Characterization of associate spaces of weighted Lorentz spaces with applications

by

Amiran Gogatishvili, Luboš Pick, and Filip Soudský (Praha)

Abstract. We characterize associate spaces of weighted Lorentz spaces $G\Gamma(p, m, w)$ and present some applications of this result including necessary and sufficient conditions for a Sobolev-type embedding into L^∞.

1. Introduction and main results. Let $(\mathcal{R}, \mu)$ be a σ-finite non-atomic measure space with $b = \mu(\mathcal{R}) \in (0, \infty]$. We denote by $\mathfrak{M}(\mathcal{R})$ the set of all μ-measurable functions on $\mathcal{R}$ whose values belong to $[-\infty, \infty]$. We also define $\mathfrak{M}_+(\mathcal{R}) = \{g \in \mathfrak{M}(\mathcal{R}) : g \geq 0\}$, and $\mathfrak{M}_0(\mathcal{R}) = \{g \in \mathfrak{M}(\mathcal{R}) : g$ is finite a.e. in $\mathcal{R}\}$.

The function space $G\Gamma(p, m, w)(\mathcal{R})$ (denoted simply by $G\Gamma(p, m, w)$ when no confusion can arise), introduced and studied in [FR2] and [FRZ], is defined as the collection of all functions $g \in \mathfrak{M}(\mathcal{R}, \mu)$ such that

$$\|g\|_{G\Gamma(p,m,w)} = \left(\int_0^b w(t) \left(\int_0^t g^*(s)^p \, ds\right)^{m/p} dt\right)^{1/m} < \infty,$$

where $m, p \in (0, \infty)$, w is a weight (that is, a positive measurable function) on $(0, b)$, and g^* is the *non-increasing rearrangement* of g, given by

$$g^*(t) = \sup\{\lambda \in \mathbb{R} : \mu(\{x \in \mathcal{R} : |g(x)| > \lambda\}) > t\} \quad \text{for } t \in (0, b).$$

We also define the *maximal non-increasing rearrangement* of g by

$$g^{**}(t) = \frac{1}{t} \int_0^t g^*(s) \, ds \quad \text{for } t \in (0, b),$$

and we note that the estimate

(1.1) $$g^*(t) \leq g^{**}(t)$$

holds universally for every $g \in \mathfrak{M}(\mathcal{R})$ and every $t \in (0, b)$.

[1]

Our main goal is to give a precise and easily-computable characterization of the norm in the *associate space* (sometimes also called the *Köthe dual*) of the space $G\Gamma(p,m,w)$. The associate space $G\Gamma(p,m,w)'$ of $G\Gamma(p,m,w)$ is defined as the collection of all functions $g \in \mathfrak{M}(\mathcal{R})$ such that

$$(1.2) \qquad \|g\|_{G\Gamma(p,m,w)'} = \sup_{\|f\|_{G\Gamma(p,m,w)} \leq 1} \int_0^b f^*(t)g^*(t)\,dt < \infty.$$

Such a result is of interest for a number of reasons. In general, an associate space is a key thing to know about any Banach function space (see definitions below). Moreover, the spaces $G\Gamma(p,m,w)$ cover several types of important function spaces and have plenty of applications. For example, if $b = \infty$, $p = 1$, $m > 1$ and $w(t) = t^{-m}v(t)$, $t \in (0,\infty)$, where v is another weight on $(0,b)$, then $G\Gamma(p,m,w)$ reduces to the space $\Gamma^m(v)$, whose norm is

$$\|g\|_{\Gamma^m(v)} = \left(\int_0^\infty g^{**}(t)^m v(t)\,dt\right)^{1/m}.$$

This space was introduced by Sawyer [Sa] who used it to describe the behavior of classical operators on Lorentz spaces and observed, among other results, that, under certain restrictions on the parameters involved, this space is the associate space of the space $\Lambda^{m'}(\tilde{v})$, introduced by Lorentz [L], where $m' = m/(m-1)$, $\tilde{v}$ is an appropriate weight, and the norm in $\Lambda^{m'}(\tilde{v})$ is given by

$$\|g\|_{\Lambda^{m'}(\tilde{v})} = \left(\int_0^\infty g^*(t)^{m'}\tilde{v}(t)\,dt\right)^{1/m'}.$$

The spaces of type Λ and Γ have been extensively investigated during the last 25 years under the common label *classical Lorentz spaces*, and an avalanche of papers by many authors devoted to their detailed study is available nowadays.

Another important example is obtained when $b = 1$, $m = 1$, $p \in (1,\infty)$ and $w(t) = t^{-1}\left(\log\frac{2}{t}\right)^{-1/p}$, $t \in (0,1)$. In this case $G\Gamma(p,m,w)$ coincides with the so-called *small Lebesgue space*, first studied by Fiorenza [F]. He proved that this space is the associate space of the so-called *grand Lebesgue space*, introduced in [IS] in connection with integrability properties of Jacobians. It was shown later by Fiorenza and Karadzhov [FK] that the norm in the small Lebesgue space can be equivalently written in the form of the norm in the $G\Gamma(p,m,w)$ space with the above-mentioned parameters and weight. For further results in this direction, see also [FR1, FR2]. Our characterization of the associate space of $G\Gamma(p,m,w)$ thus gives a new description of the grand Lebesgue space.

In [FR2] and [FRZ] the authors studied the associate spaces of the spaces $G\Gamma(p,m,w)$, but obtained only an upper bound for $\|g\|_{G\Gamma(p,m,w)'}$, moreover

under the restriction that $\mu(R) < \infty$ and either $p \neq 1$ [FR2, Theorem 6] or $m \leq p$ [FRZ, Theorem 3.2].

We are going to give a complete general characterization of the associate space of $G\Gamma(p, m, w)$ without any restrictions on the parameters involved. However, it is reasonable to adopt a general assumption that p, m and w are such that

$$(1.3) \qquad \int_0^t w(s)s^{m/p}\,ds + \int_t^b w(s)\,ds < \infty \quad \text{for every } t \in (0, b),$$

because if this requirement is not satisfied, then the "space" $G\Gamma(p, m, w)$ contains only the zero function. Under the assumption (1.3), we denote

$$(1.4) \qquad u(t) = \int_0^t w(s)s^{m/p}\,ds + t^{m/p}\int_t^b w(s)\,ds, \quad t \in (0, b).$$

The principal background tool in the proofs will be the duality results of [GP] and [Si]. It will be useful, in accordance with the terminology used in the first-mentioned paper, to call a weight w *non-degenerate* (with respect to the power function $t^{m/p}$) if (1.3) is satisfied and moreover

$$(1.5) \qquad \int_0^t w(s)\,ds = \int_t^b w(s)s^{m/p}\,ds = \infty \quad \text{for every } t \in (0, b).$$

We do not restrict our results here to non-degenerate weights, but we shall see that the characterizing conditions for degenerate weights are different from those concerning non-degenerate ones.

We shall now formulate our main theorem. Here and throughout, the symbol $\approx$ means that the two sides are bounded by each other up to multiplicative constants independent of appropriate quantities. As usual, for $p \in (1, \infty)$, we write $p' = p/(p-1)$. Throughout the paper, we use the convention $0 \cdot \infty = 0$. Another convention we use is that $b/2 = \infty$ when $b = \infty$.

THEOREM 1.1. *Assume that $0 < m, p < \infty$. Let w be a weight on $(0, b)$ such that* (1.3) *is satisfied. Let u be defined by* (1.4).

(i) *Let $0 < m \leq 1$ and $0 < p \leq 1$. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t \in (0,b/2)} g^{**}(t)\frac{t}{u(t)^{1/m}}.$$

(ii) *Let $0 < m \leq 1$ and $1 < p < \infty$. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t \in (0,b/2)} \left(\int_t^b g^{**}(s)^{p'}\,ds\right)^{1/p'}\frac{t^{1/p}}{u(t)^{1/m}}.$$

(iii) *Let* $1 < m < \infty$, $0 < p \le 1$ *and let* (1.5) *be satisfied. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \left(\int_0^{b/2} g^{**}(t)^{m'} \frac{t^{m'+m/p-1} \int_0^t w(s) s^{m/p} \, ds \int_t^b w(s) \, ds}{u(t)^{m'+1}} \, dt \right)^{1/m'}.$$

(iv) *Let* $1 < m < \infty$, $0 < p \le 1$ *and let either* $\int_0^b w(s) \, ds < \infty$ *or* $\int_0^b w(s) s^{m/p} \, ds < \infty$ *or both. Then*

$$\|g\|_{G\Gamma(p,m,w)'} \approx \left(\int_0^{b/2} g^{**}(t)^{m'} \frac{t^{m'+m/p-1} \int_0^t w(s) s^{m/p} \, ds \int_t^b w(s) \, ds}{u(t)^{m'+1}} \, dt \right)^{1/m'}$$

$$+ \frac{\limsup_{t\to 0+} g^{**}(t)}{\left(\int_0^b w(s) \, ds \right)^{1/m}} + \frac{\int_0^b g^*(s) \, ds}{\left(\int_0^b w(s) s^{m/p} \, ds \right)^{1/m}}.$$

(v) *Let* $1 < m < \infty$, $1 < p < \infty$ *and let* (1.5) *be satisfied. Then*

$$\|g\|_{G\Gamma(p,m,w)'}$$
$$\approx \left(\int_0^{b/2} \left(\int_t^b g^{**}(s)^{p'} \, ds \right)^{m'/p'} \frac{t^{m'/p+m/p-1} \int_0^t w(s) s^{m/p} \, ds \int_t^b w(s) \, ds}{u(t)^{m'+1}} \, dt \right)^{1/m'}.$$

(vi) *Let* $1 < m < \infty$, $1 < p < \infty$ *and let either* $\int_0^b w(s) \, ds < \infty$ *or* $\int_0^b w(s) s^{m/p} \, ds < \infty$ *or both. Then*

$$\|g\|_{G\Gamma(p,m,w)'}$$
$$\approx \left(\int_0^{b/2} \left(\int_t^b g^{**}(s)^{p'} \, ds \right)^{m'/p'} \frac{t^{m'/p+m/p-1} \int_0^t w(s) s^{m/p} \, ds \int_t^b w(s) \, ds}{u(t)^{m'+1}} \, dt \right)^{1/m'}$$

$$+ \frac{\left(\int_0^b g^{**}(s)^{p'} \, ds \right)^{1/p'}}{\left(\int_0^b w(s) \, ds \right)^{1/m}} + \frac{\int_0^b g^*(s) \, ds}{\left(\int_0^b w(s) s^{m/p} \, ds \right)^{1/m}}.$$

For the proof of Theorem 1.1 we will develop a simple but powerful argument based on combination of results from [GP] and [Si] with an elementary inequality involving rearrangements, contained in the next result.

THEOREM 1.2. *Assume that* $1 < p < \infty$. *Let* $g \in L^1_{\mathrm{loc}}(\mathcal{R}, \mu)$. *Then*

$$(1.6) \qquad g^{**}(t) + \left(\frac{1}{t} \int_t^b g^{**}(s)^{p'-1} g^*(s) \, ds \right)^{1/p'} \approx \left(\frac{1}{t} \int_t^b g^{**}(s)^{p'} \, ds \right)^{1/p'}$$

for every $t \in (0, b/2)$.

We shall now turn our attention to an application of Theorem 1.1 to Sobolev-type embeddings which was first pointed out in [FRZ].

A (quasi-)normed linear space X is said to be (continuously) *embedded* into another such space Y, and denoted by $X \hookrightarrow Y$, if $X \subset Y$ and the identity operator is bounded from X to Y.

Let Ω be a bounded open connected set (a domain) in $\mathbb{R}^n$, where $n \in \mathbb{N}$, $n \geq 2$. We say that Ω is a *John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0, l] \to \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq cr \quad \text{for } r \in [0, l],$$

where $\partial\Omega$ is the boundary of Ω. The class of John domains is known to include some other families of domains that are considered classical, such as domains having Lipschitz boundary or domains having the cone property. John domains arise in connection with the study of holomorphic dynamical systems and quasiconformal mappings, and they are known to support Sobolev inequalities with the same exponents as the standard Sobolev ones (see [Bo, HK, KM, CPS1]). Being a John domain is a necessary condition for a Sobolev inequality to hold on simply connected open sets in $\mathbb{R}^2$ and on more general higher-dimensional domains (see [BK]).

For $k \in \mathbb{N}$, the Sobolev space $W^k G\Gamma(p, m, w)(\Omega)$ is defined as the collection of all weakly-differentiable functions u defined on Ω such that $|\nabla^j u| \in G\Gamma(p, m, w)(\Omega)$ for every $j \in \mathbb{N} \cup \{0\}$, $j \leq k$, where $\nabla^j u$ is the jth gradient of u, $\nabla^0 u = u$ and $|\cdot|$ is the Euclidean norm. The space $W^k G\Gamma(p, m, w)(\Omega)$, endowed with the functional

$$\|u\|_{W^k G\Gamma(p,m,w)(\Omega)} = \sum_{j=0}^{k} \big\| |\nabla^j u| \big\|_{G\Gamma(p,m,w)(\Omega)},$$

is a Banach space.

It was proved in [FRZ, Lemma 1.4] that the condition

$$(1.7) \qquad t^{-1/n'} \in G\Gamma(p, m, w)'(0, b)$$

is sufficient for the Sobolev embedding

$$(1.8) \qquad W^1 G\Gamma(p, m, w)(\Omega) \hookrightarrow L^\infty(\Omega),$$

where $b = |\Omega|$. Embeddings of type (1.8) are known to have a number of applications, for example they are intimately connected with the question whether the Sobolev space is a Banach algebra (cf. e.g. [A, C, CPS2]). Our aim is to point out that, as can be deduced from our results, (1.7) is in fact not only sufficient, but also *necessary*, for (1.8) to hold. Furthermore, we shall include Sobolev embeddings of any order.

However, before we can state this result, we first need to know for which parameters p, m, w the space $G\Gamma(p, m, w)$ satisfies the axioms of rearrangement-invariant Banach function space. We say that X is a *Banach*

function space over a σ-finite measure space $(\mathcal{R}, \mu)$ if for all non-negative μ-measurable real functions f, g and $\{f_j\}_{j\in\mathbb{N}}$ on $\mathcal{R}$ and every $\lambda \geq 0$, the following properties hold:

(P1) $\|f\|_X = 0$ if and only if $f = 0$ a.e.; $\|\lambda f\|_X = \lambda\|f\|_X$; $\|f + g\|_X \leq \|f\|_X + \|g\|_X$;

(P2) $f \leq g$ a.e. implies $\|f\|_X \leq \|g\|_X$;

(P3) $f_j \nearrow f$ a.e. implies $\|f_j\|_X \nearrow \|f\|_X$;

(P4) for every $E \subset \mathcal{R}$ with $\mu(E) < \infty$ one has $\|\chi_E\|_X < \infty$;

(P5) for every $E \subset \mathcal{R}$ with $\mu(E) < \infty$ one has $\int_E f(x)\,d\mu \leq C_E\|f\|_X$ for some constant C_E independent of f.

We say that X is a *rearrangement-invariant Banach function space* if (P1)–(P5) are satisfied and moreover $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$ on $(0, b)$. Here and throughout, χ_E denotes the characteristic function of E.

We shall now state a necessary and sufficient condition for the space $G\Gamma(p, m, w)$ to be a rearrangement-invariant Banach function space. In view of applications, we restrict ourselves to the case $1 \leq p, m < \infty$. We note that the result is known for certain particular cases. We omit the details but we refer the reader to [FR2, Theorem 5].

THEOREM 1.3. *Suppose that* $1 \leq p, m < \infty$ *and let* w *be a weight on* $(0, b)$. *Then the space* $G\Gamma(p, m, w)$ *is a Banach function space if and only if*

$$(1.9) \qquad \int_0^b w(t)\min\{1, t^{m/p}\}\,dt < \infty.$$

Now we are in a position to characterize a higher-order Sobolev embedding. The results are collected in the following theorem. It will be useful to recall that Ω is a bounded domain, therefore $b < \infty$.

THEOREM 1.4. *Let* $n \in \mathbb{N}$, $n \geq 2$. *Let* $\Omega \subset \mathbb{R}^n$ *be a John domain and let* $b = |\Omega|$. *Let* $1 \leq m, p < \infty$ *and let* w *be a weight on* $(0, b)$ *such that*

$$(1.10) \qquad \int_0^b w(t)t^{m/p}\,dt < \infty.$$

Let $k \in \mathbb{N}$. *Then the Sobolev embedding*

$$(1.11) \qquad W^k G\Gamma(p, m, w)(\Omega) \hookrightarrow L^\infty(\Omega)$$

holds if and only if either $k \geq n$, *or* $k \leq n - 1$ *and one of the following conditions is satisfied:*

(i) $m = 1$, $1 \leq p < n/k$ *and*

$$\sup_{t\in(0,b/2)} \frac{t^{k/n}}{\int_0^t w(s)s^{1/p}\,ds + t^{1/p}\int_t^b w(s)\,ds} < \infty;$$

(ii) $m = 1$, $p = n/k$ and

$$\sup_{t\in(0,b/2)} \frac{t^{k/n}\big(\log\frac{b}{t}\big)^{1-k/n}}{\int_0^t w(s)s^{1/p}\,ds + t^{1/p}\int_t^b w(s)\,ds} < \infty;$$

(iii) $m = 1$, $n/k < p < \infty$ and

$$\sup_{t\in(0,b/2)} \frac{t^{1/p}}{\int_0^t w(s)s^{1/p}\,ds + t^{1/p}\int_t^b w(s)\,ds} < \infty;$$

(iv) $1 < m < \infty$, $1 \le p < n/k$, $\int_0^b w(t)\,dt = \infty$ and

$$\int_0^{b/2} \frac{t^{m'k/n+m/p-1}\int_0^t w(s)s^{m/p}\,ds\int_t^b w(s)\,ds}{\big(\int_0^t w(s)s^{m/p}\,ds + t^{m/p}\int_t^b w(s)\,ds\big)^{m'+1}}\,dt < \infty;$$

(v) $1 < m < \infty$, $p = n/k$, $\int_0^b w(t)\,dt = \infty$ and

$$\int_0^{b/2} \frac{t^{m'k/n+mk/n-1}\big(\log\frac{b}{t}\big)^{m'(1-k/n)}\int_0^t w(s)s^{mk/n}\,ds\int_t^b w(s)\,ds}{\big(\int_0^t w(s)s^{mk/n}\,ds + t^{mk/n}\int_t^b w(s)\,ds\big)^{m'+1}}\,dt < \infty;$$

(vi) $1 < m < \infty$, $n/k < p < \infty$ and

$$\int_0^{b/2} \frac{t^{m'/p+m/p-1}\int_0^t w(s)s^{m/p}\,ds\int_t^b w(s)\,ds}{\big(\int_0^t w(s)s^{m/p}\,ds + t^{m/p}\int_t^b w(s)\,ds\big)^{m'+1}}\,dt < \infty.$$

Using the results of [CPS1] one can obtain sufficient conditions for the Sobolev embedding (1.11) also for domains with worse boundary than just John domains, as long as a lower bound for their isoperimetric function is known. In many customary cases, such conditions will also be necessary in a certain broader sense. We recall that the *perimeter* of a measurable set E in Ω is given by

$$P(E,\Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial^M E),$$

where $\partial^M E$ denotes the essential boundary of E, in the sense of geometric measure theory [M, Z]. The *isoperimetric function* $I_\Omega : [0,1] \to [0,\infty]$ of Ω is then given by

$$I_\Omega(s) = \inf\{P(E,\Omega) : E \subset \Omega,\ s \le |E| \le 1/2\} \quad \text{if } s \in [0,1/2],$$

and $I_\Omega(s) = I_\Omega(1-s)$ if $s \in (1/2, 1]$. We omit the details.

In our last application of Theorem 1.1 we intend to characterize those parameters p, m and w for which the space $G\Gamma(p,m,w)$ is reflexive. This question was studied in [FRZ], where a number of results were deduced from

the assumption that $G\Gamma(p, m, w)$ is reflexive, and also a sufficient condition for reflexivity was given.

To pave the way to a characterization we shall first single out those spaces $G\Gamma(p, m, w)$ which have absolutely continuous norms. We restrict here to the case when $1 < p, m < \infty$. Such a result is of independent interest since it might be handy when compactness of operators and embeddings between function spaces is studied (see e.g. [LZ, FMP, KP, PP, Sl1, Sl2]). A Banach function space X on $(\mathcal{R}, \mu)$ is said to have *absolutely continuous norm* if for each sequence $\{E_n\}$ of μ-measurable subsets of $\mathcal{R}$ satisfying $E_n \downarrow \emptyset$ one has $\|\chi_{E_n} f\|_X \to 0$ for every $f \in X$.

THEOREM 1.5. *Let $1 < p, m < \infty$ and let w be a weight on $(0, b)$. Then the space $G\Gamma(p, m, w)$ has absolutely continuous norm if and only if at least one of the following conditions holds:*

$$(1.12) \qquad b < \infty,$$

$$(1.13) \qquad \int_0^b t^{m/p} w(t)\, dt = \infty.$$

Our next theorem shows that for the associate space of $G\Gamma(p, m, w)$, the absolute continuity of norm is granted unconditionally.

THEOREM 1.6. *Let $1 < p, m < \infty$ and let w be a weight on $(0, b)$. Then the associate space to $G\Gamma(p, m, w)$ has an absolutely continuous norm.*

Now we can state our last result. Again, some particular cases are known [FR2, Theorem 5].

THEOREM 1.7. *Let $1 < p, m < \infty$ and let w be a weight on $(0, b)$. Then the space $G\Gamma(p, m, w)$ is reflexive if and only if at least one of the conditions (1.12) and (1.13) holds.*

EXAMPLES 1.8. (a) If $b < \infty$, $0 < m < \infty$, $1 \le p < \infty$ and $\int_0^b w(s)\, ds < \infty$, then it is not difficult to verify that the space $G\Gamma(p, m, w)$ degenerates to the Lebesgue space L^p (regardless of m). Indeed, on the one hand, we have

$$\|g\|_{G\Gamma(p,m,w)} = \left(\int_0^b w(t) \left(\int_0^t g^*(s)^p\, ds \right)^{m/p} dt \right)^{1/m}$$

$$\le \left(\int_0^b g^*(s)^p\, ds \right)^{1/p} \left(\int_0^b w(t)\, dt \right)^{1/m}$$

$$= C \|g\|_{L^p}$$

with $C = \left(\int_0^b w(t)\, dt \right)^{1/m} < \infty$, while, on the other hand, due to the mono-

tonicity of g^* and positivity of w, one has

$$\|g\|_{G\Gamma(p,m,w)} = \left(\int\limits_0^b w(t)\left(\int\limits_0^t g^*(s)^p\,ds\right)^{m/p}dt\right)^{1/m}$$

$$\geq \left(\int\limits_0^{b/2} g^*(s)^p\,ds\right)^{1/p}\left(\int\limits_{b/2}^b w(t)\,dt\right)^{1/m}$$

$$\geq c\|g\|_{L^p}$$

with $c = 2^{-p}(\int_{b/2}^b w(t)\,dt)^{1/m} > 0$. A simple argument shows that, for this choice of parameters, we have $u(t) \approx t^{m/p}$, and it is easy to check that the appropriate choice of part (i), (ii), (iv) or (vi) of Theorem 1.1 yields $\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{p'}}$ for every measurable function g. For example, if $p = 1$ and $1 < m < \infty$, then by Theorem 1.1(iv) we obtain

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{p'}} + \|g\|_{L^\infty} + \|g\|_{L^1} \approx \|g\|_{L^\infty},$$

since $b < \infty$. We note that cases (iii) and (v) of Theorem 1.1 are inapplicable here since (1.5) is false.

(b) If $1 \leq p < \infty$ and $\int_0^b w(s)\,ds < \infty$ but $b = \infty$, then the upper bound for $\|g\|_{G\Gamma(p,m,w)}$ from (a) still applies, but the lower bound does not work, since, in accord with our convention, $b/2 = \infty$, and therefore the integral $\int_{b/2}^b w(t)\,dt$ is zero. Thus, the inclusion $L^p \subset G\Gamma(p,m,w)$ still holds, but the converse need not be satisfied.

(c) We shall now analyze the situation when

$$0 < p < \infty, \quad m > p, \quad m \geq 1, \quad w(t) = t^{-m/p} \quad \text{for every } t \in (0,b).$$

Then

$$\|g\|_{G\Gamma(p,m,w)} = \left(\int\limits_0^b \left(\frac{1}{t}\int\limits_0^t g^*(s)^p\,ds\right)^{m/p}dt\right)^{1/m}.$$

Therefore, by the classical Hardy inequality (see e.g. [BS, Chapter 3, Lemma 3.9]) together with (1.1) we get

$$\|g\|_{G\Gamma(p,m,w)} \approx \left(\int\limits_0^b g^*(t)^m\,dt\right)^{1/m},$$

whence, for this choice of parameters, the space $G\Gamma(p,m,w)$ always degenerates to the Lebesgue space L^m. We shall now check that the results deduced from Theorem 1.1 are consistent with the classical duality relations between Lebesgue spaces. It will be useful to note that $u(t) \approx t$ for every $t \in (0,b)$.

First, let $m = 1$ and $0 < p < 1$. Then Theorem 1.1(i) implies that

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t\in(0,b/2)} g^{**}(t) = \|g\|_{L^\infty},$$

as required.

Next, assume that $p = 1$, $1 < m < \infty$ and $b = \infty$. Then, obviously, (1.5) is satisfied. Hence Theorem 1.1(iii) applies, and we get

$$\|g\|_{G\Gamma(p,m,w)'} \approx \left(\int_0^\infty g^{**}(t)^{m'} \frac{t^{m'+m-1} \cdot t \cdot t^{1-m}}{t^{m'+1}} \, dt \right)^{1/m'} \approx \|g\|_{L^{m'}},$$

by the Hardy inequality and (1.1), again.

If $p = 1$, $1 < m < \infty$ and $b < \infty$, then

$$\int_0^t w(s) \, ds = \infty \quad \text{for every } t \in (0, \infty)$$

but

$$\int_t^b w(s) s^{m/p} \, ds < \infty \quad \text{for every } t \in (0, \infty),$$

hence (1.5) is not satisfied. Consequently, we have to use Theorem 1.1(iv) this time. We get

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{m'}} + b^{-1/m} \|g\|_{L^1}.$$

Because $b < \infty$, we have, by Hölder's inequality, $\|g\|_{L^1} \leq b^{1/m} \|g\|_{L^{m'}}$.

Thus, altogether, we again obtain

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{m'}},$$

as desired.

