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MASTER THESIS

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Skewness in theory of portfolio optimization and efficiency

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Šikmost v teorii optimalizace a eficience portfolia

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Abstrakt: Diplomová práce se zabývá modely pro volbu optimálního portfolia. Oproti klasickému přístupu použití střední hodnoty výnosu portfolia a míry rizika portfólia jako kritérií zahrnujeme mezi kritéria také šikmost. Dále jsou formulovány modely, které měří eficienci portfolia definovanou jako vzdálenost portfolia od Paretovy eficientní hranice. Součástí práce je aplikace modelů na finanční data titulů vyskytujících se v burzovním indexu NASDAQ-100 elektronické burzy NASDAQ. Závěr práce obsahuje vzájemné porovnání optimálních portfolií použitých modelů, a také porovnání s triviálními portfolii a samotným indexem NASDAQ-100.

Klíčová slova: šikmost, optimální portfolio, Paretovská eficience, míra eficience portfolia

Title: Skewness in theory of portfolio optimization and efficiency

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Abstract: In this thesis we study models, which search for an optimal portfolio from a set of stocks. On the contrary to the classical approach focusing only on expected return and variance, we examine models where an additional criterion of skewness is included. Furthermore we formulate a model for measuring performance of a portfolio defined as the distance from the Pareto efficient frontier. In numerical experiments we apply the models on historical prices and stock data from the electronic stock market NASDAQ. We analyze the stock data from companies listed in the index NASDAQ-100. We conclude by comparing of optimal portfolios created using different models among each other, with trivial single-stock portfolios and the with NASDAQ-100 index itself.

Keywords: skewness, optimal portfolio, Pareto efficiency, portfolio performance

Contents

Introduction

At the stock market, investors' intention is to create a portfolio with the highest possible return and simultaneously to achieve the lowest risk of loss. These are contradictory requirements and a trade-off is necessary. The optimal portfolio of assets can be composed in different ways according to investors' risk aversion and approach to expected return.

Many models of portfolio optimization are based on the expected return of the portfolio and variance as the risk measure. In this thesis we will include the skewness of a portfolio as another criterion into the models. The investors' attitude towards the skewness is not as evident as their attitude towards the expected return and the variance. The higher skewness of the portfolio's return signifies a lower probability of a large negative return. That means, that is only natural, when the investors prefer positive skewness.

In the first chapter, we introduce Mean-Variance framework for portfolio optimization and basic notation. The Markowitz model for searching for the optimal portfolio is presented as a baseline optimization model.

In the second chapter, we establish the concept of a Mean-Variance-Skewness optimal portfolio and optimization models based on this idea.

The third chapter deals not only with the models, which search for an optimal portfolio, but also with methods for measuring the performance of a given portfolio.

The fourth chapter contains the numerical part of this thesis. Selected models are applied to the stock data and an approximation of the efficient frontier from each model is shown.

In the last chapter, we summarize the models and compare them from the theoretical and computational point of view.

Chapter 1

Mean-Variance Portfolio Selection Problem

Harry M. Markowitz was the first who introduced a portfolio model in Markowitz (1952) and Markowitz (1959). Markowitz model chooses an optimal portfolio and is considered a foundation stone of the portfolio theory. Its main principle is diversification, which reduces the risk of the whole portfolio by investing in variety of assets disposable in the market.

In this thesis we assume that the market is efficient. There are some assumptions to be held: there are no transaction and tax costs, the assets are marketable and infinitely divisible, all relevant information is equally available for each investor and the investment time is one period (see Dupačová et al. (2002)). We expect rational behavior of investors, which means they choose a portfolio with the highest expected return among all portfolios with the same risk or a portfolio with the smallest risk among all portfolios with the same expected return.

Unless otherwise stated, shortsales are disabled and we do not include risk free assets.

The existence of transaction or tax cost can influence the investor's decision. When costs are fixed regardless of the amount invested, it is more profitable to invest in less titles. Consequently, the resulting portfolio is less profitable and more risky than in case of assuming no costs. Other situation is, when costs are float. In that case they can be proportinally included in weights of particular assets.

In stated models we consider amounts invested in particular assets not in units or lots, but in proportions of an initial investor's amount. In real world the amount possible to invest is not infinite (if short sales are disabled) and also the amount of assets possible to buy is bounded. Therefore in our models we require the assumption of infinite divisibility, although it is hardly satisfied in reality.

In this chapter we consider only expected return and variance of returns to formulate a model. We search for an optimal Mean-Variance portfolio (MV).

1.1 Notation

Let's denote random variable R_j as a relative return of an asset $j = 1, \dots, n$ and $\mathbf{R} = (R_1, \dots, R_n)^T$ as a vector of returns. We seek for a portfolio $\mathbf{x} = (x_1, \dots, x_n)^T$, where weight x_j expresses, how much the investor invests in an asset j.

Since short sales are excluded, we assume weights of assets are non-negative $x_j \geq 0$. Investor disposes an amount of 1, which means $\sum_{i=1}^{n} x_i \leq 1$. Imposing condition, that the whole amount is invested in portfolio, we get the weights $x_i,\ i = 1, \dots, n$ in form proportions of the total amount invested into corresponding assets.

$$\sum_{i=1}^{n} x_i = 1 \tag{1.1}$$

Let's denote $\Im$ as a set of admissible portfolios identified by the weights:

$$\Im = \{\mathbf{x} \in \mathbb{R}^n; \sum_{i=1}^{n} x_i = 1, \mathbf{x} \geq 0\} \tag{1.2}$$

Return of a portfolio is a random variable given by the sum of random returns of particular assets multiplied by corresponding asset's weights:

$$R(\boldsymbol{x}) = \sum_{j=1}^{n} R_j x_j.$$

The expected return of the portfolio is calculated:

$$E[R(\boldsymbol{x})] = E\left[\sum_{j=1}^{n} R_j x_j\right] = \sum_{j=1}^{n} E[R_j] x_j. \tag{1.3}$$

A parameter μ_0 represents investor's required minimal expected return, which creates the following contraint:

$$E[R(\boldsymbol{x})] \geq \mu_0. \tag{1.4}$$

The variance of the portfolio $\boldsymbol{x}$ is calculated:

$$\begin{aligned} Var[R(\boldsymbol{x})] &= E\left[\left\{\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right\}^2\right] = \\ &= E[(R(\boldsymbol{x}) - (E[R(\boldsymbol{x})])^2] = \sum_{i,j=1}^{n} \Omega_{ij} x_i x_j = \mathbf{x}^T \Omega \mathbf{x}, \end{aligned} \tag{1.5}$$

where $\Omega_{i,j} = \text{cov}(R_i, R_j),\ i, j = 1, \ldots, n.$

1.2 The Markowitz Model

Investor's preference is to have the highest possible return with the lowest possible risk. These two requirements goes against each other; the higher return of a portfolio often means the higher risk and vice-versa. Simulataneous minimizing variance of the portfolio's return and maximazing expected return of the portfolio's return is a problem of multiobjective optimization. To solve this optimization problem we can fix one of the criteria and the second minimize/maximize. Hence, we can either minimize risk and fix the expected return or maximize expected return and fix the risk. Markowitz focused at the first possibility and he minimizes risk, while setting a condition of required minimum expected return of the portfolio.

There are several options, which risk measure can be used for evaluation the risk of the portfolio. For example: variance, mean absolute deviation or Value-at-Risk. Since the Markowitz model uses variance, the problem of searching for an optimal portfolio is a problem of quadratic programming. The model can be formulated as follows:

$$\begin{aligned} \min \quad & Var[R(\boldsymbol{x})] \\ \text{s.t} \quad & E[R(\boldsymbol{x})] \geq \mu_0, \\ & \sum_{j=1}^{n} x_j = 1, \\ & x_j \geq 0, \qquad j = 1 \ldots n. \end{aligned} \tag{1.6}$$

Parameter μ_0 reflects the investor's preference on the expected return of the portfolio. But it must be chosen prudently, because the model is feasible only when μ_0 satisfies:

$$\min\{E[R_i], i = 1, \dots, n\} \leq \mu_0 \leq \max\{E[R_i], i = 1, \dots, n\}$$

The model 1.7 can be rewritten as:

$$\begin{aligned}
\min \quad & \sum_{i,j=1}^{n} \Omega_{ij} x_i x_j \\
\text{s.t} \quad & \sum_{j=1}^{n} E[R_j] x_j \geq \mu_0, \\
& \sum_{j=1}^{n} x_j = 1, \\
& x_j \geq 0, \qquad j = 1 \dots n
\end{aligned} \tag{1.7}$$

Chapter 2

Mean-Variance-Skewness Portfolio Selection Problem

In this chapter we add skewness to the standard Mean-Variance framework. We will search for an optimal Mean-Variance-Skewness portfolio (MVS). There are many studies i.e. Lau et al. (1990) or Campbell and Hentschel (1992), which show, that stocks returns are not normaly distributed in general. Skewness measures the assymetry of the distribution around the mean of the random variable. Investors prefer positive skewness, since it means, that the probability of large negative return is lower contrary to portfolio returns with non-positive skewness under the same expected return and skewness.

Let $SK[R(\boldsymbol{x})]$ denotes skewness[1] of return of a portfolio:

$$\begin{aligned} SK[R(\boldsymbol{x})] &= E\left[\left\{\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right\}^3\right] = E[(R(\boldsymbol{x}) - E[R(\boldsymbol{x})])^3] = \\ &= \sum_{i,j,k=1}^{n} x_i x_j x_k E\big[(R_i - E[R_i])(R_j - E[R_j])(R_k - E[R_k])\big] = \\ &= \sum_{i,j,k=1}^{n} x_i x_j x_k cSK_{i,j,k} \end{aligned} \tag{2.1}$$

where

$$cSK_{i,j,k} = E\big[(R_i - E[R_i])(R_j - E[R_j])(R_k - E[R_k])\big]$$

are co-skewnesses for $i, j, k = 1, \ldots, n$, where n is again number of assets.

[1] The skewness of random variable X is mostly defined as $(E[X - EX]^3)/(\sqrt{E[X - EX]})^3$. Contrary to the skewness defined in (2.1), which is the third central moment of the random variable, this is normalized by third power of standard deviation. In this thesis we defined models and derived further results using skewness from (2.1).

Since the covariance matrix Ω is a symmetric matrix with a dimension (n, n), there are $\binom{n+1}{2} = n(n+1)/2$ coefficients to be computed. The coskewness tensor has rank (n, n, n), so number of elements to be computed is $\binom{n+2}{3} = n(n+1)(n+2)/6$. By adding skewness to the optimization model, we obtain a problem of nonlinear (cubic) programming.

2.1 Motivation For Skewness – Taylor's Series

Let's begin with a definition of the utility function. In case of portfolio theory, the utility means a welfare from investments. The vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$ represents two different portfolios with amounts invested in assets $j = 1, \ldots, n$. The unequality $\boldsymbol{x} \succsim \boldsymbol{y}$ means that $\mathbf{x}$ is weakly preferred to $\mathbf{y}$, $\boldsymbol{x} \succ \boldsymbol{y}$ means that $\mathbf{x}$ is preferred to $\mathbf{y}$ (Dupačová et al. (2002)) and indifference $\boldsymbol{x} \sim \boldsymbol{y}$ means, thath $\boldsymbol{x} \succsim \boldsymbol{y}$ and $\boldsymbol{x} \precsim \boldsymbol{y}$.

Definition 1. *The utility function (ordinal utility function) is defined as $U : \mathbb{R} \to \mathbb{R}$:*

$$(U(R(\boldsymbol{x})) \geq U(R(\boldsymbol{y})) \Leftrightarrow \mathbf{x} \succsim \mathbf{y}) \wedge (U(R(\boldsymbol{x})) = U(R(\boldsymbol{y})) \Leftrightarrow \mathbf{x} \sim \mathbf{y}),$$

where $R(\boldsymbol{x})$ is return of the portfolio $\boldsymbol{x}$.

Let $u : \mathbb{R} \to \mathbb{R}$ and $R(\boldsymbol{x})$ be such an investor's utility function, which is infinitely differentiable. The representation of the utility function in a form of Taylor series around its expected value follows:

$$\begin{aligned} u(R(\boldsymbol{x})) = {} & u\left(E[R(\boldsymbol{x})]\right) + u'(E[R(\boldsymbol{x})])(R(\boldsymbol{x}) - E[R(\boldsymbol{x})]) + \\ & + \frac{u''(E[R(\boldsymbol{x})])}{2!}(R(\boldsymbol{x}) - E[R(\boldsymbol{x})])^2 + \frac{u'''(E[R(\boldsymbol{x})])}{3!}(R(\boldsymbol{x}) - E[R(\boldsymbol{x})])^3 \\ & + \sum_{k=4}^{\infty} \frac{u^{(k)}(E[R(\boldsymbol{x})])}{k!}(R(\boldsymbol{x}) - E[R(\boldsymbol{x})])^k \end{aligned} \tag{2.2}$$

Let $m^k(\boldsymbol{x})$ denotes k-th central moment of $R(\boldsymbol{x})$. Further we assume, that for all $k = 1, \ldots$ the central moments $m^k(\boldsymbol{x}) < \infty$. The expectation of (2.2) is:

$$E[u(R(\boldsymbol{x}))] = u(E[R(\boldsymbol{x})]) + \frac{u''(E[R(\boldsymbol{x})])}{2!}Var(\boldsymbol{x}) + \frac{u'''(E[R(\boldsymbol{x})])}{3!}SK(\boldsymbol{x}) + $$
$$+ \sum_{k=4}^{\infty} \frac{u^{(k)}(E[R(\boldsymbol{x})])}{k!} m^k(\boldsymbol{x}). \tag{2.3}$$

The approximation 2.3 up to the third order:

$$E[u(r(\boldsymbol{x}))] \cong u(E[R(\boldsymbol{x})]) + \frac{u''(E[R(\boldsymbol{x})])}{2!}Var(\boldsymbol{x}) + \frac{u'''(E[R(\boldsymbol{x})])}{3!}SK(\boldsymbol{x}) \tag{2.4}$$

is an increasing function of skewness of the portfolio ($SK(\boldsymbol{x})$) for a decreasingly risk averse investor (see Arditti (1967))). This corresponds to: $u''(.) \leq 0$ and $u'''(.) > 0$ as the investor's utility function is decreasing in the variance and increasing in skewness of portfolio's return distribution.

