3.02)	(2.76)	(3.02)	
1/32	1/20	3	2	9.731E-07	1.675E-10	3.214E-07	1.679E-10	3.214E-07	3.302E-01
(EOC)				(3.00)	(1.40)	(3.01)	(1.46)	(3.01)	
1/64	1/20	3	2	1.216E-07	4.787E-09	4.036E-08	4.787E-09	4.036E-08	3.320E-01
(EOC)				(3.00)	(-4.84)	(2.99)	(-4.83)	(2.99)	
1/128	1/20	3	2	2.456E-08	4.964E-08	4.990E-08	4.965E-08	4.990E-08	2.032E+00
(EOC)				(2.31)	(-3.37)	(-0.31)	(-3.37)	(-0.31)	

Table 3.11: Variant (3.84) with norm (EO)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/8	1/20	1	2	2.409E-02	1.742E-06	1.050E-02	1.969E-06	1.050E-02	4.357E-01
1/16	1/20	1	2	1.218E-02	3.575E-06	5.228E-03	3.578E-06	5.228E-03	4.294E-01
(EOC)				(0.98)	(-1.04)	(1.01)	(-0.86)	(1.01)	
1/32	1/20	1	2	6.114E-03	1.386E-06	2.616E-03	1.386E-06	2.616E-03	4.278E-01
(EOC)				(0.99)	(1.37)	(1.00)	(1.37)	(1.00)	
1/64	1/20	1	2	3.063E-03	3.324E-07	1.309E-03	3.324E-07	1.309E-03	4.274E-01
(EOC)				(1.00)	(2.06)	(1.00)	(2.06)	(1.00)	
1/128	1/20	1	2	1.533E-03	1.721E-07	6.551E-04	1.721E-07	6.551E-04	4.273E-01
(EOC)				(1.00)	(0.95)	(1.00)	(0.95)	(1.00)	
1/8	1/20	2	2	1.557E-03	1.041E-07	6.243E-04	1.175E-07	6.243E-04	4.011E-01
1/16	1/20	2	2	3.936E-04	1.143E-07	1.580E-04	1.144E-07	1.580E-04	4.015E-01
(EOC)				(1.98)	(-0.13)	(1.98)	(0.04)	(1.98)	
1/32	1/20	2	2	9.891E-05	2.699E-08	3.979E-05	2.699E-08	3.979E-05	4.023E-01
(EOC)				(1.99)	(2.08)	(1.99)	(2.08)	(1.99)	
1/64	1/20	2	2	2.479E-05	1.590E-09	9.987E-06	1.590E-09	9.987E-06	4.029E-01
(EOC)				(2.00)	(4.09)	(1.99)	(4.09)	(1.99)	
1/128	1/20	2	2	6.205E-06	8.682E-10	2.502E-06	8.682E-10	2.502E-06	4.032E-01
(EOC)				(2.00)	(0.87)	(2.00)	(0.87)	(2.00)	
1/8	1/20	3	2	6.258E-05	8.651E-09	2.015E-05	8.706E-09	2.015E-05	3.221E-01
1/16	1/20	3	2	7.795E-06	7.676E-10	2.481E-06	7.685E-10	2.481E-06	3.183E-01
(EOC)				(3.01)	(3.49)	(3.02)	(3.50)	(3.02)	
1/32	1/20	3	2	9.731E-07	8.564E-11	3.083E-07	8.566E-11	3.083E-07	3.168E-01
(EOC)				(3.00)	(3.16)	(3.01)	(3.17)	(3.01)	
1/64	1/20	3	2	1.216E-07	1.121E-10	3.844E-08	1.121E-10	3.844E-08	3.162E-01
(EOC)				(3.00)	(-0.39)	(3.00)	(-0.39)	(3.00)	
1/128	1/20	3	2	2.456E-08	7.918E-10	4.875E-09	7.919E-10	4.875E-09	1.985E-01
(EOC)				(2.31)	(-2.82)	(2.98)	(-2.82)	(2.98)	

Table 3.12: Variant (3.84) with norm (DG)

h	τ	p	q	$\lvert e_{h\tau} \rvert_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	2	3.172E+00	1.843E-03	4.506E+00	7.112E-03	4.506E+00	1.421E+00
1/12	1/100	1	2	2.129E+00	2.195E-03	5.086E+00	1.662E-02	5.086E+00	2.389E+00
(EOC)				(0.58)	(-0.25)	(-0.17)	(-1.22)	(-0.17)	
1/24	1/100	1	2	1.193E+00	2.322E-03	4.712E+00	2.461E-02	4.712E+00	3.949E+00
(EOC)				(0.84)	(-0.08)	(0.11)	(-0.57)	(0.11)	
1/48	1/100	1	2	6.137E-01	1.744E-03	4.520E+00	1.658E-02	4.520E+00	7.365E+00
(EOC)				(0.96)	(0.41)	(0.06)	(0.57)	(0.06)	
1/6	1/100	2	2	1.940E+00	2.120E-03	2.927E+00	1.017E-02	2.927E+00	1.509E+00
1/12	1/100	2	2	7.918E-01	1.476E-03	2.196E+00	1.220E-02	2.196E+00	2.774E+00
(EOC)				(1.29)	(0.52)	(0.41)	(-0.26)	(0.41)	
1/24	1/100	2	2	2.384E-01	6.456E-04	1.283E+00	1.388E-02	1.283E+00	5.382E+00
(EOC)				(1.73)	(1.19)	(0.78)	(-0.19)	(0.78)	
1/48	1/100	2	2	6.285E-02	3.561E-04	6.713E-01	1.461E-02	6.715E-01	1.068E+01
(EOC)				(1.92)	(0.86)	(0.93)	(-0.07)	(0.93)	
1/6	1/100	3	2	1.151E+00	1.289E-03	1.929E+00	9.951E-03	1.929E+00	1.676E+00
1/12	1/100	3	2	2.875E-01	4.808E-04	9.181E-01	1.322E-02	9.182E-01	3.194E+00
(EOC)				(2.00)	(1.42)	(1.07)	(-0.41)	(1.07)	
1/24	1/100	3	2	4.629E-02	1.697E-04	3.014E-01	1.435E-02	3.017E-01	6.518E+00
(EOC)				(2.63)	(1.50)	(1.61)	(-0.12)	(1.61)	
1/48	1/100	3	2	6.188E-03	4.098E-05	8.056E-02	1.496E-02	8.197E-02	1.325E+01
(EOC)				(2.90)	(2.05)	(1.90)	(-0.06)	(1.88)	

Table 3.13: Equation (3.85)-(3.86) for (EOC_h) with norm ($L^2(L^2)$)

h	τ	p	q	$\lvert e_{h\tau} \rvert_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	2	3.171E+00	2.088E-05	6.622E-02	3.021E-03	6.629E-02	2.090E-02
1/12	1/100	1	2	2.129E+00	1.844E-05	3.768E-02	9.525E-03	3.887E-02	1.826E-02
(EOC)				(0.58)	(0.18)	(0.81)	(-1.66)	(0.77)	
1/24	1/100	1	2	1.193E+00	6.937E-06	1.750E-02	1.446E-02	2.270E-02	1.903E-02
(EOC)				(0.84)	(1.41)	(1.11)	(-0.60)	(0.78)	
1/48	1/100	1	2	6.137E-01	3.796E-06	8.401E-03	1.327E-02	1.571E-02	2.560E-02
(EOC)				(0.96)	(0.87)	(1.06)	(0.12)	(0.53)	
1/6	1/100	2	2	1.940E+00	1.194E-05	2.795E-02	3.930E-03	2.823E-02	1.455E-02
1/12	1/100	2	2	7.919E-01	4.876E-06	1.041E-02	7.936E-03	1.309E-02	1.653E-02
(EOC)				(1.29)	(1.29)	(1.43)	(-1.01)	(1.11)	
1/24	1/100	2	2	2.384E-01	1.430E-06	3.033E-03	1.193E-02	1.231E-02	5.165E-02
(EOC)				(1.73)	(1.77)	(1.78)	(-0.59)	(0.09)	
1/48	1/100	2	2	6.285E-02	1.137E-07	7.927E-04	1.381E-02	1.384E-02	2.202E-01
(EOC)				(1.92)	(3.65)	(1.94)	(-0.21)	(-0.17)	
1/6	1/100	3	2	1.151E+00	6.119E-06	1.398E-02	4.052E-03	1.455E-02	1.264E-02
1/12	1/100	3	2	2.875E-01	1.600E-06	3.309E-03	8.430E-03	9.056E-03	3.150E-02
(EOC)				(2.00)	(1.93)	(2.08)	(-1.06)	(0.68)	
1/24	1/100	3	2	4.629E-02	1.214E-07	5.417E-04	1.229E-02	1.230E-02	2.658E-01
(EOC)				(2.63)	(3.72)	(2.61)	(-0.54)	(-0.44)	
1/48	1/100	3	2	6.188E-03	5.984E-09	7.225E-05	1.395E-02	1.395E-02	2.255E+00
(EOC)				(2.90)	(4.34)	(2.91)	(-0.18)	(-0.18)	

