Charles University in Prague
Faculty of Mathematics and Physics

DOCTORAL THESIS

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Weighted rearrangement-invariant spaces and their basic properties

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Study programme: Mathematical analysis
Specialization: Real complex and functional analysis

Prague 2015

I would like to thank to all who supported me during my studies. Let me name here at few who helped me most. First of all my thanks belong to professor Luboš Pick for his patience and willingness. His worthy advice especially concerning the form of papers helped a lot. I would also like to thank to Mr. Jiří Červa who taught me English when I was young. Another ones to be thanked are my parents, for the continuous support during all my studies. And final thank is directed to my colleagues doctoral students of analysis for friendly atmosphere in our office and helpful advice whenever it was needed.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Váhové prostory funkcí invariantní vůči přerovnání a jejich základní vlastnosti

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Abstrakt: Tato práce se věnuje klasickým Lorentzovým prostorům. Tyto prostory jsou předmětem intenzivního studia již od 50. let. Za tu dobu našly mnoho aplikací a to především v oblasti parciálních diferenciálních rovnic a teorii interpolací. Práce samotná se skládá z úvodu a pěti článků. První článek studuje vlastnosti zobecněných Gamma prostorů. Druhý podává alternativní důkaz charakterizace normovatelnosti Λ prostorů. Třetí článek se věnuje charakterizaci linearity a quasi-normovanosti r.i. svazů. Další pak podává alternativní důkaz charakterizace normovatelnosti Lorentzoých prostorů typu Γ. Poslední článek se pak charakterizuje vnoření mezi prostory $G\Gamma$.

Klíčová slova: Váha, Banachův prostor funkcí, vnoření, normovatelnost

Title: Weighted rearrangement-invariant spaces and their basic properties

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Abstract: In this thesis we shall provide the reader with results in the field of classical Lorentz spaces. These spaces have been studied since the 50's and have many applications in partial differential equations and interpolation theory. This work includes five papers. First paper studies the properties of Generalized Gamma spaces. Second paper provides an alternative proof of normability characterization of classical Lorentz spaces. The third paper discus conditions of linearity and quasi-norm property of rearrangement-invariant lattices. The following paper gives a characterization of normability of Gamma spaces. And finally the last paper characterizes the embeddings between $G\Gamma$ spaces.

Keywords: Weight, Banach function space, Embedding, Normability

Contents

1. Introduction to rearrangement-invariant spaces

1. Introduction

In the beginning of the twentieth century, the Lebesgue measure and the Lebesgue integral were introduced in [**22**]. Many branches of the temporary mathematics were reformed. Remarkable changes were made in the theory of ordinary and partial differential equations. The concept of weak solutions to a given problem required new methods. As a natural response, the Lebesgue spaces L^p were introduced in 1910 by F. Riesz (see [**30**]). These spaces were thoroughly investigated afterward and many results including all their functional properties and boundedness of operators on these spaces were obtained. On a finite measure space, function spaces L^p form a scale. In 1931 W. Orlicz came up with a much finer scale of Orlicz spaces L^{Φ}. This scale brought new results especially in Sobolev-type of embeddings which found their use in PDEs. In the 1930's the notion of the non-increasing rearrangement of a function was defined and became popular soon afterward. In 1950 the classical Lorentz spaces entered the stage (see [**23**]). In the 1960's Lorentz spaces with power weights were investigated and found many applications in the interpolation theory and PDEs. Later on even finer scales like Lorentz-Zygmund spaces were introduced. Meanwhile since many new measure-based function spaces were discovered some general type of structure was needed to reflect their common properties. In this view a concept of Banach spaces seemed to be too general. Therefore in 1956 W. A. J. Luxemburg came up with Banach function spaces (see [**25**]).

Still many questions in this field remained open. For instance the boundedness of some classical operators (like the Hardy-Littlewood maximal operator, the Hilbert transform, the Riesz potential....), embeddings between these spaces and normability. First papers trying to answer some of these questions appeared in 1990. In [**1**] authors were discussing the boundedness of maximal operator and later on the very same year E. Sawyer in his famous paper [**31**] characterized the associated spaces and using his duality result he characterized one case of embeddings between Λ spaces. He also discovered some equivalent conditions under which Λ is a Banach space. In this paper Lorentz space of type Γ appeared as the associated space to a Λ space. An avalanche of papers in the 90's followed. First V.D. Stepanov and G. Sinnamon in [**33, 32**] and J. Soria and M. J. Carro in [**7**] characterized embeddings between Λ spaces. Further break-through was made by A. Gogatishvili and L. Pick in [**15**] where most of the cases of embeddings between Γ and Λ spaces were characterized. The investigation was definitely closed in [**3**] (simpler alternative proof can be found in [**8**]) . Also the functional properties of these spaces were characterized by this time (see [**6, 5, 21**]). New direction of research was started in [**11**]. In this paper the authors proved equivalence of norm in Grand Lebesgue space and new type of weighted Lorentz space which they called $G\Gamma$ (generalized Gamma spaces). For more details see Subsection4.3. They investigated these spaces further on (see [**12, 13**]). However they did not obtain complete results. Many questions remained open. Another possible direction was shown in [**15**] where more general form of Gamma spaces with inner weight was considered. In the papers included in this thesis we solve some of these problems and we shall also obtain some general type of results related to rearrangement-invariant lattices.

This thesis is structured as follows. The first chapter is an introduction. In the following section we first define basic notions and recall basic theorems related to theory of Banach function spaces, non-increasing rearrangement and maximal function. Most of these results can be found in [**2**]. The third section is an introduction to weighted Hardy-type inequalities and Lorentz spaces. The fourth section provides the reader with some applications of Lorentz spaces. The last section gives a summary of results contained in this thesis. The papers in the exact form in which they were published, accepted or submitted follow in the chronological order.

2. Rearrangement-invariant function spaces, basic definitions

Let us start with basic notions. Let $(\mathcal{R}, \mu)$ be a measure space. We call it a *non-atomic measure space* if for every single point $x \in \mathcal{R}$ we have

$$\mu(\{x\}) = 0.$$

We say that $(\mathcal{R}, \mu)$ is σ-finite if there exists a countable set α such that $\{A_i\}_{i\in\alpha}$ have finite measure and

$$\bigcup_{i\in\alpha} A_i = \mathcal{R}.$$

Now let $(\mathcal{R}, \mu)$ be a σ-finite measure space, define the norm in space $L^p(\mathcal{R})$ (or just L^p if there are no doubts about the underlying measure space) by

$$\|f\|_p := \left(\int_{\mathcal{R}} |f|^p \, d\mu\right)^{\frac{1}{p}},$$

for $0 < p < \infty$ and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\mathcal{R}} |f|.$$

Definition 2.1. Let $(\mathcal{R}, \mu)$ be a σ-finite measure space. Define

(i)
$$\mathcal{M}(\mathcal{R}) := \{f : \mathcal{R} \to [-\infty, \infty], f \quad \text{measurable}\}$$
and we shall denote by $\mathcal{M}^+(\mathcal{R})$ the set of all non-negative $f \in \mathcal{M}$.

(ii) We call w a weight if $w \in \mathcal{M}^+(\mathcal{R})$. Moreover, when underlying measure space is not specified we mean a weight on $(0, \infty)$ equipped with the Lebesgue measure.

(iii) Let $f \in \mathcal{M}(\mathcal{R})$. Then we define its *distribution function* by
$$f_*(t) := \mu\{|f| > t\},$$
the *non-increasing rearrangement of f* by
$$f^*(t) := \inf\{s : f_*(s) \le t\},$$
and the *maximal function* by
$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\, ds, \quad t \in (0, \infty).$$

Note that the norm in the Lebesgue spaces in fact depends only on the measure of level sets. Let us describe this statement in detail. Observe that f and f^* have the same distribution function. Therefore, by Fubini theorem we have

$$\begin{aligned}\|f\|_p^p &= \int_{\mathcal{R}} |f|^p \, d\mu = \int_0^\infty \{|f|^p > t\}\, dt \\ &= \int_0^\infty f_*(t^{\frac{1}{p}})\, dt = p\int_0^\infty s^{p-1} f_*(s)\, ds = \int_0^\infty f^*(s)^p\, ds.\end{aligned} \tag{2.1}$$

Let us call two functions defined on the same underlying measure space having the same distribution functions *equimeasurable*.