2.2 Mean-Variance-Skewness Model

When searching for an optimal portfolio from assets $i = 1, \ldots, n$, we want to maximize the expected return and skewness and minimize variance of the return of a portfolio. That is a task of multi-objective optimization, where the components of the model conflict against each other.

There are many ways how to solve this task. One commonly chosen option is to minimize variance while fixing expected return and skewness. The model can be formulated as follows:

$$\begin{aligned} \min \quad & Var[R(\boldsymbol{x})] \\ \text{s.t} \quad & E[R(\boldsymbol{x})] \geq \mu_0, \\ & SK[R(\boldsymbol{x})] \geq \tau_0, \\ & \sum_{j=1}^{n} x_j = 1, \\ & x_j \geq 0, \qquad j = 1 \ldots n \end{aligned} \tag{2.5}$$

Same as in the mean-variance Markowitz model (1.7), parameter μ_0 is the investor's minimal required expected return. In addition, there is parameter τ_0, which represents required minimal skewness. One can choose to maximize expected return or skewness, but the option (2.5) corresponds better to investors reasoning and is common for mean-variance models.

For the purpose of calculation, the model can be rewritten as:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\ \text{s.t} \quad & \sum_{j=1}^{n}E[R_j]x_j \geq \mu_0, \\ & \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} cSK_{i,j,k}x_i x_j x_k \geq \tau_0, \\ & \sum_{j=1}^{n} x_j = 1, \\ & x_j \geq 0, \qquad j = 1\dots n \end{aligned} \tag{2.6}$$

The next model, we introduce, is based on expected value of taylor approximation of utility function in (2.4). Contrary to (2.5), it does not fix any of variables expected return, variance or skewness, but maximize their linear combination with parameters $\mu, \rho, \tau > 0$, given by investor.

$$\begin{aligned} \max \quad & \mu E[R(\boldsymbol{x})] - \rho Var[R(\boldsymbol{x})] + \tau Sk[R(\boldsymbol{x})] \\ \text{s.t} \quad & \sum_{j=1}^{n} x_j = 1, \\ & x_j \geq 0, \qquad j = 1\dots n \end{aligned} \tag{2.7}$$

The model (2.7) is nonlinear optimization problem. It corresponds to negative marginal utility for variance and positive marginal utility for expected return and skewness[2]. The ratio $\frac{\rho}{\mu}$ represents degree of absolute risk aversion of investor, the ratio $\frac{\tau}{\rho}$ represents degree of investor's prudence.

[2]Investor's utility function decreases while variance of the portfolio's return increases and investor's utility function increases while the expected return of the portfolio increases.

The formulation of (2.7) is called *Aggregate function approach* (Messac et al. (2000)), where the objective function is an aggregation of some conflicting criteria. In this case the structure of aggregation is weighted sum of three central moment of the portfolio's relative return.

Admissible portfolios, lending, borrowning Until now we assumed, that short sales are disabled a no risk free asset is included. How to put the assumptions concerning the risk free asset and shortsales into the formulation of a model?

For lending, but no borrowing (shortsales excluded, risk free asset's weight $x_{rf} \in \mathbb{R}$):

$$\sum_{i=1}^{n} x_i + x_{rf} = 1,\ x_i \geq 0,\ i = 1, \ldots, n, rf.$$

For no lending and no borrowing (shortsales excluded, no risk free asset):

$$\sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,\ i = 1, \ldots, n.$$

For lending and borrowing (shortsales included, risk free asset presence):

$$x_i \in \mathbb{R},\ i = 1, \ldots, n, rf.$$

2.3 The Polymial Goal Programming Model

Another approach to solve the multiobjective optimization problem is from Lai (1991), where he introduced *Polynomial Goal Programming* approach (PGP). Lai focuses at searching for a feasible MVS portfolio by maximizing both, expected return and skewness for a given level of variance.

In this section the presence of a risk free asset is assumed. Further short sales are allowed, it means that investor can borrow assets, sell them and later buy them back, in case he/she expects decreasing returns (*bear strategy*). The portfolio has weights $\boldsymbol{x}' = (x_1, \ldots, x_n, x_{rf})^T,\ \boldsymbol{x}' \in \mathbb{R}^{n+1}$, the vector of returns is then $R' = (R_1, \ldots, R_n, R_{rf})^T$. The return of risk free asset R_{rf} has zero covariances and coskewnesses with all other assets including itself[3]. As a result, the three first central moments of return of a portfolio are computed as follows:

[3]Return of a risk free asset R_{rf} is non-random, it is often denoted as r_{rf}.

$$E[R(\boldsymbol{x}')] = \sum_{i=1}^{n} x_i E[R_i] + x_{rf} R_{rf}, \tag{2.8}$$

$$Var[R(\boldsymbol{x}')] = E\big[R(\boldsymbol{x}') - E[R(\boldsymbol{x}')]\big]^2 = \sum_{i,j=1}^{n} x_i x_j \ \Omega_{i,j}, \tag{2.9}$$

$$Sk[R(\boldsymbol{x}')] = E\big[R(\boldsymbol{x}') - E[R(\boldsymbol{x}')]\big]^3 = \sum_{i,j,k=1}^{n} x_i x_j x_k \ cSK_{i,j,i}. \tag{2.10}$$

where $\Omega_{i,j}$ are the same covariances as in (1.5) and $cSK_{i,j,k}$ the same coskewnesses as in (2.1). Therefore $Var[R(\boldsymbol{x}')] = Var[R(\boldsymbol{x})]$ and $Sk[R(\boldsymbol{x}')] = Sk[R(\boldsymbol{x})]$.

It is not always possible to maximize both variables, expected return and skewness, simultaneously. The model PGP covers such compromise and introduces weights α, β, which represents investor's preferences. If the investor is more return-oriented, the parameter α is higher than β. Analogically the parameter β for skewness is higher than α for more skewness-oriented investors.

Since the shorting is included, the expected return of portfolio $E[R(\boldsymbol{x}')]$ is unbounded. To guarantee the model is feasible and the optimal solusion exists, Lai (1991) imposed additional restriction, namely rescaling the variance to unit variance.

For given parameters $\alpha, \beta \in \mathbb{R}$ the PGP model is defined:

$$\begin{aligned} PGP(\alpha,\beta) = \min_{x_i} \Big\{ & d_1^{\alpha} + d_3^{\beta}; d_1 = Z_1^{\star} - E[R(\boldsymbol{x}')], \\ & d_3 = Z_3^{\star} - Sk[R(\boldsymbol{x})], Var[R(\boldsymbol{x})] = 1 \Big\}, \end{aligned} \tag{2.11}$$

with

$$Z_1^{\star} = \max_{x_i} \Big\{ E[R(\boldsymbol{x}')]; Var[R(\boldsymbol{x})] = 1 \Big\} \tag{2.12}$$

and

$$Z_3^{\star} = \max_{x_i} \Big\{ Sk[R(\boldsymbol{x})]; Var[R(\boldsymbol{x})] = 1 \Big\} \tag{2.13}$$

The PGP model contains three separated portfolio optimization problems. First, maximizing expected return by fixing unit variance. Second, maximizing skewness by fixing unit variance. And last, the main model PGP minimizes diferrences of expected return and skewness from maximized value of expected return, resp. skewness calculated beforehand.

Considering not only the unit variance, but a target variance $V_0 > 0$, the extension of PGP Model from (2.11) was introduced in Briec et al. (2013). For given parameters $\alpha, \beta \in \mathbb{R}$ and variance level $V_0 > 0$, the generalized PGP model is defined:

$$\begin{aligned} PGP^{V_0}(\alpha,\beta) = \min_{x_i}\Big\{ d_1^{\alpha} + d_3^{\beta}; d_1 = Z_1^{\star}(V_0) - E[R(\boldsymbol{x}')], \\ d_3 = Z_3^{\star}(V_0) - Sk[R(\boldsymbol{x})], Var[R(\boldsymbol{x})] = V_0 \Big\}, \end{aligned} \tag{2.14}$$

with

$$Z_1^{\star}(V_0) = \max_{x_i} \Big\{ E[R(\boldsymbol{x}')]; Var[R(\boldsymbol{x})] = V_0 \Big\} \tag{2.15}$$

and

$$Z_3^{\star}(V_0) = \max_{x_i} \Big\{ Sk[R(\boldsymbol{x})]; Var[R(\boldsymbol{x})] = V_0 \Big\} \tag{2.16}$$

To guarantee the generalized PGP model is feasible, the given level of variance V_0 must be between the maximal and minimal value of observed variance of individual assets.

Chapter 3

Measuring Portfolio's Performance

Until now we were searching for optimal portolio $\boldsymbol{x}$, given the returns of all single assets. Let us consider a situation, that for given portfolio $\boldsymbol{x}$, we are interested to find out, whether the portfolio is efficient, and in case of inefficiency, how inefficient the portfolio is. As a measure of portfolio performance we will define the distance between portfolio and Pareto efficient frontier.

The Pareto frontier contains such portfolios, that cannot be better in one criterion, unless they are worse in another criterion. In this thesis we focus on minimalizing variance, while fixing other two criteria. Therefore Pareto efficient portfolios are such portfolios, which cannot have lower variance, unless the expected return or skewness is lower.

In this chapter we again ignore the presence of a risk free asset and assume that short sales are disabled.

3.1 The Variance Ratio Model

Mean-variance portfolio is possible to visualize in two dimensional plot, where the efficient frontier can be plotted as a curve. By adding skewness, a third dimension into portfolio framework, the efficient frontier becomes a surface.

Joro and Na (2006) came with an approach of measuring MVS portfolio efficiency. They showed, that it is sufficient for each asset to consider its projection into the efficient frontier instead of searching for the whole frontier. Based on the distance between the point of portfolio (image of the portfolio in MVS space) and its projection onto the efficient frontier, it is possible to measure the performance of the particular portfolio.

To describe their approach (Joro and Na (2006)), let's imagine a two dimensional space with dimensions for expected return and variance of the portfolio. We consider assets $i = 1, \dots, n$ and denote A_i a trivial portfolio containing only i'th asset. Then ${A_i}^\star$ is the projection of A_i onto the efficient frontier with the same expected return. For the purpose of measuring efficiency they defined a variance ratio:

$$\theta = \frac{\sigma_{i\star}^2}{\sigma_i^2}, \; i = 1 \dots, n, \tag{3.1}$$

where $\sigma_{i\star}^2$ is the variance of the projection ${A_i}^\star$, and ${\sigma_i}^2$ is the variance of the asset i. Since the expected return remains the same and the variance satisfies $\sigma_{i\star}^2 \leq \sigma_i^2$, θ is from the interval $[0, 1]$. The equality $\sigma_{i\star}^2 = \sigma_i^2$ and $\theta = 1$ occurs, when portfolio A_i lies on the frontier.

The following model of measuring efficiency using ratio θ was suggested in terms of *Data Envelopment Analysis* (DEA), which is a non-parametric tool for accessing efficiency. DEA is expressed by optimization problem with multiple inputs and multiple outputs (Charnes et al. (2013), Chapter 2).

First, we introduce a basic input-oriented formulation of DEA:

$$\begin{aligned}
\min \quad & \theta - \epsilon \, (\sum_{r=1}^{m} s_r^+ + \sum_{i=1}^{s} s_i^-) \\
\text{s.t} \quad & \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{r0}, \quad r = 1, \dots, s \\
& \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = \theta \, x_{i0}, \quad i = 1, \dots, m \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& s^+, s^- \geq 0
\end{aligned} \tag{3.2}$$

where x_{ij} is i-th input, y_{rj} is r-th output, s^+ is a vector of output slack variables, s^- is a vector of input slack variables. An infinitesimal constant ϵ allows the optimization program involving slack varibles to minimize θ simultaneously with maximizing sum of slack variables. The variable θ measures possible improvement in all inputs for the given level of outputs.

Let's return to the previous portfolio notation. Parameters μ_0, ρ_0, τ_0 are expected return, variance, respectively skewness of a portfolio under evaluation. Taking variances ratio (3.1) as θ, return and skewness as outputs, variance as input and an admissible portfolio $\boldsymbol{x} \in \Im$ as λ, we get a model presented in Joro and Na (2006):

$$\begin{aligned} \min \quad & \theta - \epsilon(s_1 + s_2 + s_3) \\ \text{s.t} \quad & E[R(\boldsymbol{x})] - s_1 = \mu_0 \\ & Var[R(\boldsymbol{x})] + s_2 = \theta\rho_0 \\ & Sk[R(\boldsymbol{x})] - s_3 = \tau_0 \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \qquad j = 1 \dots n \end{aligned} \tag{3.3}$$

When the resulting $\theta \in [0, 1)$, the portfolio under evaluation is inefficient. In case of $\theta = 1$, we distinguish two situations. One occurs, when all slack variables s_1, s_2, s_3 are equal to 0. In that case the porfolio is efficient. In the case when at least one slack variable $s_i \neq 0$, then the portfolio is inefficient and the slack variables identify the closest efficient portfolio.

