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Ricci flow and geometric analysis on manifolds

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Abstract: This thesis discusses basis aspects of the Ricci flow on manifolds with a view towards the ambient space construction. We start with the background review of the Riemannian geometry and parabolic partial differential equations, and the Ricci flow problem on manifolds is established. Then we aim towards the formulation of the Ricci flow problem on ambient spaces and provide several basic examples. There are two main parts: the first consists of general theory needed to formulate our problem and strategy, while the second part consists of particular calculations associated with the Ricci flow problem.

Keywords: Ricci flow, Ambient space, Ambient metric, Poincaré-Einstein metric.

Dedication.

I would like to thank my thesis supervisor doc. RNDr. Petr Somberg Ph.D. for his patient and helpful approach.

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Contents

Introduction

The ambient metric construction associates to a given manifold equipped with a conformal structure, $(M, [g])$, a pseudo-Riemannian manifold of two dimensions higher equipped with a Ricci flat metric and called the ambient metric space. A key component in the construction of the ambient metric space is the Poincaré-Einstein metric space of one dimension higher than M, [AM].

The ambient metric (or, equivalently the Poincaré-Einstein metric) is determined by a non-linear boundary valued problem for a representative metric in the conformal class $[g]$, thereby its coefficients are rather complicated, and in general not known, functions of metric invariants associated to a representative of $[g]$. Thus, it is desirable, to have yet another intrinsic and independent characterization of the ambient metric. We initiate one possible direction to potentially gain such a characterization, namely we apply the Ricci flow procedure to a given ambient (or, equivalently the Poincaré-Einstein) metric. In the light of the fact that the ambient metric is Ricci flat (or, Einstein) and hence related to specific behavior of the Ricci flow, this might result in remarkable conclusions.

The aim of the present thesis is to establish basic results about the Ricci flow on manifolds with emphasis on the ambient metric and the Poincaré-Einstein metric. The Ricci flow on a Riemannain manifold is a geometrical flow that deforms the metric on the Riemannian manifold with respect to negative of the Ricci curvature of the manifold.

The ambient metric is a pseudo-Riemannian metric on a product of the initial conformal Riemannian manifold $(M, [g])$, $\mathbb{R}$ and $\mathbb{R}_+$, and it is constructed such that it is Ricci flat (up to some order, respectively, depending on the parity of the dimension). The Poincaré-Einstein metric can then be constructed out of the ambient metric. Alternatively, the Poincaré-Einstein metric may be constructed as a metric that is Einstein with Einstein constant equal to minus the dimension of the initial manifold. The Ricci flow problem combines both notions together. We ask for a metric that flows, in the normalized time $t = 1$ for the Ricci flow, to the Poincaré-Einstein metric of the initial conformal Riemannian manifold $(M, [g])$. We note that the same reasoning works with the pseudo-Riemannian manifolds as well, however we shall work only with Riemannian manifolds.

Because we will work in the setting of Riemannian manifolds, we need a background in the Riemannian geometry as in [RF] and [CL]. Secondly, since the statement of the problem revolves around solutions of partial differential equations, we will need some theory for partial differential equations. In particular, we will need theory of parabolic partial differential equations and follow [EV]. In establishing the ambient metric and the Poincaré-Einstein metric, we follow [AM]. We carry over some results of [RF] in formulating the existence and uniqueness statements for the Ricci flow on manifolds.

Let us briefly comment on the structure of our thesis. The first chapter establishes basic definitions in Riemannian geometry and some formulas for calculating Riemann curvature. It consists of a compilation of definitions from [RF] and [CL]. The second chapter aims to provide definitions and basic properties of the ambient metric and the Poincaré-Einstein metric for a conformal Riemannian manifold, and also their relationship. The chapter consists of a compilation of results from

[AM]. In the third chapter we discuss the linear parabolic partial differential equations, their generalization to quasilinear equations in the non-linear setting. Moreover, we discuss a generalization of partial differential equations to manifold and provide an existence statement for this class of partial differential equations. This chapter follows [RF], "Comments on existence theory for parabolic PDE". The next chapter establishes the Ricci flow on a Riemannian manifold. Although the Ricci flow equation is not parabolic, we use a workaround called "deTurck trick" to prove the existence of its solutions using theory of parabolic partial differential equations. We also prove a uniqueness statement for the Ricci flow and finish this chapter by demonstrating simple examples of the Ricci flow on manifolds. In this chapter we follow again [RF], "Existence theory for the Ricci flow". The last, fifth chapter, deals with the Ricci flow problem and we provide a theoretical solution of the problem. We follow by showing two approaches that finish with an explicit unique solution in the case when the initial manifold is euclidean space. We also provide the system of partial differential equations needed to be solved to find a general solution in the curved case. Secondly, we provide a simplified system of partial differential equations obtained by simplifying assumptions on the flow metrics.

To summarize, the main contribution of this thesis is an attempt to find a solution to the Ricci flow problem for ambient (or, the Poincaré-Einstein) metrics in general, and simplifying the associated system of equations by imposing some conditions on the solution. Secondly, there is an explicit solution of the Ricci flow problem for the euclidean case.

1. Background in Riemannian geometry

Definition 1. *We say that a map $\psi : U \to \mathbb{R}^n$, where U is a subset of $\mathbb{R}^n$, is a diffemorphism if ψ is injective and both ψ and ψ^{-1} are smooth.*

A basic building concept in geometry is that of a manifold.

Definition 2. *Let M be a set. A (n-dimensional) chart on M is a pair (U, φ), where $U \subset M$ is an open subset and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset in $\mathbb{R}^n$.*

For (U_1, φ_1), (U_2, φ_2) two n-dimensional charts on M, the mapping $\varphi_1 \circ \varphi_2^{-1}$ defined on $\varphi_2(U_1 \cap U_2)$ is called the transition function associated to these two maps. When $U_1 \cap U_2 = \emptyset$, we regard the transition function to be trivial mapping satisfying all conditions of compatibility defined later on.

Two n-dimensional charts (U_1, φ_1), (U_2, φ_2) are compatible if $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are open and the transition function defined between them is a diffeomorphism.

An atlas on M is given by a set of n-dimensional charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that every two maps in the atlas are compatible and $\cup_{\alpha \in A} U_\alpha = M$.

On the other hand, given an atlas consisting of n-dimensional charts on a set M, we induce a topology on M in the following way: a set $X \subset M$ is open if and only if for every charts (U, φ) from the atlas the set $\varphi(X \cap U)$ is an open subset of $\mathbb{R}^n$.

A manifold of dimension n is a set M equipped with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ consisting of n-dimensional charts, such that with respect to the topology on M induced by the atlas the topological space M is Hausdorff and second-countable.

A (n-dimensional) chart (U, φ) is compatible with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A} \cup (U, \varphi)$ is an atlas on M.

A differential structure is defined as an atlas that is maximal with respect to inclusions of n-dimensional charts. A submanifold X of the manifold M is defined as a subset $X \subset M$, equipped with the structure of a manifold such that the induced topology is the same as the topology given by the restriction from M to X of the topology induced by the manifold structure on M.

Any manifold can always be equipped with a differentiable structure, because we can complete an atlas to a differentiable structure.

Definition 3. *We say that two (n-dimensional) charts (U_1, φ_1), (U_2, φ_2) compatibly oriented provided that the determinant of the Jacobi matrix of associated transition function is positive on $\varphi_2(U_1 \cap U_2)$. A manifold M is orientable if there exists an atlas such that every two maps of the atlas are compatibly oriented.*

Next, we provide a definition of manifold with boundary.

Definition 4. *The half-space H^n is defined as $H^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_1 \leq 0\}$. The boundary of H^n is given by $\partial H^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_1 = 0\}$, and its topology is given by restriction of the standard topology on $\mathbb{R}^n$.*

A function f defined on a subset X of H^n is smooth if there exists an open neighborhood $X \subset U \subset \mathbb{R}^n$ and a smooth function $\tilde{f}$ defined on U such that $\tilde{f}|_X = f$. The mapping $f : X \to \mathbb{R}^n$ is smooth if all of its components are smooth functions. The derivatives of f at a point in H^n are defined as the derivatives of the extension $\tilde{f}$. We note that this extension is depends only on values of f on H^n, so the derivatives are independent of the choice of extension $\tilde{f}$.

For $U_1, U_2 \subset H^n$ open subsets, we define a diffeomorphism $\psi : U_1 \to U_2$ as an injective mapping of U_1 onto U_2 for which both ψ and ψ^{-1} are smooth. For a set M, an n-dimensional chart on M is a pair (U, φ), where U is a subset of M and $\varphi : U \to H^n$ is an injective mapping onto an open set in H^n. The transition function between two n-dimensional charts is defined in the same way as before. Two n-dimensional charts (U_1, φ_1), (U_2, φ_2) are compatible if $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are open in H^n and the transition function defined between them is a diffeomorphism.

The definition of atlas is again the same as above. The topology is defined with respect to H^n in the following way: a set $X \subset M$ is open if and only if for every chart (U, φ) from the atlas the set $\varphi(X \cap U)$ is an open subset of H^n.

A manifold of dimension n with boundary is now defined in the same way as above, replacing the definitions by generalized version allowing boundary. A point $m \in M$ is a boundary point if there exists a chart (U, φ) in the atlas such that $\varphi(m) \in \partial H^n$. We call the set of all boundary points of M as the boundary of M and denote it by ∂M.

We remark that the definiton of a boundary for manifold is independent of a chart.

Definition 5. *Let M, M' be two manifolds of dimension n and n' and let $f : M \to M'$ be a mapping. We will say that the mapping f is smooth if for every map (U, φ) from the atlas of M and for every map (U', φ') from the atlas of M' the mapping $\varphi' \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(U')) \to \mathbb{R}^{n'}$ is smooth. A function $f : M \to \mathbb{R}$ is smooth if f is a smooth mapping between manifolds M and $\mathbb{R}$. A mapping $\psi : M \to M'$ is a diffeomorphism if ψ is bijective and both ψ and ψ^{-1} are smooth. Two manifolds M, M' are diffeomorphic if there exists a diffeomorphism between M and M'.*

Definition 6. *A Lie group is a group G on which there is given structure of a manifold such that the operations of the group multiplication and the group inverse are smooth. where the multiplication is viewed as a smooth mapping from the product manifold of $G \times G$ to G.*

Definition 7. *Let M be a manifold of dimension n, $m \in M$ and $c : (-\epsilon, \epsilon) \to M$ a smooth mapping for some $\epsilon > 0$ satisfying $c(0) = m$. Let $\mathcal{C}^\infty(M)$ denote smooth functions on M. We say that a linear mapping $L : \mathcal{C}^\infty(M) \to \mathbb{R}$ is a tangent vector to the mapping c at the point m if $L(f) = (f \circ c)'(0)$ for every function $f \in \mathcal{C}^\infty(M)$. We define the tangent space $T_m M$ of M in m as the set of all tangent vectors to all mappings c satisfying the previous definition on M. The tangent bundle TM of M is defined as $\cup_{m \in M} T_m M$. The manifold M is equipped with an atlas consisting of charts $\varphi : U \to \mathbb{R}^n$. We define the atlas on TM as consisting of charts $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ defined as $\tilde{\varphi}(x, v^i \partial_i) = (\varphi(x), v^1, \ldots, v^n)$.*

Definition 8. *A vector field X on M is a smooth mapping $X : M \to TM$ such that $\pi \circ X = id$ on M, where π is the projection $\pi : TM \to M$ defined by $\pi(L) = m$ for $L \in T_m M$. We will denote the space of all smooth vector fields on M by $\mathfrak{X}(M)$.*

Definition 9. *Let M be a manifold and X, Y be two vector fields on M. Then their Lie bracket is a vector field, whose value on a smooth function $f \in \mathcal{C}^\infty(M)$ is defined as $[X, Y](f) = (X \circ Y)(f) - (Y \circ X)(f)$.*

Definition 10. *The cotangent space of M at the point m is defined as the dual $T_m^* M$ to the space $T_m M$. The cotangent bundle of M is $\cup_{m \in M} T_m^*$.*

Definition 11. *Let M, M' be manifolds of dimensions n and n', respectively, $m \in M$ and $f : M \to M'$ a mapping of manifolds, $L \in T_m M$. The push-forward (or differential) of f at L, $f_*(m) : T_m M \to T_{f(m)} M'$, is defined as $[f_*(m)(L)](g) = L(g \circ f)$, where $g \in \mathcal{C}^\infty(M')$. Let $\alpha \in T_{f(m)}^* M'$. The pull-back of f at α is defined as $f^*(m) : T_{f(m)}^* M' \to T_m^* M$, $[f^*(m)(\alpha)](L) = \alpha[f_*(m)(L)]$.*

Definition 12. *Let M be a manifold of dimension n. A smooth (k, l)-tensor field T on M is mapping that assigns to $m \in M$ a tensor $T(m) \in T_m M \otimes \cdots \otimes T_m M \otimes T_m^* M \otimes \cdots \otimes T_m^* M$. There are k factors of $T_m M$ and l factors of $T_m^* M$ such that for every chart (U, φ) from the atlas of M are all components smooth.*

Definition 13. *Let M be a manifold of dimension n. A $(0, k)$-tensor field T is a differential form of degree k if $T(m) \in \wedge^k(T_m^* M)$ for all $m \in M$, which means that for every permutation $\pi \in S_k$ the following holds:*

$$T(m)(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(k)}) = sign(\pi) T(m)(v_1, v_2, \ldots, v_k), \tag{1.1}$$

where $v_1, v_2, \ldots, v_k \in T_m M$. We denote the space of differential forms of degree k on M by $\mathcal{E}^k(M)$, and the space of all differential forms on M by $\mathcal{E}(M) = \bigoplus_{k=0}^n \mathcal{E}^k(M)$. We will say shortly differential k-form for a differential form of degree k.

Definition 14. *Let α be a differential k-form and β be a differential l-form. The exterior product $\alpha \wedge \beta$ is defined as*

$$[\alpha \wedge \beta](v_1, v_2, \ldots, v_{k+l}) = \\ \frac{1}{k!l!} \sum_{\pi \in S_{k+l}} sign(\pi) \alpha(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(k)}) \beta(v_{\pi(k+1)}, v_{\pi(k+2)}, \ldots, v_{\pi(k+l)}). \tag{1.2}$$

Definition 15. *The exterior derivative d on M is defined as the unique linear differential operator $d : \mathcal{E}^k(M) \to \mathcal{E}^{k+1}(M)$ for $k = 0, 1, \ldots, n$, satisfying*
(1) For $f \in \mathcal{E}^0(M)$, the exterior derivative df is defined as the differential of f,
(2) $d^2 = 0$,
(3) For $\omega \in \mathcal{E}^k(M)$ and $\rho \in \mathcal{E}^l(M)$, the exterior derivative of the exterior product is given by the graded Leibniz rule

$$d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^k \omega \wedge (d\rho). \tag{1.3}$$

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Definition 16. *Let $k \geq 0$ and $X, X_1, X_2, \ldots, X_{k-1} \in \mathfrak{X}(M)$. Then we define a contraction by vector field X for $\omega \in \mathcal{E}^k(M)$ as a linear mapping $\iota_X : \mathcal{E}^k(M) \to \mathcal{E}^{k-1}(M)$ by $(\iota_X \omega)(X_1, X_2, \ldots, X_{k-1}) = \omega(X, X_1, X_2, \ldots, X_{k-1})$. The Lie derivative $\mathcal{L}_X$ is the operator $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$.*

Definition 17. *Let M, N be smooth manifolds and $f : M \to N$ be a smooth mapping of manifolds. We say that f is a submersion at $m \in M$ if $f_*(p) : T_m M \to T_{f(m)} N$ is surjective. We say that f is a submersion if it is a submersion at each $m \in M$.*

Definition 18. *Let E, N be smooth manifolds and let $\pi : E \to N$ be a smooth surjective mapping of manifolds. We say that a pair (E, π) is a fibered manifold over N if π is surjective and it is a submersion. We define a morphism of fibered manifolds (E, π), (E', π') as a pair (f, f_0), where $f : E \to E'$, $f_0 : N \to N'$ are smooth mappings of manifolds and for every point $y \in E$ holds*

$$\pi' \circ f(y) = f_0 \circ \pi(y). \tag{1.4}$$

A morphism of fibered manifolds (f, f_0) is an isomorphism if there exists another morphism of fibered manifolds (g, g_0) from (E', π) to (E, π) such that $f \circ g = id$ and $g \circ f = id$ inducing $f_0 \circ g_0 = id$ and $g_0 \circ f_0 = id$.

We remark that a product of manifolds is an example of a fibered manifold.

Definition 19. *A structure (E, N, π, F) is a fibered bundle provided E and N are smooth manifolds, $\pi : E \to N$ is a smooth mapping of manifolds and E is a fibered manifold over N equipped with a local trivialization, by which we mean that for every point $x \in N$ there exists a neighborhood $x \in U \subset N$ and an isomorphism of fibered manifolds $\psi : \pi^{-1}(U) \to U \times F$ such that for every point $y \in \pi^{-1}(U)$ holds*

$$\pi(y) = \pi_1 \circ \psi(y), \tag{1.5}$$

where $\pi_1 : U \times F \to U$ denotes projection on the first factor.

Definition 20. *Let (E, N, π, F) be a fibered bundle. A section of the bundle on an open set $U \subset N$ is defined as a smooth mapping $X : U \to \pi^{-1}(U)$ such that $\pi \circ X = id$. We denote the space of all sections of a bundle on an open set $U \subset N$ as $\Gamma(U)$.*

Definition 21. *Let (E, N, π, F) be a fibered bundle and $e_1, \ldots, e_n \in \Gamma(E)$. A coordinate frame $\{e_1, \ldots, e_n\}$ is defined as a basis for the space $\Gamma(E)$.*

Definition 22. *A structure (E, N, π, F) is a principal fibered bundle with structure group G provided (E, N, π, F) is a fibered bundle equipped with a Lie group G together with smooth right action of G on E, satisfying that the action is free, transitive and preserves fibers, which means that if $x \in P_y$ then $xg \in P_y$ for some $y \in N$ and $\forall g \in G$.*

Definition 22 implies that F is diffeomorphic to G and N is diffeomorphic to E/G.

Definition 23. *A structure (E, N, π, F) is a fibered vector bundle with fiber F provided (E, N, π, F) is a fibered bundle, F is a finite-dimensional real vector space satisfying the following conditions: $\forall x \in N$ E_x has the structure of a finite-dimensional vector space and there exists an atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ on E such that φ_α is a linear isomorphism in the second variable.*

Definition 23 implies that $E_x \simeq F$ for each $x \in N$. For simplicity of notation we will refer to fibered principal bundles and fibered vector bundles as principal bundles and vector bundles.

Definition 24. *We define the rank of a vector bundle (E, N, π, F) as the dimension of F. Vector bundles of rank 1 are called line bundles.*

Definition 25. *Let (E, N, π, F) be a fibered bundle, M a manifold and $f : M \to N$ a smooth mapping between manifolds. We then define the pullback bundle (E', M, π', F) by $E' = \{(m, h) | f(m) = \pi(h), m \in M, h \in E\} \subset M \times E$.*

Definition 26. *Let (E, N, π, F) be a principal bundle with structure group G and V be a representation of G equipped with left action of G. The associated vector bundle (E', N, π', V) is defined as follows: we define right action of G on $E \times V$ by $(h, v)g = (hg, g^{-1} \cdot v)$ for $h \in E$, $v \in V$ and $g \in G$, then we define $E' = (E \times V)/G = \{(hg, g^{-1} \cdot v) | e \in E, v \in V, g \in G\}$.*

Definition 27. *Let M be a manifold of dimension n and let g be a symmetric non-degenerate $(0, 2)$-tensor field on M of signature (p, q), where $p + q = n$. We say that g is a pseudo-Riemannian metric on M and (M, g) is a pseudo-Riemannian manifold. A Lorentz metric g on M is a pseudo-Riemannian metric on M of signature $(p, 1)$. A Riemannian metric g is defined as a pseudo-Riemannian metric on M satisfying the additional requirement that it is positive definite and a Riemannian manifold is then defined as a pair (M, g) consisting of a manifold M and a Riemannian metric g.*

In the following we state several definitions and theorems for Riemannian manifolds, noting that these hold in the case when the manifold M is pseudo-Riemannian.