Definition 2.2. Let $\|\cdot\|_X : \mathcal{M}(\mathcal{R}) \to [0, \infty]$ be a functional. Consider the following properties.

(P1) The functional $\|\cdot\|_X$ is a norm.

(P2) If $|f(x)| \le |g(x)|$ for μ-a.e. $x \in \mathcal{R}$, then $\|f\|_X \le \|g\|_X$.

(P3) If $0 \leq f_n(x) \uparrow f(x)$ for μ-a.e. $x \in \mathcal{R}$, then $\|f_n\|_X \uparrow \|f\|_X$.
(P4) Let $E \subset \mathcal{R}$ and $\mu(E) < \infty$, then $\|\chi_E\|_X < \infty$.
(P5) Let $\mu(E) < \infty$ then there exists C_E such that

$$\int_E |f| \leq C_E \|f\chi_E\|_X .$$

(P6) $f^* = g^*$ then $\|f\|_X = \|g\|_X$.

Set

$$X := \{f \in \mathcal{M}(\mathcal{R}) : \|f\|_X < \infty\}.$$

If $\|\cdot\|_X$ satisfies

(i) (P1)-(P5) then we call X a *Banach function space* (sometimes we shall use BFS for short);
(ii) (P1)-(P6) then we call X a *rearrangement-invariant Banach function space*;
(iii) if the functional is positively homogeneous (P2), (P6) are satisfied and $\|f\|_X = 0$ if and only if $f = 0$ μ-a.e., then we call X a *rearrangement-invariant lattice*.
(iv) If (P2)-(P4) are satisfied, the functional is positively homogeneous and instead of (P1) we have only

$$\|f + g\|_X \leq C(\|f\|_X + \|g\|_X),$$

then we call X a *quasi Banach function space* (or QBFS for short).

Examples 2.3. (i) L^p spaces are r.i. Banach function spaces for $p \geq 1$. Indeed (P6) was checked in (2.1). (P1) is Minkowski inequality (see [**24**]). (P2) and (P4) are observed immediately, (P3) follows from the monotone convergence theorem. Let E be a set of finite measure. We have

$$\int_E |f| \, d\mu = \int_{\mathcal{R}} |f\chi_E| \, d\mu \leq \|\chi_E\|_{p'} \|f\|_p .$$

Set $C_E := \|\chi_E\|_{p'}$ and we obtain (P5)

(ii) Let $0 < p < 1$. Consider L^p space. We have

$$\|f + g\|_p \leq \left(\int_{\mathcal{R}} (|f| + |g|)^p\right)^{\frac{1}{p}} \leq \left(\int_{\mathcal{R}} |f|^p + |g|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1}(\|f\|_p + \|g\|_p). \tag{2.2}$$

On the other hand by plugging into (2.2) characteristic functions of disjoint sets of the same measure, one can immediately see that the constant on the right-hand side is the best possible and therefore $\|\cdot\|_p$ is not a norm. But on the other hand one can readily check that (P2)-(P4) are satisfied, while (P5) fails. Therefore L^p is a quasi Banach function space for $p < 1$.

(iii) Let w be a weight function on $\mathcal{R}$. Then the *weighted Lebesgue space* given by

$$\|f\|_{L^p_w(\mathcal{R})} := \left(\int_{\mathcal{R}} |f|^p w \, d\mu\right)^{\frac{1}{p}}$$

satisfies (P1),(P2) and (P3). But (P6) fails unless w is not a constant function. (P4) and (P5) are dependent on the weight function w. If $w \in L^1_{loc}$ then we have (P4). If $w^{-\frac{1}{p'}} \in L^1_{loc}$, we have

$$\int_E |f| \, d\mu \leq \int_{\mathcal{R}} |f| w^{\frac{1}{p}} w^{-\frac{1}{p}} \chi_E \, d\mu \leq \|f\chi_E\|_{L^p_w(\mathcal{R})} \left(\int_E w^{-\frac{1}{p'}}\right)^{\frac{1}{p'}} .$$

(iv) Let $\mathcal{R} = \mathbb{N}$ and let μ be the counting measure. Define

$$\|x\|_Y := \sup_{n\in\mathbb{N}} |x_n| + \limsup_{n\in\mathbb{N}} |x_n|$$

One can readily check that (P1), (P2), (P4), (P5) and even (P6) hold. The only axiom that fails is (P3).

Let $p \in [1,\infty]$. We shall use the symbol p' for a quantity satisfying
$$\frac{1}{p} + \frac{1}{p'} = 1.$$
Riesz's representation theorem for L^p ($p \in [1,\infty)$) spaces asserts that if $H \in (L^p)^*$ then there exist $h \in L^{p'}$ such that
$$H(f) = \int_0^\infty h(s)f(s)\,d\,s.$$
Now, using the saturated Hölder inequality, we obtain
$$\|H\|_{(L^p)^*} = \|h\|_{p'}.$$
This leads to the definition of *associated space*. Given a Banach function space X we may define
$$\|f\|_{X'} := \sup_{\|h\|_X \le 1} \int_{\mathcal{R}} fh\,d\,\mu.$$
The associated space does not always coincide with the dual space. A very simple counter-example is the space L^∞. However, if we associate a function $f \in X'$ with the functional
$$h \mapsto \int_{\mathcal{R}} fh\,d\,\mu.$$
then we have $X' \subset X^*$. It is not difficult to verify that if X is a Banach function space then the associated space is a Banach function space as well (see [**2**, Theorem 2.2]). We may also consider second associated space $X'' = (X')'$. If X is a BFS then X'' coincides with X and we have $\|f\|_{X''} = \|f\|_X$ (for proof see [**2**, Theorem 2.7]).

Definition 2.4 (Embedding, optimal constant of embedding). Let X_0, X_1 be r.i. lattices of functions in $\mathcal{M}(\mathcal{R})$.

(i) We say that X_0 is *continuously embedded* (or just embedded for short) into X_1, writing
$$X_0 \hookrightarrow X_1,$$
if there exists $0 < C < \infty$, such that
$$\|f\|_{X_1} \le C\,\|f\|_{X_0} \quad \forall f \in \mathcal{M}(\mathcal{R}).$$

(ii) Set
$$\mathrm{Opt}(X_0, X_1) := \sup_{f \in \mathcal{M}(\mathcal{R})} \frac{\|f\|_{X_1}}{\|f\|_{X_0}}.$$
with the convention $\frac{\infty}{\infty} = \frac{0}{0} = 0$.

Let us mention that if X, Y are two BFS and $X \subset Y$ then $X \hookrightarrow Y$ (see [**2**, Theorem 1.8]).