The model (3.3) can be rewritten as follows:

$$\begin{aligned} \min \quad & \theta - \epsilon(s_1 + s_2 + s_3) \\ \text{s.t} \quad & \sum_{i=1}^{n} E[R_i]x_i - s_1 = \mu_0 \\ & \sum_{i,j=1}^{n} \Omega_{i,j}x_ix_j + s_2 = \theta\rho_0 \\ & \sum_{i,j,k=1}^{n} CSk_{i,j,k}x_ix_jx_k - s_3 = \tau_0 \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \qquad j = 1 \dots n \end{aligned} \tag{3.4}$$

For the purpose of calculation in Chapter 4 and comparison of MVS model and MV model without skewness, we define MV Variance Ratio model, which is formed from the model (3.4) by omititng the skewness criterion:

$$\begin{aligned} \min \quad & \theta - \epsilon(s_1 + s_2) \\ \text{s.t} \quad & \sum_{i=1}^{n} E[R_i]x_i - s_1 = \mu_0 \\ & \sum_{i,j=1}^{n} \Omega_{i,j}x_i x_j + s_2 = \theta\rho_0 \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \qquad j = 1 \dots n \end{aligned} \tag{3.5}$$

3.2 The Shortage Function Model

In this section we introduce shortage function in Mean-Variance-Skewness framework according to Briec et al. (2007). Most of derivation and results come from the prior Briec et al. (2004), where the shortage function was derived and applied in two dimensional mean-variance framework. First, let's introduce a notation, mostly taken from these two sources.

For given porfolio $\boldsymbol{x}$, the function representing its expected return, variance and skewness, is a vector $\Phi : \mathbb{R}^n \to \mathbb{R}^3$:

$$\Phi(\boldsymbol{x}) = (E[R(\boldsymbol{x})], Var[R(\boldsymbol{x})], SK[R(\boldsymbol{x})]). \tag{3.6}$$

For admissible portfolios $\Im$, defined in (1.2), we denote a Mean-Variance-Skewness representation of portfolios as

$$\aleph = \{\Phi(\boldsymbol{x}); \boldsymbol{x} \in \Im\} \tag{3.7}$$

Extension of the set $\aleph$ by adding a cone is denoted as a portfolio disposal representation:

$$\mathfrak{DR} = \aleph + (\mathbb{R}_+ \times (-\mathbb{R}_+) \times \mathbb{R}_+) \tag{3.8}$$

For the purpose of the definition of weakly efficient frontier for mean-variance skewness portfolio, it is useful to rewrite the set of disposal representation of portfolios (3.8) as follows:

$$\begin{aligned} \mathfrak{DR} = \{(E, V, S) \in \mathbb{R}^3;\ \exists \boldsymbol{x} \in \Im, \\ (E, -V, S) \leq (E[R(\boldsymbol{x})], -Var[R(\boldsymbol{x})], SK[R(\boldsymbol{x})])\} \end{aligned} \tag{3.9}$$

It can be shown (see Briec et al. (2004)), that portfolio disposal representation $\mathfrak{DR}$ satisfies free disposal rule (monotonicity):

$$\forall y \in \mathfrak{DR},\ \forall y' \in \mathbb{R}^3 : y' \leq y \Rightarrow y' \in \mathfrak{DR} \tag{3.10}$$

Definition 2. *The weakly efficient frontier is defined as*

$$\partial^M(\Im) = \{(E, V, S); (E^\star, -V^\star, S^\star) > (E, -V, S) \Rightarrow (E^\star, V^\star, S^\star) \notin \mathfrak{DR}\}$$

Definition 2 says, that the weakly efficient frontier is a subset of MVS portfolios, which are not dominated by any other portfolio.

Definition 3. *The set of weakly efficient portfolios is denoted:*

$$\Theta^M(\Im) = \{\boldsymbol{x} \in \Im; \Phi(\boldsymbol{x}) \in \partial^M(\Im)\}$$

Figure 3.1 is taken from Briec et al. (2004) and illustrates the set of admissible portfolios (V,E) from $\Im$ in Mean-Variance framework. The set $\mathfrak{R}$ denotes the set of disposal MV portfolio representation analogical to $\mathfrak{DR}$ for MVS portfolios. In Mean-Variance-Skewness framework the set $\mathfrak{DR}$ is an subset of $\mathbb{R}^3$ and the

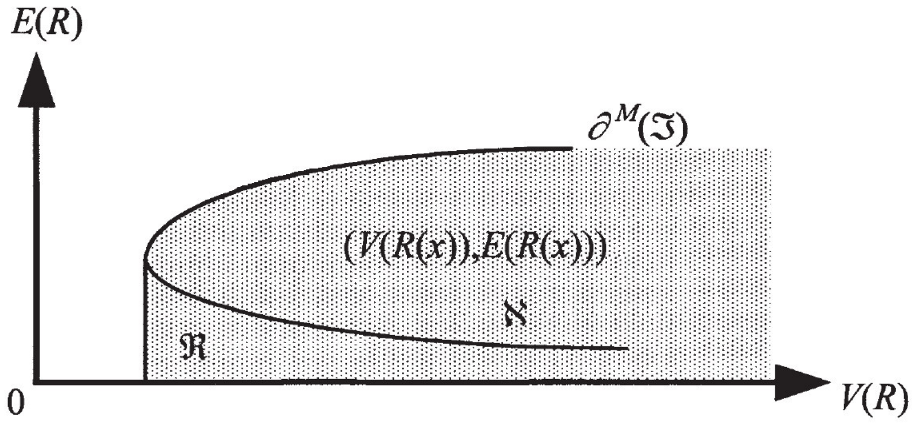

Figure 3.1: Mean-Variance illustration of the set of disposal portolio representation and weakly efficient frontier. Figure borrowed from Briec et al. (2004).

weakly efficient frontier is a surface.

3.2.1 Shortage Function

Now we can introduce a shortage function:

Definition 4. *Let* $g = (g_E, -g_V, g_S) \in \mathbb{R}_+ \times (-\mathbb{R}_+) \times \mathbb{R}_+$. *The function* $S_g : \Im \to \mathbb{R}_+$ *defined as* $S_g(\boldsymbol{x}) = \sup\{\delta;\ \Phi(\boldsymbol{x}) + \delta g \in \mathfrak{DR}\}$ *is the shortage function for portfolio* $\boldsymbol{x}$ *in the direction of vector* g.

The shortage function measures the distance in given direction g between an image of a portfolio $\boldsymbol{x}$ in MVS space and the Pareto efficient frontier. Let's formulate basic properties of the shortage function.

Proposition 1. *Shortage function* S_g *in the direction* g *defined on* $\Im$ *satisfies the following properties:*

(i) If $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$[1] *and* $\boldsymbol{x} \in \Im \Rightarrow S_g(\boldsymbol{x}) < \infty$

(ii) If $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$, *then* $S_g(\boldsymbol{x}) = 0 \Leftrightarrow \boldsymbol{x} \in \Theta^M(\Im)$ *(weak efficiency)*

(iii) S_g *is MVS weakly monotonic, i.e,*

$$(E[R(\boldsymbol{x}')], -Var[R(\boldsymbol{x}')], Sk[R(\boldsymbol{x}')]) \leq (E[R(\boldsymbol{x})], -Var[R(\boldsymbol{x})], Sk[R(\boldsymbol{x})])$$

implies, that $0 \leq S_g(x) \leq S_g(x')$

(iv) If $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$, *then* S_g *is continuous.*

Proof. (i) From the definition 3.8 of disposal representation set, if $\boldsymbol{x} \in \Im$ then the subset

$$\left\{(E^{'}, V^{'}, S^{'}) \in \mathfrak{DR}; (E^{'}, -V^{'}, S^{'}) \geq (E[R(\boldsymbol{x})], -Var[R(\boldsymbol{x})], Sk[R(\boldsymbol{x})]\right\}$$

is bounded, since the set of admissible portfolios $\Im$ defined in (1.2) is bounded. Then it follows, that $S_g(\boldsymbol{x}) < \infty$.

(ii) Assume that $\boldsymbol{x} \notin \partial^M(\Im)$. Then there exists portfolio $(E^{'}, V^{'}, S^{'}) \in \mathfrak{DR}$ such that

$$(E^{'}, -V^{'}, S^{'}) > (E[R(\boldsymbol{x})], -V[R(\boldsymbol{x})], S[R(\boldsymbol{x})])$$

Then from Definition 4 of shortage function and assumption $g_E > 0$ or $g_V > 0$ or $g_S > 0$ directly follows, that $S_g > 0$.

Thereupon, $S_g = 0 \Rightarrow \boldsymbol{x} \in \partial^M(\Im)$.

To prove the second implication, let's assume $S_g > 0$, $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$ and $\boldsymbol{x} \in \Im$. Consider a point of portfolio in MVS:

$$\Phi(\boldsymbol{x}') = (E(R(\boldsymbol{x})) + S_g(\boldsymbol{x})g_E, V(R(\boldsymbol{x})) - S_g(\boldsymbol{x})g_V, SK(R(\boldsymbol{x})) + S_g(\boldsymbol{x})g_S)$$

then we get

$$(E(R(\boldsymbol{x})) + S_g(\boldsymbol{x})g_E, -V(R(\boldsymbol{x})) + S_g(\boldsymbol{x})g_V, SK(R(\boldsymbol{x})) + S_g(\boldsymbol{x})g_S) >$$

$$> (E(R(\boldsymbol{x})), -V(R(\boldsymbol{x})), SK(R(\boldsymbol{x}))),$$

From definition of S_g we know, that $\Phi(\boldsymbol{x}') \in \mathfrak{DR}$, which implies, that portfolio $\boldsymbol{x}$ does not lie on the efficient frontier, so $\boldsymbol{x} \notin \Theta^M(\Im)$.

(iii) See Luenberger (1995).

(iv) Let's consider function $T : \mathfrak{DR} \to \mathbb{R}_+$ defined as

$$T(E, V, S) = \sup\{\delta; (E + \delta g_E, V - \delta g_V, S + \delta g_S) \in \mathfrak{DR}\}],$$

[1] $\mathbb{R}^3_{++}$ means that at least one element is positive and all of the elements are non-negative: $(g_E, g_V, g_S) > 0$, we talk then about strict positivity of the vector $g = (g_E, g_V, g_S)$

assuming $(g_E, g_V, g_S) \in \mathbb{R}^3_{++}$. Since $\mathfrak{DR}$ satisfies free disposal rule (monotonicity), and expected return, variance and skewness are continous with respect to $\boldsymbol{x}$, function T is continuous. Hence S_g is continuous (Briec et al. (2007)).

□

In other words, Proposition 1 tells, that shortage function for admissible portfolio $\boldsymbol{x}$ is always definite and non-negative, which follows from the Definition 4 of S_g. When the shortage function is equal to 0, the portfolio is weakly efficient. When portfolio $\boldsymbol{x}'$ is dominated by portfolio $\boldsymbol{x}$, which means, that portfolio $\boldsymbol{x}$ is closer to efficient frontier in a direction g than $\boldsymbol{x}'$ (in terms of euclidean distance in MVS space), then $0 \leq S_g(x) \leq S_g(x^{'})$. When $g = (g_E, g_V, g_S) \in \mathbb{R}^3_{++}$ the shortage function is continuous.

3.2.2 Computation Of Shortage Function

Again portfolio can be compound from a sample of assets $i = 1, \ldots, n$. Let consider a specific portfolio $\boldsymbol{y} = (y_1, \ldots, y_n)^T$, $\sum_{i=1}^n y_i = 1$, whose performace we want to measure. Further we consider a direction vector $g = (g_E, g_V, g_S) \in \mathbb{R}^3_{++}$.

The disposal representation set $\mathfrak{DR}$ expressed in (3.9) together with Definition 4 specified the form of the model for computation the shortage function. We seek for a portfolio $\boldsymbol{x}$, whose distance in a given direction g from a given benchmark portfolio $\boldsymbol{y}$, we maximize. Additionally the condition of admissibility of the portfolio $\boldsymbol{x}$ has to be satisfied.

We denote the model of calculating the shortage function as *MVS Shortage model* and formulate it as follows:

$$
\begin{aligned}
\max_{x_i} \quad & \delta \\
\text{s.t} \quad & E[R(\boldsymbol{y})] + \delta g_E \leq E[R(\boldsymbol{x})] \\
& Var[R(\boldsymbol{y})] - \delta g_V \geq Var[R(\boldsymbol{x})] \\
& Sk[R(\boldsymbol{y})] + \delta g_S \leq Sk[R(\boldsymbol{x})] \\
& \sum_{i}^{n} x_i = 1 \\
& x_i \geq 0, \qquad i = 1 \ldots n
\end{aligned}
\tag{3.11}
$$

If $\delta > 0$, the evaluated portfolio $\boldsymbol{y}$ is inefficient and its image in MVS space is below the efficient frontier. If $\delta = 0$ and $\boldsymbol{y} \in \Im$, then the portfolio is a part of the weakly efficient frontier. The model is infeasible, when $\Phi(\boldsymbol{y})$ of evaluated portfolio $\boldsymbol{y}$ is outside of set $\mathfrak{DR}$ (above the weakly efficient frontier).

The model 3.11 can be rewritten as:

$$\begin{aligned}
\max_{x_i} \quad & \delta \\
\text{s.t} \quad & \sum_{i=1}^{n} y_i E[R_i] + \delta g_E \leq \sum_{i=1}^{n} x_i E[R_i] \\
& \sum_{i,j=1}^{n} \Omega_{i,i}\ y_i y_j - \delta g_V \geq \sum_{i,j=1}^{n} \Omega_{i,j}\ x_i x_j \\
& \sum_{i,j,k=1}^{n} CSk_{i,i,i}\ y_i y_j y_k + \delta g_S \leq \sum_{i,j,k=1}^{n} CSk_{i,j,k}\ x_i x_j x_k \\
& \sum_{i}^{n} x_i = 1 \\
& x_i \geq 0, \qquad i = 1 \dots n
\end{aligned} \tag{3.12}$$

The Shortage model (3.12) is nonlinear (cubic) optimization problem. According to Briec et al. (2007) it can not be formulated as a standard convex problem of nonlinear programming. Therefore it is necessary to show, that local optimum is also a global optimal solution.