Let $1 < p < m < \infty$ and $b = \infty$. Then (1.5) holds and we can use Theorem 1.1(v). We obtain

$$\|g\|_{G\Gamma(p,m,w)'} \approx \left(\int_0^\infty \left(\int_t^\infty g^{**}(s)^{p'} \, ds \right)^{m'/p'} \frac{t^{m'/p+m/p-1} \cdot t \cdot t^{1-m/p}}{t^{m'+1}} \, dt \right)^{1/m'}$$

$$\approx \left(\int_0^\infty \left(\frac{1}{t} \int_t^\infty g^{**}(s)^{p'} \, ds \right)^{m'/p'} \, dt \right)^{1/m'}.$$

We claim that

$$(1.14) \qquad \left(\int_0^\infty \left(\frac{1}{t} \int_t^\infty g^{**}(s)^{p'} \, ds \right)^{m'/p'} \, dt \right)^{1/m'} \approx \|g\|_{L^{m'}}.$$

The lower bound is easy, we only have to observe that

$$\left(\int_0^\infty \left(\frac{1}{t} \int_t^\infty g^{**}(s)^{p'} \, ds \right)^{m'/p'} \, dt \right)^{1/m'} \geq \left(\int_0^\infty \left(\frac{1}{t} \int_t^{2t} g^{**}(s)^{p'} \, ds \right)^{m'/p'} \, dt \right)^{1/m'}$$

$$\geq \left(\int_0^\infty g^{**}(2t)^{m'} \, dt \right)^{1/m'} \approx \|g\|_{L^{m'}},$$

where the last relation follows by a simple change of variables. As for the upper bound, we first claim that there exists a positive constant C such that, for every $t \in (0, \infty)$ and every $g \in \mathfrak{M}(\mathcal{R})$, one has

$$(1.15) \qquad \left(\int_t^\infty g^{**}(s)^{p'}\,ds\right)^{1/p'} \le C\left(\int_t^\infty g^{**}(s)^{m'}(s-t)^{m'/p'-1}\,ds\right)^{1/m'}.$$

Clearly, (1.15) will follow once we show that

$$(1.16) \qquad \int_t^\infty h^*(s)\,ds \le C\left(\int_t^\infty h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds\right)^{p'/m'}$$

for some $C > 0$, every $t \in (0, \infty)$ and every $h \in \mathfrak{M}(\mathcal{R})$, on applying the last estimate to the particular choice $h^* = (g^{**})^{p'}$. The proof of (1.16) is similar to the classical proof of embeddings between Lorentz spaces (see e.g. [BS, Chapter 4, Proposition 4.2]). Indeed,

$$\int_t^\infty h^*(s)\,ds = \int_t^\infty h^*(s)^{m'/p'}h^*(s)^{1-m'/p'}(s-t)^{1-m'/p'}(s-t)^{m'/p'-1}\,ds$$

$$\le \left(\sup_{y\in(t,\infty)} h^*(y)(y-t)\right)^{1-m'/p'}\int_t^\infty h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds.$$

However, for every $y \in (t, \infty)$, we have

$$h^*(y)(y-t) \approx h^*(y)\left(\int_t^y (s-t)^{m'/p'-1}\,ds\right)^{p'/m'}$$

$$\le \left(\int_t^y h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds\right)^{p'/m'}$$

$$\le \left(\int_t^\infty h^*(s)^{m'/p'}(s-t)^{m'/p'-1}\,ds\right)^{p'/m'}.$$

So, combining the last two estimates, we get (1.16), hence also (1.15). Now, using (1.15) and the Fubini theorem, we arrive at

$$(1.17) \qquad \left(\int_0^\infty \left(\frac{1}{t}\int_t^\infty g^{**}(s)^{p'}\,ds\right)^{m'/p'}\,dt\right)^{1/m'}$$

$$\le C\left(\int_0^\infty t^{-m'/p'}\int_t^\infty g^{**}(s)^{m'}(s-t)^{m'/p'-1}\,ds\,dt\right)^{1/m'}$$

$$= C\left(\int_0^\infty g^{**}(s)^{m'}\int_0^s t^{-m'/p'}(s-t)^{m'/p'-1}\,dt\,ds\right)^{1/m'}.$$

Changing variables, we get, for every fixed $s \in (0, \infty)$,

$$\int_0^s t^{-m'/p'} (s-t)^{m'/p'-1} \, dt = \int_0^1 y^{-m'/p'} (1-y)^{m'/p'-1} \, dy.$$

Thus, denoting

$$K = \int_0^1 y^{-m'/p'} (1-y)^{m'/p'-1} \, dy,$$

we obtain

$$\int_0^s t^{-m'/p'} (s-t)^{m'/p'-1} \, dt \leq K \quad \text{for every } s \in (0, \infty).$$

Plugging this into (1.17), we get the upper bound in (1.14). Altogether, also in this case, we conclude that

$$\|g\|_{G\Gamma(p,m,w)'} \approx \|g\|_{L^{m'}}.$$

Finally, let $1 < p < m < \infty$ and $b < \infty$. Then

$$\int_0^b w(s) s^{m/p} \, ds < \infty,$$

hence the weight is degenerate, and we have to use Theorem 1.1(vi). The first term on the right-hand side is equivalent to $\|g\|_{L^{m'}}$ just as in the preceding case, and the last one is obviously equivalent to $\|g\|_{L^1}$. Furthermore, the middle term disappears.

(d) If $1 < p < \infty$, $m = 1$, $b = 1$ and $w(t) = t^{-1} \left(\log \frac{2}{t}\right)^{-1/p}$, then $G\Gamma(p,m,w)$ coincides with the small Lebesgue space ([F], [FK]). Hence, Theorem 1.1 provides a new characterization of the grand Lebesgue space.

(e) A similar functional to the one in Theorem 1.1(ii) appears in [CP2, Theorem 1.2] in connection with a sharp Sobolev embedding into a Morrey space. Spaces generated by similar functionals are also treated in [Kr].

2. Proofs

Proof of Theorem 1.2. Fix $g \in L^1_{\text{loc}}(\mathcal{R}, \mu)$ and $t \in (0, b/2)$. Then

$$\frac{1}{t} \int_t^b g^{**}(s)^{p'} \, ds \geq \frac{1}{t} \int_t^{2t} g^{**}(s)^{p'} \, ds$$

$$= \frac{1}{t} \int_t^{2t} s^{-p'} \left(\int_0^s g^*(y) \, dy\right)^{p'} ds$$

$$\geq \frac{1}{t} \left(\int_0^t g^*(s) \, ds\right)^{p'} \int_t^{2t} \frac{ds}{s^{p'}}$$

$$= c g^{**}(t)^{p'}$$

with $c = (p-1)(1 - 2^{1-p'})$. Since the estimate

$$\frac{1}{t}\int\limits_t^b g^{**}(s)^{p'-1}g^*(s)\,ds \le \frac{1}{t}\int\limits_t^b g^{**}(s)^{p'}\,ds$$

follows immediately from (1.1), we obtain

$$g^{**}(t) + \left(\frac{1}{t}\int\limits_t^b g^{**}(s)^{p'-1}g^*(s)\,ds\right)^{1/p'} \le C\left(\frac{1}{t}\int\limits_t^b g^{**}(s)^{p'}\,ds\right)^{1/p'}$$

with C depending only on p. Conversely, integrating by parts, we get

$$\frac{1}{t}\int\limits_t^b g^{**}(s)^{p'-1}g^*(s)\,ds = \frac{1}{t}\int\limits_t^b \frac{1}{s^{p'-1}}\left(\int\limits_0^s g^*(y)\,dy\right)^{p'-1}g^*(s)\,ds$$

$$= \frac{1}{p't}\left(\lim_{s\to b_-}\frac{1}{s^{p'-1}}\left(\int\limits_0^s g^*(y)\,dy\right)^{p'} - \frac{1}{t^{p'-1}}\left(\int\limits_0^t g^*(s)\,ds\right)^{p'}\right)$$

$$+ \frac{1}{pt}\int\limits_t^b g^{**}(s)^{p'}\,ds$$

$$\ge \frac{1}{pt}\int\limits_t^b g^{**}(s)^{p'}\,ds - \frac{1}{p'}g^{**}(t)^{p'},$$

hence

$$g^{**}(t) + \left(\frac{1}{t}\int\limits_t^b g^{**}(s)^{p'-1}g^*(s)\,ds\right)^{1/p'} \ge c'\left(\frac{1}{t}\int\limits_t^b g^{**}(s)^{p'}\,ds\right)^{1/p'}$$

with a suitable $c' > 0$. The assertion now follows from the combination of both estimates. ∎

Proof of Theorem 1.1. Assume first that $b = \infty$. Rewriting the norm in (1.2) in a more convenient way and setting $h^* = (f^*)^p$, we get

$$\|g\|_{G\Gamma(p,m,w)'} = \sup_{f\not\equiv 0}\frac{\int_0^b f^*(t)g^*(t)\,dt}{\|f\|_{G\Gamma(p,m,w)}}$$

$$= \sup_{f\not\equiv 0}\frac{\int_0^b f^*(t)g^*(t)\,dt}{\left(\int_0^b w(t)\left(\int_0^t f^*(s)^p\,ds\right)^{m/p}\,dt\right)^{1/m}}$$

$$= \sup_{h\not\equiv 0}\frac{\int_0^b h^*(t)^{1/p}g^*(t)\,dt}{\left(\int_0^b w(t)\left(\int_0^t h^*(s)\,ds\right)^{m/p}\,dt\right)^{1/m}}$$

$$= \sup_{h\not\equiv 0}\frac{\int_0^b h^*(t)^{1/p}g^*(t)\,dt}{\left(\int_0^b h^{**}(t)^{m/p}t^{m/p}w(t)\,dt\right)^{1/m}}.$$

Raising this to the power p, we arrive at

$$\|g\|_{G\Gamma(p,m,w)'}^p = \sup_{h \not\equiv 0} \frac{\left(\int_0^b h^*(t)^{1/p} g^*(t)\, dt\right)^p}{\left(\int_0^b h^{**}(t)^{m/p} t^{m/p} w(t)\, dt\right)^{p/m}}.$$

Let $0 < m \le 1$. Then, by a slight modification of [GP, Theorem 4.2(i)] and its proof, we obtain

$$\|g\|_{G\Gamma(p,m,w)'}^p \approx \sup_{t \in (0,b)} \frac{\left(\int_0^t g^*(s)\, ds\right)^p}{u(t)^{p/m}}.$$

Taking the pth root, we get

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t \in (0,b)} g^{**}(t) \frac{t}{u(t)^{1/m}}.$$

Since $b = \infty$, and therefore, by our convention, also $b/2 = \infty$, this completes the proof of (i).

If $1 < p < \infty$ and $0 < m \le 1$, then [GP, Theorem 4.2(iii)] yields

$$\|g\|_{G\Gamma(p,m,w)'}^p \approx \sup_{t \in (0,b)} \frac{\left(\int_0^t g^*(s)\, ds\right)^p + t\left(\int_t^b \left(\int_0^s g^*(y)\, dy\right)^{p'-1} g^*(s) s^{1-p'}\, ds\right)^{p-1}}{u(t)^{p/m}}$$

$$= \sup_{t \in (0,b)} \frac{t^p g^{**}(t)^p + t\left(\int_t^b g^{**}(s)^{p'-1} g^*(s)\, ds\right)^{p-1}}{u(t)^{p/m}}.$$

Thus,

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t \in (0,b)} \left(g^{**}(t) + \left(\frac{1}{t}\int_t^b g^{**}(s)^{p'-1} g^*(s)\, ds\right)^{1/p'}\right) \frac{t}{u(t)^{1/m}}.$$

By Theorem 1.2, this yields

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t \in (0,b)} \left(\int_t^b g^{**}(s)^{p'}\, ds\right)^{1/p'} \frac{t^{1/p}}{u(t)^{1/m}},$$

establishing (ii).

Now assume that $1 < m < \infty$, $0 < p \le 1$, and (1.5) is satisfied. Then, using [GP, Theorem 4.2(ii)], we get

$$\|g\|_{G\Gamma(p,m,w)'}^p$$

$$\approx \left(\int_0^b \sup_{y \in (t,b)} \left(\int_0^y g^*(\tau)\, d\tau\right)^{m'} y^{-m'/p} \frac{t^{m'/p+m/p-1} \int_0^t w(s) s^{m/p}\, ds \int_t^b w(s)\, ds}{u(t)^{m'+1}}\, dt\right)^{p/m'}$$

$$\approx \left(\int_0^b \sup_{y \in (t,b)} g^{**}(y)^{m'} y^{m'(p-1)/p} \frac{t^{m'/p+m/p-1} \int_0^t w(s) s^{m/p}\, ds \int_t^b w(s)\, ds}{u(t)^{m'+1}}\, dt\right)^{p/m'}.$$

Since $p \leq 1$, the expression $g^{**}(y)^{m'} y^{m'(p-1)/p}$ is in fact non-increasing on (t, b), hence it takes its largest value at t. With a little algebra, (iii) follows.

Next, let $1 < p, m < \infty$ and let (1.5) hold. Then, by [GP, Theorem 4.2(iv)], we have

$$\|g\|_{G\Gamma(p,m,w)'}^p$$
$$\approx \left(\int_0^b \frac{\left(\left(\int_0^t g^*(s)\, ds \right)^{p'} + t^{p'-1} \int_t^b \left(\int_0^s g^*(y)\, dy \right)^{p'-1} g^*(s) s^{1-p'}\, ds \right)^{m'/p'}}{u(t)^{m'+1}} \right.$$
$$\left. \times t^{m/p-1} \int_0^t s^{m/p} w(s)\, ds \int_t^b w(s)\, ds\, dt \right)^{p/m'}$$
$$\approx \left(\int_0^b \left(g^{**}(t) + \left(\frac{1}{t} \int_t^b g^{**}(s)^{p'-1} g^*(s)\, ds \right)^{1/p'} \right)^{m'} \right.$$
$$\left. \times \frac{t^{m/p-1} \int_0^t s^{m/p} w(s)\, ds \int_t^b w(s)\, ds}{u(t)^{m'+1}}\, dt \right)^{p/m'}.$$

By Theorem 1.2, this implies

$$\|g\|_{G\Gamma(p,m,w)'}^p$$
$$\approx \left(\int_0^b \left(\frac{1}{t} \int_t^b g^{**}(s)^{p'}\, ds \right)^{m'/p'} \frac{t^{m/p+m'-1} \int_0^t s^{m/p} w(s)\, ds \int_t^b w(s)\, ds}{u(t)^{m'+1}}\, dt \right)^{p/m'},$$

and (v) follows on taking the pth root.

If $1 < m < \infty$ and (1.5) is violated, then, in order to prove the statements (iv) and (vi), the results of [GP] cannot be used directly, because degenerate weights are not treated there. In this case we have either to use the result of Sinnamon [Si] or modify the argument in [GP]. We omit the technical details.

Now let $b < \infty$. Then, in order to finish the proof of (i), we need to show that

$$\sup_{t \in (0,b)} g^{**}(t) \frac{t}{u(t)^{1/m}} \approx \sup_{t \in (0,b/2)} g^{**}(t) \frac{t}{u(t)^{1/m}}.$$

To this end, denote

$$K = \left(\frac{u(b/3)}{\int_0^{b/2} w(s) s^{m/p}\, ds} \right)^{1/m}.$$

Then, for every $t \in [b/2, b)$, one has

$$\frac{t}{u(t)^{1/m}} < \frac{b}{\left(\int_0^{b/2} w(s) s^{m/p}\, ds \right)^{1/m}} = 3K \frac{b/3}{u(b/3)^{1/m}}.$$

Thus, using also the fact that g^{**} is non-increasing on $(0, b)$, we get, for every $t \in [b/2, b)$,

$$g^{**}(t) \frac{t}{u(t)^{1/m}} \leq 3K g^{**}(b/3) \frac{b/3}{u(b/3)^{1/m}} \leq 3K \sup_{t \in (0, b/2)} g^{**}(t) \frac{t}{u(t)^{1/m}}.$$

Consequently,

$$\sup_{t \in (0,b)} g^{**}(t) \frac{t}{u(t)^{1/m}} \leq \max\{1, 3K\} \sup_{t \in (0, b/2)} g^{**}(t) \frac{t}{u(t)^{1/m}}.$$

Since the converse inequality is trivial, this completes the proof of (i). The proof of the remaining statements is analogous and therefore omitted. ∎

Proof of Theorem 1.3. First, the 'only if' part of the assertion follows simply on testing the norm in $G\Gamma(p, m, w)$ on characteristic functions of sets of finite measure.

Let us prove the 'if' part. All the assertions in (P1) except the triangle inequality are obvious. Fix $t \in (0, b)$ and let f, g be μ-measurable real functions on $\mathcal{R}$. Denote

$$E_t = \{x \in \mathcal{R} \colon f(x) + g(x) > (f + g)^*(t)\}.$$

Then $\mu(E_t) \leq t$ [BS, Chapter 2, Proposition 1.7]. Combining this fact with the Minkowski inequality for the norm in the space $L^p(E_t)$ and the Hardy–Littlewood inequality [BS, Chapter 2, Theorem 2.2], we obtain

$$\left(\int_0^t (f + g)^*(s)^p \, ds \right)^{1/p} = \left(\int_{E_t} (f + g)(s)^p \, d\mu \right)^{1/p}$$

$$\leq \left(\int_{E_t} f(s)^p \, d\mu \right)^{1/p} + \left(\int_{E_t} g(s)^p \, d\mu \right)^{1/p}$$

$$\leq \left(\int_0^{\mu(E_t)} f^*(s)^p \, ds \right)^{1/p} + \left(\int_0^{\mu(E_t)} g^*(s)^p \, ds \right)^{1/p}$$

$$\leq \left(\int_0^t f^*(s)^p \, ds \right)^{1/p} + \left(\int_0^t g^*(s)^p \, ds \right)^{1/p}.$$

Therefore,

$$\|f + g\|_{G\Gamma(p,m,w)} = \left(\int_0^b \left(\int_0^t (f + g)^*(s)^p \, ds \right)^{m/p} w(t) \, dt \right)^{1/m}$$

$$= \left(\int_0^b \|(f + g)^*\|_{L^p(0,t)}^m w(t) \, dt \right)^{1/m}$$

$$\leq \left\| \|g^*\|_{L^p(0,t)} + \|f^*\|_{L^p(0,t)} \right\|_{L_w^m(0,b)}$$

$$\leq \left\| \|g^*\|_{L^p(0,t)} \right\|_{L^m_w(0,b)} + \left\| \|f^*\|_{L^p(0,t)} \right\|_{L^m_w(0,b)}$$

$$= \|f\|_{G\Gamma(p,m,w)} + \|g\|_{G\Gamma(p,m,w)},$$

as desired.

Next, (P2) follows immediately from the definition and (P3) from the Monotone Convergence Theorem applied first to the inner integral and then on the outer one.

As for (P4) and (P5), let E be a subset of $\mathcal{R}$ of finite measure. Then

$$\|\chi_E\|_{G\Gamma(p,m,w)} = \left(\int_0^b \min(t, \mu(E))^{m/p} w(t)\, dt \right)^{1/m} < \infty,$$

which establishes (P4).

Finally, if $b = \infty$ and f is a non-negative measurable function on $\mathcal{R}$, then

$$\left(\int_0^b \left(\int_0^t f^*(s)^p\, ds \right)^{m/p} w(t)\, dt \right)^{1/m} \geq \left(\int_{\mu(E)}^b \left(\int_0^{\mu(E)} f^*(s)^p\, ds \right)^{m/p} w(t)\, dt \right)^{1/m}$$

$$\geq \left(\int_{\mu(E)}^b \left(\int_E f(s)^p\, ds \right)^{m/p} w(t)\, dt \right)^{1/m}$$

$$= \|f\|_{L^p(E)} \left(\int_{\mu(E)}^b w(t)\, dt \right)^{1/m}$$

$$\geq C_E \|f\|_{L^1(E)} \left(\int_{\mu(E)}^b w(t)\, dt \right)^{1/m},$$

for an appropriate C_E, while, when $b < \infty$, we have

$$\left(\int_0^b \left(\int_0^t f^*(s)^p\, ds \right)^{m/p} w(t)\, dt \right)^{1/m} \geq \left(\int_{b/2}^b \left(\int_0^{b/2} f^*(s)^p\, ds \right)^{m/p} w(t)\, dt \right)^{1/m}$$

$$= \left(\int_0^{b/2} f^*(s)^p\, ds \right)^{1/p} \left(\int_{b/2}^b w(t)\, dt \right)^{1/m}$$

$$\geq \frac{1}{2} \left(\int_0^b f^*(s)^p\, ds \right)^{1/p} \left(\int_{b/2}^b w(t)\, dt \right)^{1/m}$$

$$\geq C_E \|f\|_{L^1(E)} \left(\int_{\mu(E)}^b w(t)\, dt \right)^{1/m},$$

showing (P5) again. ■

Proof of Theorem 1.4. The assumption (1.10) obviously implies (1.9). Therefore, we know from Theorem 1.3 that $G\Gamma(m, p, w)(0, b)$ is a rearrangement-invariant Banach function space. We can thus apply [CPS1, Theorem 6.1] (for the first-order case see also [CP1, Theorem 3.5]), which states that the Sobolev embedding (1.11) is equivalent to the condition

$$(2.1) \qquad\qquad t^{-1+k/n} \in G\Gamma(m, p, w)'(0, b).$$

So, we only have to analyze when (2.1) is satisfied.

First note that if $k \geq n$, then in fact obviously

$$t^{-1+k/n} \in L^\infty(0, b),$$

which immediately implies (2.1), since, by a classical fact, the space L^∞ is embedded into any rearrangement-invariant space over a finite-measure space (and we have $b < \infty$ here).

Assume now that $k \leq n - 1$. We then denote $g(t) = t^{-1+k/n}$ for $t \in (0, b)$ and note that $g^{**} \approx g^* = g$ on $(0, b)$.

Let $m = 1$. Then it follows from Theorem 1.1(i)&(ii) that

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t \in (0, b/2)} \frac{t^{k/n}}{u(t)}$$

if $p = 1$, and

$$\|g\|_{G\Gamma(p,m,w)'} \approx \sup_{t \in (0, b/2)} \left(\int_t^b g^{**}(s)^{p'} \, ds \right)^{1/p'} \frac{t^{1/p}}{u(t)}$$

if $p \in (1, \infty)$. Now, a calculation shows that, for $t \in (0, b/2)$, we have

$$\left(\int_t^b g^{**}(s)^{p'} \, ds \right)^{1/p'} \approx \left(\int_t^b s^{(-1+k/n)p'} \, ds \right)^{1/p'}$$

$$\approx \begin{cases} t^{k/n - 1/p} & \text{if } 1 < p < n/k, \\ \left(\log \frac{b}{t} \right)^{1-k/n} & \text{if } p = n/k, \\ 1 & \text{if } p \in (n/k, \infty). \end{cases}$$

This establishes (i)–(iii). The remaining three statements can be proved in an analogous way. ∎

Proof of Theorem 1.5. Assume first that (1.12) holds. Let $\{E_n\}$ be a sequence of μ-measurable subsets of $\mathcal{R}$ with $E_n \downarrow \emptyset$, and let $f \in G\Gamma(p, m, w)$. Then

$$\|f\chi_{E_n}\|_{G\Gamma(p,m,w)}^m = \int_0^b \left(\int_0^t (f\chi_{E_n})^*(s)^p \, ds \right)^{m/p} w(t) \, dt$$

$$= \int_0^b \left(\int_0^{\min(t, \mu(E_n))} f^*(s)^p \, ds \right)^{m/p} w(t) \, dt.$$

Since $b < \infty$, $E_n \downarrow \emptyset$ implies $\mu(E_n) \downarrow 0$. Therefore,

$$\lim_{n \to \infty} \int\limits_0^{\min(t, \mu(E_n))} f^*(s)^p \, ds = 0$$

for all $t \in (0, b)$. Consequently,

$$\lim_{n \to \infty} \left(\int\limits_0^t (f\chi_{E_n})^*(s)^p \, ds \right)^{m/p} w(t) = 0.$$

By the Dominated Convergence Theorem with $(\int_0^t f^*(s)^p \, ds)^{m/p} w(t)$ as an integrable majorant, we obtain $\|f\chi_{E_n}\|_{G\Gamma(p,m,w)} \to 0$, as desired.

Assume now that (1.13) is satisfied and $b = \infty$. Then, by the assumption, for every $f \in G\Gamma(p, m, w)$ and $k \in \mathbb{N}$, the set $F_k = \{x \in \mathcal{R} : f(x) \geq 1/k\}$ has finite measure. Let $\{E_n\}$ be a sequence of μ-measurable subsets of $\mathcal{R}$ satisfying $E_n \downarrow \emptyset$. Set $f_n = f\chi_{E_n}$, $f_{n,k} = f_n\chi_{F_k}$, and choose $\varepsilon > 0$. Then

$$\|f_n\|_{G\Gamma(p,m,w)} \leq \|f_n - f_{n,k}\|_{G\Gamma(p,m,w)} + \|f_{n,k}\|_{G\Gamma(p,m,w)}.$$

Fix $k \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$,

$$\|f_n - f_{n,k}\|_{G\Gamma(p,m,w)}^m = \int\limits_0^\infty \left(\int\limits_0^t (|f - f\chi_{F_k}|\chi_{E_n})^*(s)^p \, ds \right)^{m/p} w(t) \, dt$$

$$\geq \int\limits_0^\infty \left(\int\limits_0^t (|f - f\chi_{F_k}|\chi_{E_{n+1}})^*(s)^p \, ds \right)^{m/p} w(t) \, dt$$

$$= \|f_{n+1} - f_{n+1,k}\|_{G\Gamma(p,m,w)}^m.$$

Now, for a change, fix $n \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$,

$$\|f_n - f_{n,k}\|_{G\Gamma(p,m,w)}^m \leq \int\limits_0^\infty \left(\int\limits_0^t (\min(f(y), 1/k))^*(s)^p \, ds \right)^{m/p} w(t) \, dt.$$

For every $t > 0$ we clearly have

$$\lim_{k \to \infty} \left(\int\limits_0^t (\min(f(s), 1/k))^p \, ds \right)^{m/p} w(t) = 0.$$

Therefore,

$$\lim_{k \to \infty} \|f_n - f_{n,k}\|_{G\Gamma(p,m,w)} = 0,$$

by the Dominated Convergence Theorem. Observe that, for every $k \in \mathbb{N}$,

$$\lim_{n \to \infty} \|f_{n,k}\|_{G\Gamma(p,m,w)} = 0,$$

which follows from the first part of the proof since the sets F_k have finite measure.

We first choose $k \in \mathbb{N}$ such that $\|f_1 - f_{1,k}\|_{G\Gamma(p,m,w)} < \varepsilon$. With this k now fixed, we find $n_0 \in \mathbb{N}$ such that $\|f_{n,k}\|_{G\Gamma(p,m,w)} < \varepsilon$ for all $n > n_0$. Then

$$
\begin{aligned}
\|f_n\|_{G\Gamma(p,m,w)} &\leq \|f_n - f_{n,k}\|_{G\Gamma(p,m,w)} + \|f_{n,k}\|_{G\Gamma(p,m,w)} \\
&\leq \|f_{1,k} - f_1\|_{G\Gamma(p,m,w)} + \|f_{n,k}\|_{G\Gamma(p,m,w)} \\
&\leq 2\varepsilon,
\end{aligned}
$$

establishing the 'if' part of the theorem.

To prove the 'only if' part, assume that $b = \infty$ and $\int_0^\infty w(t)t^{m/p}\,dt < \infty$. Since $\mathcal{R}$ is σ-finite, there exists a sequence of finite-measure sets $\{D_n\}$ satisfying $D_n \uparrow \mathcal{R}$. For $n \in \mathbb{N}$, define $E_n = \mathcal{R} \setminus D_n$, and set $f \equiv 1$ on $\mathcal{R}$. Then $E_n \downarrow \emptyset$ and, for every $n \in \mathbb{N}$, $(f\chi_{E_n})^* \equiv 1$ on $(0,\infty)$. Therefore, for every $n \in \mathbb{N}$, we have

$$
\|f\|_{G\Gamma(p,m,w)} = \|f\chi_{E_n}\|_{G\Gamma(p,m,w)} = \int\limits_0^\infty w(t)t^{m/p}\,dt,
$$

which means, due to the assumption, that f belongs to $G\Gamma(p,m,w)$ but does not have absolutely continuous norm. ∎

Proof of Theorem 1.6. Let $p, m \in (1,\infty)$ and let w be a weight on $(0,b)$. Assume first that (1.5) holds. Then, by Theorem 1.1(v),

$$
\|g\|_{(G\Gamma(p,m,w))'}
$$

$$
\approx \left(\int\limits_0^{b/2} \left(\int\limits_t^b g^{**}(s)^{p'}\,ds \right)^{m'/p'} \frac{t^{m'/p+m/p-1} \int_0^t w(s)s^{m/p}\,ds \int_t^b w(s)\,ds}{u(t)^{m'+1}}\,dt \right)^{1/m'}.
$$

Let $\{E_n\}$ be a sequence of sets such that $E_n \downarrow \emptyset$. Denote $f_n = f\chi_{E_n}$ and $F_n(t) = \int_0^t f_n^*(s)\,ds$, $t \in (0,b)$. For every $f \in G\Gamma(p,m,w)'$, the right side of the last displayed formula is finite. Therefore, by the Dominated Convergence Theorem, it only suffices to verify that

$$
\lim_{n\to\infty} F_n(t) = 0 \quad \text{for every } t \in (0,\infty).
$$

Fix $t \in (0,b)$. Then the sequence $F_n(t)$ is non-increasing. Therefore the limit $\lim_{n\to\infty} F_n(t)$ exists. Suppose that $\lim_{n\to\infty} F_n(t) > \varepsilon$ for some $\varepsilon > 0$. Then the sets

$$
P_n = \left\{ s \in (0,t)\colon f_n^*(s) > \frac{\varepsilon}{2t} \right\}
$$

have positive measure. Clearly, $P_n \supset P_{n+1}$ for every $n \in \mathbb{N}$. Moreover,

$$
\int\limits_{(0,t)\setminus P_n} f_n^*(s)\,ds \leq \frac{\varepsilon}{2},
$$

hence

$$\int\limits_{P_n} f_n^*(s)\,ds \geq \frac{\varepsilon}{2}.$$

Furthermore, if $|P_n| \to 0$ then

$$\int\limits_{P_n} f^*(s)\,ds \geq \int\limits_{P_n} f_n^*(s)\,ds \geq \frac{\varepsilon}{2},$$

which is impossible due to the absolute continuity of the Lebesgue integral. So, the only option left is $|\bigcap P_n| > 0$. That, however, leads to a contradiction since

$$\Big|\bigcap P_n\Big| = \mu\{x \in \mathcal{R} \colon f_n(x) > \varepsilon \text{ for every } n \in \mathbb{N}\}.$$

Therefore $\lim_{n\to\infty} F_n(t) = 0$.

