Charles University in Prague

Faculty of Social Sciences

Institute of Economic Studies

MASTER THESIS

Practical usage of optimal portfolio diversification using maximum entropy principle

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Declaration of Authorship

1. Hereby I declare that I have compiled this master thesis independently, using only the listed literature and sources.
2. I declare that the thesis has not been used for obtaining another title.
3. I agree on making this thesis accessible for study and research purposes.

Prague, July 30, 2015

Signature

Acknowledgments

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Furthermore, I would also like to thank my family who were supporting me and providing me with the inspiration throughout my studies.

Abstract

This thesis enhances the investigation of the principle of maximum entropy, implied in the portfolio diversification problem, when portfolio consists of stocks. Entropy, as a measure of diversity, is used as the objective function in the optimization problem with given side constraints. The principle of maximum entropy, by the nature itself, suggests the solution for two problems; it reduces the estimation error of inputs, as it has a shrinkage interpretation and it leads to more diversified portfolio. Furthermore, improvement to the portfolio optimization is made by using design-free estimation of variance-covariance matrices of stock returns. Design-free estimation is proven to provide superior estimate of large variance-covariance matrices and for data with heavy-tailed densities. To asses and compare the performance of the portfolios, their out-of-sample Sharpe ratios are used. In nominal terms, the out-of-sample Sharpe ratios are almost always lower for the portfolios, created using maximum entropy principle, than for 'classical' Markowitz's efficient portfolio. However, this out-of-sample Sharpe ratios are not statistically different, as it was tested by constructing studentized time-series bootstrap confidence intervals.

JEL Classification	C13, C44, C61, J11,
Keywords	Portfolio selection; Entropy measure; Sharpe ratio; Design-free variance-covariance matrix

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Abstrakt

Tato diplomová práce rozšiřuje výzkum principu maximální entropie, aplikovaného v problému diversifikace portfolia, kdy portfolio se skládá z akcií. Entropie, jako měřítko diverzity, je používána jako cílová funkce v problému optimizace s danými dodatečnými omezeními. Princip maximální entropie, svou samotnou povahou, navrhuje řešení pro dva problémy; snižuje chybu odhadu vstupů, protože má smrštění interpretace a vede k díversifikovanějšímu portfoliu. Navíc, zlepšení optimizace portfolia se provádí pomocí defign-free odhadu variačních-kovariačních matic akciových výnosů. Prokázáno, že design-free odhad poskytuje vynikající odhad Pro velké variační-covariační matice a pro data s nenulovým koeficientem assymetíe. Mimo-vzorkové

Sharpovy koeficienty jsou použity pro posouzení a porovnání výkonnosti portfolií. V nominálním vzjáření, mimo-vzorkové Sharpovy koeficienty skoro pořád nížší pro porfolia, vytvořených pomocí principu maximální entropie, než pro "klasické" efektivní portfolia Markowitza. Nicméně, tyto out-of-sample Sharpovy koeficienty nejsou statisticky rozdílné jak to bylo testováno vytvořením studentizovaného intervalu spolehlivostí bootstrapu časových řád.

Klasifikace JEL C13, C44, C61, J11,
Klíčová slova Portfolio selection; Entropy measure; Sharpe ratio; Design-free variance-covariance matrix

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Acronyms

CE Cross entropy portfolio with prior weights corresponding to the minimum variance portfolio

CEQ Certainty equivalent return

CVaR Conditional Value at Risk

DJIA Dow Jones Industrial Average

EQ Equally weighted portfolio

iid Independent and identically distributed random variables

MinV Minimum variance portfolio

MV Mean-Variance efficient portfolio

SE Cross entropy portfolio with prior weights corresponding to the equally weighted portfolio

SR Sharpe Ratio

VaR Value at Risk

Master Thesis Proposal

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Proposed Topic:

Practical usage of optimal portfolio diversification using maximum entropy.

Motivation:

This thesis enhances the investigation of the principle of maximum entropy, implied in the portfolio diversification problem, when portfolio consists of stocks. Entropy, as a measure of diversity, is used as the objective function in the optimization problem with given side constraints. The principle of maximum entropy, by the nature itself, suggests the solution for two problems; it reduces the estimation error of inputs, as it has a shrinkage interpretation and it leads to more diversified portfolio. Furthermore, improvement to the portfolio optimization is made by using design-free estimation of variance-covariance matrices of stock returns.

Hypotheses:

1. Portfolio weights are shrunk toward the most diversified portfolio (with equal weights to all assets)
2. Maximum entropy approach leads to higher Sharpe ratio out of sample comparatively to the classical MV.
3. The Sharpe ratios of MV portfolio and those, obtained using maximum entropy approach, are not statistically different.

Methodology:

In order to conduct empirical investigation I plan to write Matlab programs, using which I will be able to construct and compare different portfolios based on the work of Bera and Park (2008). The degree of diversification of the portfolios will be checked based on the values of entropy, as it present good measure of how well-diversified portfolio is (Rernholz 2002, p. 36). Using ‘rolling window’ technique I will compare the performance of different portfolio for cases of daily and weekly holding periods. In order to test, where Sharpe ratios are statistically different I use the test proposed by Memmel (2003) if the out-of-sample returns are normally distributed or I will use studentized time-series bootstrap confidence intervals method, proposed by Ledoif and Wolf (2008)

Outline:

1. Introduction.
2. Drawbacks of MV approach.
3. Introducing the program.
4. Investigating the results of the program.
5. Comparison to MV approach.
6. Conclusions.

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Author **Supervisor**

Chapter 1

Introduction.

Mean-Variance (MV) portfolio creation, introduced by Markovitz (1952) has been one of the most popular methods used in solving the problem of diversification the wealth among risky assets. It gives good results in a sense, that investors are able to incorporate the risk in the decision making problem of how to allocate their wealth. The optimization problem is structured in a way that investor can set the target return of the portfolio and choose the one with lowest risk; or set the boundaries for the risk which investor is willing to accept and choose the one with highest return.

The standard way to represent this problem mathematically is:

$$\min_{\pi} \pi' \Sigma \pi, \qquad s.t.\ \mathbb{E}(\pi R) = \pi' m = \mu_0,\ \pi' 1_N = 1,$$

where $m,\ \Sigma,\ \mu_0,\ 1_N$ are mean returns on the assets, variance-covariance matrix of assets returns, target return and $N \times 1$ vector of ones respectively.

Despite the popularity of this methods, it has some severe drawbacks. First of all, MV method leads to the portfolio, which is highly concentrated on a few assets, which is contrary to the principle of the diversification. Second is that MV methods is known for its poor out-of-sample performance. Another thing about this method is that it assumes the possibility of negative weights for some assets in the portfolio, namely short-selling, whereas most asset managers are not allowed to sell short. If the last issue may be appropriate in some cases, like hedge funds, previous two are always viewed as undesirable in the classical MV method (Bera and Park 2008).

Finally, Michaud (1989) claims, that MV optimization is an “error maximizer”, in a sense, that due to statistical error in the mean returns and variance-covariance matrix, MV optimization overweights (underweights) securities that have large (low) estimated mean return, negative (positive) correlation and/or small (large) variance.

To tackle the problem of estimation error and thus, to improve MV portfolio, a number of methods were proposed, some of which are discussed below.

One way to improve MV method, is to impose factor model for estimation the variance-covariance matrix in order to reduce the number of free parameters. Sharpe (1963) proposed a single-factor model. In this case, the number of parameter to be estimated is substantially reduced form $\frac{N(N+1)}{2} - N$ to $3N+1$, where N is a number of assets in the portfolio. The drawback of single-factor model is that it leads to biased estimates of the variance-covariance matrix of the returns. To tackle this problem one can include more factors. Bera and Park (2008) showed, that is this case, number of parameters to be estimated can be reduced to $K(K+1)/2+N(K+2)$, where K is the number of the factors and N is the number of variables, namely the number of assets in the portfolio.

Imposing constraints on the portfolio, such as non-negative constraints on the portfolio weights, helps in reducing estimation error. It has been shown by Jagannathan and Ma (2003), that imposing short-sales constraint works in the same way as shrinking the extreme values of the variance-covariance matrix.

Aforementioned methods improve the performance of the portfolio, but as it can be seen, they require some additional information or, as some researchers have done, usage of the Bayesian probability theory. Investors will face the problem of finding relevant target values, as in case of shrinkage estimators with predetermined target values for returns and variance-covariance matrix. In case of using the Bayesian theory, choosing the reasonable prior distribution for the parameters in the model often becomes a challenging task.

Bera and Park (2008) proposed to use entropy approach to optimal portfolio selection, on which we will concentrate and will investigate how efficient it is in real life comparatively to the classical approach. The main idea of this approach is to use entropy measure as the objective function for the optimization problem. In this case weights of the portfolio are considered as the probability mass function of a random variable. Maximizing the entropy measure for such a random variable, subject to a given constraints, allows us to obtain well diversified portfolio. Besides, using entropy as the objective function guarantees non-negative weights for the assets in the portfolio, as they are seen as probabilities, which by definition take non-negative values. If we use classical notion of the entropy from the Information theory, we shrink the weight for the assets toward the most diversified portfolio, namely equally-weighted. This type of the entropy measure is also called the Shannon entropy. We also can use the notion of the cross-entropy, if we have some prior weights, toward which we want to shrink weight of the portfolio. By minimizing the cross-entropy subject to the given constraints, such as required return , we can obtain a portfolio, weights of which are as close as possible to the target.

In all methods, mentioned above we are relying on well estimated input variables, namely estimated mean returns of the assets and their estimated variance-covariance matrix. It is one of the most challenging part not only in portfolio creation problem. This becomes even bigger problem, when you are dealing with high number of variables. Number of parameters, which have to be estimated, grows as a quadratic function of the number of variables; and as it will be seen later, we will construct our portfolio out of 30 assets. Abadir et al. (2010) proposed the method, that leads to superior estimate of large variance-covariance matrix.

Using Matlab programs, written based on theoretical postulations of Bera and Park (2008) and Abadir et al. (2010), we create portfolios using maximum entropy principle and compare this portfolios and their performance to MV portfolios based on the rolling window technique for daily and weekly holding periods. Furthermore, we tested whether the out-of-sample Sharpe ratios, which are of our main interest, are statistically different by constructing studentized time-series bootstrap confidence intervals, introduced by Ledoif and Wolf (2008a).

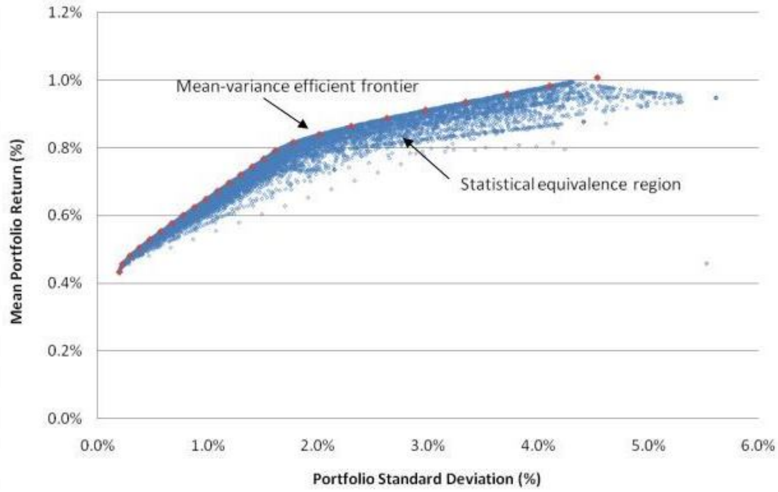
Rest of the thesis is structured as follows: Chapter 2 provides the literature review regarding the existing methods in portfolio optimization problem. Chapter 3 describes theoretical part of the models and concepts, used in this thesis. Chapter 5 contains data description together with empirical models and discussions of the empirical findings. Finally, the conclusion for this thesis is presented in Chapter 6.

Chapter 2
Literature review.

Over the recent years, numerous studies and researches were published related to the portfolio optimization. Having the Markovitz as the pioneer of the portfolio selection problem, researches have proposed many ways to improve it together with the completely new methods for the portfolio optimization.

Starting from 1989, Michaud (1989) has supported his resampled efficiency method for portfolio creation. Using bootstrapping, Michaud developed a method of constructing "statistically equivalent" efficient frontiers. This methods is meant to tackle the problem of instability and ambiguity of the classical MV portfolio in the way that, for example, rebalancing to the new portfolio is not required if it is in the statistically equivalent region. On the figure below are depicted the efficient frontier of classical Markovitz method and its statistically equivalent region.

Figure 2.1.: Mean-variance efficient frontier and statistical equivalence region.



Source: Delcourt and Petitjean (2011).

The empirical research of Delcourt and Petitjean (2011) shows, that resampled portfolio has better diversification properties comparatively to the MV portfolio. However, the out-of-sample performance of the resamling strategy tends to outperform the classical MV approach only in case of very small estimation window. Moreover, the distribution of weights in the bootstrapped sample tends to be skewed. In this case the sample mean can not represent the location parameter of the distribution correctly(Scherer 2002).

Another way to improve MV portfolio is to use shrinkage estimator, researched by James and Stein (1961), Sharpe (2003) and Frost and Savarino (1986) among others. The idea is to shrink the estimated mean returns and variance-covariance matrix to some predetermined target values. For mean values shrinkage estimator has the following form:

$$\mu_s = \delta\mu_0 + (1-\delta)\bar{\mu},$$

where μ_0 is a target constant and $\bar{\mu}$ is the sample means, $\delta \in (0,1)$. Shrinkage estimator of the variance-covariance matrix is:

$$\Sigma_s = \delta S + (1-\delta)\Sigma,$$

where S is a target estimate for the variance-covariance matrix and Σ is sample variance-covariance matrix, $\delta \in (0,1)$ (Frost and Savarino 1986).

Usually target mean values are chosen as the highest return among all assets in the portfolio and target matrix is usually a highly structured estimator, like single-factor matrix, derived by Sharpe (1963).

Recent work of DeMiguel et al. (2011) investigates extensively shrinkage estimators of moments and shrinkage estimator of portfolio weights toward the predetermined targets, like equally weighted portfolio or minimum-variance portfolio and their calibration. Authors showed, that the shrinkage techniques help to reduce the estimation error and thus to improve the performance of the portfolio. However, most of the well structured shrinkage estimators, presented in DeMiguel et al. (2011), require the assumption of the iid normality of the returns on financial securities, whereas Sheikh and Qiao (2010) provided evidence, that most of the time returns are not normally distributed.

Factor models for estimation of the variance-covariance matrix are used to reduce the number of free parameters. Sharpe (1963) proposed a single-factor model, in which the number of parameters to be estimated is substantially reduced form $\frac{N(N+1)}{2} - N$ to $3N + 2$, where N is a number of assets in the portfolio.

In the work of Gökgöz (2009), single-factor model together with three-factor and characteristic models are developed and compared to the "classical" MV optimization.

For single factor model Gökgöz (2009) used The Capital Asset Pricing Model (CAPM), developed initially by Sharpe (1964). In this case asset returns are described by its systematic risk (market beta) and its expected returns can be described mathematically as:

$$E(R_i) = R_f + \beta_i [E(R_m) - R_f],$$

where, $E(R_i)$ is expected return on asset i, R_f is risk-free rate of return, $E(R_m)$ is expected return on the market portfolio and β_i is a specific measure of the non-diversifiable risk, which assesses the responsiveness of the expected excess returns on the asset i to the expected excess returns on the market portfolio. β-coefficient can be obtained from the simple linear regression analysis using historical data.

The drawback of single-factor model is that it leads to biased estimates of the variance-covariance matrix of the returns (Bera and Park 2008). To tackle this problem one can include more factors. In this case number of parameters to be estimated can be reduced to $K(K+1)/2 + N(K+2)$, where K is the number of the factors and N is the number of variables, namely the number of assets in the portfolio. For this Gökgöz (2009) investigated three-factor model, developed by Fama and French (1996). In this model expected asset returns are described by the sensitivity to the expected market returns, the difference between the return on a portfolio of small stocks and the return on a portfolio of the large stock and moreover, the difference between the return on a portfolio with high book-to-market ratio stocks and the return on a portfolio with low book-to-market ratio. Mathematically, it can be presented as:

$$E(R_i) - R_f = \beta_{im}[E(R_m) - R_f] + \beta_{is} E(SMB) + \beta_{ih} E(HML),$$

where, $E(R_i) - R_f$ is expected excess return of the asset i over risk-free rate on the return, $E(R_m) - R_f$ is expected excess return on the market portfolio over the risk-free rate of return, $E(SMB)$ is expected difference of the returns on the portfolios of big and small stocks, $E(HML)$ is expected difference of the returns on the portfolios with high and low book-to-market ratio stocks, β_{im}- sensitivity of the assets' excess returns onto the market's excess returns, β_{is}- sensitivity of the assets' excess returns onto expected SMB returns and β_{ih} represents sensitivity of the assets' excess returns onto expected HML returns.