Proposition 2. *Assume that $\boldsymbol{x}$ is not a strict local maximum of the skewness on $\Im$. If $(\delta, \boldsymbol{x})$ is a local optimum of Shortage model* (3.11)*, then it is a global solution.*

Proof. Available as a part of online appendix to Briec et al. (2007).

For computational purposes we state also MV Shortage model (without the condition corresponding to the skewness criteria). The model as well as the properties of the two dimensional shortage function were derived in Briec et al. (2004).

$$
\begin{aligned}
\max_{x_i} \quad & \delta \\
\text{s.t} \quad & E[R(\boldsymbol{y})] + \delta g_E \leq E[R(\boldsymbol{x})] \\
& Var[R(\boldsymbol{y})] - \delta g_V \geq Var[R(\boldsymbol{x})] \\
& \sum_{i}^{n} x_i = 1 \\
& x_i \geq 0, \qquad i = 1 \dots n
\end{aligned}
\tag{3.13}
$$

3.2.3 Choice Of The Direction Vector

The choice of the direction vector should reflect the investor's preferences. When the assumption of strict positivity of vector g is satisfied, the shortage function simultanously decreases the given portfolio in variance and increases it in expected return and skewness. It measures possible improvement of the benchmark portfolio. By choice of the elements of direction vector g, we can also pick the variable, which should be minimize/maximize and fix other variables. For $(g_E, 0, 0)$ we obtain model of maximizing return, for $(0, g_V, 0)$ model of minimizing variance and $(0, 0, g_S)$ model of maximizing skewness.

When the elements of direction vector are less or equal to zero, it cannot be guaranteed that shortage function correctly indicates the efficiency, since the shortage function does not have to be continuous (see (ii) in Proposition 1).

If an element of direction vector g is negative it would contradict with utility theory derived in Chapter 2. The cases of one positive element and two zero elements, mentioned in this section, increase the chance of projection the portfolio to the nonconvex part of a frontier (in horizontal or vertical direction), whereas the shortage function is equal to zero.

Briec et al. (2007) suggests to take as a direction vector the absolute value of the expected return, the variance and the skewness itself: $|E[R(\boldsymbol{x})]|$, $|Var[R(\boldsymbol{x})]|$ and $|SK[R(\boldsymbol{x})]|$. The absolute value is necessary for the condition of strict positive elements of the direction vector g. The shortage function then measures the maximum percentage of improvement in expected return and skewness and reduction in variance.

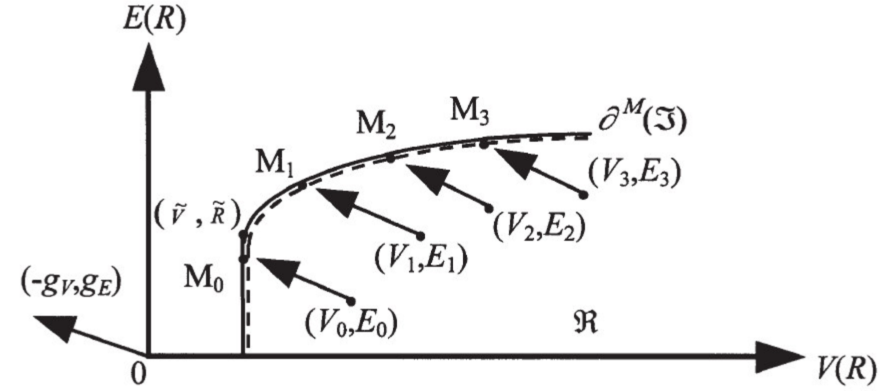

Figure 3.2: Illustration of the direction for shortage function (Briec et al. (2004)).

The figure 3.2, stated in Briec et al. (2004), shows an illustration of projections of inefficient portfolios in common direction for MV portfolio. For inefficient portfolios $(V_0, E_0), \ldots, (V_3, E_3)$, where E_i and V_i are expected return and variance of the portfolio, optimal portfolios $M_0, \ldots, M_3$ are calculated using the MV Shortage model (3.11) and the same direction vector $g = (g_E, g_V)$. From the formulation of the model is obvious, that the projection is in direction $(g_E, -g_V)$. By symbol $\mathfrak{R}$ is denoted two dimensional representation set of MV portfolios (analogy to three dimensional $\mathfrak{DR}$ stated in 3.8). The portfolio denoted in the picture as (V_0, E_0) is projected onto the vertical part of the frontier. This projection is only weakly efficient, since there exists a porfolio with the same risk, but higher expected return.

For MVS portfolio the situation is analogical. Set $\mathfrak{DR}$ is a three-dimensional subset of $\mathbb{R}^3$ and portfolios are projected onto the surface of weakly efficient frontier.

Chapter 4

Modelling

In this chapter we apply the models described in the previous chapters and find optimal solutions for some initial investor's preferences. By generating series of portfolio's projections onto the efficient frontier, we draw two and three dimensional representation of the Mean-Variance and Mean-Variance-Skewness efficient frontier.

To solve the optimization models we used GAMS *(The General Algebraic Modeling System)*, a high-level modelling system for mathematical programming and optimization. GAMS is designed for linear, nonlinear and mixed integer optimization problems. In GAMS we used solver CONOPT. All other tasks as preparation of the data, analysis of the results, graphical outputs and other computations was executed in *Wolfram Mathematica* 10.

The stated results from the models are rounded on 4 to 6 decimal digits such that, the results have still predicative values. The non-rounded results can be found in electronic appendix in *Mathematica* script.

4.1 Data

Data used in this thesis come from *National Association of Securities Dealers Automated Quotations* (NASDAQ), which is an American only electronic stock exchange. NASDAQ is the second largest stock exchange in the world by market capitalization[1].

The data contains daily historical dividend-adjusted and split-adjusted stock prices of titles from NASDAQ-100. This includes 100 largest non-financial com-

[1] The World Federation of Exchanges [http://www.world-exchanges.org].

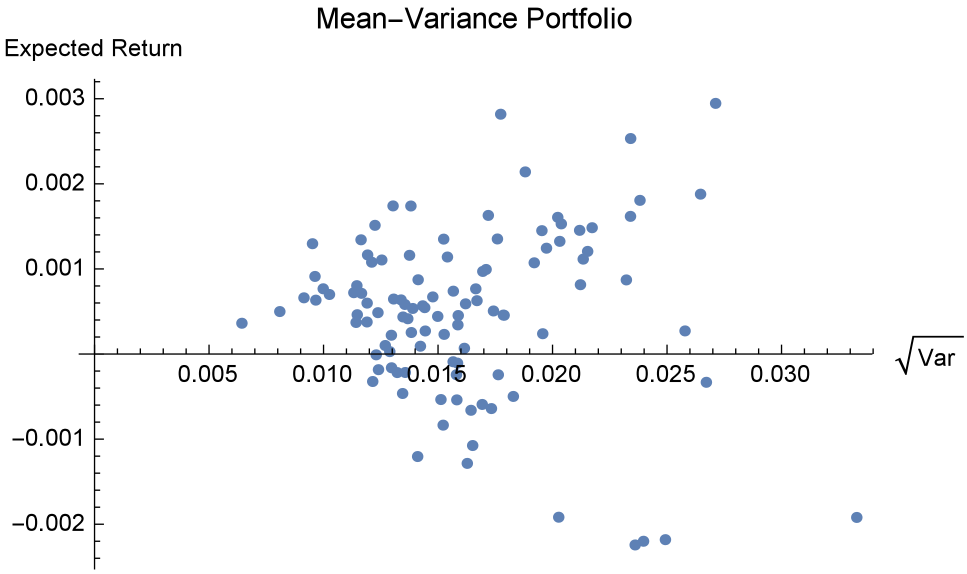

Figure 4.1: Trivial portfolios of individual assets in MV space.

panies listed in NASDAQ with 109 stocks in total — a few companies issued more than one stock. We ommitted three of them (KHC, LILA, LILAK), because of their very short history. We used prices of 106 stocks in a period from 11.7.2014 till 13.7.2015, i.e. 253 business days — one full year.

For the purpose of searching for an optimal portfolio we calculate relative changes on a daily basis. We denote $P_i(t)$ the price of the share $i = 1, \dots, n$ in time t. The relative return of i-th asset for a period $(t, t+1)$ is:

$$r_i(t) = \frac{P_i(t+1) - P_i(t)}{P_i(t)}, \ \ t = 1, \dots, 252$$

Basic descriptive statistics for all input assets used in the models: expected return, variance and skewness (defined in (2.1)) are provided in Table A.2, A.3 and A.4. Extreme values of these statistics can be found in Table 4.1. According to investor's utility function, we are interested in following values: the highest return 0.0029 has the asset MNST (*Monster Beverage Corporation*), the highest skewness 0.00014 has SIAL (*Sigma-Aldrich Corporation*) and the lowest variance 0.00004 has DTV (*DIRECTV*).

	min		max	
Mean	-0.002243	SNDK	0.002946	MNST
Variance	0.000041	DTV	0.001107	VIP
Skewness	-0.000042	SNDK	0.000144	SIAL

Table 4.1: Extreme values of expected return, variance and skewnes among individual stocks.

4.2 The Variance Ratio Model

In this section we focus on the Mean-Variance-Skewness model (3.3), where we minimize the variance ratio θ from (3.1) for a given portfolio and its projection.

4.2.1 Index NASDAQ-100

As a target portfolio we take NASDAQ-100 index. The weights of the individual stocks are given by their market capitalization. The Table A.1 in Appendix A contains the weights calculated for the slightly modified index, from 106 stocks.

We calculated both MVS and MV Variance Ratio model subject to the target portfolio. Optimal portfolios for both models consist of the same number of 19 stocks. The non-zero weights can be found in Table 4.2. First two columns are rounded, the difference between them begins from the eighth order as we can see in the third column. The weights from model MSV does not diverse much from weights of the model MV. Also the minimized objective variable θ defined in (3.1), which measures the portfolio performance is equal for optimal portfolio from both models and θ is equal to 0.06867 (9 digits accuracy).

Table 4.3 contains expected return, variance and skewness of the target portfolio (NASDAQ-100 index). Optimal portfolios found using MV and MVS Variance Ratio model did not increase in expected return, but decreased significantly in variance and increased in skewness. The NASDAQ-100 portfolio is inefficient.

4.2.2 Other Target Portfolios

We study improvement of the first three central moments of return of a portfolio also in other situations. We consider a target portfolio, where the weights of all assets are identical: $x_i = 1/106,\ i = 1, \dots, 106$. The expected return of the target portfolio is 0.00049 and again the optimal portfolios does not improve the original portfolio in expected return, it remains the same. Optimal portfo-

assets	MV	MVS	MVS-MV
ALTR	0.0191	0.0191	1.5×10^{-8}
CHKP	0.0314	0.0314	4×10^{-9}
COST	0.0303	0.0303	1.5×10^{-8}
DTV	0.4590	0.4590	2.6×10^{-8}
EA	0.0172	0.0172	3×10^{-9}
ESRX	0.0092	0.0092	0.
ISRG	0.0013	0.0013	0.
MDLZ	0.0226	0.0226	-4×10^{-9}
MNST	0.0083	0.0083	-8×10^{-9}
NFLX	0.0198	0.0198	8×10^{-9}
REGN	0.0074	0.0074	9×10^{-9}
ROST	0.0375	0.0375	-7×10^{-9}
SBUX	0.0422	0.0422	-1.2×10^{-8}
SIAL	0.0896	0.0896	5×10^{-9}
SPLS	0.0102	0.0102	-2×10^{-8}
SRCL	0.0954	0.0954	-2.1×10^{-8}
VRSK	0.0751	0.0751	6×10^{-9}
WFM	0.0161	0.0161	-7×10^{-9}
YHOO	0.0081	0.0081	-4×10^{-9}

Table 4.2: Weights of optimal portfolios according to MV and MVS Variance Ratio models with the target NASDAQ-100 portfolio. The models chose the same set of assets with only a small difference in weights. Stocks not listed in the table have weights equal to 0.

	NASDAQ-100	MV optimal portfolio	MVS optimal portfolio
E[R(x)]	0.00078	0.00078	0.00078
Var[R(x)]	0.00044	0.00003	0.00003
Sk[R(x)]	-2.59×10^{-7}	3.3×10^{-8}	3.3×10^{-8}

Table 4.3: Expected return, variance, skewness of optimal portfolios (index portfolio).

lios found by the models are better in variance and skewnes with respect to the investor's utility function. See Table 4.5. The objective θ is in both models MV and MVS equal to 0.380855. Both, MV and MVS optimal portfolios computed for the same-weight target, has the same number of non-zero weights: 21. Table 4.4 contains the weights and also the difference between them.

We also examined a Variance Ratio Model's behavior, in situation when the expected return of the target portfolio is negative. The central moments of the optimal porfolios found by models can be seen in Table 4.6. The improvement occurs in all three central moments of the portfolio. Minimized objective variable θ is equal to 0.380415.

4.2.3 Representation of Efficient Frontier Using Trivial Portfolios

We calculated 106 models with the initial parameters μ_o, ρ_0, τ_0 (as target) equal to the first three central moments of the trivial portfolios for each asset $i = 1, \ldots, 106$. As an example of GAMS code, this particular programme is on view in Appendix B. All other source codes are available in electronic appendix.

