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Report on the PhD thesis

"On fluids with pressure-dependent viscosity flowing through a porous medium"

by Joseph Žabenský

The thesis under review is concerned with mathematical models for incompressible fluids with a pressure dependent viscosity. This is motivated by experimental evidence of this effect. The author studies several nonlinear partial differential equations (PDEs) modelling this effects from a continuum mechanical perspective. To be precise he shows existence of weak solutions assuming generalizations of Darcy's law, Stokes' law or generalized Navier Stokes equations. All three models describe the fluid flow by its velocity field $\boldsymbol{v}$ and the hydrodynamical pressure function p. The results are contained in three papers which already appeared in well-accepted international journals.
To find weak solutions to a nonlinear PDE one approximates the equation/system, shows a priori estimates and tries to pass to limit. Difficulties always appear in the nonlinear terms, in particular if they are not compact. The author uses monotonicity arguments and/or the Lipschitz truncation to solve this sophisticated problem.
The first result is concerned with a generalization of the Darcy-Forchheimer equation. The balance of linear momentum is described by

$$\nabla p + \boldsymbol{m} = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{f}$ is an external body force. This describes the flow through a porous medium. Here $\boldsymbol{m}$ represents the interaction force between a fluid and a rigid body. In the easiest case one can assume a linear relation between $\boldsymbol{m}$ and $\boldsymbol{v}$. A more realistic assumption is $\boldsymbol{m} = \alpha(|\boldsymbol{v}|)|\boldsymbol{v}|$ supposed in the Darcy-Forchheimer's model. The author generalizes this by supposing $\boldsymbol{m} = \alpha(p, |\boldsymbol{v}|)|\boldsymbol{v}|$ which is the main novelty of the paper. In fact, even an implicit relation between $\boldsymbol{m}, \boldsymbol{v}$ and p, formulated via a maximal monotone graph, is included in the theory. The author uses a quasi-compressible approximation. Here the regularization perturbs the divergence-free constraint for the velocity in order to get a better control of the pressure. The limit procedure in the approximated system is performed using the monotonicity of $\boldsymbol{m}$ with respect to $\boldsymbol{v}$ and compactness of the pressure. In fact a three layer-scheme is applied and the passage to the limit has to be shown three times.
The second result is concerned with a PDE system which combines the models of Stokes,

Darcy, Forchheimer and Brinkman. The balance of momentum reads as

$$-\mathrm{Div}\Big(\big(2\nu(p,|\boldsymbol{Dv}|)\big)\boldsymbol{Dv}\Big)+\beta(p,|\boldsymbol{v}|,|\boldsymbol{Dv}|)\boldsymbol{v}+\nabla p=\boldsymbol{f}. \tag{2}$$

Hereby the viscosity coefficient ν has $(r-2)$-growth with respect to $|\boldsymbol{Dv}|$ and β can have different polynomial growth rates with respect to p, $|\boldsymbol{v}|$ and $|\boldsymbol{Dv}|$. In order to show the existence of a weak solution, the first step is again to gain a suitable approximation (which is again done via a quasi-compressible Ansatz). First, the author shows strong convergence of the pressure. This mainly follows from the assumption that ν varies only slightly in p. The limit passage in the viscosity with respect to $\boldsymbol{Dv}$ shall be performed using monotonicity of the diffusive part. This only works directly if $\beta=0$ (this case was already studied before). The problem can be overcome by testing with the Lipschitz truncation of the solution. This step requires a great technical care and is the bulk of the proof.
The last result is concerned with generalized Navier-Stokes equations, where the time evolution of the flow is governed by the system

$$\partial_t\boldsymbol{v}+\mathrm{Div}(\boldsymbol{v}\otimes\boldsymbol{v})-\mathrm{Div}\Big(\big(2\nu(p,|\boldsymbol{Dv}|)\big)\boldsymbol{Dv}\Big)+\nabla p=\boldsymbol{f}. \tag{3}$$

Here the viscosity growth like $|\boldsymbol{Dv}|^{r-2}$ where $r>\frac{2d}{d+2}$. This is the best known bound even without pressure dependence in the viscosity. As for the situation above only stationary flows have been studied in this range. For non-stationary flows the bound $r>\frac{2d+2}{d+2}$ was the best known one so far. The reason for such a general result is the application of a parabolic Lipschitz truncation which has been developed very recently and has been successfully applied to non-stationary models for generalized Newtonian fluids. The Lipschitz truncation method is very delicate in the non-stationary setting. In particular the pressure function has to be introduced and decomposed in accordance to the terms appearing in the equation. Due to the choice of Navier's boundary conditions the pressure actually exists as a function (that is different from the Dirichlet case). However, the fact that the viscosity depends on the pressure leads to additional difficulties here. One part of the pressure corresponds to the convective term and hence is compact. The other part behaves as the stress deviator and can be treated in the same spirit as for (2). Finally, the author is able to pass to the limit in the approximated problem and the existence of a weak solution to (3) is shown.

To summarize, the thesis is clearly and well-written and presents an interesting and actual research topic with applications in physics and engineering. The results are on a high scientific level and show the author's ability for creative scientific work. Hence it is my pleasure to recommend the faculty to award a doctoral degree to Joseph Žabenský.

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