Definition 28. *An affine connection ∇ on M is a bilinear mapping $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which assigns to two vector fields X, Y a vector field denoted by $\nabla_X Y$ satisfying*
(1) $\nabla_{fX} Y = f \nabla_X Y$,
(2) $\nabla_X fY = (df)(X)Y + f \nabla_X Y$,
for all $f \in \mathcal{C}^\infty(M)$.

Definition 29. *Let (M, g) be a Riemannian manifold equipped with a connection ∇, let α be a differential 1-form on M and X, Y vector fields on M. We define covariant differentiation as*

$$[\nabla_X \alpha](Y) = X(\alpha(Y)) - \alpha[\nabla_X Y]. \tag{1.6}$$

Definition 30. *Let (M, g) be a Riemannian manifold. The Levi-Civita connection ∇ is the unique affine connection on M satisfying:*
(1) $\nabla g = 0$,
(2) ∇ is torsion free, which means that $\nabla_X Y - \nabla_Y X = [X, Y]$ for all X, Y vector fields on M.

Definition 31. *Let (M, g) be a Riemannian manifold and ∇ its Levi-Civita connection. Let X, Y, Z, W be vector fields on M. Then we define the Riemannian curvature tensor as*

$$R(X,Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W, \tag{1.7}$$

the curvature function

$$R_m(X,Y,Z,W) = g(R(X,Y)W, Z), \tag{1.8}$$

the Ricci curvature

$$Ric(X,Y) = tr\ R_m(X,-,Y,-) \tag{1.9}$$

and the scalar curvature by

$$R = \ tr\ Ric(-,-). \tag{1.10}$$

When we talk about the Ricci curvature with respect to different metrics g_α, we denote particular Ricci curvature with respect to the metric g as $Ric(g)$. We shall also write $R_{ijkl} = R_m(E_i, E_j, E_k, E_l)$, where $E_i = \frac{\partial}{\partial x_i}$. The same notation will be used for the Ricci curvature $Ric_{ij} = Ric(E_i, E_j)$.

Definition 32. *A Riemannian manifold (M, g) is Einstein with Einstein constant J if*

$$Ric = Jg, \tag{1.11}$$

where Ric is the Ricci curvature of (M, g). We say that (M, g) is Ricci-flat if it is Einstein with Einstein constant $J = 0$.

Definition 33. *The Christoffel symbols on a Riemannian manifold (M, g) of dimension n (also called the Levi-Civita connection coefficients) are smooth functions Γ^k_{ij} such that*

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k, \quad 1 \leq i, j, k \leq n, \tag{1.12}$$

where $E_i = \frac{\partial}{\partial x_i}$ are coordinate vector fields.

The Christoffel symbols for the Levi-Civita connection may be calculated as

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{mi}}{\partial x_j} + \frac{\partial g_{mj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m}\right), \tag{1.13}$$

where g^{km} denotes the inverse of the metric g_{km}, and the Riemannian curvature tensor of the Levi-Civita connection may be calculated as

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x_j \partial x_k} + \frac{\partial^2 g_{jk}}{\partial x_i \partial x_l} - \frac{\partial^2 g_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 g_{jl}}{\partial x_i \partial x_k}\right) + g_{mp}(\Gamma^m_{il}\Gamma^p_{jk} - \Gamma^m_{ik}\Gamma^p_{jl}). \tag{1.14}$$

In the case when the Riemannian manifold is also a Lie group and the vector fields E_i form a basis of the Lie algebra of the Lie group, the Christoffel symbols may be calculated as

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \big(g([E_i, E_j], E_m) - g([E_i, E_m], E_j) - g([E_j, E_m], E_i)\big). \tag{1.15}$$

Definition 34. *The divergence operator for* $T \in \Gamma(Sym^2T^*M)$ *is defined as* $\delta(T) = -tr_{12}\nabla T$*, where* tr_{12} *means to take the trace in the first and the second entry of* ∇T*. The gravitation tensor is defined as* $G(T) = T - \frac{1}{2}(trT)g$*, and its divergence is given by* $\delta G(T) = \delta T + \frac{1}{2}d(trT)$*.*

Theorem 1. *Let (M,g) be a Riemannian manifold equipped with a family of metrics* $g(t)$ *parametrized by* $t \in \mathbb{R}_+$*, where* $g(t)|_{t=0} = g$ *and let* $T \in \Gamma(Sym^2T^*M)$ *independent of* t *and* $X \in \mathfrak{X}(M)$*. We set* $h = \frac{\partial g(t)}{\partial t}$*. Then the following holds:*

$$(\frac{\partial}{\partial t}\delta G(h))X = -T((\delta G(h)^{\#}, X) + \langle h, \nabla T(-,-,X) - \frac{1}{2}\nabla_X T\rangle. \tag{1.16}$$

We refer to [RF] for the proof of the theorem.

Definition 35. *For* $X, Y \in \mathfrak{X}(M)$ *and* ω *a differential* 1*-form, the musical isomorphisms* $\flat, \sharp$ *between* TM *and* T^*M *are defined as*

$$X^{\flat}(Y) = g(X, Y), \tag{1.17}$$

and $\omega^{\sharp}$ *is defined as the vector field such that*

$$\omega(Y) = g(\omega^{\sharp}, Y). \tag{1.18}$$

Definition 36. *Let* (M, g) *be a Riemannian manifold of dimension* n*. A geodesic on* M *is defined as a smooth curve* $c : [a, b] \to M$*, in local coordinates* $(x_1, x_2, \ldots, x_n)$*, satisfying the differential equation*

$$\frac{d^2c^k}{dt^2} + \Gamma_{ij}^k \frac{dc^i}{dt}\frac{dc^j}{dt} = 0, \quad 1 \le i, j, k \le n. \tag{1.19}$$

Definition 37. *Let* (M, g) *be an oriented Riemannian manifold of dimension* n*. The volume form* ω *is defined in local coordinates* $(x_1, x_2, \ldots, x_n)$ *on* $U \subset M$ *as* $\omega = \sqrt{|\det g|}dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$*, where* $dx^1, dx^2, \ldots, dx^n$ *are differential forms of degree* 1 *that form a basis for* T^*M *restricted to* U*. The Riemannian volume of* M *is defined as*

$$V = \int_M \omega. \tag{1.20}$$

Definition 38. *Let* (M, g) *be a Riemannian manifold,* X, Y *vector fields on* M *and* h *a tensor field of type* $(0, 2)$*. The Lichnerowicz Laplacian is defined as*

$$\begin{aligned}[\Delta_L h](X, Y) = (\Delta h)(X, Y) - h(X, Ric(Y)) - h(Y, Ric(X)) \\ +2tr\, h(R(X, -)Y, -).\end{aligned} \tag{1.21}$$

Definition 39. *Let* (M, g) *be a Riemannian manifold of dimension* n*. The tensor* P_{ij} *is defined as*

$$P_{ij} = \frac{1}{n-2}(Ric_{ij} - \frac{R}{2(n-1)}g_{ij}), \tag{1.22}$$

where R *denotes the scalar curvature of* (M, g) *and* Ric_{ij} *denotes components of the Ricci curvature of* (M, g)*.*

The Weyl tensor is defined as

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(Ric_{ik}g_{jl} - Ric_{il}g_{jk} + Ric_{jl}g_{ik} - Ric_{jk}g_{il}) + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{1.23}$$

The Cotton tensor is defined as

$$C_{ijk} = P_{ij,k} - P_{ik,j}, \tag{1.24}$$

where by notation $P_{ij,k}$ we mean $\frac{\partial}{\partial x_k}P_{ij}$.

The Bach tensor is defined as

$$B_{ij} = C_{ijk,}{}^{k} - P^{kl}W_{kijl}, \tag{1.25}$$

where by notation $C_{ijk,}{}^{k}$ we mean $g^{km}C_{ijk,m}$.

Definition 40. *Let (M, g) be a Riemannian manifold and let X, Y be vector fields on M. Let ∇ denote the Levi-Civita connection on M. We define the second covariant derivative as*

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}. \tag{1.26}$$

We define the connection Laplacian Δ as taking trace in the first and second entries of the second covariant derivative.

Definition 41. *A quasilinear differential operator is a differential operator that is linear in the highest order derivative.*

2. Ambient and Poincaré-Einstein metric

In this chapter, we give a brief introduction to the construction of the ambient and the Poincaré-Einstein metrics. Later on, the Poincaré-Einstein metric will be used to formulate the Ricci flow problem. Here we follow the exposition [AM], chapters $1-4$.

We shall start with a conformal Riemannian manifold of dimension n, and construct certain $(n+2)$-dimensional pseudo-Riemannian manifold called the ambient space of M. This space is equipped with a Lorentz metric called ambient metric $\tilde{g}$, and it allows to construct conformal invariants associated with the conformal manifold. The ambient metric is homogeneous with respect to a family of dilations on the $(n+2)$-dimensional space. The quotient by the action of dilations yields a $(n+1)$-dimensional metric called Poincaré-Einstein metric g_+.

This construction is motivated by the construction of the ambient space for the flat model of conformal geometry on S^n. We set

$$Q(x) = -dx_0^2 + \sum_{k=1}^{n+1} dx_k^2, \tag{2.1}$$

the standard quadratic form of Lorentz signature on $\mathbb{R}^{n+1,1}$ for $x_0 > 0$, and

$$\mathcal{N} = \{x \in \mathbb{R}^{n+1,1} | x \neq 0, Q(x) = 0\}, \tag{2.2}$$

the null cone of Q. We may identify S^n with the space of lines in $\mathcal{N}$, given by the projectivization $\pi : \mathcal{N} \to S^n$. The ambient metric $\tilde{g}$ for S^n is given by restricting Q to a neighborhood of $\mathcal{N} \subset \mathbb{R}^{n+1,1}$. In particular, $\tilde{g}$ annihilates the radial vector field

$$X = \sum_{k=0}^{n+1} x_k \partial_k \in T_x\mathcal{N} \tag{2.3}$$

and $\tilde{g}$ is non-degenerate on $T_x\mathcal{N}/\langle X\rangle \simeq T_{\pi(x)}S^n$. Various choices of x on a line in $\mathcal{N}$ result in conformally equivalent inner products on $T_{\pi(x)}S^n$. The Lorentz group $O(n+1,1,\mathbb{R})$ acts linearly on $\mathbb{R}^{n+1,1}$ by isometries of $\tilde{g}$, preserving the subspace $\mathcal{N}$ and realizes the group of conformal isometries of S^n. Restricting the quadratic form Q to a hyperboloid $\mathcal{H}$ rather than to the null cone $\mathcal{N}$,

$$\mathcal{H} = \{x \in \mathbb{R}^{n+1,1} | Q(x) = -1\}, \tag{2.4}$$

results in the Poincaré-Einstein metric g_+ for S^n. The Poincaré-Einstein metric is the hyperbolic metric of constant sectional curvature -1. If we identify the one sheet hyperboloid $\mathcal{H}$ with the unit ball in $\mathbb{R}^{n+1}$, we may write the Poincaré-Einstein metric as

$$g_+ = 4(1-|x|^2)^{-2} \sum_{k=1}^{n+1} dx_k^2. \tag{2.5}$$

In the induced action, the Lorentz group $O(n+1,1,\mathbb{R})$ acts linearly by isometries of g_+, preserving $\mathcal{H}$, hereby realizing the isometry group of the hyperbolic space.

In this section let M denote a Riemannian manifold of dimension $n \geq 2$ equipped with a conformal class of metrics $[g]$ represented by a metric g. The conformal class $[g]$ is given by all metrics $\hat{g}$ satisfying $\hat{g} = e^f g$ for some smooth function $f \in \mathcal{C}^\infty(M)$.

Definition 42. *We define the metric bundle* $(\mathcal{G}, M, \pi, \mathbb{R}_+)$ *as the space of all pairs* (h, x)*, where* $x \in M$ *and* h *is a bilinear symmetric form on* $T_x(M)$*, that satisfies the homogenity condition* $h = s^2 g_x$ *for some* $s > 0$*, where* g_x *is the bilinear symmetric form induced on* T_xM *by the metric* g*. Dilations* $\delta_s : \mathcal{G} \to \mathcal{G}$ *of the metric bundle are defined as* $\delta_s(h, x) = (s^2 h, x)$*. We note that the space* $\mathcal{G}$ *together with the projection* π *and dilations* δ_s *is a* $\mathbb{R}_+$*-bundle. We define the vector field* $T = \frac{d}{ds}|_{s=1}\delta_s$*, which is the infinitesimal generator of the dilations* δ_s*. The data given by dilations on the metric bundle are equivalent to giving a symmetric 2-tensor* g_0 *defined by* $g_0(X, Y) = h(\pi_* X, \pi_* Y)$*, where* $X, Y \in T_{(h,x)}\mathcal{G}$ *and* $\pi_* : T\mathcal{G} \to TM$ *is the differential of* π*. The tensor* g_0 *satisfies a homogenity condition with respect to the dilations, particularly* $\delta_s^* g_0 = s^2 g_0$*, where* δ_s^* *is the pull-back of* δ_s*.*

The metric bundle $\mathcal{G}$ is independent of the choice of representative of the conformal class $[g]$. By fixing a representative g, we may identify $(t, x) \in \mathbb{R}_+ \times M$ with $(t^2 g_x, x) \in \mathcal{G}$. We obtain a trivialization of the bundle $\mathcal{G}$ associated with the representative g. In this identification, the dilations are given by $\delta_s(t, x) = (st, x)$, the radial vector field is given by $X = t\partial_t$ and the homogeneous 2-tensor is given by $g_0 = t^2 \pi^* g$. The metric g is a smooth section of the bundle $\mathcal{G}$. The image of this section is the submanifold at $t = 1$. Picking a representative g determines also a horizontal subspace $\mathcal{H}_z \subset T_z\mathcal{G}$ for every $z \in \mathcal{G}$, which satisfies $\mathcal{H}_z = \ker(dt)_z$.

We may identify $\mathcal{G} \times \mathbb{R}$ with $\mathbb{R}_+ \times M \times \mathbb{R}$ by taking another representative $\hat{g} \in [g]$ in the following way: we again identify $(\hat{t}, x) \in \mathbb{R}_+ \times M$ with $(\hat{t}^2 \hat{g}_x, x) \in \mathcal{G}$. We obtain another trivialization of the bundle $\mathcal{G}$. These two trivializations satisfy the relation $\hat{t} = e^{-f(x)} t$, where $(t, x) \in \mathbb{R}_+ \times M$ and $f \in \mathcal{C}^\infty(M)$. Let $(x_1, x_2, \ldots, x_n)$ be local coordinates on an open set $x \in U \subset M$. Then the metric g is given as $g = g_{ij} dx_i dx_j$ and $(t, x_1, x_2, \ldots, x_n)$ are local coordinates on $\pi^{-1}(U)$. The tensor g_0 is given by $g_0 = t^2 g_{ij}(x) dx_i dx_j$ and the horizontal subspace $\mathcal{H}_z$ is exactly the span of $\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$. We may extend the dilations δ_s to $\delta_s : \mathcal{G} \times \mathbb{R} \to \mathcal{G} \times \mathbb{R}$ by only acting on the first factor. This also extends the vector field T to $\mathcal{G} \times \mathbb{R}$. We can define an embedding of $\mathcal{G}$ into $\mathcal{G} \times \mathbb{R}$ by $i(z) = (z, 0)$ for $z \in \mathcal{G}$. Then $(t, x_1, x_2, \ldots, x_n, r)$ are local coordinates on $\mathcal{G} \times \mathbb{R}$.

We follow [AM] in using the subscript 0 to denote the t-component, the subscript ∞ to denote the r-component and the subscript lower case letters to denote $x_1, x_2, \ldots, x_n$-components.

In general, the ambient metric $\tilde{g}$ of a conformal Riemannian manifold M is defined as a solution to a system of partial differential equations with a boundary condition and satisfying a homogenity property. The ambient metric is a metric defined in a neighborhood of $\mathcal{G}$ in $\mathcal{G} \times \mathbb{R}$ and homogeneous with respect to dilations on $\mathcal{G} \times \mathbb{R}$. The system of partial differential equations asserts that the ambient metric is Ricci-flat and its boundary value is given by the initial conformal manifold. This boundary value problem is singular because the pullback of the ambient metric to the initial manifold is degenerate. For this reason we consider only formal power series solutions to the system. The existence of solutions of the system depends on the parity of the dimension n of the initial conformal metric M. In the case when the dimension n is odd, there exists a formal power series solution of the system to any order, unique up to an isomorphism. On the other hand, for $n \geq 4$ even we get a solution up to the order $n/2-1$, again unique up to an isomorphism, and the existence of a solution beyond this order is obstructed by the so called obstruction tensor.This is a conformally invariant natural trace-free symmetric 2-tensor. In the case of a non-trivial obstruction a formal power series soluton may be still constructed by introducing log terms into the series. For $n = 4$, the obstruction tensor is the Bach tensor.

The content of this section including proofs can be found in [AM].

Definition 43. *A pre-ambient space for $(M, [g])$, where $[g]$ is a conformal class of signature (p, q) on M, is a pair $(\tilde{\mathcal{G}}, \tilde{g})$, where:*
(1) $\tilde{\mathcal{G}}$ is a dilation-invariant open neighborhood of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$;
(2) $\tilde{g}$ is a smooth metric of signature $(p+1, q+1)$ on $\tilde{\mathcal{G}}$, where the signature of the initial metric g is (p, q);
(3) $\tilde{g}$ is homogeneous of degree 2 on $\tilde{\mathcal{G}}$ $(\delta_s^* \tilde{g} = s^2 g)$;
(4) The pullback $i^*\tilde{g}$ is the tautological tensor g_0 on $\mathcal{G}$.
If $(\tilde{\mathcal{G}}, \tilde{g})$ is a pre-ambient space, the metric $\tilde{g}$ is called the ambient metric. If the dimension n of M is odd or $n = 2$, then a pre-ambient space $(\tilde{\mathcal{G}}, \tilde{g})$ is called an ambient space for $(M, [g])$, provided we have
(5) Ric$(\tilde{g})$ vanishes to infinite order at every point of $\mathcal{G} \times \{0\}$.
In order to define ambient space for even dimensions, we first define a symmetric 2-tensor field S_{IJ} on an open neighborhood of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$. We write $S_{IJ} = \mathrm{O}^+_{IJ}(r^m)$ if: (i) $S_{IJ} = \mathrm{O}(r^m)$
(ii) For each point $z \in \mathcal{G}$, the symmetric 2-tensor $i^*(r^{-m}S)(z)$ is of the form $\pi^* s$ for some symmetric 2-tensor s at $x = \pi(z) \in M$ satisfying tr $_{g_x} s = 0$. The symmetric 2-tensor S_{IJ} is allowed to depend on z, not just on x.

In terms of components relative to a choice of representative metric $g \in [g]$, $S_{IJ} = \mathrm{O}^+_{IJ}(r^m)$ if and only if all components satisfy $S_{IJ} = \mathrm{O}_{IJ}(r^m)$ and if in addition one has that S_{00}, S_{0i} and $g^{ij}S_{ij}$ are $\mathrm{O}(r^{m+1})$. The condition $S_{IJ} = \mathrm{O}^+_{IJ}(r^m)$ is easily seen to be preserved by diffeomorphisms ϕ defined on a neighborhood of $\mathcal{G} \times \{0\}$ in $\mathcal{G} \times \mathbb{R}$ satisfying $\phi|_{\mathcal{G}\times\{0\}} = id$.
Now suppose $(\tilde{\mathcal{G}}, \tilde{g})$ is a pre-ambient space for $(M, [g])$, with $n = dim\ M$ even and $n \geq 4$. We say that $(\tilde{\mathcal{G}}, \tilde{g})$ is an ambient space for $(M, [g])$, provided we have
(5') Ric$(\tilde{g}) = \mathrm{O}^+_{IJ}(r^{n/2-1})$.
If $(\tilde{\mathcal{G}}, \tilde{g})$ is an ambient space, the metric $\tilde{g}$ is called an ambient metric.