The weights of the optimal portfolios are omitted due to the large output, which could not be well-arranged and synoptic. Instead we provide table containing information about numbers of assets in particular projections. Table 4.7 shows number of non-zero weights in the optimal portfolios, where trivial portfolios $x_i, i = 1, \ldots, 106$ were taken as targets. There are 75 stocks which do not occur in any solution of MV model and 73, which do not occur in solutions of MVS model. Other 31, respectively 33 assets vary in the optimal portfolios. In 54 cases the optimal MV portfolios is compound from the same assets as the MVS. In 47 cases is the optimal MV portfolio composed from more assets then MVS optimal portfolio. For 5 target portfolios (AVGO, BBBY, EA, MNST, VRTX) is the situation reverse, so the MV optimal portfolio is composed from more assets than MVS optimal portfolio. In many cases MVS models creates more diverse

asset	MV	MVS	MVS-MV
ALTR	0.0135	0.0135	1.9×10^{-9}
CHKP	0.0260	0.0260	2.9×10^{-9}
COST	0.0202	0.0202	-5.0×10^{-9}
DTV	0.4922	0.4922	7.0×10^{-9}
FOXA	0.0015	0.0015	0
GMCR	0.0221	0.0221	0
GRMN	0.0054	0.0054	-9.9×10^{-10}
LBTYK	0.0030	0.0030	0
MAT	0.0166	0.0166	$- 2.0 \times 10^{-9}$
MDLZ	0.0332	0.0332	9.9×10^{-10}
NFLX	0.0100	0.0100	9.9×10^{-10}
NTAP	0.0055	0.0055	3.9×10^{-10}
ROST	0.0043	0.0043	-9.9×10^{-10}
SBAC	0.0023	0.0023	9.9×10^{-10}
SBUX	0.0214	0.0214	9.9×10^{-10}
SIAL	0.0810	0.0810	9.9×10^{-10}
SPLS	0.0002	0.0002	0
SRCL	0.1386	0.1386	1.9×10^{-9}
VRSK	0.0605	0.0605	-6.9×10^{-9}
WFM	0.0285	0.0285	6.0×10^{-9}
YHOO	0.0142	0.0142	-4.0×10^{-9}

Table 4.4: Weights of optimal portfolios according to MV and MVS Variance Ratio models with the target same-weight portfolio. The models chose the same set of assets with only a small difference in weights. Stocks not listed in the table have weights equal to 0.

	Target	MV optimal portfolio	MVS optimal portfolio
E[R(x)]	0.00049	0.00049	0.00049
Var[R(x)]	0.00008	0.00003	0.00003
Sk[R(x)]	-1.93×10^{-7}	3.2×10^{-8}	3.2×10^{-8}

Table 4.5: Expected return, variance, skewness of optimal portfolios (same-weights-target).

	Target	MV optimal portfolio	MVS optimal portfolio
E[R(x)]	-0.001	0.00044	0.00044
Var[R(x)]	0.00008	0.00003	0.00003
Sk[R(x)]	-1.93×10^{-7}	3.3×10^{-8}	3.3×10^{-8}

Table 4.6: Expected return, variance, skewness of optimal portfolios (target with negative expected return).

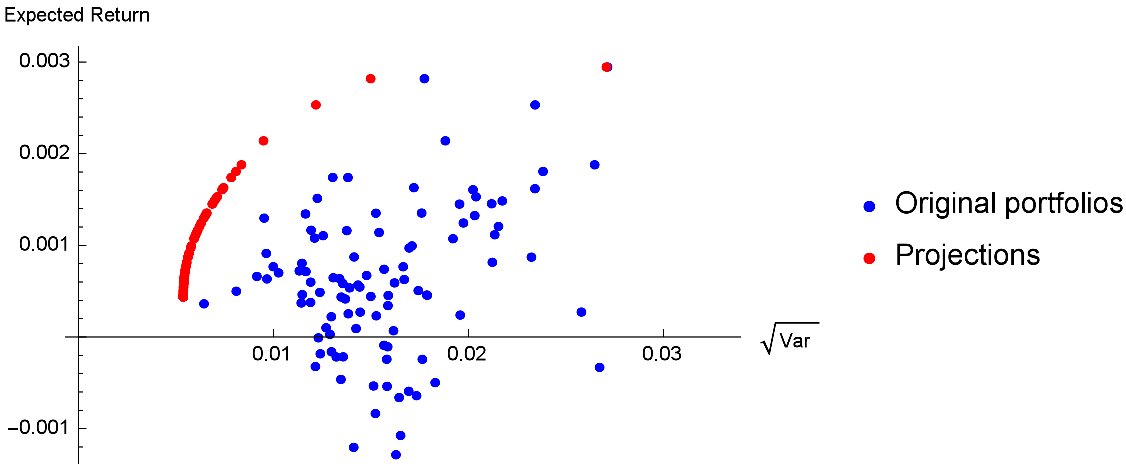

Figure 4.2: Optimal portfolios from MV Variance Ratio Model

portfolios than MV models.

Tables 4.8 and 4.9 contain a Variance ratio θ for each stock from a trivial portfolio targets computed in models without skewness (MV θ) and with skewness (MVS θ). The last column contains resulting θ for MV Variance Ratio model, where the target is our found MVS portfolio. The lower the value θ of MV model is, the larger is the distance of the target portfolio from its optimal portfolio in terms of variance. The value θ of the MV model corresponding to MNST is equal to 1, which says, that this trivial portfolio is efficient in MV space. All other single-asset portfolios are inefficient according to the MV Variance ratio model. The title MNST has the highest return among the other assets, but in the model with skewness, is not evaluated as efficient.

Figure 4.2 shows the original trivial portfolios, together with the resulting optimal portfolios from the MV Variance ratio model (without skewness equation) in the MV space. The projections of the original portfolios represent the efficient frontier.

Figure 4.3 contains again original portfolios and the projection points calculated with MVS models (including skewness). The plot is for comparison also displayed in two dimensional MV space. Not all the portfolios are projected onto the MV efficient frontier as in Figure 4.2. It is due to the inclusion of skewness criterion in a model. The efficient frontier is not a curve, but a surface in MVS space.

Figure 4.4 shows additional optimal MV portfolios, where targets were the already optimized portfolios using MVS. That means, the blue points are origi-

portfolio	MV	MVS	portfolio	MV	MVS	portfolio	MV	MVS
AAL	19	18	EA	4	5	NVDA	20	18
AAPL	19	19	EBAY	20	19	NXPI	16	15
ADBE	20	18	ESRX	19	19	ORLY	16	16
ADI	18	18	EXPD	19	18	PAYX	20	20
ADP	19	18	FAST	19	19	PCAR	19	19
ADSK	19	19	FB	17	17	PCLN	19	18
AKAM	20	20	FISV	17	17	QCOM	19	19
ALTR	16	2	FOX	19	19	QVCA	21	18
ALXN	19	16	FOXA	19	19	REGN	13	10
AMAT	19	19	GILD	17	17	ROST	16	16
AMGN	19	19	GMCR	19	18	SBAC	19	19
AMZN	17	17	GOOG	19	18	SBUX	17	17
ATVI	19	18	GOOGL	19	18	SIAL	17	1
AVGO	8	9	GRMN	19	19	SIRI	19	19
BBBY	18	19	HSIC	19	18	SNDK	19	19
BIDU	19	18	ILMN	19	16	SPLS	16	16
BIIB	19	16	INTC	19	18	SRCL	22	22
BRCM	17	15	INTU	19	19	STX	19	19
CA	19	19	ISRG	19	15	SYMC	19	19
CELG	17	17	KLAC	19	19	TRIP	19	12
CERN	18	18	LBTYA	20	18	TSCO	16	15
CHKP	19	18	LBTYK	19	19	TSLA	20	20
CHRW	19	18	LLTC	19	19	TXN	19	19
CHTR	19	18	LMCA	19	19	VIAB	19	19
CMCSA	19	19	LMCK	19	19	VIP	19	14
CMCSK	19	19	LRCX	19	19	VOD	19	19
COST	20	18	LVNTA	22	22	VRSK	19	18
CSCO	21	18	MAR	19	19	VRTX	17	19
CTRX	17	15	MAT	19	19	WBA	19	19
CTSH	19	19	MDLZ	20	18	WDC	19	19
CTXS	19	18	MNST	1	2	WFM	19	18
DISCA	19	18	MSFT	19	18	WYNN	19	19
DISCK	19	18	M	19	19	XLNX	19	19
DISH	19	18	MYL	17	11	YHOO	20	20
DLTR	17	16	NFLX	16	12			
DTV	19	18	NTAP	19	19			

Table 4.7: Number of titles in optimal portfolios

Name	MVS θ	MV θ	MV* θ	Name	MVS θ	MV θ	MV* θ
AAL	0.07714	0.04342	0.57651	DTV	0.72697	0.69877	0.96168
AAPL	0.19975	0.19709	0.98670	EA	0.77215	0.71585	0.92708
ADBE	0.20704	0.14166	0.68862	EBAY	0.25227	0.15813	0.62683
ADI	0.23467	0.12251	0.52206	ESRX	0.25388	0.22835	0.89943
ADP	0.31868	0.31472	0.98757	EXPD	0.19015	0.17927	0.94368
ADSK	0.11497	0.11497	0.99999	FAST	0.15962	0.15962	0.99999
AKAM	0.19843	0.13558	0.68324	FB	0.18599	0.18599	0.99999
ALTR	0.68584	0.11522	0.20441	FISV	0.45684	0.45684	0.99999
ALXN	0.12381	0.06845	0.55288	FOX	0.17174	0.17174	0.99999
AMAT	0.10076	0.10076	0.99999	FOXA	0.15690	0.15690	0.99999
AMGN	0.20070	0.15482	0.77144	GILD	0.10179	0.10179	1.00001
AMZN	0.27886	0.10246	0.36742	GMCR	0.13025	0.07027	0.55412
ATVI	0.20056	0.11109	0.56271	GOOG	0.21346	0.19582	0.91891
AVGO	0.38944	0.27119	0.69636	GOOGL	0.20947	0.18793	0.89927
BBBY	0.17478	0.16401	0.93841	GRMN	0.10593	0.10593	0.99999
BIDU	0.15049	0.07538	0.51700	HSIC	0.39652	0.30434	0.76752
BIIB	0.22556	0.07936	0.35181	ILMN	0.16291	0.09531	0.58503
BRCM	0.52555	0.12345	0.23491	INTC	0.20971	0.11074	0.54309
CA	0.20416	0.20416	0.99999	INTU	0.26142	0.24099	0.92185
CELG	0.13988	0.13988	1.00000	ISRG	0.45958	0.11472	0.24962
CERN	0.26313	0.26313	1.00001	KLAC	0.11779	0.11779	0.99999
CHKP	0.25852	0.23413	0.90567	LBTYA	0.17764	0.13964	0.78789
CHRW	0.15284	0.14265	0.93449	LBTYK	0.15921	0.15921	0.99999
CHTR	0.16614	0.11453	0.69810	LLTC	0.16534	0.16534	0.99999
CMCSA	0.21987	0.21987	1.00001	LMCA	0.19099	0.19099	0.99999
CMCSK	0.23349	0.23349	0.99999	LMCK	0.17195	0.17195	0.99999
COST	0.41730	0.34711	0.83179	LRCX	0.10516	0.10516	1.00000
CSCO	0.32670	0.18888	0.59058	LVNTA	0.09536	0.09536	0.99999
CTRX	0.62479	0.12180	0.19495	MAR	0.18836	0.17253	0.91594
CTSH	0.10939	0.10939	0.99999	MAT	0.10901	0.10901	0.99999
CTXS	0.17995	0.15441	0.86129	MDLZ	0.29725	0.21964	0.74576
DISCA	0.10589	0.08634	0.82000	MNST	0.77500	1.00000	0.13506
DISCK	0.12351	0.09615	0.78424	MSFT	0.18308	0.12861	0.71086
DISH	0.26552	0.12391	0.48421	MU	0.04645	0.04645	0.99999
DLTR	0.34974	0.33252	0.95074	MYL	0.44176	0.10289	0.23292

Table 4.8: Efficiency measure - Variance ratio model I

Name	MVS θ	MV θ	MV* θ
NFLX	0.27527	0.09986	0.36278
NTAP	0.12624	0.12624	0.99999
NVDA	0.12327	0.09030	0.73972
NXPI	0.21359	0.10018	0.46903
ORLY	0.45685	0.36225	0.79292
PAYX	0.35320	0.35320	0.99999
PCAR	0.17382	0.17382	0.99999
PCLN	0.15206	0.11564	0.76683
QCOM	0.12463	0.12463	0.99999
QVCA	0.26870	0.15020	0.57059
REGN	0.28526	0.25502	0.89400
ROST	0.40500	0.32219	0.79553
SBAC	0.25967	0.20618	0.79402
SBUX	0.35990	0.31723	0.88144
SIAL	1.00000	0.10533	0.10533
SIRI	0.22152	0.22152	0.99999
SNDK	0.05180	0.05180	0.99999
SPLS	0.17619	0.13273	0.75334

Name	MVS θ	MV θ	MV* θ
SRCL	0.44260	0.44260	1.00000
STX	0.10686	0.10686	0.99999
SYMC	0.13848	0.13848	0.99999
TRIP	0.30535	0.04044	0.16757
TSCO	0.42919	0.18758	0.43705
TSLA	0.05846	0.05846	1.00001
TXN	0.15094	0.15094	0.99999
VIAB	0.14511	0.14511	0.99999
VIP	0.13139	0.02605	0.22957
VOD	0.15868	0.15868	1.00000
VRSK	0.37222	0.28306	0.76048
VRTX	0.11617	0.08310	0.71533
WBA	0.11518	0.11518	0.99999
WDC	0.11540	0.11540	0.99999
WFM	0.24959	0.11474	0.47755
WYNN	0.05023	0.05023	0.99999
XLNX	0.09291	0.09291	0.99999
YHOO	0.09061	0.09061	1.00000