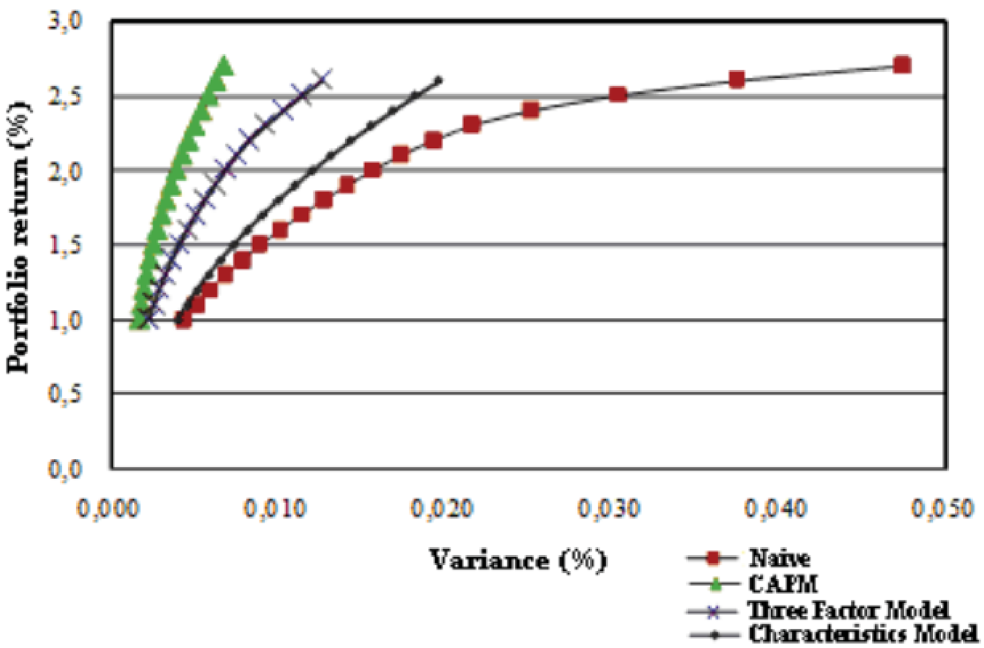
An alternative to Three factor model, which is argued to assess cross-sectional relation better, is Characteristic model (Daniel et al. 2001). The expected return on asset i is calculated in the following way:

$$E(R_{it}) = a + b\tilde{\theta}_{it},$$

where a is the intercept, obtained from the linear regression and b is sensitivity onto the the firm's characteristics $\tilde{\theta}$.

Gökgöz (2009) concluded that mean-variance optimization with estimated inputs, obtained from single-factor or multi-asset models, results in portfolios with smaller expected risk. It can be seen from the provided plot of efficient sets, given below:

Figure 2.2.: Efficient sets.



Source: Gökgöz (2009).

The resulting efficient sets differ drastically, especially in case of single-factor model (CAPM). Obtaining estimates from a single-factor model results in efficient set with much lower risk, but as it was mentioned before, this model is likely to be biased as it assumes an equilibrium situation. Two other methods (three-factor model and characteristics model) tend to converge to the solution of MV approach, but as it can be seen, a lot of additional work has to be done in order to construct them. Developing additional factors, such as portfolios which consist of a big (small) stocks, portfolios of stocks with a high (low) book-to-market ratio or asset's characteristics may become a tedious task. Moreover, coming back to the issue of biased estimator, despite the lowest expected risk in single-factor model, using daily or weekly data leads to additional systematic bias (Koller et al. 2010, p. 250).

Avramov and Zhou (2010) performed an investigation of the portfolio creation using Bayesian theory. Authors stated that Bayesian approach allows one to incorporate the information of macro conditions, beliefs and uncertainty in a sense of prior density.

In classical mean-variance portfolio optimization the expected utility is maximized, under the assumption that estimated input parameters are equal to the true ones, namely:

$$\max_{w}[U(w)|\theta = \hat{\theta}],$$

where $\hat{\theta}$ is the vector of estimated parameters: expected return of the portfolio and its variance, θ is a vector of true parameters. $U(w)$ is the utility quadratic function given as:

$$U(w) = E[R_p] - \frac{\gamma}{2}Var[R_p] = w'\mu - \frac{\gamma}{2}w'Vw.$$

Bayesian approach allows to treat θ as a random quantity and to make the assumption only about its probability distribution. Avramov and Zhou (2010) investigated different types of prior beliefs on the parameter θ. Under the diffuse prior, optimal portfolio weights are:

$$\hat{w}^B = \frac{1}{\gamma}\left(\frac{T-N-2}{T+1}\right)\hat{V}^{-1}\hat{\mu}.$$

In case of classical MV optimization optimal weights are given as: $\hat{w} = \frac{1}{\gamma}\hat{V}^{-1}\hat{\mu}$. This additional multiplier, for the case of Bayesian approach, puts less weights on risky assets, when the portfolio has in it risk-free asset, which is due to the fact that under diffuse prior weights are adjusted in order to capture the uncertainty about parameters estimates.

However, authors concluded that using diffuse prior results in not that significant difference in the result and some informative prior, which incorporates macro conditions or asset pricing theory comes in better use. For this reason authors analyzed asset pricing priors, which initially were introduced by Pástor and F. (2000). Such models allow to integrate investors belief that cross section dispersion in expected returns may be explained by asset pricing model (Avramov and Zhou 2010).

In all aforementioned methods, risk of the portfolio is assessed through its standard deviation. An alternative method to measure how risky is the portfolio is Value at Risk (VaR). VaR can be described as the worst expected loss over the given period at a given confidence level (Danielsson 2007).

One of the most attractive arguments to use VaR model is that it assesses the downside risk. Assuming, for example, when the price of the stock rises quickly. This increases the variance of that stock, and consequently risk related to it, if one uses mean-variance optimization. However, investors are not distressed by such movements.

It is worth to mention, that efficient frontiers, obtained from portfolio optimization using VaR can not be compared directly to efficient frontiers, obtained from MV optimization. This is logical inference, as this models use different notions for assessing the risk of a portfolio, in case of mean-variance optimization risk is presented as the standard deviation of the portfolio while in portfolio optimization using VaR, risk is presented as the VaR of the portfolio with the corresponding confidence level.

As Gaivoronski and Pflug (2004-2005) showed, portfolio optimization with VaR as an objective function can be presented mathematically as:

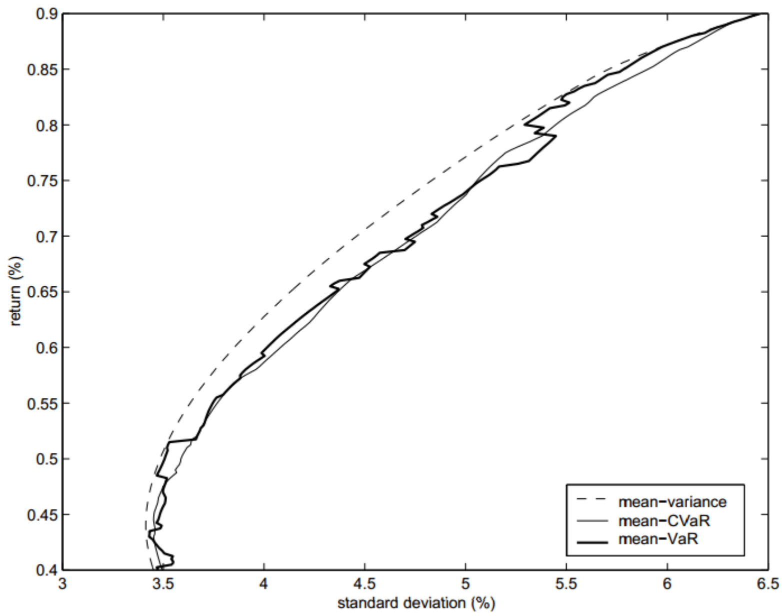
$$\min_x R(x'\xi),\ s.t.\ x'\mathbb{E}(\xi) \geq \mu,\ x'1 = 1,\ x \geq 0, \tag{2.1}$$

where the objective function $R(W) = VAR(W) = \mathbb{E}(W) - Q_\alpha(W)$. $Q_\alpha(W)$ is the α-quantile of the returns.

However, VaR has severe drawbacks as an objective function for a portfolio optimization. First of all, VaR is non-sub-additive, meaning that for any two portfolios X and Y : $VAR(X+Y) > VAR(X) + VAR(Y)$, which is counter-intuitive to the notion of the diversification. Second problem of using VaR is the fact, that it does not have to have one local minimum, meaning that it is non-convex. This complicates the optimization tremendously (Gaivoronski and Pflug 2004-2005).

To overcome aforementioned problems, Conditional Value at Risk (CVaR) can be taken as the objective function for the portfolio optimization. CVaR is proved to be convex and is sub-additive (Rockafellar and S. 2000; 2002). This makes the the portfolio optimization way easier. In this case the objective function for the optimization 2.1 is of the form: $R(W) = VAR(W) = \mathbb{E}(W) - C_{\alpha}(W)$, where "$C_{\alpha}(W)$ is the expected loss given that the loss is greater that the VaR at that level"(Gaivoronski and Pflug 2004-2005).

Figure 2.3.: Boundary of mean-variance feasible set and images of mean-VaR boundary and mean-CVaR boundary.



Source: Gaivoronski and Pflug (2004-2005).

As it can be seen, feasible efficient frontier are very close to each other. Taking into account the results of Delcourt and Petitjean (2011), efficient sets for mean-VaR and mean-CVaR may lay in the statistically equivalent region.

Gaivoronski and Pflug (2004-2005) concluded that mean-VaR optimization comes in better use, when investor is willing to assess the potential down-side risk of the portfolio and wants it to be incorporated into the optimization. However, the optimization problem for mean-VaR and mean-CVaR portfolios often become to be challenging task and it has to be carried out in further investigations.

Bera and Park (2008) proposed to use cross-entropy measure as the objective function in the portfolio optimization. Entropy is a measure of uncertainty and is proved by the authors to provide elegant way to assess portfolio's diversity. This method may be interpreted as the shrinkage estimator of the portfolio weights towards the predetermined values, for example equally-weighted portfolio or minimum-variance portfolio. Rather than shrink estimated inputs like mean returns and variance-covariance matrix, which are then used in the optimization, information theoretic approach to portfolio creation shrinks weights directly in the optimization, as it is incorporated in the objective function.

Maximum entropy approach to portfolio creation guarantees no short positions in the portfolio, as weights are seen as the discrete probability mass, which by definition can not take negative values. Moreover, this method may become of a great use for investors as the prior target weights in optimization problem can be chosen manually, based on investors beliefs and preferences.

The general representation of the objective function to be minimized is cross-entropy measure (Golan et al. 1996, p.31):

$$CE(\pi|q) = \sum_{i=1}^{N} \pi_i ln(\pi_i/q_i) \approx \sum_{i=1}^{N} \frac{1}{q_i} (\pi_i - q_i)^2 .$$

In minimization problem we choose set of weights $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ so the cross-entropy between them and initial weights $q = (q_1, q_2, \ldots, q_N)$ has its minimum value, given additional constrains. Bera and Park (2008) argued, that by using CE as an objective function in portfolio optimization, one puts emphasis on small allocations in the portfolio, so they are adjusted more than the large ones, which possibly lead to more diversified portfolio.

Information theoretic approach, as any other method for portfolio creation, relies on well-estimated input parameters, such as mean returns and variance-covariance matrix. Noting, that the main source of the estimation error is variance-covariance matrix, which is quite intuitive as the the number of parameters in variance-covariance matrix to be estimated, increases quadratically with the increase in the number of variables.

Abadir et al. (2010) proposed Design-free estimate of large variance matrix. The uniqueness of this method is that it does not rely on any assumptions about the data. Abadir et al. (2010) showed, that their approach reduces estimation error, especially in case of large variance-covariance matrices and/or when sample size is small relatively to the number of variables (assets in the portfolio).

Both, information approach to the portfolio creation and Design-free estimate of variance-covariance matrix are described in more details in the next chapter, as they are of the main interest in this master thesis and are used in the empirical analysis.

Chapter 3
Methodology

3.1. Markovitz's Mean-Variance portfolio selection

Markovitz' Mean-Variance(MV) approach is one of the most popular methods for the portfolio creation, which consists of risky assets. Markovitz (1952) introduced a notion, based on which investors minimize the variance of the portfolio subject to the target expected return rather than to create the portfolio with the highest return, in other words, investor considers expected return as a desirable thing and the corresponding variance as undesirable thing. The principle, that investor should create the portfolio with the highest return is rejected as it leads the portfolio, which includes only that asset, which has the highest mean return among others, no matter what the variance of the returns on this asset is. If there are more than one asset with the same return, then any combination of these assets in the portfolio is as good as the portfolio, constructed from only one of these assets. Thereby, such principle for portfolio selection contradicts to investment behavior, as the return of the portfolio is not the only thing investors are interested in. This created the principle of risk in the portfolio.

Let $R = (R_1, R_2, \cdots, R_N)' = (r_1 - r_f, r_2 - r_f, \cdots, r_N - r_f)'$ be the excess returns on N risky assets, where r_i is the return on the i-th asset and r_f is the return on the risk-free asset. Expected returns are $E(R) = m = (m_1, m_2, \cdots, m_N)'$ and variance-covariance matrix $Var(R) = \Sigma$ of dimension $N \times N$. Portfolio is defined as a vector of weights $\pi = (\pi_1, \pi_2, \cdots, \pi_N)'$. This weight have to satisfy the condition: $\pi' 1_N = 1$, where 1_N is the vector of ones of dimension $N \times 1$. This condition is crucial and represents the investor's wealth allocation among the risky assets. Having this and the desirable return of the portfolio, MV approach stipulates solving of the following minimization problem:

$$\min_{\pi} \pi'\Sigma\pi, \quad s.t.\ E(\pi'R) = \pi'm = \mu_0, \quad \pi'1_N = 1, \tag{3.1}$$

where $(\pi'\Sigma\pi)$ is the the variance of the portfolio and μ_0 is desirable return of this portfolio.

Merton (1972) derived, that with the Lagrangian multipliers:

$$\gamma = \frac{C\mu_0 - A}{D}, \qquad \nu = \frac{B - A\mu_0}{D},$$

the solution for the optimization problem 3.1 is given by:

$$\hat{\pi} = (\frac{\mu_0}{B})\Sigma^{-1}m,$$

$$\sigma^2_{\hat{\pi}} = \hat{\pi}'\Sigma\hat{\pi} = \frac{C\mu_0^2 - 2A\mu_0 + B}{D},$$

where $A = 1_N'\Sigma^{-1}m,\ B = m'\Sigma m,\ C = 1_N'\Sigma^{-1}1_N,\ D = BC - A^2$.

Therefore, the efficient frontier for the MV portfolio can be presented as:

$$\left(\frac{D}{C}\right)\sigma^2_{\hat{\pi}} - \left(\mu_0 - \frac{A}{C}\right)^2 = \frac{D}{C^2}.$$

MV approach provides us with a solid logic of how the creation of the portfolio should be seen, but it has it's well known drawbacks. One of them is that MV approach leads to the portfolio with weights which are highly concentrated only on few assets and changes in the estimated returns can influence the weights in portfolio dramatically. This is due to its estimation mechanism (Kolusheva 2008).

3.2. Maximum Entropy Principle in Portfolio Diversification

In Information theory, entropy is called the measure of disorder for a random variable with a discrete probability distribution $p = (p_1, p_2, \cdots, p_N)$ that takes N values. Bera and Park (2008) suggested to use this concept in the portfolio creation problem. If we consider allocation of weights for the assets of the portfolio $\pi = (\pi_1, \pi_2, \cdots, \pi_N)$

with restrictions, that all weights are greater or equal to zero and their sum is equal to one, we can treat this weights as a discrete probability distribution for a random variable. One of the ways to look on it is to use Shannon entropy as an objective function in portfolio optimization problem.

Shannon entropy

Introduced by Shannon (1948) it is defined as:

$$SE(\pi) = -\sum_{i=1}^{N} \pi_i ln(\pi_i). \tag{3.2}$$

In case, when we look on the probabilities π_i as on the weights for the assets in the portfolio it gives us a measure of the disorder in the portfolio (measure of portfolio's diversification). $SE(\pi)$ reaches its maximum value of $ln(N)$ and minimum of 0 in case, when $\pi_i = 1/N$, for all $i = 1, \ldots, N$ and $\pi_i = 1$ for only one i, respectively. So, we can see here the logic, when $SE(\pi)$ equals to its maximum, the uncertainty about what value (from the set of possible outcomes) random variable will have (equally weighted portfolio) is the the highest. Whereas, in case when $SE(\pi) = 0$ there is no uncertainty about the outcome for the "random" variable (portfolio consists only from one asset). As it was proposed by Bera and Park (2008), we will have the entropy as our objective function and using it we will try to obtain well-diversified portfolio with given side conditions.

In case, when we use Shannon's entropy measure we shrink our portfolio weights toward an equally weighted portfolio. Shannon's entropy is a special case of more general measure of the uncertainty: cross-entropy, that is also called Kullback-Leiber information criteria, defined as :

$$CE(p,q) = KLIC(p,q) = \sum_{i=1}^{N} p_i ln \frac{p_i}{q_i},$$

where $q = (q_1, q_2, \cdots, q_N)$ is the reference probability distribution and $p = (p_1, p_2, \cdots, p_N)$ is the distribution which we want to shrink toward the q. So, by minimizing $CE(p,q)$ we are trying to adjust the probability distribution $p = (p_1, p_2, \cdots, p_N)$, in a way, that the uncertainty about a random variable with the probability distribution p would be as close as possible to the value of uncertainty of this random variable if its probability distribution was $q = (q_1, q_2, \cdots, q_N)$.

Preliminary approach Once again, we can consider vector of the portfolio weights $\pi = (\pi_1, \pi_2, \cdots, \pi_N)$ as a the probability mass function for a random variable. If we are asked to select the portfolio, namely, to determine portfolio weights for an investor, who provides us with the initial desirable mean value of the portfolio return, say μ_0, we can state the optimization problem as (Bera and Park 2008)

$$\max_{\{\pi_{i=1}^N\}} \left(\text{-} \sum_{i=1}^{N} \pi_i ln \pi_i \right), \tag{3.3}$$

s.t.

$$\sum_{i=1}^{N} m_i \pi_i = \mu_0, \qquad \sum_{i=1}^{N} \pi_i = 1, \tag{3.4}$$

where $m_i,\ i = 1, \cdots, N$ is sample mean return on asset i. Condition $\sum_{i=1}^{N} \pi_i = 1$ ensures (in content of the probability theory), that we have the proper mass function for a random variable.

Lagrangian function for aforementioned maximization problem is:

$$\mathscr{L} = -\sum_{i=1}^{N} \pi_i ln \pi_i + \gamma \left(\sum_{i=1}^{N} m_i \pi_i - \mu_0 \right) - \lambda \left(\sum_{i=1}^{N} \pi_i - 1 \right).$$

The solution is given as:

$$\hat{\pi}_i = \frac{1}{\Omega(\gamma)} exp[-\gamma \hat{m}_i], \qquad i = 1, 2, \cdots, N, \tag{3.5}$$

where $\Omega(\gamma) = \sum_{i=1}^{N} exp[-\gamma m_i]$.