Definition 2.5 (Sum and intersection of spaces). Let X_0, X_1 be r.i. lattices of functions in $\mathcal{M}(\mathcal{R})$. We define

(i)
$$X_0 + X_1 := \left\{f \in \mathcal{M}(\mathcal{R});\ \|f\|_{X_0+X_1} < \infty\right\},$$
where
$$\|f\|_{X_0+X_1} = \inf_g \left(\|g\|_{X_0} + \|f-g\|_{X_1}\right)$$

(ii)
$$X_0 \cap X_1 := \left\{f \in \mathcal{M}(\mathcal{R})\, \|f\|_{X_0 \cap X_1} = \|f\|_{X_1} + \|f\|_{X_0} < \infty\right\},$$

Remarks 2.6. (i) Let X be any Banach function space then we have
$$L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty.$$

(ii) If moreover $\mu(\mathcal{R}) < \infty$, then

$$L^\infty \hookrightarrow X \hookrightarrow L^1.$$

(iii) It is shown in [**29**, Chapter 7, Remark 7.12.4] that

$$\|f\|_{L^1+L^\infty} = \int_0^1 f^*(t)\,dt = \sup_{\mu(E)=1} \int_E |f|.$$

Now we may consider operators between r.i. lattices. We say that an operator T is bounded from X to Y if and only if

$$\|Tf\|_Y \le C\,\|f\|_X$$

and write $T : X \to Y$. The least possible constant of this inequality is the *norm* of the operator T from X to Y. If T' is an operator such that

$$\int_{\mathcal{R}} (Tf) g\,d\mu = \int_{\mathcal{R}} f(T'g)\,d\mu,$$

we call T' a *conjugated operator of T*. Now note that $T : X \to Y$ if and only if $T' : X' \to Y'$. Let $(\mathcal{R}, \mu)$, $(\mathcal{S}, \nu)$ be two measure spaces. We shall say that an operator $T : A \to B$ where $A \subset \mathcal{M}(\mathcal{R})$ and $B \subset \mathcal{M}(\mathcal{S})$ are linear sets, is *quasi-linear* if there exists $K \in (0, \infty)$ such that

$$T(f+g) \le K(Tf + Tg) \quad \forall f, g \in A.$$

One of the most significant of these operators is the *Hardy-Littlewood maximal operator*. Let $f : \mathbb{R}^n \to \mathbb{R}$, define

$$Mf(x) := \sup_{B \ni x} \fint_B |f(y)|\,dy, \tag{2.3}$$

where the supremum on the left-hand side is taken over all balls containing x. Now, by the Lebesgue differentiation theorem we conclude that

$$|f(x)| \le Mf(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

This operator first appeared in 1930 in famous paper of Hardy and Littlewood (see[**18**]) in one dimensional case. In 1939 it was defined for arbitrary dimension by Wiener. It was shown by Riesz in one dimension in 1933 and later by Wiener in the general case that

$$(Mf)^* \lesssim f^{**}. \tag{2.4}$$

The converse inequality

$$f^{**} \lesssim (Mf)^*, \tag{2.5}$$

was proved in 1967 by Herz.

Definition 2.7. Given a r.i. lattice X we may define it's *fundamental function*, by

$$\varphi_X(t) := \|E\|_X,$$

where $E \subset \mathcal{R}$ is an arbitrary measurable set with $\mu(E) = t$.

Note that there is a close relation between a fundamental function of a Banach function space X and it's associated space X' which reads

$$\varphi_X(t)\varphi_{X'}(t) = t \quad \forall t \in (0, \infty). \tag{2.6}$$

For a proof see [**2**, Theorem 5.2., pg 66]

Definition 2.8. Let X be a Banach function space. We say that $f \in X$ has an *absolutely continuous norm in* X if for every sequence of $E_n \subset \mathcal{R}$ such that $\chi_{E_n} \downarrow 0$ a.e. then

$$\|f\chi_{E_n}\|_X \downarrow 0.$$

Denote the set of all functions with absolutely continuous norm by X_a. If $X_a = X$ we say that X *has an absolutely continuous norm*.

Theorem 2.9. *Let X be a Banach function space. Then X is reflexive if and only if both X and X' have absolutely continuous norm.*

Define the norm in *weighted Lebesgue space* by

$$\|f\|_{L^p_w} := \left(\int_0^\infty f(s)^p w(s)\, d s\right)^{\frac{1}{p}}.$$

Definition 2.10. Let $\varphi : [0,\infty) \to [0,\infty)$ be a function satisfying

(Y1) $\varphi(0) = 0$;
(Y2) $\varphi(s) > 0$ for $s > 0$;
(Y3) φ is right-continuous;
(Y4) φ is non-decreasing on $[0,\infty)$;
(Y5) $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

Then we call

$$\Phi(t) := \int_0^t \varphi(s)\, d s$$

a *Young function*.

Definition 2.11 (Orlicz Spaces). Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous increasing function with $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. Given a $f \in \mathcal{M}(\mathcal{R})$ define

$$L^\varphi := \left\{ f \in \mathcal{M}(\mathcal{R}) : \exists\lambda \in (0,\infty) : \int_{\mathcal{R}} \varphi\left(\lambda|f|\right) d\mu < \infty \right\}.$$

If Φ is a Young function we call L^Φ an *Orlicz space*. Define the norm in *Orlicz space* by

$$\|f\|_{L^\Phi} := \inf\left\{ \lambda : \int_{\mathcal{R}} \Phi\left(\frac{|f|}{\lambda}\right) d\mu \le 1 \right\}.$$

Note that the Orlicz spaces are another examples of r.i. BFS. Now we may also define *weighted Orlicz spaces*, which generalize both Lebesgue weighted and Orlicz spaces, whose norm is defined by

$$\|f\|_{L^\varphi_w} := \inf\left\{ \lambda : \int_{\mathcal{R}} \varphi\left(\frac{|f|}{\lambda}\right) w\, d\mu \le 1 \right\}, \tag{2.7}$$

where $w \in \mathcal{M}^+(\mathcal{R})$ is a weight.

3. Hardy inequality and Classical Lorentz spaces

Hardy's inequality is a very old problem. It's first appearance took place in 1915 when G. H. Hardy studied it's discrete form for the power 2. He needed to find out when the inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)^2 \le C\sum_{n=1}^{\infty} a_n^2$$

holds. This inequality in more general form $p \in (1, \infty)$ (instead of 2) was proved in 1920. The same year it's integral form

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f(s)\,ds\right)^p dt\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty f(t)^p\,dt\right)^{\frac{1}{p}} \tag{3.1}$$

was proved. It was also shown that the constant on the right-hand side is sharp. Hardy's inequality was further investigated. Finally the most general weighted form of Hardy inequality

$$\left(\int_0^\infty \left(\int_0^t f(s)\,ds\right)^p w(t)\,dt\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty f(t)^q w(t)\,dt\right)^{\frac{1}{q}}. \tag{3.2}$$

was introduced. We may also consider it's conjugated variant in the form of

$$\left(\int_0^\infty \left(\int_t^\infty f(s)\,ds\right)^p w(t)\,dt\right)^{\frac{1}{p}} \leq C\left(\int_0^\infty f(t)^q w(t)\,dt\right)^{\frac{1}{q}}. \tag{3.3}$$

Curious reader may find characterization of both inequalities and their optimal constants for instance in [**29**]. These inequalities are very important tool when studying inequalities between weighted spaces. Lorentz spaces, Lorentz Karamata spaces and others are covered by more general structure called *Classical Lorentz spaces*. Their norms are defined in the following way.

Definition 3.1 (Classical Lorentz spaces). Let $0 < p < \infty$ and let v be a weight. Define

(i)

$$\|f\|_{\Lambda_v^p} = \left(\int_0^\infty f^*(s)^p v(s)\,ds\right)^{\frac{1}{p}}$$

and

$$\|f\|_{\Lambda_v^\infty} := \operatorname*{ess\,sup}_{t>0} f^*(t)v(t).$$

(ii)

$$\|f\|_{\Gamma_v^p} = \left(\int_0^\infty f^{**}(s)^p v(s)\,ds\right)^{\frac{1}{p}}$$

and

$$\|f\|_{\Gamma_v^\infty} := \operatorname*{ess\,sup}_{t>0} f^{**}(t)v(t)$$

We may also consider *Orlicz-Sobolev space* which is a generalization of Lorentz spaces and whose norm is defined by

$$\|f\|_{\Lambda_{\varphi,w}} := \|f^*\|_{L_w^\varphi} \tag{3.4}$$

Lorentz spaces of the type Λ were first introduced in [**23**]. In the research of functional properties and the embeddings between Lorentz spaces, the following statements will be very useful.

Theorem 3.2 (Hardy Lemma). *Let* $f, g \in L^1_{loc}[0, \infty)$ *such that*

$$\int_0^t f(s)\,ds \leq \int_0^t g(s)\,ds \quad \forall t \in (0, \infty)$$

then for all w non-increasing measurable functions the following holds

$$\int_0^t f(s)w(s)\,ds \leq \int_0^t g(s)w(s)\,ds.$$

For proof see [**2**, Proposition 3.6].