Table 4.9: Efficiency measure - Variance ratio model II

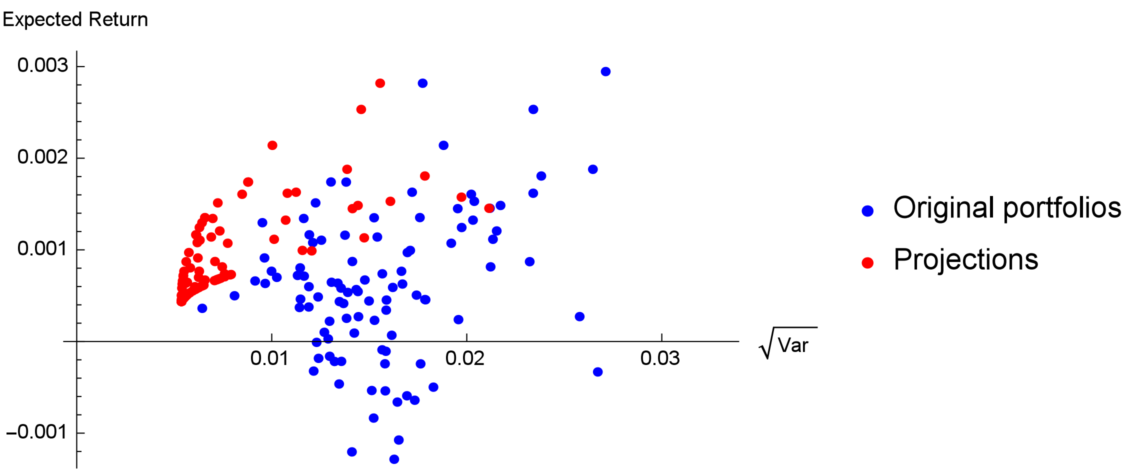

Figure 4.3: Optimal portfolios from MVS Variance Ratio Model

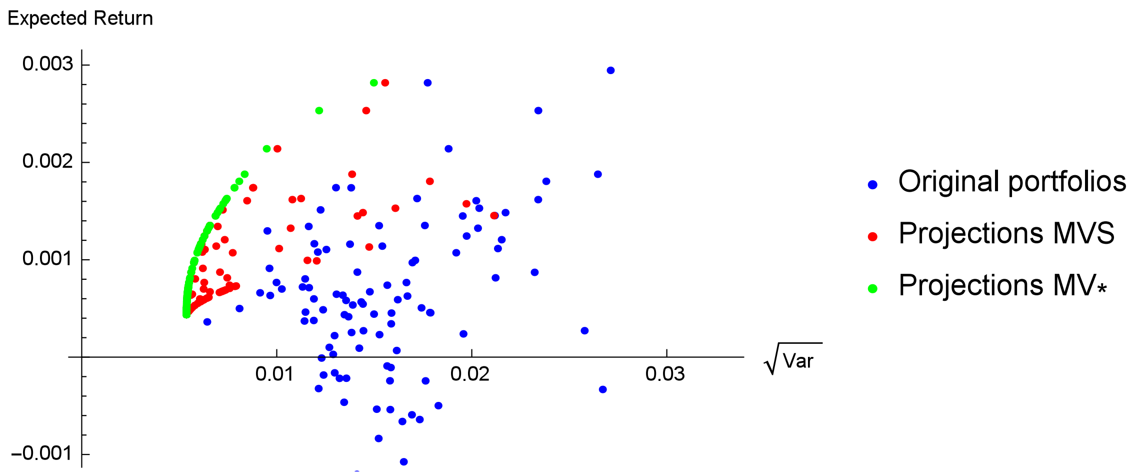

Figure 4.4: MV projections of MVS optimal portfolios

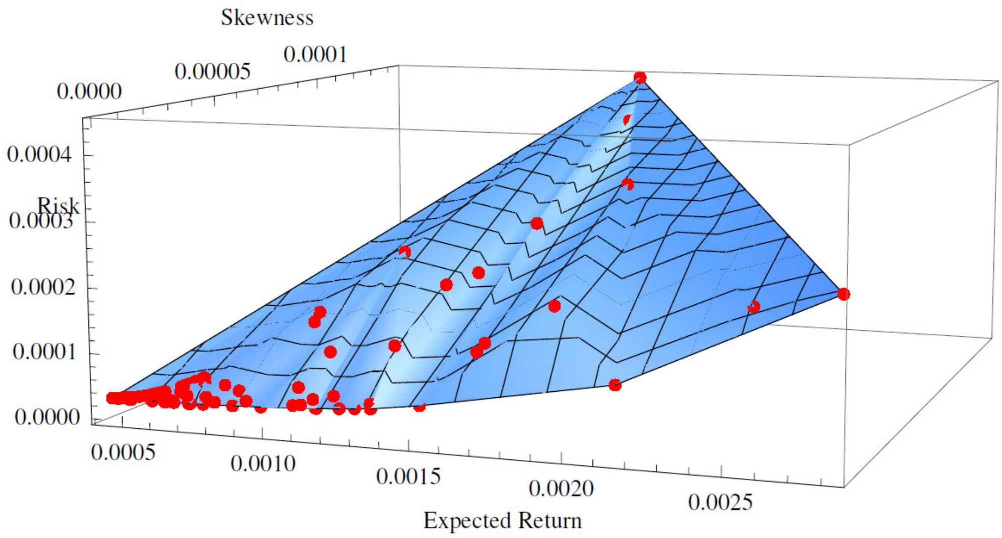

Figure 4.5: MVS Variance ratio model in 3D

nal trivial portfolios, red points are their projections using MVS Variance Ratio Model and green points are the previous projections optimized using MV Variance Ratio Model.

Figure 4.5 shows red points as the MVS optimal portfolios found by Variance Ratio Model in three dimensional MVS space, together with the surface representing the approximated weakly efficient frontier.

The points of optimal portfolios found by MVS Variance Ratio model are not equidistantly distributed. The surface generated using *Wolfram Mathematica*'s function *ListPlot3D* is not smooth. For the purpose of better graphical representation of the three dimensional MVS weakly efficient frontier, we generated a grid of one hundred equidistant admissible portfolios in MVS space. Afterwards we found optimal portfolios for each of these points and computed the expected return, variance, and skewness of the optimal portfolios. The geometric representation of a part of the efficient frontier in three dimensional space MVS is shown in Figure 4.6.

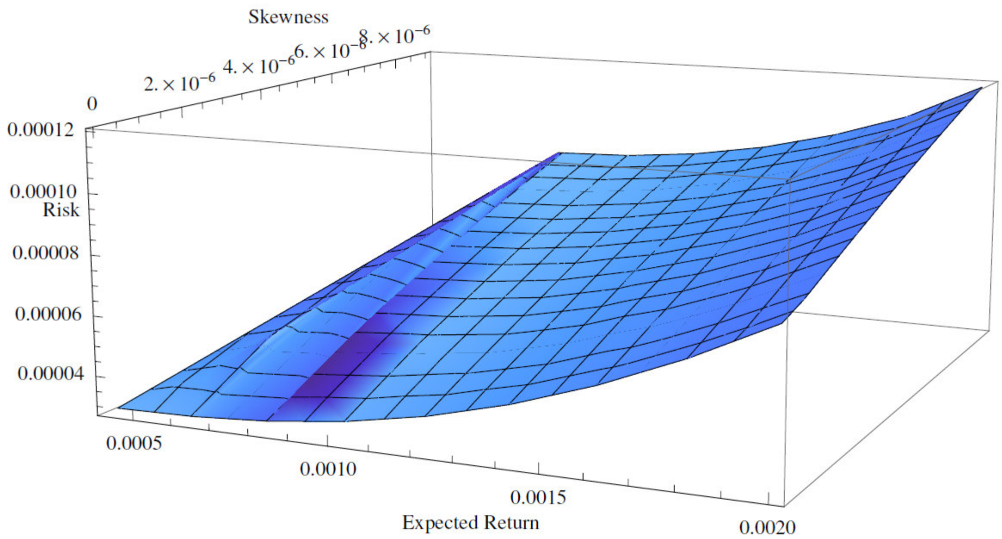

Figure 4.6: 3D representation of MVS efficient frontier

4.3 The Shortage Model

Model 3.11 maximizes objective δ, which measures a distance between a given portfolio $\boldsymbol{y}$ and the weakly efficient frontier. Vector $g = (g_E, g_V, g_S)$ determines the direction of the improvement of the portfolio.

In the section 3.2.3 we discussed the choice of the direction vector. We calculate the shortage model for different choices of vector g and compare the results. First we choose a target portfolio.

4.3.1 NASDAQ-100 Target

As a direction vector g we take the absolute value of the expected return, the variance and the skewness of the portfolio itself: $|E[R(\boldsymbol{x})]|$, $|Var[R(\boldsymbol{x})]|$ and $|SK[R(\boldsymbol{x})]|$. Table 4.10 shows the statistics for the target portfolio and then for optimal portfolios found by MV and MVS model. We can see increase of expected return and decrease of variance for both models. Skewness is much lower in MV model, since there is no criteria for skewness included. Skewness of the MVS portfolio remains the same.

	Target	MV optimal portfolio	MVS optimal portfolio
E[R(x)]	0.00049	0.00078	0.00084
Var[R(x)]	0.00008	0.00003	0.00008
Sk[R(x)]	0.00008	3.3×10^{-8}	0.00008
δ		0.0687	0.0724

Table 4.10: Optimal portfolios of Shortage model - target NASDAQ-100 I

The optimal portfolio obtained by MV shortage model is composed from 19 assets, the MVS optimal portfolio from 40 assets. The resulting shortage function can be found in the last row of the Table 4.10.

Now, we set the direction vector to $g = (1, 1, 1)$. Table 4.11 gives the resulting statistics. The MV shortage model improved the target portfolio in expected return and variance, but not in skewness.

	Target	MV optimal portfolio	MVS optimal portfolio
E[R(x)]	0.00049	0.00084	0.00078
Var[R(x)]	0.00008	0.00003	0.00009
Sk[R(x)]	0.00008	0.00003	0.00009
δ		0.00006	$< 10^{-9}$

Table 4.11: Optimal portfolios of Shortage model - target NASDAQ-100 II

4.3.2 Same-Weights Target

Let $g = (|E[R(\boldsymbol{x})]|, |Var[R(\boldsymbol{x})]|, |SK[R(\boldsymbol{x})]|)$ and $\boldsymbol{y}$ be same-weights target portfolio, where the weights of all titles are identical: $y_i = 1/106,\ i = 1, \ldots, 106$. The MV and MVS Shortage models evaluate the portfolios with $\delta = 0.6562$. Both optimal portfolios are composed from 17 assests and also the first three central moments are the same. In Table 4.12 are shown the computed values.

For the same target we now consider a direction vector $g = (1, 1, 1)$. From Table 4.13 we can see, that from both models the variance and skewness are al-

	Target	MV optimal portfolio	MVS optimal portfolio
E[R(x)]	0.00078	0.0014185	0.0014185
Var[R(x)]	0.00009	0.0000456	0.0000456
Sk[R(x)]	0.00009	7×10^{-8}	7×10^{-8}
δ		0.65622	0.65622

Table 4.12: Optimal portfolios of Shortage model - Same-weights target I

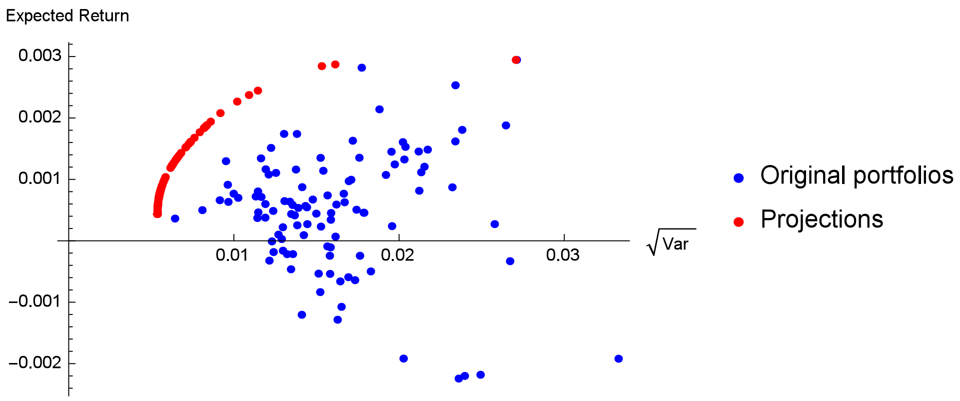

Figure 4.7: MV Shortage model for trivial portfolios in direction (1,1)

most the same (they differ from the -7th order).

	Target	MV optimal portfolio	MVS optimal portfolio
E[R(x)]	0.00078	0.00053	0.00049
Var[R(x)]	0.00009	0.00003	0.00008
Sk[R(x)]	0.00009	0.00003	0.00008
δ		0.00005	$< 10^{-9}$

Table 4.13: Optimal portfolios of Shortage model - Same-weights target II

4.3.3 Graphical Representation

Figure 4.7 shows the projections calculated using Mean-Variance Shortage model from trivial portfolios. As a direction vector is taken $g = (1, 1, 1)$.

The same trivial portfolios with the direction vector g=(1,1,1) is projected by MVS Shortage model. Figure 4.8 displayed these MVS optimal portfolio in MVS space. The optimal points does not reach the weakly efficient frontier in MV space. Again, that is because of inclusion of the third criteria of skewness in the model.

Figure 4.9 shows the previous projections of trivial portfolios in the direction $g = (1, 1, 1)$ three-dimensionally. Red points represent the MVS optimal portfolios.

