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DIPLOMA THESIS

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Elliptic equations in nonreflexive function spaces

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Název práce: Eliptické rovnice v nereflexivních prostorech funkcí
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Abstrakt: V práci modifikujeme všeobecně známý problém minimální plochy do speciálního tvaru, kde dvojka v exponentu je nahrazena obecným pozitivním parametrem. K upravenému problému zavedeme čtyři pojmy řešení v nereflexivním Sobolevově prostoru a v prostoru funkcí s omezenou variací. Prozkoumáme vztahy mezi těmito pojmy a ukážeme, že některé z nich jsou ekvivalentní a některé jsou slabší. Poté budeme hledat podmínky potřebné k dokázaní existence řešení problému ve smyslu zavedených definic. Poukážeme na to, že v prostorech funkcí s omezenou variací řešení existuje pro libovolný konečný parametr a pokud přidáme jisté podmínky na parametr, pak řešení existuje i v Sobolevově prostoru. Také uvedeme protipříklad ukazující, že řešení v Sobolevově prostoru nemusí existovat v případě nekonvexní oblasti.
Klíčová slova: nelineární eliptické rovnice, monotónní operátor, BV prostory, nereflexivní prostory funkcí, variační počet, rovnice minimální plochy

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Abstract: In the work we modify the well-known minimal surface problem to a very special form, where the exponent two is replaced by a general positive parameter. To the modified problem we define four notions of solution in nonreflexive Sobolev space and in the space of functions of bounded variation. We examine the relationships between these notions to show that some of them are equivalent and some are weaker. After that we look for assumptions needed to prove the existence of solution to the problem in the sense of definitions provided. We outline that in the setting of spaces of functions of bounded variation the solution exists for any positive finite parameter and that if we accept some restrictions on the parameter then the solution exists in the Sobolev space, too. We also provide counterexample indicating that if the domain is non-convex, the solution in Sobolev space need not exist.
Key words: nonlinear elliptic equations, monotone operator, BV spaces, nonreflexive function spaces, calculus of variations, minimal surface equation

Contents

Introduction

Finding the surface of least area among those bounded by a given curve is a fundamental problem in the calculus of variations. It is sometimes known as Plateau's problem, after the blind physicist who did beautiful experiments with soap films and bubbles. Since the 18^{th} century, many mathematicians had put efforts into solving it as it revived several times until they found a satisfactory solution only in recent decades. More details can be found in Giusti (1984).

The case of primary interest is when the surface is considered to be the graph of a function $u(x)$ defined on some open set $\Omega \subset \mathbb{R}^n$. Then the surface lies in the cylinder $\mathcal{Q} = \Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ and has dimension n. If u is a smooth function, the area of its graph is given by

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\,\mathrm{d}x,$$

where ∇u denotes gradient of u. Therefore u minimizes the area if and only if it is a solution to the minimal surface equation

$$-\mathrm{div}\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \text{ in } \Omega.$$

A natural question is whether a solution to the Dirichlet problem exists - a solution to the minimal surface equation taking prescribed values on the boundary of Ω. In fact, this problem is not generally solvable. When $n = 2$, a solution exists for arbitrary data if Ω is convex, but may fail to exist without the convexity of the domain, even if the boundary data φ is smooth and has arbitrarily small absolute value. However, if we do not characterize the class of functions competing to minimize the area $\mathcal{A}(u)$ by the boundary condition $u = \varphi$ but rather we introduce it in the functional, we look for a minimum of

$$\mathcal{I}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\,\mathrm{d}x + \int_{\partial\Omega} |u - \varphi|\,\mathrm{d}H_{n-1},$$

where H_{n-1} stands for the $(n-1)$-dimensional Hausdorff measure. That is defined to be the infimum over all countable covers by sets $U_i \subset \mathbb{R}^n$ satisfying $\mathrm{diam} U_i < \delta$, where $\delta > 0$ is a real number. In the Chapter 3 we show that the solution to the Dirichlet problem also minimizes $\mathcal{I}$, and, on the other hand, the new functional always reaches its minimum in the class $BV(\Omega)$ - of functions of bounded variation in Ω (we say more about BV spaces in the next chapter). As stated in Miranda (1964), u minimizes the area $\mathcal{A}(u)$ in Ω if and only if its subgraph

$$U = \{(x, t) \in \mathcal{Q};\, t < u(x)\}$$

is a set of least perimeter in $\mathcal{Q}$.