We continue by formulating the notion of equivalence for pairs of pre-ambient spaces.

Definition 44. *Let $(\tilde{\mathcal{G}}_1, \tilde{g}_1)$ and $(\tilde{\mathcal{G}}_2, \tilde{g}_2)$ be two pre-ambient spaces associated to $(M, [g])$. We say that $(\tilde{\mathcal{G}}_1, \tilde{g}_1)$ and $(\tilde{\mathcal{G}}_2, \tilde{g}_2)$ are ambient-equivalent if there exist open sets $\mathcal{U}_1 \subset \tilde{\mathcal{G}}_1, \mathcal{U}_2 \subset \tilde{\mathcal{G}}_2$ and a diffeomorphism $\Phi : \mathcal{U}_1 \to \mathcal{U}_2$, with the following properties:*
(1) $\mathcal{U}_1$ and $\mathcal{U}_2$ both contain $\mathcal{G} \times \{0\}$;
(2) $\mathcal{U}_2$ and $\mathcal{U}_2$ are dilation-invariant and Φ commutes with dilations;
(3) The restriction of Φ to $\mathcal{G} \times \{0\}$ is the identity map;

(4) If $n = dim\ M$ is odd, then $\tilde{g}_1 - \Phi^*\tilde{g}_2$ vanishes to infinite order at every point of $\mathcal{G} \times \{0\}$.
(4') If $n = dim\ M$ is even, then $\tilde{g}_1 - \Phi^*\tilde{g}_2 = \mathrm{O}^+_{IJ}(r^{n/2})$.

The relation of ambient-equivalence is an equivalence. Next we formulate the existence theorem for ambient spaces.

Theorem 2. *Let $(M, [g])$ be a smooth conformal manifold of dimension $n \geq 2$. Then there exists an ambient space for $(M, [g])$ and any two ambient spaces for $(M, [g])$ are ambient-equivalent.*

Now, we will establish an additional characterization for ambient spaces.

Theorem 3. *Let $(\tilde{\mathcal{G}}, \tilde{g})$ be a pre-ambient space for $(M, [g])$. There is a dilation-invariant open subset $\mathcal{U} \subset \tilde{\mathcal{G}}$ containing $\mathcal{G} \times \{0\}$ such that the following three conditions are equivalent.*
(1) $\tilde{\nabla} T = id$ on $\mathcal{U}$;
(2) $\iota_{2T}\tilde{g} = d(||T||^2)$ on $\mathcal{U}$;
(3) For each $p \in \mathcal{U}$, the parametrized dilation orbit $s \to \delta_s p$ is a geodesic for $\tilde{g}$.
In (1), $\tilde{\nabla}$ denotes the covariant derivative with respect to the Levi-Civita connection of $\tilde{g}$. So $\tilde{\nabla} T$ is a (1,1)-tensor on $\mathcal{U}$, and the requirement is that it be the identity endomorphism of TM at each point. In (2), $||T||^2 = \tilde{g}(T, T)$. In (3), ι_{2T} denotes contraction by the vector field $2T$.

Definition 45. *A pre-ambient space $(\tilde{\mathcal{G}}, \tilde{g})$ for $(M, [g])$ will be said to be straight if the equivalent properties of theorem 3 hold with $\mathcal{U} = \mathcal{G}$. In this case, the pre-ambient metric $\tilde{g}$ is also said to be straight.*

The following theorem asserts that an ambient space satisfying the straightness condition always exists.

Theorem 4. *Let $(M, [g])$ be a smooth manifold of dimension $n > 2$ equipped with a conformal class. Then there exists a straight ambient space for $(M, [g])$. Moreover, if $\tilde{g}$ is any ambient metric for $(M, [g])$, there is a straight ambient metric $\tilde{g}'$ such that if*
(1) n is odd, then $\tilde{g} - \tilde{g}'$ vanishes to infinite order at $\mathcal{G} \times \{0\}$;
(2) if n is even, then $\tilde{g} - \tilde{g}' = \mathrm{O}^+_{IJ}(r^{n/2})$.

Next, we formulate a condition to choose a representative ambient space in a special form and two theorems that guarantee the existence of an ambient space.

Definition 46. *A pre-ambient space $(\tilde{\mathcal{G}}, \tilde{g})$ for $(M, [g])$ is said to be in normal form relative to g if the following three conditions hold:*
(1) For each fixed $z \in \mathcal{G}$, the set of all $r \in \mathbb{R}$ such that $(z, r) \in \tilde{\mathcal{G}}$ is an open interval I_z containing 0.
(2) For each $z \in \mathcal{G}$, the parametrized curve $I_z : r \to (z, r)$ is a geodesic for the metric $\tilde{g}$.
(3) Let us write (t, x, r) for a point of $\mathbb{R}_+ \times M \times \mathbb{R} \simeq \mathcal{G} \times \mathbb{R}$ under the identification induced by g, as discussed in the beginning of this chapter.
Then, at each point $(t, x, 0) \in \mathcal{G} \times \{0\}$, the metric tensor $\tilde{g}$ takes the form

$$\tilde{g} = g_0 + 2tdtdr. \tag{2.6}$$

Theorem 5. *Let $(M,[g])$ be a smooth manifold equipped with a conformal class, let g be a representative of the conformal class, and let $(\tilde{\mathcal{G}},\tilde{g})$ be a pre-ambient space for $(M,[g])$. Then there exists a dilation-invariant open set $\mathcal{U}\subset\mathcal{G}\times\mathbb{R}$ containing $\mathcal{G}\times\{0\}$ for which there is a unique diffeomorphism Φ from $\mathcal{U}$ into $\tilde{\mathcal{G}}$, such that Φ commutes with dilations, $\Phi|_{\mathcal{G}\times\{0\}}$ is the identity map, and such that the pre-ambient space $(\mathcal{U},\Phi^*\tilde{g})$ is in the normal form relative to g.*

Theorem 6. *Let M be a smooth manifold of dimension $n\geq 2$ and g a smooth metric on M.*
(A) There exists an ambient space $(\tilde{\mathcal{G}},\tilde{g})$ for $(M,[g])$, which is in the normal form relative to g.
(B) Suppose that $(\tilde{\mathcal{G}}_1,\tilde{g}_1)$ and $(\tilde{\mathcal{G}}_2,\tilde{g}_2)$ are two ambient spaces for $(M,[g])$, both of which are in the normal form relative to g. If n is odd, then $\tilde{g}_1-\tilde{g}_2$ vanishes to infinite order at every point of $\mathcal{G}\times\{0\}$. If n is even, then $\tilde{g}_1-\tilde{g}_2=\mathrm{O}^+_{IJ}(r^{n/2})$.

We formulate two more theorems, that make the form of the ambient metric more specific.

Theorem 7. *Let $(\tilde{\mathcal{G}},\tilde{g})$ be a pre-ambient space for $(M,[g])$, where $\tilde{\mathcal{G}}$ has the property that for each $z\in\mathcal{G}$, the set of all $r\in\mathbb{R}$ such that $(z,r)\in\tilde{\mathcal{G}}$ is an open interval I_z containing 0. Let g be a metric in the conformal class, with associated identification $\mathbb{R}_+\times M\times\mathbb{R}\simeq\mathcal{G}\times\mathbb{R}$. Then $(\tilde{\mathcal{G}},\tilde{g})$ is in normal form relative to g if and only if one has on $\tilde{\mathcal{G}}$:*

$$\tilde{g}_{0\infty}=t,\quad \tilde{g}_{i\infty}=0,\quad \tilde{g}_{\infty\infty}=0. \tag{2.7}$$

Theorem 8. *Suppose $n\geq 2$. Let the pre-ambient space $(\tilde{\mathcal{G}},\tilde{g})$ be in normal form relative to a representative metric g. The following conditions are equivalent.*
(1) $\tilde{g}_{00}=2r$ and $\tilde{g}_{0i}=0$.
(2) For each $p\in\tilde{\mathcal{G}}$, the dilation orbit $s\to\delta_s p$ is a geodesic for $\tilde{g}$.
(3) $\iota_{2T}\tilde{g}=d(||T||^2)$
(4) The infinitesimal dilation field T satisfies $\tilde{\nabla}T=id$.

We remark that by 43, condition 4 the coefficients $\tilde{g}_{ij}$ are given by $\tilde{g}_{ij}=t^2\hat{g}_{ij}$, where $\hat{g}_{ij}$ is a formal power series in r, for which the coefficients are determined by vanishing of the Ricci curvature. The process of calculating these coefficients is recursive, where the separate steps solve the system of partial differential equations that assert the the Ricci curvature vanishes as $\mathrm{O}(r^k)$, $k=1,2,\dots$. We conclude that by Definitions 7 and 8, under suitable conditions (straightness and normal form) the ambient metric is given by a power series, where the coefficients are determined by vanishing of the Ricci curvature. Particularly, the ambient metric is given by

$$\tilde{g}=t^2\hat{g}_{ij}(x,r)+2tdtdr+2rdt^2, \tag{2.8}$$

where $\hat{g}_{ij}(x,0)=g_{ij}(x)$.

Definition 47. *Let $(M,[g])$ be a conformal Riemannian manifold. We may view the metric bundle as a principal bundle with structure group $\mathbb{R}_+$. Irreducible representations of $\mathbb{R}_+$ induce associated line bundles on M. We denote $\mathcal{E}[w]$ the line bundle induced by representation of weight $-w/2$. A tensor density of weight w is then defined as a section of the associated line bundle $\mathcal{E}[w]$.*

As noted earlier, for $n \geq 4$ even the existence of solutions for $\hat{g}_{ij}$ beyond order $n/2-1$ is generally obstructed by the so-called obstruction tensor $\mathcal{O}$, which we will now define. Let $Q = \tilde{g}(T,T)$, where T is the infinitesimal dilation vector field. We restrict Q to $\mathcal{G} \times \{0\}$. Particularly,

$$Q = 2rt^2 \mod \mathrm{O}(r^{n/2-1}). \tag{2.9}$$

Now since $\mathrm{Ric}(\tilde{g})_{IJ} = \mathrm{O}^+_{IJ}(r^{n/2-1})$, $Q^{1-n/2}\mathrm{Ric}(\tilde{g})_{IJ}$ is a tensor field on $\mathcal{G}$, it defines a symmetric (0,2)-tensor density on M of weight $2-n$. Taking a metric g in the conformal class $[g]$, we may view it as a section of the bundle $\mathcal{G}$. Evaluating the tensor field $Q^{1-n/2}\mathrm{Ric}(\tilde{g})_{IJ}$ at g, we obtain a (0,2)-tensor on M, which will be denoted as $Q^{1-n/2}\mathrm{Ric}(\tilde{g})_{IJ}|_g$. Now we define the obstruction tensor of g as

$$\mathcal{O}_{IJ} = c_n Q^{1-n/2}\mathrm{Ric}(\tilde{g}_{IJ})|_g, \tag{2.10}$$

where

$$c_n = (-1)^{n/2-1}\frac{2^{n-2}(n/2-1)!^2}{n-2}. \tag{2.11}$$

When $\tilde{g}$ is in normal form relative to g, the above formula reduces to

$$\mathcal{O}_{IJ} = 2^{1-n/2}c_n r^{1-n/2}\mathrm{Ric}(\tilde{g})_{IJ}|_{r=0}. \tag{2.12}$$

In this case when g is conformally equivalent to an Einstein metric, the obstruction tensor $\mathcal{O}$ is zero. This follows easily from the last identity for $\mathcal{O}_{IJ}$ in normal form relative to g. When $n=4$ the obstruction tensor is the Bach tensor and for $n=6$ it it is equal to

$$\begin{aligned} \mathcal{O} &= B_{ij,k}{}^{k} - 2W_{kijl}B^{kl} - 4P_k^k B_{ij} + 8P^{kl}C_{ijk,l} - 4C_i^{kl}C_{ljk} \\ &+ 2C_i^{kl}C_{jkl} + 4P^k_{k,l}C^l_{ij} - 4W_{kijl}P^k_m P^{ml}, \end{aligned} \tag{2.13}$$

where B_{ij} denotes the Bach tensor, C_{ijk} denotes the Cotton tensor and W_{ijkl} denotes the Weyl tensor.

When the obstruction tensor is nonzero, there are no formal power series solutions for $\tilde{g}$ beyond the order $n/2$. For $n \geq 4$, n even, we may continue the process of a determining a solution for $\tilde{g}$ beyond the order $n/2$ by introducing log terms into the formal power series. When n is odd, we introduce half-integral power terms into the formal power series. In both cases the solution is no longer unique.

We continue by providing a definiton of the Poincaré-Einstein metric on a conformal manifold and discuss its properties. We shall denote again by $(M,[g])$ a Riemannian manifold of dimension $n \geq 2$ with a conformal class of metrics $[g]$ represented by a metric g.

Definition 48. *Let M_+ be a manifold with boundary $\partial M_+ = M$ and let r be a defining function for the boundary of M_+, by which we mean that r is smooth on M_+, positive on M_+^0 (the interior of M_+) and $r=0$ on ∂M_+, $dr \neq 0$ on ∂M_+. We say that a smooth metric g_+ on the interior of M_+ of signature $(p+1,q)$ is conformally compact if $r^2 g_+$ extends smoothly to M_+ and $r^2 g_+|_M$ is non-degenerate as a quadratic form. Moreover, we say that a conformally compact metric g_+ has conformal infinity $(M,[g])$ if $r^2 g_+|_{TM} \in [g]$.*

The conditions in the previous definition are independent of a choice of the defining function r. Now we will identify M_+ with an open neighborhood of $M \times \{0\}$ in $M \times [0, \infty)$. The second coordinate factor in this decomposition will be labeled as r.

Definition 49. *Let $S_{\alpha\beta}$ be a symmetric (0,2)-tensor field on an open neighborhood of $M \times \{0\}$ and let $m \geq 0$. We will write $S_{\alpha\beta} = \mathrm{O}^+_{\alpha\beta}(r^m)$ if $S = \mathrm{O}(r^m)$ and $tr(i^*(r^{-m}S)) = 0$ on M, where $i : M \to M \times [0, \infty)$ is the inclusion given by $i(x) = (x, 0)$, g is a metric in the conformal class $[g]$ and in the trace is taken with respect to the metric g.*

Now we are ready to introduce the notion of the Poincaré-Einstein metric.

Definition 50. *A Poincaré-Einstein metric for the conformal class $(M, [g])$ is a conformally compact metric g_+ of signature $(p + 1, q)$ defined on the interior of M_+, where M_+ is an open neighborhood of $M \times \{0\}$ in $M \times [0, \infty)$ such that g_+ has conformal infinity $(M, [g])$ and the following condition is satisfied:*
(1) if n is odd or $n = 2$ we require that $Ric(g_+)_{\alpha\beta} + ng_+$ vanishes to infinite order on M.
(2) If $n \geq 4$ and n is even we require that $Ric(g_+)_{\alpha\beta} + ng_+ = \mathrm{O}^+_{\alpha\beta}(r^{n-2})$.

We remark that we may alternatively define a Poincaré-Einstein metric g_- with signature $(p, q + 1)$ and with the term $\mathrm{Ric}(g_-) - ng_-$ instead of the term $\mathrm{Ric}(g_+) + ng$ in the condition. This is equivalent to taking $g_- = -g_+$ and the signature (q, p) instead of (p, q).

Definition 51. *A conformally compact metric g_+ is asymptotically hyperbolic if $|\frac{dr}{r}|_{g_+} = 1$ on M_+.*

Poincaré-Einstein metric is always hyperbolic. We will formulate a notion of normal form for asymptotically hyperbolic metrics in order to establish a connection between the ambient metric and the Poincaré-Einstein metric.

Definition 52. *We will say that an asymptotically hyperbolic metric g_+ is in normal form relative to a metric g in the conformal class $[g]$ if $g_+ = r^{-2}(dr^2 + g_r)$, where g_r is a 1-parametric family of metrics on M of signature (p, q) satisfying $g_r|_{r=0} = g$.*

Theorem 9. *Let g_+ be an asymptotically hyperbolic metric on M^0_+ and let $g \in [g]$. Then there exists an open neighborhood $\mathcal{U}$ of $M \times \{0\}$ in $M \times [0, \infty]$ on which there is a unique diffeomorphism Φ from $\mathcal{U}$ into M_+ such that $\Phi|_M = id_M$, and such that $\Phi^* g_+$ is in normal form relative to g on $\mathcal{U}$.*

We will need one more definition.

Definition 53. *We will call an asymptotically hyperbolic metric g_+ on M^0_+ even if $r^2 g_+$ is the restriction to M_+ of a smooth metric h on an open set $\mathcal{V} \subset M \times (-\infty, \infty)$ containing M_+ (under the identification), such that $\mathcal{V}$ and h are invariant under the transformation $T(r) = -r$. Moreover we will call a diffeomorphism $\Phi : M_+ \to M \times [0, \infty)$ satisfying $\Phi|_{M \times \{0\}} = id_{M \times \{0\}}$ even if Φ is the restriction of a diffeomorphism of $\mathcal{V}$ which commutes with T.*

We remark that if g_+ is an even asymptotically hyperbolic metric and Φ is an even diffeomorphism, then $\Psi^* g_+$ is even. Now we will formulate a theorem that guarantees existence of a Poincaré-Einstein metric and a theorem for existence of a Poincaré-Einstein metric in normal form.

Theorem 10. *Let $(M,[g])$ be a smooth manifold of dimension $n \geq 2$, equipped with a conformal class $[g]$. Then there exists an even Poincaré-Einstein metric for $(M,[g])$. Moreover, if g_+^1 and g_+^2 are two even Poincaré-Einstein metrics for $(M,[g])$ defined on $(M_+^1)^0$ and $(M_+^2)^0$, resp., then there are open subsets $\mathcal{U}^1 \subset M_+^1$ and $\mathcal{U}^2 \subset M_+^2$ containing $M \times \{0\}$ and even diffeomorphism $\Phi : \mathcal{U}^1 \to \mathcal{U}^2$ such that $\Psi|_{M\times\{0\}}$ is the identity map, and such that:*
(a) If $n = \dim M$ is odd, then $g_+^1 - \Phi^* g_+^2$ vanishes to infinite order at every point of $M \times \{0\}$.
(b) If $n = \dim M$ is even, then $g^1_{+\alpha\beta} - \Phi^* g^2_{+\alpha\beta} = \mathrm{O}^+_{\alpha\beta}(r^{n-2})$.

Theorem 11. *Let M be a smooth manifold of dimension $n \geq 2$ and g a smooth metric on M.*
(A) There exists an even Poincaré-Einstein metric g_+ for $(M,[g])$ which is in normal form relative to g.
(B) Suppose that g_+^1 and g_+^2 are even Poincaré-Einstein metrics for $(M,[g])$, both of which are in normal form relative to g. If n is odd, then $g_+^1 - g_+^2$ vanishes to infinite order at every point of $M \times \{0\}$. If n is even, then $g^1_{+\alpha\beta} - g^2_{+\alpha\beta} = \mathrm{O}^+_{\alpha\beta}(r^{n-2})$.