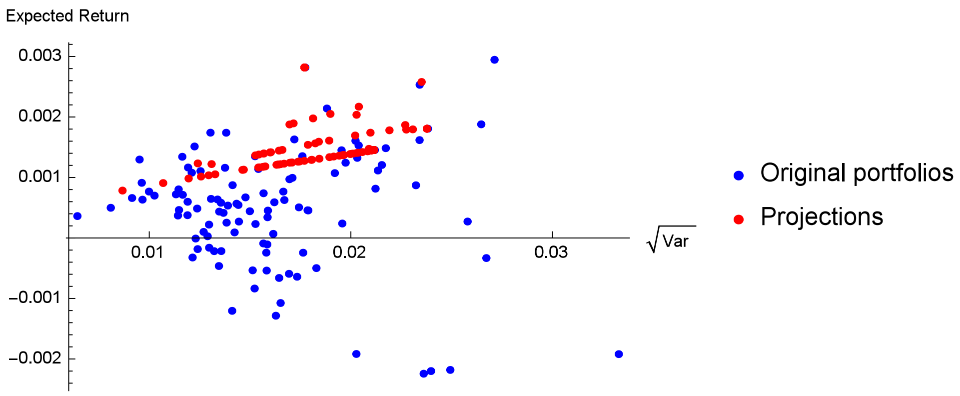

Figure 4.8: MVS Shortage model for trivial porfolios in direction (1,1,1)

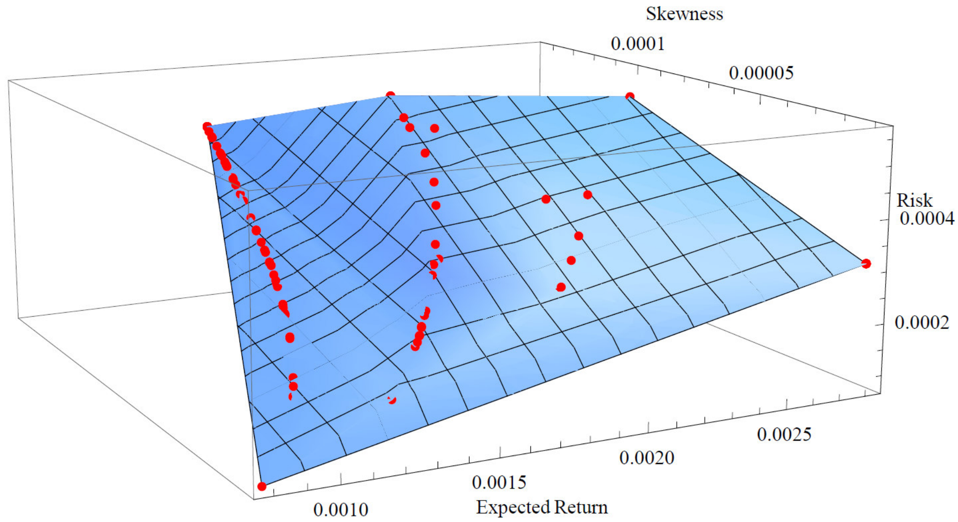

Figure 4.9: Shortage Model - direction (1,1,1) - optimal points from trivial portfolios

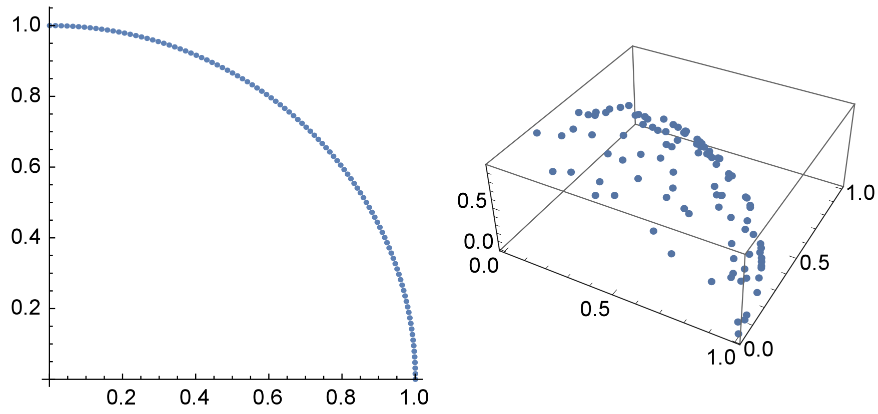

Figure 4.10: Generated direction for MV and MVS model

4.3.4 Approximation of the Efficient Frontier Using Shortage Model

As a target porfolio we take portfolio $\boldsymbol{y} = (\frac{1}{n}, \ldots, \frac{1}{n})$, where each asset has the same weight. The expected return, variance and skewness of such portfolio $\boldsymbol{y}$ is $\Phi(\boldsymbol{y}) = (4.876511 * 10^{-4}, 7.552755 * 10^{-5}, -1.92736 * 10^{-7})$.

For the purpose of generating the greatest possible part of the efficient frontier under the satisfied condition of strict positivity for direction vector g, we took as target such portfolio $\boldsymbol{y}$, with the lowest positive expected return and skewness and the highest variance. $\Phi(\boldsymbol{y}) = (0.00013, 0.013, 0.00005)$.

Futher we generated 100 directions $g = (g_E, g_V)$ for the MV Shortage model, that they cover angle $[0, 2\pi]$, for illustration see Figure 4.10. In order to have all vector normalized to 1, we computed directions as follows:

$$g_i = (\cos(\alpha_i), \sin(\alpha_i)),\ \ \alpha_i = \frac{\pi}{2 * 100} * i, i = 1, \ldots, 100. \tag{4.1}$$

For MVS model, we generated random directions for uniformly distributed points on the part of unit sphere, where all elements of a direction vector are positive. Directions for $\theta_i \in R[0, \frac{\pi}{2}], u_i \in R[0, 1],\ \ i = 1. \ldots, 100$ are computed as:

$$g_i = (\sqrt{1 - {u_i}^2} \cos(\theta_i), \sqrt{1 - {u_i}^2} \sin(\theta_i), u_i). \tag{4.2}$$

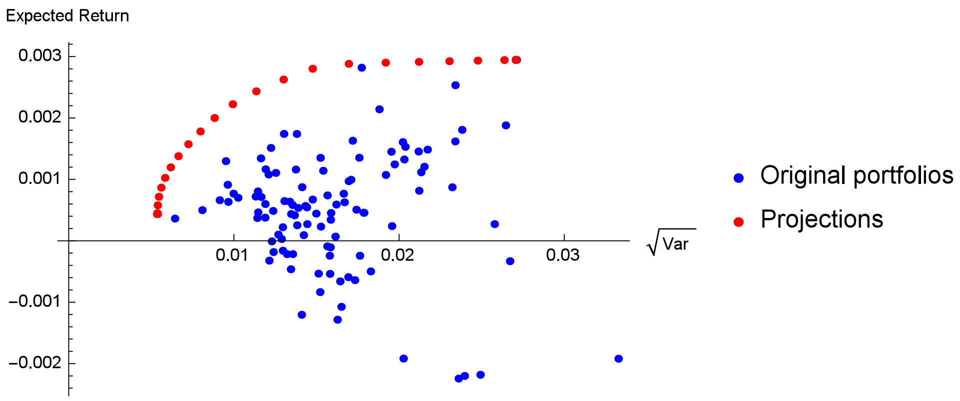

Figure 4.11: Shortage model generated direction (2D) from one point

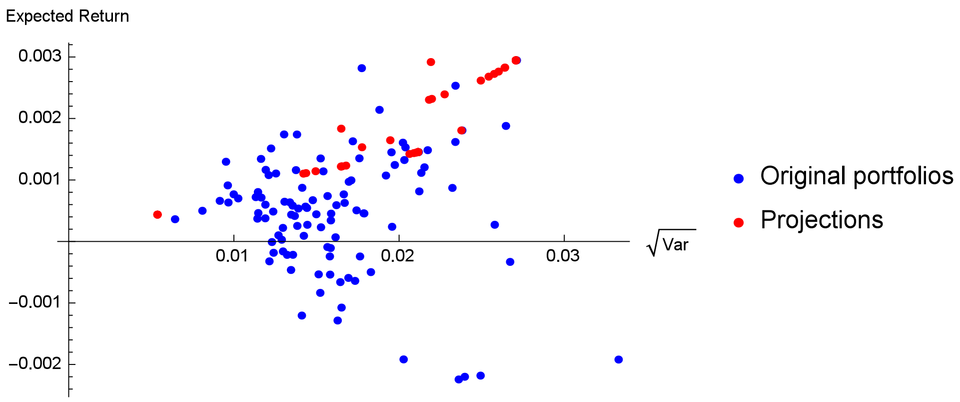

Figure 4.12: Shortage model generated direction (3D sphere) from one point in 2D

Using directions from (4.1), we calculated Mean-Variance Shortage model for a target portfolio $\boldsymbol{y}$, wtih $\Phi(\boldsymbol{y}) = (0.00013, 0.013, 0.00005)$. The projections are displayed in Figure 4.11.

The MV Shortage model approximates the Pareto MV efficient frontier. By including skewness, the projections from MVS Shortage model in directions (4.2), displayed in two dimensional space does not form an efficient MV frontier, see Figure 4.12.

Next Figure 4.13 shows these projected MVS optimal porfolios in 3D space. Additionaly, there are added optimal portfolios calculated by MV Shortage model also in generated directions. These red points indicate prolonged surface of the MVS frontier. Note the missing surface between the MV efficient frontier (red points) and MVS efficient surface. This is because of the minimal required skewness in MVS model was higher than the skewness of the optimal portfolios found on MV efficient frontier.

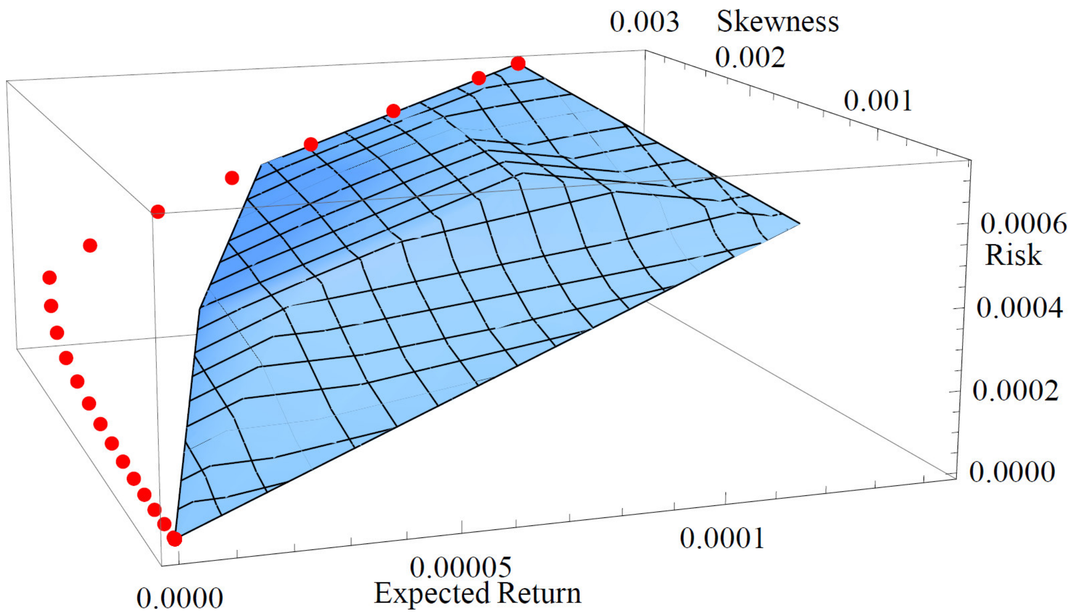

Figure 4.13: Shortage model generated direction (3D sphere) from one point in 3D

Chapter 5

Summary Of The Mean-Variance-Skewness Models

In this chapter we summarize theoretical and experimental findings, which we examined along this thesis. All stated models find an optimal potfolio. On the top of this, Variance Ratio model and Shortage model also provide a measure of portfolio's performance, based on the distance between a given portfolio and the Pareto efficient frontier, resp. the weakly efficient frontier.

In following sections we summarize three main models, we stated in this thesis. We also remind the three-dimensional analogy to the Markowitz model, which we defined in (2.5). This model minimizes the variance of the portfolio's return and takes the investor's preferences into account as criteria of minimal required return and minimal required skewness.

The model (2.7), called also an Indirect MVS utility function, simultaneously minimizes variance and maximizes expected return of a porfolio. Investor's preferences are reflected in corresponding coefficients, which can be interpreted as absolute risk aversion and prudence of the investor.

5.1 The Variance Ratio Model

The MVS Variance ratio model, defined in (3.3), is based on an idea of the portfolio's projection onto the Pareto efficient frontier and its formulation results from Data Envelopment Analysis. For a given portfolio $\boldsymbol{y}$ the Variance ratio model projects $\boldsymbol{y}$ onto the Pareto efficient frontier.

This model reflects the investor's preferences only by the choice of portfolio

$\boldsymbol{y}$, i.e of the first three central moments of the portfolio $\boldsymbol{y}$, which can be undestand as criteria of required minimal expected return and minimal skewness and maximal allowed variance of the return of an optimal portfolio.

The improvement of the portfolio's utility function for an investors is only in horizontal direction (of variance), unless the criteria corresponding to expected return or skewnees is not satisfied. Then the models search for the higher expected return and skewness.

We compared the results for MVS and MV Variance Ratio models. In most cases the optimal porfolios obtained using MVS models was composed from less or equal number of assets as the MV portfolio. The expected return, variance and skewness of the resulting portfolios from both models MV and MVS were very similar. They differs from order of 10^{-9} just as the objective functions, which is considered to be an efficiency measure.

5.2 The Shortage Model

The Mean-Variance-Skewness shortage model in (3.11) is based on Definition 4 of Shortage function and the disposal represantion set $\mathfrak{DR}$. The model maximized the objective function δ, which corresponds to the shortage function. Intputs for the model are a target portfolio $\boldsymbol{y}$ and strictly positive direction vector g. Choice of the direction vector expresses the investor's preference concerning expected return, variance and skewness of the return of his/her portfolio. Vector g determines the direction of the projection of the target portfolio $\boldsymbol{y}$ onto the Pareto efficient frontier.

For shortage functions $S_{g_1},\ g_1 = (g_E, g_V)$ defined in mean-variance space and $S_{g_2},\ g_2 = (g_E, g_V, \cdot)$ defined in mean-variance-skewness space holds, that $S_{g_1} < S_{g_2}$ as one more constraint conserning skewness is added.

The Shortage model compared to the Variance Ratio model better reflects the investor's preferences. The investor can choose the direction vector and also the target portfolio. The target portfolio, as by Variance ratio model, can be understand as criteria of required minimal expected return and minimal skewness and maximal allowed variance of the return of an optimal portfolio.