If (1.5) is violated, then the above proof works just as well, the only extra observation we have to make is that all functions in $L^{p'}$ and in L^1 have absolutely continuous norms. ∎

Proof of Theorem 1.7. A Banach function space X is reflexive if and only if both X and its associate space X' have absolutely continuous norm [BS, Chapter 1, Corollary 4.4]. Thus, the assertion follows from Theorems 1.5 and 1.6. ∎

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Amiran Gogatishvili
Institute of Mathematics
Academy of Sciences of the Czech Republic
Žitná 25
115 67 Praha 1, Czech Republic
E-mail: gogatish@math.cas.cz

Luboš Pick, Filip Soudský
Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: pick@karlin.mff.cuni.cz
soudsky@karlin.mff.cuni.cz

NORMABILITY OF LORENTZ SPACES—
AN ALTERNATIVE APPROACH

AMIRAN GOGATISHVILI, FILIP SOUDSKÝ, Praha

Abstract. We study normability properties of classical Lorentz spaces. Given a certain general lattice-like structure, we first prove a general sufficient condition for its associate space to be a Banach function space. We use this result to develop an alternative approach to Sawyer's characterization of normability of a classical Lorentz space of type Λ. Furthermore, we also use this method in the weak case and characterize normability of Λ_v^∞. Finally, we characterize the linearity of the space Λ_v^∞ by a simple condition on the weight v.

Keywords: weighted Lorentz space; weighted inequality; non-increasing rearrangement; Banach function space; associate space

MSC 2010: 46E30

1. INTRODUCTION

Classical Lorentz spaces were introduced by Lorentz in 1951 in [6]. Their normability and duality properties have been intensively studied since 1990 when Sawyer in [7] determined when a classical Lorentz space of type Λ is equivalent to a Banach space. It turns out that a classical Lorentz space of type Λ need not in general be normable and even does not have to be necessarily a linear set (see [3]), similarly for the space of weak type.

In this paper we present an alternative approach to this problem, using duality methods based on properties of associate spaces to rather general structures. In

our first main result we characterize when the set defined as an associate space to a certain structure of lattice type has the properties required by the definition of the so-called Banach function norm (definitions are given in Section 2 below). We then apply this general result to the specific case of the classical Lorentz space, obtaining thereby a new proof of Sawyer's result. We then turn our attention to the classical Lorentz space of weak type, studied for example in [2] and [4]. We give a necessary and sufficient condition for the normability of this space.

The paper is structured as follows. In the following section we give some background material and fix notation. In Section 3 we recall the results of general nature concerning Banach function spaces. In Section 4 we state and prove our main results concerning the classical Lorentz spaces. Finally, in Section 5 we state and prove our results concerning weak-type spaces.

2. Preliminaries

Throughout the paper we shall always consider a σ-finite nonatomic underlying measure space $(\mathbb{R}, \mu)$. The symbol $\mathcal{M}(\mathbb{R})$ will always be used to denote the set of all real-valued μ-measurable functions on $\mathbb{R}$. For $f \in \mathcal{M}(\mathbb{R})$ we shall consider the *distribution function* defined by

$$\lambda_f(s) := \mu(\{|f| > s\}), \quad s \in (0, \infty),$$

the *nonincreasing rearrangement* of f defined by

$$f^*(s) := \inf\{\lambda_f \leqslant s\}, \quad s \in (0, \infty),$$

and

$$f^{**}(s) := \frac{1}{s} \int_0^s f^*(t) \, dt, \quad s \in (0, \infty).$$

The set of all simple functions on $\mathbb{R}$ will be denoted by

$$\mathcal{S}(\mathbb{R}) := \left\{ f \colon f = \sum_{i=1}^n a_i \chi_{A_i} \colon \mu(A_k) < \infty \right\}.$$

Moreover, if $\mu(\mathbb{R}) < \infty$, then we set $f^*(s) := 0$ for $s > \mu(\mathbb{R})$. The expression *weight* will always refer to a locally integrable nonnegative function defined on $(0, \infty)$, positive on $(0, \delta)$ for some $\delta > 0$ and with $v(s) = 0$ for $s \in (\mu(\mathbb{R}), \infty)$. In the following text we shall also use capitals U, V, W for functions defined as

$$U(t) := \int_0^t u(s) \, ds,$$

$$V(t) := \int_0^t v(s) \, ds,$$

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and
$$W(t) := \int_0^t w(s) \, \mathrm{d}s, \quad t \in (0, \infty).$$

The symbol p' will always denote the associate exponent to $p \in (1, \infty)$ defined by $p' = p/(p-1)$.

Definition 2.1. Let $(\mathbb{R}, \mu)$ be a nonatomic σ-finite measure space. Let us consider a functional $\|\cdot\|_X \colon \mathcal{M}(\mathbb{R}) \to [0, \infty]$ and set $X := \{f \in \mathcal{M}(\mathbb{R}) \colon \|f\|_X < \infty\}$. Let us consider the following properties.

(P1) $\|\cdot\|_X$ is a norm on X.
(P2) If $|f| \geqslant |g|$ a.e., then $\|f\|_X \geqslant \|g\|_X$.
(P3) If $0 \leqslant f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$.
(P4) $\|\chi_E\|_X < \infty$, whenever $\mu(E) < \infty$.
(P5) For every set E of a finite measure, there exists a constant C_E such that
$$\|f\chi_E\|_X \geqslant C_E \int_E |f| \, \mathrm{d}\mu.$$

(P6) If $f^*(s) = g^*(s)$ for every $s \in (0, \mu(\mathbb{R}))$, then $\|f\|_X = \|g\|_X$.

We call X

(1) a *Banach function space* if (P1)–(P5) are satisfied;
(2) a *rearrangement-invariant Banach function space* if (P1)–(P6) are satisfied;
(3) a *rearrangement-invariant lattice* if $\|\cdot\|_X$ is a positively homogeneous functional and (P2), (P3) and (P6) are satisfied.

Remark 2.1. If $\|\cdot\|_X$ satisfies (P2), it easily follows that $|f| = |g|$ implies $\|f\|_X = \|g\|_X$.

Definition 2.2. Let $\|\cdot\|_X \colon \mathcal{M}(\mathbb{R}) \to [0, \infty]$ be a functional. For $f \in \mathcal{M}(\mathbb{R})$ define
$$\|f\|_{X'} := \sup_{g \in X} \frac{\int_{\mathbb{R}} fg \, \mathrm{d}\mu}{\|g\|_X}$$
and
$$\|f\|_{X''} := \sup_{g \in X'} \frac{\int_{\mathbb{R}} fg \, \mathrm{d}\mu}{\|g\|_{X'}}$$
(following the convention $0/0 = \infty/\infty = 0$).

Definition 2.3. Let $\|\cdot\|_X$ have the properties (P2), (P3) and (P6). For $t \in (0, \infty)$ we define the fundamental function by
$$\varphi_X(t) := \|\chi_E\|_X, \quad \text{where } \mu(E) = t.$$

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Definition 2.4. Let $\|\cdot\|_X, \|\cdot\|_Y \colon \mathcal{M}(\mathbb{R}) \to [0, \infty]$ and let

$$X := \{f \in \mathcal{M}(\mathbb{R}) \colon \|f\|_X < \infty\}$$

and

$$Y := \{f \in \mathcal{M}(\mathbb{R}) \colon \|f\|_Y < \infty\}.$$

Define

$$\mathrm{Opt}(X, Y) := \sup_{f \in X} \frac{\|f\|_Y}{\|f\|_X}$$

(following the convention $0/0 = \infty/\infty = 0$).

3. General duality theorems

We first present a simple sufficient condition for the identity $X = X''$. This result is of independent interest but also will be very useful for the proofs in the next chapters.

Theorem 3.1. *Let $\|\cdot\|_X \colon \mathcal{M}(\mathbb{R}) \to [0, \infty]$ be a functional with the following properties.*

(1) *If $\|f\|_X = \|\|f\|\|_X$.*
(2) *$\|\chi_E\|_X < \infty$ whenever $\mu(E) < \infty$.*
(3) *For every E of finite measure there exists $\infty > C_E > 0$ such that*

$$C_E \|f\chi_E\|_X \geqslant \int_E |f|\,\mathrm{d}\mu.$$

Then the functional $\|\cdot\|_{X'}$ is a Banach function norm.

Moreover, $\|\cdot\|_X$ is equivalent to a Banach function norm if and only if $\|\cdot\|_X \approx \|\cdot\|_{X''}$.

P r o o f. Let us first assume $\|\cdot\|_X \approx \|\cdot\|_{X''}$. We shall verify that $\|\cdot\|_{X'}$ is a Banach function norm. Let $f_1, f_2 \in X'$ and $g \in X$, obviously

$$\frac{\int_{\mathbb{R}}(f_1 + f_2)g\,\mathrm{d}\mu}{\|g\|_X} = \frac{\int_{\mathbb{R}} f_1 g\,\mathrm{d}\mu}{\|g\|_X} + \frac{\int_{\mathbb{R}} f_2 g\,\mathrm{d}\mu}{\|g\|_X} \leqslant \sup_{g \in X}\frac{\int_{\mathbb{R}} f_1 g\,\mathrm{d}\mu}{\|g\|_X} + \sup_{g \in X}\frac{\int_{\mathbb{R}} f_2 g\,\mathrm{d}\mu}{\|g\|_X}.$$

Passing to the supremum on the left-hand side proves

$$\|f_1 + f_2\|_{X'} \leqslant \|f_1\|_{X'} + \|f_2\|_{X'}.$$

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If $\mu(\{|f| > 0\}) > 0$, then there exists $\varepsilon > 0$ such that $\mu(\{|f| > \varepsilon\}) > 0$. Let us consider $A \subset \{|f| > \varepsilon\}$ with $\mu(A) > 0$. Then

$$0 < \frac{\int_{\mathbb{R}} \varepsilon \chi_A \, \mathrm{d}\mu}{\|\chi_A\|_X} \leqslant \frac{\int_{\mathbb{R}} f \chi_A \cdot \mathrm{sgn}(f) \, \mathrm{d}\mu}{\|\chi_A\|_X} \leqslant \|f\|_{X'}.$$

Since the homogeneity is obvious, we have that $\|\cdot\|_{X'}$ is a norm. Now, if $|f| \geqslant |g|$ a.e., then (due to assumption (1)) for every $h \in X$ we have $\|h\|_X = \||h|\,\mathrm{sgn}(f)\|_X$, and therefore

$$\frac{\int_{\mathbb{R}} gh \, \mathrm{d}\mu}{\|h\|_X} \leqslant \frac{\int_{\mathbb{R}} |gh| \, \mathrm{d}\mu}{\|h\|_X} \leqslant \frac{\int_{\mathbb{R}} |f||h| \, \mathrm{d}\mu}{\|h\|_X} = \frac{\int_{\mathbb{R}} f \, \mathrm{sgn}(f) |h| \, \mathrm{d}\mu}{\|h\|_X}$$

$$= \frac{\int_{\mathbb{R}} f \, \mathrm{sgn}(f) |h| \, \mathrm{d}\mu}{\||h|\,\mathrm{sgn}(f)\|_X} \leqslant \|f\|_{X'}.$$

Passing to the supremum over h on the left-hand side gives (P2) for X'. Property (P3) is an easy consequence of the monotone convergence theorem. Let $\mu(E) < \infty$ and let C_E be the constant from property (3) of X. Then

$$\frac{\int_{\mathbb{R}} \chi_E g \, \mathrm{d}\mu}{\|g\|_X} \leqslant C_E < \infty.$$

Passing to the supremum over $g \in X$ we obtain (P4). Choose E with $\mu(E) < \infty$ and $g \in X'$ such that

$$\int_E g \, \mathrm{d}\mu = \int_{\mathbb{R}} g \chi_E \chi_E \, \mathrm{d}\mu \leqslant \|g \chi_E\|_{X'} \|\chi_E\|_X = C_E \|g \chi_E\|_{X'}$$

and that proves (P5) for X'. If X' is a BFS then X'' is also a BFS.

Let us now assume that $\|\cdot\|_X$ is equivalent to some Banach function norm $\|\cdot\|_Y$. Then, obviously $\|\cdot\|_{X'} \approx \|\cdot\|_{Y'}$. And hence $\|\cdot\|_{X''} \approx \|\cdot\|_{Y''} = \|\cdot\|_Y \approx \|\cdot\|_X$. The proof is complete. $\qquad\square$

Lemma 3.1. *Define* $X := \{f \in \mathcal{M}(\mathbb{R}): \|f\|_X < \infty\}$, *where* $\|\cdot\|_X$ *satisfies the conditions of Theorem 3.1. Then* $X \hookrightarrow X''$.

P r o o f. The proof is analogous to the one in [1], Theorem 2.7. $\qquad\square$

Lemma 3.2. Let X_0, X_1, Y be rearrangement invariant lattices. Let (X_0, X_1) be a compatible couple. Then

$$\mathrm{Opt}(X_0 + X_1, Y) \approx \mathrm{Opt}(X_0, Y) + \mathrm{Opt}(X_1, Y).$$

P r o o f.

$$\mathrm{Opt}(X_0 + X_1, Y) = \sup_f \frac{\|f\|_Y}{\inf_{f = f_1 + f_2} (\|f_1\|_{X_0} + \|f_2\|_{X_1})}.$$

We search for the optimal constant of the embedding

(3.1) $$\|f\|_Y \leqslant C(\|f_1\|_{X_1} + \|f - f_1\|_{X_0}),$$

where f, f_1 are arbitrary measurable functions. Since we have the assumption (P2), the following holds

$$\|f\|_Y \leqslant \|(|f_1| + |f - f_1|)\|_Y.$$

Therefore, to prove (3.1) it is enough, in fact, to prove

$$\|(|f_1| + |f - f_1|)\|_Y \leqslant C(\||f_1|\|_{X_1} + \||f - f_1|\|_{X_0}).$$

Thus we may suppose $f \geqslant 0$, $f_1 \geqslant 0$, and $f - f_1 \geqslant 0$. We have

$$\frac{1}{2}(\|f_1\|_Y + \|f - f_1\|_Y) \leqslant \|f\|_Y \leqslant \|f_1\|_Y + \|f - f_1\|_Y.$$

Since

$$\sup_{f_1, f_2 \geqslant 0} \frac{\|f_1\|_Y + \|f_2\|_Y}{\|f_1\|_{X_0} + \|f_2\|_{X_1}} \approx \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_0}} + \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_1}},$$

the inequality $\gtrsim$ is obtained immediately, since the sum of the two suprema on the right-hand side is equivalent to its maximum, which is attained if we set $f_1 = 0$ or $f_2 = 0$. Since the other inequality is obvious, we have

$$\mathrm{Opt}(X_0 + X_1, Y) = \sup_{f \geqslant g \geqslant 0} \frac{\|g\|_Y + \|f - g\|_Y}{(\|g\|_{X_0} + \|f - g\|_{X_1})} \approx \sup_{f_1, f_2 \geqslant 0} \frac{\|f_1\|_Y + \|f_2\|_Y}{\|f_1\|_{X_0} + \|f_2\|_{X_1}}$$

$$\approx \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_0}} + \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_1}} = \mathrm{Opt}(X_0, Y) + \mathrm{Opt}(X_1, Y).$$

$\square$

Definition 4.1. Let $p \in (1, \infty)$, $\mu(\mathbb{R}) = \infty$ and let v be a weight. Define

$$\Lambda_v^p := \left\{ f \in \mathcal{M} \colon \|f\|_{\Lambda_v^p} := \left(\int_0^\infty f^*(s)^p v(s) \, \mathrm{d}s \right)^{1/p} < \infty \right\},$$

$$\Gamma_v^p := \left\{ f \in \mathcal{M} \colon \|f\|_{\Gamma_v^p} := \left(\int_0^\infty f^{**}(s)^p v(s) \, \mathrm{d}s \right)^{1/p} < \infty \right\},$$

$$\Lambda_v^\infty := \left\{ f \in \mathcal{M}(\mathbb{R}) \colon \|f\|_{\Lambda_v^\infty} := \operatorname*{ess\,sup}_{s>0} f^*(s) v(s) < \infty \right\},$$

and

$$\Gamma_v^\infty := \left\{ f \in \mathcal{M}(\mathbb{R}) \colon \|f\|_{\Gamma_v^\infty} := \operatorname*{ess\,sup}_{s>0} f^{**}(s) v(s) < \infty \right\}.$$

Remark 4.1. Note that the spaces Λ_v^∞ and Γ_v^∞ generalize the spaces of type $\Lambda_v^{p,\infty}$ and $\Gamma_v^{p,\infty}$ (see [2] for the definition). Indeed, we have

$$\|f\|_{\Lambda_{V^p}^\infty} = \|f\|_{\Lambda_v^{p,\infty}}$$

and

$$\|f\|_{\Gamma_{V^p}^\infty} = \|f\|_{\Gamma_v^{p,\infty}}.$$

Remark 4.2. Usually, X may be called rearrangement invariant laticce only if, in addition to our assumptions on this structure, it is a linear set. But that would cause certain troubles in this case, because for an arbitrary weight v, Λ_v^p and $\Lambda_v^{p,\infty}$ do not have to be linear sets (for an equivalent condition on weight for which Λ_v^p is a linear space see [3]). Consider for instance the case of $\mathbb{R} = (-\infty, \infty)$ Λ_v^p with $\mu(\mathbb{R}) = \infty$, where

$$v(s) := \sum_{n=1}^\infty n! \, \chi_{(n,n+1)}(s), \quad s \in (0, \infty),$$

and functions f, g with $\operatorname{supt} f \cap \operatorname{supt} g = \emptyset$. For instance, set

$$f(s) := \sum_{n=1}^\infty \frac{1}{n^2 n!} \chi_{(n,n+1)}(s), \quad g(s) := \sum_{n=1}^\infty \frac{1}{n^2 n!} \chi_{(-n-1,-n)}(s).$$

Then clearly

$$f^*(s) = g^*(s) = \sum_{n=1}^\infty \frac{1}{n^2 n!} \chi_{(n,n+1)}(s).$$

Therefore $f, g \in \Lambda_v^p$, but $f + g \notin \Lambda_v^p$. But in the following text by abuse of language we shall call the Λ_v^p spaces.

For a weight v, $p \in (1, \infty)$ and $t \in (0, \infty)$, let us recall the fundamental function for spaces Λ and Γ. We have

$$\varphi_{\Lambda_v^p}(t) = \left(\int_0^\infty \chi_{(0,t)} v(s) \, ds \right)^{1/p} = V(t)^{1/p}.$$

As for Γ_v^p, let $\mu(E) = t$. We have

$$(\chi_E)^{**}(s) = \frac{1}{s} \int_0^s \chi_{(0,t)}(\xi) \, d\xi = \min \left\{ 1, \frac{t}{s} \right\}.$$

Therefore

$$\varphi_{\Gamma_v^p}(t) = \left(\int_0^t v(s) \, ds + t^p \int_t^\infty \frac{v(s)}{s^p} \, ds \right)^{1/p} \approx V^{1/p}(t) + t \left(\int_t^\infty \frac{v(s)}{s^p} \, ds \right)^{1/p}.$$

Similarly

$$\varphi_{\Lambda_v^{p,\infty}}(t) = \sup_{s>0} \chi_{(0,t)}(s) V(s)^{1/p} = V(t)^{1/p}$$

and

$$\varphi_{\Gamma_v^{p,\infty}} = \sup_{s>0} (\chi_{(0,t)})^{**}(s) V(s)^{1/p} \approx V(t)^{1/p} + t \sup_{s>t} \frac{V(s)^{1/p}}{s}.$$

Remark 4.3. Let us check that $\|\cdot\|_{\Lambda_v^p}$ satisfies the assumptions of Theorem 3.1. The assumption (1) is obviously satisfied. The assumption (2) demands the fundamental function to be finite. For this it is sufficient to have $v \in L_{\text{loc}}^1$. The characterization of the assumption (3) is described in the next proposition.

Proposition 4.1. *The functional $\|\cdot\|_{\Lambda_v^p}$ satisfies assumption (3) from Theorem 3.1 if and only if*

$$\int_0^{\min\{1,\mu(\mathbb{R})\}} \frac{t^{p'-1}}{V(t)^{p'-1}} \, dt < \infty.$$

P r o o f. Choose $E \subset \mathbb{R}$ with $\mu(E) < \infty$. We need to show

$$\left(\int_0^\infty (\chi_E f)^*(s)^p v(s) \, ds \right)^{1/p} \geqslant C \int_{\mathbb{R}} \chi_E f \, d\mu.$$

This inequality holds if and only if there exists $0 < C < \infty$ such that

$$\left(\int_0^{\mu(E)} f^*(s)^p v(s) \, ds \right)^{1/p} \geqslant C \int_0^{\mu(E)} f^*(s) \, ds,$$

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for every $f \in \mathcal{M}(\mathbb{R})$. This is equivalent to the embedding

$$\Lambda_v^p \hookrightarrow \Lambda_1^1,$$

which holds (see [2]) if and only if

$$\int_0^{\min\{1,\mu(\mathbb{R})\}} \frac{t^{p'-1}}{V(t)^{p'-1}} \, dt < \infty.$$

$\square$

Now for a weight v define

$$(4.1) \qquad v_a(s) = s^{p'} \frac{v(s)}{V(s)^{p'}}.$$

Then, since the embedding of type $\Gamma \hookrightarrow \Lambda$ has already been characterized in [5], Theorem 4.2, we have

$$\|f\|_{(\Gamma_{v_a}^{p'})'} = \mathrm{Opt}(\Gamma_{v_a}^{p'}, \Lambda_f^1) \approx \left(\int_0^\infty f^{**}(t)^p v_{aa}(t) \, dt \right)^{1/p},$$

where

$$(4.2) \qquad v_{aa}(t) = \frac{t^{p+p'+1} \int_0^t s^{p'} v(s) V(s)^{-p} \, ds \left[V(t)^{1-p'} - V(\infty)^{1-p'} \right]}{\left(\int_0^t s^{p'} v(s) V^{-p'}(s) \, ds + t^p \left[V(t)^{1-p'} - V(\infty)^{1-p'} \right] \right)^{p'+1}}.$$

Lemma 4.1. *Let $1 < p \leqslant q < \infty$. Then the following holds:*

$$\mathrm{Opt}(\Gamma_v^p, \Lambda_w^q) := \sup_{f \in \Gamma_p^v} \frac{\|f\|_{\Lambda_w^q}}{\|f\|_{\Gamma_v^p}} \approx \sup_{t>0} \frac{\varphi_{\Lambda_w^q}(t)}{\varphi_{\Gamma_v^p}(t)}.$$

P r o o f. Obviously

$$\sup_{f \in \Gamma_p^v} \frac{\|f\|_{\Lambda_w^q}}{\|f\|_{\Gamma_v^p}} \geqslant \sup_{t>0} \frac{\varphi_{\Lambda_w^q}(t)}{\varphi_{\Gamma_v^p}(t)}.$$

It is enough to realize that on the right-hand side we are taking the supremum over the characteristic functions of sets of finite measure.

From [5], page 24, we obtain

$$\mathrm{Opt}(\Gamma_{v,u}^p, \Lambda_w^q) \approx \sup_{t>0} \frac{W(t)^{1/q}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s) \, ds \right)^{1/p}},$$

in this particular case with $u(t) = 1$ and $U(t) = t$. Hence we get

$$\mathrm{Opt}(\Gamma_v^p, \Lambda_w^q) \approx \sup_{t>0} \frac{W(t)^{1/q}}{\left(V(t) + t^p \int_t^\infty s^{-p} v(s) \, ds \right)^{1/p}} \approx \sup_{t>0} \frac{\varphi_{\Lambda_w^q}(t)}{\varphi_{\Gamma_v^p}(t)}.$$

$\square$

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Lemma 4.2. *Let v be a weight. If we set $X := \Lambda_v^p$, then the following conditions are equivalent.*

(1) $\mathrm{Opt}(X'', X) < \infty$.

(2) $\int_0^t s^{p'} v(s) V^{-p'}(s)\,\mathrm{d}s \lesssim t^{p'} V^{1-p'}(t)$.