Theorem 3.3 (Hardy-Littlewood). *Let* $f, g \in \mathcal{M}(\mathcal{R})$ *then we have*

$$\int_{\mathcal{R}} fg \, d\mu \leq \int_0^\infty f^*(s) g^*(s) \, ds.$$

We shall observe that the operator $f \mapsto f^{**}$ is sublinear. Indeed, let be measurable set such that $E_t \subset \{|f+g| \geq (f+g)^*(t)\}$ and $\mu(E_t) = t$. Using the Hardy-Littlewood theorem, one can observe that

$$\int_0^t (f+g)^*(s) \, ds = \int_{E_t} |f| \, d\mu + \int_{E_t} |g| \, d\mu = \int_{\mathcal{R}} |f| \chi_{E_t} \, d\mu + \int_{\mathcal{R}} |g| \chi_{E_t} \, d\mu \leq \int_0^t f^*(s) \, ds + \int_0^t g^*(s) \, ds.$$

hence we obtain

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t) \quad \forall t \in (0, \infty). \tag{3.5}$$

One more formula we shall use in the following text is the *duality principle*. The proof of this statement is a simple exercise that can be done using saturated Hölder inequality. Let $p \in (1, \infty)$ then

$$\left(\int_0^\infty h^p w \right)^{\frac{1}{p}} = \sup_{g \geq 0} \frac{\int_0^\infty hg}{\left(\int_0^\infty g^{p'} w^{1-p'} \right)^{\frac{1}{p'}}}. \tag{3.6}$$

We also have

$$\sup_{g \geq 0} \frac{\int_0^\infty hg}{\int_0^\infty \frac{g}{w}} = \operatorname*{ess\,sup}_{t>0} hw.$$

Now note that the functional associating function with it's non-increasing rearrangement is not sublinear. This means that we do not know if $\|\cdot\|_{\Lambda^p_v}$ is a norm even in the case of $p \geq 1$. Therefore the following questions could come on readers mind.

Question 3.4. (i) For which setting is $\|\cdot\|_{\Lambda^p_w}$ a norm?
(ii) For what weights is $\|\cdot\|_{\Lambda^p_w}$ at least equivalent to a norm?
(iii) What are the optimal constants of embeddings of types $\Lambda \hookrightarrow \Lambda$, $\Lambda \hookrightarrow \Gamma$, $\Gamma \hookrightarrow \Gamma$ and $\Gamma \hookrightarrow \Lambda$?
(iv) For what type of weights are $\|\cdot\|_{\Lambda^p_w}$, $\|\cdot\|_{\Gamma^p_w}$ at least quasi-norms?
(v) For what weights are spaces $\|\cdot\|_{\Lambda^p_w}$

Let us go through history and answer every single of these elementary questions. The first question for $p \geq 1$ was answered in the very beginning by Lorentz (see [**23**]). He found out that $\|\cdot\|_{\Lambda^p_w}$ is a norm if and only if w is non-increasing. The equivalence to a norm is however much more difficult question. This question was solved by Sawyer in his famous paper (see [**31**]). Not only this question was considered here also the dual space was discussed. More precisely Sawyer proved the following equivalence.

$$\sup_{f \in \mathcal{M}_+^\downarrow(0,\infty)} \frac{\int_0^\infty gf}{\left(\int_0^\infty f^p w \right)^{\frac{1}{p}}} \approx \left[\int_0^\infty \left(\int_t^\infty \frac{g}{V} \right)^{p'} v \right]^{\frac{1}{p'}} \approx \left(\int_0^\infty \left(\int_0^t g \right)^{p'} \frac{v}{V^{p'}} \right) + \frac{\|g\|_1}{\|v\|_1^{\frac{1}{p}}}, \tag{3.7}$$

for $p > 1$ and arbitrary weight w. This equivalence provides a key step in characterization of the optimal constant of the embedding $\Lambda^q_w \hookrightarrow \Lambda^p_v$. Indeed, we have

$$\sup_{f \in \mathcal{M}_+^\downarrow(0,\infty)} \frac{\left(\int_0^\infty f^p v \right)^{\frac{1}{p}}}{\left(\int_0^\infty f^q w \right)^{\frac{1}{q}}} \overset{h=f^p}{=} \left(\sup_{h \in \mathcal{M}_+^\downarrow(0,\infty)} \frac{\int_0^\infty hv}{\left(\int_0^\infty h^r w \right)^{\frac{1}{r}}} \right)^{\frac{1}{p}},$$

where $r = \frac{q}{p}$. (3.7) together with Hardy lemma implies the following theorem.

Theorem 3.5. *Let* w, v *be weights and let* $0 < p \leq q < \infty$ *then the equality*

$$\left(\int_0^\infty f^*(s)^q w(s) \, ds \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^*(s)^p v(s) \, ds \right)^{\frac{1}{p}} \tag{3.8}$$

holds if and only if

$$A \approx \sup_{t>0} \frac{V(t)^{\frac{1}{p}}}{W(t)^{\frac{1}{q}}} < \infty.$$

Moreover the optimal constant of (3.8) *is equivalent to* A.

Now, one can observe that (3.7) provides the characterization of the associate space to $X = \Lambda^p_v$ and we have

$$\|f\|_{X'} = \left(\int_0^\infty f^{**}(s) \frac{s^{p'} v(s)}{V(s)^{p'}} \, ds \right)^{\frac{1}{p'}}.$$

We can see that the spaces of type Γ appear very naturally as associated spaces to the spaces of type Λ. Also if we study boundedness of maximal operator on the space lambda by (2.4) and (2.5) we have

$$\|Mf\|_{\Lambda^p_v} \approx \|f\|_{\Gamma^p_v}.$$

Therefore we have to study embeddings of the type $\Lambda \hookrightarrow \Gamma$. Another connection between Λ and Γ spaces can be found in the following theorem (see [**31**]) for proof.

THEOREM 3.6. *Let* $1 < p < \infty$ *then the following conditions are equivalent*

(i) Λ^p_v *is a Banach space (which means* $\|\cdot\|_{\Lambda^p_v} \approx \|\cdot\|_X$ *where the functional on the right-hand side is a Banach function norm.)*

(ii) $\Lambda^p_v = \Gamma^p_v$ *in the sense of equivalent norms*

(iii)
$$V(t)^{\frac{1}{p}} \int_0^t \frac{s^{p'-1}}{V(s)^{p'-1}} \, ds \le Ct$$
for some $0 < C < \infty$ *and all* $t \in (0, \infty)$.

(iv)
$$t^p \int_t^\infty \frac{v(s)}{s^p} \, ds \le CV(t),$$
for some $0 < C < \infty$ *and all* $t \in (0, \infty)$

The condition (iv) is called B_p condition and if w satisfy B_p we write $w \in B_p$. These results started a series of papers dealing with embeddings between Lorentz spaces. The embedding (3.8) in the case of $q < p$ was characterized in [**34**]. We can also consider embeddings of the following types

$$\Lambda^p_v \hookrightarrow \Gamma^q_w, \tag{3.9}$$

$$\Gamma^p_v \hookrightarrow \Lambda^q_w, \tag{3.10}$$

$$\Gamma^p_v \hookrightarrow \Gamma^q_w. \tag{3.11}$$

The embedding (3.9) was characterized in [**31**] for $1 < p \le q < \infty$, in [**7**] for $0 < p \le 1$, in [**31**] for $1 < q < p < \infty$, in [**7**] for $0 < q < 1 < p < \infty$. Case $0 < q < p = 1$ was characterized in [**32**] and the last case of $0 < q < p < 1$ was solved in [**33**]. Embedding (3.10) is fully described in more general form in [**15**]. The last embedding can be found in the more general form again in [**15**] and the last case $(0 < q < p < 1)$ in [**3**]. Concerning the other questions the quasi-norm property of Λ spaces was characterized in [**6**] and equivalent condition on linearity were given in [**9**]. Let us conclude this two results in one theorem.