The objective functions of the Shortage model and the Variance Ratio model give the measure of a porfolio's performance. In case of Shortage Model, it is the the variable δ, which is maximized. The lower the resulting δ is, the better the target portfolio $\boldsymbol{y}$ is according to investor's preferences. For Variance ratio model the efficiency measure is a variance ratio θ defined in (3.1), which is from interval $[0, 1]$. The higher the θ, the better the target portfolios in terms of risk aversion of the investor.

5.3 The Polynomial Goal Programming Model

The Polynomial Goal Programming model (2.14) is another approach to solve the multiobjective optimization problem for Mean-Variance-Skewness portfolio. The optimization is divided into three separate problems. First is maximization of expected return for fixed variance, second is maximization of skewness for fixed variance and the last is minimization differences between the resulting objective values from the first two problems and corresponding expected return resp. skewness of the portfolio's return. The differences in the objective function (for expected value and skewness) are polynoms of the degree α, respectively β.

When using PGP model investor chooses the variance as a fix value and the resulting optimal portfolio, in case of feasibility, does have the same variance. The investor can affect the computation of an optimal porfolio by the specification of parameters α and β. When $\alpha > \beta$ then he/she prefers high expected return to high skewness.

Conclusion

In this thesis we studied optimization models for portfolio selection. Contrary to the classical portfolio optimization models based on the expected return of the portfolio and a risk measure, we included skewness as an additional criterion. For all our models considered in this thesis we used variance as the risk measure and the third central moment of portfolio's return as skewness.

As the main motivation for including skewness into the models we showed the link between third derivative of the utility function and skewness. Skewness of portfolios's return has a positive marginal utility for an investor. The probability of large negative return decreases with higher skewness.

We described and compared Mean-Variance-Skewness optimization models and focused mainly on the Variance Ratio model and the Shortage model. These two models provide also a measure of portfolio's performance.

In the experimental part of the thesis we applied the models to the historical stock data of NASDAQ-100 index taken from the world biggest electronic stock market NASDAQ. We compared optimal portfolios computed using models with and without inclusion of the skewness. We provided a graphical representation of the Pareto efficient frontier for these models in 2D Mean-Variance and 3D Mean-Variance-Skewness space.

Portfolios created by optimization models with skewness criterion achieved lower or equal expected return and higher or equal variance than the Mean-Variance model. The inclusion of skewness allows investors with high level of absolute prudence to compose portfolios, which has lower probability of large negative return. We consider the Shortage model to be the most suitable among the stated models from the investor's perspective, since the construction of the model allows to set the minimal required expected return and skewness, variance and also to choose a direction for improvement of the portfolio.

Appendix A

Additional tables

AAL	0.0075	EA	0.0069	NVDA	0.0039
AAPL	0.21	EBAY	0.0126	NXPI	0.0111
ADBE	0.0055	ESRX	0.005	ORLY	0.0026
ADI	0.0045	EXPD	0.0012	PAYX	0.002
ADP	0.0029	FAST	0.0015	PCAR	0.0027
ADSK	0.0032	FB	0.0446	PCLN	0.0161
AKAM	0.0024	FISV	0.0018	QCOM	0.017
ALTR	0.023	FOX	0.0015	QVCA	0.0013
ALXN	0.0055	FOXA	0.0072	REGN	0.0074
AMAT	0.0057	GILD	0.0227	ROST	0.0028
AMGN	0.0138	GMCR	0.0026	SBAC	0.0014
AMZN	0.022	GOOG	0.0215	SBUX	0.0093
ATVI	0.0056	GOOGL	0.0181	SIAL	0.0041
AVGO	0.0206	GRMN	0.0013	SIRI	0.0028
BBBY	0.0033	HSIC	0.0008	SNDK	0.0031
BIDU	0.0227	ILMN	0.0063	SPLS	0.0019
BIIB	0.0118	INTC	0.0304	SRCL	0.0012
BRCM	0.0148	INTU	0.0031	STX	0.0027
CA	0.001	ISRG	0.0014	SYMC	0.0028
CELG	0.0083	KLAC	0.0012	TRIP	0.0019
CERN	0.0021	LBTYA	0.0017	TSCO	0.0025
CHKP	0.0036	LBTYK	0.0031	TSLA	0.0193
CHRW	0.0014	LLTC	0.0021	TXN	0.0244
CHTR	0.0029	LMCA	0.0021	VIAB	0.0045
CMCSA	0.0155	LMCK	0.0007	VIP	0.0002
CMCSK	0.003	LRCX	0.0032	VOD	0.0024
COST	0.0104	LVNTA	0.0003	VRSK	0.0011
CSCO	0.0121	MAR	0.0027	VRTX	0.0045
CTRX	0.0079	MAT	0.0023	WBA	0.0204
CTSH	0.0041	MDLZ	0.0049	WDC	0.0036
CTXS	0.0024	MNST	0.0019	WFM	0.0045
DISCA	0.002	MSFT	0.0322	WYNN	0.0063
DISCK	0.0006	M	0.0178	XLNX	0.0032
DISH	0.0016	MYL	0.007	YHOO	0.0142
DLTR	0.0045	NFLX	0.0345		
DTV	0.0109	NTAP	0.0032		

Table A.1: NASDAQ-100 index – weights

stock	Mean	Variance	Skewness
AAL	0.0003	0.0007	0.1701
AAPL	0.0012	0.0002	0.0654
ADBE	0.0006	0.0002	0.5522
ADI	0.0007	0.0002	1.0255
ADP	0.0006	0.0001	0.1094
ADSK	-0.0001	0.0003	-0.0604
AKAM	0.0007	0.0002	0.5361
ALTR	0.0018	0.0006	6.8606
ALXN	0.0008	0.0004	0.3773
AMAT	-0.0006	0.0003	-0.5293
AMGN	0.0011	0.0002	0.4602
AMZN	0.0013	0.0004	1.8212
ATVI	0.0006	0.0003	0.7361
AVGO	0.0025	0.0005	0.8258
BBBY	0.0006	0.0002	0.1115
BIDU	0.0002	0.0004	0.5276
BIIB	0.0011	0.0005	1.3337
BRCM	0.0014	0.0004	5.4033
CA	0.0004	0.0001	-0.2707
CELG	0.0014	0.0003	-0.1572
CERN	0.0012	0.0001	-0.1664
CHKP	0.0008	0.0001	0.2978
CHRW	0.0001	0.0002	0.0889
CHTR	0.0005	0.0003	0.3851
CMCSA	0.0007	0.0001	-0.1506
CMCSK	0.0007	0.0001	-0.1877
COST	0.0009	0.0001	1.0135
CSCO	0.0005	0.0002	1.4339
CTRX	0.0015	0.0004	7.2135
CTSH	0.0008	0.0003	-0.9871
CTXS	0.0004	0.0002	0.2098
DISCA	-0.0005	0.0003	0.1197
DISCK	-0.0006	0.0003	0.1786
DISH	0.0002	0.0002	1.3185
DLTR	0.0015	0.0001	0.3424
DTV	0.0004	0.	0.6357
EA	0.0028	0.0003	1.3399
EBAY	0.0009	0.0002	0.9468
ESRX	0.0011	0.0002	0.3328

Table A.2: Descriptive Statistics I.

stock	Mean	Variance	Skewness
EXPD	0.0001	0.0002	0.1105
FAST	-0.0005	0.0002	-0.7645
FB	0.0014	0.0002	-0.0944
FISV	0.0013	0.0001	0.0348
FOX	-0.0002	0.0002	-0.1774
FOXA	-0.0002	0.0002	-0.2228
GILD	0.0012	0.0004	-1.6708
GMCR	-0.0019	0.0004	0.3927
GOOG	-0.0003	0.0001	0.1729
GOOGL	-0.0002	0.0002	0.2002
GRMN	-0.0011	0.0003	-1.3878
HSIC	0.0008	0.0001	1.1823
ILMN	0.0011	0.0004	0.5604
INTC	0.0001	0.0003	0.8218
INTU	0.0011	0.0001	0.2859
ISRG	0.001	0.0003	4.2668
KLAC	-0.0001	0.0002	-0.8125
LBTYA	0.0005	0.0002	0.3032
LBTYK	0.0004	0.0002	-0.0197
LLTC	-0.0002	0.0002	-0.5024
LMCA	0.	0.0002	0.0118
LMCK	0.0002	0.0002	0.0082
LRCX	0.0006	0.0003	-0.536
LVNTA	0.0005	0.0003	-1.2375
MAR	0.0006	0.0002	0.1577
MAT	-0.0013	0.0003	-0.0652
MDLZ	0.0005	0.0001	0.7801
MNST	0.0029	0.0007	5.7487
MSFT	0.0004	0.0002	0.4271
MU	-0.0022	0.0006	-1.4702
MYL	0.0015	0.0005	1.7774
NFLX	0.0019	0.0007	0.7373
NTAP	-0.0005	0.0002	-1.3744

Table A.3: Descriptive Statistics II.

stock	Mean	Variance	Skewness
NVDA	0.0005	0.0003	0.2131
NXPI	0.0016	0.0005	1.0614
ORLY	0.0017	0.0002	1.4831
PAYX	0.0007	0.0001	-0.2319
PCAR	0.	0.0002	-0.3442
PCLN	-0.0002	0.0002	0.2627
QCOM	-0.0008	0.0002	-1.9396
QVCA	0.0005	0.0002	1.1211
REGN	0.0021	0.0004	0.358
ROST	0.0017	0.0002	1.2251
SBAC	0.0006	0.0001	0.5277
SBUX	0.0013	0.0001	0.6616
SIAL	0.0015	0.0004	15.2405
SIRI	0.0004	0.0001	-0.2582
SNDK	-0.0022	0.0006	-3.1975
SPLS	0.0016	0.0004	0.414
SRCL	0.0005	0.0001	-0.1649
STX	-0.0007	0.0003	-0.5493
SYMC	0.0003	0.0002	-0.4542
TRIP	-0.0003	0.0007	2.4842
TSCO	0.0016	0.0003	3.1705
TSLA	0.0009	0.0005	-0.0637
TXN	0.0003	0.0002	-0.7659
VIAB	-0.0012	0.0002	-0.4916
VIP	-0.0019	0.0011	0.6633
VOD	0.0006	0.0002	-0.0651
VRSK	0.0007	0.0001	1.0779
VRTX	0.0012	0.0005	0.2468
WBA	0.001	0.0003	-2.0058
WDC	-0.0005	0.0002	-0.1566
WFM	0.0003	0.0003	1.2167
WYNN	-0.0022	0.0006	-0.9238
XLNX	-0.0002	0.0003	-2.2024
YHOO	0.0005	0.0003	-0.2759

Table A.4: Descriptive Statistics III.

Appendix B

GAMS script

Skript in GAMS for the MVS Variance Ratio Model. This program projects trivial portfolios onto the efficient frontier.

```
Sets
t cas /1*253/
j akcie /MU, INTC, AAPL, ODP, FB, QQQ, SIRI, MSFT, CY, CSCO,.../
ALIAS (j,i,k,c);

*loading data, settings for d,dd
$onecho > taskin.txt
par=d rng=A1:T253
par=dd rng=A1:T253
$offecho

Parameter d(t,j)
dd(t,c) ;
$call GDXXRW.EXE gams-adjusted.xls @taskin.txt
$GDXIN gams-adjusted.gdx
$LOAD d  dd
$GDXIN
* ------------------------------------------;
Parameter
r(j) ocekavany vynos;
r(j) = sum(t,d(t,j))/CARD(t);

Parameter
V(j,i) rozptylova matice;
V(j,i)= sum(t, (d(t,j)-r(j))*(d(t,i)-r(i))/ CARD(t) );

Parameter
```

```
CSK(j,i,k) tensorova matice sikmost;
CSK(j,i,k)=sum(t,(d(t,j)-r(j))*(d(t,i)-r(i)*(d(t,k)-r(k)))/CARD(t));
* ----------------------------------------;
Parameter
rr(c) return of particular asset c;
rr(c)= sum(t,dd(t,c))/CARD(t);

Parameter
VV(c) variance of particular asset c;
VV(c)= sum(t, (dd(t,c)-rr(c))*(dd(t,c)-rr(c))/ CARD(t) );

Parameter
CCSK(c) variance of particular asset c;
CCSK(c)=sum(t,(dd(t,c)-rr(c))*(dd(t,c)-rr(c))*(dd(t,c)-rr(c))/CARD(t));

Scalar
m0 return of asset under evaluation / 0.05 /
s0 variance of asset under evaluation /0.001/
k0 skewness of asset under evaluation  /0.02 /;

variable
theta objective function;

Positive variables
x(j) portfolio weights;
* ----------------------------------------;
Equations
er expected return
va variance
sk  skewness
bc budget constraint ;

er.. sum(j, r(j)*x(j)) =g= m0;
va.. sum((j, i), x(j)*V(j,i)*x(i) ) =l= theta*s0;
sk.. sum((j,i,k), CSK(j,i,k)*x(j)*x(i)*x(k) )=g= k0 ;
bc.. sum(j, x(j)) =e= 1;
* ----------------------------------------;
Model MVSJORO mean-variance-skewness model / er,va,sk,bc/ ;

* save results into a text file;
File Result /Res_JORO.txt/;
```

```
put result;
* ----------------------------------------;
Loop(c,
s0=VV(c);
k0=CCSK(c);
m0=rr(c);

Solve MVSJORO using NLP minimizing theta;
Display theta.l, x.l;

*put gamma ' ';
Loop(i,put x.l(i):6:3 ' ');
put theta.l;
put sum(j, x.l(j)*r(j) ):12:9;
put sum((j,i),x.l(j)*V(j,i)*x.l(i)):12:9;
put sum((j,i,k), x.l(j)*CSK(j,i,k)*x.l(i)*x.l(k)):12:9;
put /;
)
```

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