P r o o f. According to [7], Theorem 1, we have:

$$\|f\|_{(\Lambda_v^p)'} \approx \|f\|_{\Gamma_{v_a}^{p'}} + \frac{\|f\|_1}{\|v\|_1}.$$

In the case of $v \notin L^1$, we have

$$(\Lambda_v^p)' = \Gamma_{v_a}^{p'},$$

where v_a is defined by (4.1). If $v \in L^1$, then

$$(\Lambda_v^p)' = \Gamma_{v_a}^{p'} \cap L^1.$$

In the case of $v \notin L^1$, (1) is satisfied if and only if

$$(\Gamma_{v_a}^{p'})' = \Gamma_{v_{aa}}^{p} \hookrightarrow \Lambda_v^p$$

holds (where v_{aa} is defined by (4.2)). In the case of $v \in L^1$, this occurs if and only if

$$(\Gamma_{v_a}^{p'} \cap L^1)' = (\Gamma_{v_{aa}}^{p} + L^\infty) \hookrightarrow \Lambda_v^p.$$

For $v \notin L^1$ we therefore need to check if

$$\mathrm{Opt}(\Gamma_{v_{aa}}^{p}, \Lambda_v^p) < \infty.$$

In the case of $v \in L^1$, we need to verify whether

$$\mathrm{Opt}(\Gamma_{v_{aa}}^{p} + L^\infty, \Lambda_v^p) \approx \mathrm{Opt}(\Gamma_{v_{aa}}^{p}, \Lambda_v^p) + \mathrm{Opt}(L^\infty, \Lambda_v^p) < \infty.$$

But $v \in L^1$ implies

$$\mathrm{Opt}(L^\infty, \Lambda_v^p) = \sup_{t>0} V(t)^{1/p} = \|v\|_1^{1/p} < \infty,$$

therefore in both cases it is necessary and sufficient to check that

$$\mathrm{Opt}(\Gamma_{v_{aa}}^{p}, \Lambda_v^p) < \infty.$$

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Due to a well known theorem (see [1], Theorem 5.2, 66) we have

$$\varphi_{X''}(t) = \frac{t}{\varphi_{X'}(t)}.$$

From [5], Theorem 4.2, we know it is enough to show that

$$\varphi_{\Lambda_v^p}(t) \lesssim \varphi_{\Gamma_{v_{aa}}^p}(t) \approx \frac{t}{\varphi_{\Gamma_{v_a}^{p'}}(t)}.$$

Therefore, we need the following inequality

$$\varphi_{\Lambda_v^p}(t) \lesssim \frac{t}{\varphi_{\Gamma_{v_a}^{p'}}(t)}.$$

We have

$$\varphi_{\Lambda_v^p}(t) = V(t)^{1/p},$$

and

$$\varphi_{\Gamma_{v_a}^{p'}}^{p'}(t) = V_a(t) + t^{p'} \int_t^\infty \frac{v_a(s)}{s^{p'}} \, \mathrm{d}s.$$

This implies

$$\varphi_{X'}(t) \approx V_a(t)^{1/p'} + t \left(\int_t^\infty \frac{v_a(s)}{s^{p'}} \, \mathrm{d}s \right)^{1/p'}.$$

Hence we have

$$\varphi_{X'}(t) \approx \left(\int_0^t s^{p'} \frac{v(s)}{V^{p'}(s)} \, \mathrm{d}s \right)^{1/p'} + t \left(\int_t^\infty \frac{v(s)}{V^{p'}(s)} \, \mathrm{d}s \right)^{1/p'}$$

$$\approx \left(\int_0^t s^{p'} \frac{v(s)}{V^{p'}(s)} \, \mathrm{d}s \right)^{1/p'} + tV^{-1/p}(t) - tV^{-1/p}(\infty).$$

This occurs if and only if

$$V^{1/p}(t) \lesssim \frac{t}{\left(\int_0^t s^{p'} v(s) V^{-p'}(s) \, \mathrm{d}s \right)^{1/p'} + tV^{-1/p}(t) - tV^{-1/p}(\infty)},$$

which is equivalent to

$$\left(\int_0^t s^{p'} \frac{v(s)}{V^{p'}(s)} \, \mathrm{d}s \right) + t^{p'} \left(V^{1-p'}(t) - V^{1-p'}(\infty) \right) \lesssim t^{p'} V^{1-p'}(t),$$

and the latter holds if and only if

$$\int_0^t s^{p'} \frac{v(s)}{V^{p'}(s)} \, \mathrm{d}s \lesssim t^{p'} V^{1-p'}(t).$$

The proof is complete. □

Theorem 4.1. *The following conditions are equivalent.*

(1) *Functional* $\|\cdot\|_{\Lambda_v^p}$ *is equivalent to a Banach function norm.*

(2) $\int_0^t s^{p'} v(s) V^{-p'}(s)\,\mathrm{d}s \lesssim t^{p'} V^{1-p'}(t)$, $t \in (0, \infty)$.

(3) $\int_0^t s^{p'-1} V^{-p'+1}(s)\,\mathrm{d}s \lesssim t^{p'} V^{1-p'}(t)$, $t \in (0, \infty)$.

P r o o f. Let us first show the equivalence of the second and the third condition. $(2) \Leftrightarrow (3)$: Clearly

$$\int_0^t \frac{s^{p'-1}}{V^{p'-1}(s)}\,\mathrm{d}s \approx \int_0^t s^{p'-1}\left(\int_s^\infty \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z + V^{1-p'}(\infty)\right)\mathrm{d}s$$

$$\approx \int_0^t \int_0^z s^{p'-1}\,\mathrm{d}s \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z + \int_t^\infty \int_0^t s^{p'-1}\,\mathrm{d}s \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z$$

$$+ t^{p'} V^{1-p'}(\infty) =: I + II + III.$$

Now, since all three terms on the right-hand side are nonnegative, we have

$$I \approx \int_0^t s^{p'} \frac{v(s)}{V^{p'}(s)}\,\mathrm{d}s \leqslant \int_0^t \frac{s^{p'-1}}{V^{p'-1}(s)}\,\mathrm{d}s.$$

Therefore $(3) \Rightarrow (2)$. For the converse implication, let us recall that since V is increasing, we have

$$III = t^{p'} V^{1-p'}(\infty) \leqslant t^{p'} V^{1-p'}(t).$$

Furthermore, we have

$$II = \int_0^t s^{p'-1}\,\mathrm{d}s \int_t^\infty \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z \approx t^{p'} \int_t^\infty \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z$$

$$\approx t^{p'}(V^{1-p'}(t) - V^{1-p'}(\infty)) \lesssim t^{p'} V^{1-p'}(t).$$

Therefore, if (2) is satisfied, we have $I \lesssim t^{p'} V^{1-p'}(t)$ and also $II + III \lesssim t^{p'} V^{1-p'}(t)$ and that implies $I + II + III \lesssim t^{p'} V^{1-p'}(t)$, which is nothing else but (3).

Now let us show the implication $(2) \Rightarrow (1)$. First note that if (2) is satisfied, then (3) is satisfied as well and hence also

$$\int_0^1 \frac{s^{p'-1}}{V^{p'-1}(s)}\,\mathrm{d}s < \infty.$$

Therefore by Proposition 4.1, the assumption (3) in Theorem 3.1 is satisfied in the case of $X = \Lambda_v^p$ (we shall use this identity till the end of the proof). Since all weights are defined as locally integrable positive functions, we also have the assumption (2) in Theorem 3.1, and as the reader can easily check, the assumption (1) in Theorem 3.1

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is satisfied as well. Theorem 3.1 claims that $\|\cdot\|_{\Lambda_v^p}$ is equivalent to a BFN if and only if $\|\cdot\|_X \approx \|\cdot\|_{X''}$. Let us first recall that the inequality $\|\cdot\|_{X''} \lesssim \|\cdot\|_X$ is trivially satisfied. It remains to investigate when $\|\cdot\|_X \lesssim \|\cdot\|_{X''}$ occurs. If the condition (2) is satisfied, we only use Lemma 4.2 and obtain $\mathrm{Opt}(X'', X) < \infty$, which gives the desired inequality.

Now let $\|\cdot\|_X$ be equivalent to a Banach function norm. Therefore the assumptions (2) and (3) in Theorem 3.1 have to be satisfied. And hence we have $\mathrm{Opt}(X'', X) < \infty$. If we use Lemma 4.2, we obtain (2). This completes the proof. $\qquad\square$

5. Normability of lambda spaces, case $p = \infty$

In order to meet the assumption (2) in Theorem 3.1, we need the weight function v to be essentialy bounded on every finite interval $(0, t)$. This follows from the fact that for E, with $\mu(E) = t < \infty$, we demand

$$(5.1) \qquad \tilde{v}(t) := \|\chi_E\|_{\Lambda_v^\infty} = \operatorname*{ess\,sup}_{s>0} \chi_{(0,t)}(s)v(s) = \operatorname*{ess\,sup}_{0<s<t} v(s) < \infty.$$

In the following text we shall assume that this assumption is satisfied. And the weight $\tilde{v}$ will always be defined by (5.1).

Lemma 5.1. *Let v be a weight. Then*

$$\operatorname*{ess\,sup}_{s>0} f^*(s)v(s) = \operatorname*{ess\,sup}_{s>0} \tilde{v}(s)f^*(s),$$

for every measurable f.

P r o o f. This proposition can be found in [4], but for the sake of completeness let us present a short proof. We have

$$\operatorname*{ess\,sup}_{s>0} f^*(s)\tilde{v}(s) = \operatorname*{ess\,sup}_{s>0} f^*(s) \operatorname*{ess\,sup}_{t<s} v(t)$$
$$\leqslant \operatorname*{ess\,sup}_{s>0} \operatorname*{ess\,sup}_{s>t>0} v(t)f^*(t) = \operatorname*{ess\,sup}_{s>0} v(s)f^*(s).$$

Since the opposite inequality is trivially satisfied, the proof is complete. $\qquad\square$

Theorem 5.1. *Let v be a weight. Then the following conditions are equivalent.*

(1) *Functional $\|\cdot\|_{\Lambda_v^\infty}$ is equivalent to a Banach function norm.*

(2) $\sup\limits_{t>0} \tilde{v}(t)t^{-1} \int_0^t \mathrm{d}z/\tilde{v}(z) < \infty.$

(3) $\Lambda_v^\infty = \Gamma_v^\infty$ *(in the sense of equivalent norms).*

P r o o f. Let us show the equivalence of (1) and (2). Denote $X := \Lambda_v^\infty$. By Lemma 5.1 we have

$$\|f\|_{\Lambda_v^\infty} = \|f\|_{\Lambda_{\tilde{v}}^\infty}.$$

Since the space $(\mathbb{R}, \mu)$ is nonatomic and therefore resonant, we may express the dual norm as

$$\|f\|_{X'} = \sup_{g \in X} \frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\|g\|_X} = \sup_{g \in \Lambda_{\tilde{v}}^\infty} \frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\|g\|_{\Lambda_{\tilde{v}}^\infty}}.$$

We claim that

$$(5.2) \qquad \sup_{g \in X} \frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\|g\|_X} = \int_0^\infty f^*(s)\frac{\mathrm{d}s}{\tilde{v}(s)}.$$

For the inequality "$\geqslant$" we may just choose $g \in \mathcal{M}(\mathbb{R})$ such that $g^* = 1/\tilde{v}$. For the opposite inequality, just realize that

$$\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s = \int_0^\infty \frac{f^*(s)}{\tilde{v}(s)}g^*(s)\tilde{v}(s)\,\mathrm{d}s \leqslant \int_0^\infty \frac{f^*(s)}{\tilde{v}(s)}\,\mathrm{d}s \operatorname*{ess\,sup}_{s>0} g^*(s)\tilde{v}(s).$$

Let us compute the functional $\|\cdot\|_{X''}$. We have

$$(5.3) \qquad \|f\|_{X''} = \sup_{g \in X'} \frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\int_0^\infty g^*(s)\,\mathrm{d}s/\tilde{v}(s)}.$$

We claim that

$$(5.4) \qquad \|f\|_{X''} \approx \sup_{t>0} \frac{\int_0^t f^*(s)\,\mathrm{d}s}{\int_0^t \mathrm{d}s/\tilde{v}(s)} = \sup_{t>0} f^{**}(t) \frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)}.$$

Indeed, the inequality $\lesssim$ is an immediate consequence of Hardy's lemma (see [1], Proposition 3.6). The opposite inequality trivially follows by taking $g = \chi_{(0,t)}$ in (5.3).

Now according to Lemma 3.1, we need to show $\|\cdot\|_X \lesssim \|\cdot\|_{X''}$. This holds if and only if the optimal constant of the inequality

$$(5.5) \qquad \operatorname*{ess\,sup}_{t>0} f^*(t)\tilde{v}(t) \leqslant C \sup_{t>0} f^{**}(t) \frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)}$$

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is finite. Testing this inequality on the set of simple functions yields

$$\operatorname*{ess\,sup}_{s>0} \chi_{(0,t)}(s)\tilde{v}(s) = \tilde{v}(t) \lesssim \sup_{s>0} \frac{\min(s,t)}{\int_0^s \mathrm{d}z/\tilde{v}(z)}.$$

We have

$$\sup_{s>0} \frac{\min(t,s)}{\int_0^s \mathrm{d}z/\tilde{v}(z)} = \max\left(\sup_{s<t} \frac{s}{\int_0^s \mathrm{d}z/\tilde{v}(z)}, \sup_{s>t} \frac{t}{\int_0^s \mathrm{d}z/\tilde{v}(z)}\right)$$

$$=: \max\left(\sup_{0<s<t} G(s), \sup_{s>t} H(s)\right).$$

Fix t. The function $H(s)$ is clearly decreasing. We also claim that $G(s)$ is nondecreasing. Indeed, we have

$$G(s) = \left(\frac{1}{s}\int_0^s \frac{\mathrm{d}z}{\tilde{v}(z)}\right)^{-1}$$

and since the mean value of a nonincreasing function is also nonincreasing, we obtain the claim. From the monotonicity of these functions, we may conclude

$$\sup_{s>0} \frac{\min(s,t)}{\int_0^s \mathrm{d}z/\tilde{v}(z)} = \frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)}.$$

Now, using these facts in (5.5), we obtain that the condition (2) is necessary.

Concerning the sufficiency, we have

(5.6) $$\operatorname*{ess\,sup}_{t>0} f^*(t)\tilde{v}(t) \leqslant \operatorname*{ess\,sup}_{t>0} f^{**}(t)\tilde{v}(t) \lesssim \operatorname*{ess\,sup}_{t>0} f^{**}(t)\frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)}.$$

It remains to show that (P5) holds. Let $E \subset \mathbb{R}$ be a measurable set, such that $\mu(E) < \infty$. By Hardy-Littlewood-Polya and Hölder inequality, we have

$$\int_E |f|\,\mathrm{d}\mu = \int_0^{\mu(E)} (f\chi_E)^*(s) \leqslant \int_0^{\mu(E)} \frac{\mathrm{d}s}{\tilde{v}(s)} \operatorname*{ess\,sup}_{t>0} (f\chi_E)^*(t)\tilde{v}(t).$$

Now set

$$C_E := \int_0^{\mu(E)} \frac{\mathrm{d}s}{\tilde{v}(s)}.$$

Since the condition (2) holds, the constant C_E is finite. Thus the assumption (3) in Theorem 3.1 is satisfied and therefore the condition (2) is sufficient. The equivalence of (1) and (2) is proved.

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Let us now assume (2) is satisfied. From (5.6) we have

$$\|f\|_{\Gamma_v^\infty} \lesssim \|f\|_{X''} \leqslant \|f\|_X.$$

Since the opposite inequality is trivially satisfied and the condition (2) implies

$$(5.7) \qquad \int_0^t \frac{\mathrm{d}z}{\tilde{v}(z)} < \infty, \quad \text{for every } t \in (0, \infty),$$

the condition (3) holds.

For the implication (3) $\Rightarrow$(1), it suffices to verify that Γ_v^∞ is a BFS. The only axiom that is not obvious is (P5). In order to see that (P5) holds, just realize that the function $f \in \mathcal{M}(\mathbb{R})$ such that $f^*(s) = 1/\tilde{v}(s)$ belongs to the space Γ_v^∞. $\qquad\square$

Let us remind that according to Remark 4.2, Λ_v^p does not have to be a linear set. A characterization of weight for which Λ_v^p is a linear set was given in [3] for $1 \leqslant p < \infty$. The authors also gave an equivalent condition on weight for which $\Lambda_v^{1,\infty}$ is a linear set. Let us present now similar characterization for the case of Λ_v^∞.

Theorem 5.2. *Let v be a weight. Then the set Λ_v^∞ is linear if and only if*

$$(5.8) \qquad \tilde{v}(2s) \lesssim \tilde{v}(s), \quad s \in (0, \infty).$$

P r o o f. Denote $X := \Lambda_v^\infty$. Due to Lemma 5.1 we have $X = \Lambda_{\tilde{v}}^\infty$. Let us first suppose that (5.8) is violated. Then there exists a sequence t_n such that

$$(5.9) \qquad \tilde{v}(2t_n) \geqslant 2^n \tilde{v}(t_n).$$

We may, without loss of generality, suppose that t_n is either increasing or decreasing. And also without loss of generality suppose that $t_1 < \mu(\mathbb{R})/2$. In the case of $\mu(\mathbb{R}) = \infty$ it is trivial, otherwise, if $\mu(\mathbb{R}) < \infty$ one can see that $t_n \to 0$, so for certain n_0 we have $t_n < \mu(\mathbb{R})/2$ for all $n > n_0$. Now, taking t_{n+n_0} instead of t_n does the job. Let us first suppose t_n is increasing. Because we have $t_1 \leqslant \mu(\mathbb{R})/2$, we may choose $f, g \in \mathcal{M}(\mathbb{R})$ such that

$$\operatorname{supt}(f) \cap \operatorname{supt}(g) = \emptyset$$

and

$$(5.10) \qquad f^*(s) = g^*(s) = \sum_{n=1}^\infty \frac{1}{\tilde{v}(t_n)} \chi_{(t_{n-1}, t_n]}(s).$$

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Then clearly $\|f\|_X = \|g\|_X = 1$. Choose $n \in \mathbb{N}$. We have

$$\|f + g\|_X = \operatorname*{ess\,sup}_{s>0} \sum_{k=1}^{\infty} \frac{1}{\tilde{v}(t_k)} \chi_{(2t_{k-1}, 2t_k]} \tilde{v}(s)$$

$$\geqslant \operatorname*{ess\,sup}_{s>0} \frac{1}{\tilde{v}(t_n)} \chi_{(2t_{n-1}, 2t_n]}(s) \tilde{v}(s) = \frac{\tilde{v}(2t_n)}{\tilde{v}(t_n)} \geqslant 2^n.$$

Since n is an arbitrary natural number, we obtain $f + g \notin \Lambda_v^{\infty}$. Now, let the sequence t_n be decreasing. Since we have $t_1 \leqslant \mathbb{R}/2$, we can find f, g with disjoint supports such that

$$f^*(s) = g^*(s) = \sum_{j=1}^{\infty} \frac{1}{\tilde{v}(t_j)} \chi_{(t_j, t_{j-1})}(s).$$

If we use the similar calculation as in the first case, we obtain that $f + g \notin \Lambda_v^{\infty}$.

Now, let us suppose (5.8) holds. Choose $f, g \in X$. We have

$$\|f + g\|_X \leqslant \operatorname*{ess\,sup}_{s>0} \left(f^*\left(\frac{s}{2}\right) + g^*\left(\frac{s}{2}\right) \right) \tilde{v}(s)$$

$$= \operatorname*{ess\,sup}_{s>0} (f^*(s) + g^*(s)) \tilde{v}(2s) \lesssim \|f\|_X + \|g\|_X.$$

Therefore $f + g \in \Lambda_v^{\infty}$. The proof is complete. $\qquad\square$

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Authors' addresses: A m i r a n G o g a t i s h v i l i, Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `gogatish@math.cas.cz`; F i l i p S o u d s k ý, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `soudsky@karlin.mff.cuni.cz`.

NOTE ON LINEARITY OF REARRANGEMENT-INVARIANT SPACES

FILIP SOUDSKÝ[1*]

ABSTRACT. For some rearrangement-invariant functional $\|\cdot\|_X$ having the lattice property, we give a characterization of linearity of the set $\{f : \|f\|_X < \infty\}$. Afterwards we apply this general abstract theorem in the case of Orlicz-Lorentz spaces.

1. INTRODUCTION

Rearrangement-invariant spaces play important role in analysis and its applications. They first appeared in the 1930's. Since then they have found a number of important applications for example on partial differential equations and theory of Sobolev spaces. They have been intensively studied since 1950's starting with the famous pioneering paper [7], in which the so-called *classical Lorentz spaces* were introduced. It was also shown in that very paper, however, that the functional which governs these spaces is not necessarily a norm in general. In certain cases these "spaces" do not even have to be linear sets. The main reason for this fact is that the operator which associates a measurable function with its non-increasing rearrangement is not sub-additive. Many authors have studied functional properties of these spaces. In 1990, Sawyer in [9] characterized the normability of classical Lorentz spaces. Many papers, with characterizations of linearity (see [4]) quasi-normability (see [3]) and normability ([2]) followed, see also ([5]).

General properties of Banach lattices were also studied in [6]. In this paper the notion of space symetrization was considered. In the following we shall adopt a similar approach when studying the problem of linearity of a rearrangement-invariant lattice.

Our principal goal in this paper is to establish necessary and sufficient conditions for a rearrangement-invariant lattice to be linear set. It turns out that under these circumstances such conditions depend on finiteness of the dilation operator.

We also point out some applications of this result including an alternative approach to the characterization of linearity of Orlicz-Lorentz, which enjoy the above-mentioned properties. In case of linearity, the result is known ([4]) but assuming Δ_2-condition of the function φ. In our paper we shall present stronger version of this theorem without this restriction.

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2. Preliminaries and main theorem

Let $(\mathcal{R}, \mu)$ a non-atomic, σ-finite measure space. Denote the set of all real-valued μ-measurable functions on $\mathcal{R}$ by $\mathcal{M}(\mathcal{R})$. In the special case when $\mathcal{R} = (0, \infty)$, we write $\mathcal{M}(0, \infty)$. For $f \in \mathcal{M}(\mathcal{R})$ we shall define the *distribution function* by

$$f_*(t) := \mu \{|f| > t\}$$

and the *non-increasing rearrangement* of f by

$$f^*(t) := \inf \{s \in [0, \infty) : f_*(s) \leq t\}, \quad t \in [0, \infty).$$

For $a > 0$, we denote the *dilation operator* E_a by

$$E_a g(t) = g(a^{-1}t) \quad \text{for } g \in \mathcal{M}(0, \infty) \text{ and } t \in (0, \infty).$$

It is known that the operation $f \mapsto f^*$ is not sub-additive, instead we have the following inequality

$$(f + g)^*(s) \leq E_2 f^*(s) + E_2 g^*(s) \quad \text{for every } f, g \in \mathcal{M}(\mathcal{R}) \text{ and } s \in (0, \infty). \quad (2.1)$$

We shall also use the term *weight* for a positive locally integrable function defined on $(0, \infty)$. For a weight w we shall define function W by the following formula

$$W(t) := \int_0^t w(s)\, \mathrm{d}\, s, \quad t \in [0, \infty).$$

When a functional $\|\cdot\|_X : \mathcal{M}(\mathcal{R}) \to [0, \infty]$ is given, we denote

$$X := \{f \in \mathcal{M}(\mathcal{R}) : \|f\|_X < \infty\}.$$

Definition 2.1. We call X a *rearrangement-invariant (r.i. for short) lattice* if the following conditions are satisfied:

(P1) If $f^* = g^*$ then $\|f\|_X = \|g\|_X$.
(P2) If $|f| \leq |g|$ μ-a.e. then $\|f\|_X \leq \|g\|_X$.
(P3) $\|af\|_X = |a| \|f\|_X$.

We call $\|\cdot\|_X$ a *quasi-norm* if (P1)–(P3) hold and, moreover, the inequality

$$\|f + g\|_X \leq C (\|f\|_X + \|g\|_X)$$

holds for some $C \in (0, \infty)$ independent on f and g. If X is an r.i. lattice and there exists a functional

$$\|\cdot\|_{\bar{X}} : \mathcal{M}(0, \infty) \to [0, \infty]$$

satisfying

$$\|f\|_X = \|f^*\|_{\bar{X}},$$

we say that $\|\cdot\|_{\bar{X}}$ is the *representation functional* of X.

We shall use the following immediate consequence of the *Hardy's lemma* ([1, Proposition 3.6, pg 56], see also [9]): for given weights w, v we have

$$\sup_{f \in \mathcal{M}(\mathcal{R})} \frac{\int_0^\infty f^*(s)w(s)\, \mathrm{d}\, s}{\int_0^\infty f^*(s)v(s)\, \mathrm{d}\, s} = \sup_{t > 0} \frac{W(t)}{V(t)}.$$

Although the following lemma is a simple observation and it is kind of folklore, let us provide convenience to a reader by listing it with a proof, which is a little bit technical.

Lemma 2.2. *Let $(\mathcal{R}, \mu)$ be a non-atomic σ-finite measure space. Let $h \in \mathcal{M}(0, \mu(\mathcal{R}))$ be a non-negative, non-increasing and right-continuous function. Then there exists a function $f \in \mathcal{M}(\mathcal{R})$ such that $f^* = h$.*

Proof. Let us first suppose that h is a simple function. Let

$$h = \sum_{i=1}^{l} a_i I_i,$$

where I_i are disjointed intervals. Since $(\mathcal{R}, \mu)$ is non-atomic, there exist $A_i \subset \mathcal{R}$ disjointed with $\mu(A_i) = |I_i|$ (see [1, Lemma 2.5., pg 46]). If we set

$$f := \sum_{i=1}^{l} a_i \chi_{A_i},$$

we have $f^* = h$ as desired.

For general non-increasing nonnegative function h we shall find simple functions h_n such that $0 \le h_n \uparrow h$ and f_n such that $f_{n+1} \ge f_n$ and $f_n^* = h_n$. Then if we define

$$f := \lim_n f_n,$$

we have $f^* = h$ (by [1, Proposition 1.7, pg 41]).

Now, let us conctruct such a sequence in the following way. For $k, l \in \mathbb{N}$ set

$$H_l^k := \left\{ \frac{l}{2^k} < h \le \frac{l+1}{2^k} \right\}$$

and find $F_l^k \subset \mathcal{R}$ such that $\mu(F_l^k) = |H_l^k|$, $F_j^k \cap F_i^k = \emptyset$ for $i \ne j$ and $F_{2l}^{k+1} \cup F_{2l+1}^{k+1} = F_l^k$. Define

$$f_n := \sum_{i=1}^{n2^n - 1} \frac{i}{2^n} \chi_{F_i^n} + n\chi_{F_n},$$

where $F_n = \bigcup_{j \ge n} F_j^1$ and set

$$h_n := f_n^*.$$

One redily checks that such a sequence has the required properties. $\square$

Our main result is the statement (i) in the following theorem.

Theorem 2.3. *Let X be an r.i. lattice for which there exists a representation functional $\|\cdot\|_{\bar{X}}$.*

(i) *Assume that the space*

$$\bar{X} := \{ f \in \mathcal{M}(0, \infty) : \|f\|_{\bar{X}} < \infty \}$$

is a linear set. Then the space X is a linear set if and only if the following implication holds:

$$\text{if } \|f^*\|_{\bar{X}} < \infty, \text{ then } \|E_2 f^*\|_{\bar{X}} < \infty. \tag{2.2}$$

(ii) *Assume that $\|\cdot\|_{\bar{X}}$ is a quasi-norm. Then $\|\cdot\|_X$ is a quasi-norm if and only if there exists a positive constant C such that*

$$\|E_2 f^*\|_{\bar{X}} \leq C \|f^*\|_{\bar{X}}. \tag{2.3}$$

(iii) *Assume that $\|\cdot\|_{\bar{X}}$ is a norm. Then $\|\cdot\|_X$ is a norm if and only if*

$$\|E_2 f^*\|_{\bar{X}} \leq 2 \|f^*\|_{\bar{X}}. \tag{2.4}$$

Proof. (i) Assume first that (2.2) holds. Let there be f, g such that $\|f\|_X < \infty$ and $\|g\|_X < \infty$. Then, by (2.1) and (2.2), we get

$$\|f + g\|_X = \|(f + g)^*\|_{\bar{X}} \leq \|E_2 f^* + E_2 g^*\|_{\bar{X}} < \infty.$$

Conversely, let f be such that $\|f^*\|_{\bar{X}} < \infty$ but $\|E_2 f^*\|_{\bar{X}} = \infty$. Let us first suppose that $\mu(\mathcal{R}) = \infty$. Then there exists two sets of infinite measure $E, M \subset \mathcal{R}$ such that $E \cap M = \emptyset$. Now (E, μ) and (M, μ) are two measure spaces therefore according to Lemma 2.2 there exist functions $\tilde{h} \in \mathcal{M}(E)$ and $\tilde{g} \in \mathcal{M}(M)$, with

$$\tilde{g}^* = \tilde{h}^* = f^*.$$

Let us extend them by zero at the rest of $\mathcal{R}$ to functions h and g. We have

$$g^* = h^* = f^*$$

and

$$(g + h)_*(t) = 2f_*(t).$$

Hence

$$(g + h)^*(s) = E_2 f^*(s) \quad \text{for } s \in (0, \infty).$$

Therefore, $g, h \in X$ but $g + h \notin X$. Consequently, X is not a linear set.

In the case when $r := \mu(\mathcal{R}) < \infty$, we find a μ-measurable set $E \subset \mathcal{R}$ such that

$$\left\{ |f| > f^*\left(\frac{r}{2}\right) \right\} \subset E \subset \left\{ |f| \geq f^*\left(\frac{r}{2}\right) \right\}, \tag{2.5}$$

and $\mu(E) = \frac{r}{2}$. Then there exists h, g with disjointed supports satisfying

$$h^* = g^* = (f\chi_E)^*$$

and we have

$$\|h + g\|_X = \|(h + g)^*\|_{\bar{X}} \geq \|E_2 f^*\|_{\bar{X}}.$$

Therefore, $g, h \in X$ but $g + h \notin X$, hence, again, X is not a linear set.