Now everything is set up to discuss the construction of a Poincaré-Einstein metric from a pre-ambient metric. Let $(\tilde{\mathcal{G}}, \tilde{g})$ be a straight pre-ambient space for $(M,[g])$. Using Theorem 8, that $||T||^2$ vanishes exactly up to first order on $\mathcal{G} \times \{0\} \subset \tilde{\mathcal{G}}$. We define the hypersurface $\mathcal{H} = \tilde{\mathcal{G}} \cap \{||T||^2 = -1\}$. Every dilation orbit of $\tilde{\mathcal{G}}$ intersects the hypersurface $\mathcal{H}$ exactly at one point because $||T||^2$ is homogeneous of degree 2 with respect to the dilations. We extend the projection $\pi : \mathcal{G} \to M$ to $\pi : \tilde{\mathcal{G}} \subset \mathcal{G} \times \mathbb{R} \to M \times \mathbb{R}$ by acting only on the first factor. Define the mapping $\chi : M \times \mathbb{R} \to M \times [0,\infty)$ as $\chi(x,r) = (x, \sqrt{2|r|})$. Then there exists an open neighborhood M_+ of $M \times \{0\}$ in $M \times [0,\infty)$ such that $\chi \circ \pi|_{\mathcal{H}} : \mathcal{H} \to M_+^0$ is a diffeomorphism.

Theorem 12. *If $(\tilde{\mathcal{G}}, \tilde{g})$ is a straight pre-ambient space for $(M,[g])$ and $\mathcal{H}$ and M_+ are as above, then*

$$g_+ := ((\chi \circ \pi|_{\mathcal{H}})^{-1})^* \tilde{g}|_{\mathcal{H}} \tag{2.14}$$

is an even asymptotically hyperbolic metric with conformal infinity $(M,[g])$. If $\tilde{g}$ is in normal form relative to a metric $g \in [g]$, then g_+ is also in normal form relative to g. Every even asymptotically hyperbolic metric g_+ with conformal infinity $(M,[g])$ is of the form (2.14) for some straight pre-ambient metric $\tilde{g}$ for $(M,[g])$. If g_+ is in normal form relative to g, then $\tilde{g}$ can be taken to be in normal form relative to g, and in this case $\tilde{g}$ on $\{||T||^2 \leq 0\}$ is uniquely determined by g_+.

We remark that in general there are many straight pre-ambient metrics $\tilde{g}$ such that 2.14 gives the same Poincaré-Einstein metric g_+.

We conclude this section by giving the formal power series form of an even Poincaré-Einstein metric g_+ in normal form relative to the representative $g \in [g]$ associated to $(M, [g])$ represented by g. Refer to [AM]. The Poincaré-Einstein metric is given by

$$g_+ = \frac{1}{r^2}(g_r + dr^2), \tag{2.15}$$

where g_r is a 1-parameter family of smooth Riemannian metrics satisfying $g_r|_{r=0} = g$

$$g_r = g^{(0)} + r^2 g^{(2)} + \cdots + r^n g^{(n)}, \quad g^{(0)} = g, \tag{2.16}$$

where $g^{(0)} = g$ and $g^{(2)} = -P_{ij}$, $P_{ij} = tf(\frac{\partial \tilde{g}_{ij}}{\partial r})|_{r=0}$, where tf denotes the trace-free part (also see 39) and $\tilde{g}$ is the ambient metric for g_{ij}. In the case when $(M, [g])$ is conformally equivalent to a Einstein metric g, the formula for g_r reduces to

$$g_r = g^{(0)} + r^2 g^{(2)}, \quad g^{(0)} = g, \tag{2.17}$$

where $g^{(0)} = g$. We will not consider the generalized Poincaré-Einstein metrics.

We remark that we may again generalize the notion of Poincaré-Einstein metric to a generalized Poincaré metric by replacing smoothness with a weaker condition and requiring that $\mathrm{Ric}(g_+) - ng_+$ vanishes to infinite order. Notions of straightness and normal form again apply, and correspond to introducing log terms into the power series. We write a straight generalized Poincaré-Einstein in normal form as a power series involving powers of r and $\log(r)$. Since we will work only with Poincaré-Einstein metrics, we will not formulate results about generalized Poincaré-Einstein metrics.

3. Parabolic PDE

3.1 Existence theory

In this section we define linear partial differential equations of parabolic type and discuss their generalisation to a non-linear setting on manifolds. We aim to formulate the existence and uniqueness statements for parabolic partial differential equations.

Definition 54. *We consider a class of second order partial differential equations on an open set $\Omega \subset \mathbb{R}^n$ of the form*

$$\frac{\partial u}{\partial t} = a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u + b_i\frac{\partial}{\partial x_i}u + cu, \quad i, j = 1, \ldots, n \tag{3.1}$$

where $u : \Omega \times I \to \mathbb{R}$ is a smooth function, where $I \subset \mathbb{R}$ is an open interval and $a_{ij}, b_i, c : \Omega \to \mathbb{R}$ are smooth functions. Equation (3.1) *is called parabolic if there exists $\lambda \in \mathbb{R}_+$, such that*

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2. \tag{3.2}$$

We extend this notion to closed manifolds:

Definition 55. *Let M be a closed manifold, $I \subset \mathbb{R}$ an open interval and $u : M \times I \to \mathbb{R}$ a function. Consider the equation*

$$\frac{\partial u}{\partial t} = L(u), \tag{3.3}$$

where $L : C^\infty(M) \to C^\infty(M)$ is a second order differential operator that can be written in a coordinate chart $(x_1, x_2, \ldots, x_n)$ on some $U \subset M$ as

$$L(u) = a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u + b_i\frac{\partial}{\partial x_i}u + cu, \tag{3.4}$$

where a_{ij}, b_i, c are smooth functions defined on U. We call the equation 3.3 parabolic, if there exists $\lambda > 0$, such that

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2. \tag{3.5}$$

Now we will state an alternative definition of parabolic equations which will be useful later in proving the existence theorem for Ricci flow.

Definition 56. *For the second order differential operator L in equation* (3.3) *we define the principal symbol $\sigma(L) : T^*(M) \to \mathbb{R}$ as*

$$\sigma(L)(x, \xi) = a_{ij}(x)\xi_i\xi_j. \tag{3.6}$$

The principal symbol is independent of the choice of coordinates (see [RF], page 54 for a third alternative definition). Equation (3.3) *is now parabolic, if $\sigma(L)(x, \xi) > 0$ for all $(x, \xi) \in T^*(M)$. See again [RF] for the equivalence of these definitons.*

In particular when $L = \Delta$ the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{d}}\frac{\partial}{\partial x_i}(\sqrt{d}g^{ij}\frac{\partial}{\partial x_j}) = g^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \text{LOTs}, \tag{3.7}$$

where LOTs denotes lower order terms and $d = det(g_{ij})$, the principal symbol equals

$$\sigma(\Delta)(x,\xi) = g^{ij}\xi_i\xi_j = |\xi|^2 > 0. \tag{3.8}$$

We have demonstrated that the heat equation $\frac{\partial u}{\partial t} = \Delta u$ is parabolic according to Definition 55.

Next step is generalising our definition of parabolicity to sections of vector fiber bundles. We will again define the principal symbol in this setting. We will use lower-case greek letters to label indices associated with coordinate frames to distinguish them from indices associated with coordinate charts.

Definition 57. *Let E be a smooth vector fiber bundle over a closed manifold M and $I \subset \mathbb{R}$ be an open subset. Let $\Pi : M \times I \to M$ denote projection on the first factor. Now we consider the pullback bundle $\Pi^*(E)$ with base space $M \times I$. Let $v \in \Gamma(\Pi^*(E))$. Choosing a coordinate frame $\{e_1, e_2, \dots, e_k\}$, we may write v as*

$$v = v^\alpha e_\alpha. \tag{3.9}$$

Now, we consider the equation

$$\frac{\partial v}{\partial t} = L(v), \tag{3.10}$$

where L is a second order linear differential operator $L : \Gamma(E) \to \Gamma(E)$, which may be written in a coordinate chart $(x_1, x_2, \dots, x_n)$ on some $U \subset M$ and a coordinate frame $\{e_1, e_2, \dots, e_k\}$ on E as

$$L(v) = [a_{ij}^{\alpha\beta}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v^\beta + b_i^{\alpha\beta}\frac{\partial}{\partial x_i}v^\beta + c^{\alpha\beta}v^\beta]e_\alpha, \quad i,j = 1,\dots,n, \tag{3.11}$$

for $a_{ij}, b_i, c : U \to E$ smooth functions. At this point, we may define the principal symbol of the second order differential operator in equation (3.10)*. Let $\Pi : T^*(M) \to M$ denote the bundle projection of the vector fiber bundle $T^*(M)$. Then the pullback bundle $\Pi^*(E)$ is a vector bundle over $T^*(M)$ with fiber at $(x,\xi) \in T^*(M)$ equal to E_x (fiber of the bundle E). The principal symbol $\sigma(L) : \Pi^*(E) \to \Pi^*(E)$ is a vector fiber bundle homomorphism defined as*

$$\sigma(L)(x,\xi)v = (a_{ij}^{\alpha\beta}\xi_i\xi_j)e_\alpha. \tag{3.12}$$

We call the equation (3.10) *parabolic, if there exists $\lambda > 0$ such that*

$$\langle\sigma(L)(x,\xi)v, v\rangle \geq \lambda|\xi|^2|v|^2. \tag{3.13}$$

Definition 58. *We consider a nonlinear partial differential equation of the form*

$$\frac{\partial v}{\partial t} = P(v), \tag{3.14}$$

where $P : \Gamma(E) \to \Gamma(E)$ *is a second order quasilinear differential operator, which may be given locally in a coordinate chart* $(x_1, x_2, \dots, x_n)$ *on* M *and a coordinate frame* $\{e_1, e_2, \dots, x_n\}$ *on* E *as*

$$P(v) = [a_{ij}^{\alpha\beta}(x, v, \nabla v)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}v^\beta + b_i^\alpha(x, v, \nabla v)]e_\alpha, \quad i, j = 1, \dots, n. \tag{3.15}$$

Given $w \in \Gamma(E)$, *we call the equation* (3.14) *parabolic at* w, *if the linearisation of equation* (3.14) *at* w, *which is given by the equation*

$$\frac{\partial v}{\partial t} = [\frac{\partial}{\partial t}P(w)]v \tag{3.16}$$

is parabolic by definition 57.

We finish the section by formulating the existence and uniqueness statements for solutions of parabolic differential equations.

Theorem 13. *Let us consider the equation* (3.14). *Let* $w \in \Gamma(E)$. *If the equation is parabolic at* w *and the functions* $a_{ij}^{\alpha\beta}, b_i^\alpha$ *are smooth for* $i, j = 1, \dots, n$ *and* $\alpha, \beta = 1. \dots, n$, *then there exists* $\epsilon > 0$ *and* $v \in \Gamma(\Pi^*(E))$ *for* $t \in [0, \epsilon]$, *such that*

$$\frac{\partial v}{\partial t} = P(v), \quad v(0) = w \tag{3.17}$$

for $t \in [0, \epsilon]$. *Second, uniqueness also holds: suppose that* $\frac{\partial v}{\partial t} = P(v)$ *and* $\frac{\partial \tilde{v}}{\partial t} = P(\tilde{v})$ *for* $t \in [0, \epsilon]$. *If either* $v(0) = \tilde{v}(0)$ *or* $v(\epsilon) = \tilde{v}(\epsilon)$, *then* $v(t) = \tilde{v}(t)$ *for all* $t \in [0, \epsilon]$.

See [EV], Second-order parabolic equations, Section 7.1 Theorem 7, page 367 for the proof of the theorem in the case when $M = \mathbb{R}^n$.

4. Ricci Flow on manifolds

4.1 Basics and existence theory

We introduce the notion of the Ricci flow and discuss its existence and uniqueness for closed manifolds. We follow [RF] chapter 5, Existence theory for the Ricci flow. Note that the assumption of closedness on the manifold M may be removed, see [SH], [CZ] and [TO] for existence results.

Definition 59. *Let (M, g) be a closed Riemannian manifold. Ricci flow of the manifold (M, g) is a 1-parametric family of metrics $g(t)$ on M (where $I = [0, a]$ is a closed interval for some $a \in \mathbb{R}_+$ and $t \in I$) fulfilling the Ricci flow equation*

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)), \quad t \in I, \tag{4.1}$$

together with the boundary condition

$$g(t)|_{t=0} = g. \tag{4.2}$$

Recall that $Ric(g(t))$ is the Ricci curvature of $(M, g(t))$.

Since under multiplication of the metric by a positive constant the Ricci tensor remains the same, we may choose an arbitrary constant coefficient for the Ricci flow equation. In this case we will use 2 as the constant coefficient in the Ricci flow equation.

We note that the Ricci flow equation fits Definition 58, however it is not parabolic (see [RF], pages 59-60). This will be solved by a workaround called "DeTurck trick" (see [RF], section 5.2). Consider $T \in \Gamma(\mathrm{Sym}^2 T^*(M))$ smooth and positive definite. We will also denote by T the invertible map $T : \Gamma(T^*(M)) \to \Gamma(T^*(M))$ induced by T. For a Riemannian manifold (M, g) equipped with a smooth family of metrics $g(t)$ parametrized by $t \in [0, \epsilon]$ we define operators

$$Q = -2\mathrm{Ric}(g(t)) \tag{4.3}$$

and

$$P(g(t)) = -2\mathrm{Ric}(g(t)) + \mathcal{L}_{(T^{-1}\delta G(T))^{\#}} g(t), \tag{4.4}$$

where $\mathcal{L}$ is the Lie derivative, δ is the divergence with respect to $g(t)$, G is the gravitation tensor (see Definition 34) and $\#$ is the musical isomorphism. Note that $P(g(t)) = Q + \mathcal{L}_{(T^{-1}\delta G(T))^{\#}} g$. The strategy of the DeTurck trick is first to show that the equation

$$\frac{\partial g(t)}{\partial t} = P(g(t)) \tag{4.5}$$

is parabolic and then to modify this solution to obtain the Ricci flow equation.

In order to demonstrate that the equation (4.5) is parabolic, we need to calculate its principal symbol. First, we set $h = \frac{\partial g(t)}{\partial t}$. Using Theorem 1, we get

$$(\frac{\partial}{\partial t}\delta(G(T)))Z = -T((\delta G(h))^{\#}, Z) + \langle h, \nabla T(.,.,Z) - \frac{1}{2}\nabla_Z T\rangle, \tag{4.6}$$

where $\langle , \rangle$ denotes pairing with respect to g, so

$$\frac{\partial}{\partial t} T^{-1}\delta G(T) = -\delta G(h) + \dots, \tag{4.7}$$

where $\dots$ consists of terms that do not depend on derivatives of h, while $\delta G(h)$ depends on first derivatives of h. So we have

$$\frac{\partial}{\partial t}\mathcal{L}_{(T^{-1}\delta G(T))^{\#}} g = -\mathcal{L}_{(\delta G(T))^{\#}} g + \dots, \tag{4.8}$$

where the Lie derivative term contains one second derivative of h and the $\dots$ terms contain h and first derivative of h. Now, using theorem 17, we calculate

$$\begin{aligned} \frac{\partial}{\partial t} P(g(t)) &= [\frac{\partial}{\partial t} Q(g) + D\mathcal{L}_{(T^{-1}\delta G(T))^{\#}} g]h \\ &= (\Delta_L h + \mathcal{L}_{(\delta G(h))^{\#}} g) - \mathcal{L}_{(\delta G(h))^{\#}} g + \dots \\ &= \Delta_L h + \dots, \end{aligned} \tag{4.9}$$

where $\Delta_L h$ is the Lichnerowicz Laplacian of h. Using (3.8), we see the principal symbol $\sigma(DP(g))(x,\xi)h = |\xi|^2 h$, so the equation (4.5) is parabolic for any initial metric g. We may now use Theorem 13 in equation (4.5). For any initial metric g there exists $\epsilon > 0$ and a solution $g(t)$ of equation (4.5) satisfying the boundary condition

$$g(t)|_{t=0} = g. \tag{4.10}$$

The second step of the DeTurck trick consists of a modification of the solution to (4.5) by a diffeomorphism to obtain the Ricci flow. We begin by defining for each t a smooth vector field given by $X = -(T^{-1}\delta G(T))^{\#}$. This vector field generates a diffeomorphism $\psi_t : M \times I \to M$ given by the equation

$$X(\psi_t(y), t)f = \frac{\partial (f \circ \psi_t)}{\partial t}(y) \tag{4.11}$$

for any $f : M \to \mathbb{R}$ a smooth function on M and $y \in M$. Next we define a modified family of metrics $\hat{g}(t) = \psi_t^*(g(t))$. Using the definition of Lie derivative, we see that

$$\frac{\partial \hat{g}}{\partial t} = \psi_t^*(\frac{\partial g(t)}{\partial t} + \mathcal{L}_X g), \tag{4.12}$$

so we may calculate

$$\begin{aligned} \psi_t^*(\frac{\partial g(t)}{\partial t} + \mathcal{L}_X g) &= \psi_t^*(-2\mathrm{Ric}(g(t)) + \mathcal{L}_{(T^{-1}\delta G(T))^{\#}} g + \mathcal{L}_{-(T^{-1}\delta G(T))^{\#}} g) \\ &= -2\psi_t^*(\mathrm{Ric}(g(t))) \\ &= -2\mathrm{Ric}(\hat{g}(t)) \end{aligned} \tag{4.13}$$

and

$$\hat{g}(t)|_{t=0} = \psi_t^*(g(t)|_{t=0}) = g, \tag{4.14}$$

so by (4.12) $\hat{g}(t)$ is the Ricci flow. We formulate the result as a theorem.

Theorem 14. *Let M be a closed Riemannian manifold equipped with metric g. There exists $\epsilon > 0$ and a smooth family of metrics $g(t)$ for $t \in [0, \epsilon]$, such that*

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)), \tag{4.15}$$

fulfilling the boundary condition

$$g(t)|_{t=0} = g. \tag{4.16}$$

Now we will discuss uniqueness of the Ricci flow. Let $g(t)^1$ and $g(t)^2$ be two solutions of the Ricci flow equation (4.1) with boundary condition (4.2). These solutions exist from Theorem 14 for some ϵ^1 and ϵ^2. We will work on the time domain given by the minimum of ϵ^1, ϵ^2, which we denote by ϵ. First, we choose a metric T on M and solve the partial differential equation given by

$$\frac{\partial \psi_t^i}{\partial t} = (T^{-1}\delta G(T))^{\#}, \tag{4.17}$$

for $i = 1, 2$ (the divergence is taken with respect to $g^1(t), g^2(t)$), satisfying the boundary condition

$$\psi_t^i|_{t=0} = \text{id}, \tag{4.18}$$

where $\psi_t^i : M \times [0, \epsilon] \to M$. We may assume that ψ_t^i is a diffeomorphism for every $t \in [0, \epsilon]$ and note that $(\psi_t^i)_*(g_t^i)$ satisfies the equation (4.5) for $i = 1, 2$ with the same boundary condition. Using the uniqueness part of Theorem 13, we get that $\psi_t^1 = \psi_t^2$ for $t \in [0, \epsilon]$, since ψ_t^1, ψ_t^2 are invertible.

We again formulate the result.

Theorem 15. *Let M be a closed Riemannian manifold with metric g and let $g^1(t), g^2(t)$ be solutions of the Ricci flow equation* (4.1) *with boundary condition* (4.2) *for (M, g), where $t \in [0, \epsilon]$ for some $\epsilon > 0$. If $g^1(t) = g^2(t)$ for some $t \in [0, \epsilon]$, then $g^1(t) = g^2(t)$ for all $t \in [0, \epsilon]$.*

Since we have formulated existence and uniqueness, we may formulate the definition of Ricci flow on a maximal time interval $[0, T]$.

That is, either $T = \infty$ or in the case when $T < \infty$, there does not exist $\epsilon > 0$ for which a solution of the Ricci flow equation extends from $[0, T]$ to $[0, T + \epsilon]$.

Finally, we will formulate theorems on evolution of Riemannian volume and Ricci curvature under the Ricci flow.