THEOREM 3.7. *Let* $p \in (0, \infty)$ *and let* w *be a weight. Then the following conditions are equivalent.*

(i) $\|\cdot\|_{\Lambda^p_w}$ *is a quasi-norm.*

(ii) Λ^p_v *is a linear set.*

(iii) *There exists a constant* C *such that* $V(2t) \le CV(t)$ *for all* $t \in (0, \infty)$.

A generalization of Gamma space-$\Gamma_u^p(w)$, whose norm was defined by

$$\|f\|_{\Gamma_u^p(w)} := \left(\int_0^\infty f_u^{**}(s)^p v(s)\, d s\right)^{\frac{1}{p}},$$

where $p \in (0, \infty)$ and

$$f_u^{**}(t) := \frac{1}{U(t)} \int_0^t f^*(s) u(s)\, d s$$

(where $f_u^{**}(t)$ is called a *maximal function with respect to* u) was introduced in [**15**]. Now note that for general weight u the functional $f \mapsto f_u^{**}$ does not have to be sublinear. Therefore we may not assume $\Gamma_u^p(v)$ is a space for arbitrary weight u and $p \geq 1$ (in fact this is not true). For this space it's embedding to Λ spaces were proved in [**15**]. Converse embeddings were characterized in [**14**]. Another way of generalization was made in [**11**]. The authors were motivated by study of the norm in *Grand Lebesgue space*. This space arises from the problems in PDEs (for more information see section4.3). However authors proved the equivalence between this type of space and the *Generalized Gamma Space* for certain setting of parameters. Let $p, m \in (0, \infty)$ and let w be a weight. Then define the norm in generalized gamma space by

$$\|f\|_{G\Gamma_w^{p,m}} := \left[\int_0^\infty \left(\int_0^t f^*(s)^p\, d s\right)^{\frac{m}{p}} w(t)\, d t\right]^{\frac{1}{m}}. \tag{3.12}$$

Two more articles by these authors (see [**12, 13**]) were devoted to the study of it's functional properties (continuity of the norm, characterization of associated norm and reflexivity). However authors did not reach the complete results.

4. Applications of Lorentz spaces

4.1. Interpolation theory. Let us note that if $(\mathcal{R}, \mu)$ is a finite measure space, then $L^p(\mathcal{R})$-spaces form a scale. Indeed, Hölder inequality implies

$$L^q(\mathcal{R}) \hookrightarrow L^p(\mathcal{R}),$$

provided $q \geq p$. This L^p scale plays very important in the interpolation theory. Let us present a very classical result to illustrate what do we mean.

Theorem 4.1 (Riesz-Thorin's interpolation theorem). *Let $(\mathcal{R}, \mu)$, $(\mathcal{S}, \nu)$ be σ-finite measure spaces. And let T be a linear operator defined on $L^{p_0} + L^{p_1}$ (see Definition2.5), $T : L^{p_0}(\mathcal{R}) \to L^{q_0}$ and $T : L^{p_1} \to L^{q_1}$. If p, q are numbers satisfying*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \text{ for some } \theta \in (0,1)$$

Then $T : L^p \to L^q$.

This theorem is very useful in proving boundedness of many operators. It suffices to prove it for two spaces on the scale and we have boundedness for all the spaces between. But there exists even a finer scale. If we define

$$\|f\|_{L^{p,q}} := \left\| t^{\frac{1}{p}-\frac{1}{q}} f^* \right\|_q$$

we obtain the norm in *Lorentz Spaces* (in fact in case $q > p$ this is not a norm). Again let $\mu(\Omega) < \infty$. It is not so difficult to show that

$$L^{p,q}(\mathcal{R}) \hookrightarrow L^{p,r}(\mathcal{R}) \quad \text{for} \quad q < r,$$

and

$$L^{p_0,q_0} \hookrightarrow L^{p_1,q_1} \quad \text{if} \quad p_0 > p_1.$$

We can see that $L^{p,q}$ spaces provides finer scale then L^p. Let us demonstrate their crucial meaning in interpolation on a simple example. Let us study boundedness of maximal operator from L^p to L^p. Simple observation shows that $M : L^\infty \to L^\infty$. But it is also relatively easy to realize that $M : L^1 \not\to L^1$. So we may not use the Riesz-Thorin interpolation theorem. On the other hand, using Vitali covering theorem see ([**24**]) one can prove that

$$t|\{Mf > t\}| \le C(n) \|f\|_1 \quad \forall t \in (0, \infty).$$

Since $C(n)$ is independent of t we may pass to supremum over all $t \in (0, \infty)$ and deduce

$$\|Mf\|_{1,\infty} \le C \|f\|_1 .$$

Now, in the terms of Lorentz spaces we have more powerful tool - Marcinkiewicz interpolation theorem, which reads as follows.

Theorem 4.2 (Marcinkiewicz interpolation theorem). *Let $(\mathcal{R}, \mu)$, $(\mathcal{S}, \nu)$ be two σ-finite measure spaces. Let $1 \le p_0 < p_1 \le \infty$ and $1 \le q_0, q_1 \le \infty$. Let T be a quasilinear operator such that T is defined on $L^{p_0,1} + L^{p_1,1}$, $T : L^{p_0,1} \to L^{q_0,\infty}$ and $T : L^{p_1,1} \to L^{q_1,\infty}$. If p, q are numbers satisfying*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

for some $\theta \in (0, 1)$. Then $T : L^{p,r} \to L^{q,r}$, for any $1 \le r \le \infty$.

Using this theorem one can see that for every $p \ge 1$ there exists C such that

$$\|Mf\|_p \le C \|f\|_p .$$

4.2. Optimal Sobolev embeddings. Sobolev spaces are basic instrument in the theory of partial differential equations and calculus of variations. It is a natural space in which solutions of these equations should be found. Let us provide the reader with a definition of this space.

Definition 4.3. Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in L^1_{loc}(\Omega)$ let g_α be a measurable function such that

$$\int_\Omega f(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g_\alpha(x)\varphi(x)\, dx, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

where α is a multi-index and $\mathcal{D}(\Omega)$ is the space of all smooth functions with compact support in Ω. Then we call g_α *weak α-partial derivative of* f.

Now the norm in Sobolev space is defined in the following way

$$\|u\|_{W^{k,p}} := \sum_{|\alpha| \le k, \alpha \in I} \|D^\alpha u\|_{L^p} ,$$

where I is the set of multi-indexes. To prove existence of solutions of certain type of PDEs it is always good to have some embeddings theorems like the following one.

Theorem 4.4 (Gagliardo-Nirenberg-Sobolev). *Let $\Omega \subset \mathbb{R}^n$ be an open domain, $1 \le p < n$ and set $p^* := \frac{np}{n-p}$. Let $u \in W^{1,p}_0(\Omega)$ then there exists $C = C(p, n)$ such that*

$$\|u\|_{p^*} \le C \|Du\|_p .$$

This theorem can be sharpened, if we consider finer scale of Lorentz spaces. The sharpened version of Gagliardo-Nirenberg theorem reads as follows.

Theorem 4.5 (Peetre). *Let $1 \le p < n$ and set $p^* := \frac{np}{n-p}$ then there exists $C = C(n, p)$ such that*

$$\|u\|_{L^{p^*,q}} \le C \|Du\|_{L^{p,q}} .$$

This embedding is not only sharper than the classical one. If we consider any rearrangement-invariant function space Y such that

$$W^{1,p} \hookrightarrow Y \tag{4.1}$$

then $L^{p^*,p} \hookrightarrow Y$ (see [**28, 10**]). So the space $L^{p^*,p}$ is in some sense optimal range space for (4.1) in the class of r.i. BFS.

REMARKS 4.6. (i) Note that in the case of $q > p$ the functional is not a norm itself.
(ii) For $p > 1$ the functional $f \mapsto \|f\|_{L^{p,q}}$ is equivalent to

$$\|f\|_{L(p,q)} := \left\| t^{\frac{1}{p} - \frac{1}{q}} f^{**} \right\|_q ,$$

which is a norm.