Solution 3.5 is a probability mass function of an exponential distribution and as it is known, probabilities for such distribution are strictly non-negative (no short-selling). This result gives us the portfolio, which is shrunk towards the equally weighted portfolio given the required portfolio's mean return. 'In this sense, resulting portfolio weights are maximum diversified portfolio given mean constraint without considering risk (variance)' Bera and Park (2008).

If we want to include restricting conditions for risk (variance-covariance matrix), optimization problem will be (in matrix form):

$$\max_{\pi} \left(\text{-} \pi' ln \pi \right)$$

subject to

$$\pi' \hat{m} \geq \mu_0, \quad \sqrt{\pi' \hat{\Sigma} \pi} \leq \sigma_0, \quad \pi \geq 0, \quad \pi' 1_N = 1,$$

where $\hat{\Sigma}$ is a variance-covariance matrix of the returns assets. Inequality conditions are more appropriate in this case, as investors are usually not interested in having the exact return and risk of the portfolio, rather they prefer to have some boundaries for this values.

General approach Following Bera and Park (2008) we can state more general statement of the problem. Using cross-entropy we can state the following minimization problem:

$$\min_{\pi} CE(\pi|q) = \min_{\pi} \sum_{i=1}^{N} \pi_i ln \left(\frac{\pi_i}{q_i} \right) \tag{3.6}$$

subject to

$$\mathbb{E}U(\pi, R, \lambda) \geq \tau, \quad \pi \geq 0, \quad \pi' 1_N = 1,$$

where λ is the risk aversion parameter, τ determines investor's belief in the estimated expected utility $U(\pi, R, \lambda)$.

Moving next to the investigation of the utility function. Lets assume a vector R , which has a distribution function $F(R)$. Lets define $\xi \equiv \mathbb{E}U(\hat{\pi}, R, \lambda)$, where $\hat{\pi}$ is obtained from the following maximization problem:

$$\hat{\pi} = arg \max_{\pi} \; \mathbb{E}U(\pi, \widetilde{R}, \lambda) \tag{3.7}$$

s.t.

$$\pi' 1_N = 1, \quad \pi \geq 1,$$

where $\widetilde{R}$ is random sample of size T, obtained from the empirical distribution $\hat{F}(R)$. Using resampling method we will obtain B portfolios. Therefore, investor's strength of belief can be expressed as the r-th quantile of the distribution of ξ, $0 < r < 1$. If the distribution function of ξ is $G(\xi)$, which is obtained using B values of the maximized expected utility from problem 3.7, then

$$\tau = G^{-1}(r) = \xi_r.$$

Bera and Park (2008) proposed to use the following quadratic expected utility function to solve the problem 3.7:

$$\max_{\pi} \mathbb{E}U(\pi, R, \lambda) = \max_{\pi} \left[\pi' m - \frac{\lambda}{2} - \frac{\lambda}{2} \pi' \Sigma \pi \right] \tag{3.8}$$

subject to

$$\pi' 1_N = 1, \quad \pi \geq 1,$$

Using bootstrap or Monte Carlo methods for re-sampling R from the empirical distribution $\hat{F}(R)$ following results can be calculated:

$$\breve{\pi}_{(b)} = arg \max_{\pi} \left[\pi' \tilde{m}_{(b)} - \frac{\lambda}{2} \pi' \tilde{\Sigma}_{(b)} \pi \right],$$

$$\xi_{(b)} = \breve{\pi}'_{(b)} \hat{m} - \frac{\lambda}{2} \breve{\pi}' \hat{\Sigma} \pi,$$

where $\hat{m}$ and $\hat{\Sigma}$ are sample mean and sample variance-covariance matrix, estimated form the original data R. $\tilde{m}$ and $\tilde{\Sigma}$ are calculated from the simulated data $\tilde{R}_{(b)}$. Empirical distribution of ξ is estimated based on the sets $\xi_{(b)}$, $b = 1 \cdots B$.

Now our minimization problem for cross-entropy can be stated as :

$$\min_{\pi} \sum_{i=1}^{N} \pi_i ln(\pi_i / q_i), \tag{3.9}$$

subject to

$$\pi' \hat{m} - \frac{\lambda}{2} \pi' \hat{\Sigma} \pi \geq \hat{G}^{-1}(r), \quad \pi \geq 0, \quad \pi' 1_N = 1, \tag{3.10}$$

where $\hat{G}(\cdot)$ denoted the empirical distribution of ξ.

Minimization problem can be solved by the classical gradient based routine if we assume that the utility function in 3.8 is smooth function[1].

[1] A function that has derivatives of all orders is called a smooth function.

Generalized cross entropy method So far, in the information theoretic approach we considered the case, when assets managers are not allowed to sell short. Although, this is often the case in real life, we may want to have the option of short selling in our portfolio. Bera and Park (2008) proposed to use Generalized cross entropy method, initially introduced by Golan et al. (1996), for portfolio creation, with which we will get familiar right now.

The problem, that arises in case when we want to allow short selling is that the objective function of the optimization problem 3.6 may not exist, as it contains the $ln(\cdot)$ function which is defined only for non-negative values. Let $p_i = (p_{i1}, p_{i2}, \cdots, p_{iM})$, $i = 1, \cdots, N$ be a discrete probability distribution for each asset in the portfolio that is over the set $[l,m]$, a equally distanced discrete points $z = (z_{1,} z_{2,} \cdots, z_M)$. Namely, this means that the portfolio weights are defined as:

$$\pi = Zp = \begin{bmatrix} z' & 0 & 0 & 0 & 0 \\ 0 & z' & 0 & 0 & 0 \\ 0 & 0 & z' & 0 & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & 0 & z' \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_N \end{bmatrix}.$$

Here we face the problem, how should the probability distribution p be defined. To tackle this one can start by noting that the weights for MV efficient portfolio have to be in the set $[\pi | \pi = Zp,\ p \in P]$. From this we should be able to obtain the probability distribution p given a set of equally distanced points z.

Therefore, after finding weights for MV efficient portfolio, set of points z may be find from the following problem:

$$p = Z^{-1}\pi,$$

subject to

$$\sum_{i=1}^{M} p_i = 1,\ i = 1 \cdots N,\ (z_1, z_2, \cdots, z_N)\ \textit{are equally distanced on the set}\ [l,m].$$

If the quadratic expected utility is considered, the following GCE minimization problem, which allows for sort-selling, can be presented:

$$\min_{p \in P} \sum_{i=1}^{N} \sum_{m=1}^{M} p_{im} ln(p_{im}/\omega_{im}) \tag{3.11}$$

subject to

$$(Zp)'\hat{m} - \frac{\lambda}{2}(Zp)'\hat{\Sigma}(Zp) \geq \hat{G}^{-1}(r) \tag{3.12}$$

$$p_i' 1_M = 1,\ i = 1, 2, \cdots, N, \tag{3.13}$$

$$(Zp)' 1_N = 1, \tag{3.14}$$

where $\hat{G}(\cdot)$ is the empirical distribution function of the maximized expected utility ξ and $\omega_i = (\omega_{i1,}\omega_{i2}, \cdots, \omega_{iM})$ $i = 1, \cdots N$ is a discrete prior probability distribution for each target weight q_i over z. Solving the problem 3.11-3.14 the following weights for the portfolio can be calculated as:

$$\hat{\pi}_i = z'\hat{p}_i = \sum_{m=1}^{M} z_m \hat{p}_{im}.$$

Another problem is that before we want to solve 3.11 we have to determine a discrete prior distribution $\omega_i = (\omega_{i1,}\omega_{i2}, \cdots, \omega_{iM}),\ i = 1, \cdots N$. Bera and Park (2008) suggested to find it from the solution of the optimization problem:

$$\max_{\omega_i} \left(\text{-} \sum_{m=1}^{M} \omega_{im} ln(\omega_{im}) \right),$$

subject to

$$\sum_{m=1}^{M} z_m \omega_{im} = q_i, \quad \sum_{m=1}^{M} \omega_{im} = 1.$$

3.3. Design-free estimation of variance-covariance matrix

In previous sections we discussed some of the methods for the creation of the portfolio, but all of this methods rely on well estimated input variables, such as vector of mean returns and variance-covariance matrix.

Essentially, estimation of variance-covariance matrix can be challenging task. Recall, that we will construct our portfolios using the components of Dow Jones Industrial Average (30 publicly traded companies). When dealing with large number of assets in the portfolio, values in variance-covariance matrix are very imprecise and matrix operations (which we use in our optimization problems, i.e. inverse) are hard to implement. To resolve this problem, we use the method for the estimation of the variance-covariance matrix, introduced by Abadir et al. (2010) which we will review below.

Let the variance-covariance be defined as:

$$\hat{\Sigma} = \widehat{var(x)} = \frac{1}{n} X' M_n X,$$

where $M_n = I_n - \frac{1}{n} i_n i_n'$, i_n is a vector of ones of length n and X is a matrix of dimension $n \times k$ of the observed returns on the assets of the portfolio. Furthermore, we can decompose this matrix as:

$$\hat{\Sigma} = \hat{P} \hat{\Lambda} \hat{P}', \tag{3.15}$$

where $\hat{\Lambda}$ is the diagonal matrix of eigenvalues of $\hat{\Sigma}$ and $\hat{P}$ is the orthogonal matrix. Rearranging the last equation leads to the following result:

$$\hat{\Lambda} = \hat{P}' \hat{\Sigma} \hat{P} = diag(\widehat{var}(\hat{p}_1 x), \cdots, \widehat{var}(\hat{p}_k x)) \tag{3.16}$$

The idea, Abadir et al. (2010) came up with is that we can use only a portion of the data (say m random observations) to estimate $\hat{P}$ and then use the rest (remaining $n - m$ observations) to re-estimate Λ.

The estimation procedure starts with splitting matrix X into two parts:

$$X' = (X_1', X_2'), \tag{3.17}$$

where $X_1'$is of dimension $m \times k$ and X_2' is of dimension $(n-m) \times k$. After this calculate the variance-covariance based on the first sub-sample of the observation (X_1'), namely:

$$\hat{\Sigma}_1 = \frac{1}{m} X_1' M_m X_1 = \hat{P}_1 \hat{\Lambda}_1 \hat{P}_1'. \tag{3.18}$$

From the last equation we estimate $\hat{P}_1$. Next step is to estimate the diagonal matrix of eigenvalues from the remaining observations:

$$dg\left(\widehat{var}(\hat{P}_1'x)\right) = dg\left(\hat{P}_1'\hat{\Sigma}_2\hat{P}_1'\right) = \frac{1}{n-m}dg\left(\hat{P}_1'X_2'M_{n-m}X_2\hat{P}_1\right) = \tilde{\Lambda} \tag{3.19}$$

and at the end, using the equation 3.15 we can obtain the new estimator of variance-covariance matrix as:

$$\tilde{\Sigma} = \hat{P}\tilde{\Lambda}\hat{P}' = \hat{P}dg\left(\hat{P}_1'\hat{\Sigma}_2\hat{P}_1'\right)\hat{P}'. \tag{3.20}$$

The problem, one will face is what value for m should be chosen. Abadir et al. (2010) proposed computationally exhausting but efficient, as authors showed, method to determine m.

The main idea of this method is to incorporate bootstrapping into the minimization problem. By creating a bootstrap sample ($X_b = (x_1^b, x_2^b, \cdots, x_n^b)$ by re-sampling using the replacement) from original data we can calculate $\hat{\Sigma}_b$ and $\tilde{\Sigma}_{m,b}$.

Then we can calculate the average bootstrap values for the corresponding variables:

$$\hat{\Sigma}_B = \frac{n}{(n-1)B}\sum_{b=1}^{B}\hat{\Sigma}_b, \qquad \tilde{\Sigma}_{m,B} = \frac{1}{B}\sum_{b=1}^{B}\tilde{\Sigma}_{m,B}.$$

The minimization problem, that incorporates bootstrap part, stated above, is defined in the following way:

$$m = arg\min_{m\in\mathbb{M}}\frac{1}{B}\sum_{b=1}^{B}\left\|vech(\tilde{\Sigma}_{m,b} - \hat{\Sigma}_B)\right\|_2^2,$$

where $\|\alpha\|_2 = (\sum_{i=1}^{j}|\alpha|^2)^{1/2}$ denotes the second norm for $l-$dimensional vector α, $vech(\cdot)$ denotes the half-vectorization of the matrix, $\mathbb{M} = (m_1, m_2, \cdots, m_M)$ is a grid of the possible values for m. Structure of $\mathbb{M}$ (how many values it consists of) depends mainly on the computational resources (Abadir et al. 2010).

3.4. Sharpe ratio.

The main critique, used in this work, for comparison the performance of different portfolios is Sharpe ratio, which is presented below.

Revised by Sharpe (1994), ratio is well known measure of the performance of an investment. It can be interpreted as excess return over the benchmark portfolio per unit of deviation of this excess return. Mathematically:

$$SR_p = \frac{E[R_P - R_B]}{\sqrt{var[R_P - R_B]}},$$

where R_P is the rate of returns on the portfolio, R_B is the rate of returns on the benchmark portfolio (i.e. risk-free rate of return). $\sqrt{var[R_P - R_B]}$ is the standard deviation of the excess returns. This ratio tells us by how much we are compensated (in terms of excess returns) for the risk taken, as we give up on the benchmark portfolio.

3.5. Hypothesis testing with the Sharpe ratio.

As in the case of variance-covariance matrix, Sharpe ramytio has to be estimated from the past historical data. Therefore, in order to compare the performance of two portfolios over the the same period, in terms of their Sharpe ratios, one has to conduct a test for this. In such a way we can conclude, that Sharpe ratios of two different portfolios are statistically different at some level of confidence.

Originally proposed by Jobson and Korkie (1981) and adjusted by Memmel (2003) method, which exploits the assumption, that out-of-sample returns on the portfolios are iid normally distributed, has the next setup:

Let $\hat{\mu}_j$, $\hat{\mu}_k$, $\hat{\sigma}_j^2$, $\hat{\sigma}_k^2$, $\hat{\sigma}_{j,k}$ be means, variances and covariance, respectively, of the out-of-sample returns of two corresponding portfolios. The null hypothesis H_0 : $\hat{\mu}_j/\hat{\sigma}_j - \hat{\mu}_k/\hat{\sigma}_k = 0$ can be tested, using the test statistics:

$$\hat{z}_{j,k} = \frac{\hat{\sigma}_k \hat{\mu}_j - \hat{\sigma}_j \hat{\mu}_k}{\sqrt{\hat{\zeta}}},$$

where

$$\zeta = \frac{1}{T-W}\left(2\hat{\sigma}_j^2\hat{\sigma}_k^2 - 2\hat{\sigma}_j\hat{\sigma}_k\hat{\sigma}_{j,k} + \frac{1}{2}\hat{\mu}_j^2\hat{\sigma}_k^2 + \frac{1}{2}\hat{\mu}_k^2\hat{\sigma}_j^2 - \frac{\hat{\mu}_j\hat{\mu}_j}{\hat{\sigma}_j\hat{\sigma}_k}\hat{\sigma}_{j,k}^2\right).$$

This statistic is asymptotically normally distributed under the assumption, that out-of-sample returns on the portfolio are iid normal. However, as itoften happens, out-of-sample returns on the portfolio are not iid normally distributed.

Ledoif and Wolf (2008a) proposed the improved method of assessing the statistical difference between the Sharpe ratios of two different portfolios. Authors were able to show that their method provides robust results for the returns, which have tails heavier than normal distribution.

The test is carried out by constructing a studentized bootstrap confidence intervals for the difference between two Sharpe ratios and concluding that the ratios are not statistically different if zero is contained in the given confidence interval[2].

Below is given the theoretical postulation, given by Ledoif and Wolf (2008a), of how one should proceed with the test using a studentized time series bootstrap confidence intervals:

Let $\hat{\mu}_j$, $\hat{\mu}_k$ be a sample means of the out-of-sample returns. Furthermore, let $\hat{\gamma}_j = E(r_j^2)$, $\hat{\gamma}_k = E(r_k^2)$ be the sample uncentered second moments of the out-of-sample returns on the portfolios j and k, respectively.

Let $\hat{v} = (\hat{\mu}_j, \hat{\mu}_k, \hat{\gamma}_j, \hat{\gamma}_k)'$ be a vector, defined for the sample of a data and the difference between the Sharpe ratios is defined as a function of $\hat{v}$:

$$\hat{\triangle} = f(\hat{v}),$$

where function f is of a form:

$$f(a,b,c,d) = \frac{a}{\sqrt{c-a^2}} - \frac{b}{\sqrt{d-b^2}}.$$

Next, we impose the assumption that

$$\sqrt{T}(\hat{v} - v) \xrightarrow{d} N(0, \Psi),$$

where Ψ is unknown symmetric positively semi-definite matrix.

Implying delta method to the last expression:

$$\sqrt{T}(\hat{\triangle} - \triangle) \xrightarrow{d} N(0, \nabla' f(v) \Psi \nabla f(v)),$$

where

[2]In my empirical research I use Matlab program (Ledoif and Wolf 2008b), because as it will be seen later, the out-of-sample returns in my empirical study are not normally distributed.

$$\nabla' f(a,b,c,d) = \left(\frac{c}{(c-a^2)^{1.5}},\ -\frac{d}{(d-b^2)^{1.5}},\ -\frac{1}{2}\frac{a}{(c-a^2)^{1.5}},\ \frac{1}{2}\frac{b}{(d-b^2)^{1.5}} \right).$$

Having a consistent estimate $\hat{\Psi}$ of Ψ, the standard error for $\hat{\triangle}$ is:

$$s(\hat{\triangle}) = \sqrt{\frac{\nabla' f(v)\Psi\nabla f(v)}{T}}. \tag{3.21}$$

In order to obtain $\hat{\Psi}$ one can use kernel estimation in the following manner:

Let $\hat{v}^* = (\hat{\mu}_j^*, \hat{\mu}_k^*, \hat{\gamma}_j^*, \hat{\gamma}_k^*)'$ be a vector obtained from the bootstrap data and $l = \lfloor T/b \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part of a number, T is a number of observed returns in the sample and b is a block size, which is required in order to use circular block bootstrap.