Theorem 16. *Let (M, g) be a Riemannian manifold and let $g(t)$ be a solution of the Ricci flow equation* (4.1) *with boundary condition* (4.2) *for (M, g), for $t \in [0, \epsilon]$ for some $\epsilon > 0$. Let $V(t)$ be the Riemannian volume parametrized by $t \in [0, \epsilon]$ associated to the family of metrics $g(t)$. We set $q(t) = \frac{\partial g(t)}{\partial t}$. Then the following holds*

$$\frac{\partial}{\partial t} dV = \frac{1}{2}(tr\ q(t))dV, \tag{4.19}$$

from which follows that the following holds

$$\frac{\partial}{\partial t} V(t) = -\int_M R dV, \tag{4.20}$$

for $t \in [0, \epsilon]$.

Theorem 17. *Let (M, g) be a Riemannian manifold and let $g(t)$ be a solution of the Ricci flow equation* (4.1) *with boundary condition* (4.2) *for (M, g), where $t \in [0, \epsilon]$ for some $\epsilon > 0$. Let $Ric(t)$ denote the Ricci curvature tensor of $g(t)$ and let $q(t) = \frac{\partial g(t)}{\partial t}$. The Ricci curvature tensor $Ric(t)$ evolves according to*

$$\frac{\partial}{\partial t} Ric(t) = -\frac{1}{2}\Delta_L q(t) - \frac{1}{2}\mathcal{L}_{(\delta G(q(t)))^\#} g, \tag{4.21}$$

for $t \in [0, \epsilon]$, where Δ_L is the Lichnerowicz Laplacian defined by (1.21).

Theorem 18. *Let (M, g) be a Riemannian manifold and let $g(t)$ be a solution of the Ricci flow equation* (4.1) *with boundary condition* (4.2) *for (M, g), for $t \in [0, \epsilon]$ for some $\epsilon > 0$. Let Ric_{ij} denote components of the Ricci curvature tensor of g and R_{ijkl} denote components of the Riemann curvature tensor of g. Let $Ric_{ij}(t)$ denote components of the Ricci curvature tensor of $g(t)$. Then the components of the Ricci curvature tensor $Ric_{ij}(t)$ evolve according to*

$$\frac{\partial}{\partial t} Ric_{ij}(t) = \Delta Ric_{ij} - 2g^{kl} Ric_{ki} Ric_{lj} + 2g^{kl} g^{pq} R_{ikjp} Ric_{lq}, \tag{4.22}$$

for $t \in [0, \epsilon]$, where Δ is the connection Laplacian. We will use $\langle Ric_{i-}, Ric_{j-}\rangle$ to denote $g^{kl} Ric_{ki} Ric_{lj}$ and $\langle R_{i-j-}, Ric_{--}\rangle$ to denote $g^{kl} g^{pq} R_{ikjp} Ric_{lq}$.

We refer to [RF] for the proofs.

4.2 Examples

As a first example we consider $\mathbb{R}^n$ with euclidean metric $g_{ij} = \delta_{ij} dx_i dx_j$. Since it is Ricci-flat, the manifold remains unchanged by evolution of the Ricci flow.

In the case when the initial Riemannian manifold (M, g) is Einstein with Einstein constant J, the evolution of the Ricci flow is particularly simple. Because

$$\mathrm{Ric}(g) = Jg, \tag{4.23}$$

the solution $g(t)$ for the Ricci flow equation (4.1), satisfying the boundary condition

$$g(t)|_{t=0} = g, \tag{4.24}$$

is given by

$$g(t) = (1 - 2Jt)g, \tag{4.25}$$

by the fact that the Ricci tensor is invariant under uniform scalings of the initial metric.

The unit sphere $(\mathbb{S}^n, g)$ for $n > 1$ has $J = (n - 1)$, so the evolution is

$$g(t) = (1 - 2(n - 1)t)g, \tag{4.26}$$

and the unit sphere collapses to a point at time $t = \frac{1}{2(n-1)}$, where the metric degenerates.

Next, we take a hyperboloid with hyperbolic metric (of constant sectional curvature -1) $(\mathbb{H}^n, g)$. This time the Einstein constant is $J = -(n-1)$, and the evolution is given by

$$g(t) = (1 + 2(n-1)t)g. \tag{4.27}$$

The manifold expands with t increasing, which can be seen from the evolution of the Riemannian volume, Theorem 16

$$V(t) = V + (n-1)t, \tag{4.28}$$

where V is the Riemannian volume of $(\mathbb{H}^n, g)$.

Our last example is a 1-parameter family of perturbed 3-spheres $(\mathbb{S}^3, g), g \equiv g(\epsilon)$, where

$$g(\epsilon) = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $\epsilon > 0$. For $\epsilon = 1$ we get the case covered in (4.26). We use the isomorphism of $\mathbb{S}^3$ with the Lie group $SU(2)$ to simplify the calculations. The Lie algebra $su(2)$of $SU(2)$ has a basis X, Y, Z

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \tag{4.29}$$

satisfying commutator relations

$$[X, Y] = 2Z, \quad [Z, X], = 2Y \quad [Y, Z] = 2X. \tag{4.30}$$

Using the formula (1.15), we calculate the connection coefficients

$$\begin{aligned} \nabla_X X &= 0, \quad \nabla_X Y = (2-\epsilon)Z, \quad \nabla_X Z = -(2-\epsilon)Y, \\ \nabla_Y X &= -\epsilon Z, \quad \nabla_Y Y = 0, \quad \nabla_Y Z = X, \\ \nabla_Z X &= \epsilon Y, \quad \nabla_Z Y = -X, \quad \nabla_Z Z = 0, \end{aligned} \tag{4.31}$$

and calculate components of the Riemann curvature tensor

$$\begin{aligned} R_{XY}X &= -\epsilon^2 Y, \quad R_{XZ}X = -\epsilon^2 Z, \quad R_{YZ}X = 0, \\ R_{XY}Y &= \epsilon X, \quad R_{XZ}Y = 0, \quad R_{YZ}Y = -(4-3\epsilon)Z, \\ R_{XY}Z &= 0, \quad R_{XZ}Z = \epsilon X, \quad R_{YZ}Z = (4-3\epsilon)Y. \end{aligned} \tag{4.32}$$

The components of the Ricci tensor are then equal to

$$\begin{aligned} &\mathrm{Ric}_{XX} = 2\epsilon^2, \quad \mathrm{Ric}_{YY} = 2(2-\epsilon), \quad \mathrm{Ric}_{ZZ} = 2(2-\epsilon), \\ &\mathrm{Ric}_{XY} = 0, \quad \mathrm{Ric}_{YZ} = 0, \quad \mathrm{Ric}_{XZ} = 0. \end{aligned} \tag{4.33}$$

Since the Ricci tensor is diagonal, the solution $g(t)$ to the Ricci flow equation (4.1) is constant in off-diagonal components. By the boundary condition $g(t)|_{t=0} = g$ we get that the solution $g(t)$ is diagonal, so it is in the form

$$g(t) = \begin{pmatrix} \epsilon(t)\rho(t) & 0 & 0 \\ 0 & \rho(t) & 0 \\ 0 & 0 & \rho(t) \end{pmatrix} \tag{4.34}$$

for some smooth functions $\epsilon(t)$, $\rho(t)$ satisfying the boundary conditions given by $(\epsilon(t)\rho(t))|_{t=0} = \epsilon$ and $\rho(t)|_{t=0} = 1$. Then the Ricci flow equation (4.1) is equivalent to the system of partial differential equations

$$\begin{aligned} \frac{\partial \rho(t)}{\partial t} &= -4(2-\epsilon(t)), \\ \frac{\partial \epsilon(t)\rho(t)}{\partial t} &= \epsilon(t)\frac{\partial \rho(t)}{\partial t} + \frac{\partial \epsilon(t)}{\partial t}\rho(t) = -4\epsilon^2(t). \end{aligned} \tag{4.35}$$

The second equation in (4.35) gives

$$\begin{aligned} \frac{\partial \epsilon(t)}{\partial t}\rho(t) &= 8\epsilon(t)(1-\epsilon(t)), \\ \frac{\partial \epsilon(t)}{\partial t} &= 8\frac{\epsilon(t)(1-\epsilon(t))}{\rho(t)}. \end{aligned} \tag{4.36}$$

We will calculate dependence of $\rho(t)$ on $\epsilon(t)$, using the implicit function theorem. In the following we assume that $\epsilon(t) \neq 1$ for $\mathbb{R}_+$. Calculating the partial derivative of $\rho(t)$ with respect to $\epsilon(t)$, we get

$$\frac{\partial \rho(t)}{\partial \epsilon(t)} = \frac{\frac{\partial \rho(t)}{\partial t}}{\frac{\partial \epsilon(t)}{\partial t}} = -\frac{4(2-\epsilon(t))\rho(t)}{8\epsilon(t)(1-\epsilon(t))} = -\frac{(2-\epsilon(t))\rho(t)}{2\epsilon(t)(1-\epsilon(t))}. \tag{4.37}$$

The differential equation (4.37) is separable. We may write

$$\frac{d\rho(t)}{\rho(t)} = -\frac{2-\epsilon(t)}{2\epsilon(t)(1-\epsilon(t))}d\epsilon(t). \tag{4.38}$$

Integrating (4.38) by t, we get

$$\log|\rho(t)| = \log|\epsilon(t)| - \frac{1}{2}\log|1-\epsilon(t)| + C_1, \tag{4.39}$$

where C_1 is an integration constant.

Equation (4.39) implies

$$\rho(t) = \frac{C_2\epsilon(t)}{\sqrt{|1-\epsilon(t)|}}, \tag{4.40}$$

where $C_2 = \exp|C_1|$.

Substituting (4.40) into (4.36) we get

$$\frac{\partial \epsilon(t)}{\partial t} = C_3\epsilon^2(t)\sqrt{|1-\epsilon(t)|}, \tag{4.41}$$

where $C_3 = \frac{12}{C_2}$.

The differential equation (4.41) is again separable. Integration by t using substitution $\sqrt{1-\epsilon(t)} = u$ gives

$$-\frac{1}{2}(\log|\frac{1+\sqrt{1-\epsilon(t)}}{1-\sqrt{1-\epsilon(t)}}| + \frac{2\sqrt{1-\epsilon(t)}}{\epsilon(t)}) = C_3t + C_4, \tag{4.42}$$

where C_4 is an integration constant.

5. The Ricci flow problem for Poincaré-Einstein metrics

5.1 Formulation of the problem

Let (M, g) be a Riemannian manifold. We are interested in finding metrics $h(0)$ that flow to the Poincaré-Einstein metric h_+ of g at a finite time (normalized to $t = 1$). Although the Poincaré-Einstein metric forms a manifold with boundary on $M \times I$, we will only work in the interior of the manifold. We will not consider generalized Poincaré-Einstein metrics. We will refer to this problem as the Ricci flow problem for (M, g).

Namely, we consider 1-parameter family of smooth metrics $h(t)$ on $M \times I$, parametrized by $t \in [0, \epsilon]$ for some $\epsilon > 0$, satisfying the Ricci flow equation

$$\frac{\partial h(t)}{\partial t} = -2\mathrm{Ric}(h(t)), \tag{5.1}$$

together with the terminal (boundary) condition

$$h(t)|_{t=1} = h_+. \tag{5.2}$$

Here t denotes the coordinate on $[0, \epsilon]$, r denotes the coordinate on I as in chapter 2, $x_1, \ldots, x_n$ denote coordinates on M. A solution of the Ricci flow problem is an initial metric occuring in the family $h(t)$ at the time $t = 0$. Finding a solution to the Ricci flow problem is equivalent to finding a solution to backward Ricci flow, that flows in a finite time in the following way: we consider the backward Ricci flow given by

$$\frac{\partial h(t)}{\partial t} = 2\mathrm{Ric}(h(t)), \tag{5.3}$$

together with the boundary condition

$$h(t)|_{t=1} = h_+, \tag{5.4}$$

for $t \in [1 - \tilde{\epsilon}, 1]$, for some $\tilde{\epsilon} > 0$. For a discussion on the backward Ricci flow, see [RF] pages 71-73. It is discussed there that a solution of the backward Ricci flow is generally not unique. It follows that a solution of the Ricci flow problem is generally not unique. For simplification we will assume that the obstruction tensor $\mathcal{O}$ for the metric g is zero, so we do not consider generalized Poincaré-Einstein metrics with log terms in the formal power series.

By (2.15) and (2.16) we can write the Taylor expansion in r of the Poincaré-Einstein metric

$$h_+ = \frac{1}{r^2}(h_r + dr^2)$$

using

$$h_r = h^{(0)} + r^2 h^{(2)} + \ldots + r^n h^{(n)} + \ldots, \tag{5.5}$$

where $h^{(0)} = g$. We remark that $\frac{1}{r^2}(h^{(0)} + dr^2)$, $\frac{1}{r^2}(h^{(0)} + dr^2) + h^{(2)}$, $\frac{1}{r^2}(h^{(0)} + dr^2) + h^{(2)} + r^2 h^{(4)}, \ldots$ and so on are smooth Riemannian metrics on $M \times I$. We shall consider a 1-parameter family of metrics in the general form $h(t) \equiv h_r(t)$

$$h(t) = \frac{1}{r^2} h^{(0)}(t) + h^{(2)}(t) + r^2 h^{(4)}(t) + \ldots + r^n h^{(n+2)}(t) + \ldots \tag{5.6}$$

fulfilling the boundary condition for $t = 1$:

$$h^{(0)}(t)|_{t=1} = h^{(0)} = g + dr^2, \quad h^{(2k)}(t)|_{t=1} = h^{(2k)}. \tag{5.7}$$

Aim 1. *In the thesis we adopt the following strategy: we solve the problem modulo $o(r^{2k})$ for $k \in \mathbb{N}_0$, taking first $\frac{1}{r^2}(h^{(0)} + dr^2)$ and solving for $\frac{1}{r^2}h^{(0)}(t)$ that satisfies the Ricci flow equation*

$$\frac{1}{r^2}\frac{\partial h^{(0)}(t)}{\partial t} = -2Ric(\frac{1}{r^2}h^{(0)}(t)), \tag{5.8}$$

together with the boundary condition

$$(\frac{1}{r^2}h^{(0)}(t))|_{t=1} = \frac{1}{r^2}(h^{(0)} + dr^2), \tag{5.9}$$

by direct computation. In the next step we take $\frac{1}{r^2}(h^{(0)} + dr^2) + h^{(2)}$ and solve for $\frac{1}{r^2}h^{(0)}(t) + h^{(2)}(t)$ satisfying the Ricci flow equation

$$\frac{\partial(\frac{1}{r^2}h^{(0)}(t) + h^{(2)}(t))}{\partial t} = -2Ric(\frac{1}{r^2}h^{(0)}(t) + h^{(2)}(t)), \tag{5.10}$$

together with the boundary condition

$$(\frac{1}{r^2}h^{(0)}(t) + h^{(2)}(t))|_{t=1} = \frac{1}{r^2}(h^{(0)} + dr^2) + h^{(2)}, \tag{5.11}$$

by computation for this metric and so on. In particular we shall consider

$$h^{(2s)}(t) = \begin{pmatrix} a^{(2s)}(t) & b_i^{(2s)}(t) \\ b_i^{(2s)}(t) & g_{ij}^{(2s)}(t) \end{pmatrix} \tag{5.12}$$

for $s \in \mathbb{N}_0$, where $a^{(2s)}(t)$ is a function $a^{(2s)}(t, x_1, \ldots, x_n) \equiv a^{(2s)}(t)$, $b_i^{(2s)}(t)$ is a vector-valued function $b_i^{(2s)}(t, x_1, \ldots, x_n) \equiv b_i^{(2s)}(t)$ and $g_{ij}^{(2s)}(t)$ is a matrix-valued function $g_{ij}^{(2s)}(t, x_1, \ldots, x_n) \equiv g_{ij}^{(2s)}(t)$. In particular, this means that in the first step, taking $\frac{1}{r^2}(h^{(0)} + dr^2)$, the boundary condition for $h^{(0)}(t)$ is set to $a^{(0)}(1, x_1, \ldots, x_n) = 1$, $b_i^{(0)}(1, x_1, \ldots, x_n) = 0$ for $i = 1, \ldots, n$ and $g_{ij}^{(0)}(1, x_1, \ldots, x_n) = g_{ij}$ (the initial metric) for $i, j = 1, \ldots, n$.

We remark that (5.6) is not the most general form of solution of (5.1) with boundary condition (5.2), however it is the most general form that includes all terms with nonzero boundary condition. Generally there may exist solutions of (5.1) with terms of negative lower than -2 or odd powers of r. These terms vanish at $t = 1$ by (5.2).

Next we will explain how we will calculate $h^{(2s)}(t)$ for $s \in \mathbb{N}_0$. We will only discuss the calculation of $h^{(0)}(t)$. We consider the family of metrics $h^{(0)}(t)$ given by the Taylor series in $t-1$

$$h^{(0)}(t) = h^{(0)}_{(0)} + (t-1)h^{(0)}_{(1)} + \frac{(t-1)^2}{2!}h^{(0)}_{(2)} + \ldots, \tag{5.13}$$

where $h^{(0)}_{(0)} = h^{(0)}(1) = h^{(0)} + dr^2$.

We do the same for the Ricci curvature tensor $\mathrm{Ric}(h(t))$ of the family of metrics $h^{(0)}(t)$

$$\mathrm{Ric}(h^{(0)}(t)) = \mathrm{Ric}_{(0)} + (t-1)\mathrm{Ric}_{(1)} + \frac{(t-1)^2}{2!}\mathrm{Ric}_{(2)} + \ldots, \tag{5.14}$$

where $\mathrm{Ric}_{(0)} = \mathrm{Ric}(h^{(0)}(1)) = \mathrm{Ric}(h^{(0)})$.

In the following we will use Theorem 18 to calculate partial derivatives of $h^{(0)}(t)$ with respect to t. We will retain the notation from Theorem 18.

For the simplification of notation we will write $a^{(0)}(t) \equiv a(t)$, $b_i^{(0)}(t) \equiv b_i(t)$ for $i = 1, \ldots, n$ and $g_{ij}^{(0)}(t) \equiv g_{ij}(t)$ for $i, j = 1, \ldots, n$.

We will devote the rest of this chapter to the discussion of solutions of the system (5.1). First, we will show that under a certain assumption a simple solution exists with $b_i = 0$ for $i = 1, \ldots, n$, and show that under more restrictive conditions a unique solution exists for $\mathbb{R}^n$ equipped with euclidean metric. Secondly, we will show how to proceed in the case of a general initial Riemannian manifold $(M, h^{(0)})$

We shall refer to $\frac{1}{r^2}h^{(0)}(t)$ simply as h to simplify our notation. Note that we do not use r as a summation index.

5.2 Simplified case of the Ricci flow problem

The case of $b_i = 0$ for all $i = 1, \ldots, n$ substantially simplifies our calculations due to the block-diagonal form of the metric (5.12). The most general form of this type of 1-parameter family of metrics is

$$h = \frac{1}{r^2}\begin{pmatrix} a & 0 \\ 0 & g_{ij} \end{pmatrix},$$

with

$$a \equiv a(t, x_1, \ldots, x_n) \equiv a(t),\ g_{ij} \equiv g_{ij}(t, x_1, \ldots, x_n) \equiv g_{ij}(t), \tag{5.15}$$

for a smooth and invertible function and g_{ij} smooth, invertible as a matrix, with its inverse

$$h^{-1} = r^2\begin{pmatrix} a^{-1} & 0 \\ 0 & g^{ij} \end{pmatrix},$$

where $g^{ij} = (g_{ij})^{-1}$, for $i, j = 1, \ldots, n$.