4.3. Integrability of Jacobian. Let us consider the following situation. If $\Omega \subset \mathbb{R}^n$ is a given domain and let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$. Denote the Jacobian of f by

$$J_f = \det(Df).$$

The Jacobian function arises in many various fields of mathematics such as the theory of measure and integration, degree of a mapping, quasi-conformal mapping theory, nonlinear elasticity... Now one may ask what is the necessary or sufficient condition for local integrability of Jacobian. First thing that comes to mind is the condition

$$\int_\Omega |Df|^n \, dx < \infty.$$

This is clearly a sufficient condition. But is it also a necessary condition? What is the mildest sufficient condition on Df for local integrability of J_f? If we assume that the mapping f is *orientation preserving* which means

$$J_f \geq 0 \quad \text{a.e.}$$

then we have that

$$\int_\Omega J_f \log(e + J_f) \, dx \leq C(n) \int_\Omega |Df|^n \, dx.$$

This result was proved in [**26**]. Therefore this condition is strong enough to assure that $J_f \in (L \log L)_{loc}(\Omega)$, where $L \log L$ is the Orlicz space with Young function given by

$$\Phi(t) = t \log(t + e).$$

Now reader might be curious about some better result in the other direction. This result came soon in [**19**] and it reads as follows.

THEOREM 4.7. *Let $B \subset \mathbb{R}^n$ be a domain, $f \in W^{1,1}(3B, \mathbb{R}^n)$ (where $3B$ is a ball with same center as B but with radius three times larger). Let f be an orientation preserving mapping. Moreover let $f \in W^{1,s}(3B, \mathbb{R}^n)$ for all $s < n$, such that*

$$\sup_{1 \leq s < n} \left[(n - s) \int_{3B} |Df|^s \, dx \right]^{\frac{n}{s}} .$$

Then $J_f \in L^1_{loc}(3B)$ and the following estimate holds

$$⨍_B |J_f| \, dx \leq C(n) \sup_{1 \leq s < n} \left[(n - s) ⨍_{3B} |Df|^s \, dx \right]^{\frac{n}{s}} .$$

So if we define

$$\|f\|_{p)} := \sup_{1\le s<p} \left[(p-s)\fint_{3B} |f|^s \, dx\right]^{\frac{p}{s}}, \quad \text{for} \quad p \in (0,\infty),$$

which is the norm in *Grand Lebesgue space*, we obtain that if f is orientation preserving and $Df \in L^{p)}_{loc}$ then $J_f \in L^1_{loc}$. In the paper **[11]** the authors proved that this norm is equivalent to a norm in Generalized Gamma space. They proved the following equivalence

$$\|f\|_{p)} \approx \int_0^1 \left[\int_0^t f^*(s)^p \, ds\right]^{\frac{1}{p}} \frac{dt}{t \log^{\frac{1}{p}}\left(\frac{1}{t}\right)}. \tag{4.2}$$

2. Summary of papers attached to the thesis

1. Paper 1: Characterization of associate spaces of weighted Lorentz spaces with applications

This paper is devoted to the study of generalized gamma spaces for definition see (3.12). These spaces appeared in [**11**]. Their connection to Grand Lebesgue spaces was obtained here (see (4.2)). There were two more papers about this type of spaces (see [**12, 13**]). In these papers the authors were trying to characterize the associated norm, and to obtain characterization of reflexivity, they also proved that $G\Gamma_w^{p,m}$ is a Banach function space for $1 \leq p \leq m \leq \infty$. Concerning the associated norm authors proved only one-sided estimates that did not meet. Using these results they obtained reflexivity of these spaces for $p \geq 2$. In our paper (see [**17**]) we improved the results. To avoid emptiness of the space we have to assume that

$$\int_0^t w(s) s^{\frac{m}{p}}\, ds + \int_t^b w(s)\, ds < \infty \quad \text{for every } t \in (0,b), \tag{1.1}$$

First we obtain the following characterization of the associated norm.

Theorem 1.1. *Assume that* $m, p \in (0,\infty)$ *and let* w *be a weight on* $(0,\infty)$ *such that* (1.1) *is satisfied. Let* $X = G\Gamma(p,m,w)$ *and let* X' *be its associate space. Denote*

$$u(t) := \int_0^t w(s) s^{\frac{m}{p}}\, ds + t^{\frac{m}{p}} \int_t^b w(s)\, ds, \quad t \in (0,b).$$

(i) *Let* $0 < m \leq 1$ *and* $0 < p \leq 1$. *Then*

$$\|g\|_{X'} \approx \sup_{t \in (0,\frac{b}{2})} g^{**}(t) \frac{t}{u(t)^{\frac{1}{m}}}.$$

(ii) *Let* $0 < m \leq 1$ *and* $1 < p < \infty$. *Then*

$$\|g\|_{X'} \approx \sup_{t \in (0,\frac{b}{2})} \left(\int_t^b g^{**}(s)^{p'}\, ds \right)^{\frac{1}{p'}} \frac{t}{u(t)^{\frac{1}{m}}}.$$

(iii) *Let* $1 < m < \infty$ *and* $0 < p \leq 1$. *Then*

$$\|g\|_{X'} \approx \left(\int_0^{\frac{b}{2}} \sup_{y \in (t,b)} g^{**}(y)^{m'} y^{\frac{m'(p-1)}{p}} \frac{t^{\frac{m'}{p} + \frac{m}{p} - 1} \int_0^t w(s) s^{\frac{m}{p}}\, ds \int_t^b w(s)\, ds}{u(t)^{m'+1}}\, dt \right)^{\frac{1}{m'}}.$$

(iv) *Let* $1 < m < \infty$ *and* $1 < p < \infty$. *Then*

$$\|g\|_{X'} \approx \left(\int_0^{\frac{b}{2}} \left(\int_t^b g^{**}(s)^{p'}\, ds \right)^{\frac{m'}{p'}} \frac{t^{\frac{m'}{p} + \frac{m}{p} - 1} \int_0^t w(s) s^{\frac{m}{p}}\, ds \int_t^b w(s)\, ds}{u(t)^{m'+1}}\, dt \right)^{\frac{1}{m'}} + \lim_{t \to 0+} \frac{\left[\int_t^b g^{**}(s)^{p'}\, ds \right]^{\frac{1}{p'}}}{u^{\frac{1}{m}}(t)} t^{\frac{1}{p}}.$$

We characterized full range of parameters for which $G\Gamma_w^{p,m}$ is a Banach function space in the following theorem

Theorem 1.2. *Suppose that* $1 \leq p, m < \infty$ *and let* w *be a weight on* $(0,b)$. *Then the space* $G\Gamma(p,m,w)$ *is a Banach function space if and only if*

$$\int_0^b \min\{1, t^{\frac{m}{p}}\} w(t)\, dt < \infty. \tag{1.2}$$

Moreover we characterized the sobolev embedding

$$W^1 G\Gamma(p,m,w)(\Omega) \hookrightarrow L^\infty(\Omega). \tag{1.3}$$

THEOREM 1.3. *Let $n \in \mathbb{N}$, $n \geq 2$. Let $m, p \in [1,\infty)$ and let w be a weight on $(0,1)$ such that*

$$\int_0^1 t^{\frac{m}{p}} w(t)\,dt < \infty. \tag{1.4}$$

Then the Sobolev embedding (1.3) *holds if and only if one of the following conditions is satisfied:*

(i) $m = 1$, $p \in [1,n)$ *and*

$$\sup_{t\in(0,\frac12)} \frac{t^{\frac1n - \frac1p + 1}}{\int_0^t w(s)s^{\frac1p}\,ds + t^{\frac1p}\int_t^1 w(s)\,ds} < \infty;$$

(ii) $m = 1$, $p = n$ *and*

$$\sup_{t\in(0,\frac12)} \frac{t(\log\frac1t)^{\frac{1}{p'}}}{\int_0^t w(s)s^{\frac1p}\,ds + t^{\frac1p}\int_t^1 w(s)\,ds} < \infty;$$