Furthermore, lets define:

$$y_t^* = (r_{tj}^* - \hat{\mu}_j^*, r_{tk}^* - \hat{\mu}_k^*, r_{tj}*^2 - \hat{\gamma}_j^*, r_{tk}^{*2} - \hat{\gamma}_k^*)' \qquad t = 1,...T$$

and

$$\varsigma_i = \frac{1}{\sqrt{b}} \sum_{t=1}^{b} y_{(i-1)b+t}^* \qquad j = 1...l$$

Having this, the kernel estimate for Ψ for each bootstrap sample is computed:

$$\hat{\Psi}^* = \frac{1}{l} \sum_{i=1}^{l} \varsigma_i \varsigma_i'.$$

The bootstrap standard error $\hat{\triangle}^*$ is given by formula 3.21.

Obtaining the p-value.

Ones again, the two-sided test for the null hypothesis $H_0: \triangle = \mu_j/\sigma_j - \mu_k/\sigma_k = 0$ is calculated by contracting a block bootstrap confidence intervals with confidance level $1-\alpha$ and is rejected is zero is not contained in the interval. The computation of the p-value for this test is carried out in 3 steps:

1. Compute the original studentized test statistic:

$$d = \frac{|\hat{\triangle}|}{s(\hat{\triangle})},$$

2. Next, for each mth bootstrap sample we calculate the centered studentized statistic:

$$\tilde{d}^{*,m} = \frac{|\hat{\triangle}^{*,m} - \hat{\triangle}|}{s(\hat{\triangle}^{*,m})} \qquad m = 1, ..., M$$

where M is a number of bootstrap resamples.

3. Finally, corresponding p-value for the test statistic is computed as:

$$p - val = \frac{\#\{\tilde{d}^{*,m} \geq d\} + 1}{M + 1},$$

$\#\{\tilde{d}^{*,m} \geq d\}$ is a number of the test statistics of bootstrap resamples, which are higher or equal to the original test statistic.

Chapter 4

Empirical analysis.

In this section are present the statements for the optimization problems and performance measures together with all additional notes, which are essentially used to create Matlab programs in order to conduct the empirical analysis.

Before starting the main part of the chapter, one thing has to be mentioned. For all optimization problems, used in the empirical analysis, we use no short-selling constrain. We back this decision for two reasons. The first reason, is that most asset managers are not allowed to sell short. The second reason and the most important, is that this constraint allows to reduce the estimation error. Jagannathan and Ma (2003) concluded, that no short selling constraint helps to reduce estimation error in the covariance matrix, when one wants to use the returns of a higher frequency, which is our case, as we use daily returns in the empirical investigation below.

4.1. Data.

In order to perform empirical analysis we use daily data for companies, which are (were) components of the Dow Jones Industrial Average (DJIA).

The DJIA is one of the most watched indices in U. S., that tracks targeted stock marked activity. It consists of 30 publicly traded companies from different industries, such as oil and gas, pharmaceuticals, banking, etc. Data is taken from Yahoo! Finance.

The data is divided into two samples: first covers the period from 01 January 2000 until 18 March 2008 and second sample spans from 19 March 2008 until 02 December 2014, the date when our first empirical results were obtained. Having company “Visa” as the youngest publicly trading company in DJIA index, second sample starts from 19 March 2008, the date when Visa went public. Secondly, it is always of a great use to investigate the portfolio creation problem in periods of different macroeconomic conditions, which are happened to be captured in two samples due to financial crisis in 2008 and the consequent global recession.

It is worth to mention that the components of DJIA were changing during both periods. Therefore, to stay consistent, in a sense that set of assets, which are used to estimate the inputs for the optimization and obliquely, to sole the optimization problem, I decided to include in each sample companies that were in DJIA at the end of the corresponding period, namely 18 March 2008 and 02 December 2014. Companies and information about their presence in the portfolios for each period can be found in Appendix A, Table A.1

Raw data is daily close prices, adjusted for dividends and splits.

Furthermore, adjusted close prices have to be transformed into returns, which are required the portfolio optimization. Returns are calculated as:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \tag{4.1}$$

where P_t and $P_{t-1}$are adjusted close prices at time t and $t-1$, respectively.

This results in 2062 observations for 30 companies, of which DJIA consisted on 18 March 2008 and 1689 observations for 30 companies, of which DJIA consisted on 02 December 2014.

Leaping ahead, after conducting empirical analysis using two aforementioned samples, third sample that covers the period of economical recession in 2007-2009 in the United States of America was created. This sample spans the period from 01 December 2007 until 30 June 2009. NBER (2008) determined that the peak of economic activity was in December 2007, after 73 months of expansion and the trough in the business activity occurred in June 2009.

4.2. Portfolio optimization.

4.2.1. Design-free estimation of variance-covariance matrix.

In Literature review chapter theoretical framework for design-free estimate of variance-covariance matrix, designed by Abadir et al. (2010), was introduced. Aforementioned method is very computationally intensive, especially when one wants to use bootstrapping. Therefore, in this paper we use resampling and averaging technique.

On the first step, returns on assets are divided into two subsamples : $X'_S = (X'_{1,S}, X'_{2,S})$, where $X'_{1,S}$ is obtained by random sampling without replacement of m columns from the original sample X', and $X'_{2,S}$ is filled up with the remaining $n-m$ columns from X_S. For each sample X'_S we calculate variance-covariance matrix $\tilde{\Sigma}_{S,m}$ using the procedure described by equations 3.15-3.20 of previous chapter. Lastly, we average out obtained variance-covariance matrices:

$$\tilde{\Sigma}_{m,S} = \frac{1}{S}\sum_{s=1}^{S} \tilde{\Sigma}_{S,m}.$$

There are $\binom{n}{m}$ ways to choose m observations for first sample, but Abadir et al. (2010) concluded, that based on the simulations, it is sufficient to take $S \approx 20$ in order to calculate consistent estimate of design-free variance-covariance matrix.

4.2.2. Equally weighted portfolio.

We start with equally weighted portfolio as it is the easiest to obtain. It is known, that despite naive technique of the equally weighted portfolio creation it often has better out-of-sample performance comparatively to “classical” MV. The weights for this portfolio are obtained in a straightforward way:

$$\pi_i = 1/N, \quad i = 1, ..., N,$$

where N is the number of the assets in the portfolio. The corresponding portfolio mean expected return and risk are:

$$\mathbb{E}(\mu_p) = \pi'\hat{m},$$

$$\mathbb{E}(\sigma_p) = \sqrt{\pi'\hat{\Sigma}\pi},$$

where $\pi' = (1/N, ..., 1/N)$ is $1 \times N$ vector of weights, $\hat{m} = (m_1, m_2, ..., m_N)'$ is $N \times 1$ vector of mean returns of the assets in the portfolio and $\hat{\Sigma}$ is $N \times N$ variance-covariance matrix.

4.2.3. Minimum variance portfolio.

Second method, used in this thesis, is minimum variance portfolio optimization problem. Investor, seeking to minimize the risk of the portfolio has to solve the following optimization problem.

$$\min_{\pi} \pi' \hat{\Sigma} \pi, \quad s.t. \quad \pi \geq 0, \ \pi' 1_N = 1,$$

where 1_N is $N \times 1$ vectors of ones. The constraint itself is designed to ensure all weights sum to one, that is, allocated investor's wealth is completely distributed among the assets. By adding target return to the constraint we obtain one of the interpretations for the creation of the classical MV portfolio allocation problem.

4.2.4. Mean-Variance efficient frontier.

Another way to look on MV portfolio is a maximization of the expected return with a given maximum risk, investor is willing to take. Therefore, MV portfolio method for portfolio creation provides an elegant way to achieve an efficient allocation of risky assets such that, with a given mean return and variance covariance matrix, higher returns of the portfolio can only be achieved by taking on more risk.

Taking into account no short selling, it is ease to see that returns on MV portfolio vary between two values, mean return of minimum variance portfolio and the maximum mean return among all assets. Maximum expected return can be achieved only by allocating whole investor's wealth to one asset with the highest mean return.

Creation of all feasible MV portfolios, namely, the efficient frontier can be achieved by solving the following minimization problem:

$$\min_{\pi} \pi' \hat{\Sigma} \pi, \quad s.t. \quad \mathbb{E}(\pi' R) = \pi' \hat{m} = \mu_{0,} \ \pi' 1_N = 1,$$

where $\hat{m} = (\hat{m}_1, ..., \hat{m}_N)'$ is $N \times 1$ vector of assets mean returns and μ_0 is target return, which vary between the mean return of minimum variance portfolio and $\max(\hat{m})$.

4.2.5. Maximum Shannon entropy efficient frontier.

Moving next to information theoretic approach for the portfolio creation problem, we start with creation the efficient frontier, when objective function of the optimization problem is Shannon entropy.

Portfolios, from which the efficient frontier consists of, are obtained by solving following maximization problem:

$$\max_{\{\pi\}_{i=1}^{N}} \left(-\sum_{i=1}^{N} \pi_i \ln(\pi_i) \right),$$

subject to

$$\sum_{i=1}^{N} \hat{m}_i \pi_i \geq \mu_0, \quad \pi \geq 0, \quad \sum_{i=1}^{N} \pi_i = 1,$$

where $\hat{m}_i$ denotes sample mean return of asset.

Minimum value on the maximum Shannon entropy efficient frontier is mean return of equally weighted portfolio, as Shannon entropy has its maximum value of $\ln(N)$, when $\pi_i = 1/N$ for all i. Maximum value on the frontier is the highest mean return among all mean returns on the assets.

4.2.6. Maximum cross-entropy using weights of the minimum variance portfolio as targets.

When we want to estimate the portfolio, which is is as diversified as some predetermined portfolio, we can use cross-entropy as the objective function to be minimized. Bera and Park (2008) concluded, that by minimizing the cross-entropy measure, small allocation of the target portfolio are adjusted more than the large ones. This may result in more diversified portfolio.

For empirical investigation we use weights of the minimum variance portfolio as references adding the constrain for the minimum required return on the portfolio.

Optimization problem to be solved, is stated as:

$$\min_{\{\pi_i\}_{i=1}^{N}} CE(\pi|q) = \min_{\{\pi_i\}_{i=1}^{N}} \sum_{i=1}^{N} \pi_i \ln(\pi_i / q_i)$$

subject to

$$\sum_{i=1}^{N} \hat{m}_i \pi_i = \mu_0, \quad \pi \geq 0, \quad \sum_{i=1}^{N} \pi_i = 1,$$

where $\hat{m}_i$ denotes sample mean return on the asset i and μ_0 is a minimum required return of the portfolio.

In order to construct the efficient frontier for minimum cross-entropy portfolio with target weights set to those of minimum variance portfolio one has just to change the required return μ_0. The range, in which the required return has to be, has the minimum value, which corresponds to mean return of the minimum variance portfolio. The maximum value is the maximum mean return among all assets in the portfolio.

4.2.7. Discussion of empirical results.

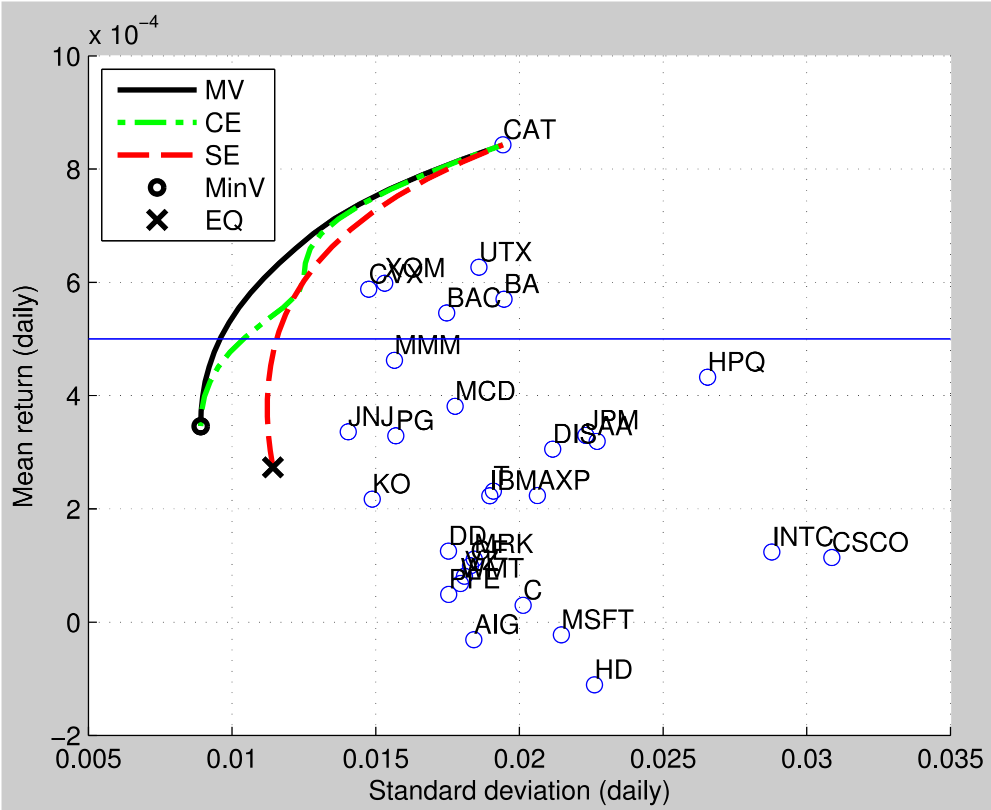
We start with the first sample, namely 01.01.2000-18.03.2008. Three efficient frontiers are depicted on the Figure 4.1; Mean-Variance efficient frontier (MV), efficient frontier for Maximum Shannon entropy portfolios (SE), where target is equally weighted portfolio and efficient frontier for Maximum cross-entropy portfolios (CE), where target is minimum-variance portfolio. (MinV) and (EQ) correspond to minimum-variance portfolio and equally weighted portfolio, respectively.

All efficient frontiers reach the highest mean daily return on the corresponding portfolio at the same point: 8.4275×10^{-4} with corresponding standard deviation: 0.0193, which is the daily mean daily return on the shares of the “Caterpillar Inc.” and daily standard deviation of its daily returns.

As it was mentioned previously, SE and CE portfolios have to reach on the other end of the efficient frontiers equally weighted and minimum-variance portfolios, respectively.

For the portfolios with high returns all three efficient frontiers give similar results in terms of daily mean returns and daily standard deviations. However, moving down to lower daily mean returns, efficient frontiers diverge and MV efficient frontier always suggests portfolios with higher daily mean returns for any given number of daily standard deviation. Starting from the point with the highest value of mean daily return, CE efficient frontier overlaps the one with MV portfolios. However, this pattern changes when the values of daily mean returns are in the range: 3.74×10^{-4} to 7.64×10^{-4}. In this interval CE efficient frontier bends toward the SE efficient frontiers and overlaps with it in the point with corresponding values of daily mean return and daily standard deviation equal to 5.79×10^{-4} and 0.0121 respectively. Blue horizontal line on the level of 0.0005 of daily mean return represents the required mean return for three portfolios: MV, SE and CE, for which corresponding performance measures are computed, namely Sharpe Ratios and Certainty Equivalent Return.

Figure 4.1.: Efficient frontiers and portfolios; design-free estimate of variance-covariance matrix; 01.01.2000-18.03.2008.



Moving next to the second sample: 19.03.2008-02.12.2014. On Figure4.2 are depicted efficient frontiers and portfolios for the same methods of the portfolio selection problems, as were used previously.

The highest possible daily mean return for all efficient frontiers is now daily mean return on the shares of “Visa Inc.”. Daily mean return for this company is 0.0012, with corresponding standard deviation of its daily returns: 0.0217. Such a good result of “Visa Inc.” could be one of the reasons, why the company became a component of DJIA index on 20.09.2013.

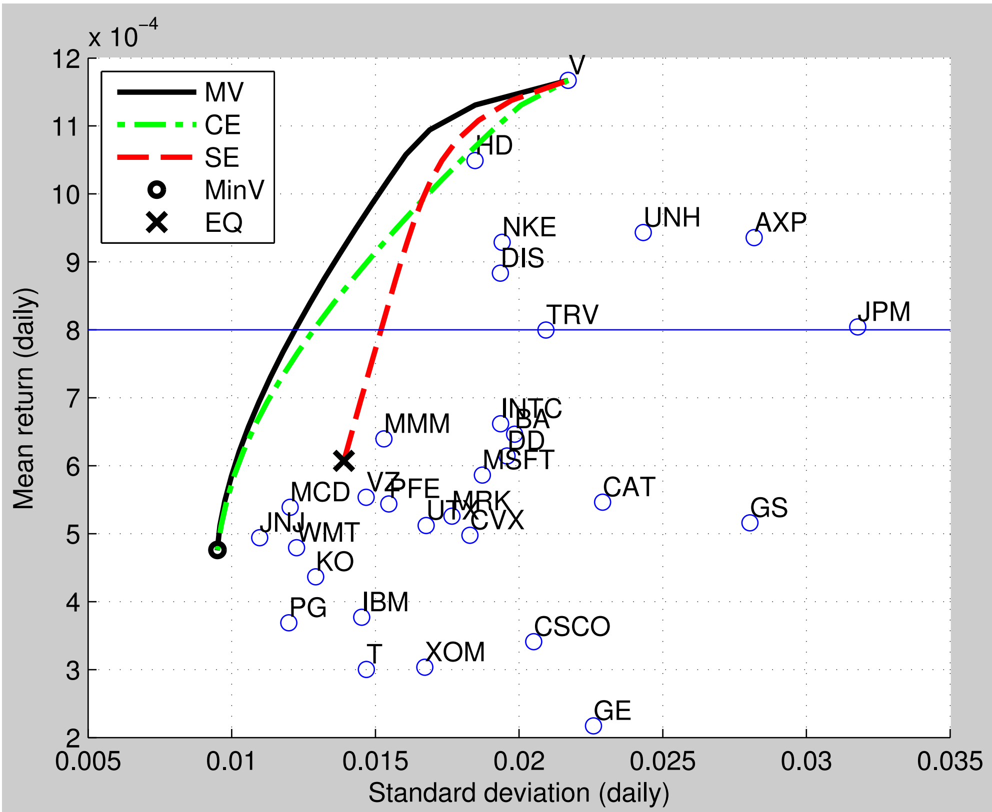
Unlike in the first sample, where CE efficient frontier almost always goes in line with MV efficient frontier, especially for a higher values of daily mean returns, in second sample CE efficient frontier shows a complete different pattern. Starting from the highest value for the daily mean return, CE eficient frontier shows steep convergence towards its minimum value for daily mean return, which corresponds to MinV portfolio. Moreover, CE efficient frontier is even below SE efficient frontier and intersects with it in the point with corresponding values of daily mean return and daily standard deviation equal to 0.001 and 0.0153 respectively.