Table 3.14: Equation (3.85)-(3.86) for (EOC_h) with norm ($L^2(H^1)$)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	2	3.172E+00	3.977E-04	6.497E-01	3.977E-04	6.497E-01	2.049E-01
1/12	1/100	1	2	2.129E+00	1.014E-04	3.734E-01	1.017E-04	3.734E-01	1.754E-01
(EOC)				(0.58)	(1.97)	(0.80)	(1.97)	(0.80)	
1/24	1/100	1	2	1.193E+00	1.075E-04	1.736E-01	1.082E-04	1.736E-01	1.455E-01
(EOC)				(0.84)	(-0.08)	(1.11)	(-0.09)	(1.11)	
1/48	1/100	1	2	6.137E-01	3.590E-05	8.336E-02	3.761E-05	8.336E-02	1.358E-01
(EOC)				(0.96)	(1.58)	(1.06)	(1.52)	(1.06)	
1/6	1/100	2	2	1.940E+00	9.583E-05	2.758E-01	9.590E-05	2.758E-01	1.422E-01
1/12	1/100	2	2	7.919E-01	5.385E-05	1.029E-01	5.427E-05	1.029E-01	1.299E-01
(EOC)				(1.29)	(0.83)	(1.42)	(0.82)	(1.42)	
1/24	1/100	2	2	2.384E-01	1.435E-05	2.992E-02	1.754E-05	2.992E-02	1.255E-01
(EOC)				(1.73)	(1.91)	(1.78)	(1.63)	(1.78)	
1/48	1/100	2	2	6.285E-02	2.994E-06	7.808E-03	1.206E-05	7.808E-03	1.242E-01
(EOC)				(1.92)	(2.26)	(1.94)	(0.54)	(1.94)	
1/6	1/100	3	2	1.151E+00	6.146E-05	1.379E-01	6.158E-05	1.379E-01	1.198E-01
1/12	1/100	3	2	2.875E-01	1.705E-05	3.252E-02	1.852E-05	3.252E-02	1.131E-01
(EOC)				(2.00)	(1.85)	(2.08)	(1.73)	(2.08)	
1/24	1/100	3	2	4.629E-02	2.004E-06	5.294E-03	1.059E-05	5.294E-03	1.144E-01
(EOC)				(2.63)	(3.09)	(2.62)	(0.81)	(2.62)	
1/48	1/100	3	2	6.188E-03	2.247E-07	7.043E-04	1.180E-05	7.044E-04	1.138E-01
(EOC)				(2.90)	(3.16)	(2.91)	(-0.16)	(2.91)	

Table 3.15: Equation (3.85)-(3.86) for (EOC_h) with norm (EO)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	2	3.172E+00	3.319E-04	6.067E-01	3.319E-04	6.067E-01	1.913E-01
1/12	1/100	1	2	2.129E+00	2.044E-04	3.417E-01	2.044E-04	3.417E-01	1.605E-01
(EOC)				(0.58)	(0.70)	(0.83)	(0.70)	(0.83)	
1/24	1/100	1	2	1.193E+00	7.574E-05	1.578E-01	7.578E-05	1.578E-01	1.322E-01
(EOC)				(0.84)	(1.43)	(1.12)	(1.43)	(1.12)	
1/48	1/100	1	2	6.137E-01	2.983E-05	7.553E-02	2.985E-05	7.553E-02	1.231E-01
(EOC)				(0.96)	(1.34)	(1.06)	(1.34)	(1.06)	
1/6	1/100	2	2	1.940E+00	1.403E-04	2.701E-01	1.404E-04	2.701E-01	1.392E-01
1/12	1/100	2	2	7.918E-01	6.131E-05	9.976E-02	6.136E-05	9.976E-02	1.260E-01
(EOC)				(1.29)	(1.19)	(1.44)	(1.19)	(1.44)	
1/24	1/100	2	2	2.384E-01	1.687E-05	2.893E-02	1.697E-05	2.893E-02	1.213E-01
(EOC)				(1.73)	(1.86)	(1.79)	(1.85)	(1.79)	
1/48	1/100	2	2	6.285E-02	4.096E-06	7.550E-03	4.233E-06	7.550E-03	1.201E-01
(EOC)				(1.92)	(2.04)	(1.94)	(2.00)	(1.94)	
1/6	1/100	3	2	1.151E+00	6.816E-05	1.364E-01	6.823E-05	1.364E-01	1.185E-01
1/12	1/100	3	2	2.875E-01	1.588E-05	3.200E-02	1.613E-05	3.200E-02	1.113E-01
(EOC)				(2.00)	(2.10)	(2.09)	(2.08)	(2.09)	
1/24	1/100	3	2	4.629E-02	2.622E-06	5.201E-03	3.262E-06	5.201E-03	1.124E-01
(EOC)				(2.63)	(2.60)	(2.62)	(2.31)	(2.62)	
1/48	1/100	3	2	6.188E-03	3.602E-07	6.917E-04	1.138E-06	6.917E-04	1.118E-01
(EOC)				(2.90)	(2.86)	(2.91)	(1.52)	(2.91)	

Table 3.16: Equation (3.85)-(3.86) for (EOC_h) with norm (DG)

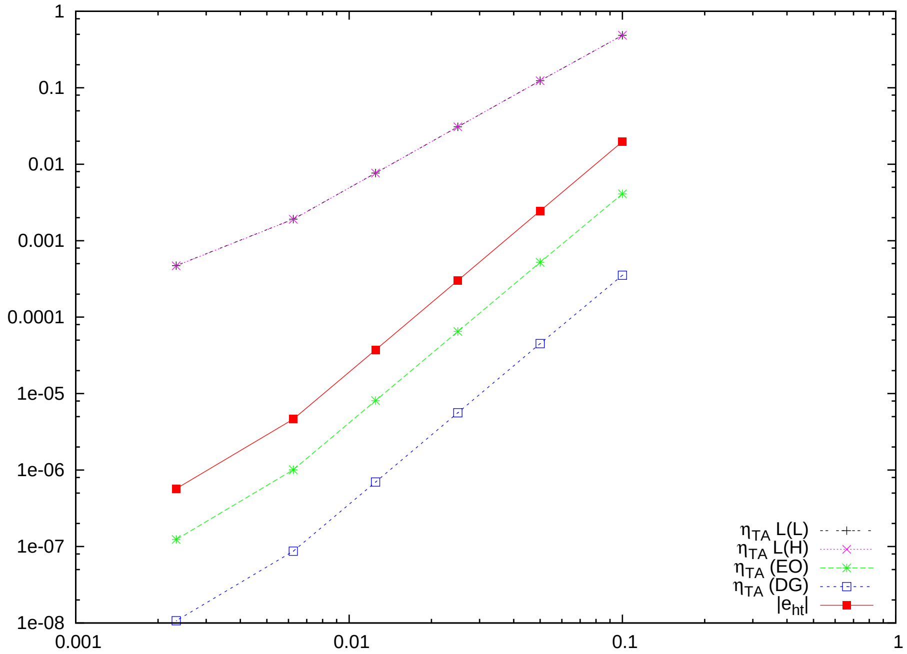

Figure 3.5: The order of convergence EOC_τ of equation (3.82)-(3.83) with IIPG variant equipped with norms $(L^2(L^2))$, $(L^2(H^1))$, (EO) and (DG), $p = 4, q = 2$.

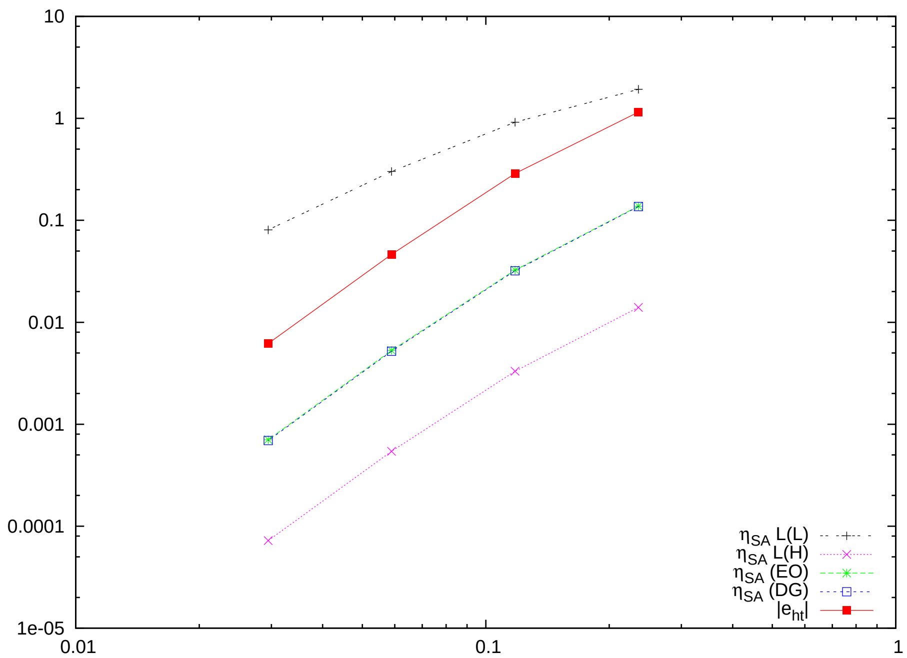

Figure 3.6: The order of convergence EOC_h of equation (3.85)-(3.86) with IIPG variant equipped with norms $(L^2(L^2))$, $(L^2(H^1))$, (EO) and (DG), $p = 3, q = 2$.