(iii) $m = 1$, $p \in (n,\infty)$ *and*

$$\sup_{t\in(0,\frac12)} \frac{t}{\int_0^t w(s)s^{\frac1p}\,ds + t^{\frac1p}\int_t^1 w(s)\,ds} < \infty;$$

(iv) $m \in (1,\infty)$, $p \in [1,n)$ *and*

$$\int_0^{\frac12} \frac{t^{\frac{m'}{n} + \frac{m}{p} - 1}\int_0^t w(s)s^{\frac{m}{p}}\,ds \int_t^1 w(s)\,ds}{\left(\int_0^t w(s)s^{\frac{m}{p}}\,ds + t^{\frac{m}{p}}\int_t^1 w(s)\,ds\right)^{m'+1}}\,dt < \infty;$$

(v) $m \in (1,\infty)$, $p = n$ *and*

$$\int_0^{\frac12} \frac{t^{\frac{m'}{p} + \frac{m}{p} - 1}(\log\frac1t)^{\frac{m'}{p'}}\int_0^t w(s)s^{\frac{m}{p}}\,ds \int_t^1 w(s)\,ds}{\left(\int_0^t w(s)s^{\frac{m}{p}}\,ds + t^{\frac{m}{p}}\int_t^1 w(s)\,ds\right)^{m'+1}}\,dt < \infty;$$

(vi) $m \in (1,\infty)$, $p \in (n,\infty)$ *and*

$$\int_0^{\frac12} \frac{t^{\frac{m'}{p} + \frac{m}{p} - 1}\int_0^t w(s)s^{\frac{m}{p}}\,ds \int_t^1 w(s)\,ds}{\left(\int_0^t w(s)s^{\frac{m}{p}}\,ds + t^{\frac{m}{p}}\int_t^1 w(s)\,ds\right)^{m'+1}}\,dt < \infty;$$

Finally we discused the absolute continuity of the norm in the original and the associated space with the following results.

THEOREM 1.4. *Let $p, m \in (1,\infty)$ and let w be a weight on $(0,b)$. Then the space $G\Gamma(p,m,w)$ has absolutely continuous norm if and only if at least one of the following conditions holds:*

$$b < \infty, \tag{1.5}$$

$$\int_0^b t^{\frac{m}{p}} w(t)\,dt = \infty. \tag{1.6}$$

THEOREM 1.5. *Let $p, m \in (1,\infty)$ and let w be a weight on $(0,b)$. Then the associate space to $G\Gamma(p,m,w)$ has an absolutely continuous norm.*

Using these two results and Theorem 2.9, we obtain the following corollary.

THEOREM 1.6. *Let $p, m \in (1,\infty)$ and let w be a weight on $(0,b)$. Then the space $G\Gamma(p,m,w)$ is reflexive if and only if at least one of the conditions* (1.5) *and* (1.6) *holds.*

2. Paper 2: Normability of Lorentz spaces - an alternative approach

In this paper we consider the problem which was originally solved by Sawyer (see Theorem 3.6). We take a duality approach, which is a very universal instrument and can be applied in many specific situations. The fundamental theorem reads as follows.

THEOREM 2.1. *Let* $\|\cdot\|_X : \mathcal{M}(\mathcal{R}) \to [0, \infty]$ *be a functional with the following properties.*

(i) $\|f\|_X = \||f|\|_X$;

(ii) $\|\chi_E\|_X < \infty$ *whenever* $\mu(E) < \infty$;

(iii) *for every* E *of finite measure there exists* $\infty > C_E > 0$ *such that*

$$C_E \|f\chi_E\|_X \geq \int_E |f| d\mu.$$

Then the functional $\|\cdot\|_{X'}$ *is a Banach function norm.*

Moreover, $\|\cdot\|_X$ *is equivalent to a Banach function norm if and only if* $\|\cdot\|_X \approx \|\cdot\|_{X''}$.

Since the associated norm to $\|\cdot\|_{\Lambda^p_w}$ is known, as well as the second dual were characterized, it remains only characterize the appropriate embedding of the second dual space. Now, we know that X' is a Banach function space and since we know that X'' is a Lorentz space of type Γ. The appropriate optimal constant of

$$X'' \hookrightarrow X \tag{2.1}$$

may be expressed in the terms of fundamental function (more precisely (2.1) is equivalent to (2.2)). Since the space X'' is rather complicated, we may simplify things by using Theorem 2.6 on the space X' to obtain that

$$\varphi_{X''}(t) \lesssim \varphi_X(t) \quad \forall t \in (0, \infty) \tag{2.2}$$

is in fact equivalent to

$$\frac{t}{\varphi_{X'}(t)} \lesssim \varphi_X(t) \quad \forall t \in (0, \infty).$$

Now by calculating the optimal constant of the last inequality we may reprove the Sawyer's result. We obtain the following characterization.

THEOREM 2.2. *Let* v *be a weight and let* $1 < p < \infty$. *The following conditions are equivalent.*

(i) *Functional* $\|\cdot\|_{\Lambda^p_v}$ *is equivalent to a Banach function norm;*

(ii)

$$\int_0^t s^{p'} \frac{v(s)}{V^{p'}(s)} ds \lesssim t^{p'} V^{1-p'}(t), \quad t \in (0, \infty);$$

(iii)

$$\int_0^t \frac{s^{p'-1}}{V^{p'-1}(s)} ds \lesssim t^{p'} V^{1-p'}(t), \quad t \in (0, \infty)$$

Using this approach we may also obtain the characterization in the weak case. The result which can be with some modifications found in [4] is the following.

THEOREM 2.3. *Let* v *be a weight. Let us set*

$$\boldsymbol{v}(t) := \operatorname*{ess\,sup}_{s \leq t} v(s).$$

Then the following conditions are equivalent.

(i) *Functional* $\|\cdot\|_{\Lambda^\infty_v}$ *is equivalent to a Banach function norm;*

(ii)

$$\sup_{t>0} \frac{\mathfrak{v}(t)}{t} \int_0^t \frac{dz}{\mathfrak{v}(z)} < \infty;$$

(iii)

$$\Lambda_\nu^\infty = \Gamma_\nu^\infty;$$

in the sense of equivalent norms.

3. Paper 3: Note on linearity of rearrangement-invariant spaces

In this paper we consider a general r.i. lattice (see Definition 2.2). In this very general setting we study when a r.i. lattice is a linear space it turns out that the characterization via dilation operator on the representation space can be made. We proved the following theorem.

Theorem 3.1. *Let* X *be an r.i. lattice for which there exists a representation functional* $\|\cdot\|_X$.

(i) *Assume that the space*

$$X := \{f \in \mathcal{M}(0,\infty) : \|f\|_X < \infty\}$$

is a linear set. Then the space X *is a linear set if and only if the following implication holds:*

$$\text{if } \|f^*\|_X < \infty, \text{ then } \|E_2 f^*\|_X < \infty. \tag{3.1}$$

(ii) *Assume that* $\|\cdot\|_X$ *is a quasi-norm. Then* $\|\cdot\|_X$ *is a quasi-norm if and only if there exists a positive constant* C *such that*

$$\|E_2 f^*\|_X \leq C \|f^*\|_X . \tag{3.2}$$

(iii) *Assume that* $\|\cdot\|_X$ *is a norm. Then* $\|\cdot\|_X$ *is a norm if and only if*

$$\|E_2 f^*\|_X \leq 2 \|f^*\|_X . \tag{3.3}$$

We used these results in the case of Orlicz-Sobolev lattices. First step is to prove one lemma, which can be deduces from [**27**] but since the proof in this book is quite technical and typing style is difficult to read let us present the lemma with a short proof (which was not included in the paper).