Moreover, in the second sample, mean daily return on EQ portfolio is higher than corresponding value for MinV portfolio. This is not a case for the first sample.

Another notable difference between two sample is that in the second sample, pairs (mean daily return, daily standard deviation) of the assets, from which the portfolios are created, are located closer to the efficient frontiers. For instance, mean daily return on "The Home Depot, Inc." (HD) is practicaly the same as for the portfolio on SE efficient frontier; and both of the portfolios have the same daily standard deviation.

Blue horizontal line on the level of 0.0008, again, represents the target daily mean return, for which the portfolios and their corresponding performances are compared. Now it is not 0.0005, as we want to incorporate all types of investigated portfolios, in particular SE portfolio, which has its lowest value of daily mean return equal to 0.000607.

Figure 4.2.: Efficient frontiers and portfolios; design-free estimate of variance-covariance matrix; 19.03.2008-02.12.2014.



MV efficient portfolio selection is often criticized for its heavily concentrated weights only on few assets. Maximum entropy principle imposes "direct" shrinkage of weights to some predetermined values, which in our case are weights of equally weighted and minimum-variance portfolios. Two following tables are provided in order the investigate and compare the allocation of the assets in the portfolios.

In Table 4.1 we compare three portfolios, MV, CE, SE, which are obtained using data from the first sample for a given required daily mean return: 0.0005. From pie chart we can see that for case of MV portfolio only 12 components out of 30 are in the portfolio. The heaviest weight is 26 per cent and there are another 3 stocks, which have weights heavier than 10 per cent. In case of CE portfolio, even from pie chart, we can conclude that it fails to resolve the problem of heavily concentrated weights, as only one single stock makes 53 per cent of the portfolio. Pie chart for SE portfolio, in contrast, shows that 80 per cent of the investor's wealth can be well-diversified among the given stocks and remaining 20 per cent should be invested in a single stock.

Shannon entropy, presented in equation 3.2, has a great explanatory power in the assessment of the degree of diversification of a portfolio, after the later is obtained using some portfolio selection method (Rernholz 2002, p. 36). For instance, for equally weighted portfolio, Shannon entropy reaches its maximum value $ln(N)$, where N is a number of assets. In our case, the maximum values of Shannon entropy is: $ln(30) = 3.4012$. Although the pie charts in Table 4.1 may give some interpretation about how well the portfolios are diversified, comparatively to each other, it is good to have a single measure presented for each portfolio. Therefore we can see, based on Shannon entropy measure, that CE is the least diversified portfolio with corresponding value of 1.751 and the SE, as it is expected, is the most-diversified portfolio with corresponding value of 2.9524.

Table 4.1.: Comparison of the portfolios; design-free estimate of variance-covariance matrix; 19.03.2008-02.12.2014.

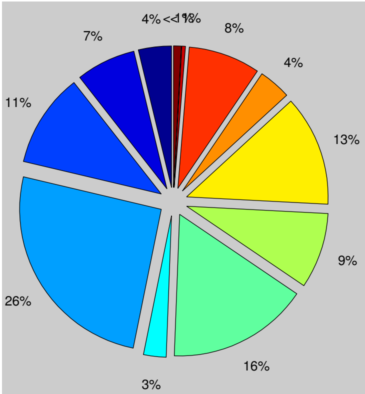
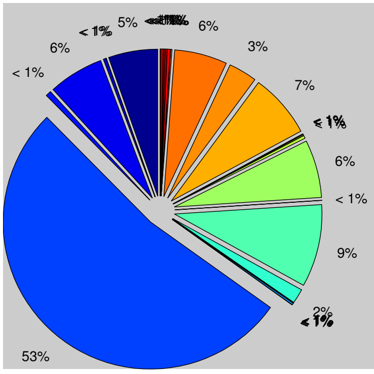
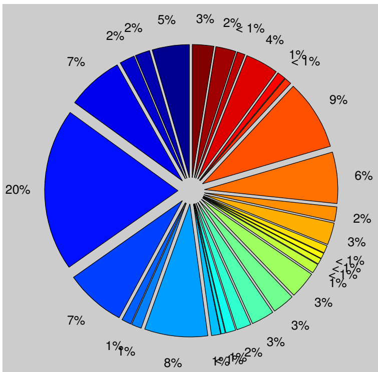
	Weights	Mean Return	Standard deviation	Shannon Entropy
MV		0.0005	0.0095	2.1474
CE		0.0005	0.0105	1.7510
SE		0.0005	0.0115	2.9524

Table 4.2 provides the comparison between MV, CE and SE portfolios in case when the second sample is used and with the required daily mean return at the level of 0.0008. Here MV portfolio shows just the exact property, for which it is criticized; only 7 out of 30 stocks are in MV portfolio and three of them have weights heavier or equal to 20 per cent. CE portfolio brings slight improvement, in terms of Shannon entropy, to the allocation of the stocks in the portfolio. But it still has the problem of

extreme positions, 38 per cent of the portfolio correspond to a single stock; 4 stocks make up 85 per cent of CE portfolio. SE portfolio, in contrast to CE portfolio, has 5 stocks with the heaviest weights, which make up 43 per cent of SE portfolio. Both, pie chart and Shannon entropy, suggest that SE portfolio is way more diversified comparatively to two other portfolio; but its standard deviation comes at a price.

Table 4.2.: Comparison of the portfolios; design-free estimate of variance-covariance matrix; 19.03.2008-02.12.2014.

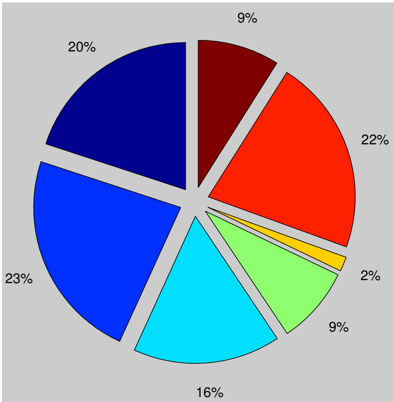
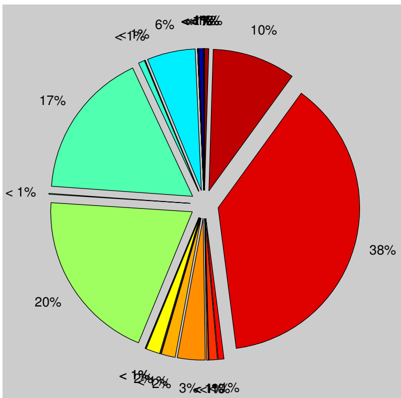
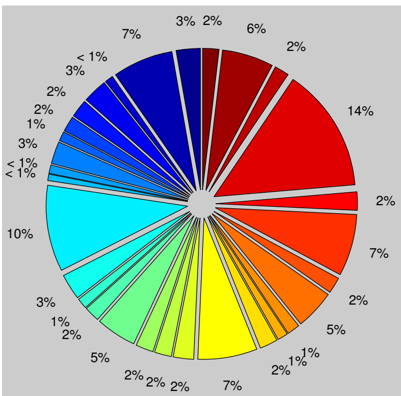
	Weights	Mean Return	Standard deviation	Shannon Entropy
MV		0.0008	0.0123	1.7884
CE		0.0008	0.0129	1.8255
SE		0.0008	0.0153	3.0925

Table 4.1 and 4.2 give an easy interpretation of the structure of portfolios, in case when different portfolio creation methods are used. But investors are essentially interested is portfolio's performance, to which we proceed in the next section.

4.3. Performance of the portfolios.

In order to compare different methods for the portfolio optimization we use "rolling window" technique. Windows lengths are $W = 300,\ 500,\ 1000,\ 1500$. Estimates of the mean values and variance-covariance matrix may change over time, or during different periods. Consequently, the solution for the portfolios, obtained by different methods, may change also. "Rolling window" technique allows to incorporate this issue and provides better estimates for the portfolios' performance.

The procedure is divided into several steps Kolusheva (2008):

1. Starting at time $t = M$ one has to estimate the parameters, such as mean returns and variance-covariance matrix, over the estimation window M. For example, when $M = 300$, mean returns and variance-covariance matrix of the assets are estimated over the first 300 days.
2. Next, one has to solve the constrained optimization problem for each of the methods for portfolio creation (MV, SE, EQ, MinV, CE). The result is the optimal weights, mean return and risk for each of the portfolios.
3. In case, when out-of-sample performance has to be obtained, one more measure has to be calculated. Having optimal weights for each of the portfolio methods, the return on the corresponding portfolio in the period $t + 1$ is calculated. For example, when $M = 300$, on the first iteration, the return for the period $t = 301$ is calculated on the portfolios, which were obtained based on the estimated inputs of first 300 days.

As the name appellation "rolling window" suggests, second iteration involves adding the return for each asset in the data set and dropping the earliest return. This keeps the estimation window fixed. Procedure is repeated until we reach the end of the data set, namely, until $t = T$ in case of in-sample performance measure and until $t = T - 1$ in case of out-of-sample portfolio measure.

4.3.1. Sharpe Ratio (SR).

The first measure of the portfolio performance is Sharpe Ratio (SR). In-sample SR represents the historic average risk-adjusted return, namely, the historic average return of the portfolio per each unit of risk, related to this portfolio. It is calculated in the following way (Bera and Park 2008):

$$SR_{in} = \frac{1}{T-W}\sum_{t=W}^{T}\frac{\hat{\pi}_t'\hat{m}_t}{\sqrt{\hat{\pi}_t'\hat{\Sigma}_t\hat{\pi}_t}},$$

where $\hat{m}_t$, $\hat{\Sigma}_t$ are, respectively, estimated mean return and variance-covariance matrix for window $[t-W+1,\ t]$ and $\hat{\pi}_t$ denotes optimal portfolio weights for this window.

To calculate the out-of-sample SR we need to obtain the return on the portfolio for the period, following the last observation in the window W, keeping the estimated weights, obtained form the constrained optimization problem over window W. Therefore, the portfolio return at time $t+1$ is $\hat{\mu}_{t+1} = \hat{\pi}_t' R_{t+1}$, where R_{t+1} are the returns at time $t+1$.

The mathematical representation of the out-of-sample SR is:

$$\tilde{m} = \frac{1}{(T-W)}\sum_{t=W}^{T}\hat{\mu}_t, \tag{4.2}$$

$$\tilde{\sigma}^2 = \frac{1}{(T-W-1)}\sum_{t=W}^{T}(\hat{\mu}_t-\tilde{m})^2, \tag{4.3}$$

$$SR_{out} = \frac{\tilde{m}}{\tilde{\sigma}}.$$

$\tilde{m}$, $\tilde{\sigma}^2$ are, respectively, sample mean of the out-of-sample returns on portfolio and corresponding variance of this returns.

4.3.2. Certainty equivalent return (CEQ).

Second performance measure of the portfolio is Certainty equivalent return, which represents the average minimum risk-free return for which investor are willing to abandon the risky portfolio and invest in asset with this risk-free return.

the in-sample and the out-of-sample averages of CEQ can be defined in the following way:

$$CEQ_{in} = \frac{1}{(T-W)} \sum_{t=W}^{T} \left(\hat{\pi}_t' \hat{m}_2 - \frac{\lambda}{2} \hat{\pi}_t' \hat{\Sigma}_t \hat{\pi}_t \right),$$

$$CEQ_{out} = \tilde{m} - \frac{\lambda}{2} (\tilde{\sigma})^2,$$

where $\tilde{m}$ and $\tilde{\sigma}$ are defined in equations 4.2 and 4.3 respectively.

Both, CEQ_{in} and CEQ_{out} require the value of the risk aversion of the investor. In this thesis we consider only $\lambda = 0,1$. This is caused by two reasons, first and the main is that Bera and Park (2008) concluded that the results for CEQ measures are quite similar for different values of λ, ranging from $0,07$ to 1 and the second reason is, that the evaluation itself is very computationally intensive and takes a lot of time.

4.3.3. Discussion of empirical results.

In order to investigate the performance of the MV portfolio and portfolios, which are created using maximum entropy principle, target value either for expected mean return or risk has to be chosen, so the comparison will be conducted on the same “level”; this concerns MV, SE, and CE portfolios. All results are provided for target values of daily mean returns equal to 0.0005 for the first sample: 01.01.2000-18.03.2008; 0.0008 for the second sample: 19.03.2008-02.12.2014 and 0.0005 for the period of economic recession.

We firts summarize the result of portfolios’ performance for the first sample, which are presented in Table 4.3. In Appendix B, Table B.1 one can find the results for the same period in case, when “classical” estimation of variance-covariance matrix is used. Both table have performance measures for the portfolios, which are rebalanced daily. In other words, every next day optimal weights for all presented portfolios are calculated and are hold for one day before the rebalancing takes place.

When W=300, in-sample CEQs of CE and MV have the highest value of 0.000496 and SE has the second highest value. Regarding in-sample SR, MV has the highest value of 0.0712; the lowest value is of EQ: 0.0440. Out-of-sample performance is the best for MV in terms of SR and CEQ. SE’s out-of-sample SR is 0.0235 and it is the lowest among all presented portfolios. We found that for all portfolios, except MinV, performance measures are higher or equal when “classical” estimate of variance-covariance matrix is used comparatively to values in Table 4.3 for W=300.

Next, moving to W=500; MV has the highest SR for both cases, in- and out-of-sample. In-sample SR of SE is 0.0555 and is the third highest value, after those of MV and of CE. However, once again, out-of-sample performance of SE is the worst in both terms, SR and CEQ, comparatively to other portfolios. MV has the highest out-of-sample performance with corresponding values of 0.0384 and 0.000315 for SR and CEQ, respectively.

Considering the case W=1000, MV has the best in- and out-of-sample performance in terms of SR and CEQ. SE, nevertheless outperforms EQ in terms of both, out-of-sample SR and out-of-sample CEQ. Therefore, EQ performs the worst comperatively to other portfolios.

Taking the largest windows size W =1500. As it was for previous cases, MV has the best performance in terms of in-sample SR and in-sample CEQ; although in-sample CEQ of CE has the same value as MV’s, which is 0.000497. For in-sample case, EQ has the worst performance in both terms, SR and CEQ. For out-of-sample case, it is ones again SE, which has the worst performance. The best performing portfolio in terms of out-of-sample SR is now MinV with the corresponding value equal to 0.0754.

Table 4.3.: In- and out-of-sample performance of the portfolios; design-free estimate of variance-covariance matrix; daily rebalancing; 01.01.2000-18.03.2008.

	W=300				W=500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0440	0.000313	0.0285	0.000313	0.0416	0.000314	0.0302	0.000320
MinV	0.0517	0.000320	0.0334	0.000281	0.0483	0.000327	0.0341	0.000288
MV	0.0712	0.000496	0.0363	0.000297	0.0686	0.000496	0.0384	0.000315
SE	0.0572	0.000494	0.0235	0.000233	0.0555	0.000494	0.0286	0.000276
CE	0.0691	0.000496	0.0348	0.000290	0.0666	0.000496	0.0345	0.000283
	W=1000				W=1500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0399	0.000358	0.0412	0.000329	0.0288	0.000312	0.0427	0.000384
MinV	0.0489	0.000371	0.0632	0.000427	0.0429	0.000365	0.0754	0.000542
MV	0.0643	0.000497	0.0693	0.000469	0.0590	0.000497	0.0718	0.000519
SE	0.0506	0.000494	0.0491	0.000394	0.0446	0.000494	0.0413	0.000377
CE	0.0625	0.000496	0.0641	0.000444	0.0572	0.000497	0.0651	0.000487

Daily rebalancing may be a reasonable decision for investors. For instance, daily rebalancing allows one to asses and respond quickly, in order to minimize the losses from stocks, which experience decrease in price day after day. However, some investors, i.e. corporate portfolio managers, may be more conservative regarding how often they change their positions. For this in Table 4.4 we describe the results for portfolios' performances for the first sample when holding period of the portfolios is one week. Furthermore, we still use daily data for our optimization problems, therefore in Table 4.4 only out-of-sample performance measures are presented, as in-sample measures are the same as in Table4.3.

For W=300, in Table 4.4, the highest SR and CEQ are of MV with the corresponding values of 0.0881 and 0.0014, respectively. In contrast to the case when daily rebalancing is used, SR of SE using weekly rebalancing has higher value than SR of EQ, however CEQ values of SE and EQ are equal: 0.0012. The worst performance, in terms of CEQ, is of MinV portfolio and is equal to 0.0011.

In case when W=500 the highest SR is of MV and it is equal to 0.0893, however CEQ of MV is the same as corresponding value of CE. EQ has the worst performance among presented portfolios, its CEQ is 0.0011, which is the same as CEQ of SE.

SR of MV in case when W=1000 is 0.1507 and it is the highest values for performance comparatively to other portfolios. The worst performance is of EQ with the values of 0.0788 and 0.0011 for SR and CEQ, respectively. We can see the improvement in CEQ of SE as it is for the first time equal to CEQ of MinV, which is 0.0016 .

Moving next to the case W=1500. Here the highest SR is of MinV: 0.1556. This macomparitevilyy be caused by the fact, that each rolling window, for which optimal portfolios are calculated, now contain the data for the time, when global financial crises started to take place. So the portfolio, which minimizes the risk and is essentially concerned only about the risk, may be more profitable; especially when the holding period is one week and not one day. EQ has the lowest profitability comparatively to other portfolios, its SR is 0.0793 and CEQ is 0.0013.