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	5	2	3.840E-01	2.337E-06	3.255E-03	2.364E-06	3.255E-03	8.476E-03
1/12	1/100	5	2	3.460E-02	5.876E-08	1.828E-04	6.833E-07	1.828E-04	5.282E-03
(EOC)				(3.47)	(5.31)	(4.15)	(1.79)	(4.15)	
1/24	1/100	5	2	1.878E-03	2.468E-09	2.164E-05	1.116E-06	2.167E-05	1.154E-02
(EOC)				(4.20)	(4.57)	(3.08)	(-0.71)	(3.08)	
1/6	1/100	6	2	2.142E-01	3.096E-07	1.104E-03	4.871E-07	1.104E-03	5.156E-03
1/12	1/100	6	2	1.194E-02	4.256E-08	1.024E-04	6.912E-07	1.024E-04	8.573E-03
(EOC)				(4.16)	(2.86)	(3.43)	(-0.51)	(3.43)	
1/24	1/100	6	2	1.017E-03	1.132E-10	2.677E-06	1.116E-06	2.900E-06	2.851E-03
(EOC)				(3.55)	(8.55)	(5.26)	(-0.69)	(5.14)	

Table 3.17: Equation (3.85)-(3.86) with norm (EO)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	5	2	3.840E-01	1.745E-06	3.239E-03	1.768E-06	3.239E-03	8.436E-03
1/12	1/100	5	2	3.460E-02	1.219E-07	1.816E-04	3.034E-07	1.816E-04	5.248E-03
(EOC)				(3.47)	(3.84)	(4.16)	(2.54)	(4.16)	
1/24	1/100	5	2	1.878E-03	3.110E-09	2.149E-05	2.102E-07	2.149E-05	1.144E-02
(EOC)				(4.20)	(5.29)	(3.08)	(0.53)	(3.08)	
1/6	1/100	6	2	2.142E-01	4.800E-07	1.100E-03	5.618E-07	1.100E-03	5.137E-03
1/12	1/100	6	2	1.194E-02	2.416E-08	1.019E-04	2.815E-07	1.019E-04	8.531E-03
(EOC)				(4.16)	(4.31)	(3.43)	(1.00)	(3.43)	
1/24	1/100	6	2	1.017E-03	6.862E-10	2.663E-06	2.102E-07	2.671E-06	2.626E-03
(EOC)				(3.55)	(5.14)	(5.26)	(0.42)	(5.25)	

Table 3.18: Equation (3.85)-(3.86) with norm (DG)

3.4.5 Numerical experiments

In this section, we test the robustness of the selected norms with respect to spatial polynomial degree p and diffusion coefficient ε. In other words, we ask: Do the properties (3.43) hold for higher spatial degree p? or Do they hold with smaller diffusion coefficient ε?
We have chosen the norms (EO) and (DG). To test the norms, we carried out computations for spatial polynomial degree $p = 5, 6$ for equations (3.85)-(3.86) . The results are presented in tables 3.17 and 3.18.
We also test the two norms for equation (3.85)-(3.86) with different diffusion coefficients $\varepsilon = 0.001, 0.0001$; the results are shown in tables 3.19, 3.20, 3.21 and 3.22.
The properties hold for higher approximate polynomial degree p, but they do not hold for very small diffusion coefficients ε.

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	1	1.524E+01	1.192E-03	2.125E+00	1.297E-03	2.125E+00	1.394E-01
1/12	1/100	1	1	1.459E+01	7.963E-04	1.476E+00	1.232E-03	1.476E+00	1.012E-01
(EOC)				(0.06)	(0.58)	(0.53)	(0.07)	(0.53)	
1/24	1/100	1	1	1.328E+01	5.869E-04	9.968E-01	1.866E-03	9.968E-01	7.506E-02
(EOC)				(0.14)	(0.44)	(0.57)	(-0.60)	(0.57)	
1/6	1/100	2	1	1.498E+01	9.052E-04	1.672E+00	1.252E-03	1.672E+00	1.116E-01
1/12	1/100	2	1	1.385E+01	6.858E-04	1.083E+00	1.487E-03	1.083E+00	7.817E-02
(EOC)				(0.11)	(0.40)	(0.63)	(-0.25)	(0.63)	
1/24	1/100	2	1	1.145E+01	3.549E-04	5.975E-01	2.292E-03	5.975E-01	5.218E-02
(EOC)				(0.27)	(0.95)	(0.86)	(-0.62)	(0.86)	
1/6	1/100	2	2	1.499E+01	8.655E-04	1.674E+00	8.679E-04	1.674E+00	1.117E-01
1/12	1/100	2	2	1.386E+01	5.432E-04	1.086E+00	5.632E-04	1.086E+00	7.832E-02
(EOC)				(0.11)	(0.67)	(0.62)	(0.62)	(0.62)	
1/24	1/100	2	2	1.146E+01	2.667E-04	6.063E-01	3.910E-04	6.063E-01	5.288E-02
(EOC)				(0.27)	(1.03)	(0.84)	(0.53)	(0.84)	
1/6	1/100	3	2	1.500E+01	6.456E-04	1.380E+00	6.554E-04	1.380E+00	9.197E-02
1/12	1/100	3	2	1.324E+01	4.554E-04	8.236E-01	5.103E-04	8.236E-01	6.221E-02
(EOC)				(0.18)	(0.50)	(0.74)	(0.36)	(0.74)	
1/24	1/100	3	2	9.796E+00	1.865E-04	3.919E-01	4.130E-04	3.919E-01	4.000E-02
(EOC)				(0.43)	(1.29)	(1.07)	(0.31)	(1.07)	

Table 3.19: Equation (3.85)-(3.86) with $\varepsilon = 10^{-3}$ with norm (EO)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	1	4.500E+01	2.665E-03	4.665E+00	2.775E-03	4.665E+00	1.037E-01
1/12	1/100	1	1	5.220E+01	2.822E-03	4.605E+00	3.430E-03	4.605E+00	8.820E-02
(EOC)				(-0.21)	(-0.08)	(0.02)	(-0.31)	(0.02)	
1/24	1/100	1	1	4.993E+01	1.808E-03	3.388E+00	4.406E-03	3.388E+00	6.785E-02
(EOC)				(0.06)	(0.64)	(0.44)	(-0.36)	(0.44)	
1/6	1/100	2	1	4.673E+01	2.448E-03	4.483E+00	3.009E-03	4.483E+00	9.594E-02
1/12	1/100	2	1	5.061E+01	2.504E-03	3.749E+00	4.268E-03	3.749E+00	7.408E-02
(EOC)				(-0.12)	(-0.03)	(0.26)	(-0.50)	(0.26)	
1/24	1/100	2	1	5.082E+01	1.651E-03	2.441E+00	5.553E-03	2.441E+00	4.804E-02
(EOC)				(-0.01)	(0.60)	(0.62)	(-0.38)	(0.62)	
1/6	1/100	2	2	5.516E+01	2.066E-03	4.496E+00	2.072E-03	4.496E+00	8.151E-02
1/12	1/100	2	2	6.179E+01	2.016E-03	3.788E+00	2.109E-03	3.788E+00	6.131E-02
(EOC)				(-0.16)	(0.03)	(0.25)	(-0.03)	(0.25)	
1/24	1/100	2	2	4.690E+01	1.303E-03	2.489E+00	2.042E-03	2.489E+00	5.306E-02
(EOC)				(0.40)	(0.63)	(0.61)	(0.05)	(0.61)	
1/6	1/100	3	2	4.624E+01	2.386E-03	4.502E+00	2.410E-03	4.502E+00	9.736E-02
1/12	1/100	3	2	4.319E+01	2.018E-03	3.376E+00	2.260E-03	3.376E+00	7.818E-02
(EOC)				(0.10)	(0.24)	(0.42)	(0.09)	(0.42)	
1/24	1/100	3	2	5.130E+01	1.456E-03	2.008E+00	2.541E-03	2.008E+00	3.914E-02
(EOC)				(-0.25)	(0.47)	(0.75)	(-0.17)	(0.75)	

Table 3.20: Equation (3.85)-(3.86) with $\varepsilon = 10^{-4}$ with norm (EO)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	1	1.524E+01	1.039E-03	2.257E+00	1.369E-03	2.257E+00	1.481E-01
1/12	1/100	1	1	1.460E+01	8.301E-04	1.442E+00	1.289E-03	1.442E+00	9.878E-02
(EOC)				(0.06)	(0.32)	(0.65)	(0.09)	(0.65)	
1/24	1/100	1	1	1.328E+01	5.320E-04	9.401E-01	1.129E-03	9.401E-01	7.079E-02
(EOC)				(0.14)	(0.64)	(0.62)	(0.19)	(0.62)	
1/6	1/100	2	1	1.498E+01	9.426E-04	1.760E+00	1.659E-03	1.760E+00	1.174E-01
1/12	1/100	2	1	1.385E+01	6.997E-04	1.080E+00	1.552E-03	1.080E+00	7.799E-02
(EOC)				(0.11)	(0.43)	(0.70)	(0.10)	(0.70)	
1/24	1/100	2	1	1.145E+01	2.971E-04	5.851E-01	1.411E-03	5.851E-01	5.109E-02
(EOC)				(0.27)	(1.24)	(0.88)	(0.14)	(0.88)	
1/6	1/100	2	2	1.499E+01	8.349E-04	1.762E+00	8.405E-04	1.762E+00	1.176E-01
1/12	1/100	2	2	1.386E+01	5.145E-04	1.083E+00	5.360E-04	1.083E+00	7.814E-02
(EOC)				(0.11)	(0.70)	(0.70)	(0.65)	(0.70)	
1/24	1/100	2	2	1.146E+01	3.221E-04	5.937E-01	3.661E-04	5.937E-01	5.178E-02
(EOC)				(0.27)	(0.68)	(0.87)	(0.55)	(0.87)	
1/6	1/100	3	2	1.500E+01	6.520E-04	1.430E+00	6.745E-04	1.430E+00	9.532E-02
1/12	1/100	3	2	1.324E+01	4.687E-04	8.228E-01	5.240E-04	8.228E-01	6.216E-02
(EOC)				(0.18)	(0.48)	(0.80)	(0.36)	(0.80)	
1/24	1/100	3	2	9.796E+00	2.416E-04	3.872E-01	3.300E-04	3.872E-01	3.953E-02
(EOC)				(0.43)	(0.96)	(1.09)	(0.67)	(1.09)	