Lemma 3.1. *Let* φ *and* w *be as in Definition 2.11. Then*

(i) $\|\cdot\|_{L^\varphi_w}$ *has lattice property,*
(ii) L^φ_w *is a linear set,*
(iii) *if, moreover, there exists* $\alpha > 1$ *such that*

$$\varphi(\alpha s) > 2\varphi(s) \quad \forall s \in (0,\infty), \tag{3.4}$$

then $\|\cdot\|_{L^\varphi_w}$ *is a quasi-norm.*

Proof. Statement (i) is obvious, since φ is increasing.
(ii) We have

$$\begin{aligned}
&\int_0^\infty \varphi\left(\frac{f(s)+g(s)}{\lambda}\right) w(s)\,ds \\
&= \int_{\{f\geq g\}} \varphi\left(\frac{f(s)+g(s)}{\lambda}\right) w(s)\,ds + \int_{\{f<g\}} \varphi\left(\frac{f(s)+g(s)}{\lambda}\right) w(s)\,ds \\
&\leq \int_{\{f\geq g\}} \varphi\left(\frac{2f(s)}{\lambda}\right) w(s)\,ds + \int_{\{f<g\}} \varphi\left(\frac{2g(s)}{\lambda}\right) w(s)\,ds \\
&\leq 2\max\left\{\int_0^\infty \varphi\left(\frac{2f(s)}{\lambda}\right) w(s)\,ds, \int_0^\infty \varphi\left(\frac{2g(s)}{\lambda}\right) w(s)\,ds\right\}.
\end{aligned} \tag{3.5}$$

Therefore, if $f, g \in L^\varphi_w$, then there exist $\lambda_f, \lambda_g > 0$ for which

$$\int_0^\infty \varphi\left(\frac{f(s)}{\lambda_f}\right) w(s)\,ds < \infty \qquad \int_0^\infty \varphi\left(\frac{g(s)}{\lambda_g}\right) w(s)\,ds < \infty.$$

Now if we set

$$\lambda_{f+g} := 2\max\{\lambda_f, \lambda_g\},$$

then we have

$$\int_0^\infty \varphi\left(\frac{f(s)+g(s)}{\lambda_{f+g}}\right) w(s)\,ds < \infty.$$

Therefore, $f + g \in L^\varphi_w$. This establishes assertion (ii).

(iii) Assume that the additional condition (3.4) is satisfied. Without loss of generality we may suppose that $\|f\|_{L^\varphi_w} \ge \|g\|_{L^\varphi_w}$. Thus, due to (3.5), we have

$$\left\{\lambda : \int_0^\infty \varphi\left(\frac{f(s)+g(s)}{\lambda}\right) w(s)\,ds < 1\right\} \supset \left\{\lambda : \int_0^\infty \varphi\left(\frac{2f(s)}{\lambda}\right) w(s)\,ds < \frac{1}{2}\right\}$$
$$\supset \left\{\lambda : \int_0^\infty \varphi\left(\frac{2\alpha f(s)}{\lambda}\right) w(s)\,ds < 1\right\}.$$

Therefore,

$$\|f+g\|_{L^\varphi_w} \le 2\alpha \max\left\{\|f\|_{L^\varphi_w}, \|g\|_{L^\varphi_w}\right\} \le 2\alpha(\|f\|_{L^\varphi_w} + \|g\|_{L^\varphi_w}),$$

showing the validity of assertion (iii). The proof is complete. □

Combining this lemma with our theorem yields the following characterization of linearity and also quasi-norm property (which was not included in the paper since it was known). The extended version of the theorem is the following.

Theorem 3.2. *Let φ and w have the same properties as in Definition 2.11. Then the following conditions are equivalent.*

(i) *$\Lambda_{\varphi,w}$ is linear;*
(ii)

$$\sup_{t>0} \frac{W(2t)}{W(t)} < \infty.$$

If moreover $\varphi(2t) \le C\varphi(t)$ where C is independent of t, both conditions are equivalent to quasi-norm property of $\|\cdot\|_{\Lambda_{\varphi,w}}$.

4. Paper 4: Normability of gamma spaces

In this paper we focus on the question for which setting is the functional $\|\cdot\|_{\Gamma^p_w}$ equivalent to a norm. One can observe that since the operator $f \mapsto f^{**}$ is sub-additive for $p \ge 1$ the question is trivial. We shall therefore restrict to case $p < 1$. The result can be deduced from a more complicated theorem in [**20**]. Our proof has the advantage that it is straightforward and less technical. First we prove the following lemma.

Lemma 4.1. *Let X be a linear vector space. Let $\sigma : X \to [0, \infty)$ be a positively homogeneous functional. Then the following conditions are equivalent*

(i) *σ is equivalent to a norm;*

(ii) *there exists a constant* C, *independent on* N, *such that*

$$\sigma\left(\sum_{k=1}^{N} f_k\right) \le C\sum_{k=1}^{N}\sigma(f_k),$$

for all $f_k \in X$.

We used this lemma to prove the main theorem.

THEOREM 4.1. *Let* $0 < p < 1$ *and let* w *be a weight. Then the following conditions are equivalent.*

(i) *The space* Γ^p_w *is normable.*

(ii) *Both* $w(s)$ *and* $w(s)s^{-p}$ *are integrable on* $(0,\infty)$.

(iii) *The identity*

$$\Gamma^p_w = L^1 + L^\infty$$

holds in the sense of equivalent norms.

4.1. Paper 5: Embeddings of classical Lorentz spaces involving weighted integral means. In this paper we study the optimal constants of the embeddings

$$G\Gamma^{m_1,p_1}_{w_1,u_1} \hookrightarrow G\Gamma^{m_2,p_2}_{w_2,u_2}, \tag{4.1}$$

where

$$G\Gamma^{m,p}_{w,u} := \left\{ f \in \mathcal{M}(\mathcal{R}) : \|f\|_{G\Gamma^{m,p}_{w,u}} \right\},$$

and

$$\|f\|_{G\Gamma^{m,p}_{w,u}} := \left(\int_0^\infty \left[\int_0^t f^*(s)^p u(s)\,d\,s\right]^{\frac{m}{p}} w(s)ds\right)^{\frac{1}{m}}.$$

The optimal constants of such and embeddings are characterized in the case of $m_2 > p_2$ in such a case we use the duality principle ((3.6)) and Fubini theorem and obtain the following equation for the optimal constant of (4.1).

$$C_{opt} = \sup_{f\downarrow}\left[\sup_{h\ge 0}\frac{\int_0^\infty h(t)\int_0^t f(s)^p u(s)\,d\,s w(t)\,d\,t}{\|h\|_{L^{p_2'}_{w_2^{1-p'}}}\|f\|_{G\Gamma^{m_1,p_1}_{w_1,u_1}}}\right]^{\frac{1}{p_2}}$$

$$= \sup_{h\ge 0}\left[\sup_{f\downarrow}\frac{\int_0^\infty \int_s^\infty h(t)w(t)\,d\,t f(s)^p u(s)\,d\,s}{\|h\|_{L^{p_2'}_{w_2^{1-p'}}}\|f\|_{G\Gamma^{m_1,p_1}_{w_1,u_1}}}\right]^{\frac{1}{p_2}}.$$

Now using some results from Hardy-type inequalities (see for instance [**16, 15**]) and after technical calculation we arrive to the desired characterization. The result itself is so technical that we shall skip it in the introduction. Let us note that in the most simple case, if $m_2 > p_2$, $p_1 \le p_2$ and $m_1 \le p_2$, we have

$$C_{opt} \approx \sup_{t>0}\frac{\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,d\,s + U_2(t)^{\frac{m_2}{p_2}}\int_t^\infty w_2(s)\,d\,s\right)^{\frac{1}{m_2}}}{\left(\int_0^t U_1(s)^{\frac{m_1}{p_1}} w_1(s)\,d\,s + U_1(t)^{\frac{m_2}{p_2}}\int_t^\infty w_1(s)\,d\,s\right)^{\frac{1}{m_2}}}.$$

If $p_2 = m_2$, the space degenerates to Lorentz space of type Λ and the embeddings are easy consequences of results in [**15**]. If $p_2 > m_2$ the problem remains open.

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