Table 4.4.: Out-of-sample performance of the portfolios; design-free estimate of variance-covariance matrix; weekly rebalancing; 01.01.2000-18.03.2008.

	W=300		W=500		W=1000		W=1500	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0551	0.0012	0.0564	0.0011	0.0788	0.0011	0.0793	0.0013
MinV	0.0680	0.0011	0.0675	0.0011	0.1311	0.0016	0.1556	0.0020
MV	0.0881	0.0014	0.0893	0.0013	0.1507	0.0018	0.1551	0.0020
SE	0.0619	0.0012	0.0658	0.0012	0.1030	0.0016	0.0842	0.0014
CE	0.0833	0.0013	0.0834	0.0013	0.1398	0.0017	0.1435	0.0019

Moving to the next sample: 19.03.2008-02.12.2014. Recall, that in order to investigate the performance of the MV portfolio and portfolios, which are created using maximum entropy principle, target value either for expected mean return or risk has to be chosen, so the comparison will be conducted on the same “level”; this concerns MV, SE, and CE portfolios. For the second sample we choose target daily mean return of the portfolios at the level of 0.0008.

Starting with Table 4.5 we describe the results for in- and out-of-sample performance of the portfolios when rebalancing takes place every day. In Appendix B Table B.3 in- and out-of-sample performance of the portfolios is the presented, when “classical” estimate of variance-covariance matrix is used.

In case when W=300, the the best in-sample performance is shown by MV, its in-sample SR and CEQ are 0.1076 and 0.000793, respectively. the worst performing, for in-sample case, is EQ with corresponding values of 0.0772 and 0.000717 for in-sample SR and CEQ. However, out-of-sample performance of MV is the worst among all portfolios. Even EQ 's out-of-sample CEQ is 0.000787, what is 0.000153 higher than the corresponding value of MV. This shows how the changes in estimated daily mean returns and variance-covariance matrix, resulted by global crisis and the following global recession, result in having extreme MV weights and therefore it poor out-of-sample performance even in case of daily rebalancing. The highest out-of-sample SR is of MinV: 0.0915; whereas out-of-sample CEQ value is much higher for SE, with corresponding value of 0.000832 that is the highest out-of-sample CEQ.

Moving next to W=500. Again, the best in-sample performance is of MV. MinV and EQ are the least performing portfolio, in terms of in-sample SR and CEQ measures. Out-of-sample SR of MinV is the highest and is 0.0788, however, in terms of out-of-sample CEQ, EQ perfroms the best, with corresponding value of 0.000666.

Considering the case W=1000, MV has the best in- and out-of-sample performance in terms of SR and CEQ. This may be caused by the fact, that not all windows, starting from the very first, contain the period of recovery after global financial crisis. SE outperforms EQ in terms of both, out-of-sample SR and out-of-sample CEQ. EQ performs the worst in terms of in-sample SR, however the worst in-sample performance in terms of CEQ is one of the MinV that is 0.000565.

Taking the largest windows size W =1500. As it was in all previous cases, in-sample performance of MV is the best, with corresponding values of 0.0676 and 0.000796 for SR and CEQ, respectively. SE's out-of-sample performance is the worsts if terms of SR and CEQ. Out-of-sample SR and CEQ of MinV are the highest among all considered portfolios.

Table 4.5.: In- and out-of-sample performance of the portfolios; design-free estimate of variance-covariance matrix; daily rebalancing; 19.03.2008-02.12.2014.

	W=300				W=500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0772	0.000717	0.0845	0.000787	0.0731	0.000714	0.0723	0.000666
MinV	0.0779	0.000539	0.0915	0.000631	0.0774	0.000562	0.0788	0.000549
MV	0.1076	0.000793	0.0822	0.000634	0.1036	0.000795	0.0734	0.000554
SE	0.0829	0.000789	0.0868	0.000832	0.0789	0.000791	0.0626	0.000597
CE	0.1022	0.000792	0.0892	0.000711	0.0980	0.000794	0.0724	0.000565
	W=1000				W=1500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0653	0.000705	0.1068	0.000728	0.0491	0.000652	0.1135	0.000676
MinV	0.0720	0.000565	0.1041	0.000597	0.0514	0.000492	0.1285	0.000680
MV	0.0923	0.000796	0.0605	0.000400	0.0676	0.000796	0.0975	0.000623
SE	0.0708	0.000793	0.1018	0.000722	0.0544	0.000792	0.0798	0.000532
CE	0.0861	0.000795	0.0775	0.000504	0.0632	0.000795	0.0888	0.000570

Recall that some investors, i.e. corporate portfolio managers, may be more conservative regarding how often they change their positions. For this in Table 4.6 we describe the results for portfolios' performances for the second sample when holding period of the portfolio is one week. Daily data is still used for the optimization problems, therefore in Table 4.6 only out-of-sample performance measures are presented, as in-sample measures are the same as in Table4.5.

W=300. CE has the best performance in both terms, SR and CEQ, with the corresponding values of 0.1936 and 0.0029. MV has the second highest value of out-of-sample SR, however in terms of CEQ, SE and EQ perform better. Out-of-sample SR and CEQ of SE are only higher in comparison to those of EQ.

Considering W=500, now MV's out-of-sample SR has the highest value of 0.1685. All three portfolios, MV, SE and CE have the same value of out-of-sample CEQ equal to 0.0024. EQ, although has the second smallest value of out-of-sample CEQ, outperforms all other portfolios in terms of out-of-sample CEQ, which is equall to 0.0036.

Moving next to W=1000. For the first time the best out-of-sample performance is shown by EQ with the corresponding value of 0.2113. The second best out-of-sample performance is one of MinV. MV performs the worst among all portfolios, which once again proves how changes in the estimated input parameters makes MV an "error maximizer" (Michaud 1989) and results in poor out-of-sample performance. SE has out-of-sample CEQ equal to one of EQ and it is the highest value of 0.0029.

Lastly for W=1500, the highest out-of-sample SR is of MinV and is equal to 0.2728; it's out-of-sample CEQ of 0.0029 is the same as a corresponding value of MV. SE performs the worst in terms of out-of-sample SR and out-of-sample CEQ. EQ performs better than any portfolio, obtained by using maximum entropy principle, namely SE and CE.

Table 4.6.: Out-of-sample performance of the portfolios; design-free estimate of variance-covariance matrix; weekly rebalancing; 19.03.2008-02.12.2014.

	W=300		W=500		W=1000		W=1500	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.1708	0.0031	0.1494	0.0036	0.2113	0.0029	0.2274	0.0027
MinV	0.1895	0.0025	0.1659	0.0022	0.2091	0.0024	0.2728	0.0028
MV	0.1914	0.0028	0.1685	0.0024	0.1472	0.0019	0.2280	0.0028
SE	0.1800	0.0033	0.1333	0.0024	0.2077	0.0029	0.1673	0.0022
CE	0.1936	0.0029	0.1642	0.0024	0.1779	0.0023	0.2028	0.0026

Summarizing the discussion above: (i) In each sample, performance of MV is the best in terms of both SR and CEQ among all considered portfolios for in-sample case. Moreover, it has the best out-of-sample performance, for the first sample, except from the case of the largest W=1500, when each and every rolling window captures the period when Global financial crisis started to take place. Therefore, we can conclude that MV in case of both, daily and weekly rebalancing, has the best performance under moderate financial conditions with no extreme uncertainty (volatility) that precedes 2008 financial crisis (Schwert 2011). In second period, MV no more has its leading position, in terms of out-of-sample case, yielding not only to MinV and CE but even to EQ and SE, for some cases of W. (ii) Most of the time CE's values of performance are between those of MinV and MV, which is expected as CE shrinks weights from MV to those of MinV (Bera and Park 2008). Nevertheless, for the second sample, CE performs the best when we consider weekly rebalancing, in case when W=300. (iii) SE showed poor in-sample performance in the first sample, having most of the time corresponding values of SR and CEQ even lower than those of naive EQ in case of daily rebalancing. The out-of-sample performance of SE in the first period is only better than of EQ. The same works for the second period: SE performs poorly comparatively to other portfolio, having the performance worse than

EQ portfolio in many cases.

4.3.3.1. Testing the equality of the Sharpe ratios.

In subsection above we compared the performance of the portfolios in terms of in- and out-of-sample SR and CEQ. However, all comparisons were made based on the nominal terms.

Because the input parameters, namely mean returns and variance-covariance matrix, are estimated, one is always exposed to some degree of estimation error, which in term affects performance measures. To incorporate this issue we conduct the test, in order to compare whether out-of-sample SR of different portfolios are statistically different.

Memmel (2003) proposed the test, described in methodology, which assumes the normal distribution of the out-of-sample portfolios' returns; however, as it is shown in Appendix C, Jarque-Bera test suggests that all considered out-of-sample returns, except only two cases, are not normally distributed. Recall that Jarque-Bera's Null hypothesis that the data are from normal distribution.

Therefore, we used method, proposed by Ledoif and Wolf (2008a), which is also described in methodology chapter. Using the program, provided by the authors, we test the null hypothesis, that the difference between two Sharpe Ratios is equal to zero using studentized time series bootstrap confidence intervals. Confidence intervals are set to 95%. In order to provide robust results we use 10000 as a number of bootstrap repetitions.

Table 4.7 and Table 4.8 provide the results for two periods in case when daily rebalancing is used. Corresponding results for weekly rebalancing are in Appendix D, as they are very similar to those, given below.

As we can see from the Table 4.7, out-of-sample SR of MV is never statistically different from those of SE and CE. Moreover, all three portfolio, MV,SE and CE have out-of-sample SR's that are not statistically different from those of EQ. The only two exceptions are the case when we compare out-of-sample SRs of MV and CE to the corresponding value of EQ for W=1000. In first case, this SRs are 0.0643 and 0.399 for MV and EQ, respectively; second case is when SRs are equal to 0.0625 and 0.0399 for CE and EQ, respectively.

Table 4.7.: Tests for the difference in the Sharpe ratios of the portfolios; design-free estimate of variance-covariance matrix; daily holdings; 01.01.2000-18.03.2008.

Description	W	test stat.	p-value	Decision H
$SR_{MV} = SR_{EQ}$	300	0.8529	0.404	can not reject H0
	500	0.5998	0.558	can not reject H0
	1000	2.121	0.0392	reject H0
	1500	1.49	0.162	can not reject H0
$SR_{MV} = SR_{SE}$	300	1.007	0.324	can not reject H0
	500	0.6187	0.533	can not reject H0
	1000	1.568	0.127	can not reject H0
	1500	1.381	0.182	can not reject H0
$SR_{MV} = SR_{CE}$	300	0.4945	0.627	can not reject H0
	500	0.7333	0.469	can not reject H0
	1000	1.056	0.305	can not reject H0
	1500	0.5841	0.572	can not reject H0
$SR_{SE} = SR_{EQ}$	300	0.3815	0.713	can not reject H0
	500	0.5111	0.615	can not reject H0
	1000	1.186	0.239	can not reject H0
	1500	0.2426	0.81	can not reject H0
$SR_{CE} = SR_{EQ}$	300	0.6727	0.508	can not reject H0
	500	0.536	0.594	can not reject H0
	1000	2.189	0.0349	reject H0
	1500	1.321	0.208	can not reject H0

The same situation can be observed for the second period. Tests results are presented in Table 4.8. For W=1000 out-of-sample SR of MV is statistically different from the corresponding values of EQ and CE, which is the only case, when out-of-sample SR of MV has the lowest value among all considered portfolios.

Table 4.8.: Tests for the difference in the Sharpe ratios of the portfolios; design-free estimate of variance-covariance matrix; daily rebalancing; 19.03.2008-02.12.2014.

Description	W	test stat.	p-value	Decision H
$SR_{MV} = SR_{EQ}$	300	0.1571	0.877	can not reject H0
	500	0.06822	0.944	can not reject H0
	1000	2.168	0.0348	reject H0
	1500	0.3932	0.717	can not reject H0
$SR_{MV} = SR_{SE}$	300	0.3416	0.737	can not reject H0
	500	0.7045	0.489	can not reject H0
	1000	2.005	0.0534	can not reject H0
	1500	0.5391	0.606	can not reject H0
$SR_{MV} = SR_{CE}$	300	1.226	0.227	can not reject H0
	500	0.1388	0.888	can not reject H0
	1000	1.977	0.0484	reject H0
	1500	0.5653	0.585	can not reject H0
$SR_{SE} = SR_{EQ}$	300	0.3849	0.709	can not reject H0
	500	2.206	0.0306	can not reject H0
	1000	0.6736	0.513	can not reject H0
	1500	1.6	0.18	can not reject H0
$SR_{CE} = SR_{EQ}$	300	0.4292	0.675	can not reject H0
	500	0.00773	0.994	can not reject H0
	1000	1.791	0.0842	can not reject H0
	1500	0.7793	0.472	can not reject H0

4.4. The US economic recession of 2007-2009.

This section serves as an addendum to the empirical analysis, described above. After the investigation of the portfolios performances for two aforementioned periods, few questions arose. First of all, we would like to investigate in more details the influence of economic recession of 2007-2009 on the portfolios performance. Although, it is clear that recession has negative impact on the performance of the portfolio as it causes lower or negative returns on the assets and moreover, it causes high volatility. However, the thing we are interested in is whether the performances of our portfolios, obtained using different methods, are on the same positions comparatively to each other as in empirical analysis, conducted above. Second of all, we want to investigate the influence of smaller rolling windows in the performance measures mechanisms. Again, we are interested in whether one portfolio would perform better than other if the estimation error is rather high, which would be the case for small rolling windows, used in estimation procedure. Lastly, in this section we compare out-of-sample performance of the portfolios, given that the holding period is one month. By this we move one step toward real life situations as frequent rebalancing may be undesirable and most of the time is costly.

In the empirical investigation below we incorporate aforementioned issues all at once. This is done in the following way:

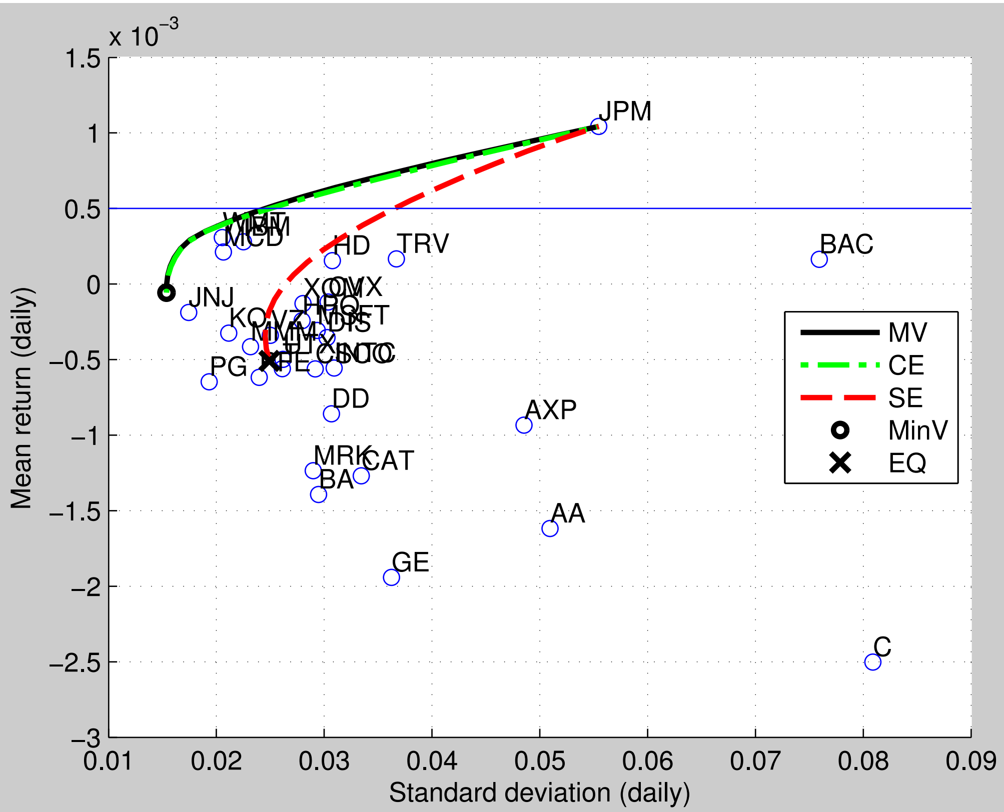
- We create separate sample of data, used in empirical analysis, for the period of U.S. economic recessions of 2007-2009. This sample spans the period from 01 December 2007 until 30 June 2009. NBER (2008) determined that the peak of economic activity was in December 2007, after 73 months of expansion and the trough in business activity occurred in June 2009.
- The length of the windows in "rolling window" technique for investigation the performance of the portfolios are now set to 60, 90, 120, and 200 days.
- Out-of-sample performance of the portfolios is now also assessed for the case, when holding period is one month.

Before starting the discussion, for convenience, till the end of this section we will refer to the first sample (01.01.2000-18.03.2008) and all the empirical analysis, conducted using the data from it, as to **sample(i)** and to the second period (19.03.2008-02.12.2014), also including all empirical analysis using data from it, as to the **sample(ii)**.

On Figure 4.3 are depicted all the efficient frontiers and portfolio using the data from period of economic recession, which were already investigated for sample(i) and sample(ii). Namely, Mean-Variance efficient frontier (MV), efficient frontier for Maximum Shannon entropy portfolios (SE), efficient frontier for Maximum cross-entropy portfolios (CE), Minimun- variance portfolio (MinV) and Equally-weighted portfolio (EQ).

The highest possible return for all efficient frontiers is daily mean return on the shares of “JPMorgan Chase & Co.”. Daily mean return for this company is 0.00106, with corresponding standard deviation of its daily returns: 0.051. Majority of the pairs (mean daily return, daily standard deviation) of the assets, from which the portfolios are created, are located close to the efficient frontiers. Moreover, in contrast to sample(i) and sample(ii), some of the assets are even located on the left side from the SE efficient frontier, suggesting that investing in one of those assets would have lower risk and accompanied with higher daily mean return than on the portfolio on SE efficient frontier. For instance, such stocks as “McDonald’s”, “IBM Co.” and “Wal-Mart Stores Inc.” are very close to MV efficient frontier and produce better results than the portfolio on SE efficient frontier, in terms of daily mean return and daily standard deviation.