For proof of the statement (ii) see [6, Lemma 1.4]. The proof of (iii) is analogous to that of (ii). $\square$

Remark 2.4. Let $\delta > 1$. Then the conditions (2.2) and (2.3) can be respectively replaced by

$$\|f^*\|_{\bar{X}} < \infty \quad \text{then } \|E_\delta f^*\|_{\bar{X}} < \infty$$

and

$$\|E_\delta f^*\|_{\bar{X}} \leq C \|f^*\|_{\bar{X}}.$$

3. Applications

In this section we shall illustrate the results obtained on the particular example of Lorentz-Orlicz spaces. We start with a general definition of a general structure that covers such spaces. These spaces first appeared in [8, Definition 7.2.].

Definition 3.1. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function with $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$. Let w be a weight. Then we define the functional

$$\|f\|_{\Lambda_{\varphi,w}} := \inf \left\{ \lambda : \int_0^\infty \varphi \left(\frac{f^*(s)}{\lambda} \right) w(s) \, ds \leq 1 \right\}$$

and the set

$$\Lambda_{\varphi,w} := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Lambda_{\varphi,w}} < \infty \right\}.$$

We note that, clearly, $\Lambda_{\varphi,w}$ is an r.i. lattice. Furthermore, the representation functional is therefore defined as follows:

$$\|f\|_{L_w^\varphi} := \inf \left\{ \lambda : \int_0^\infty \varphi \left(\frac{|f(s)|}{\lambda} \right) w(s) \, d\,s \leq 1 \right\}, \qquad f \in \mathcal{M}(0, \infty).$$

Note that

$$L_w^\varphi = \left\{ f \in \mathcal{M}(0, \infty) : \exists \lambda \in (0, \infty) : \int_0^\infty \varphi \left(\frac{|f(s)|}{\lambda} \right) w(s) \, d\,s < \infty \right\}.$$

The following lemma is a classical result (for a more general form see [8])

Lemma 3.2. *Let φ and w be as in Definition 3.1. Then*

(i) $\|\cdot\|_{L_w^\varphi}$ *has lattice property,*
(ii) L_w^φ *is a linear set.*

The following theorem is known in a weaker form (with the additional assumption of $\varphi \lesssim E_2\varphi$) (see [4]). We shall point an alternative proof based on Theorem 2.3 removing the assumption.

Theorem 3.3. *Let φ and w have the same properties as in Definition 3.1. Then the following conditions are equivalent.*

(i) $\Lambda_{\varphi,w}$ *is linear;*
(ii)

$$\sup_{t>0} \frac{W(2t)}{W(t)} < \infty.$$

Proof. According to Lemma 3.2, the representation space meets the assumptions of Theorem 2.3. Let us first proof that (ii) implies (i). We have

$$\int_0^\infty \varphi\left(\frac{f^*\left(\frac{s}{2}\right)}{\lambda}\right) w(s)\,\mathrm{d}\,s = 2\int_0^\infty \varphi\left(\frac{f^*(t)}{\lambda}\right) w(2t)\,\mathrm{d}\,t$$

$$\leq 2\sup_f \left(\frac{\int_0^\infty \varphi\left(\frac{f^*(t)}{\lambda}\right) w(2t)\,\mathrm{d}\,t}{2\int_0^\infty \varphi\left(\frac{f^*(t)}{\lambda}\right) w(t)\,\mathrm{d}\,t}\right) \int_0^\infty \varphi\left(\frac{f^*(t)}{\lambda}\right) w(t)\,\mathrm{d}\,t$$

$$= \sup_{t>0} \frac{W(2t)}{W(t)} \int_0^\infty \varphi\left(\frac{f^*(t)}{\lambda}\right) w(t)\,\mathrm{d}\,t$$

$$\leq C\int_0^\infty \varphi\left(\frac{f^*(t)}{\lambda}\right) w(t)\,\mathrm{d}\,t,$$

$$(3.1)$$

where the last inequality follows immediately from Hardy's lemma. This proves (i) via Theorem 2.3(i).

Now, let us assume condition (ii) is violated. Then we may pick a sequence $\{t_n\}$ such that

$$\frac{W(2t_n)}{W(t_n)} > 4^n \quad \text{for all } n \in \mathbb{N}.$$

Since the function $\frac{W(2t)}{W(t)}$ is continuous, and therefore locally bounded on $(0,\infty)$, we may assume that either $t_n \uparrow \infty$ or $t_n \downarrow 0$. Let us first suppose $t_n \uparrow \infty$. Since in this case the weight cannot be integrable, we may assume (by a picking suitable sub-sequence if necessary) that

$$W(t_n) \geq 2W(t_{n-1}), \quad W(2t_n) > 2W(2t_{n-1}), \quad \text{and} \quad \int_{t_{k-1}}^{t_k} w(s)\,\mathrm{d}\,s \uparrow.$$

Then

$$\frac{W(2t_n) - W(2t_{n-1})}{W(t_n) - W(t_{n-1})} \geq c4^n \quad \text{for some } c > 0 \text{ and all } n \in \mathbb{N}.$$

For our technical convenience, we set $t_0 := 0$. Now, let us define a sequence of functions $\{f_n\} \subset \mathcal{M}(\mathcal{R})$ with pairwise disjoint supports and such that

$$f_n^* = \chi_{(0,t_n-t_{n-1})}\varphi^{-1}\left(\left(2^n \int_{t_{n-1}}^{t_n} w\right)^{-1}\right) \quad \text{for } n \in \mathbb{N}.$$

Set

$$f := \sum_{i=1}^\infty f_n.$$

Calculation shows that

$$\int_0^\infty \varphi(f^*(s))w(s)\,\mathrm{d}\,s = \sum_{k=1}^\infty \int_{t_{k-1}}^{t_k} \varphi\left(\varphi^{-1}\left(\left(2^k \int_{t_{k-1}}^{t_k} w\right)^{-1}\right)\right) w(s)\,\mathrm{d}\,s$$

$$= \sum_{k=1}^\infty 2^{-k} = 1.$$

On the other hand, we have

$$\int_0^\infty \varphi(E_2 f^*(s)) w(s) \, \mathrm{d}\, s = 2 \int_0^\infty \varphi(f^*(t)) w(2t) \, \mathrm{d}\, t$$

$$\geq \int_{t_{k-1}}^{t_k} \frac{w(2t)}{2^k W(t_k) - W(t_{k-1})} \, \mathrm{d}\, t$$

$$\geq \frac{W(2t_k) - W(2t_{k-1})}{2^k (W(t_k) - W(t_{k-1}))} \geq c 2^k$$

for some $c > 0$ and all $k \in \mathbb{N}$. Therefore, $E_2 f^* \notin L_w^\varphi$, which implies that $\Lambda_{\varphi,w}$ is not a linear set. On the other hand, if $t_k \downarrow 0$, then we may suppose that

$$W(t_{n-1}) \geq 2W(t_n), \quad W(2t_{n-1}) \geq 2W(2t_n) \quad \text{and} \quad \int_{t_n}^{t_{n-1}} w(s) \, \mathrm{d}\, s \downarrow .$$

Now we find f_n with disjointed supports such that

$$f_n^* = \chi_{(0,t_{n-1}-t_n)} \varphi^{-1} \left(\left(2^n \int_{t_n}^{t_{n-1}} w \right)^{-1} \right).$$

We have

$$\|f\|_{\Lambda_{\varphi,w}} = \sum_{n=2}^\infty \int_{t_n}^{t_{n-1}} \frac{w(t)}{2^n (W(t_{n-1}) - W(t_n))} \, \mathrm{d}\, t$$

$$= \sum_{n=2}^\infty 2^{-n} = \frac{1}{2}.$$

On the other hand,

$$\|E_2 f^*\|_{L_w^\varphi} = 2 \sum_{n=2}^\infty \int_{t_n}^{t_{n-1}} \frac{w(2t)}{2^{-n} (W(t_{n-1}) - W(t_n))}$$

$$\geq 2 \int_{t_n}^{t_{n-1}} \frac{w(2t)}{2^{-n} (W(t_{n-1}) - W(t_n))} \geq 2^n,$$

whence $E_2 f^* \notin L_w^\varphi$. This shows that $\Lambda_{\varphi,w}$ is not linear, which is a contradiction. The proof is complete. □

Acknowledgement

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[1] Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University Prague, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: filip.soudsky@karlin.mff.cuni.cz

NORMABILITY OF GAMMA SPACES

FILIP SOUDSKÝ

ABSTRACT. We give a full characterization of normability of Lorentz spaces Γ_w^p.

1. INTRODUCTION AND THE MAIN RESULT

In this paper we present a complete characterization of those parameters p and w, where $p \in (0, 1)$ and w is a nonnegative measurable function (weight), for which the corresponding classical Lorentz space $\Gamma^p(w)$ (the precise definition is given below) is normable. By this we mean that the functional $\| \cdot \|_{\Gamma^p(w)}$ is equivalent to a norm. We in fact prove two characterizations, quite different in nature. One of them is a certain integrability condition on the weight while the other states that the corresponding space coincides with the space $L^1 + L^\infty$. The proofs are based on a combination of discretization and weighted norm inequalities.

This result is in fact known as it can be derived from the quite complicated argument concerning a copy of the ℓ^p space treated in [4]. However, we present a new elementary proof which does not go beyond the scoop of the classical Lorentz spaces.

We recall that classical Lorentz spaces of type Λ were first introduced by Lorentz in 1951 ([5]) while their modification of type Γ was developed first in 1990 by Sawyer ([6]) in connection with their crucial duality properties. These spaces proved to be extremely useful for a wide range of applications and have been studied ever since by many authors ([1], [3], [8], [7]...). Normability of spaces of type Λ has been characterized long time ago (see [6] and [2]).

The result is a contribution to the long-standing research of functional properties such as linearity, (quasi)-normability etc., of classical Lorentz spaces of various types (see, e.g.).

During the whole paper, the underlying measure space $(\mathcal{R}, \mu)$ shall be always non-atomic and σ-finite with $\mu(\mathcal{R}) = \infty$. We shall also use the symbol $\mathcal{M}(\mathcal{R})$ for the set of all real-valued measurable funtions defined on $\mathcal{R}$. For a measurable, real-valued function f on such a space, a *non-increasing rearrangement* of f is defined by

$$f^*(t) := \inf \{s : \mu (\{|f| > s\}) \le t\},$$

while the *maximal function* of f is given by

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \mathrm{d}s.$$

Throughout all of this paper the expression *weight* will always be used for positive, measurable function defined on $(0, \infty)$.

Definition 1. Let $0 < p < \infty$ and let w be a weight. Set

$$\Lambda_w^p := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Lambda_v^p} := \left(\int_0^\infty f^*(s)^p w(s) \mathrm{d}s \right)^{\frac{1}{p}} < \infty \right\};$$

and

$$\Gamma_w^p := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{\Gamma_v^p} := \left(\int_0^\infty f^{**}(s)^p w(s) \mathrm{d}s \right)^{\frac{1}{p}} < \infty \right\}.$$

Furthermore in the following text we shall use notation $X := \Gamma_v^p$. In order to avoid the technical difficulties, we shall assume that w is locally integrable and

$$(1.1) \qquad\qquad \int_a^\infty w(s) s^{-p} \mathrm{d}s < \infty,$$

for all $a > 0$. We may also assume this without loss of generality, since if $w \notin L_{\mathrm{loc}}^1$ or (1.1) is not satisfied, then $\Gamma_v^p = \{0\}$. In the following text function W will be defined as

$$W(t) := \int_0^t w(s) \mathrm{d}s.$$

We recallthat the space $L^1 + L^\infty$ space consists of all functions $f \in \mathcal{M}(\mathcal{R})$ for which there exists a decomposition $f = g + h$ such that $g \in L^1$ and $f \in L^\infty$ equiped with the norm

$$\|f\|_{L^1 + L^\infty} := \int_0^1 f^*(s) \mathrm{d}s.$$

Let us also recall the definition of norm in weighted Lebesgue space on $(0, \infty)$ which shall be also used in the proof, namely

$$\|f\|_{L_w^p} := \left(\int_0^\infty |f(s)|^p w(s) \mathrm{d}s \right)^{\frac{1}{p}}.$$

Our main result reads as follows.

Theorem 1. *Let $0 < p < 1$ and let w be a weight. Then the following conditions are equivalent.*

  (i) *The space Γ_w^p is normable.*
 (ii) *Both $w(s)$ and $w(s) s^{-p}$ are integrable on $(0, \infty)$.*
(iii) *The identity*

$$\Gamma_w^p = L^1 + L^\infty$$

  holds in the sense of equivalent norms.

2. Proof of Theorem1

Lemma 1. *Let X be a linear vector space. Let $\sigma : X \to [0, \infty)$ be a positively homogenous functional. Then the following conditions are equivalent*

  (i) *σ is equivalent to a norm;*

(ii) *there exists a constant C, independent on N, such that*

$$\sigma\left(\sum_{k=1}^{N} f_k\right) \le C\sum_{k=1}^{N}\sigma(f_k),$$

for all $f_k \in X$.

Proof of Lemma 1. First let us suppose that (i) holds. Denote the equivalent norm by ϱ. Then we have

$$\sigma\left(\sum_{k=1}^{N} f_k\right) \le C\varrho\left(\sum_{k=1}^{N} f_k\right) \le C\sum_{k=1}^{N}\varrho(f_k) \le C\sum_{k=1}^{N}\sigma(f_k).$$

Now, suppose that (2) holds. Denote

$$\varrho(f) := \inf\left(\sum_{k=1}^{N}\sigma(f_k)\right),$$

where the infimum on the right-hand side is taken over all finite decompositions of f i.e.

$$(2.1) \qquad \sum_{k=1}^{N} f_k = f.$$

Then obviously

$$\varrho(f) \le \sigma(f),$$

for all $f \in X$. On the other hand, for all f_k satisfying (2.1) we have

$$C\left(\sum_{k=1}^{N}\sigma(f_k)\right) \ge \sigma(f).$$

Passing to the infimum on the left-hand side gives

$$C\varrho(f) \ge \sigma(f).$$

Now, take $f_1, f_2 \in X$. Let

$$\sum_{k=1}^{N_1} f_k^1 = f_1, \quad \sum_{k=1}^{N_2} f_k^2 = f_2,$$

then

$$\varrho(f_1 + f_2) \le \sum_{k=1}^{N_1}\sigma(f_k^1) + \sum_{k=1}^{N_2}\sigma(f_k^2).$$

By passing to the infimum on the right-hand side we obtain the triangle inequality for ϱ. □

Proof of Theorem 1. Let us first prove that (i) implies (ii). We shall give an indirect proof. Suppose that (ii) is not true. Then either

$$(2.2) \qquad \int_0^\infty w(s)\mathrm{d}s = \infty$$

or

$$(2.3) \qquad \int_0^\infty s^{-p}w(s)\mathrm{d}s = \infty.$$

First, note that if $w \in \mathcal{B}_p$ then $\|\cdot\|_X \approx \|\cdot\|_{\Lambda^p_w}$. Since the functional $\|\cdot\|_{\Lambda^p_w}$ is not normable for $p < 1$ (as was shown in [2]), neither is $\|\cdot\|_X$. This allows us to focus on the case when $w \notin \mathcal{B}_p$. Therefore we may suppose that there exists a sequence $\{a_n\}_{n=1}^\infty$ such that

$$(2.4) \qquad a_n^p \int_{a_n}^\infty w(s)s^{-p}\mathrm{d}s \geq 2^n W(a_n).$$

Now let us define

$$(2.5) \qquad H(t) := \frac{t^p \int_t^\infty w(s)s^{-p}\mathrm{d}s}{W(t)}.$$

Since H is continuous on $(0,\infty)$ and therefore bounded on every $[c,d] \subset (0,\infty)$, we may without loss of generality (by choosing appropriate sub-sequence) assume that either $a_n \downarrow 0$ or $a_n \uparrow \infty$. Now, let us consider three cases

   (1) $a_n \uparrow \infty$;
   (2) $a_n \downarrow 0$ and (2.3) holds;
   (3) $a_n \downarrow 0$, (2.2) holds and $\sup\limits_{t>1} H(t) < \infty$ (We can assume this otherwise it is in fact case 1).

Case 1

Now, if $a_n \uparrow \infty$, we may again without loss of generality suppose that

$$(2.6) \qquad \int_{a_{n+1}}^\infty w(s)s^{-p}\mathrm{d}s \leq \frac{1}{2}\int_{a_n}^\infty w(s)s^{-p}\mathrm{d}s.$$

Fix $N \in \mathbb{N}$. Pick $\{f_k\}_{k=1}^N$, such that

   (1) $\mathrm{supp}(f_k) \subset \mathrm{supp}(f_k)$
   (2) $f_k^*(s) = q_k\chi_{(0,a_k)}$, where

$$q_k = \left(a_k^p \int_{a_k}^\infty w(s)s^{-p}\mathrm{d}s\right)^{-\frac{1}{p}}.$$

Then (2.6) gives

$$(2.7) \qquad \int_{a_n}^\infty w(s)s^{-p}\mathrm{d}s \leq 2\int_{a_n}^{a_{n+1}} w(s)s^{-p}\mathrm{d}s.$$

Note that

$$f_k^{**}(s) = q_k\left(\chi_{(0,a_k)} + a_k s^{-1}\chi_{(a_k,\infty)}\right).$$

Now, by (2.4) we have

$$(2.8) \qquad \begin{aligned} \|f_k\|_X &= q_k\left(W(a_k) + a_k^p\int_{a_k}^\infty w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} \\ &\leq q_k\left(2a_k^p\int_{a_k}^\infty w(s)s^{-p}\mathrm{d}s\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}. \end{aligned}$$

Calculate

$$\left\| \sum_{k=1}^{N} f_k \right\|_X \geq \left\| \sum_{k=1}^{N} f_k^{**} \chi_{(a_k, a_{k+1})} \right\|_{L_w^p}$$

$$= \left(\sum_{k=1}^{N} q_k^p a_k^p \int_{a_k}^{a_{k+1}} w(s) s^{-p} \mathrm{d}s \right)^{\frac{1}{p}}$$

$$\geq 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N} q_k^p a_k^p \int_{a_k}^{\infty} w(s) s^{-p} \mathrm{d}s \right)^{\frac{1}{p}}$$

$$= 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N} 1 \right)^{\frac{1}{p}} \approx N^{\frac{1}{p}}.$$

The third inequality follows from (2.7), while the next one from (2.4). Therefore by Lemma 1 we obtain that $\|\cdot\|_X$ cannot be equivalently normed.

Case 2

Suppose (2.3) holds. If $a_n \downarrow 0$, define $a_0 = \infty$. We may without loss of generality suppose that

(2.9) $$\int_{a_{n+1}}^{\infty} w(s) s^{-p} \mathrm{d}s \geq 2 \int_{a_n}^{\infty} w(s) s^{-p} \mathrm{d}s.$$

Fix $N \in \mathbb{N}$. Now, let us pick $\{f_k\}_{k=1}^N$ with the following properties

(1) $\operatorname{supp}(f_{k+1}) \subset \operatorname{supp}(f_k)$
(2) $f_k^* = q_k \chi_{(0, a_k)}$, where

$$q_k = \left(a_k^p \int_{a_k}^{\infty} w(s) s^{-p} \mathrm{d}s \right)^{-\frac{1}{p}}.$$

The same calculation as in (2.8) gives

$$\|f_k\| \leq 2^{\frac{1}{p}}.$$

Now, by (2.9), we have

(2.10) $$\int_{a_{n+1}}^{a_n} w(s) s^{-p} \mathrm{d}s \geq \frac{1}{2} \int_{a_{n+1}}^{\infty} w(s) s^{-p} \mathrm{d}s.$$

Calculate

$$\left\| \sum_{k=1}^{N} f_k \right\|_X \geq \left\| \sum_{k=1}^{N-1} f_{k+1}^{**} \chi_{(a_{k+1},a_k)} \right\|_{L_w^p}$$

$$= \left(\sum_{k=1}^{N-1} q_{k+1}^p a_{k+1}^p \int_{a_{k+1}}^{a_k} w(s) s^{-p} \mathrm{d}s \right)^{\frac{1}{p}}$$

$$\geq 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N-1} q_{k+1}^p a_{k+1}^p \int_{a_{k+1}}^{\infty} w(s) s^{-p} \mathrm{d}s \right)^{\frac{1}{p}}$$

$$= 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N-1} 1 \right)^{\frac{1}{p}} \approx N^{\frac{1}{p}},$$

where the third inequality follows from (2.10). Therefore, by Lemma 1, the functional is not normable.

Case 3

Now, suppose that the condition (2.2) holds. Again, if we can choose $\{a_n\}_{n=1}^{\infty}$ satisfying (2.4) and such that $a_n \uparrow \infty$, we may use the same calculation as in the previous one. Now if there is no such a sequence, then the function $H(t)$ (where H is defined in (2.5)) is bounded on $[1, \infty)$. Set

$$C := 1 + \sup_{t>1} H(t).$$

Fix $N \in \mathbb{N}$. Since w is not in L^1, we may choose $\{a_k\}_{k=1}^{\infty}$ such that

(2.11) $W(a_{k+1}) \geq 2W(a_k),$

and $a_1 > 1$. Observe that

(2.12) $\int_{a_{k-1}}^{a_k} w(s) \mathrm{d}s \geq \frac{1}{2} W(a_k),$

for $k = 1, \ldots, N$ Find a sequence $\{f_k\}_{k=1}^{N}$ such that
    (1) $\mathrm{supp}(f_k) \subset \mathrm{supp}(f_{k+1})$;
    (2) $f_k^*(s) = b_k \chi_{(0,a_k)}$, where $b_k = W^{-\frac{1}{p}}(a_k)$.
For technical reasons, set $a_0 := 0$. We have

$$\|f_k\|_X = W^{-\frac{1}{p}}(a_k) \left(W(a_k) + a_k \int_{a_k}^{\infty} w(s) s^{-p} \mathrm{d}s \right)^{\frac{1}{p}}$$

$$\leq W^{-\frac{1}{p}}(a_k) \left[W(a_k)(1 + \sup_{t>1} H(t)) \right]^{\frac{1}{p}}$$

$$= C^{\frac{1}{p}}.$$

Calculate

$$\left\|\sum_{k=1}^{N} f_k\right\|_X \geq \left\|\sum_{k=1}^{N} \chi_{(a_{k-1},a_k)} b_k\right\|_{L_w^p}$$

$$= \left(\sum_{k=1}^{N} b_k^p \int_{a_k}^{a_{k+1}} w(s) \mathrm{d}s\right)^{\frac{1}{p}}$$

$$\geq 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N} b_k^p W(a_k)\right)^{\frac{1}{p}}$$

$$= 2^{-\frac{1}{p}} \left(\sum_{k=1}^{N} 1\right)^{\frac{1}{p}} = N^{\frac{1}{p}}.$$

The third inequality follows from (2.12).

Now, let us prove that (ii) implies (iii). We shall prove that if (ii) is satisfied then

(2.13) $$B \int_0^1 f^*(s) \mathrm{d}s \leq \|f\|_X \leq A \int_0^1 f^*(s) \mathrm{d}s,$$

where

$$A := \left[\int_0^1 w(s) s^{-p} \mathrm{d}s \left(1 + \frac{\int_1^\infty w(s) \mathrm{d}s}{\int_1^\infty w(s) \mathrm{d}s}\right)\right]^{\frac{1}{p}}$$

and

$$B := \left(\int_0^1 w(s) \mathrm{d}s\right)^{-\frac{1}{p}}.$$

We have

$$\|f\|_X^p = \int_0^1 f^{**}(s)^p w(s) \mathrm{d}s + \int_1^\infty f^{**}(s)^p w(s) \mathrm{d}s =: \mathrm{I+II}.$$

Let us first estimate the second term by the first one

$$\int_1^\infty f^{**}(s)^p w(s) \mathrm{d}s \leq f^{**}(1)^p \int_0^1 w(s) \mathrm{d}s \left(\frac{\int_1^\infty w(s) \mathrm{d}s}{\int_0^1 w(s) \mathrm{d}s}\right)$$

$$\leq \left(\frac{\int_1^\infty w(s) \mathrm{d}s}{\int_0^1 w(s) \mathrm{d}s}\right) \int_0^1 f^{**}(s)^p w(s) \mathrm{d}s.$$

Now estimate

$$\int_0^1 f^{**}(s)^p w(s) \mathrm{d}s = \int_0^1 w(s) s^{-p} \left(\int_0^s f^*(z) \mathrm{d}z\right)^p \mathrm{d}s$$

$$\leq \int_0^1 w(s) s^{-p} \mathrm{d}s \left(\int_0^1 f^*(z) \mathrm{d}z\right)^p.$$

Due to this two estimates we have

$$\|f\|_X^p \leq A^p \left(\int_0^1 f^*(s) \mathrm{d}s\right)^p.$$

On the other hand note that

$$\left(\int_0^1 f^*(s)\mathrm{d}s\right)^p = f^{**}(1)^p = \left(\int_0^1 w(s)\mathrm{d}s\right)^{-1} f^{**}(1)^p \int_0^1 w(s)\mathrm{d}s$$

$$\leq B^p \int_0^1 f^{**}(s)^p w(s)\mathrm{d}s \leq B^p \|f\|_X^p .$$

Therefore the desired equivalence (see 2.13) holds. $\qquad\square$

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: soudsky@karlin.mff.cuni.cz

EMBEDDINGS OF CLASSICAL LORENTZ SPACES INVOLVING WEIGHTED INTEGRAL MEANS

AMIRAN GOGATISHVILI, MARTIN KŘEPELA, LUBOŠ PICK AND FILIP SOUDSKÝ

ABSTRACT. We characterize embeddings between two classical Lorentz spaces of type Gamma defined with respect to two different weighted means. In particular, we give two-sided estimates of the optimal constant C in the inequality

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s)\, ds \right)^{\frac{m_2}{p_2}} w_2(t)\, dt \right)^{\frac{1}{m_2}} \le C \left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s)\, ds \right)^{\frac{m_1}{p_1}} w_1(t)\, dt \right)^{\frac{1}{m_1}},$$

where $m_1, m_2, p_1, p_2 \in (0, \infty)$, u_1, u_2, w_1, w_2 are weights on $(0, \infty)$ and $p_2 < m_2$. The most innovative part consists of the fact that possibly different inner weights u_1 and u_2 are allowed. Proofs are based on a combination of duality techniques with various kinds of weighted inequalities for iterated operators some of which are known and others are proved here.

1. INTRODUCTION AND THE MAIN RESULT

In this paper we study the weighted inequalities of the form

$$(1.1) \quad \left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s)\, ds \right)^{\frac{m_2}{p_2}} w_2(t)\, dt \right)^{\frac{1}{m_2}} \le C \left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s)\, ds \right)^{\frac{m_1}{p_1}} w_1(t)\, dt \right)^{\frac{1}{m_1}},$$

where m_1, m_2, p_1, p_2 are positive real numbers and u_1, u_2, w_1, w_2 are *weights*, that is, measurable non-negative functions on $(0, \infty)$. The inequality is required to hold with a positive constant C depending only on m_1, m_2, p_1, p_2 and for all scalar measurable functions f defined on a σ-finite measure space $(\mathcal{R}, \mu)$. By f^* we denote the *non-increasing rearrangement* of f, given by

$$f^*(t) = \sup\{\lambda \in \mathbb{R} \colon \mu(\{x \in \mathcal{R} \colon |f(x)| > \lambda\}) > t\} \quad \text{for } t \in (0, \infty).$$

Our main goal is to establish easily verifiable necessary and sufficient conditions on the parameters $m_1, m_2, p_1, p_2 \in (0, \infty)$ and the weights u_1, u_2, w_1, w_2 for which (1.1) holds and to give plausible two-sided estimates of the optimal constant C. We develop a method based on combination of duality techniques with a wide array of estimates of optimal constants in various weighted inequalities involving iterated integral and supremum operators, either applied to any non-negative measurable function or restricted to the cone of non-increasing functions. Plenty of such estimates are known but we shall need also some new ones which we shall state and prove.