Table 3.21: Equation (3.85)-(3.86) with $\varepsilon = 10^{-3}$ with norm (DG)

h	τ	p	q	$\|e_{h\tau}\|_{L^2(H^1)}$	η_A	η_S	η_T	η_{ST}	i_X
1/6	1/100	1	1	4.500E+01	3.639E-03	8.062E+00	4.945E-03	8.062E+00	1.792E-01
1/12	1/100	1	1	5.221E+01	3.025E-03	5.552E+00	5.597E-03	5.552E+00	1.063E-01
(EOC)				(-0.21)	(0.27)	(0.54)	(-0.18)	(0.54)	
1/24	1/100	1	1	4.993E+01	1.878E-03	3.459E+00	6.180E-03	3.459E+00	6.928E-02
(EOC)				(0.06)	(0.69)	(0.68)	(-0.14)	(0.68)	
1/6	1/100	2	1	4.673E+01	3.412E-03	6.469E+00	6.800E-03	6.469E+00	1.384E-01
1/12	1/100	2	1	5.061E+01	2.503E-03	4.272E+00	7.543E-03	4.272E+00	8.441E-02
(EOC)				(-0.12)	(0.45)	(0.60)	(-0.15)	(0.60)	
1/24	1/100	2	1	5.082E+01	1.544E-03	2.513E+00	7.503E-03	2.513E+00	4.946E-02
(EOC)				(-0.01)	(0.70)	(0.77)	(0.01)	(0.77)	
1/6	1/100	2	2	5.516E+01	3.390E-03	6.487E+00	3.426E-03	6.487E+00	1.176E-01
1/12	1/100	2	2	6.179E+01	1.899E-03	4.316E+00	2.257E-03	4.316E+00	6.985E-02
(EOC)				(-0.16)	(0.84)	(0.59)	(0.60)	(0.59)	
1/24	1/100	2	2	4.690E+01	1.401E-03	2.562E+00	2.492E-03	2.562E+00	5.462E-02
(EOC)				(0.40)	(0.44)	(0.75)	(-0.14)	(0.75)	
1/6	1/100	3	2	4.624E+01	3.148E-03	5.928E+00	3.282E-03	5.928E+00	1.282E-01
1/12	1/100	3	2	4.319E+01	2.060E-03	3.706E+00	2.827E-03	3.706E+00	8.581E-02
(EOC)				(0.10)	(0.61)	(0.68)	(0.22)	(0.68)	
1/24	1/100	3	2	5.130E+01	1.286E-03	2.044E+00	3.120E-03	2.044E+00	3.985E-02
(EOC)				(-0.25)	(0.68)	(0.86)	(-0.14)	(0.86)	

Table 3.22: Equation (3.85)-(3.86) with $\varepsilon = 10^{-4}$ with norm (DG)

4. Algorithm Definition

We accept the computed approximate solution $\tilde{u}_{h\tau}$ if the space-time algebraic residual estimator $\eta_{STA}(\tilde{u}_{h\tau})$ is below given tolerance $\omega > 0$, i.e.

$$\eta_{STA}(\tilde{u}_{h\tau}) \leqslant \omega. \tag{4.1}$$

Using the Lemma 3.3, we can divide the tolerance to the time level m and one space-time element as follows

$$\eta_{STA}^{m}(\tilde{u}_{h\tau}^{m}) \leqslant \omega_m, \quad \omega_m = \omega\sqrt{\frac{\tau_m}{T}}, \quad m = 1, \dots, r \tag{4.2}$$

$$\eta_{STA}^{m,\mu}(\tilde{u}_{h\tau}^{m\mu}) \leqslant \omega_{m,\mu}, \quad \omega_{m,\mu} = c_S\left(\frac{|K|}{|\Omega|}\right)^{1/2}\omega_m, \quad \mu \in E_m,\ c_S = 0.5. \tag{4.3}$$

We want to construct an adaptive algorithm with the following properties:

- the computed approximate solution satisfies $\eta_{STA}(\tilde{u}_{h\tau}) \leqslant \omega$,
- the computational time is as small as possible.

To ensure the second property, we require:

(i) the degree of freedom (DOF) is minimum, where DOF is defined in (2.30) for a scalar problem with p, q fixed,

(ii) small number of changing time steps τ_m and mesh $\mathcal{T}_{h,m}$ at time level m,

(iii) the iteration process (2.43) is not "oversolved".

We note that the refinement on mesh $\mathcal{T}_{hm}$ is more complicated than the refinement of the time. If we change the elements and their faces Γ, the matrix $\mathbb{C}$ from (2.39) is changed by $\tilde{a}_{hm}, \tilde{b}_{hm}, J_{hm}$ and also the form l_{hm} will be changed. In contrast, if we change the time step τ_m, the forms remain the same and the change is in integral $\int_{I_m}$. Re-meshing $\mathcal{T}_{hm}$ slows the computation, and so we prefer the change of τ_m to the change of elements $K \in \mathcal{T}_{hm}$.

To ensure (ii), we set the adaptation of the time step and the adaptation of the mesh $\mathcal{T}_{hm}$. To ensure (iii), we specify when the iteration process should terminate. The settings are as follows:

Adaptation of the time step τ_m As presented in [10], we have found a relation between η_{TA}^m and η_{SA}^m, and their connection to an a priori error estimate. Let us compute a sequence of approximate solutions with a decreasing time step τ_m. Numerical experiments show that when the inequality

$$\eta_{TA}^m \leqslant c_T\eta_{SA}^m, \quad c_T = 0.01, \tag{4.4}$$

is satisfied, the error $\|e\|_{L^2(I_m;H^1(\Omega))}$ stops decreasing. We have shown that $\|e\|_{L^2(I_m;H^1(\Omega))}$ may stop decreasing before the inequality $\eta_{TA}^m \leqslant 0.01\eta_{SA}^m$ applies.
Therefore, it can be efficient to stop refining the time step τ_m earlier, but we do

not know how much earlier. We can terminate the time adaptation manually, that is, let ω_T be a tolerance and if $\eta_{TA}^m \leqslant \omega_T$, then accept the time step τ_m. This method is guaranteed to work.
We have found the relation (4.4) and the a priori error estimate, so we may want to stop refining the time step once $\eta_{TA}^m \leqslant c_T \eta_{SA}^m,\ c_T \in [0.01, 0.1]$. Indeed, we face two scenarios:

- If $\eta_{TA}^m > c_T \eta_{SA}^m$, then we will find the optimal time step (to get smaller residual η_*^m) as follows:
  First, numerically verify that $\eta_{TA}^m(\tilde{u}_{h\tau}^m) = c_0 \tau_m^{q+1}$ for a constant $c > 0$.
  Therefore, for the current time step, we have

$$\frac{\eta_{TA}^m(\tilde{u}_{h\tau}^m)}{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)} = \frac{c_0 \tau_m^{q+1}}{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)}. \tag{4.5}$$

  For the optimal time step, we want

$$1 = \frac{c_0 \tau_{m,opt}^{q+1}}{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)}. \tag{4.6}$$

  The value $\eta_{SA}^m(\tilde{u}_{h\tau}^m)$ varies negligibly between the time steps τ_m and $\tau_{m,opt}$. Thus,

$$\frac{\eta_{TA}^m(\tilde{u}_{h\tau}^m)}{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)} = \frac{c_0 \tau_m^{q+1}}{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)} \frac{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)}{c_0 \tau_{m,opt}^{q+1}} \Rightarrow \tau_{m,opt} = \tau_m \left(\frac{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)}{\eta_{TA}^m(\tilde{u}_{h\tau}^m)} \right)^{1/q+1}. \tag{4.7}$$

  Hence, we take $\tau_m := \tau_{m,opt}$ and we carry out the Newton iteration (2.43) again.

- If $\eta_{TA}^m \leqslant c_T \eta_{SA}^m$, then we expect that a smaller time step does not affect the a priori error estimate, so the time step τ_m is accepted.

We also use inequality (4.4) to express that we prefer a refinement in the time dimension rather than a refinement in the space dimension. In fact, we only refine the spatial mesh $\mathcal{T}_{hm}$ if the time refinement does not produce the required approximate solution. In this way, we have $\eta_{SA}^m \approx \eta_{STA}^m$ because the space residual error η_{SA}^m outweighs the time residual error η_{TA}^m.

Adaptation of the mesh $\mathcal{T}_{hm}$ When a computed approximate solution $\tilde{u}_{h\tau}^m$ does not yield a space-time algebraic residual estimator η_{STA} below given tolerance ω_m, even though the time step was chosen to satisfy the criterion (4.4), we change the mesh $\mathcal{T}_{hm}$ and compute the approximate solution again. On every element $K \in \mathcal{T}_{hm}$, we can change degrees p, q or the shape of K. In other words, we can enlarge or refine p, q on every element K, or adapt K itself (i.e., reconstruct K and neighbours to one bigger element or refine K to four elements). For simplicity, this thesis considers p, q unchanged, and adapts the mesh $\mathcal{T}_{hm}$ by which we mean adapting the shape of elements $K \in \mathcal{T}_{hm}$. The technique to adapt the mesh $\mathcal{T}_{hm}$ was described in [9] for *hp-adaptive method* or in [11] and [12] for *anisotropic mesh adaptation*.