We begin by calculating the Christoffel symbols and components of the Riemann curvature tensor of the metric h using (1.13) and (1.14),

$$\begin{aligned}
\Gamma^r_{rr} &= -\frac{1}{r}, \quad \Gamma^k_{rr} = -\frac{1}{2}g^{kp}\frac{\partial a}{\partial x_p}, \\
\Gamma^r_{ri} &= \frac{1}{2a}\frac{\partial a}{\partial x_i}, \quad \Gamma^k_{ri} = -\frac{1}{r}\delta^{ki}, \\
\Gamma^r_{ij} &= \frac{1}{ra}g_{ij}, \quad \Gamma^k_{ij} = \Gamma^k_{ij}(g),
\end{aligned} \tag{5.16}$$

for $i, j = 1, \ldots, n$, where $\Gamma^k_{ij}(g)$ is the Christoffel symbol of the metric g.

$$\begin{aligned}
&R_{rrir} = 0, \quad R_{kril} = \frac{1}{r^3}\left(\frac{\partial g_{ki}}{\partial x_l} - \frac{\partial g_{kl}}{\partial x_i} + g_{lq}\Gamma^q_{ki}(g) - g_{iq}\Gamma^q_{kl}(g)\right), \\
&R_{rijr} = \frac{1}{r^4}g_{ij} + \frac{1}{2r^2}\left(\frac{\partial^2 a}{\partial x_i \partial x_j} - \frac{1}{2a}\frac{\partial a}{\partial x_i}\frac{\partial a}{\partial x_j} - \frac{\partial a}{\partial x_p}\Gamma^p_{ij}(g)\right), \\
&R_{kijl} = \frac{1}{r^2}R_{kijl}(g) + \frac{1}{ar^4}(g_{ij}g_{kl} - g_{kj}g_{il}),
\end{aligned} \tag{5.17}$$

for $i, j = 1, \ldots, n$, where $R_{kijl}(g)$ is the Riemann curvature tensor of the metric g.

The components of the Ricci curvature tensor of the metric h are equal to

$$\begin{aligned}
\mathrm{Ric}_{rr} &= h^{kl}R_{rkrl}, \\
\mathrm{Ric}_{ri} &= h^{kl}R_{rkil}, \\
\mathrm{Ric}_{ij} &= h^{rr}R_{irjr} + h^{kl}R_{ikjl}.
\end{aligned} \tag{5.18}$$

for $i, j = 1, \ldots, n$.

Since $b_i = 0$, the second equation implies

$$\frac{\partial g_{ki}}{\partial x_l} - \frac{\partial g_{kl}}{\partial x_i} + g_{lq}\Gamma^q_{ki}(g) - g_{iq}\Gamma^q_{kl}(g) = 0. \tag{5.19}$$

Once the equation (5.19) is satisfied, we may set $b_i = 0$ and solve the Ricci flow equation (4.1). The equation $b_i = 0$ implies $R_{kril} = 0$ and so $\mathrm{Ric}_{ri} = 0$.

Using Theorem 18 we set up the Ricci flow equation. We calculate components of the connection Laplacian of the Ricci tensor and the traces of the Ricci tensor in formula (4.22).

$$
\begin{aligned}
\Delta \mathrm{Ric}_{rr} &= \frac{r^2}{a}(\frac{\partial^2}{\partial r^2}\mathrm{Ric}_{rr} + \frac{4}{r}\frac{\partial}{\partial r}\mathrm{Ric}_{rr} + \frac{2}{r^2}\mathrm{Ric}_{rr}) \\
&\quad + r^2 g^{kk}(\frac{\partial^2}{\partial x_k}\mathrm{Ric}_{rr} - 2\Gamma^r_{rk}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rr} - 2(\frac{\partial}{\partial x_k})\mathrm{Ric}_{rr} \\
&\quad - 2\Gamma^r_{rk}(\frac{\partial}{\partial x_k}\mathrm{Ric}_{rr}) + \frac{r}{a}(\frac{\partial}{\partial r}\mathrm{Ric}_{rr} + \frac{2}{r}\mathrm{Ric}_{rr}) \\
&\quad + \frac{r}{a} g^{kk}(g_{kk}(\frac{\partial}{\partial r}\mathrm{Ric}_{rr} + \frac{2}{r}\mathrm{Ric}_{rr}) + \Gamma^p_{kk}\frac{\partial}{\partial x_p}\mathrm{Ric}_{rr} - \Gamma^p_{kk}\Gamma^r_{pr}\mathrm{Ric}_{rr}), \\
\Delta \mathrm{Ric}_{ri} &= 0, \\
\Delta \mathrm{Ric}_{ij} &= \frac{r^2}{a}(\frac{\partial^2}{\partial r^2}\mathrm{Ric}_{ij} + \frac{4}{r}\frac{\partial}{\partial r}\mathrm{Ric}_{ij} + \frac{2}{r^2}\mathrm{Ric}_{ij}) \\
&\quad + r^2 g^{kk}(\frac{\partial^2}{\partial x_k^2}\mathrm{Ric}_{ij} - (\frac{\partial}{\partial x_k}\Gamma^p_{ik})\mathrm{Ric}_{pj} - (\frac{\partial}{\partial x_k}\Gamma^p_{jk})\mathrm{Ric}_{ip} \\
&\quad - \Gamma^p_{ik}(\frac{\partial}{\partial x_k}\mathrm{Ric}_{pj}) - \Gamma^p_{jk}(\frac{\partial}{\partial x_k}\mathrm{Ric}_{ip}) - \Gamma^p_{ki}(\frac{\partial}{\partial x_k}\mathrm{Ric}_{pj}) \\
&\quad - \Gamma^p_{kl}(\frac{\partial}{\partial x_k}\mathrm{Ric}_{ip}) + \Gamma^q_{jk}\Gamma^p_{ik}\mathrm{Ric}_{pq} + \Gamma^q_{kp}\Gamma^p_{ik}\mathrm{Ric}_{qj} \\
&\quad + \Gamma^q_{ki}\Gamma^p_{jk}\mathrm{Ric}_{qp} + \Gamma^q_{kp}\Gamma^p_{jk}\mathrm{Ric}_{iq}) + \frac{r}{a}(\frac{\partial}{\partial r}\mathrm{Ric}_{ij} + \frac{2}{r}\mathrm{Ric}_{ij}) \\
&\quad + \frac{r}{a} g^{kk}(g_{kk}(\frac{\partial}{\partial r}\mathrm{Ric}_{ij} + \frac{2}{r}\mathrm{Ric}_{ij}) + \Gamma^p_{kk}\Gamma^q_{pi}\mathrm{Ric}_{qj} + \Gamma^p_{kk}\Gamma^q_{pj}\mathrm{Ric}_{iq}),
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
\langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{r-}\rangle &= \frac{r^2}{a}(\mathrm{Ric}_{rr})^2, \\
\langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{i-}\rangle &= 0, \\
\langle \mathrm{Ric}_{i-}, \mathrm{Ric}_{j-}\rangle &= r^2 g^{kl}\mathrm{Ric}_{ki}\mathrm{Ric}_{lq}, \\
\langle R_{r-r-}, \mathrm{Ric}_{--}\rangle &= r^4 g^{kl} g^{pq} R_{rkrp}\mathrm{Ric}_{lq}, \\
\langle R_{r-i-}, \mathrm{Ric}_{--}\rangle &= 0, \\
\langle R_{i-j-}, \mathrm{Ric}_{--}\rangle &= \frac{r^4}{a^2} R_{irjr}\mathrm{Ric}_{rr} + r^4 g^{kl} g^{pq} R_{ikjp}\mathrm{Ric}_{lq},
\end{aligned}
\tag{5.21}
$$

for $i, j = 1, \ldots, n$.

We consider $h(t)$ and $\mathrm{Ric}(h(t))$ as a Taylor series in $t-1$ as in (5.13) and (5.14). The Ricci flow equation is given by (5.1). The coefficients of the power series for the Ricci curvature tensor $\mathrm{Ric}(h(t))$ are given by $\mathrm{Ric}_{(j)} = \frac{1}{j!}\frac{\partial^j}{\partial t^j}\mathrm{Ric}(h(t))$ and can be calculated by Theorem 18. We have already calculated $\mathrm{Ric}_{(0)}$ in (5.18). We will only discuss calculation of $\mathrm{Ric}_{(1)}$.

By Theorem 18 we have

$$
\begin{aligned}
\mathrm{Ric}_{(1),rr} &= \Delta(\mathrm{Ric}_{rr}) - 2\langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{r-}\rangle + 2\langle R_{r-r-}, \mathrm{Ric}_{--}\rangle, \\
\mathrm{Ric}_{(1),ri} &= 0, \\
\mathrm{Ric}_{(1),ij} &= \Delta(\mathrm{Ric}_{ij}) - 2\langle \mathrm{Ric}_{i-}, \mathrm{Ric}_{j-}\rangle + 2\langle R_{i-j-}, \mathrm{Ric}_{--}\rangle,
\end{aligned}
\tag{5.22}
$$

where by Ric we mean $\mathrm{Ric}_{(0)} = \mathrm{Ric}(h(1))$. The traces are taken with respect to the metric $h_{(0)}$.

We can calculate $\mathrm{Ric}_{(2)}$ from $\mathrm{Ric}_{(1)}$ again by using Theorem 18.

Next we will show the particular example of a solution in the case when the initial manifold (M, g) is the euclidean space equipped with euclidean metric. By imposing certain restrictions we obtain a unique solution of the Ricci flow problem.

Assuming a and g_{ij} to be independent of the variables $x_1, \ldots, x_n$, all partial derivatives $\frac{\partial a}{\partial x_i}$ and $\frac{\partial g_{ij}}{\partial t}$ are trivial. This means that a and g_{ij} are constant. Respecting the boundary condition (5.7) we will assume that $a = 1$ and $g_{ij} = \delta_{ij}$. We remark that the equation (5.19) is obviously satisfied in the case when M is the euclidean space equipped with euclidean metric. In this case the system simplifies as follows:

$$\begin{aligned} R_{rrir} &= 0, \quad R_{kril} = 0, \\ R_{rijr} &= \frac{1}{r^4}\delta_{ij}, R_{ijkl} = \frac{1}{r^4}(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}), \end{aligned} \tag{5.23}$$

$$\mathrm{Ric}_{rr} = -\frac{n}{r^2}, \quad \mathrm{Ric}_{ri} = 0, \quad \mathrm{Ric}_{ij} = -\frac{n}{r^2}\delta_{ij}. \tag{5.24}$$

$$\begin{aligned} \Delta\mathrm{Ric}_{rr} &= 0, \\ \Delta\mathrm{Ric}_{ri} &= 0, \\ \Delta\mathrm{Ric}_{ij} &= 0. \end{aligned} \tag{5.25}$$

$$\begin{aligned} \langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{r-} \rangle &= \frac{n^2}{r^2}, \\ \langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{i-} \rangle &= 0, \\ \langle \mathrm{Ric}_{i-}, \mathrm{Ric}_{j-} \rangle &= \frac{n^2}{r^2}\delta_{ij}, \\ \langle R_{r-r-}, \mathrm{Ric}_{--} \rangle &= \frac{n^2}{r^2}, \\ \langle R_{r-i-}, \mathrm{Ric}_{--} \rangle &= 0, \\ \langle R_{i-j-}, \mathrm{Ric}_{--} \rangle &= \frac{n^2}{r^2}\delta_{ij}. \end{aligned} \tag{5.26}$$

By Theorem 18 and equations (5.25), (5.26) we have $\frac{\partial}{\partial t}\mathrm{Ric}(h(t)) = 0$. This means that the Ricci flow equation is given in component expansion by

$$\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial t}a(t) &= \frac{2n}{r^2}, \\ \frac{1}{r^2}\frac{\partial}{\partial t}b_i(t) &= 0, \\ \frac{1}{r^2}\frac{\partial t}{\partial t}g_{ij}(t) &= \frac{2n}{r^2}\delta_{ij}, \end{aligned} \tag{5.27}$$

for $i, j = 1, \dots n$, where a and g_{ij} are constant. The unique solution with respect to the boundary condition (5.7) is

$$\begin{aligned} a(t) &= 2n(t-1)+1, \\ b_i(t) &= 0, \\ g_{ij}(t) &= (2n(t-1)+1)\delta_{ij}). \end{aligned} \tag{5.28}$$

We note that this metric degenerates at the time $t = 1 - \frac{1}{2n}$. This means that this solution (5.28) never flows from $t = 0$ to $t = 1$. We may however consider for example in dimension n equal to 3 a solution that flows to $h(1)$ in time $\frac{1}{12}$. We can obtain such a solution by evaluating $h(t)$ at $t = \frac{11}{12}$. We remark that this can be seen also from the fact that in this particular case the initial manifold $(M \times I, h(1))$ may be viewed as the Poincaré half-plane model. This space is Einstein with Einstein constant equal to $-n$. The solution of Ricci flow is simple in this case as we have seen in the example (4.27).

We finish this section by concluding that in the euclidean case $h^{(2k)} = 0$ for all $k \in \mathbb{N}$ in (5.5) , so (5.28) is a solution to the Poincaré-Einstein problem in the euclidean case.

5.3 General case $\mathrm{o}(1)$ of the Ricci flow problem

In the general case, we take

$$h = \frac{1}{r^2}\begin{pmatrix} a & b_i \\ b_i & g_{ij} \end{pmatrix}, \tag{5.29}$$

$$a \equiv a(t, x_1, \dots, x_n) \equiv a(t), \quad b_i \equiv b_i(t, x_1, \dots, x_n) \equiv b_i(t),$$
$$g_{ij} \equiv g_{ij}(t, x_1, \dots, x_n) \equiv g_{ij}(t).$$

for $i, j = 1, \dots, n$.

The inverse metric is given by

$$h^{-1} = \frac{r^2}{\tilde{k}}\begin{pmatrix} 1 & -A^{-1}b \\ -b^T A^{-1} & \tilde{k}A^{-1} + A^{-1}bb^T A^{-1} \end{pmatrix}, \tag{5.30}$$

where $A = (g_{ij})$, $A^{-1} = (g^{ij})$, $b = b_i$, b^T is the transpose of b (as a row vector) and $\tilde{k} = a - b^T A^{-1} b$. To simplify our notation, we will write components of the inverse metric as

$$h^{-1} = \frac{r^2}{\tilde{k}}\begin{pmatrix} 1 & \tilde{h}^{ri} \\ \tilde{h}^{ri} & \tilde{h}^{ij} \end{pmatrix} = r^2 \tilde{h}, \tag{5.31}$$

where in particular $\tilde{h}^{ri} = g^{li} b_l$ and $\tilde{h}^{ij} = \tilde{k} g^{ij} + g^{il} b_l g^{jk} b_k$.

For further simplification, we set

$$
\begin{aligned}
A_{ij} &= \frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i} - \frac{2g_{ij}}{r}, \\
B^k_{ij} &= \tilde{h}^{km}(\frac{\partial g_{mi}}{\partial x_j} + \frac{\partial g_{mj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m}), \\
B^r_{ij} &= \tilde{h}^{rm}(\frac{\partial g_{mi}}{\partial x_j} + \frac{\partial g_{mj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m}), \\
C^k &= \tilde{h}^{km}(\frac{4}{r}b_m + \frac{\partial a}{\partial x_m}), \\
C^r &= \tilde{h}^{rm}(\frac{4}{r}b_m + \frac{\partial a}{\partial x_m}),
\end{aligned} \tag{5.32}
$$

for $i, j = 1, \ldots, n$.

Based on this notation, the Christoffel symbols $\Gamma^k_{ij} \equiv \Gamma^k_{ij}(h)$ for the metric h are given by

$$
\begin{aligned}
\Gamma^r_{rr} &= -\frac{1}{2\tilde{k}}(\frac{2a}{r} + C^r), \\
\Gamma^k_{rr} &= -\frac{1}{2\tilde{k}}(\frac{2a}{r}\tilde{h}^{rk} + C^k) \\
\Gamma^r_{ri} &= \frac{1}{2\tilde{k}}(\frac{\partial a}{\partial x_i} - \tilde{h}^{rm}A_{im}), \\
\Gamma^k_{ri} &= \frac{1}{2\tilde{k}}(\tilde{h}^{rk}\frac{\partial a}{\partial x_i} + \tilde{h}^{km}A_{im}) \\
\Gamma^r_{ij} &= \frac{1}{2\tilde{k}}(A_{ij} + B^r_{ij}), \\
\Gamma^k_{ij} &= \frac{1}{2\tilde{k}}(\tilde{h}^{rk}A_{ij} + B^k_{ij}),
\end{aligned} \tag{5.33}
$$

for $i, j = 1, \ldots, n$.

We proceed directly to calculate the equations for partial derivatives of h using the formula (1.14) for Riemann curvature. The components of the Riemann curvature tensor $R_{ijkl} \equiv R_{ijkl}(h)$ for the metric h are given by

$$
\begin{aligned}
R_{irrj} = \; & \frac{1}{2}(\frac{1}{r^2}\frac{\partial^2 a}{\partial x_i \partial x_j} + \frac{6}{r^4}g_{ij} + \frac{2}{r^3}(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i})) \\
& + \frac{a}{4r^2\tilde{k}^2}[-(\frac{2a}{r} + C^r)(A_{ij} + B^r_{ij}) - (\frac{\partial a}{\partial x_j} + \tilde{h}^{rm}A_{jm})(\frac{\partial a}{\partial x_i} + \tilde{h}^{rm}A_{im})] \\
& + \frac{bp}{4r^2\tilde{k}^2}[-(\frac{2a}{r} + C^r)(\tilde{h}^{rp}A_{ij} + B^p_{ij}) - (\frac{\partial a}{\partial x_j} + \tilde{h}^{rm}A_{jm})(\tilde{h}^{rp}\frac{\partial a}{\partial x_i} \\
& + \tilde{h}^{mp}A_{im})] + \frac{bp}{4r^2\tilde{k}^2}[-(\frac{2\tilde{h}^{rp}a}{r} + C^p)(A_{ij} + B^r_{ij}) - (\tilde{h}^{rp}\frac{\partial a}{\partial x_j} \\
& + \tilde{h}^{mp}A_{mj})(\frac{\partial a}{\partial x_i} + \tilde{h}^{rm}A_{im})] + \frac{g_{pq}}{4r^2\tilde{k}^2}[-(\frac{2\tilde{h}^{rp}a}{r} + C^p)(\tilde{h}^{qr}A_{ij} + B^q_{ij}) \\
& -(\tilde{h}^{rp}\frac{\partial a}{\partial x_j} + \tilde{h}^{mp}A_{mj})(\tilde{h}^{rq}\frac{\partial a}{\partial x_i} + \tilde{h}^{mq}A_{mi})],
\end{aligned} \tag{5.34}
$$