Although the minimal surface problem is one of the most important problems in the calculus of variations, it falls into a much broader class of minimization problems having the linear growth in the unknown (or more generally in the gradient of the unknown). For such a class a typical framework to deal with is not the standard Sobolev space setting but rather the setting of BV functions. On the other hand, a natural question arises immediately - whether such a change of the topology is really necessary or, at least in some particular cases, one is allowed to stay in the framework of Sobolev spaces. For general functionals the answer to this question is not affirmative. However, in the thesis we shall show that under some structural assumptions on the minimizing function, we are able to build a theory that is qualitatively equal to that of the minimal surface problem. To be more specific, we shall consider the following modification of the minimal surface problem

$$-\operatorname{div}\frac{\nabla u}{(1+|\nabla u|^a)^{\frac{1}{a}}} = 0 \text{ in } \Omega, \tag{1}$$

where the exponent two was replaced by a general positive a. This problem is interesting not only from the mathematical analysis' point of view, but also plays an important role in certain parts of the continuum mechanics – the limiting strain problems. For more details concerning the physical aspects we refer to Bulíček et al. (2014); Bulíček et al. (2015). As for the analytical results, the very similar problem was already treated in Bildhauer and Fuchs (1999) and Bildhauer and Fuchs (2002) and our work only completes the available existence theory and shows that in many cases, the results obtained here are optimal.

To complete this introductory part, we will briefly describe the main results of the thesis. First, we shall discuss various notions of solution to (1) and relationships between them. Second, for the weakest notion of solution (the one in the BV spaces framework) we shall establish a robust existence theory for any parameter $a \in (0, \infty)$. Third, for $a \in (0, 2]$, we shall strengthen the result and show that one can obtain a theory that is qualitatively same as the one of the minimal surface equations, i.e., one does not need to work in the setting of BV functions once allowing non-attainment of the prescribed boundary value. This extends the results of Bildhauer and Fuchs (1999, 2002) as the authors consider bounded and smooth boundary data and we require them to be only $W^{1,1}(\Omega)$.

Finally, in the last chapter we provide a counterexample in the case $a \in (1, 2]$ that shows the optimality of the obtained results, i.e., that the standard weak solution may not exist in general and one is forced to relax the condition on attainment of the boundary value condition.

Chapter 1

Function spaces

In the theory of partial differential equations it is very important to choose the right function space in which one works. Unfortunately, in the problem we shall study, the non-existence of a classical solution is a typical phenomenon and thus the notion of classical solution needs to be relaxed. We will work with the Sobolev spaces which contain functions with not only classical, but also so called weak derivatives. Moreover, the standard notion of a weak solution will be insufficient and therefore we will be led to BV spaces.

In this chapter we recall several basic facts about Lebesgue, Sobolev and BV spaces. All these results are in detail described in Evans (2010), Lukeš and Malý (2005), Kufner et al. (1977), Giusti (1984).

The very crucial concepts we use are the Lebesgue integral and spaces of Lebesgue measurable functions on $\Omega \subset \mathbb{R}^d$, where $d \geq 2$, $d \in \mathbb{N}$ is the dimension.

Definition 1.1. Let $\Omega \subset \mathbb{R}^d$ and $p \in [1, \infty]$. We define the Lebesgue space

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R};\ f \text{ is Lebesgue measurable, } \|f\|_{L^p(\Omega)} < \infty \right\},$$

where

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \quad \text{if } p \in [1, \infty),$$
$$\|f\|_{L^p(\Omega)} := \operatorname*{ess\,sup}_\Omega |f| \quad \text{if } p = \infty.$$

Remark 1. $L^p(\Omega)$ is a Banach space, provided we identify two functions which agree almost everywhere.

1.1 Weak derivatives

For relaxing the notion of partial derivative we will use the function φ that we call a test function. For a given open $\Omega \subset \mathbb{R}^d$ this function belongs to the space of infinitely differentiable functions $\varphi : \Omega \to \mathbb{R}$ with compact support in Ω, denoted by $\mathcal{D}(\Omega)$. These properties are exactly what we need - we will use the integration by parts, so we need derivatives of φ for yet defining the notion of weak derivative of our function, and thanks to the compact support of φ the boundary terms will formally vanish. Moreover, we will work with the multiindex

$\alpha = (\alpha_1, ..., \alpha_d)$ with its order defined as $|\alpha| = \alpha_1 + ... + \alpha_d$, where $\alpha_i \in \mathbb{N}_0$ for all $i = 1, ..., d$. With the help of this notation we will use the following abbreviation

$$D^\alpha \varphi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \varphi.$$