Termination of the iteration process (2.43) We know that the algebraic residual estimator η_A is smaller than the space-time algebraic residual estimator η_{STA}. We carry out the iteration process to obtain a computed approximate solution $\tilde{u}_{h\tau}^m$ such that this function approximates the approximate solution $u_{h\tau}^m$. We can terminate the iteration process manually, that is, let ω_A be a tolerance and if $\eta_A^m \leqslant \omega_A$, then stop (2.43). This method is guaranteed to work.
Let $U^{(n)} \longleftrightarrow \tilde{u}_{h\tau}^{m,(n)}$ from the iteration process (2.43).
From previous considerations, we have $\eta_{SA}^m \approx \eta_{STA}^m$. The residual estimator η_{STA}^m is our criterion to accept an approximate solution. If we want to stop the iteration process according to our residual estimator, the relation between η_A^m and η_{SA}^m persists. We set the following termination condition for the iteration process:

$$\eta_A^m(\tilde{u}_{h\tau}^{m,(n)}) \leqslant c_A \eta_{SA}^m(\tilde{u}_{h\tau}^{m,(n)}), \quad , c_A \in [0.001, 0.1]. \tag{4.8}$$

We have $\eta_A^m \leqslant c_A \eta_{SA}^m \approx c_A \eta_{STA}^m$. In this way, we let the iteration process compute approximate solutions until the algebraic residual estimator is sufficiently small compared with the space algebraic residual estimator. Let ω_m be our tolerance at time level m. If $\eta_{STA}^m > \omega_m$, then we have to refine either the time step τ_m or the triangulation $\mathcal{T}_{hm}$ and to recompute the approximate solution. In this case, if η_A^m is very small, we will have $\tilde{u}_{h\tau}^m \approx u_{h\tau}^m$, but $u_{h\tau}^m$ is not a good solution because of the time step or the spatial mesh. The setting (4.8) prevents the iteration process from "oversolving".
An experiment was carried out in [1] to verify the stopping criterion (4.8). When inequality (4.8) was satisfied and the iteration process (2.43) was still continuing, then η_A^m decreased and η_{SA}^m remained almost constant. This suggests that from this moment, precise computations during the iteration process affect the algebraic residual estimator, but they have almost no effect on the spatial residual estimator.

Algorithm We propose the following space-time adaptive process:
Let ω be a given tolerance, $\mathcal{T}_{h0}$ an initial mesh and τ_0 an initial time step.
Let $S_{hp}^{\tau q}$ be the finite-dimensional broken Bochner space with p, q fixed.
Set $m = 1$.

1. solve the iteration process (2.43) until the stopping criterion (4.8) is satisfied, that is $\eta_A^m(\tilde{u}_{h\tau}^{m,(n)}) \leqslant c_A \eta_{SA}^m(\tilde{u}_{h\tau}^{m,(n)})$,
2. if $\eta_{TA}^m(\tilde{u}_{h\tau}^m) > c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)$ then we adapt the time step τ_m according to (4.7), i.e. $\tau_{m,opt} = \tau_m \left(\frac{c_T \eta_{SA}^m(\tilde{u}_{h\tau}^m)}{\eta_{TA}^m(\tilde{u}_{h\tau}^m)}\right)^{1/q+1}$, and go to step 1,
3. if $\eta_{STA}^m(\tilde{u}_{h\tau}^m) > \omega_m$ then we adapt the mesh $\mathcal{T}_{hm}$ and go to 1,
4. if $t_m > T$ then STOP, else $\mathcal{T}_{h,m+1} = \mathcal{T}_{hm}$, $\tau_{m+1} = \tau_{m,opt}$, $m = m + 1$ and go to 1.

5. Numerical experiments

In this chapter, we extend the scalar problem to a vector-valued problem described by the Navier-Stokes equations. The details are described in [7]. Then, we apply our proposed algorithm to simulate an interaction.

5.1 Compressible flow problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $\partial\Omega = \partial\Omega_i \cup \partial\Omega_o \cup \partial\Omega_w$ its boundary, where $\partial\Omega_i$, $\partial\Omega_o$, $\partial\Omega_w$ are the inlet, outlet, and impermeable walls parts of the boundary and these parts are disjoint. Let $Q_T = \Omega \times (0, T)$.
Let us consider the Navier-Stokes equations describing a motion of non-stationary viscous compressible fluids, where $x = (x_1, x_2) \in \Omega$, as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}_1(\boldsymbol{u})}{\partial x_1} + \frac{\partial \boldsymbol{f}_2(\boldsymbol{u})}{\partial x_2} - \frac{\partial \boldsymbol{R}_1(\boldsymbol{u}, \nabla \boldsymbol{u})}{\partial x_1} - \frac{\partial \boldsymbol{R}_2(\boldsymbol{u}, \nabla \boldsymbol{u})}{\partial x_2} = \boldsymbol{g} \quad \text{in } Q_T. \tag{5.1}$$

We consider the *initial condition*

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}^0(x) \tag{5.2}$$

and the *boundary condition*

$$\rho = \rho_D,\ \boldsymbol{v} = \boldsymbol{v}_D,\ \sum_{k=1}^{2}\left(\sum_{l=1}^{2} \tau'_{lk} n_l\right) v_k + \frac{\gamma}{Re\ Pr} \sum_{k=1}^{2} \frac{\partial \theta}{\partial x_k} n_k = 0 \quad \text{on } \partial\Omega_i, \tag{5.3a}$$

$$\sum_{k=1}^{2} \tau'_{sk} n_k = 0 \quad \text{for } s = 1, 2, \quad \sum_{k=1}^{2} \frac{\partial \theta}{\partial x_k} n_k = 0 \quad \text{on } \partial\Omega_o. \tag{5.3b}$$

On the impermeable wall $\partial\Omega_w$, the boundary condition can be defined in two different ways:

$$\boldsymbol{v} = 0, \quad \sum_{k=1}^{2} \frac{\partial \theta}{\partial x_k} n_k = 0 \quad \text{on } \partial\Omega_w \quad \textit{adiabatic boundary condition} \tag{5.3c}$$

$$\text{or} \quad \boldsymbol{v} = 0, \quad \theta = \theta_D \quad \text{on } \partial\Omega_w \tag{5.3d}$$

The following notation is used: $\rho(x, t) \in \mathbb{R}$ is the density of the fluid in position (x_1, x_2) and time t, $\boldsymbol{v}(x, t) = (v_1, v_2)(x, t) \in \mathbb{R}^2$ is the vector of velocity and $e(x, t) \in \mathbb{R}$ the energy, $\theta(x, t)$ the temperature, $p(x, t)$ the pressure, γ the Poisson adiabatic constant, Re the Reynolds number, Pr the Prandtl number, τ'_{sk} is a element of stress tensor which describes viscous influence, δ_{ij} is Kronecker symbol (if $i = j$ then $\delta_{ij} = 1$, else $\delta_{ij} = 0$).
We have

$$\begin{aligned} &\boldsymbol{u} : \mathbb{R}^3 \to \mathbb{R}^4 \\ &\boldsymbol{u}(x, t) = (u_1, u_2, u_3, u_4)^T(x, t) = (\rho, \rho v_1, \rho v_2, e)^T(x, t), \end{aligned} \tag{5.4}$$

so (5.1) is formed by 4 equations for 4 unknown quantities ρ, v_1, v_2, e. The functions $\boldsymbol{f}_s$ represent the inviscid fluxes. They are non-linear, defined as below:

$$\begin{aligned} &\boldsymbol{f}_s : \mathbb{R}^4 \to \mathbb{R}^4, \quad s = 1, 2 \\ &\boldsymbol{f}_s(\boldsymbol{u}(x, t)) = (\rho v_s, \rho v_s v_1 + \delta_{1s} p, \rho v_s v_2 + \delta_{2s} p, (e + p) v_s)^T(x, t). \end{aligned} \tag{5.5}$$

The functions $\boldsymbol{R}_s$ represent the viscous fluxes and they are non-linear, defined as

$$\begin{aligned} &\boldsymbol{R}_s : \mathbb{R}^4 \times \mathbb{R}^{4\times 4} \to \mathbb{R}^4, \quad s = 1, 2, \\ &\boldsymbol{R}_s(\boldsymbol{u}, \nabla \boldsymbol{u})(x,t) = (0, \tau'_{s1}, \tau'_{s2}, \tau'_{s1} v_1 + \tau'_{s2} v_2 + \frac{\gamma}{RePr}\frac{\partial \theta}{\partial x_s})^T(x,t), \\ &\tau'_{kl} = \frac{1}{Re}\left[(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}) - \frac{2}{3}\delta_{kl}\mathrm{div}\boldsymbol{v} \right]. \end{aligned} \tag{5.6}$$

We consider the Newtonian type of fluid accompanied by the state equation of a perfect gas and the definition of total energy, i.e.

$$\begin{aligned} p = (\gamma - 1)(e - \frac{\rho|\boldsymbol{v}|^2}{2}), \\ e = c_v \rho\theta + \frac{\rho|\boldsymbol{v}|^2}{2} \end{aligned} \tag{5.7}$$