Figure 4.3.: Efficient frontiers and portfolios; design-free estimate of variance-covariance matrix; 01.12.2007-30.06.2009.

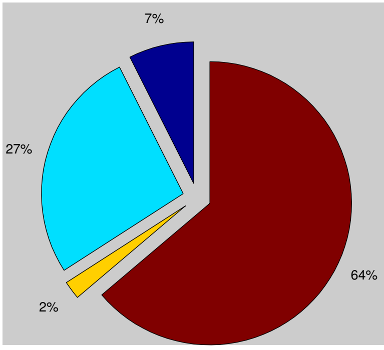
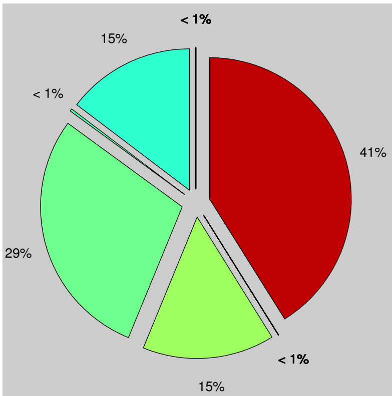
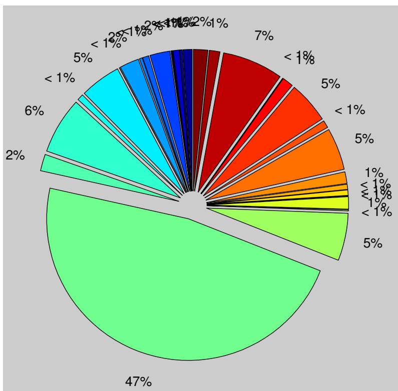


Unlike in the case of sample(i) and sample(ii), on Table 4.3 we can see that CE efficient frontier follows the same path as MV efficient frontier. However, this does not mean that the allocation of the assets in two portfolios, taken from each efficient frontier on the same level of daily mean return, has to be the same; it will be seen further in the Table 4.9. Blue horizontal line on Figure 4.3 on the level of 0.0005 of daily mean return represents the required mean return for three portfolios: MV, SE and CE, for which corresponding performance measures are computed, namely Sharpe Ratios and Certainty Equivalent Return.

In Table 4.1 we compare three portfolios, MV, CE, SE, which are obtained using data for the period of the recession for a given required daily mean return: 0.0005. From pie chart we can see that for case of MV only 4 components out of 30 are in the portfolio. 64 per cent of investor's wealth would be invested only in one asset. CE as well has 4 major components in its portfolio, however the weights are allocated more evenly with the heaviest weight being 41 per cent. Pie chart for SE suggests that 47 per cent of investor's wealth should be invested in a single asset, but remaining 53 per cent are distributed among many other assets.

Based on the values of Shannon Entropy, during economic recession CE shows more diversified portfolio that MV as Shannon Entropy increases by 0.03598. To compare, in sample(i) CE has Shannon entropy even lover than the corresponding value for MV and in sample(ii) CE shows marginal improvement of 0.0371, in terms of Shannon entropy, comparatively to MV . Both, pie chart and Shannon entropy, suggest that SE is way more diversified comparatively to two other portfolio; but once again its standard deviation comes at a price as in previous two samples.

Table 4.9.: Comparison of the portfolios; design-free estimate of variance-covariance matrix; 01.12.2007-30.06.2009.

	Weights	Mean Return	Standard deviation	Shannon Entropy
MV		0.0005	0.025	0.9498
CE		0.0005	0.0254	1.3096
SE		0.0005	0.0373	2.165

Moving now to the investigation of the performances of the portfolios. In Table 4.10 are presented in- and out-of-sample measures of SR and CEQ for different portfolio. After the description of the results, a quick conclusion is provided after each table.

Starting with W=60, in-sample CEQs of MV and CE have the highest values of 0.0394 and 0.0385, respectively; and SE has the third highest value. Regarding in-sample SR, MV has the highest value of 0.00048; the lowest value is of EQ: -0.000593. Out-of-sample performance is the best for EQ in terms of both SR and CEQ. CE's out-of-sample SR is -0.0406 and it suggests better out-of-sample performance that MV, which has the corresponding value equal to -0.0527. SE's out-of-sample SR is -0.0552 and is the lowest among all presented portfolios.

Next, moving to W=90; MV has the highest in-sample SR and CEQ. In-sample SR of SE is 0.025 and is the third highest value, after those of MV and of CE. However, once again, out-of-sample performance of SE is the worst in both terms, SR and CEQ, comparatively to other portfolios. SE has the highest out-of-sample performance with corresponding values of -0.016 and -0.000494 for SR and CEQ, respectively. Out-of-sample performance of MV is only on this place, in terms of both SR and CEQ, after EQ and CE.

Considering the case W=120, MV has the best in- and out-of-sample performance in terms of SR and CEQ. out-of-sample SR of EQ is the highest among other portfolios, however out-of-sample CEQ is higher for MinV. SE has the worst results in terms of all in- and out-of-sample performance measures. MV now outperforms CE based of the out-of-sample SR.

Taking the largest windows size W =200. As it was for the previous cases, MV has the best performance in terms of SR and CEQ for in-sample case. For in-sample case, EQ has the worst performance in both terms, SR and CEQ, however it has the best out-of-sample performance once again. For out-of-sample case, it is ones again SE, which has the worst performance.

To conclude, MV always provides the best in-sample performance. However it fails to provide good results in case of out-of-sample performance, performing worse than naive EQ and in most of the cases it performs worse than CE. SE has better in-sample performance only comparatively to EQ and MinV; out-of-sample performance of SE is always the worst.

Table 4.10.: In- and out-of-sample performance of the portfolios; design-free estimate of variance-covariance matrix; daily rebalancing; 01.12.2007-30.06.2009.

	W=60			
	In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ
EQ	-0.0199	-0.000593	-0.0159	-0.000474
MinV	-0.0279	-0.000519	-0.0269	-0.000493
MV	0.0394	0.00048	-0.0527	-0.000991
SE	0.0274	0.000465	-0.0552	-0.0014
CE	0.0385	0.000479	-0.0406	-0.000794
	W=90			
	In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ
EQ	-0.0289	-0.000883	-0.0160	-0.000494
MinV	-0.0271	-0.00052	-0.0348	-0.000653
MV	0.0375	0.000474	-0.0224	-0.000483
SE	0.025	0.000455	-0.0465	-0.0013
CE	0.0369	0.000473	-0.0203	-0.00044
	W=120			
	In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ
EQ	-0.0367	-0.0011	-0.0192	-0.000614
MinV	-0.027	-0.000531	-0.0303	-0.000601
MV	0.0313	0.000388	-0.0674	-0.0016
SE	0.0205	0.000373	-0.0729	-0.002
CE	0.0307	0.000386	-0.0676	-0.0016
	W=200			
	In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ
EQ	-0.0442	-0.0013	-0.0065	-0.000274
MinV	-0.025	-0.000459	-0.0287	-0.000647
MV	0.0199	-0.000382	-0.0493	-0.0012
SE	0.0146	0.000362	-0.0563	-0.0016
CE	0.0193	0.000378	-0.0172	-0.00048

Moving next to the case, when holding period is equal to one week. As we still use daily data for our optimization problems, in Table 4.11 only out-of-sample performance measures are presented, as in-sample measures are the same as in previous table.

For the case when W=60, as it can be seen from the Table 4.11, the best performing portfolio in terms of SR is EQ, as it was the case for daily rebalancing. However in terms of CEQ the best performing portfolio is CE with the corresponding value of -0.0015; in fact EQ has the fourth lowest value of CEQ. CE has the second best result in terms of SR.

In case when W=90 the highest SR is again of EQ and it is equal to -0.0477; EQ has also the best performance in terms of CEQ. MV has the worst performance among presented portfolios, its SR is -0.1054. SE has the third best performance results in terms of both SR and CEQ.

Considering the case W=120, MV has the worst performance in terms of SR. EQ outperforms all other portfolios in terms of both SR and CEQ, its corresponding values are -0.0575 and -0.0028. MinV showed an improvement comparatively to the cases with lower windows lengths as it has the second best performance in terms of SR and is the best performing portfolio in terms of CEQ with the corresponding values equal to -0.0028.

For the last case W=200, the portfolios, constructed using maximum entropy principle, are for the firts time the best performing portfolios. CE has the best performance, with values equal to -0.0171and -0.000728 for SR and CEQ respectively; followed, in terms of performance, by SE. The worst performance is of MinV.

Therefore, in most of the cases EQ provides the best out-of-sample performance, when rebalancing is conducted on weekly basis. MV shows a significant decline in out-of-sample performance, being sometimes the least performing portfolio. Regarding CE and SE, they showed improvements in their performance in a case of weekly rebalancing.

Table 4.11.: Out-of-sample performance of the portfolios; design-free estimate of variance-covariance matrix; weekly rebalancing; 01.12.2007-30.06.2009.

	W=60		W=90	
	SR	CEQ	SR	CEQ
EQ	-0.0413	-0.0021	-0.0477	-0.0025
MinV	-0.0723	-0.002	-0.1038	-0.003
MV	-0.0491	-0.0016	-0.1054	-0.0031
SE	-0.0687	-0.0030	-0.0935	-0.0042
CE	-0.0448	-0.0015	-0.0852	-0.0026
	W=120		W=200	
	SR	CEQ	SR	CEQ
EQ	-0.0575	-0.0031	-0.0418	-0.0026
MinV	-0.0958	-0.0028	-0.0968	-0.0033
MV	-0.1154	-0.0041	-0.0777	-0.0029
SE	-0.1084	-0.0047	-0.0257	-0.0013
CE	-0.1138	-0.0041	-0.0171	-0.000728

For the sample(i) and sample(ii) we conducted an empirical investigation of the performance of the portfolios for cases of daily and weekly holdings. However, for the case when we use the data only for the period of the economic recession we add also the case of monthly holdings. This would give us more robust insight about the performance of the portfolios and moreover, this case is closer to he real-world situations, when rebalancing of the portfolio are not frequent.

In Table 4.12 we present our empirical findings. Starting with the case W=60, we can see that EQ and MinV are the best performing portfolios as SR of SE has the highest value of -0.1329 and the second highest values of CEQ,conceding only to MinV. CE performs better that MV in terms of SR, however it has lower CEQ comparatively to MV. SE is the least performing portfolio.

Moving next to W=90, as in previouse case, EQ has the best performance in all terms. Also, MV performance is now better than of CE in both SR and CEQ terms. SR of SE is the lowest among others and is eqaul to -0.3218, with the closest portfolio, in terms of SR, being MinV with corresponding value -0.2914.

Considering W=120, CE has better performance that MV and SE, however it fails to outperform EQ and MinV. CEQ of MinV has the highest value of -0.0149. The best result, in terms of SR, is once again shown by EQ. SE has the worst performance with values of -0.3681 and -0.0318 for SR and CEQ respectively.

Lastly for W=200, SR of CE has the highest values, equal to 0.028. The closest portfolio to CE, in terms of SR, is EQ with corresponding value of -0.0655. MV is only third best performing portfolio in terms of both SR and CEQ. SE fails to outperform any other portfolios.

To conclude the results for the monthly rebalancing: EQ is almost always the best performing portfolio, followed by MinV. Only in case W=200, portfolio, constructed using maximum entropy principle, has the best performance. However, another portfolio, created using the same method, that is SE, has always the worst performing results.

Table 4.12.: Out-of-sample performance of the portfolios; design-free estimate of variance-covariance matrix; monthly rebalancing; 01.12.2007-30.06.2009.

	W=60		W=90	
	SR	CEQ	SR	CEQ
EQ	-0.1329	-0.0139	-0.1766	-0.0189
MinV	-0.2252	-0.0128	-0.2914	-0.0165
MV	-0.2248	-0.0144	-0.2873	-0.0185
SE	-0.2611	-0.0225	-0.3218	-0.0282
CE	-0.219	-0.0147	-0.319	-0.02
	W=120		W=200	
	SR	CEQ	SR	CEQ
EQ	-0.1637	-0.0184	-0.0655	-0.0084
MinV	-0.2493	-0.0149	-0.1981	-0.0117
MV	-0.2834	-0.0216	-0.1049	-0.0072
SE	-0.3681	-0.0318	-0.1146	-0.01
CE	-0.265	-0.02	0.028	-0.0018

In Appendix D we presented the results of the test for the statistical difference between SR of different portfolios, for case when the input data is from the period of economic recession. As it can be seen from the results, completely all of the tests fail to reject the Null hypothesis of statistical difference of SR ratios.

Chapter 5
Conclusion

Purpose of this thesis was to investigate the performance of portfolios, created using maximum entropy principle, proposed by Bera and Park (2008), to those created using "classical" Markovitz method. Key points, which differentiate this thesis from previous works are using disign-free estimate of variance-covariance matrix, introduces by Abadir et al. (2010); comparing the performance of the portfolios using daily, weekly and monthly holding periods; and conducting tests in order to assess whether the performances of different portfolios are statistically different.

Markovitz (1952) Mean-Variance (MV) efficient portfolio selection is a pioneer method in the problem of diversification of the wealth among risky assets. However, it was criticized for the fact, that it is highly concentrated only on a few assets due to the statistical error in estimates of means and variance-covariance matrix, which in turn results in poor out-of-sample performance (Michaud 1989). Maximum entropy principle, by its nature, tackles this problems; First, it has the shrinkage interpretation, in a sense that weights are directly shrunk towards predetermined values, second is that when equally weighted portfolio is used as a target, the optimization problems solves for the weights which are the closest to those of equally weighted portfolio and achieve required return (risk) of the portfolio Bera and Park (2008). Furthermore, we use design-free estimate of variance-covariance matrix, which leads to superior estimate of large variance-covariance matrices and for data with heavy-tailed densities (Abadir et al. 2010). In order to test the statistical difference of out-of-sample Sharpe ratios of different portfolios, we use studentized timeseries bootstrap interval method, proposed by Ledoif and Wolf (2008a), which produces robust results in case when out-of-sample returns on the portfolios are not normally distributed, which is a case for this thesis.

In order to perform empirical analysis we use daily data for companies, which are (were) components of Dow Jones Industrial Average. This data is divided into two samples: 01.01.2000-18.03.2008 and 19.03.2008-02.12.2014. Using Matlab programs, written based on the works of Bera and Park (2008) and Abadir et al. (2010), we create the portfolios using maximum entropy principle and compare this portfolios and their performance to MV portfolios based on the rolling window technique having daily and weekly holding periods. Furthermore, additional investigation is presented separately for the period of U.S. economic recession of 2007-2009.

Based on our results, MV portfolio almost always has better performance than those, created using maximum entropy principle, in case when we consider daily holdings, except in the period of the recession, when naive equally weighted portfolio has the best out-of-sample performance. Weekly rebalanicing reveals that increased holding period and periods of high market volatility drives down out-of-sample performance of MV porfolio and in some cases portoflios, created using maximum entropy principle, outperform MV.

Investigation of the performance of the portfolios during the period of the recession revealed, that unpleasant market conditions have a severe influence on MV out-of-sample performance. Portfolios, created using maximum entropy principle, in particular CE, in most of the cases provide better results, in terms of out-of-sample performance, than classical MV. However, almost in all cases, naive EQ was able to provide superior performance comparatively to all other portfolios.

To conclude , we found that although out-of-sample Sharpe ratios, as our main measure of performance, differ in nominal terms between investigated portfolios, they are not statistically different. Therefore, one can obtain well-diversified portfolio using maximum entropy principle and yet stay at the same level of out-of-sample performance as in case of 'classical' Markowitz efficient portfolio.

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Appendix A

Portfolios' components.

Table A.1.: Portfolios' components. Signs "+" and "-" represent the presence of the particular asset in the portfolios for two different periods, respectively.

#	Company	ticker	01.01.2000-18.03.2008	19.03.2008-02.12.2014	01.12.2007-30.06.2009
1.	3M Company	(MMM)	+	+	+
2.	Alcoa Inc.	(AA)	+	-	+
3.	American Express Company	(AXP)	+	+	+
4.	American International Group, Inc.	(AIG)	+	-	-
5.	AT&T Inc.	(T)	+	+	+
6.	The Boeing Company	(BA)	+	+	+
7.	Bank of America Corporation	(BAC)	+	-	+
8.	Caterpillar Inc.	(CAT)	+	+	+
9.	Chevron Corporation	(CVX)	+	+	+
10.	Cisco Systems, Inc.	(CSCO)	+	+	+
11.	Citigroup	(C)	+	-	+
12.	The Coca-Cola Company	(KO)	+	+	+
13.	E. I. du Pont de Nemours and Company	(DD)	+	+	+
14.	Exxon Mobil Corp.	(XOM)	+	+	+

#	Company	ticker	01.01.2000-18.03.2008	19.03.2008-02.12.2014	01.12.2007-30.06.2009
15.	General Electric Company	(GE)	+	+	+
16.	The Goldman Sachs Group, Inc.	(GS)	-	+	-
17.	Hewlett-Packard Company	(HPQ)	+	-	+
18.	The Home Depot, Inc.	(HD)	+	+	+
19.	Intel Corporation	(INTC)	+	+	+
20.	IBM Co.	(IBM)	+	+	+
21.	Johnson & Johnson	(JNJ)	+	+	+
22.	JPMorgan Chase & Co.	(JPM)	+	+	+
23.	McDonald's	(MCD)	+	+	+
24.	Merck & Co., Inc.	(MRK)	+	+	+
25.	Microsoft Corporation	(MSFT)	+	+	+
26.	Nike, Inc.	(NKE)	-	+	-
27.	Pfizer Inc.	(PFE)	+	+	+
28.	Procter&Gamble Co.	(PG)	+	+	+
29.	The Travelers Companies, Inc.	(TRV)	-	+	+
30.	UnitedHealth Group	(UNH)	-	+	-
31.	United Technologies Corporation	(UTX)	+	+	+
32.	Verizon Communications Inc.	(VZ)	+	+	+
33.	Visa Inc.	(V)	-	+	-
34.	Wal-Mart Stores, Inc.	(WMT)	+	+	+
35.	The Walt Disney Company	(DIS)	+	+	+

Appendix B

In- and out-of-sample performance of the portfolios; “classical” estimate of the variance-covariance matrix.