This topic requires a small introduction. Assume we are given a function $u \in \mathcal{C}^1(\Omega)$. Then, if $\varphi \in \mathcal{D}(\Omega)$, from the integration by parts formula we can see

$$\int_\Omega u \varphi_{x_i} \, \mathrm{d}x = - \int_\Omega u_{x_i} \varphi \, \mathrm{d}x, \tag{1.1}$$

for $i = 1, \ldots, d$ and φ_{x_i} denotes the partial derivative of φ according to variable x_i,

$$\varphi_{x_i}(x) = \lim_{h \to 0} \frac{\varphi(x + he_i) - \varphi(x)}{h},$$

while $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ has the only non-zero value on the i-th place.

More generally, if k is a positive integer, $u \in \mathcal{C}^k(\Omega)$ and α is a multiindex of order $|\alpha| = k$, then

$$\int_\Omega u D^\alpha \varphi \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega D^\alpha u \varphi \, \mathrm{d}x, \tag{1.2}$$

where we apply formula (1.1) k-times.

We ask whether some variant of (1.2) might still be true even if u is not k times continuously differentiable and so the expression $D^\alpha u$ has not the classical meaning. We resolve this by asking if there exists a locally integrable function v for which formula (1.2) is valid, with v replacing $D^\alpha u$.

Definition 1.2. Suppose $u, v \in L^1_{loc}(\Omega)$, and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \varphi \, \mathrm{d}x \tag{1.3}$$

for all test functions $\varphi \in \mathcal{D}(\Omega)$.

In other words, if we are given u and if there happens to exist a function v which verifies (1.3) for all φ, we say that $D^\alpha u = v$ in the weak sense. If such a function v does not exist, then u does not possess a locally integrable weak α^{th}-partial derivative.

Lemma 1.1 (Uniqueness of weak derivatives). A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of zero measure.

Proof. Assume that $v, \tilde{v} \in L^1_{loc}(\Omega)$ satisfy

$$\int_\Omega u D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \tilde{v} \varphi \, \mathrm{d}x$$

for all $\varphi \in \mathcal{D}(\Omega)$. Then

$$\int_\Omega (v - \tilde{v}) \varphi \, \mathrm{d}x = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$; whence $v - \tilde{v} = 0$ almost everywhere. □

1.2 Sobolev spaces

Definition of Sobolev spaces

Fix $p \in [1, \infty]$ and let k be a non-negative integer. We define certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces.

Definition 1.3. The Sobolev space $W^{k,p}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$, $u \in L^p(\Omega)$ such that for each multiindex α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Remark 2. Similarly as in Lebesgue spaces, we identify functions in $W^{k,p}(\Omega)$ which agree almost everywhere.

Definition 1.4. If $u \in W^{k,p}(\Omega)$, we define its norm to be

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{k,p} := \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^p \, \mathrm{d}x \right)^{1/p} \quad \text{if } p \in [1, \infty),$$
$$\|u\|_{k,p} := \sum_{|\alpha| \le k} \operatorname*{ess\,sup}_\Omega |D^\alpha u| \quad \text{if } p = \infty.$$

Definition 1.5. (i) Let $\{u^m\}_{m=1}^\infty$, $u \in W^{k,p}(\Omega)$. We say u^m converges to u in $W^{k,p}(\Omega)$, written $u^m \to u$ in $W^{k,p}(\Omega)$, provided

$$\lim_{m \to \infty} \|u^m - u\|_{W^{k,p}(\Omega)} = 0.$$

(ii) We also write $u^m \to u$ in $W^{k,p}_{loc}(\Omega)$, which means $u^m \to u$ in $W^{k,p}(V)$ for each $V \subset\subset \Omega$.

Definition 1.6. We denote by $W^{k,p}_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$ for $p \in [1, \infty)$.

Thus $u \in W^{k,p}_0(\Omega)$ if and only if there exist functions $u^m \in \mathcal{D}(\Omega)$ such that $u^m \to u$ in $W^{k,p}(\Omega)$. We interpret $W^{k,p}_0(\Omega)$ as comprising those functions $u \in W^{k,p}(\Omega)$ such that

$$\text{"} D^\alpha u = 0 \text{ on } \partial\Omega\text{" for all } |\alpha| \le k - 1.$$

Properties of Sobolev spaces

At this point we shall recall some important properties of Sobolev spaces, which are proven in the basic course on theory on partial differential equations. We will introduce these spaces as function spaces and state theorems that allow us to approximate them even up to the boundary by smooth functions. We will define operators to get extension to all $\mathbb{R}^n$ and trace on $\partial\Omega$. Finally, we shall come with equivalent characterization of Sobolev functions with the help of differential quotients.