Next, we introduce some properties of the inviscid and viscous terms. More details can be found in [5].The Euler fluxes $\boldsymbol{f}_s$ satisfy

$$\boldsymbol{f}_s(\boldsymbol{u}) = \mathbb{A}_s(\boldsymbol{u})\boldsymbol{u}, \text{ where } \mathbb{A}_s(\boldsymbol{u}) = \frac{D\boldsymbol{f}_s(\boldsymbol{u})}{D\boldsymbol{u}}, \quad s = 1, 2. \tag{5.8}$$

We introduce a vector-valued function

$$\boldsymbol{P}(\boldsymbol{u}, \boldsymbol{n}) := \boldsymbol{f}_1(\boldsymbol{u})n_1 + \boldsymbol{f}_2(\boldsymbol{u})n_2. \tag{5.9}$$

Then it can be proven that

$$\frac{D\boldsymbol{P}(\boldsymbol{u}, \boldsymbol{n})}{D\boldsymbol{u}} = \mathbb{P}(\boldsymbol{u}, \boldsymbol{n}) = \mathbb{A}_1(\boldsymbol{u})n_1 + \mathbb{A}_2(\boldsymbol{u})n_2, \tag{5.10}$$

where $\frac{D\boldsymbol{P}}{D\boldsymbol{u}} \in \mathbb{R}^{4\times 4}$ is the Jacobian matrix of the function $\boldsymbol{P}$. We will use matrix $\mathbb{P}$ in the linearization of inviscid terms. Matrix $\mathbb{P}$ is diagonalizable, so that $\mathbb{P} = \mathbb{T}\Lambda\mathbb{T}^{-1}$, where $\mathbb{T}$ is a non-singular matrix containing eigenvectors of $\mathbb{P}$ and matrix Λ contains eigenvalues of $\mathbb{P}$ on its diagonal. We define $\mathbb{P}^+$ and $\mathbb{P}^-$ as

$$\mathbb{P}^{\pm} := \mathbb{T}\Lambda^{\pm}\mathbb{T}^{-1}. \tag{5.11}$$

The viscous term can be written in the form

$$\boldsymbol{R}_s(\boldsymbol{u}, \nabla \boldsymbol{u}) = \mathbb{K}_{s1}(\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x_1} + \mathbb{K}_{s2}(\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x_2}. \tag{5.12}$$

This relation helps us to linearize the viscous term. The form of matrices $\mathbb{K}_{ij}$ can be found in [8].

5.2 Linearization of the forms in Navier-Stokes equations

In the full space-time discretization of the Navier-Stokes problem (5.1), we have to specify $\boldsymbol{u}_\Gamma^{(R)}$ on the boundary $\partial\Omega$. Details about the discretization can be found in [7], we present only the result.

For each part of the boundary, we define boundary set $\mathcal{F}^i_{h,m}, \mathcal{F}^o_{h,m}, \mathcal{F}^w_{h,m}$ as the set of all $\Gamma \in \mathcal{F}^B_{h,m}$ such that $\Gamma \in \partial\Omega_i, \Gamma \in \partial\Omega_o, \Gamma \in \partial\Omega_w$, respectively.

Linearization of the inviscid term

For faces $\Gamma \in \mathcal{F}^w_{h,m}$, we define linearized form

$$\boldsymbol{f}^{1,L}_w(\overline{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\psi}) := \mathbb{P}_w(\overline{\boldsymbol{u}}, \boldsymbol{\psi})\boldsymbol{u}, \quad \Gamma \in \mathcal{F}^w_{h,m}, \tag{5.13a}$$

where matrix $\mathbb{P}_w$ satisfies the relation $\mathbb{P}(\boldsymbol{u}, \boldsymbol{\psi}) = \mathbb{P}_w(\boldsymbol{u}, \boldsymbol{\psi})\boldsymbol{u}$ and the flow has the property $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega_w$.

$$\boldsymbol{f}^{2,L}_w(\overline{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\psi}) := \mathbb{P}^+(\overline{\boldsymbol{u}}, \boldsymbol{n})\boldsymbol{u} + \mathbb{P}^-(\overline{\boldsymbol{u}}, \boldsymbol{n})\mathcal{M}(\boldsymbol{u}), \tag{5.13b}$$

where $\boldsymbol{u}^{(R)}_\Gamma = \mathcal{M}(\boldsymbol{u}^{(L)})$ and the inviscid mirror operator is defined as $\mathcal{M}(\boldsymbol{u}) := (\rho, \rho(\boldsymbol{v} - 2(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}), e)^T$.

For faces on the inlet and outlet boundary, $\Gamma \in \mathcal{F}^{io}_{h,m}$, $\boldsymbol{u}^{(R)}_\Gamma = \mathcal{B}(\boldsymbol{u}^{(L)}, \boldsymbol{u}_{BC})$ can be defined by extrapolation corresponding with the boundary conditions (5.3).

The *linearized inviscid form* is defined as

$$\begin{aligned}\tilde{b}^L_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\psi}) := &- \sum_{K \in \mathcal{T}_{h,m}} \int_K \sum_{s=1}^{2} \mathbb{A}_s(\overline{\boldsymbol{u}})\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\psi}}{\partial x_s}\, \mathrm{d}x \\ &+ \sum_{\Gamma \in \mathcal{F}^I_{h,m}} \int_\Gamma \left(\mathbb{P}^+(\langle \overline{\boldsymbol{u}} \rangle, \boldsymbol{n})\boldsymbol{u}^{(L)} + \mathbb{P}^-(\langle \overline{\boldsymbol{u}} \rangle, \boldsymbol{n})\boldsymbol{u}^{(R)}\right) \cdot [\boldsymbol{\psi}]\, \mathrm{d}S \\ + \sum_{\Gamma \in \mathcal{F}^{io}_{h,m}} \int_\Gamma \mathbb{P}^+(\overline{\boldsymbol{u}}^{(L)}, \boldsymbol{n})\boldsymbol{u}^{(L)} \cdot \boldsymbol{\psi}\, \mathrm{d}S &+ \sum_{\Gamma \in \mathcal{F}^w_{h,m}} \int_\Gamma \boldsymbol{f}^{i,L}_w(\overline{\boldsymbol{u}}^{(L)}, \boldsymbol{u}^{(L)}, \boldsymbol{n}) \cdot \boldsymbol{\psi}\, \mathrm{d}S. \end{aligned} \tag{5.14}$$

where $i = 1, 2$ depends on the properties of the flow on the impermeable wall.

$$\tilde{b}^R_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{\psi}) := \sum_{\Gamma \in \mathcal{F}^{io}_{h,m}} \int_\Gamma \mathbb{P}^-(\overline{\boldsymbol{u}}^{(L)}, \boldsymbol{n})\mathcal{B}(\overline{\boldsymbol{u}}^{(L)}, \boldsymbol{u}_{BC}) \cdot \boldsymbol{\psi}\, \mathrm{d}S. \tag{5.15}$$

Linearization of the viscous term

On the inlet faces $\Gamma \in \mathcal{F}^i_{h,m}$ we define $\boldsymbol{u}^{(R)}_\Gamma$ as

$$\boldsymbol{u}_B := \begin{cases} (\rho_D, \rho_D v_{D,1}, \rho_D v_{D,2}, \rho_D \theta^L + \frac{1}{2}\boldsymbol{\rho}_D |v_D|^2)^T \\ \mathcal{B}(\boldsymbol{u}^{(L)}, \boldsymbol{u}_{BC}) \qquad \text{if the flow passes an airfoil} \end{cases} \tag{5.16}$$

On the impermeable wall $\partial\Omega_w$, we define $\boldsymbol{u}^{(R)}_\Gamma$ as follows

$$\boldsymbol{u}_B := \begin{cases} (\rho^L, 0, 0, \rho^L \Theta^L)^T & \text{if } \Gamma \in \mathcal{F}^w_{h,m} \text{ and the boundary condition (5.3c)} \\ (\rho^L, 0, 0, \rho^L \Theta_D)^T & \text{if } \Gamma \in \mathcal{F}^w_{h,m} \text{ and the boundary condition (5.3d),} \end{cases} \tag{5.17}$$

where ρ^L and θ^L are extrapolated functions on the boundary.