Table B.1.: In- and out-of-sample performance of the portfolios; “classical” estimate of the variance-covariance matrix; daily rebalancing; 01.01.2000-18.03.2008.

	W=300				W=500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0433	0.000313	0.0285	0.000313	0.0413	0.000314	0.0302	0.000320
MinV	0.0524	0.000328	0.0340	0.000284	0.0489	0.000334	0.0351	0.000296
MV	0.0716	0.000496	0.0386	0.000315	0.0690	0.000496	0.0393	0.000322
SE	0.0561	0.000494	0.0235	0.000233	0.0550	0.000494	0.0286	0.000276
CE	0.0692	0.000496	0.0365	0.000303	0.0668	0.000496	0.0357	0.000292
	W=1000				W=1500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0397	0.000358	0.0412	0.000329	0.0287	0.000312	0.0427	0.000384
MinV	0.0496	0.000379	0.0646	0.000436	0.0437	0.000374	0.0748	0.000538
MV	0.0645	0.000497	0.0705	0.000478	0.0590	0.000497	0.0702	0.000508
SE	0.0504	0.000494	0.0491	0.000394	0.0444	0.000494	0.0413	0.000377
CE	0.0626	0.000496	0.0648	0.000449	0.0572	0.000497	0.0652	0.000487

Table B.2.: Out-of-sample performance of the portfolios; “classical” estimate of the variance-covariance matrix; weekly rebalancing; 01.01.2000-18.03.2008

	W=300		W=500		W=1000		W=1500	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0551	0.0012	0.0564	0.0011	0.0788	0.0011	0.0793	0.0013
MinV	0.0691	0.0011	0.0695	0.0011	0.1344	0.0016	0.1543	0.0020
MV	0.0914	0.0014	0.0907	0.0014	0.1528	0.0018	0.1520	0.0020
SE	0.0619	0.0012	0.0658	0.0012	0.1030	0.0016	0.0842	0.0014
CE	0.0846	0.0013	0.0848	0.0013	0.1420	0.0017	0.1426	0.0019

Table B.3.: In- and out-of-sample performance of the portfolios; “classical” estimate of the variance-covariance matrix; daily rebalancing;19.03.2008-02.12.2014.

	W=300				W=500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0765	0.000717	0.0485	0.000787	0.0726	0.000714	0.0723	0.000666
MinV	0.0752	0.000520	0.0886	0.000609	0.0756	0.000548	0.0773	0.000539
MV	0.1074	0.000793	0.0801	0.000620	0.1034	0.000795	0.0733	0.000555
SE	0.0820	0.000789	0.0868	0.000832	0.0784	0.000791	0.0626	0.000597
CE	0.1012	0.000792	0.0868	0.000693	0.0973	0.000794	0.0717	0.000561
	W=1000				W=1500			
	In-Sample		Out-Sample		In-Sample		Out-Sample	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.0650	0.000705	0.1068	0.000728	0.0488	0.000652	0.1135	0.000676
MinV	0.0715	0.000562	0.1036	0.000594	0.0514	0.000491	0.1276	0.000680
MV	0.0920	0.000796	0.0568	0.000378	0.0675	0.000796	0.0951	0.000607
SE	0.0706	0.000792	0.1018	0.000722	0.0541	0.000792	0.0798	0.000532
CE	0.0856	0.000795	0.0769	0.000500	0.0629	0.000795	0.0877	0.000565

Table B.4.: Out-of-sample performance of the portfolios; “classical” estimate of the variance-covariance matrix; weekly rebalancing; 19.03.2008-02.12.2014

	W=300		W=500		W=1000		W=1500	
	SR	CEQ	SR	CEQ	SR	CEQ	SR	CEQ
EQ	0.1708	0.0031	0.1494	0.0036	0.2113	0.0029	0.2274	0.0027
MinV	0.1836	0.0024	0.1625	0.0021	0.2088	0.0024	0.2692	0.0028
MV	0.1895	0.0028	0.1697	0.0024	0.1401	0.0019	0.2240	0.0028
SE	0.1800	0.0033	0.1333	0.0024	0.2077	0.0029	0.1673	0.0022
CE	0.1889	0.0028	0.1623	0.0024	0.1767	0.0023	0.2012	0.0025

Appendix C

Tests for normal distribution.

Table C.1.: Test for Normal distribution of the portfolios' returns; daily holdings. 01.01.2000-18.03.2008

	W	skewness	kurtosis	Jarque-Bera test (95% conf. level)
EQ	300	0.2057	6.6428	reject H0
	500	0.3231	6.6923	reject H0
	1000	-0.1008	5.3831	reject H0
	1500	-0.1447	5.5548	reject H0
MV	300	-0.4615	8.3977	reject H0
	500	-0.3538	9.1878	reject H0
	1000	-0.2457	3.8501	reject H0
	1500	-0.4218	4.8954	reject H0
SE	300	-0.1907	5.5156	reject H0
	500	0.1265	5.4395	reject H0
	1000	-0.1670	4.5351	reject H0
	1500	-0.2641	4.8139	reject H0
CE	300	-0.4184	8.0515	reject H0
	500	-0.3068	8.0109	reject H0
	1000	-0.2167	3.9781	reject H0
	1500	-0.4096	4.9031	reject H0

Table C.2.: Test for Normal distribution of the portfolios' returns; weekly holdings. 01.01.2000-18.03.2008

	W	skewness	kurtosis	Jarque-Bera test (95% conf. level)
EQ	300	0.0262	8.0373	reject H0
	500	0.2621	8.8439	reject H0
	1000	-0.4675	3.6132	reject H0
	1500	-0.5732	3.4775	reject H0
MV	300	-0.3230	8.0320	reject H0
	500	-0.1605	10.2709	reject H0
	1000	-0.4188	3.6310	reject H0
	1500	-0.5023	4.1398	reject H0
SE	300	-0.2477	5.7189	reject H0
	500	0.2072	6.9551	reject H0
	1000	-0.4719	3.4447	reject H0
	1500	-0.5129	3.3365	reject H0
CE	300	-0.1802	7.7213	reject H0
	500	-0.0576	9.2770	reject H0
	1000	-0.4344	3.6156	reject H0
	1500	-0.5375	4.0077	reject H0

Table C.3.: Test for Normal distribution of the portfolios' returns; daily holdings. 19.03.2008-02.12.2014.

	W	skewness	kurtosis	Jarque-Bera test (95% conf. level)
EQ	300	-0.3057	6.7693	reject H0
	500	-0.3186	7.3945	reject H0
	1000	-0.1842	4.0503	reject H0
	1500	-0.5591	4.1255	reject H0
MV	300	0.3058	10.4997	reject H0
	500	-0.0598	6.5261	reject H0
	1000	-0.1504	4.5327	reject H0
	1500	-0.2969	3.4746	can not reject H0
SE	300	-0.0492	8.5824	reject H0
	500	-0.3422	7.6781	reject H0
	1000	-0.2295	4.1404	reject H0
	1500	-0.6285	4.0917	reject H0
CE	300	0.2087	10.3051	reject H0
	500	-0.1798	6.6152	reject H0
	1000	-0.2287	4.3875	reject H0
	1500	-0.5267	3.7152	reject H0

Table C.4.: Test for Normal distribution of the portfolios' returns; weekly holdings. 19.03.2008-02.12.2014.

	W	skewness	kurtosis	Jarque-Bera test (95% conf. level)
EQ	300	-0.2531	4.6920	reject H0
	500	-0.3807	4.8964	reject H0
	1000	-0.3706	3.2478	reject H0
	1500	-0.5839	4.0325	reject H0
MV	300	0.0772	5.3278	reject H0
	500	-0.2604	4.1771	reject H0
	1000	-0.3003	3.6385	reject H0
	1500	-0.1009	3.7335	can not reject H0
SE	300	-0.0461	5.2695	reject H0
	500	-0.3418	5.3680	reject H0
	1000	-0.3238	3.5453	reject H0
	1500	-0.4375	3.4648	reject H0
CE	300	0.1329	5.2517	reject H0
	500	-0.1729	4.3962	reject H0
	1000	-0.3250	3.6061	reject H0
	1500	-0.3131	3.5568	can not reject H0

Table C.5.: Test for Normal distribution of the portfolios' returns; daily holdings. 01.12.2007-30.06.2009

	W	skewness	kurtosis	Jarque-Bera test (95% conf. level)
EQ	60	0.3513	5.5741	reject H0
	90	0.3345	5.3883	reject H0
	120	0.3335	4.9839	reject H0
	200	0.3167	4.2018	reject H0
MV	60	-0.0091	9.0219	reject H0
	90	0.9347	12.2895	reject H0
	120	0.1996	10.6616	reject H0
	200	0.7489	5.9031	reject H0
SE	60	-0.1157	5.6644	reject H0
	90	-0.0568	7.5358	reject H0
	120	-0.3334	8.0651	reject H0
	200	0.0845	4.0039	reject H0
CE	60	-0.1235	8.5115	reject H0
	90	0.8831	12.3885	reject H0
	120	0.1645	10.4380	reject H0
	200	0.6814	5.1976	reject H0

Table C.6.: Test for Normal distribution of the portfolios' returns; weekly holdings. 01.12.2007-30.06.2009

	W	skewness	kurtosis	Jarque-Bera test (95% conf. level)
EQ	60	0.1861	5.5013	reject H0
	90	0.1969	5.2509	reject H0
	120	0.2248	4.9022	reject H0
	200	0.1783	3.9466	reject H0
MV	60	-0.2086	11.7809	reject H0
	90	-0.1058	7.9806	reject H0
	120	-0.3724	6.7498	reject H0
	200	0.8097	7.5806	reject H0
SE	60	-0.4392	7.4228	reject H0
	90	-0.7849	8.0689	reject H0
	120	-0.9793	7.8340	reject H0
	200	0.1403	6.5688	reject H0
CE	60	-0.2153	10.6654	reject H0
	90	-0.2115	8.3945	reject H0
	120	-0.5105	7.2930	reject H0
	200	0.7410	6.8225	reject H0

Table C.7.: Test for Normal distribution of the portfolios' returns; monthly holdings. 01.12.2007-30.06.2009

	W	skewness	kurtosis	Jarque-Bera test (95% conf. level)
EQ	60	0.3946	3.9632	reject H0
	90	0.5173	3.8447	reject H0
	120	0.4836	3.5350	reject H0
	200	0.4355	2.9841	can not reject H0
MV	60	-0.5828	5.8818	reject H0
	90	-0.2560	4.0457	reject H0
	120	-0.0829	3.8986	reject H0
	200	0.4441	3.3744	reject H0
SE	60	-0.3623	3.4611	reject H0
	90	-0.1441	3.3078	can not reject H0
	120	-0.0394	3.0771	can not reject H0
	200	0.0821	2.8843	can not reject H0
CE	60	-0.4481	5.2421	reject H0
	90	-0.2141	4.5074	reject H0
	120	-0.0770	4.0155	reject H0
	200	0.3156	3.0850	can not reject H0

Appendix D

Tests for equality of the Sharpe ratios (weekly rebalancing) .

Table D.1.: Tests for the difference in the Sharpe ratios of the portfolios; design-free estimate of variance-covariance matrix; weekly holdings; 01.01.2000-18.03.2008.

Description	W	test stat.	p-value	Decision H0
$SR_{MV} = SR_{EQ}$	300	0.8529	0.404	can not reject H0
	500	0.5998	0.558	can not reject H0
	1000	2.121	0.0392	reject H0
	1500	1.49	0.162	can not reject H0
$SR_{MV} = SR_{SE}$	300	1.007	0.324	can not reject H0
	500	0.6187	0.533	can not reject H0
	1000	1.568	0.127	can not reject H0
	1500	1.381	0.182	can not reject H0
$SR_{MV} = SR_{CE}$	300	0.4945	0.627	can not reject H0
	500	0.7333	0.469	can not reject H0
	1000	1.056	0.305	can not reject H0
	1500	0.5841	0.572	can not reject H0
$SR_{SE} = SR_{EQ}$	300	0.3815	0.713	can not reject H0
	500	0.5111	0.615	can not reject H0
	1000	1.186	0.239	can not reject H0
	1500	0.2426	0.81	can not reject H0
$SR_{CE} = SR_{EQ}$	300	0.6727	0.508	can not reject H0
	500	0.536	0.594	can not reject H0
	1000	2.189	0.0349	reject H0
	1500	1.321	0.208	can not reject H0

Table D.2.: Tests for the difference in the Sharpe ratios of the portfolios; design-free estimate of variance-covariance matrix; weekly rebalancing; 19.03.2008-02.12.2014.

Description	W	test stat.	p-value	Decision H
$SR_{MV} = SR_{EQ}$	300	0.3424	0.74	can not reject H0
	500	0.347	0.73	can not reject H0
	1000	1.421	0.159	can not reject H0
	1500	0.00646	0.996	can not reject H0
$SR_{MV} = SR_{SE}$	300	0.1725	0.865	can not reject H0
	500	0.781	0.445	can not reject H0
	1000	1.412	0.168	can not reject H0
	1500	1.008	0.347	can not reject H0
$SR_{MV} = SR_{CE}$	300	0.1878	0.863	can not reject H0
	500	0.2159	0.833	can not reject H0
	1000	1.408	0.173	can not reject H0
	1500	0.9518	0.362	can not reject H0
$SR_{SE} = SR_{EQ}$	300	1.029	0.314	can not reject H0
	500	1.684	0.101	can not reject H0
	1000	0.1764	0.865	can not reject H0
	1500	0.6093	0.577	can not reject H0
$SR_{CE} = SR_{EQ}$	300	0.6737	0.5	can not reject H0
	500	0.4321	0.669	can not reject H0
	1000	0.882	0.392	can not reject H0
	1500	0.2544	0.817	can not reject H0

Table D.3.: Tests for the difference in the Sharpe ratios of the portfolios; design-free estimate of variance-covariance matrix; daily holdings; 01.12.2007-30.06.2009.

Description	W	test stat.	p-value	Decision H
$SR_{MV} = SR_{EQ}$	60	1.191	0.274	can not reject H0
	90	0.1961	0.856	can not reject H0
	120	1.228	0.24	can not reject H0
	200	0.9391	0.381	can not reject H0
$SR_{MV} = SR_{SE}$	60	0.1076	0.917	can not reject H0
	90	0.9966	0.339	can not reject H0
	120	0.2492	0.809	can not reject H0
	200	0.163	0.883	can not reject H0
$SR_{MV} = SR_{CE}$	60	1.462	0.182	can not reject H0
	90	0.1859	0.86	can not reject H0
	120	0.02579	0.98	can not reject H0
	200	2.368	0.103	can not reject H0
$SR_{SE} = SR_{EQ}$	60	1.644	0.143	can not reject H0
	90	1.198	0.266	can not reject H0
	120	1.536	0.163	can not reject H0
	200	1.163	0.284	can not reject H0
$SR_{CE} = SR_{EQ}$	60	0.7574	0.478	can not reject H0
	90	0.1326	0.907	can not reject H0
	120	1.225	0.252	can not reject H0
	200	0.2421	0.823	can not reject H0

Table D.4.: Tests for the difference in the Sharpe ratios of the portfolios; design-free estimate of variance-covariance matrix; weekly holdings; 01.12.2007-30.06.2009.

Description	W	test stat.	p-value	Decision H
$SR_{MV} = SR_{EQ}$	60	0.1664	0.873	can not reject H0
	90	1.093	0.295	can not reject H0
	120	0.7843	0.457	can not reject H0
	200	0.3104	0.774	can not reject H0
$SR_{MV} = SR_{SE}$	60	0.464	0.654	can not reject H0
	90	0.2503	0.812	can not reject H0
	120	0.1434	0.898	can not reject H0
	200	0.8801	0.408	can not reject H0
$SR_{MV} = SR_{CE}$	60	0.238	0.818	can not reject H0
	90	0.8859	0.385	can not reject H0
	120	0.1014	0.925	can not reject H0
	200	1.658	0.152	can not reject H0
$SR_{SE} = SR_{EQ}$	60	0.6806	0.528	can not reject H0
	90	0.9106	0.404	can not reject H0
	120	1.17	0.284	can not reject H0
	200	0.1863	0.867	can not reject H0
$SR_{CE} = SR_{EQ}$	60	0.06771	0.949	can not reject H0
	90	0.6673	0.527	can not reject H0
	120	0.7624	0.463	can not reject H0
	200	0.2292	0.829	can not reject H0

Table D.5.: Tests for the difference in the Sharpe ratios of the portfolios; design-free estimate of variance-covariance matrix; monthly holdings; 01.12.2007-30.06.2009.

Description	W	test stat.	p-value	Decision H
$SR_{MV} = SR_{EQ}$	60	0.4305	0.68	can not reject H0
	90	1.002	0.342	can not reject H0
	120	1.073	0.308	can not reject H0
	200	0.15	0.879	can not reject H0
$SR_{MV} = SR_{SE}$	60	0.336	0.756	can not reject H0
	90	0.1069	0.913	can not reject H0
	120	0.9044	0.409	can not reject H0
	200	0.08078	0.939	can not reject H0
$SR_{MV} = SR_{CE}$	60	0.2288	0.831	can not reject H0
	90	1.562	0.138	can not reject H0
	120	0.4987	0.629	can not reject H0
	200	1.382	0.216	can not reject H0
$SR_{SE} = SR_{EQ}$	60	2.156	0.081	can not reject H0
	90	2.927	0.0202	can not reject H0
	120	2.341	0.0555	can not reject H0
	200	0.3241	0.769	can not reject H0
$SR_{CE} = SR_{EQ}$	60	0.4768	0.649	can not reject H0
	90	1.436	0.175	can not reject H0
	120	0.9162	0.383	can not reject H0
	200	0.4751	0.674	can not reject H0