We denote by $\mathfrak{M}(\mathcal{R}, \mu)$ the set of all μ-measurable functions on $\mathcal{R}$ whose values belong to $[-\infty, \infty]$. We also define $\mathfrak{M}_+(\mathcal{R}, \mu) = \{g \in \mathfrak{M}(\mathcal{R}, \mu) \colon g \ge 0\}$, and $\mathfrak{M}_0(\mathcal{R}, \mu) = \{g \in \mathfrak{M}(\mathcal{R}, \mu) \colon g \text{ is finite a.e. in } \mathcal{R}\}$.

The inequality (1.1) can be viewed as a continuous embedding between appropriate function spaces. We denote by $\mathrm{G\Gamma}_{u,w}^{m,p}(\mathcal{R}, \mu)$ (or just $\mathrm{G\Gamma}_{u,w}^{m,p}$ for short when no confusion can arise) the collection of all functions $f \in \mathfrak{M}(\mathcal{R}, \mu)$ such that

$$\|f\|_{\mathrm{G\Gamma}_{u,w}^{m,p}} := \left(\int_0^\infty \left(\int_0^t f^*(s)^p u(s)\, ds \right)^{\frac{m}{p}} w(t)\, dt \right)^{\frac{1}{m}} < \infty,$$

where $m, p \in (0, \infty)$ and w is a weight (on $(0, \infty)$). Under this notation, (1.1) is equivalent to the continuous embedding

$$(1.2) \qquad\qquad \mathrm{G\Gamma}_{u_1, w_1}^{m_1, p_1} \hookrightarrow \mathrm{G\Gamma}_{u_2, w_2}^{m_2, p_2}.$$

Moreover, the norm of (1.2) coincides with the optimal (smallest) constant C that renders (1.1) true.

The study of function spaces involving weights and rearrangements goes back at least to the 1950's, when the fundamental papers of Lorentz [42] and [43] first appeared. In particular, in [42], the space $\Lambda^p(v)$ was defined as the set of all $f \in \mathfrak{M}(\mathcal{R}, \mu)$ for which the functional

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty f^*(t)^p v(t) \, dt \right)^{\frac{1}{p}}$$

is finite, where $p \in (0, \infty)$ and v is a weight on $(0, \infty)$. These spaces proved to be indispensable in a wide array of disciplines of mathematical analysis, in particular in theory of interpolation, theory of operators of harmonic analysis and theory of partial differential equations. A further major breakthrough came in 1990, when Ariño and Muckenhoupt in [2] characterized the parameters $p \in (1, \infty)$ and weights v such that the Hardy–Littlewood maximal operator is bounded on $\Lambda^p(v)$. In the same year, Sawyer in [49] developed an extremely useful duality concept for spaces $\Lambda^p(v)$. Among other results he obtained a generalization of Ariño and Muckenhoupt's theorem to situation allowing two possibly different exponents and two possibly different weights and reformulated the action of the maximal operator in terms of embeddings between function spaces by introducing the space $\Gamma^p(v)$ as the family of all $f \in \mathfrak{M}(\mathcal{R}, \mu)$ for which the functional

$$\|f\|_{\Gamma^p(v)} := \left(\int_0^\infty f^{**}(t)^p v(t) \, dt \right)^{\frac{1}{p}}$$

is finite. Here f^{**} is the *maximal non-increasing rearrangement* of f, defined by

$$(1.3) \qquad\qquad f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds \quad \text{for } t \in (0, \infty).$$

It will be useful to note straightaway that

$$(1.4) \qquad\qquad f^*(t) \le f^{**}(t)$$

holds universally for every $f \in \mathfrak{M}(\mathcal{R}, \mu)$ and every $t \in (0, \infty)$, whence one always trivially has

$$\Gamma^p(v) \hookrightarrow \Lambda^p(v).$$

During the 1990's the spaces $\Lambda^p(v)$ and $\Gamma^p(v)$ were put under a serious scrutiny (under the common label *classical Lorentz spaces*) and their mutual relations for various cases of parameters and weights were characterized. It would be next to impossible to give a complete account of the literature which is available to this subject nowadays. We name at least the efforts of M. Carro, A. García del Amo, M. Gol'dman, H. Heinig, L. Maligranda, J. Martín, C. Neugebauer, R. Oinarov, J. Soria, G. Sinnamon, V.D. Stepanov and many others that resulted in the avalanche of papers of which we can mention at least [5, 8, 9, 10, 11, 25, 32, 33, 34, 35, 40, 45, 46, 50, 53, 56, 57] with an apology to the authors of papers that have not been quoted. A first survey of the situation in the subject was given in [8] where the contemporary state of the art was described. Since then, however, important new results have been obtained and the field has changed essentially.

The next significant progress was made in the early 2000's. This was mainly due to the efforts of Sinnamon [51, 52], and also to the discovery of the discretization and anti-discretization techniques by the first and the third author in [26]. Using the newly developed methods, embeddings between classical Lorentz spaces in cases that had not been known before were characterized. The final missing case was added later in [7]. Thanks to these discoveries the field could have been exploited deeper (see e.g. [6, 7, 27, 28]), and, more importantly, new function spaces involving *inner weighted means* could have been involved. In order to describe such function spaces, let us first consider the weighted version of (1.3), namely

$$(1.5) \qquad\qquad f_u^{**}(t) = \frac{1}{U(t)} \int_0^t f^*(s) u(s) \, ds, \quad t \in (0, \infty),$$

where u is a given weight on $(0, \infty)$ and

$$U(t) := \int_0^t u(s) \, ds \qquad \text{for } t \in (0, \infty).$$

Given $p \in (0, \infty)$ and another weight, v, on $(0, \infty)$, we define the space $\Gamma_u^p(v)$ as the collection of all functions $f \in \mathfrak{M}(\mathcal{R}, \mu)$ such that

$$\|f\|_{\Gamma_u^p(v)} := \left(\int_0^\infty f_u^{**}(t)^p v(t) \, dt \right)^{\frac{1}{p}} < \infty.$$

Some authors tried to recover general embedding results for classical Lorentz spaces by methods that would avoid the rather complicated discretization techniques, but only with a partial success (see e.g. [30, 31, 21]). A recent survey of necessary and sufficient conditions for embeddings of classical Lorentz spaces can be found in [48, Chapter 10].

There exists plenty of motivation for studying relations between classical Lorentz spaces in great detail. For example, in the recent work [1], several of the above-mentioned results on classical Lorentz spaces are used in order to investigate the continuity properties of local solutions to the n-Laplace equation

$$- \operatorname{div}(|\nabla u|^{n-2} \nabla u) = f(x) \quad \text{in } \Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^n$.

It should be noted that none of the above would ever have existed without the (now classical) characterizations of weights for which Hardy inequalities hold. This subject, which is, incidentally, exactly one hundred years old, is absolutely indispensable in this part of mathematics. In our proofs below we will of course heavily use it, but we shall also need characterizations of the weighted inequalities involving *supremum operators*. While some of these can be found in literature (cf. [12] and [24]), we will also need some specific iterated and combined versions which are not known and need to be established (see Section 3).

Recently, yet more modified spaces, namely those denoted by $\mathrm{G}\Gamma(p, m, v)$, have been introduced and intensively studied. Given two parameters $m, p \in (0, \infty)$ and a weight v, on $(0, \infty)$, the space $\mathrm{G}\Gamma(p, m, v)$ is defined as the the collection of all functions $f \in \mathfrak{M}(\mathcal{R}, \mu)$ such that

$$\|f\|_{\mathrm{G}\Gamma(p,m,v)} := \left(\int_0^b \left(\int_0^t f^*(s)^p \, ds \right)^{\frac{m}{p}} v(t) \, dt \right)^{\frac{1}{m}} < \infty.$$

These spaces turn out to be important among other reasons because of their intimate connection to the so-called *grand Lebesgue spaces* and their slightly younger relatives called *small Lebesgue spaces*. The grand Lebesgue space was introduced by Iwaniec and Sbordone in [36] in connection with integrability properties of Jacobians. Since it is a relatively complicated structure, it took some time before its dual was characterized. This was done by Fiorenza in [16]. In that paper also the small Lebesgue spaces were introduced. It was shown later by Fiorenza and Karadzhov in [17] that the norm in the small Lebesgue space can be equivalently expressed in terms of the functional governing the $\mathrm{G}\Gamma(p, m, w)$ space with appropriate parameters and weights. Further results in this direction were obtained e.g. in [18, 19, 20]. The associate space of $\mathrm{G}\Gamma(p, m, w)$ was then completely characterized in our earlier work [29].

One of the principal new results of [26] was the characterization of the embeddings of the form

$$(1.6) \qquad\qquad \Gamma_u^q(w) \hookrightarrow \Gamma_u^p(v),$$

where $p, q \in (0, \infty)$ and u, v, w are weights on $(0, \infty)$. Such characterization constitutes an important step ahead, but it still contains a restriction which could diminish its possible field of application to some measure. The drawback consists in the fact that the inner weight u, which determines the corresponding inner integral mean, is the same on both sides of the embedding.

On the side of applications, there exists a significant desire for two-sided estimates of optimal constants in embeddings of the type (1.6), but with *two possibly different inner weights*. This motivation arises usually in tasks that involve in some way two possibly different integral mean operators. To give at least one example, let us recall the long-time extensive research of the optimality of function spaces in Sobolev-type embeddings, carried out in many papers (cf. e.g. [15, 37, 38, 39, 13]). In particular, for instance, in considerations from [39, Theorem 3.1], where the explicit formula for the optimal rearrangement-invariant function norm in a Sobolev inequality is sought and the known implicit one is reduced to a formula involving an integral mean with respect to another weight function, characterizations of embeddings of the form (1.1) would have been useful.

Our principal aim in this paper is to investigate embeddings between Gamma-type spaces enjoying two possibly different inner weights. This is a very difficult and technically complicated task. In order to reach the goal we develop an approach consisting of a duality argument combined with estimates of optimal constants in inequalities involving iterated operators of either integral or supremal type. Detailed analysis of separate cases leads to various, quite different in nature, inequalities, of which only some are known. Interestingly, some are quite recent, such as [22], for instance. Even more interesting, some are not known at all.

Our duality approach does not work in the case $m_2 < p_2$ which therefore remains open. Let us finally note that the case $m_2 = p_2$ is not very interesting since then the space $\mathrm{G\Gamma}^{m_2,p_2}_{u_2,w_2}$ degenerates to a classical Lorentz space of type Λ for which everything is known ([26]).

Now we are in the position to state our main results.

The formulation of the statements as well as the expository of proofs can naturally be expected to be quite technical and to involve plenty of computation. There is hardly any way to avoid it. We shall therefore do our best in order to simplify the notation, shorten the formulas, and make the exposition as reader-friendly as possible.

Most of the functions we deal with is defined on $(0,\infty)$. Then the space $(\mathcal{R}, \mu)$ is just $(0,\infty)$ endowed with the one-dimensional Lebesgue measure λ_1. If that is the case, we shall write just $\mathfrak{M}$, $\mathfrak{M}_+$ and $\mathfrak{M}_0$ instead of $\mathfrak{M}((0,\infty),\lambda_1)$, $\mathfrak{M}_+((0,\infty),\lambda_1)$ and $\mathfrak{M}_0((0,\infty),\lambda_1)$, respectively. The underlying measure space will be indicated only when necessary.

Let u_1, u_2, w_1 and w_2 be weights on $(0,\infty)$ and $t \in (0,\infty)$. We will use the following notation:

$$U_1(t) = \int_0^t u_1(s)\,ds, \quad U_2(t) = \int_0^t u_2(s)\,ds, \quad W_1(t) = \int_0^t w_1(s)\,ds, \quad W_2(t) = \int_0^t w_2(s)\,ds.$$

Further, let $p_1, p_2, m_1, m_2 \in (1,\infty)$. Then we define the function φ by

$$(1.7) \qquad \varphi(t) = \int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s)\,ds + U_1(t)^{\frac{m_1}{p_1}} \int_t^\infty w_1(s)\,ds, \quad t \in (0,\infty).$$

It might be useful to notice that, for $t \in (0,\infty)$,

$$\varphi(t) = \|\chi_{(0,t)}\|_{\mathrm{G\Gamma}^{m_1,p_1}_{u_1,w_1}(0,\infty)}.$$

We furthermore define the function σ by

$$(1.8) \qquad \sigma(t) = \frac{U_1(t)^{\frac{m_1^2}{p_1(m_1-p_2)}-1} u_1(t) \int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s)\,ds \int_t^\infty w_1(s)\,ds}{\varphi(t)^{\frac{m_1}{m_1-p_2}+1}}, \quad t \in (0,\infty).$$

Our principal result is the following theorem.

Theorem 1.1. *Let $p_1, p_2, m_1, m_2 \in (1,\infty)$. Assume that $m_2 > p_2$. Let u_1, u_2, w_1 and w_2 be weights. Assume that*

- *u_1 is strictly positive, $\int_0^t u_1(s)\,ds < \infty$ for all $t \in (0,\infty)$, $\int_0^\infty u_1(s)\,ds = \infty$,*

- *$\int_0^t w_1(s)U_1(s)^{\frac{m_1}{p_1}}\,ds < \infty$, $\int_t^\infty w_1(s)U_1(s)^{\frac{m_1}{p_1}}\,ds = \infty$ for all $t \in (0,\infty)$,*

- *$\int_0^t w_1(s)\,ds = \infty$, $\int_t^\infty w_1(s)\,ds < \infty$ for all $t \in (0,\infty)$.*

We define the quantity C by

$$(1.9) \qquad C = \sup_{f \in \mathfrak{M}} \frac{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s)\,ds \right)^{\frac{m_2}{p_2}} w_2(t)\,dt \right)^{\frac{1}{m_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s)\,ds \right)^{\frac{m_1}{p_1}} w_1(t)\,dt \right)^{\frac{1}{m_1}}}.$$

In all the statements of this theorem, the constants of equivalence depend only on p_1, p_2, m_1 and m_2.

(i) *Let $m_1 \leq p_2$ and $p_1 \leq p_2$. Then*

$$C \approx \sup_{t \in (0,\infty)} \frac{\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s) \, ds + U_2(t)^{\frac{m_2}{p_2}} \int_t^\infty w_2(s) \, ds \right)^{\frac{1}{m_2}}}{\varphi(t)^{\frac{1}{m_1}}}.$$

(ii) *Let $m_1 > p_2$, $p_1 \leq p_2$ and $m_1 \leq m_2$. Then*

$$C \approx B_1 + B_2,$$

where

$$B_1 = \sup_{t \in (0,\infty)} \left(U_1(t)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^t \sigma(s) \, ds \right.$$
$$\left. + \int_t^\infty U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(s) \, ds \right)^{\frac{m_1-p_2}{m_1 p_2}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s) \, ds \right)^{\frac{1}{m_2}}$$

and

$$B_2 = \sup_{t \in (0,\infty)} \left(\int_0^t \left[\sup_{y \in (s,t)} U_2(y) U_1(y)^{-\frac{p_2}{p_1}} \right]^{\frac{m_1}{m_1-p_2}} \sigma(s) \, ds \right)^{\frac{m_1-p_2}{m_1 p_2}} \left(\int_t^\infty w_2(s) \, ds \right)^{\frac{1}{m_2}}.$$

(iii) *Let $m_1 > p_2$, $p_1 \leq p_2$ and $m_1 > m_2$. Then*

$$C \approx B_3 + B_4,$$

where

$$B_3 = \left(\int_0^\infty \left(\int_t^\infty \left[\sup_{y \in (s,\infty)} U_2(y) U_1(y)^{-\frac{p_2}{p_1}} \left(\int_t^\infty w_2(s) \, ds \right)^{\frac{2p_2}{p_2+m_2}} \right]^{\frac{m_1}{m_1-p_2}} \sigma(s) \, ds \right)^{\frac{m_1(m_1-p_2)(m_2-p_2)}{p_2(m_1-m_2)}} \right.$$

$$\times \left[\sup_{s \in (t,\infty)} U_2(s) U_1(s)^{-\frac{p_2}{p_1}} \left(\int_t^\infty w_2(s) \, ds \right)^{\frac{2p_2}{p_2+m_2}} \right]^{\frac{m_1}{m_1-p_2}}$$

$$\times \left. \left(\int_t^\infty w_2(s) \, ds \right)^{-\frac{m_1(m_1-p_2)(m_2-p_2)(p_2+m_2)}{(m_1-m_2)}} \sigma(t) \, dt \right)^{\frac{m_2 p_2(m_1-m_2)}{m_1(m_1-p_2)}}$$

and

$$B_4 = \left(\int_0^\infty \left(\int_0^t \sigma(s) \, ds \right)^{\frac{m_1(m_1-p_2)(m_2-p_2)}{p_2(m_1-m_2)}} \right.$$

$$\times \left[\sup_{y \in (t,\infty)} \left(U_2(y) U_1(y)^{-\frac{p_2}{p_1}} \right)^{\frac{m_2}{m_2-p_2}} \left(\int_t^\infty w_2(z) \, dz \right)^{\frac{p_2}{p_2+m_2}} \right]^{\frac{m_1(m_1-p_2)(m_2-p_2)}{p_2(m_1-m_2)}}$$

$$\times \left. \sigma(t) \, dt \right)^{\frac{m_2 p_2(m_1-m_2)}{m_1(m_1-p_2)}}.$$

(iv) *Let $m_1 \leq p_2$, $p_1 > p_2$ and $p_1 \leq m_2$. Then*

$$C \approx B_5 + B_6,$$

where

$$B_5 = \sup_{t \in (0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}}}{\varphi(t)^{\frac{1}{m_1}}} \sup_{s \in (t,\infty)} U_1(s)^{-\frac{1}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y) \, dy \right)^{\frac{1}{m_2}}$$

and

$$B_6 = \sup_{t \in (0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}}}{\varphi(t)^{\frac{1}{m_1}}} \sup_{s \in (t,\infty)} W_2(s)^{\frac{1}{m_2}} \left(\int_t^s U_2(y)^{\frac{p_1}{p_1-p_2}} U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y) \, dy \right)^{\frac{p_1-p_2}{p_1 p_2}}.$$

(v) *Let $m_1 \leq p_2$, $p_1 > p_2$ and $p_1 > m_2$. Then*

$$C \approx B_7 + B_8 + B_9,$$

where

$$B_7 = \sup_{t \in (0,\infty)} \frac{\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s) \, ds \right)^{\frac{1}{m_2}}}{\varphi(t)^{\frac{1}{m_1}}},$$

$$B_8 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}} \left(\int_t^\infty \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\, dy \right)^{\frac{m_2}{p_1-m_2}} U_2(s)^{\frac{m_2}{p_2}} w_2(s) U_1(s)^{-\frac{m_2}{p_1-m_2}}\, ds \right)^{\frac{p_1-m_2}{p_1 m_2}}}{\varphi(t)^{\frac{1}{m_1}}}$$

and

$$B_9 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}} \left(\int_t^\infty \left(\int_t^s U_2(y)^{\frac{p_1}{p_1-p_2}} U_1(y)^{-\frac{p_1}{p_1-p_2}}\, dy \right)^{\frac{m_2(p_1-p_2)}{p_2(p_1-m_2)}} \left(\int_s^\infty w_2(y)\, dy \right)^{\frac{m_2}{p_1-m_2}} w_2(t)\, dt \right)^{\frac{p_1-m_2}{p_1 m_2}}}{\varphi(t)^{\frac{1}{m_1}}}.$$

(vi) *Let $p_2 < m_1 < p_1 \le m_2$. Then*

$$C \approx B_{10} + B_{11} + B_{12},$$

where

$$B_{10} = \sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\, ds \right)^{\frac{m_1-p_2}{m_1 p_2}} U_1(t)^{-\frac{1}{p_1}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\, ds \right)^{\frac{1}{m_2}},$$

$$B_{11} = \sup_{t\in(0,\infty)} \left(\int_0^t U_1(s)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(s)\, ds \right)^{\frac{m_1-p_2}{m_1 p_2}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\, ds \right)^{\frac{1}{m_2}}$$

and

$$B_{12} = \sup_{t\in(0,\infty)} \left(\int_t^\infty w_2(s)\, ds \right)^{\frac{1}{m_2}} \left(\int_0^t \left(\int_s^t U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y)\, dy \right)^{\frac{m_1(p_1-p_2)}{p_1(m_1-p_2)}} \sigma(s)\, ds \right)^{\frac{m_1-p_2}{m_1 p_2}}.$$

(vii) *Let $p_2 < m_1 \le m_2 < p_1$. Then*

$$C \approx B_{11} + B_{12} + B_{13},$$

where

$$B_{13} = \sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\, ds \right)^{\frac{m_1-p_2}{m_1 p_2}} \left(\int_t^\infty U_1(s)^{-\frac{m_2}{p_1-m_2}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\, dy \right)^{\frac{m_2}{p_1-m_2}} U_2(s)^{\frac{m_2}{p_2}} w_2(s)\, ds \right)^{\frac{p_1-m_2}{p_1 m_2}}$$

$$+ \sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\, ds \right)^{\frac{m_1-p_2}{m_1 p_2}} \left(\int_t^\infty \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y)\, dy \right)^{\frac{m_2(p_1-p_2)}{p_2(m_1-m_2)}} \right.$$

$$\times \left. \left(\int_s^\infty w_2(y)\, dy \right)^{\frac{m_2}{p_1-m_2}} w_2(s)\, ds \right)^{\frac{p_1-m_2}{p_1 m_2}}$$

(viii) *Let $p_2 \le m_2 < m_1 < p_1$. Then*

$$C \approx B_{14},$$

where

$$B_{14} = \left(\int_0^\infty \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s) \, ds \right)^{\frac{m_1}{m_1-m_2}} \left(\int_t^\infty U_1(s)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(s) \, ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} U_1(t)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(t) \, dt \right)^{\frac{m_1-m_2}{m_1 m_2}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty U_1(s)^{-\frac{m_2}{p_1-m_2}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y) \, dy \right)^{\frac{m_2}{p_1-m_2}} U_2(s)^{\frac{m_2}{p_2}} w_2(s) \, ds \right)^{\frac{m_1(p_1-m_2)}{p_1(m_1-m_2)}} \right.$$

$$\left. \times \left(\int_0^t \sigma(s) \, ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \sigma(t) \, dt \right)^{\frac{m_1-m_2}{m_1 m_2}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y) \, dy \right)^{\frac{m_2(p_1-p_2)}{p_2(p_1-m_2)}} \left(\int_s^\infty w_2(y) \, dy \right)^{\frac{m_2}{p_1-m_2}} w_2(s) \, ds \right)^{\frac{m_1(p_1-m_2)}{p_1(m_1-m_2)}} \right.$$

$$\left. \times \left(\int_0^t \sigma(s) \, ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \sigma(t) \, dt \right)^{\frac{m_1-m_2}{m_1 m_2}}$$

$$+ \left(\int_0^\infty \left(\int_0^t \left(\int_s^t U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y) \, dy \right)^{\frac{p_1-p_2}{p_1}} \sigma(s) \, ds \right)^{\frac{m_2(m_1-p_2)}{p_2(m_1-m_2)}} \right.$$

$$\left. \times \left(\int_t^\infty w_2(s) \, ds \right)^{\frac{m_2}{m_1-m_2}} w_2(t) \, dt \right)^{\frac{m_1-m_2}{m_1 m_2}}.$$

The paper is organized as follows. In the next section we collect some background material and some known results which will be used in the proofs. In Section 3 we state and prove several new statements involving weighted inequalities for iterated integral and supremum operators which will also be needed in the proof of the main theorem. Section 4 then contains the proof of Theorem 1.1.

2. BACKGROUND MATERIAL

We recall the following well-known duality principle of weighted Lebesgue spaces, which will be useful in the proofs below. If $p \in (1,\infty)$, $f \in \mathfrak{M}_+$ and v is a weight on $(0,\infty)$, then

$$(2.1) \qquad \left(\int_0^\infty f(t)^p v(t) \, dt \right)^{\frac{1}{p}} = \sup_{g \in \mathfrak{M}_+} \frac{\int_0^\infty f(t) g(t) \, dt}{\left(\int_0^\infty g(t)^{p'} v(t)^{1-p'} \, dt \right)^{\frac{1}{p'}}}.$$

For the convenience of the reader, we shall first collect known results which will be used in the proof.

Theorem 2.1 ([4, Theorem 1]). *Let $1 \le p \le q \le \infty$ and let v, w be weights on $(0,\infty)$. Then the inequality*

$$(2.2) \qquad \left\| w(t) \int_0^t f(s) \, ds \right\|_{L^q(0,\infty)} \le K \, \|fv\|_{L^p(0,\infty)}$$

holds for some positive K and every $f \in \mathfrak{M}_+$ if and only if

$$(2.3) \qquad B = \sup_{t \in (0,\infty)} \left\| w \chi_{(t,\infty)} \right\|_{L^q(0,\infty)} \left\| v^{-1} \chi_{(0,t)} \right\|_{L^{p'}(0,\infty)} < \infty.$$

Furthermore, the optimal constant K in (2.2) satisfies

$$B \le K \le p^{\frac{1}{q}} (p')^{\frac{1}{p'}} B \quad \text{if } p = 1 \text{ or } q = \infty.$$

if $1 < p \le q < \infty$ and

$$B = K \quad \text{if } p = 1 \text{ or } q = \infty.$$

if $p = 1$ or $q = \infty$.

Theorem 2.2 ([4, Theorem 2])**.** *Let $1 \le p \le q \le \infty$ and let v, w be weights on $(0, \infty)$. Then the inequality*

$$(2.4) \qquad \left\| w(t) \int_t^\infty f(s) \, ds \right\|_{L^q(0,\infty)} \le K \, \|fv\|_{L^p(0,\infty)}$$

holds for some positive K and every $f \in \mathfrak{M}_+$ if and only if

$$(2.5) \qquad B = \sup_{t \in (0,\infty)} \left\| w\chi_{(0,t)} \right\|_{L^q(0,\infty)} \left\| v^{-1}\chi_{(t,\infty)} \right\|_{L^{p'}(0,\infty)} < \infty.$$

Furthermore, the optimal constant K in (2.2) satisfies

$$B \le K \le p^{\frac{1}{q}} (p')^{\frac{1}{p'}} B$$

if $1 < p \le q < \infty$ and

$$B = K$$

if $p = 1$ or $q = \infty$.

Theorem 2.3 ([44, Theorem 1.3.1])**.** *Let $1 \le q < p < \infty$ and let v, w be weights on $(0, \infty)$. Set*

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}.$$

Then the inequality

$$(2.6) \qquad \left(\int_0^\infty \left(\int_0^t f(s) \, ds \right)^q w(t) \, dt \right)^{\frac{1}{q}} \le K \left(\int_0^\infty f(t)^p v(t) \, dt \right)^{\frac{1}{p}}$$

holds for some positive K and every $f \in \mathfrak{M}_+$ if and only if

$$(2.7) \qquad A = \left(\int_0^\infty \left(\int_t^\infty w(s) \, ds \right)^{\frac{r}{q}} \left(\int_0^t v(s)^{1-p'} \, ds \right)^{\frac{r}{q'}} v(t)^{1-p'} \, dt \right)^{\frac{1}{r}}.$$

Furthermore, the optimal constant K in (2.6) satisfies

$$K \approx A.$$

Apart from Hardy inequalities, we shall also need analogous results concerning supremum operators.