$$
\begin{aligned}
R_{rrir} &= \frac{bp}{4r^2\tilde{k}^2}[-(\frac{\partial a}{\partial x_i}+\tilde{h}^{rm}A_{im})(\frac{2\tilde{h}^{rp}a}{r}+C^p)+(\frac{2a}{r}+C^r)(\tilde{h}^{rp}\frac{\partial a}{\partial x_i} \\
&\quad +\tilde{h}^{pm}A_{im})]+\frac{bp}{4r^2\tilde{k}^2}[-(\tilde{h}^{rp}\frac{\partial a}{\partial x_i}+\tilde{h}^{pm}A_{im})(\frac{2a}{r}+C^r)+(\frac{2\tilde{h}^{rp}a}{r} \\
&\quad +C^p)(\frac{\partial a}{\partial x_i}+\tilde{h}^{rm}A_{im})]+\frac{g_{pq}}{4r^2\tilde{k}^2}[-(\tilde{h}^{rp}\frac{\partial a}{\partial x_i}+\tilde{h}^{pm}A_{im})(\frac{2\tilde{h}^{qr}a}{r} \\
&\quad +C^q)+(\frac{2\tilde{h}^{rp}a}{r}+C^p)(\tilde{h}^{rq}\frac{\partial a}{\partial x_i}+\tilde{h}^{qm}A_{im})],
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
R_{mrik} &= \frac{1}{2}(\frac{1}{r^2}(\frac{\partial^2 b_i}{\partial x_k \partial x_m}-\frac{\partial^2 b_k}{\partial x_m \partial x_i})+\frac{2}{r^3}(\frac{\partial g_{mi}}{\partial x_k}-\frac{\partial g_{mk}}{\partial x_i})) \\
&\quad +\frac{a}{4r^2\tilde{k}^2}[(\frac{\partial a}{\partial x_i}+\tilde{h}^{rl}A_{li})(A_{mk}+B^r_{mk})-(\frac{\partial a}{\partial x_k}+\tilde{h}^{rl}A_{lk})(A_{mi}+B^r_{mi})] \\
&\quad +\frac{b_p}{4r^2\tilde{k}^2}[(\frac{\partial a}{\partial x_i}+\tilde{h}^{rl}A_{li})(\tilde{h}^{rp}A_{mk}+B^p_{mk})-(\frac{\partial a}{\partial x_k}+\tilde{h}^{rl}A_{lk})(\tilde{h}^{rp}A_{mi} \\
&\quad +B^p_{mi})]+\frac{b_p}{4r^2\tilde{k}^2}[(\tilde{h}^{rp}\frac{\partial a}{\partial x_i}+\tilde{h}^{lp}A_{li})(A_{mk}+B^r_{mk})-(\tilde{h}^{rp}\frac{\partial a}{\partial x_k} \\
&\quad +\tilde{h}^{lp}A_{kl})(A_{mi}+B^r_{mi})]+\frac{g_{pq}}{4r^2\tilde{k}^2}[(\tilde{h}^{rp}\frac{\partial a}{\partial x_i}+\tilde{h}^{lp}A_{li})(\tilde{h}^{rq}A_{mk}+B^q_{mk}) \\
&\quad -(\tilde{h}^{rp}\frac{\partial a}{\partial x_k}+\tilde{h}^{lp}A_{lk})(\tilde{h}^{qr}A_{mi}+B^q_{mi})],
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
R_{rijr} &= \frac{1}{2}(\frac{6}{r^4}g_{ij}+\frac{2}{r^3}(\frac{\partial b_i}{\partial x_j}+\frac{\partial b_j}{\partial x_i})+\frac{1}{r^2}\frac{\partial^2 a}{\partial x_i \partial x_j}) \\
&\quad +\frac{a}{4r^2\tilde{k}^2}[-(\frac{2a}{r}+C^r)(A_{ij}+B^r_{ij})-(\frac{\partial a}{\partial x_j}+\tilde{h}^{rl}A_{jl})(\frac{\partial a}{\partial x_i}+\tilde{h}^{rl}A_{li})] \\
&\quad +\frac{b_p}{4r^2\tilde{k}^2}[-(\frac{2a}{r}+C^r)(\tilde{h}^{rp}A_{ij}+B^p_{ij})-(\frac{\partial a}{\partial x_j}+\tilde{h}^{rl}A_{jl})(\tilde{h}^{rp}\frac{\partial a}{\partial x_i} \\
&\quad +\tilde{h}^{pl}A_{li})]+\frac{b_p}{4r^2\tilde{k}^2}[-(\frac{2\tilde{h}^{rp}a}{r}+C^p)(A_{ij}+B^r_{ij})-(\tilde{h}^{rp}\frac{\partial a}{\partial x_j} \\
&\quad +\tilde{h}^{lp}A_{jl})(\frac{\partial a}{\partial x_i}+\tilde{h}^{rl}A_{li})]+\frac{g_{pq}}{4r^2\tilde{k}^2}[-(\frac{2\tilde{h}^{rp}a}{r}+C^p)(\tilde{h}^{rq}A_{ij} \\
&\quad +B^q_{ij})-(\tilde{h}^{rp}\frac{\partial a}{\partial x_j}+\tilde{h}^{lp}A_{jl})(\tilde{h}^{rq}\frac{\partial a}{\partial x_i}+\tilde{h}^{lq}A_{li})],
\end{aligned}
\tag{5.37}
$$

$$
\begin{aligned}
R_{mijk} &= \frac{1}{2r^2}(\frac{\partial^2 g_{ij}}{\partial x_k \partial x_m} + \frac{\partial^2 g_{mk}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ik}}{\partial x_m \partial x_j} - \frac{\partial^2 g_{mj}}{\partial x_i \partial x_k}) \\
&+ \frac{a}{4r^2\tilde{k}^2}[(A_{ij} + B^r_{ij})(A_{mk} + B^r_{mk}) - (A_{ik} + B^r_{ik})(A_{mj} + B^r_{mj})] \\
&+ \frac{b_p}{4r^2\tilde{k}^2}[(A_{ij} + B^r_{ij})(\tilde{h}^{rp}A_{mk} + B^p_{mk}) - (A_{ik} + B^r_{ik})(\tilde{h}^{rp}A_{mj} + B^p_{mj})] \\
&+ \frac{b_p}{4r^2\tilde{k}^2}[(\tilde{h}^{rp}A_{ij} + B^p_{ij})(A_{mk} + B^r_{mk}) - (\tilde{h}^{rp}A_{ik} + B^p_{ik})(A_{mj} + B^r_{mj})] \\
&+ \frac{g_{pq}}{4r^2\tilde{k}^2}[(\tilde{h}^{rp}A_{ij} + B^p_{ij})(\tilde{h}^{rq}A_{mk} + B^q_{mk}) - (\tilde{h}^{rp}A_{ik} + B^p_{ik})(\tilde{h}^{rq}A_{mj} \\
&+ B^q_{mj})].
\end{aligned} \tag{5.38}
$$

Components of the Ricci curvature tensor are given by

$$
\begin{aligned}
\mathrm{Ric}_{rr} &= r^2\tilde{h}^{kl}R_{rkrl}, \\
\mathrm{Ric}_{ri} &= r^2(\frac{1}{a}R_{rrir} + \tilde{h}^{kl}R_{rkil}), \\
\mathrm{Ric}_{ij} &= r^2(\frac{1}{a}R_{irjr} + \tilde{h}^{kl}R_{ikjl}).
\end{aligned} \tag{5.39}
$$

Next we use Theorem 18 to set up the Ricci flow equation.

$$
\begin{aligned}
\Delta \mathrm{Ric}_{rr} \quad = \quad & \frac{r^2}{\tilde{k}}\big(\frac{\partial^2}{\partial r^2}\mathrm{Ric}_{rr} - 2(\frac{\partial}{\partial r}\Gamma^r_{rr})\mathrm{Ric}_{rr} - 2\Gamma^r_{rr}(\frac{\partial}{\partial r}\mathrm{Ric}_{rr}) \\
& -2(\frac{\partial}{\partial r}\Gamma^p_{rr})\mathrm{Ric}_{rp} - 2\Gamma^p_{rr}(\frac{\partial}{\partial r}\mathrm{Ric}_{rp}) - 2\Gamma^r_{rr}\frac{\partial}{\partial r}\mathrm{Ric}_{rr} \\
& -2\Gamma^p_{rr}\frac{\partial}{\partial r}\mathrm{Ric}_{rp} + 4(\Gamma^r_{rr})^2\mathrm{Ric}_{rr} + 4\Gamma^r_{rr}\Gamma^p_{rr}\mathrm{Ric}_{rp} \\
& +2\Gamma^p_{rr}\Gamma^r_{rr}\mathrm{Ric}_{rp} + 2\Gamma^q_{rr}\Gamma^p_{rr}\mathrm{Ric}_{qp} + 2\Gamma^p_{rr}\Gamma^r_{pr}\mathrm{Ric}_{pr} + 2\Gamma^p_{rr}\Gamma^q_{pr}\mathrm{Ric}_{pq}\big) \\
& +\frac{r^2}{\tilde{k}}\tilde{h}^{kk}\big(\frac{\partial^2}{\partial x_k^2}\mathrm{Ric}_{rr} - 2(\frac{\partial}{\partial x_k}\Gamma^r_{rk})\mathrm{Ric}_{rr} - 2\Gamma^r_{rk}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rr} \\
& -2(\frac{\partial}{\partial x_k}\Gamma^p_{rk})\mathrm{Ric}_{rp} - 2\Gamma^p_{rk}(\frac{\partial}{\partial x_k}\mathrm{Ric}_{rp}) - 2\Gamma^r_{rk}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rr} \\
& -2\Gamma^p_{rk}(\frac{\partial}{\partial x_k}\mathrm{Ric}_{rp}) + 4(\Gamma^r_{rk})^2\mathrm{Ric}_{rr} + 4\Gamma^r_{rk}\Gamma^p_{rk}\mathrm{Ric}_{rp} \\
& +2\Gamma^p_{rk}\Gamma^r_{rk}\mathrm{Ric}_{rp} + 2\Gamma^p_{rk}\Gamma^q_{rk}\mathrm{Ric}_{qp} + 2\Gamma^p_{rk}\Gamma^r_{pk}\mathrm{Ric}_{rr} \\
& +2\Gamma^p_{rk}\Gamma^q_{pk}\mathrm{Ric}_{qr}\big) \\
& -\frac{r^2}{\tilde{k}}(\Gamma^r_{rr}\big(\frac{\partial}{\partial r}\mathrm{Ric}_{rr} - 2\Gamma^r_{rr}\mathrm{Ric}_{rr} - 2\Gamma^p_{rr}\mathrm{Ric}_{rp}\big) \\
& +\Gamma^p_{rr}\big(\frac{\partial}{\partial x_p}\mathrm{Ric}_{rr} - 2\Gamma^r_{rp}\mathrm{Ric}_{rr} - 2\Gamma^q_{rp}\mathrm{Ric}_{rq})\big) \\
& -\frac{r^2}{\tilde{k}}\tilde{h}^{kk}\big(\Gamma^r_{kk}(\frac{\partial}{\partial r}\mathrm{Ric}_{rr} - 2\Gamma^r_{rr}\mathrm{Ric}_{rr} - 2\Gamma^p_{rr}\mathrm{Ric}_{rp}) \\
& +\Gamma^p_{kk}(\frac{\partial}{\partial x_p}\mathrm{Ric}_{rr} - 2\Gamma^r_{rp}\mathrm{Ric}_{rr} - 2\Gamma^q_{rp}\mathrm{Ric}_{qr})\big),
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
\Delta \mathrm{Ric}_{ri} \;=\; & \frac{r^2}{\tilde{k}}\big(\frac{\partial^2}{\partial r^2}\mathrm{Ric}_{ri} - (\frac{\partial}{\partial r}\Gamma^r_{rr})\mathrm{Ric}_{ri} - \Gamma^r_{rr}\frac{\partial}{\partial r}\mathrm{Ric}_{ri} \\
& -\frac{\partial}{\partial r}\Gamma^r_{ri}\mathrm{Ric}_{rr} - \Gamma^r_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{rr} - (\frac{\partial}{\partial r}\Gamma^p_{ri})\mathrm{Ric}_{rp} - \Gamma^p_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{rp} \\
& -\Gamma^r_{rr}\frac{\partial}{\partial r}\mathrm{Ric}_{ro} - \Gamma^p_{rr}\frac{\partial}{\partial r}\mathrm{Ric}_{pi} - \Gamma^r_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{rr} - \Gamma^p_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{pr} \\
& +(\Gamma^r_{rr})^2\mathrm{Ric}_{ri} + \Gamma^p_{rr}\Gamma^r_{rr}\mathrm{Ric}_{pi} + \Gamma^r_{rr}\Gamma^r_{ri}\mathrm{Ric}_{rr} + \Gamma^r_{rr}\Gamma^p_{ri}\mathrm{Ric}_{pr} \\
& +\Gamma^p_{ri}\Gamma^r_{rr}\mathrm{Ric}_{rp} + \Gamma^p_{ri}\Gamma^q_{rr}\mathrm{Ric}_{qp} + \Gamma^p_{ri}\Gamma^r_{pr}\mathrm{Ric}_{rr} + \Gamma^p_{ri}\Gamma^q_{pr}\mathrm{Ric}_{rq}\big) \\
& +\frac{r^2}{\tilde{k}}\tilde{h}^{kk}\big(\frac{\partial^2}{\partial x_k^2}\mathrm{Ric}_r i - (\frac{\partial}{\partial x_k}\Gamma^r_{kr})\mathrm{Ric}_{ri} - \Gamma^r_{kr}\frac{\partial}{\partial x_k}\mathrm{Ric}_{ri} \\
& -(\frac{\partial}{\partial x_k}\Gamma^p_{kr})\mathrm{Ric}_{pi} - \Gamma^p_{kr}\frac{\partial}{\partial x_k}\mathrm{Ric}_{pi} - (\frac{\partial}{\partial x_k}\Gamma^r_{ki})\mathrm{Ric}_{rr} \\
& -\Gamma^r_{ki}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rr} - (\frac{\partial}{\partial x_k}\Gamma^p_{ki})\mathrm{Ric}_{rp} - \Gamma^p_{ki}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rp} \\
& -\Gamma^r_{rk}\frac{\partial}{\partial x_k}\mathrm{Ric}_{ri} - \Gamma^p_{rk}\frac{\partial}{\partial x_k}\mathrm{Ric}_{pi} - \Gamma^r_{ki}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rr} \\
& -\Gamma^p_{ki}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rp} + (\Gamma^r_{kr})^2\mathrm{Ric}_{ri} + \Gamma^r_{kr}\Gamma^p_{kr}\mathrm{Ric}_{pi} + \Gamma^r_{kr}\Gamma^r_{ki}\mathrm{Ric}_{rr} \\
& +\Gamma^r_{kr}\Gamma^q_{ki}\mathrm{Ric}_{rq} + \Gamma^p_{kr}\Gamma^r_{kp}\mathrm{Ric}_{ri} + \Gamma^p_{kr}\Gamma^q_{kp}\mathrm{Ric}_{qi} + \Gamma^p_{kr}\Gamma^p_{kr}\mathrm{Ric}_{pi} \\
& +\Gamma^p_{kr}\Gamma^q_{ki}\mathrm{Ric}_{pq} + \Gamma^r_{ki}\Gamma^r_{kr}\mathrm{Ric}_{rr} + \Gamma ki^r\Gamma^p_{kr}\mathrm{Ric}_{rp} + \Gamma ki^r\Gamma^r_{kr}\mathrm{Ric}_{rr} \\
& +\Gamma^r_{ki}\Gamma^p_{kr}\mathrm{Ric}_{rp} + \Gamma^p_{ki}\Gamma^r_{kr}\mathrm{Ric}_{rp} + \Gamma^p_{ki}\Gamma^q_{kr}\mathrm{Ric}_{qp} + \Gamma^p_{ki}\Gamma^r_{kp}\mathrm{Ric}_{rr} \\
& +\Gamma^p_{ki}\Gamma^q_{kp}\mathrm{Ric}_{rq}\big) \\
& -\frac{r^2}{\tilde{k}}\big(\Gamma^r_{rr}(\frac{\partial}{\partial r}\mathrm{Ric}_{ri} - \Gamma^r_{rr}\mathrm{Ric}_{ri} - \Gamma^p_{rr}\mathrm{Ric}_{pi} - \Gamma^r_{ri}\mathrm{Ric}_{rr} - \Gamma^p_{ri}\mathrm{Ric}_{rp}) \\
& +\Gamma^p_{rr}(\frac{\partial}{\partial x_p}\mathrm{Ric}_{ri} - \Gamma^r_{pr}\mathrm{Ric}_{ri} - \Gamma^q_{pr}\mathrm{Ric}_{qi} - \Gamma^r_{pi}R_{rr} - \Gamma^q_{pi}\mathrm{Ric}_{rq})\big) \\
& -\frac{r^2}{\tilde{k}}\tilde{h}^{kk}\big(\Gamma^r_{kk}(\frac{\partial}{\partial r}\mathrm{Ric}_{ri} - \Gamma^r_{rr}\mathrm{Ric}_{ri} - \Gamma^p_{rr}\mathrm{Ric}_{pi} - \Gamma^r_{ri}\mathrm{Ric}_{rr} - \Gamma^p_{ri}\mathrm{Ric}_{pr}) \\
& +\Gamma^r_{kk}(\frac{\partial}{\partial x_p}\mathrm{Ric}_{ri} - \Gamma^r_{pr}\mathrm{Ric}_{ri} - \Gamma^q_{pr}\mathrm{Ric}_{qi} - \Gamma^r_{pi}\mathrm{Ric}_{rr} - \Gamma^q_{pi}\mathrm{Ric}_{rq})\big),
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
\Delta \mathrm{Ric}_{ij} \;=\; & \frac{r^2}{\tilde{k}}\big(\frac{\partial^2}{\partial r^2}\mathrm{Ric}_{ij} - (\frac{\partial}{\partial r}\Gamma^r_{ri})\mathrm{Ric}_{rj} - \Gamma^r_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{rj} - (\frac{\partial}{\partial r}\Gamma^p_{ri})\mathrm{Ric}_{pj} \\
& -\Gamma^p_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{pj} - (\frac{\partial}{\partial t}\Gamma^r_{rj})\mathrm{Ric}_{ir} - \Gamma^r_{rj}\frac{\partial}{\partial t}\mathrm{Ric}_{ir} - (\frac{\partial}{\partial r}\Gamma^p_{rj})\mathrm{Ric}_{ip} \\
& -\Gamma^p_{rj}\frac{\partial}{\partial r}\mathrm{Ric}_{ip} - \Gamma^r_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{rj} - \Gamma^p_{ri}\frac{\partial}{\partial r}\mathrm{Ric}_{pj} - \Gamma^r_{rj}\frac{\partial}{\partial r}\mathrm{Ric}_{ir} \\
& -\Gamma^p_{rj}\frac{\partial}{\partial r}\mathrm{Ric}_{ip} + \Gamma^r_{ri}\Gamma^r_{rr}\mathrm{Ric}_{rj} + \Gamma^r_{ri}\Gamma^p_{rr}\mathrm{Ric}_{pj} + \Gamma^r_{ri}\Gamma^r_{rj}\mathrm{Ric}_{rr} \\
& +\Gamma^r_{ri}\Gamma^p_{rj}\mathrm{Ric}_{rp} + \Gamma^p_{ri}\Gamma^r_{rp}\mathrm{Ric}_{rj} + \Gamma^p_{ri}\Gamma^q_{rp}\mathrm{Ric}_{qj} + \Gamma^p_{ri}\Gamma^r_{rj}\mathrm{Ric}_{pr} \\
& +\Gamma^p_{ri}\Gamma^q_{rj}\mathrm{Ric}_{pq} + \Gamma^r_{rj}\Gamma^r_{ri}\mathrm{Ric}_{rr} + \Gamma^r_{rj}\Gamma^p_{ri}\mathrm{Ric}_{pr} + \Gamma^r_{rj}\Gamma^r_{rr}\mathrm{Ric}_{ir} \\
& +\Gamma^r_{rj}\Gamma^p_{rr}\mathrm{Ric}_{ip} + \Gamma^p_{rj}\Gamma^r_{ri}\mathrm{Ric}_{rp} + \Gamma^p_{rj}\Gamma^q_{ri}\mathrm{Ric}_{qp} + \Gamma^p_{rj}\Gamma^r_{rp}\mathrm{Ric}_{ir} \\
& +\Gamma^p_{rj}\Gamma^q_{rp}\mathrm{Ric}_{iq}\big) \\
& +\frac{r^2}{\tilde{k}}\tilde{h}^{kk}\big(\frac{\partial^2}{\partial x_k^2}\mathrm{Ric}_{ij} - (\frac{\partial}{\partial x_k}\Gamma^r_{ki})\mathrm{Ric}_{rj} - \Gamma^r_{ki}\frac{\partial}{\partial t}\mathrm{Ric}_{rj} \\
& -(\frac{\partial}{\partial x_k}\Gamma^p_{ki})\mathrm{Ric}_{pj} + \Gamma^p_{ki}\frac{\partial}{\partial x_k}\mathrm{Ric}_{pj} - (\frac{\partial}{\partial x_k}\Gamma^r_{kj})\mathrm{Ric}_{ir} \\
& -\Gamma^r_{kj}\frac{\partial}{\partial x_k}\mathrm{Ric}_{ir} - (\frac{\partial}{\partial x_k}\Gamma^p_{kj})\mathrm{Ric}_{ip} - \Gamma^p_{kj}\frac{\partial}{\partial x_k}\mathrm{Ric}_{ip} \\
& -\Gamma^r_{ki}\frac{\partial}{\partial x_k}\mathrm{Ric}_{rj} - \Gamma^p_{ki}\frac{\partial}{\partial x_k}\mathrm{Ric}_{pj} - \Gamma^r_{kj}\frac{\partial}{\partial x_k}\mathrm{Ric}_{ir} \\
& -\Gamma^p_{kj}\frac{\partial}{\partial x_k}\mathrm{Ric}_{ip} + \Gamma^r_{ki}\Gamma^r_{kr}\mathrm{Ric}_{rj} + \Gamma^r_{ki}\Gamma^p_{kr}\mathrm{Ric}_{pj} + \Gamma^r_{ki}\Gamma^r_{kj}\mathrm{Ric}_{rr} \\
& +\Gamma^r_{ki}\Gamma^p_{kj}\mathrm{Ric}_{rp} + \Gamma^p_{ki}\Gamma^r_{kp}\mathrm{Ric}_{rj} + \Gamma^p_{ki}\Gamma^q_{kp}\mathrm{Ric}_{qj} + \Gamma^p_{ki}\Gamma^r_{kj}\mathrm{Ric}_{pr} \\
& +\Gamma^p_{ki}\Gamma^q_{kj}\mathrm{Ric}_{pq} + \Gamma^r_{kj}\Gamma^r_{ki}\mathrm{Ric}_{rr} + \Gamma^r_{kj}\Gamma^p_{ki}\mathrm{Ric}_{rp} + \Gamma^r_{kj}\Gamma^r_{kr}\mathrm{Ric}_{ir} \\
& +\Gamma^r_{kj}\Gamma^q_{kr}\mathrm{Ric}_{iq} + \Gamma^p_{kj}\Gamma^r_{ki}\mathrm{Ric}_{rp} + \Gamma^p_{kj}\Gamma^q_{ki}\mathrm{Ric}_{qp} + \Gamma^p_{kj}\Gamma^r_{kp}\mathrm{Ric}_{ir} \\
& +\Gamma^p_{kj}\Gamma^q_{kp}\mathrm{Ric}_{iq}\big) \\
& +\frac{r^2}{\tilde{k}}\big(\Gamma^r_{rr}(\frac{\partial}{\partial r}\mathrm{Ric}_{ij} - \Gamma^r_{ri}\mathrm{Ric}_{rj} - \Gamma^p_{ri}\mathrm{Ric}_{pj} - \Gamma^r_{rj}R_{ir} \\
& -\Gamma^p_{rj}\mathrm{Ric}_{ip}) + \Gamma^p_{rr}(\frac{\partial}{\partial x_p}\mathrm{Ric}_{ij} - \Gamma^r_{pi}\mathrm{Ric}_{rj} - \Gamma^q_{pi}\mathrm{Ric}_{qj} \\
& -\Gamma^r_{pj}\mathrm{Ric}_{ir} - \Gamma^q_{pj}\mathrm{Ric}_{iq})\big) \\
& +\frac{r^2}{\tilde{k}}\tilde{h}^{kk}\big(\Gamma^r_{kk}(\frac{\partial}{\partial r}\mathrm{Ric}_{ij} - \Gamma^r_{ri}\mathrm{Ric}_{rj} - \Gamma^p_{ri}\mathrm{Ric}_{pj} - \Gamma^r_{rj}\mathrm{Ric}_{ir} \\
& -\Gamma^q_{rj}\mathrm{Ric}_{iq}) + \Gamma^p_{kk}(\frac{\partial}{\partial x_p}\mathrm{Ric}_{ij} - \Gamma^r_{pi}\mathrm{Ric}_{rj} - \Gamma^q_{pi}\mathrm{Ric}_{qj} \\
& -\Gamma^r_{pj}\mathrm{Ric}_{ir} - \Gamma^q_{pj}\mathrm{Ric}_{iq}))\big), \qquad (5.42)
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{r-}\rangle &= \frac{r^2}{\tilde{k}}((\mathrm{Ric}_{rr})^2 + 2\tilde{h}^{ri}\mathrm{Ric}_{rr}\mathrm{Ric}_{ri} + \tilde{h}^{ij}\mathrm{Ric}_{ri}\mathrm{Ric}_{rj}),\\
\langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{i-}\rangle &= \frac{r^2}{\tilde{k}}(\mathrm{Ric}_{rr}\mathrm{Ric}_{ri} + \tilde{h}^{rj}\mathrm{Ric}_{rr}\mathrm{Ric}_{ij} + \tilde{h}^{rj}\mathrm{Ric}_{rj}\mathrm{Ric}_{ri} +\\
&\quad \tilde{h}^{kl}\mathrm{Ric}_{rk}\mathrm{Ric}_{li}),\\
\langle \mathrm{Ric}_{i-}, \mathrm{Ric}_{j-}\rangle &= \frac{r^2}{\tilde{k}}(\mathrm{Ric}_{ri}\mathrm{Ric}_{rj} + \tilde{h}^{rk}\mathrm{Ric}_{ri}\mathrm{Ric}_{kj} + \tilde{h}^{rk}\mathrm{Ric}_{ki}\mathrm{Ric}_{rj}\\
&\quad +\tilde{h}^{kl}\mathrm{Ric}_{ki}\mathrm{Ric}_{lj}),\\
\langle R_{r-r-}, \mathrm{Ric}_{--}\rangle &= \frac{r^4}{\tilde{k}^2}(2\tilde{h}^{ri}R_{rrri}\mathrm{Ric}_{rr} + 2\tilde{h}^{ri}\tilde{h}^{rj}R_{rirr}\mathrm{Ric}_{rj} + \tilde{h}^{ij}R_{rirj}\mathrm{Ric}_{rr}\\
&\quad +\tilde{h}^{ri}\tilde{h}^{kl}R_{rrrk}\mathrm{Ric}_{li} + \tilde{h}^{ri}\tilde{h}^{kl}R_{rirk}\mathrm{Ric}_{rl} + \tilde{h}^{kl}\tilde{h}^{pq}R_{rkrp}\mathrm{Ric}_{lq}),\\
\langle R_{r-i-}, \mathrm{Ric}_{--}\rangle &= \frac{r^4}{\tilde{k}^2}(R_{rrir}\mathrm{Ric}_{rr} + 2\tilde{h}^{rk}R_{rrir}\mathrm{Ric}_{rk} + \tilde{h}^{rk}R_{rrik}\mathrm{Ric}_{rr}\\
&\quad +\tilde{h}^{rk}R_{rkir}\mathrm{Ric}_{rr} + \tilde{h}^{rk}\tilde{h}^{rl}R_{rkil}\mathrm{Ric}_{rr} + \tilde{h}^{rk}\tilde{h}^{rl}R_{rrik}\mathrm{Ric}_{rl}\\
&\quad +\tilde{h}^{rk}\tilde{h}^{rl}R_{rkir}\mathrm{Ric}_{rl} + \tilde{h}^{rk}\tilde{h}^{rl}R_{rrir}\mathrm{Ric}_{kl} + \tilde{h}^{kl}R_{rkir}\mathrm{Ric}_{lr}\\
&\quad +\tilde{h}^{kl}R_{rrik}\mathrm{Ric}_{rl} + \tilde{h}^{pq}\tilde{h}^{rk}R_{rpik}\mathrm{Ric}_{rq} + \tilde{h}^{pq}\tilde{h}^{rk}R_{rrip}\mathrm{Ric}_{qk}\\
&\quad +\tilde{h}^{pq}\tilde{h}^{rk}R_{rpir}\mathrm{Ric}_{qr} + \tilde{h}^{pq}\tilde{h}^{cd}R_{rprc}\mathrm{Ric}_{qd}),\\
\langle R_{i-j-}, \mathrm{Ric}_{--}\rangle &= \frac{r^4}{\tilde{k}^2}(R_{irjr}\mathrm{Ric}_{rr} + \tilde{h}^{rk}R_{ikjr}\mathrm{Ric}_{rr} + \tilde{h}^{rk}R_{irjk}\mathrm{Ric}_{rr}\\
&\quad +2\tilde{h}^{rk}R_{irjr}\mathrm{Ric}_{rk} + \tilde{h}^{rk}h^{rl}R_{ikjl}\mathrm{Ric}_{rr} + \tilde{h}^{rk}\tilde{h}^{rl}R_{irjk}\mathrm{Ric}_{rl}\\
&\quad +\tilde{h}^{rk}\tilde{h}^{rl}R_{ikjr}\mathrm{Ric}_{rl} + \tilde{h}^{rl}\tilde{h}^{rl}R_{irjr}\mathrm{Ric}_{kl} + \tilde{h}^{kl}R_{ikjr}\mathrm{Ric}_{rl}\\
&\quad +\tilde{h}^{kl}R_{irjk}\mathrm{Ric}_{rl} + \tilde{h}^{pq}\tilde{h}^{rk}R_{ipjr}\mathrm{Ric}_{qk} + \tilde{h}^{pq}\tilde{h}^{rk}R_{ipjk}\mathrm{Ric}_{qr}\\
&\quad +\tilde{h}^{pq}\tilde{h}^{rk}R_{irjp}\mathrm{Ric}_{qk} + \tilde{h}^{pq}\tilde{h}^{rk}R_{ipjk}\mathrm{Ric}_{rq} + \tilde{h}^{kl}\tilde{h}^{pq}R_{ipjk}\mathrm{Ric}_{ql}).
\end{aligned}
\tag{5.43}
$$