Let matrices $\mathbb{K}^w_{sk}$ be identical with matrices $\mathbb{K}_{sk}$ except that the last row of $\mathbb{K}^w_{sk}$

is equal to zero. The *linearized viscous form* is defined as

$$\tilde{a}^L_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\psi}) := \sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{i,j=1}^{2} (\mathbb{K}_{ij}(\overline{\boldsymbol{u}}) \frac{\partial \boldsymbol{u}}{\partial x_j}) \cdot \frac{\partial \boldsymbol{\psi}}{\partial x_i} \, \mathrm{d}x$$
$$- \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \sum_{i,j=1}^{2} \left\langle \mathbb{K}_{ij}(\overline{\boldsymbol{u}}) \frac{\partial \boldsymbol{u}}{\partial x_j} \right\rangle n_i \cdot [\boldsymbol{\psi}] + \theta \left\langle \mathbb{K}^T_{ij}(\overline{\boldsymbol{u}}) \frac{\partial \boldsymbol{\psi}}{\partial x_j} \right\rangle n_i \cdot [\boldsymbol{u}] \, \mathrm{d}S$$
$$- \sum_{\Gamma\in\mathcal{F}^i_{h,m}} \int_\Gamma \sum_{i,j=1}^{2} \mathbb{K}_{ij}(\overline{\boldsymbol{u}}^{(L)}) \frac{\partial \boldsymbol{u}}{\partial x_j} n_i \cdot \boldsymbol{\psi} + \theta \mathbb{K}^T_{ij}(\overline{\boldsymbol{u}}^{(L)}) \frac{\partial \boldsymbol{\psi}}{\partial x_j} n_i \cdot \boldsymbol{u}^{(L)} \, \mathrm{d}S$$
$$- \sum_{\Gamma\in\mathcal{F}^w_{h,m}} \int_\Gamma \sum_{i,j=1}^{2} \mathbb{K}^w_{ij}(\overline{\boldsymbol{u}}^{(L)}) \frac{\partial \boldsymbol{u}}{\partial x_j} n_i \cdot \boldsymbol{\psi} + \theta (\mathbb{K}^w_{ij}(\overline{\boldsymbol{u}}^{(L)}))^T \frac{\partial \boldsymbol{\psi}}{\partial x_j} n_i \cdot \boldsymbol{u}^{(L)} \, \mathrm{d}S. \tag{5.18}$$

$$\tilde{a}^R_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{\psi}) := -\theta \sum_{\Gamma\in\mathcal{F}^i_{h,m}} \int_\Gamma \sum_{i,j=1}^{2} \mathbb{K}^T_{ij}(\overline{\boldsymbol{u}}^{(L)}) \frac{\partial \boldsymbol{\psi}}{\partial x_j} n_i \cdot \overline{\boldsymbol{u}}_B \, \mathrm{d}S$$
$$- \theta \sum_{\Gamma\in\mathcal{F}^w_{h,m}} \int_\Gamma \sum_{i,j=1}^{2} (\mathbb{K}^w_{ij}(\overline{\boldsymbol{u}}^{(L)}))^T \frac{\partial \boldsymbol{\psi}}{\partial x_j} n_i \cdot \overline{\boldsymbol{u}}_B \, \mathrm{d}S. \tag{5.19}$$

Linearization of the interior and penalty form
Let operator $\vartheta(\boldsymbol{\psi}) := (0, \psi_2, \psi_3, 0)^T$ and $\overline{\boldsymbol{u}}_B$ be defined as (5.16), $\sigma = \frac{c_W}{h_\Gamma Re}$. Then

$$\tilde{J}^L_{hm}(\boldsymbol{u}, \boldsymbol{\psi}) := \sum_{\Gamma\in\mathcal{F}^I_{h,m}} \int_\Gamma \sigma[\boldsymbol{u}]\cdot[\boldsymbol{\psi}] \, \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^i_{h,m}} \int_\Gamma \sigma \boldsymbol{u}^{(L)}\cdot\boldsymbol{\psi} \, \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^w_{h,m}} \int_\Gamma \sigma \boldsymbol{u}^{(L)}\cdot\vartheta(\boldsymbol{\psi}) \, \mathrm{d}S. \tag{5.20}$$

$$\tilde{J}^R_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{\psi}) := \sum_{\Gamma\in\mathcal{F}^i_{h,m}} \int_\Gamma \sigma \overline{\boldsymbol{u}}_B \cdot \boldsymbol{\psi} \, \mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^w_{h,m}} \int_\Gamma \sigma \overline{\boldsymbol{u}}_B \cdot \vartheta(\boldsymbol{\psi}) \, \mathrm{d}S. \tag{5.21}$$

Let us define forms which will be used in the computational part

$$\tilde{c}_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\psi}) := \tilde{a}^L_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\psi}) + \tilde{b}^L_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\psi}) + \tilde{J}^L_{hm}(\boldsymbol{u}, \boldsymbol{\psi}). \tag{5.22}$$

$$d_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{\psi}) := \tilde{a}^R_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{\psi}) + \tilde{b}^R_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{\psi}) + \tilde{J}^R_{hm}(\overline{\boldsymbol{u}}, \boldsymbol{\psi}) + \sum_K \int_K \boldsymbol{g} \cdot \boldsymbol{\psi} \, \mathrm{d}x. \tag{5.23}$$

5.3 Shock wave and isentropic vortex interaction

Let us simulate an unsteady viscous flow problem using the algorithm in Section 4. We consider an interaction between a plane weak shock wave with a single isentropic vortex. During the interaction, acoustic waves are generated and we use the algorithm to capture them.
The computational domain is $\Omega = (0,2) \times (0,2)$ with the periodic boundary

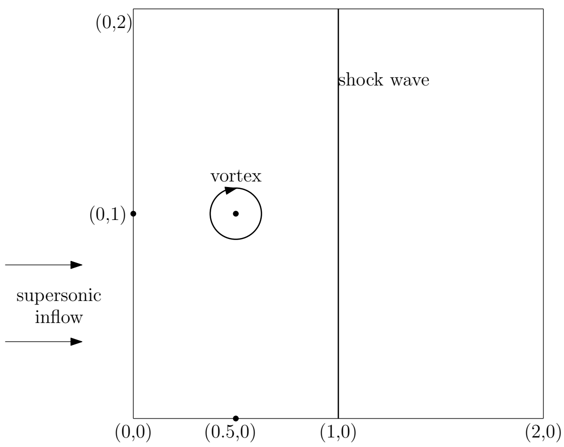

Figure 5.1: Computational domain Ω.

conditions at the top and bottom faces.

As presented in figure 5.1, a stationary plane shock is located at $x_1 = 1$ and an isolated isentropic vortex centered at (0.5,1). The prescribed pressure jump through the shock is $p_R - p_L = 0.4$, where p_L and p_R is the pressure value from the left and right of the shock wave, respectively, corresponding to the inlet Mach number $M_L = 1.1588$. The Poisson constant is $\gamma = 1.4$ and Reynolds number is $Re = 2000$.
Initial condition (quantities at $t = 0$) is given separately for the left and right side of the shock wave. For $x_1 < 1$, the density, velocity, and pressure are given as

$$\rho_L = 1,\ v_{1L} = M_L\sqrt{\gamma},\ v_{2L} = 0,\ p_L = 1.$$

In contrast, for $x_1 \geq 1$ they are defined such that

$$\rho_R = \rho_L K_1,\ v_{1R} = \frac{v_{1L}}{K_1},\ v_{2R} = 0,\ p_R = p_L K_2,$$

with $K_1 = \frac{\gamma+1}{2}\frac{M_L^2}{1+\frac{\gamma-1}{2}M_L^2},\ K_2 = \frac{2}{\gamma+1}(\gamma M_L^2 - \frac{\gamma-1}{2})$. The vortex is described by following tangential velocity:

$$v_\theta = c_1 r \exp(-c_2 r^2),$$

where

$$c_1 = \frac{u_c}{r_c},\ c_2 = \frac{1}{2r_c^2},\ r(x_1, x_2) = ((x_1 - 0.5)^2 + (x_2 - 1)^2)^{1/2}.$$

We set $r_c = 0.075$ and $u_c = 0.5$ to define the strength of the vortex. The computations are stopped at the dimensionless time $T = 0.7$.

We solve the problem with the IIPG variant, $c_W = 50$, with spatial polynomial degree $p = 3$ and time polynomial degree $q = 2$. We used the tolerance $\omega = 0.02$. The algorithm is set up with following parameters:

1. the stopping criterion (4.8) with $c_A = 0.01$,
2. the time step τ_m is adapted with $c_T = 0.1$,

3. the mesh $\mathcal{T}_{hm}$ is adapted by the anisotropic mesh adaptation with only h-refinement,

4. the norm is modified according to Remark 3.5.

The interaction of shock wave and isentropic vortex was simulated, where the refined meshes $\mathcal{T}_{hm}$ and the isolines of pressure are presented in figures 5.2 and 5.3. The acoustic wave were captured during the interaction. We can conclude that the proposed algorithm is able to simulate difficult physical phenomena.

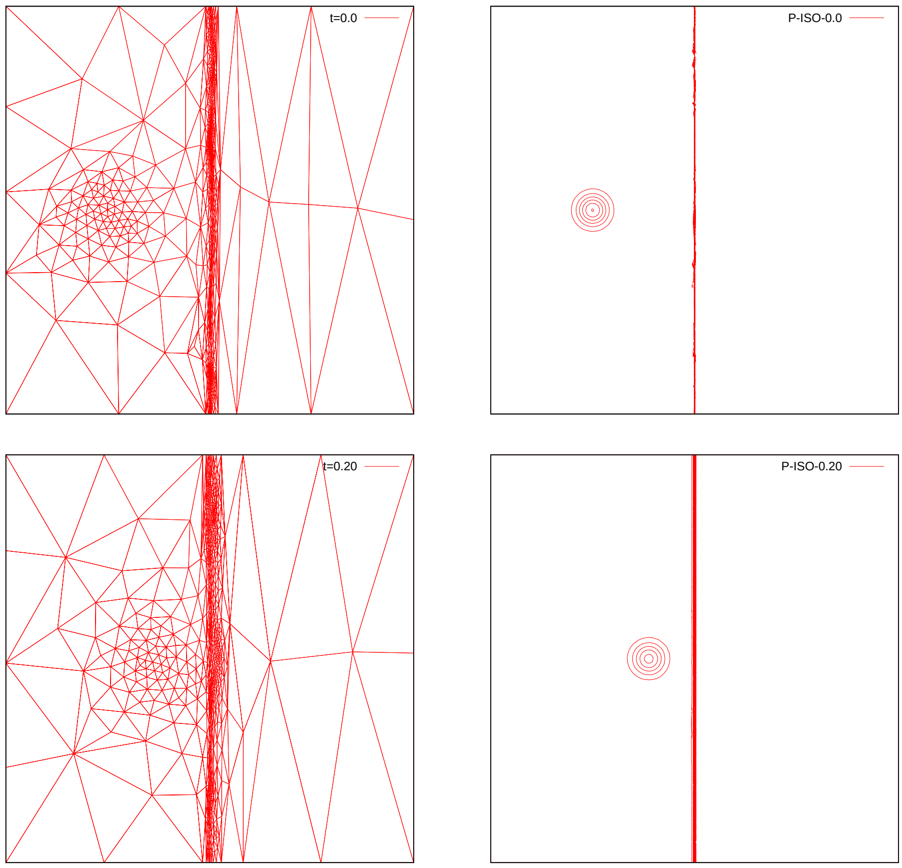

Figure 5.2: Meshes and isolines of the pressure at $t = 0.0, 0.2$.