Theorem 2.4 ([24, Theorem 3.2(i)])**.** *Let $0 < p \le q < \infty$ and let u be a continuous weight. Let v and w be weights such that $0 < \int_0^t v(s) \, ds < \infty$ and $0 < \int_0^t w(s) \, ds < \infty$ for every $t \in (0, \infty)$. Then*

$$(2.8) \qquad \left(\int_0^\infty \left(\sup_{s \in (t,\infty)} u(s) h^*(s) \right)^q w(t) \, dt \right)^{\frac{1}{q}} \le K \left(\int_0^\infty h^*(t)^p v(t) \, dt \right)^{\frac{1}{p}}$$

is satisfied for a positive constant K and every $h \in \mathfrak{M}$ if and only if

$$(2.9) \qquad B = \sup_{t \in (0,\infty)} \left(\int_0^t \sup_{y \in (s,t)} u(y)^q w(s) \, ds \right)^{\frac{1}{q}} \left(\int_0^t v(s) \, ds \right)^{-\frac{1}{p}} < \infty.$$

Moreover,

$$K \approx B.$$

Theorem 2.5 ([24, Theorem 4.4])**.** *Let u be a continuous weight and let v and w be weights such that $0 < \int_0^t v(s) \, ds < \infty$ and $0 < \int_0^t w(s) \, ds < \infty$ for every $t \in (0, \infty)$. Let $1 \le p < \infty$ and $0 < q < p$ and let r be defined by*

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}.$$

For $t \in (0, \infty)$, set

$$\sigma_p(0, t) = \begin{cases} \left(\int_0^t v(s)^{1-p'} \, ds \right)^{\frac{1}{p'}} & \text{when } 1 < p < \infty, \\ \operatorname*{ess\,sup}_{s \in (0,t)} \frac{1}{v(s)} & \text{when } p = 1. \end{cases}$$

Then the inequality

$$(2.10) \qquad \left(\int_0^\infty \left(\sup_{s \in (t,\infty)} \frac{u(s)}{s} \int_0^s g(y) \, dy \right)^q w(t) \, dt \right)^{\frac{1}{q}} \le K \left(\int_0^\infty g(t)^p v(t) \, dt \right)^{\frac{1}{p}}$$

holds for some $K > 0$ and every $g \in \mathfrak{M}_+$ if and only if

$$(2.11) \qquad A_1 = \left(\int_0^\infty \left(\int_t^\infty \left(\sup_{y \in (s,\infty)} \frac{u(y)}{y} \right)^q w(s) \, ds \right)^{\frac{r}{p}} \left(\sup_{s \in (t,\infty)} \frac{u(s)}{s} \right)^q \sigma_p(0,t)^r w(t) \, dt \right)^{\frac{1}{r}} < \infty$$

and

$$(2.12) \qquad A_2 = \left(\int_0^\infty \left(\int_0^t w(s) \, ds \right)^{\frac{r}{p}} \left[\sup_{y \in (t,\infty)} \sup_{z \in (y,\infty)} \frac{u(z)}{z} \sigma_p(0,y) \right]^r w(t) \, dt \right)^{\frac{1}{r}} < \infty.$$

Moreover, the optimal constant K in (2.10) satisfies

$$K \approx A_1 + A_2.$$

3. Weighted inequalities for integral and supremum operators

In certain stages of the proof of the main result a reformulation of conditions on weights will be required. This procedure is as usual done by an elementary computation based on integration by parts. Although it is a folklore, we sketch the simple proof of it in one of the cases for the sake of completeness.

Lemma 3.1. *Let w, u be weights. Assume that*

$$\int_0^\infty u(t) \, dt = \infty.$$

Let $0 < q < 1$. Then, for every $t \in (0, \infty)$, one has

$$W(t)^{\frac{1}{q}} + U(t) \left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}} \, ds \right)^{\frac{1-q}{q}} \approx U(t) \left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s) \, ds \right)^{\frac{1-q}{q}},$$

in which the constants of equivalence depend only on q.

Proof. Fix $t \in (0, \infty)$. Integration by parts yields

$$(3.1) \qquad \int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}} \, ds$$

$$= q \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s) \, ds + (1-q) \left(\lim_{y \to \infty} \frac{W(y)^{\frac{1}{1-q}}}{U(y)^{\frac{q}{1-q}}} - \frac{W(t)^{\frac{1}{1-q}}}{U(t)^{\frac{q}{1-q}}} \right).$$

Therefore, we immediately have

$$\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}} \, ds$$

$$\le q \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s) \, ds + (1-q) \lim_{y \to \infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}}.$$

Next,

$$\lim_{y\to\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}} \le \sup_{t\le y<\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}}$$

$$= \frac{q}{1-q} \sup_{t\le y<\infty} W(y)^{\frac{1}{1-q}} \int_y^\infty U(s)^{-\frac{1}{1-q}} u(s)\, ds$$

$$\le \frac{q}{1-q} \sup_{t\le y<\infty} \int_y^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds$$

$$= \frac{q}{1-q} \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds.$$

Altogether, we obtain

$$(3.2) \qquad \int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}}\, ds \le 2q \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds.$$

We also have

$$W(t)^{\frac{1}{1-q}} = W(t)^{\frac{1}{1-q}} U(t)^{\frac{q}{1-q}} U(t)^{-\frac{q}{1-q}}$$

$$= \frac{1-q}{q} W(t)^{\frac{1}{1-q}} U(t)^{\frac{q}{1-q}} \int_t^\infty U(s)^{-\frac{1}{1-q}} u(s)\, ds$$

$$\le \frac{1-q}{q} U(t)^{\frac{q}{1-q}} \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds.$$

Raising the inequality to $\frac{1-q}{q}$, we get

$$(3.3) \qquad W(t)^{\frac{1}{q}} \le \left(\frac{1-q}{q}\right)^{\frac{1-q}{q}} U(t) \left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds\right)^{\frac{1-q}{q}}.$$

Altogether, (3.2) and (3.3) imply

$$W(t)^{\frac{1}{q}} + U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\, ds\right)^{\frac{1-q}{q}} \le C_q U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds\right)^{\frac{1-q}{q}}$$

in which

$$C_q = \left(\frac{1-q}{q}\right)^{\frac{1-q}{q}} + (2q)^{\frac{1-q}{q}}.$$

Conversely, by (3.1) again, we have

$$\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds$$

$$\le \frac{1}{q} \int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\, ds + \left(\frac{1-q}{q}\right)^{\frac{1-q}{q}} \frac{W(t)^{\frac{1}{1-q}}}{U(t)^{\frac{q}{1-q}}}.$$

Raising this estimate to $\frac{1-q}{q}$ and multiplying it by $U(t)$, we obtain

$$U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\, ds\right)^{\frac{1-q}{q}}$$

$$\le \left(\frac{1}{q}\right)^{\frac{1-q}{q}} U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\, ds\right)^{\frac{1-q}{q}} + \left(\frac{1-q}{q}\right)^{\frac{1-q}{q}} W(t)^{\frac{1}{q}}.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.2. *Assume that $1 < p \le q < \infty$. Let v, w be weights on $(0,\infty)$ and let u be a continuous weight on $(0,\infty)$. Denote*

$$K = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\sup_{s\in(t,\infty)} u(s) \int_s^\infty g(y)\, dy\right)^q w(t)\, dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\, ds\right)^{\frac{1}{p}}}.$$

Then

$$K \approx A,$$

where

$$A = \sup_{t \in (0,\infty)} \Big(\int_0^t w(s) \sup_{y \in (s,t)} u(y)^q \, ds \Big)^{\frac{1}{q}} \Big(\int_t^\infty v(s)^{1-p'} \, ds \Big)^{\frac{1}{p'}}.$$

Proof. Assume that $K < \infty$. Fix $t \in (0,\infty)$. Then, by duality of weighted Lebesgue spaces, there exists a function $g \in L^{p'}(v)(t,\infty)$ such that

$$\Big(\int_t^\infty v(s)^{1-p'} \, ds \Big)^{\frac{1}{p'}} \le 2 \Big(\int_t^\infty g(s) \, ds \Big) \Big(\int_t^\infty g(s)^p v(s) \, ds \Big)^{-\frac{1}{p}}.$$

Therefore,

$$\Big(\int_0^t w(s) \sup_{y \in (s,t)} u(y)^q \, ds \Big)^{\frac{1}{q}} \Big(\int_t^\infty v(s)^{1-p'} \, ds \Big)^{\frac{1}{p'}}$$

$$\le 2 \Big(\int_0^t w(s) \sup_{y \in (s,t)} u(y)^q \, ds \Big)^{\frac{1}{q}} \Big(\int_t^\infty g(s) \, ds \Big) \Big(\int_t^\infty g(s)^p v(s) \, ds \Big)^{-\frac{1}{p}}$$

$$\le 2 \Big(\int_0^t w(s) \Big(\sup_{y \in (s,t)} u(y) \int_y^\infty g(z) \, dz \Big)^q \, ds \Big)^{\frac{1}{q}} \Big(\int_t^\infty g(s)^p v(s) \, ds \Big)^{-\frac{1}{p}}$$

$$\le 2K.$$

Passing to supremum over $t \in (0,\infty)$, we get $A \le 2K$.

Conversely, assume that $A < \infty$. Assume first that, for every $\varepsilon > 0$, one has

(3.4)
$$\int_0^\varepsilon v(s)^{1-p'} \, ds = \infty.$$

Let $g \in L^p(v)$. Then the function $h(t) = \int_t^\infty g(s) \, ds$, $t \in (0,\infty)$, is non-increasing. Hence, applying Theorem 2.4, we obtain

$$\Big(\int_0^\infty \Big(\sup_{s \in (t,\infty)} u(s) \int_s^\infty g(y) \, dy \Big)^q w(t) \, dt \Big)^{\frac{1}{q}}$$

$$\lesssim \sup_{t \in (0,\infty)} \Big(\int_0^t \sup_{y \in (s,t)} u(y)^q w(s) \, ds \Big)^{\frac{1}{q}} \Big(\int_0^t \Big(\int_s^\infty v(y)^{1-p'} \, dy \Big)^{-p} v(s)^{1-p'} \, ds \Big)^{-\frac{1}{p}}$$

$$\times \Big(\int_0^\infty \Big(\int_s^\infty g(y) \, dy \Big)^p \Big(\int_s^\infty v(z)^{1-p'} \, dz \Big)^{-p} v(s)^{1-p'} \, ds \Big)^{\frac{1}{p}},$$

that is,

$$\Big(\int_0^\infty \Big(\sup_{s \in (t,\infty)} u(s) \int_s^\infty g(y) \, dy \Big)^q w(t) \, dt \Big)^{\frac{1}{q}} \lesssim A \Big(\int_0^\infty \Big(\int_s^\infty g(y) \, dy \Big)^p \Big(\int_s^\infty v(z)^{1-p'} \, dz \Big)^{-p} v(s)^{1-p'} \, ds \Big)^{\frac{1}{p}}.$$

By Theorem 2.2, one has

$$\Big(\int_0^\infty \Big(\int_s^\infty g(y) \, dy \Big)^p \Big(\int_s^\infty v(z)^{1-p'} \, dz \Big)^{-p} v(s)^{1-p'} \, ds \Big)^{\frac{1}{p}} \lesssim \Big(\int_0^\infty g(t)^p v(t) \, dt \Big)^{\frac{1}{p}}.$$

Altogether, it follows that

$$\Big(\int_0^\infty \Big(\sup_{s \in (t,\infty)} u(s) \int_s^\infty g(y) \, dy \Big)^q w(t) \, dt \Big)^{\frac{1}{q}} \lesssim A \Big(\int_0^\infty g(t)^p v(t) \, dt \Big)^{\frac{1}{p}},$$

proving that $K \lesssim A$.

When (3.4) is not satisfied, we get the result in a similar manner by an approximation procedure. We omit the details for the sake of brevity. The proof is complete. □

Theorem 3.3. *Assume that $1 < q < p < \infty$. Let v, w be weights on $(0, \infty)$ and let u be a continuous weight on $(0, \infty)$. Assume moreover that*

$$\int_t^\infty v(s)^{1-p'} ds < \infty \quad \text{for every } t \in (0, \infty).$$

Denote

$$K = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\sup_{s \in (t, \infty)} u(s) \int_s^\infty g(y)\, dy \right)^q w(t)\, dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\, ds \right)^{\frac{1}{p}}}.$$

Denote moreover

$$\Psi(t) = \left(\int_t^\infty v(s)^{1-p'} ds \right)^{\frac{1}{p'+1}}, \quad t \in (0, \infty).$$

Then

$$K \approx A_1 + A_2,$$

where

$$A_1 = \left(\int_0^\infty \left(\int_t^\infty \left[\sup_{y \in (s, \infty)} u(y) \Psi(y)^2 \right]^q w(s)\, ds \right)^{\frac{p}{p-q}} \Psi(t)^{-\frac{p}{p-q}-p'} v(t)^{1-p'}\, dt \right)^{\frac{p-q}{q}}$$

and

$$A_2 = \left(\int_0^\infty \left(\int_0^t w(s)\, ds \right)^{\frac{q}{p-q}} \left[\sup_{y \in (t, \infty)} u(y)^p \Psi(y)^{2p-1} \right]^{\frac{q}{p-q}} w(t)\, dt \right)^{\frac{p-q}{pq}}.$$

Proof. Define the operator T by

$$Tg(t) = \sup_{s \in (t, \infty)} u(s) g(s), \quad g \in \mathfrak{M}_+, \ t \in (0, \infty).$$

Then T satisfies the conditions

$$T(\lambda g) = \lambda Tg \quad \text{for every } \lambda \geq 0 \text{ and } g \in \mathfrak{M}_+,$$
$$Tf \leq Tg \text{ a.e.} \quad \text{if} \quad f \leq g \text{ a.e,}$$
$$T(g + \lambda \chi_{(0,\infty)}) \leq Tg + \lambda T\chi_{(0,\infty)} \quad \text{for every } \lambda \geq 0 \text{ and } g \in \mathfrak{M}_+.$$

Thus, by [22, Corollary 3.5], we obtain

$$K \approx K_1 + K_2,$$

where

$$K_1 = \sup_{h \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left[\sup_{s \in (t,\infty)} u(s) \Psi(s)^2 \left(\int_0^s h(y)^p\, dy \right)^{\frac{1}{p}} \right]^q w(t)\, dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty h(s)^p \Psi(s)\, ds \right)^{\frac{1}{p}}},$$

$$K_2 = \left(\int_0^\infty \left[\sup_{s \in (t,\infty)} u(s) \Psi(s)^2 \right]^q w(t)\, dt \right)^{\frac{1}{q}} \Psi(0)^{-\frac{1}{p}}.$$

Substituting $g = h^p$, we can write

$$K_1^p = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left[\sup_{s \in (t,\infty)} u(s)^p \Psi(s)^{2p} \int_0^s g(y)\, dy \right]^{\frac{q}{p}} w(t)\, dt \right)^{\frac{p}{q}}}{\int_0^\infty g(s) \Psi(s)\, ds}.$$

Using Theorem 2.5, we obtain

$$K_1 \approx K_3 + K_4,$$

where

$$K_3^p = \left(\int_0^\infty \left(\int_t^\infty \left[\sup_{y \in (s,\infty)} u(y) \Psi(y)^2 \right]^q w(s)\, ds \right)^{\frac{q}{p-q}} \left[\sup_{y \in (t,\infty)} u(y) \Psi(y)^2 \right]^q \Psi(t)^{-\frac{q}{p-q}} w(t)\, dt \right)^{\frac{p-q}{q}}$$

and

$$K_4^p = \left(\int_0^\infty \left(\int_0^t w(s)\, ds \right)^{\frac{q}{p-q}} \left[\sup_{y \in (t,\infty)} \frac{1}{\Psi(y)} \sup_{\tau \in (y,\infty)} u(\tau)^p \Psi(\tau)^{2p} \right]^{\frac{q}{p-q}} w(t)\, dt \right)^{\frac{p-q}{q}}.$$

Note that
$$K \approx K_2 + K_3 + K_4.$$

Now, by an argument analogous to Lemma 3.1, we obtain
$$K_2 + K_3 \approx A_1.$$

Finally, interchanging the suprema in the last formula and using the fact that the function $\frac{1}{\Psi}$ is nondecreasing on $(0, \infty)$, we in fact have
$$K_4^p = \left(\int_0^\infty \left(\int_0^t w(s) \, ds \right)^{\frac{q}{p-q}} \left[\sup_{y \in (t,\infty)} u(y)^p \Psi(y)^{2p-1} \right]^{\frac{q}{p-q}} w(t) \, dt \right)^{\frac{p-q}{q}},$$

showing, on taking roots,
$$K_4 \approx A_2.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Theorem 3.4. *Assume that $1 < p, q < \infty$. Let u, v be weights on $(0, \infty)$ and let w be a continuous weight on $(0, \infty)$. Denote*

$$K = \sup_{g \in \mathfrak{M}_+} \frac{\sup\limits_{t \in (0,\infty)} w(t) \left(\int_t^\infty \left(\int_0^s g(y) \, dy \right)^q u(s) \, ds \right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s) \, ds \right)^{\frac{1}{p}}}.$$

(i) *If $p \le q$, then*
$$K \approx A_1,$$

where
$$A_1 = \sup_{t \in (0,\infty)} w(t) \sup_{s \in (t,\infty)} \left(\int_s^\infty u(y) \, dy \right)^{\frac{1}{q}} \left(\int_0^s v(y)^{1-p'} \, dy \right)^{\frac{1}{p'}}.$$

(ii) *If $q < p$, then*
$$K \approx A_2,$$

where
$$A_2 = \sup_{t \in (0,\infty)} w(t) \left(\int_t^\infty \left(\int_s^\infty u(y) \, dy \right)^{\frac{q}{p-q}} \left(\int_0^s v(y)^{1-p'} \, dy \right)^{\frac{q(p-1)}{p-q}} u(s) \, ds \right)^{\frac{p-q}{pq}}.$$

Proof. We can rewrite K in the form

$$(3.5) \qquad K = \sup_{t \in (0,\infty)} w(t) \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^s g(y) \, dy \right)^q u(s) \chi_{(t,\infty)}(s) \, ds \right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s) \, ds \right)^{\frac{1}{p}}}.$$

(i) Assume that $p \le q$. Fix $t \in (0, \infty)$. Then, by a trivial modification of Theorem 2.1, we get

$$\sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^s g(y) \, dy \right)^q u(s) \chi_{(t,\infty)}(s) \, ds \right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s) \, ds \right)^{\frac{1}{p}}} \approx \sup_{s \in (0,\infty)} \left(\int_s^\infty u(y) \chi_{(t,\infty)}(y) \, dy \right)^{\frac{1}{q}} \left(\int_0^s v(y)^{1-p'} \, dy \right)^{\frac{1}{p'}}.$$

Working out the right-hand side, we get

$$\sup_{s \in (0,\infty)} \left(\int_s^\infty u(y) \chi_{(t,\infty)}(y) \, dy \right)^{\frac{1}{q}} \left(\int_0^s v(y)^{1-p'} \, dy \right)^{\frac{1}{p'}}$$
$$\approx \sup_{s \in (0,t)} \left(\int_t^\infty u(y) \, dy \right)^{\frac{1}{q}} \left(\int_0^s v(y)^{1-p'} \, dy \right)^{\frac{1}{p'}} + \sup_{s \in (t,\infty)} \left(\int_s^\infty u(y) \, dy \right)^{\frac{1}{q}} \left(\int_0^s v(y)^{1-p'} \, dy \right)^{\frac{1}{p'}}.$$

Now the supremum in the first term is obviously attained at $s = t$, hence

$$\sup_{s \in (0,\infty)} \Big(\int_s^\infty u(y) \chi_{(t,\infty)}(y)\, dy \Big)^{\frac{1}{q}} \Big(\int_0^s v(y)^{1-p'}\, dy \Big)^{\frac{1}{p'}}$$

$$\approx \Big(\int_t^\infty u(y)\, dy \Big)^{\frac{1}{q}} \Big(\int_0^t v(y)^{1-p'}\, dy \Big)^{\frac{1}{p'}} + \sup_{s \in (t,\infty)} \Big(\int_s^\infty u(y)\, dy \Big)^{\frac{1}{q}} \Big(\int_0^s v(y)^{1-p'}\, dy \Big)^{\frac{1}{p'}}.$$

Now the first term on the right is clearly majorized by the second, so we finally obtain

$$\sup_{g \in \mathfrak{M}_+} \frac{\Big(\int_0^\infty \big(\int_0^s g(y)\, dy \big)^q u(s) \chi_{(t,\infty)}(s)\, ds \Big)^{\frac{1}{q}}}{\Big(\int_0^\infty g(s)^p v(s)\, ds \Big)^{\frac{1}{p}}} \approx \sup_{s \in (t,\infty)} \Big(\int_s^\infty u(y)\, dy \Big)^{\frac{1}{q}} \Big(\int_0^s v(y)^{1-p'}\, dy \Big)^{\frac{1}{p'}}.$$

Altogether, plugging this into (3.5) yields $K \approx A_1$, which establishes the assertion in the case (i).

(ii) Assume that $q < p$. Then, following the same argument as above and using Theorem 2.3 instead of Theorem 2.1, we obtain

$$K \approx \sup_{t \in (0,\infty)} w(t) \Big(\int_t^\infty u(y)\, dy \Big)^{\frac{1}{q}} \Big(\int_0^t v(s)^{1-p'}\, ds \Big)^{\frac{1}{p'}}$$

$$+ \sup_{t \in (0,\infty)} w(t) \Big(\int_t^\infty \big(\int_s^\infty u(y)\, dy \big)^{\frac{p}{p-q}} \big(\int_0^s v(y)^{1-p'}\, dy \big)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'}\, ds \Big)^{\frac{p-q}{pq}}.$$

By a slight modification of Lemma 3.1 we can prove that the last sum is equivalent to A_2. The proof is complete. $\qquad\square$

Theorem 3.5. *Assume that $1 < p, q < \infty$. Let u, v be weights on $(0, \infty)$ and let w be a continuous weight on $(0, \infty)$. Denote*

$$K = \sup_{g \in \mathfrak{M}_+} \frac{\sup\limits_{t \in (0,\infty)} w(t) \Big(\int_t^\infty \big(\int_s^\infty g(y)\, dy \big)^q u(s)\, ds \Big)^{\frac{1}{q}}}{\Big(\int_0^\infty g(s)^p v(s)\, ds \Big)^{\frac{1}{p}}}.$$

(i) *If $p \le q$, then*

$$K \approx A_1,$$

where

$$A_1 = \sup_{x \in (0,\infty)} w(x) \sup_{t \in (x,\infty)} \Big(\int_x^t u(s)\, ds \Big)^{\frac{1}{q}} \Big(\int_t^\infty v(s)^{1-p'}\, ds \Big)^{\frac{1}{p'}}.$$

(ii) *If $q < p$, then*

$$K \approx A_2,$$

where

$$A_2 = \sup_{x \in (0,\infty)} w(x) \Big(\int_x^\infty \big(\int_x^s u(y)\, dy \big)^{\frac{p}{p-q}} \big(\int_s^\infty v(y)^{1-p'}\, dy \big)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'}\, ds \Big)^{\frac{p-q}{pq}}.$$

Proof. (i) Let $p \le q$. Then

$$K = \sup_{t \in (0,\infty)} w(t) \sup_{g \in \mathfrak{M}_+} \frac{\Big(\int_0^\infty \big(\int_s^\infty g(y)\, dy \big)^q u(s) \chi_{(t,\infty)}(s)\, ds \Big)^{\frac{1}{q}}}{\Big(\int_0^\infty g(s)^p v(s)\, ds \Big)^{\frac{1}{p}}}$$

$$\approx \sup_{t \in (0,\infty)} w(t) \sup_{x \in (0,\infty)} \Big(\int_0^x u(s) \chi_{(t,\infty)}(s)\, ds \Big)^{\frac{1}{q}} \Big(\int_x^\infty v(s)^{1-p'}\, ds \Big)^{\frac{1}{p'}}$$

$$= \sup_{t \in (0,\infty)} w(t) \sup_{x \in (t,\infty)} \Big(\int_t^x u(s)\, ds \Big)^{\frac{1}{q}} \Big(\int_x^\infty v(s)^{1-p'}\, ds \Big)^{\frac{1}{p'}},$$

as desired.

(ii) Now let $q < p$. Then

$$K \approx \sup_{t \in (0,\infty)} w(t) \Big(\int_0^\infty \Big(\int_0^s u(y) \chi_{(t,\infty)}(y) \, dy \Big)^{\frac{p}{p-q}} \Big(\int_s^\infty v(y)^{1-p'} \, dy \Big)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'} \, ds \Big)^{\frac{p-q}{pq}}$$

$$= \sup_{t \in (0,\infty)} w(t) \Big(\int_t^\infty \Big(\int_t^s u(y)(y) \, dy \Big)^{\frac{p}{p-q}} \Big(\int_s^\infty v(y)^{1-p'} \, dy \Big)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'} \, ds \Big)^{\frac{p-q}{pq}}.$$

The proof is complete. $\qquad\square$

Theorem 3.6. *Assume that $p, q, m \in (1,\infty)$ and $q < m$. Let u, v, w be weights on $(0,\infty)$. We define*

$$K = \sup_{g \in \mathfrak{M}_+} \frac{\Big(\int_0^\infty \Big(\int_t^\infty \Big(\int_s^\infty g(y) \, dy \Big)^q u(s) \, ds \Big)^{\frac{m}{q}} w(t) \, dt \Big)^{\frac{1}{m}}}{\Big(\int_0^\infty g(s)^p v(s) \, ds \Big)^{\frac{1}{p}}}.$$

(i) *Let $p \le q$. Then*

$$K \approx A_1,$$

where

$$A_1 = \sup_{x \in (0,\infty)} \Big(\int_x^\infty v(s)^{1-p'} \, ds \Big)^{\frac{1}{p'}} \Big(\int_0^x \Big(\int_s^x u(y) \, dy \Big)^{\frac{m}{q}} w(s) \, ds \Big)^{\frac{1}{m}}.$$

(ii) *Let $q < p$ and $p \le m$. Then*

$$K \approx A_1 + A_2,$$

where

$$A_2 = \sup_{x \in (0,\infty)} \Big(\int_x^\infty \Big(\int_s^\infty u(y) \, dy \Big)^{\frac{p}{p-q}} \Big(\int_s^\infty v(y)^{1-p'} \, dy \Big)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'} \, ds \Big)^{\frac{p-q}{pq}} \Big(\int_0^x w(s) \, ds \Big)^{\frac{1}{m}}.$$

(iii) *Let $q < p$ and $m < p$. Then*

$$K \approx A_3 + A_4,$$

where

$$A_3 = \Big(\int_0^\infty \Big(\int_x^\infty \Big(\int_s^\infty u(y) \, dy \Big)^{\frac{p}{p-q}} \Big(\int_s^\infty v(y)^{1-p'} \, dy \Big)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'} \, ds \Big)^{\frac{m(p-q)}{q(p-m)}} W(s)^{\frac{m}{p-m}} w(s) \, ds \Big)^{\frac{p-m}{pm}}$$

and

$$A_4 = \Big(\int_0^\infty \Big(\int_x^\infty v(s)^{1-p'} \, ds \Big)^{\frac{p(m-1)}{p-m}} \Big(\int_0^x \Big(\int_s^x u(y) \, dy \Big)^{\frac{m}{q}} w(s) \, ds \Big)^{\frac{p}{p-m}} v(x)^{1-p'} \, dx \Big)^{\frac{p-m}{pm}}.$$

Proof. We first observe that, by duality of weighted Lebesgue spaces and the Fubini theorem, we have

$$K = \sup_{g \in \mathfrak{M}_+} \sup_{h \in \mathfrak{M}_+} \frac{\Big(\int_0^\infty h(t) \int_t^\infty \Big(\int_s^\infty g(y) \, dy \Big)^q u(s) \, ds \, dt \Big)^{\frac{1}{q}}}{\Big(\int_0^\infty g(s)^p v(s) \, ds \Big)^{\frac{1}{p}} \Big(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}} \, ds \Big)^{\frac{m-q}{mq}}}$$

$$= \sup_{h \in \mathfrak{M}_+} \sup_{g \in \mathfrak{M}_+} \frac{\Big(\int_0^\infty \Big(\int_s^\infty g(y) \, dy \Big)^q \int_0^s h(t) \, dt \, u(s) \, ds \Big)^{\frac{1}{q}}}{\Big(\int_0^\infty g(s)^p v(s) \, ds \Big)^{\frac{1}{p}} \Big(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}} \, ds \Big)^{\frac{m-q}{mq}}}$$

Let $p \le q$. Then, using an appropriate weighted Hardy inequality, interchanging suprema and applying the Fubini theorem, we get

$$K \approx \sup_{h \in \mathfrak{M}_+} \frac{\displaystyle\sup_{x \in (0,\infty)} \Big(\int_0^x u(s) \int_0^s h(t) \, dt \, ds \Big)^{\frac{1}{q}} \Big(\int_x^\infty v(s)^{1-p'} \, ds \Big)^{\frac{1}{p'}}}{\Big(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}} \, ds \Big)^{\frac{m-q}{mq}}}$$

$$= \sup_{x \in (0,\infty)} \Big(\int_x^\infty v(s)^{1-p'} \, ds \Big)^{\frac{1}{p'}} \Big[\sup_{h \in \mathfrak{M}_+} \frac{\int_0^x h(s) \int_s^x u(y) \, dy \, ds}{\Big(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}} \, ds \Big)^{\frac{m-q}{mq}}} \Big]^{\frac{1}{q}}.$$

The assertion in the case (i) now follows from the duality of weighted Lebesgue spaces.