We proceed exactly as in the 5.2. We again consider $h(t)$ and $\mathrm{Ric}(h(t))$ as Taylor series in $t-1$ as in (5.13) and (5.14).

The Ricci flow equation is given by (5.1).

The coefficients of the Taylor series equation can be calculated from $\mathrm{Ric}_{(j)} = \frac{1}{j!}\frac{\partial^j}{\partial t^j}\mathrm{Ric}(h(t))$ and from Theorem 18. We will again only show calculation of the linear coefficients $\mathrm{Ric}_{(1)}$ of the Ricci flow equation in power series in t.

By Theorem 18 we have

$$
\begin{aligned}
\mathrm{Ric}_{(1),rr} &= \Delta(\mathrm{Ric}_{rr}) - 2\langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{r-}\rangle + 2\langle R_{r-r-}, \mathrm{Ric}_{--}\rangle,\\
\mathrm{Ric}_{(1),ri} &= \Delta(\mathrm{Ric}_{ri}) - 2\langle \mathrm{Ric}_{r-}, \mathrm{Ric}_{i-}\rangle + 2\langle R_{r-i-}, \mathrm{Ric}_{--}\rangle,\\
\mathrm{Ric}_{(1),ij} &= \Delta(\mathrm{Ric}_{ij}) - 2\langle \mathrm{Ric}_{i-}, \mathrm{Ric}_{j-}\rangle + 2\langle R_{i-j-}, \mathrm{Ric}_{--}\rangle,
\end{aligned}
\tag{5.44}
$$

where Ric means $\mathrm{Ric}_{(0)}$ and the traces are taken with respect to $h_{(0)}$.

5.4 Reduced system for general case o(1) of Ricci flow problem

The Ricci flow equation (4.1) written in terms of Theorem 18 depends on the Riemann curvature tensor and Ricci curvature tensor of the metric h. By imposing restrictions on the solution of the Ricci flow equation, we simplify the formulas for the Riemann curvature tensor and Ricci curvature tensor of h. This leads to a simplification of the Ricci flow equation.

We assume that $\tilde{h}^{ri} = 0$ for $i = 1, \ldots, n$. This leads to $\tilde{h}^{ij} = \tilde{k}g^{ij}$ and to $C^r = 0$, $B^r_{ij} = 0$ for $i, j = 1, \ldots, n$. Also $C^k = \tilde{k}g^{km}\frac{\partial a}{\partial x_m}$ and $\tilde{h}^{ri} = 0$ for $i = 1, \ldots, n$ implies $\tilde{k} = a$. The Riemann curvature tensor of the metric h reduces to

$$
\begin{aligned}
R_{irrj} &= \frac{1}{2}(\frac{1}{r^2}\frac{\partial^2 a}{\partial x_i \partial x_j} + \frac{6}{r^4}g_{ij} + \frac{2}{r^3}(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i})) \\
&\quad -\frac{1}{4r^2 a}[\frac{2a}{r}A_{ij} + \frac{\partial a}{\partial x_j}\frac{\partial a}{\partial x_i}] \\
&\quad -\frac{g_{pq}}{4ar^2}(g^{pm}\frac{\partial a}{\partial x_m}B^q_{ij} + ag^{mp}A_{mj}g^{lq}A_{li}),
\end{aligned} \tag{5.45}
$$

$$
R_{rrir} = \frac{g_{pq}}{4ar^2}(g^{pm}\frac{\partial a}{\partial x_m}g^{ql}A_{li} - g^{qm}\frac{\partial a}{\partial x_m}g^{pl}A_{li}), \tag{5.46}
$$

$$
\begin{aligned}
R_{mrik} &= \frac{1}{2}(\frac{1}{r^2}(\frac{\partial^2 b_i}{\partial x_k \partial x_m} - \frac{\partial^2 b_k}{\partial x_m \partial x_i}) + \frac{2}{r^3}(\frac{\partial g_{mi}}{\partial x_k} - \frac{\partial g_{mk}}{\partial x_i})) \\
&\quad +\frac{1}{4r^2 a}(\frac{\partial a}{\partial x_i}A_{mk} - \frac{\partial a}{\partial x_k}A_{mi}) + \frac{g_{pq}}{4ar^2}(g^{lp}A_{li}B^q_{mk} - g^{lp}A_{lk}B^q_{mi}),
\end{aligned} \tag{5.47}
$$

$$
\begin{aligned}
R_{rijr} &= \frac{1}{2}(\frac{6}{r^4}g_{ij} + \frac{2}{r^3}(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i}) + \frac{1}{r^2}\frac{\partial^2 a}{\partial x_i \partial x_j}) \\
&\quad -\frac{1}{4r^2 a}(\frac{2a}{r}A_{ij} + \frac{\partial a}{\partial x_i}\frac{\partial a}{\partial x_j}) \\
&\quad -\frac{g_{pq}}{4r^2 a}(g^{pm}\frac{\partial a}{\partial x_m}B^q_{ij} + ag^{pl}A_{lj}g^{qm}A_{mi}),
\end{aligned} \tag{5.48}
$$

$$
\begin{aligned}
R_{mijk} &= \frac{1}{2r^2}(\frac{\partial^2 g_{ij}}{\partial x_k \partial x_m} + \frac{\partial^2 g_{mk}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ik}}{\partial x_m \partial x_j} - \frac{\partial^2 g_{mj}}{\partial x_i \partial x_k}) \\
&\quad +\frac{1}{4r^2 a}(A_{ij}A_{mk} - A_{ik}A_{mj}) \\
&\quad +\frac{g_{pq}}{4r^2 a^2}(B^p_{ij}B^q_{mk} - B^p_{ik}B^q_{mj}).
\end{aligned} \tag{5.49}
$$

for $i, j = 1, \ldots, n$.

The Ricci curvature tensor is given by

$$
\begin{aligned}
\mathrm{Ric}_{rr} &= r^2\tilde{h}^{kl}R_{rkrl},\\
\mathrm{Ric}_{ri} &= r^2(\frac{1}{a}R_{rrir} + \tilde{h}^{kl}R_{rkil}),\\
\mathrm{Ric}_{ij} &= r^2(\frac{1}{a}R_{irjr} + \tilde{h}^{kl}R_{ikjl}),
\end{aligned} \tag{5.50}
$$

for $i, j = 1, \ldots, n$.

We will consider the particular case when the initial manifold (M, g) is the euclidean space with euclidean metric. We obtain a unique solution by imposing additional conditions on the solution.

Let us assume that a, b_i and g_{ij} are independent of the variables $x_1, \ldots, x_n$. Then all partial derivatives $\frac{\partial a}{\partial x_k}$, $\frac{\partial b_i}{\partial x_k}$ and $\frac{\partial g_{ij}}{\partial x_k}$ for $i, j, k = 1, \ldots, n$ are trivial. This means that a, b_i, g_{ij} are constant for $i, j = 1, \ldots, n$. Respecting the boundary condition (5.7) we will assume that $a = 1$, $b_i = 0$ and $g_{ij} = \delta_{ij}$ for $i, j = 1, \ldots, n$. This already implies $\tilde{h}^{ri} = g^{ik}b_k = 0$ for $i = 1, \ldots, n$, $\tilde{k} = 1$ and $\tilde{h}^{ij} = g^{ij}$ for $i, j = 1, \ldots, n$. Then the components of the Riemann curvature for the metric h reduce

$$
\begin{aligned}
R_{rrir} &= 0, \quad R_{kril} = 0,\\
R_{rijr} &= \frac{1}{r^4}, R_{ijkl} = \frac{1}{r^4}(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}),
\end{aligned} \tag{5.51}
$$

for $i, j = 1, \ldots, n$.

Components of the Ricci curvature tensor for the metric h are equal to

$$
\mathrm{Ric}_{rr} = -\frac{n}{r^2}, \quad \mathrm{Ric}_{ri} = 0, \quad \mathrm{Ric}_{ij} = -\frac{n}{r^2}\delta_{ij}, \tag{5.52}
$$

for $i, j = 1, \ldots, n$.

We finish this section by remarking that the Ricci flow equation is exactly the same as in Section 5.2, so the solutions coincide. The solution of (5.1) is given by (5.28)

5.5 Another possibility for the Ricci flow problem

Let (M, g) be a Riemannian manifold, and consider the Poincaré-Einstein metric h_+ of (M, g) on $M \times I$ such that the obstruction tensor $\mathcal{O}$ is zero. We will again label the coordinate on I by r.

We may now consider the same setting as for the Ricci flow problem formulated in previous sections, with one exception that the flow coordinate t is not an additional one but is given by r. This means that when reversing the Ricci flow problem, we ask for a symmetric (2,0) tensor T_{ij} such that

$$
\frac{\partial}{\partial r}h_r(r, x_1, \ldots, x_n) = T(r, x_1, \ldots, x_n), \tag{5.53}
$$

where h_r is a part of the formal power series of the Poincaré-Einstein metric (2.16)

$$h_r = h^{(0)} + r^2 h^{(2)} + \ldots + r^n h^{(n)} + \ldots. \tag{5.54}$$

Calculating the partial derivative of h_r with respect to r, we get

$$T_{ij} = 2r h_{ij}^{(2)} + 4r^3 h_{ij}^{(4)} + \ldots + n r^{n-1} h_{ij}^{(n)} + \ldots. \tag{5.55}$$

In particular, $h_{ij}^{(2)} = -P_{ij}$ (see, (2.16)). We consider the 1-parameter family of metrics $h(t)$ on M with $t = r$. Then the tensor T_{ij} defines a geometrical flow by the equation

$$\frac{\partial h_{ij}}{\partial t} = T_{ij}. \tag{5.56}$$

Following (1.22), we may view P_{ij} as the Ricci tensor multiplied by a constant with a correction term. In this way, the flow (5.56) may be viewed as prescribing the Ricci flow with specific correction terms and multiplied by a constant.

Conclusion

In the thesis we formulated and in certain case found the Ricci flow problem for the ambient and the Poincaré-Einstein metric construction. Since the solution is given by approximations up to some order in terms of the transversal coordinate to the base manifold M, we discuss only the first order approximation to the the actual solution. Let us briefly summarize the main achievements in our thesis. In section 5.2, we introduce a simplified version of the general system of equations equivalent to the Ricci flow problem, suppressing one of the components of the metric. However, due to suppressing one of the metric components the solution is valid only for initial metrics satisfying certain condition on the Christoffel symbols. In particular, our result holds for the euclidean space and provides a unique solution. In the next section, we provide a general system of differential equations for any initial metric. In section 5.4, we investigate a system of equations that is valid for any initial metric, but is already simplified by assuming some property of the solution. After imposing an extra condition, we arrive at the same unique solution as in the simplified case that is valid for the euclidean space.

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