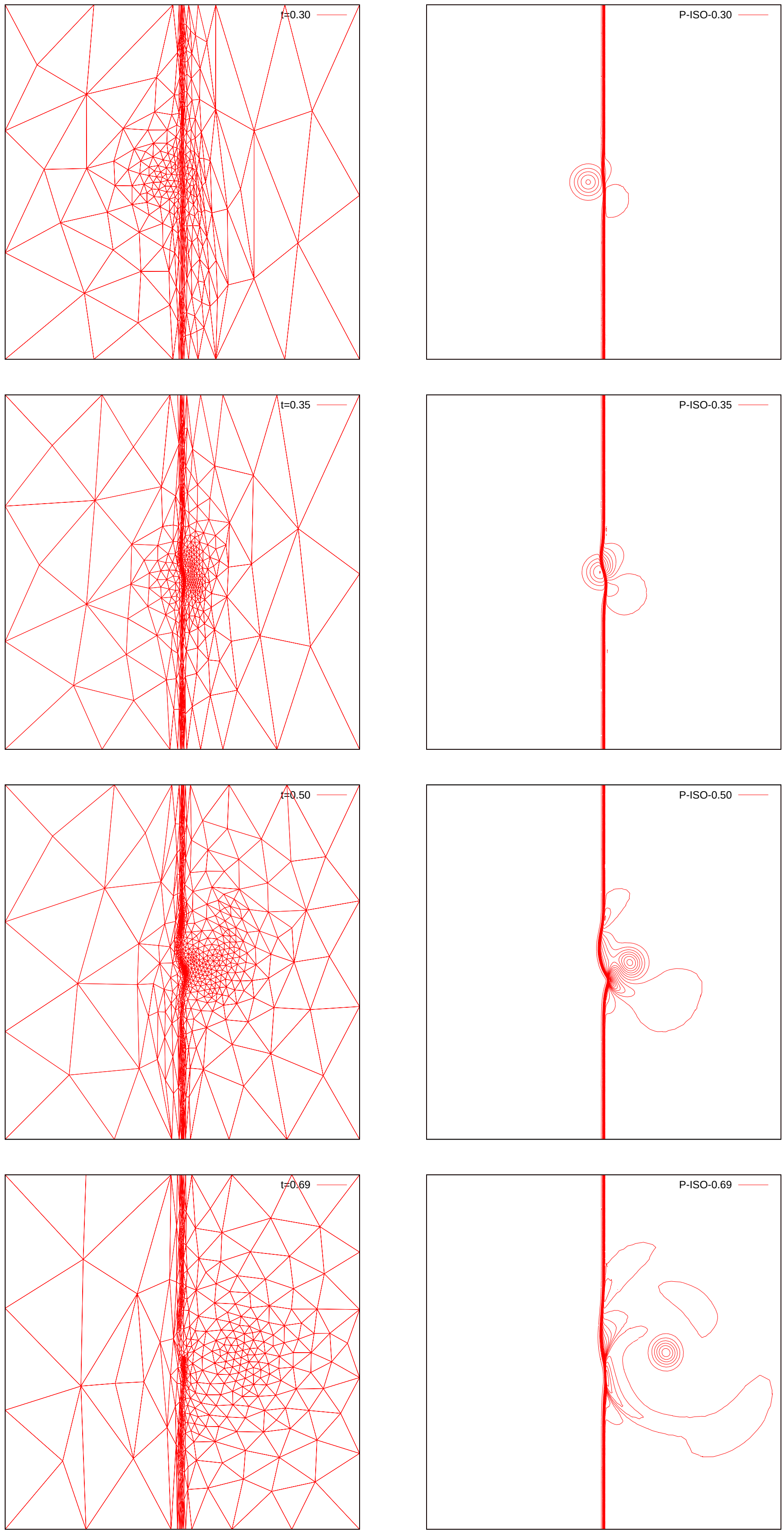

Figure 5.3: Meshes and isolines of the pressure at $t = 0.3, 0.35, 0.5, 0.7$.

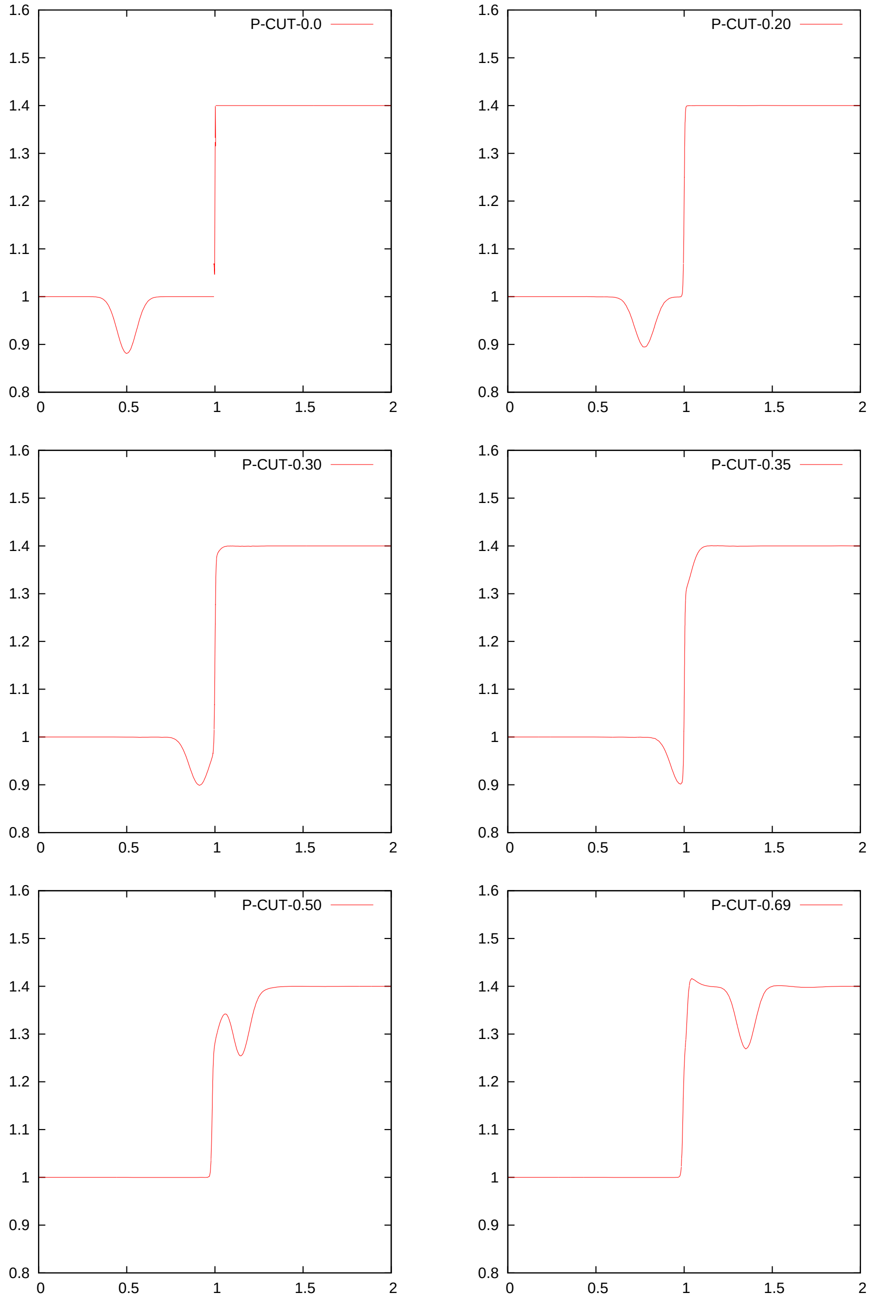

Figure 5.4: The pressure distribution along the line $x_2 = 1$ at dimensionless time $t = 0.0, 0.2, 0.3, 0.35, 0.5, 0.7$.

Conclusion

We developed an adaptive algorithm for the numerical solution of nonlinear convection-diffusion equations. The algorithm is based on the space-time discontinuous Galerkin method and residual estimates, which are able to distinguish the spatial, temporal, and algebraic errors. We have proposed several variants of the estimators and compared them numerically. Some of them, depending on the chosen norm, satisfied natural properties. Moreover, we demonstrated the robustness with respect to the polynomial approximation degree and the input data. Finally, the example of shock-vortex interaction demonstrates a potential of the presented algorithm. Further work is needed in the numerical analysis of our theory.

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