Let now $q < p$. Then, by the Hardy inequality, we have

$$K \approx \sup_{h \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^x u(s) \int_0^s h(t)\, dt\, ds \right)^{\frac{p}{p-q}} \left(\int_x^\infty v(s)^{1-p'}\, ds \right)^{\frac{p(q-1)}{p-q}} v(x)^{1-p'}\, dx \right)^{\frac{p-q}{pq}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\, ds \right)^{\frac{m-q}{mq}}}$$

$$= \left(\sup_{h \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^x h(t) \int_t^x u(s)\, ds\, dt \right)^{\frac{p}{p-q}} \left(\int_x^\infty v(s)^{1-p'}\, ds \right)^{\frac{p(q-1)}{p-q}} v(x)^{1-p'}\, dx \right)^{\frac{p-q}{p}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\, ds \right)^{\frac{m-q}{m}}} \right)^{\frac{1}{q}}.$$

Now, in the case (ii) the assertion follows from [46, Theorem 1.1] and in the case (iii) from [46, Theorem 1.2]. □

Theorem 3.7. *Assume that $m, p, q \in (1, \infty)$ and let u, v, w be weights on $(0, \infty)$. Assume that $q < m$. We denote*

$$K = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s g(y)\, dy \right)^q u(s)\, ds \right)^{\frac{m}{q}} w(t)\, dt \right)^{\frac{1}{m}}}{\left(\int_0^\infty g(s)^p v(s)\, ds \right)^{\frac{1}{p}}}.$$

(i) *If $p \leq q < m$, then*

$$K \approx \sup_{x \in (0,\infty)} W(x)^{\frac{1}{m}} \left(\int_x^\infty u(s)\, ds \right)^{\frac{1}{q}} \left(\int_0^x v(s)^{1-p'}\, ds \right)^{\frac{1}{p'}}$$

$$+ \sup_{x \in (0,\infty)} \left(\int_x^\infty \left(\int_s^\infty u(y)\, dy \right)^{\frac{m}{q}} w(s)\, ds \right)^{\frac{1}{m}} \left(\int_0^x v(s)^{1-p'}\, ds \right)^{\frac{1}{p'}}.$$

(ii) *If $q < p \leq m$, then*

$$K \approx \sup_{x \in (0,\infty)} \left(\int_x^\infty \left(\int_s^\infty u(y)\, dy \right)^{\frac{m}{q}} w(s)\, ds \right)^{\frac{1}{m}} \left(\int_0^x v(s)^{1-p'}\, ds \right)^{\frac{1}{p'}}$$

$$+ \sup_{x \in (0,\infty)} W(x)^{\frac{1}{m}} \left(\int_x^\infty \left(\int_s^\infty u(y)\, dy \right)^{\frac{p}{p-q}} \left(\int_0^s v(y)^{1-p'}\, dy \right)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'}\, ds \right)^{\frac{p-q}{pq}}.$$

(iii) *If $q < m < p$, then*

$$K \approx \left(\int_0^\infty \left(\int_0^x v(s)^{1-p'}\, ds \right)^{\frac{m(p-1)}{p-m}} \left(\int_x^\infty \left(\int_s^\infty u(y)\, dy \right)^{\frac{m}{q}} w(s)\, ds \right)^{\frac{m}{p-m}} \left(\int_x^\infty u(y)\, dy \right)^{\frac{m}{q}} w(x)\, dx \right)^{\frac{p-m}{mp}}$$

$$+ \left(\int_0^\infty \left(\int_x^\infty \left(\int_s^\infty u(y)\, dy \right)^{\frac{p}{p-q}} \left(\int_0^s v(y)^{1-p'} \right)^{\frac{p(q-1)}{p-q}} v(s)^{1-p'}\, ds \right)^{\frac{m(p-q)}{q(p-m)}} W(x)^{\frac{m}{p-m}} w(x)\, dx \right)^{\frac{p-m}{mp}}.$$

Proof. The proof can be done in the same way as that of Theorem 3.6. □

4. Proof of Theorem 1.1

Now we can proceed with the proof of our main result.

Proof of Theorem 1.1. As the first step of our analysis we will express the value of C in a modified way. For every fixed $g \in \mathfrak{M}_+$, set

$$A(g) = \sup_{h \in \mathfrak{M}_+} \frac{\left(\int_0^\infty h^*(t)^{\frac{p_2}{p_1}} u_2(t) \int_t^\infty g(s)\, ds\, dt \right)^{\frac{p_1}{p_2}}}{\left(\int_0^\infty h_{u_1}^{**}(t)^{\frac{m_1}{p_1}} w_1(t) U_1(t)^{\frac{m_1}{p_1}}\, dt \right)^{\frac{p_1}{m_1}}},$$

where we apply the notation introduced in (1.5). We claim that

$$(4.1) \qquad C = \sup_{g \in \mathfrak{M}_+} \frac{A(g)^{\frac{1}{p_1}}}{\left(\int_0^\infty g(t)^{\frac{m_2}{m_2 - p_2}} w_2(t)^{-\frac{p_2}{m_2 - p_2}} \, dt \right)^{\frac{m_2 - p_2}{m_2 p_2}}}.$$

Indeed, fix $f \in \mathfrak{M}$. Since $\frac{m_2}{p_2} > 1$, we can apply (2.1) to $p = \frac{m_2}{p_2}$ and $v = w_2$. Then $p' = \frac{m_2}{m_2 - p_2}$ and $1 - p' = -\frac{p_2}{m_2 - p_2}$, and so we get

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s) \, ds \right)^{\frac{m_2}{p_2}} w_2(t) \, dt \right)^{\frac{1}{m_2}} = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty g(t) \int_0^t f^*(s)^{p_2} u_2(s) \, ds \, dt \right)^{\frac{1}{p_2}}}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2 - p_2}} w_2(s)^{-\frac{p_2}{m_2 - p_2}} \, ds \right)^{\frac{m_2 - p_2}{m_2 p_2}}}.$$

By the Fubini theorem, this turns into

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s) \, ds \right)^{\frac{m_2}{p_2}} w_2(t) \, dt \right)^{\frac{1}{m_2}} = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(s)^{p_2} u_2(s) \int_s^\infty g(t) \, dt \, ds \right)^{\frac{1}{p_2}}}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2 - p_2}} w_2(s)^{-\frac{p_2}{m_2 - p_2}} \, ds \right)^{\frac{m_2 - p_2}{m_2 p_2}}}.$$

Plugging this into (1.9), we get

$$C = \sup_{f \in \mathfrak{M}} \frac{1}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s) \, ds \right)^{\frac{m_1}{p_1}} w_1(t) \, dt \right)^{\frac{1}{m_1}}} \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(s)^{p_2} u_2(s) \int_s^\infty g(t) \, dt \, ds \right)^{\frac{1}{p_2}}}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2 - p_2}} w_2(s)^{-\frac{p_2}{m_2 - p_2}} \, ds \right)^{\frac{m_2 - p_2}{m_2 p_2}}}.$$

On interchanging suprema, this yields

$$C = \sup_{g \in \mathfrak{M}_+} \frac{1}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2 - p_2}} w_2(s)^{-\frac{p_2}{m_2 - p_2}} \, ds \right)^{\frac{m_2 - p_2}{m_2 p_2}}} \sup_{f \in \mathfrak{M}} \frac{\left(\int_0^\infty f^*(s)^{p_2} u_2(s) \int_s^\infty g(t) \, dt \, ds \right)^{\frac{1}{p_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s) \, ds \right)^{\frac{m_1}{p_1}} w_1(t) \, dt \right)^{\frac{1}{m_1}}}.$$

Now, for a change, fix $g \in \mathfrak{M}_+$. Substituting $f^* = (h^*)^{\frac{1}{p_1}}$, we can write

$$\sup_{f \in \mathfrak{M}} \frac{\left(\int_0^\infty f^*(s)^{p_2} u_2(s) \int_s^\infty g(t) \, dt \, ds \right)^{\frac{1}{p_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s) \, ds \right)^{\frac{m_1}{p_1}} w_1(t) \, dt \right)^{\frac{1}{m_1}}} = \sup_{h \in \mathfrak{M}} \frac{\left(\int_0^\infty h^*(t)^{\frac{p_2}{p_1}} u_2(t) \int_t^\infty g(s) \, ds \, dt \right)^{\frac{1}{p_2}}}{\left(\int_0^\infty h_{u_1}^{**}(t)^{\frac{m_1}{p_1}} w_1(t) U_1(t)^{\frac{m_1}{p_1}} \, dt \right)^{\frac{1}{m_1}}}.$$

The quantity on the right-hand side now equals $A(g)^{\frac{1}{p_1}}$. This establishes (4.1).

We next observe that, for every fixed $g \in \mathfrak{M}_+$, one has

$$A(g) = \sup_{h \in \mathfrak{M}} \frac{\left(\int_0^\infty h^*(t)^q w(t) \, dt \right)^{\frac{1}{q}}}{\left(\int_0^\infty h_u^{**}(t)^p v(t) \, dt \right)^{\frac{1}{p}}}$$

with

$$(4.2) \qquad p = \frac{m_1}{p_1}, \ q = \frac{p_2}{p_1}$$

and

$$(4.3) \qquad w(t) = u_2(t) \int_t^\infty g(s) \, ds, \quad v(t) = w_1(t) U_1(t)^{\frac{m_1}{p_1}}, \quad u(t) = u_1(t), \qquad t \in (0, \infty).$$

Hence, under this notation, $A(g)$ is equal to the norm of the embedding

$$(4.4) \qquad \Gamma_u^q(w) \hookrightarrow \Lambda^p(v).$$

The remaining part of proof will be split into separate cases accordingly to the assertions (i)-(viii) of the theorem. The reason for the separation stems from the fact that each of the cases requires its own intrinsic techniques. The techniques dramatically differ one from another. It is important to note that at this stage, in all the cases, the (still fixed) function g is a part of the weight w.

(i) Assume that $m_1 \leq p_2$ and $p_1 \leq p_2$. Then $0 < p \leq q < \infty$ and $1 \leq q < \infty$. Therefore, it follows from [26, Theorem 4.2(i)] that

$$(4.5) \qquad A(g) \approx \sup_{t \in (0,\infty)} \frac{W(t)^{\frac{1}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{1}{p}}}.$$

By (4.3) and the Fubini theorem we have, for every $t \in (0,\infty)$,

$$(4.6) \qquad W(t) = \int_0^t u_2(s) \int_s^\infty g(y)\, dy\, ds = \int_0^t g(s) U_2(s)\, ds + U_2(t) \int_t^\infty g(s)\, ds.$$

Plugging (4.6), (4.2) and (4.3) into (4.5), we get

$$A(g) \approx \sup_{t \in (0,\infty)} \frac{\left(\int_0^t g(s) U_2(s)\, ds + U_2(t) \int_t^\infty g(s)\, ds\right)^{\frac{p_1}{p_2}}}{\left(\int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s)\, ds + U_1(t)^{\frac{m_1}{p_1}} \int_t^\infty w_1(s)\, ds\right)^{\frac{p_1}{m_1}}}.$$

Raising this to $\frac{p_2}{p_1}$, using (1.7) and the subadditivity of supremum, we obtain

$$A(g)^{\frac{p_2}{p_1}} \approx \sup_{t \in (0,\infty)} \varphi(t)^{-\frac{p_2}{m_1}} \int_0^t g(s) U_2(s)\, ds + \sup_{t \in (0,\infty)} \varphi(t)^{-\frac{p_2}{m_1}} U_2(t) \int_t^\infty g(s)\, ds.$$

Combined with (4.1), this yields

$$C \approx C_1 + C_2,$$

where

$$C_1^{p_2} = \sup_{g \in \mathfrak{M}_+} \frac{\displaystyle\sup_{t \in (0,\infty)} \varphi(t)^{-\frac{p_2}{m_1}} \int_0^t g(s) U_2(s)\, ds}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2 - p_2}} w_2(s)^{-\frac{p_2}{m_2 - p_2}}\, ds\right)^{\frac{m_2 - p_2}{m_2}}}$$

and

$$C_2^{p_2} = \sup_{g \in \mathfrak{M}_+} \frac{\displaystyle\sup_{t \in (0,\infty)} \varphi(t)^{-\frac{p_2}{m_1}} U_2(t) \int_t^\infty g(s)\, ds}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2 - p_2}} w_2(s)^{-\frac{p_2}{m_2 - p_2}}\, ds\right)^{\frac{m_2 - p_2}{m_2}}}.$$

Substituting $f = g U_2$, we get

$$C_1^{p_2} = \sup_{f \in \mathfrak{M}_+} \frac{\displaystyle\sup_{t \in (0,\infty)} \varphi(t)^{-\frac{p_2}{m_1}} \int_0^t f(s)\, ds}{\left(\int_0^\infty f(s)^{\frac{m_2}{m_2 - p_2}} U_2(s)^{-\frac{m_2}{m_2 - p_2}} w_2(s)^{-\frac{p_2}{m_2 - p_2}}\, ds\right)^{\frac{m_2 - p_2}{m_2}}}.$$

Thus, applying Theorem 2.1 to

$$p = \frac{m_2}{m_2 - p_2}, \quad q = \infty, \quad w = \varphi^{-\frac{p_2}{m_1}}, \quad v = U_2^{-1} w_2^{-\frac{p_2}{m_2}},$$

we get

$$C_1^{p_2} = \sup_{t \in (0,\infty)} \varphi(t)^{-\frac{p_2}{m_1}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\, ds\right)^{\frac{p_2}{m_2}}.$$

Finally, applying Theorem 2.2 to

$$p = \frac{m_2}{m_2 - p_2}, \quad q = \infty, \quad w = \varphi^{-\frac{p_2}{m_1}} U_2, \quad v = w_2^{-\frac{p_2}{m_2}},$$

we arrive at

$$C_2^{p_2} = \sup_{t \in (0,\infty)} \varphi(t)^{-\frac{p_2}{m_1}} U_2(t) \left(\int_t^\infty w_2(s)\, ds\right)^{\frac{p_2}{m_2}}.$$

The assertion of the theorem in case (i) follows.

Assume now that $m_1 > p_2$ and $p_1 \leq p_2$. Then $1 \leq q < p < \infty$. Thus, by [26, Theorem 4.2(ii)], one has

$$(4.7) \qquad A(g) \approx \left(\int_0^\infty \frac{U(t)^{\frac{pq}{p-q}} \left[\sup_{y \in [t,\infty)} U(y)^{-\frac{pq}{p-q}} W(y)^{\frac{p}{p-q}} \right] V(t) \int_t^\infty U(s)^{-p} v(s)\, ds}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds \right)^{\frac{p}{p-q}+1}} \, d(U^p(t)) \right)^{\frac{p-q}{pq}} .$$

Inserting (4.6), (4.2) and (4.3) into (4.7) and using (4.1) and also the notation from (1.8), we get

$$C \approx C_3 + C_4$$

with

$$C_3^{p_2} = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left[\sup_{s \in (t,\infty)} U_1(s)^{-\frac{p_2}{p_1}} \int_0^s g(y) U_2(y)\, dy \right]^{\frac{m_1}{m_1-p_2}} \sigma(t)\, dt \right)^{\frac{m_1-p_2}{m_1}}}{\left(\int_0^\infty g(t)^{\frac{m_2}{m_2-p_2}} w_2(t)^{-\frac{p_2}{m_2-p_2}} \, dt \right)^{\frac{m_2-p_2}{m_2}}}$$

and

$$C_4^{p_2} = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left[\sup_{s \in (t,\infty)} U_2(s) U_1(s)^{-\frac{p_2}{p_1}} \int_s^\infty g(y)\, dy \right]^{\frac{m_1}{m_1-p_2}} \sigma(t)\, dt \right)^{\frac{m_1-p_2}{m_1}}}{\left(\int_0^\infty g(t)^{\frac{m_2}{m_2-p_2}} w_2(t)^{-\frac{p_2}{m_2-p_2}} \, dt \right)^{\frac{m_2-p_2}{m_2}}} .$$

Writing $h = g U_2$, we get

$$C_3^{p_2} = \sup_{h \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left[\sup_{s \in (t,\infty)} U_1(s)^{-\frac{p_2}{p_1}} \int_0^s h(y)\, dy \right]^{\frac{m_1}{m_1-p_2}} \sigma(t)\, dt \right)^{\frac{m_1-p_2}{m_1}}}{\left(\int_0^\infty h(t)^{\frac{m_2}{m_2-p_2}} U_2(t)^{-\frac{m_2}{m_2-p_2}} w_2(t)^{-\frac{p_2}{m_2-p_2}} \, dt \right)^{\frac{m_2-p_2}{m_2}}} .$$

This quantity can be evaluated with the help of [24, Theorem 4.1(ii)]. We get

$$C_3^{p_2} \approx \sup_{x \in (0,\infty)} \left(U_1(x)^{-\frac{p_2}{p_1} \frac{m_1}{m_1-p_2}} \int_0^x \sigma(s)\, ds + \int_x^\infty U_1(s)^{-\frac{p_2}{p_1} \frac{m_1}{m_1-p_2}} \sigma(s)\, ds \right)^{\frac{m_1-p_2}{m_1}} \left(\int_0^x U_2(s)^{\frac{m_2}{p_2}} w_2(s)\, ds \right)^{\frac{p_2}{m_2}}$$

if $m_1 \leq m_2$, and

$$C_3^{p_2} \approx \left(\int_0^\infty \left(\int_t^\infty U_1(s)^{-\frac{p_2}{p_1} \frac{m_1}{m_1-p_2}} \sigma(s)\, ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} U_1(s)^{-\frac{p_2}{p_1} \frac{m_1}{m_1-p_2}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\, ds \right)^{\frac{m_1}{m_1-p_2}} \sigma(t)\, dt \right)^{\frac{p_2(m_1-m_2)}{m_1 m_2}}$$

$$+ \left(\int_0^\infty \left(\int_0^t \sigma(s)\, ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \left[\sup_{y \in (t,\infty)} U_1(y)^{-\frac{p_2}{p_1}} \left(\int_0^y U_2(z)^{\frac{m_2}{p_2}} w_2(z)\, dz \right)^{\frac{p_2}{m_2}} \right]^{\frac{m_1 m_2}{p_2(m_1-m_2)}} \sigma(t)\, dt \right)^{\frac{p_2(m_1-m_2)}{m_1 m_2}}$$

if $m_1 > m_2$.

As for C_4, assume first that $m_1 \leq m_2$. Then by Theorem 3.2, applied to the parameters

$$p = \frac{m_2}{m_2 - p_2}, \quad q = \frac{m_1}{m_1 - p_2}, \quad w = \sigma, \quad u = U_2 U_1^{-\frac{p_2}{p_1}}, \quad v = w_2^{-\frac{p_2}{m_2-p_2}},$$

we get

$$C_4^{p_2} \approx \sup_{x \in (0,\infty)} \left(\int_0^x \left[\sup_{y \in (s,x)} U_2(y) U_1(y)^{-\frac{p_2}{p_1}} \right]^{\frac{m_1}{m_1-p_2}} \sigma(s)\, ds \right)^{\frac{m_1-p_2}{m_1}} \left(\int_x^\infty w_2(s)\, ds \right)^{\frac{p_2}{m_2}} .$$

In the case $m_2 < m_1$ we adopt an analogous argument involving Theorem 3.3.

Combining all the estimates obtained, we finally establish the assertions of the theorem in the cases (ii) and (iii).

Before we plunge into the proof of the remaining cases we have to observe an important difference compared to those that have been treated already. At this stage it is important to note that, in our approach, the crucial fact is the manner in which the function g appears (somewhat implicitly) in the

conditions. The function g comes to play through the weight w, therefore also through its primitive function W. In the remaining cases we have (by [26, Theorem 4.2(iii) and (iv)]),

$$(4.8) \qquad A(g) \approx \sup_{t \in (0,\infty)} \frac{W(t)^{\frac{1}{q}} + U(t) \left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}} \, ds \right)^{\frac{1-q}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s) \, ds \right)^{\frac{1}{p}}}$$

if $0 < p \le q < 1$, and

$$(4.9) \qquad A(g) \approx \left(\int_0^\infty \frac{\left[W(t)^{\frac{1}{1-q}} + U(t)^{\frac{q}{1-q}} \int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}} \, ds \right]^{-\frac{p(q-1)}{p-q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s) \, ds \right)^{\frac{p}{p-q}+1}} \times$$

$$\times V(t) \int_t^\infty U(s)^{-p} v(s) \, ds \, d(U^p(t)) \right)^{\frac{p-q}{pq}}$$

if $0 < q < 1$ and $0 < q < p$. Using Lemma 3.1, we obtain

$$A(g) \approx \sup_{t \in (0,\infty)} \frac{U(t) \left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s) \, ds \right)^{\frac{1-q}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s) \, ds \right)^{\frac{1}{p}}} < \infty$$

if $0 < p \le q < 1$, and

$$A(g) \approx \left(\int_0^\infty \frac{\left[U(t)^{\frac{q}{1-q}} \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s) \, ds \right]^{-\frac{p(q-1)}{p-q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s) \, ds \right)^{\frac{p}{p-q}+1}} V(t) \int_t^\infty U(s)^{-p} v(s) \, ds \, d(U^p(t)) \right)^{\frac{p-q}{pq}}$$

if $0 < q < 1$ and $0 < q < p$.

Assume now that $m_1 \le p_2$ and $p_1 > p_2$. Then, using (4.1), (4.2), (4.3), (4.4) and (4.6), we get

$$C \approx C_5 + C_6,$$

where

$$C_5^{p_2} = \sup_{g \in \mathfrak{M}_+} \frac{\displaystyle\sup_{t \in (0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}} \left(\int_t^\infty \left(\int_0^s h(y) \, dy \right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s) \, ds \right)^{\frac{p_1-p_2}{p_1}}}{\left(\int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s) \, ds + U_1(t)^{\frac{m_1}{p_1}} \int_t^\infty w_1(s) \, ds \right)^{\frac{p_2}{m_1}}}}{\left(\int_0^\infty h(s)^{\frac{m_2}{m_2-p_2}} U_2(s)^{-\frac{m_2}{m_2-p_2}} w_2(s)^{-\frac{p_2}{m_2-p_2}} \, ds \right)^{\frac{m_2-p_2}{m_2}}}$$

and

$$C_6^{p_2} = \sup_{g \in \mathfrak{M}_+} \frac{\displaystyle\sup_{t \in (0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}} \left(\int_t^\infty \left(\int_s^\infty g(y) \, dy \right)^{\frac{p_1}{p_1-p_2}} U_2(s)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s) \, ds \right)^{\frac{p_1-p_2}{p_1}}}{\left(\int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s) \, ds + U_1(t)^{\frac{m_1}{p_1}} \int_t^\infty w_1(s) \, ds \right)^{\frac{p_2}{m_1}}}}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2-p_2}} w_2(s)^{-\frac{p_2}{m_2-p_2}} \, ds \right)^{\frac{m_2-p_2}{m_2}}}.$$

Applying Theorem 3.4 to the parameters

$$p = \frac{m_2}{m_2 - p_2}, \quad q = \frac{p_1}{p_1 - p_2}$$

and the weights u, v, w, defined for $t \in (0, \infty)$ by

$$u(t) = U_1(t)^{-\frac{p_1}{p_1-p_2}} u_1(t), \qquad v(t) = U_2(t)^{-\frac{m_2}{m_2-p_2}} w_2(t)^{-\frac{p_2}{m_2-p_2}}$$

and

$$w(t) = \frac{U_1(t)^{\frac{p_2}{p_1}}}{\left(\int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s) \, ds + U_1(t)^{\frac{m_1}{p_1}} \int_t^\infty w_1(s) \, ds \right)^{\frac{p_2}{m_1}}},$$

and then Theorem 3.5 to the parameters

$$p = \frac{m_2}{m_2 - p_2}, \ q = \frac{p_1}{p_1 - p_2}$$

and the weights u, v, w, defined for $t \in (0, \infty)$ by

$$u(t) = U_2(t)^{\frac{p_1}{p_1 - p_2}} U_1(t)^{-\frac{p_1}{p_1 - p_2}} u_1(t), \qquad v(t) = w_2(t)^{-\frac{p_2}{m_2 - p_2}}$$

and

$$w(t) = \frac{U_1(t)^{\frac{p_2}{p_1}}}{\left(\int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s) \, ds + U_1(t)^{\frac{m_1}{p_1}} \int_t^\infty w_1(s) \, ds \right)^{\frac{p_2}{m_1}}},$$

we get the assertions of the theorem in cases (iv) and (v).

Finally assume that $p_2 < m_1$ and $p_2 < p_1$. Then, using again (4.1), (4.2), (4.3), (4.4) and (4.6), we have

$$C \approx C_7 + C_8,$$

where

$$C_7^{p_2} = \sup_{h \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s h(y) \, dy \right)^q u(s) \, ds \right)^{\frac{m}{q}} w(t) \, dt \right)^{\frac{1}{m}}}{\left(\int_0^\infty h(s)^p v(s) \, ds \right)^{\frac{1}{p}}}$$

with

$$p = \frac{m_2}{m_2 - p_2}, \quad m = \frac{m_1}{m_1 - p_2}, \quad q = \frac{p_1}{p_1 - p_2},$$

and, for $t \in (0, \infty)$,

$$u(t) = U_1(t)^{-\frac{p_1}{p_1 - p_2}} u_1(t), \qquad v(t) = U_2(t)^{-\frac{m_2}{m_2 - p_2}} w_2(t)^{-\frac{p_2}{m_2 - p_2}}, \qquad w(t) = \sigma(t),$$

and

$$C_8^{p_2} = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty g(y) \, dy \right)^q u(s) \right)^{\frac{m}{q}} w(t) \, dt \right)^{\frac{1}{m}}}{\left(\int_0^\infty g(s)^p v(s) \, ds \right)^{\frac{1}{p}}},$$

where p, m, q are the same as above and the weights u, v, w are defined for $t \in (0, \infty)$ by

$$u(t) = U_1(t)^{-\frac{p_1}{p_1 - p_2}} U_2(t)^{\frac{p_1}{p_1 - p_2}} u_1(t), \qquad v(t) = w_2(t)^{-\frac{p_2}{m_2 - p_2}}, \qquad w(t) = \sigma(t).$$

Therefore, C_7 can be evaluated using Theorem 3.7 and C_8 using Theorem 3.6. □

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A. Gogatishvili, Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic

E-mail address: `gogatish@math.cas.cz`

M. Křepela, Karlstad University, Faculty of Health, Science and Technology, Department of Mathematics and Computer Science, 651 88 Karlstad, Sweden

E-mail address: `martin.krepela@kau.se`

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: `pick@karlin.mff.cuni.cz`

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: `soudsky@karlin.mff.cuni.cz`