Sobolev spaces as function spaces and approximation theorems

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be open and k be a non-negative integer. Then the following statements hold*

(i) $W^{k,p}(\Omega)$ *is the Banach space for* $p \in [1, \infty]$,

(ii) $W^{k,p}(\Omega)$ *is separable space if and only if* $p \in [1, \infty)$,

(iii) $W^{k,p}(\Omega)$ *is reflexive space if and only if* $p \in (1, \infty)$.

The space $W^{1,1}(\Omega)$ we will work with is therefore a non-reflexive separable Banach space.

Definition 1.7 (Domain of the class $\mathcal{C}^{k,\lambda}$)**.** Let $k \in \mathbb{N}_0$ and $\lambda \in [0, 1)$. We say that a set $\Omega \subset \mathbb{R}^d$ is of the class $\mathcal{C}^{k,\lambda}$ if and only if the following holds: There exists $\alpha, \beta > 0$, N orthogonal systems and $a_i \in \mathcal{C}^{k,\lambda}([-\alpha, \alpha]^{d-1})$ for $i = 1, \dots, N$ such that after a possible change of coordinates and defining $x' := (x_1, \dots, x_{d-1})$ we have

- for all $i = 1, \dots, N$

$$\Omega^i_+ := \{x \in \mathbb{R}^d;\, x' \in (-\alpha, \alpha)^{d-1} \text{ and } x_d - \beta < a_i(x') < x_d\} \subset \Omega, \tag{1.4}$$

$$\Omega^i_- := \{x \in \mathbb{R}^d;\, x' \in (-\alpha, \alpha)^{d-1} \text{ and } x_d < a_i(x') < x_d + \beta\} \subset \mathbb{R} \setminus \overline{\Omega}, \tag{1.5}$$

$$\partial\Omega^i := \{x \in \mathbb{R}^d;\, x' \in (-\alpha, \alpha)^{d-1} \text{ and } a_i(x') = x_d\} \subset \partial\Omega, \tag{1.6}$$

- for Ω^i given as

$$\Omega^i := \Omega^i_+ \cup \Omega^i_- \cup \partial\Omega^i$$

  there holds

$$\partial\Omega \subset \bigcup_{i=1}^N \Omega^i.$$

According to the definition above, we call Ω Lipschitz if $\Omega \in \mathcal{C}^{0,1}$.

Theorem 1.2 (Global approximation by smooth functions up to the boundary)**.** *Let $p \in [1, \infty)$ and Ω be Lipschitz. Then for all $u \in W^{1,p}(\Omega)$ there exists $u^m \in \mathcal{C}^\infty(\bar{\Omega})$ such that $u^m \to u$ in $W^{1,p}(\Omega)$.*

Definition 1.8 (Continuous and compact embedding)**.** Let X, Y be normed vector spaces. We say that X is continuously embedded in Y, denote $X \hookrightarrow Y$, if $X \subset Y$ and there exists a constant $c \geq 0$ such that

$$\|x\|_Y \leq c\|x\|_X$$

for all $x \in X$. We say that X is compactly embedded in Y, denote $X \hookrightarrow\hookrightarrow Y$, if $X \hookrightarrow Y$ and every sequence that is bounded in X has a subsequence that converges strongly in Y.

Theorem 1.3 (Embedding Theorem)**.** *Let Ω be Lipschitz, $p \in [1, \infty)$, $p^\# := \frac{dp}{d-p}$ and $\alpha := 1 - \frac{d}{p}$. Then*

- *for $p \in [1, d)$: $W^{1,p}(\Omega) \hookrightarrow L^{p^\#}(\Omega)$,*
- *for $p = d$: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$,*

- *for* $p \in (d, \infty)$*:* $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega})$.

Moreover,

- *for* $p \in [1, d)$*:* $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for all* $q \in [1, p^{\#})$,
- *for* $p = d$*:* $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for all* $q \in [1, \infty)$,
- *for* $p \in (d, \infty)$*:* $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^{0,\beta}(\overline{\Omega})$ *for all* $\beta \in [0, \alpha)$.

We are able to extend functions in the Sobolev space $W^{1,p}(\Omega)$ to become functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$. We call Eu the extension of u to $\mathbb{R}^n$.

Theorem 1.4 (Extension Theorem)**.** *Consider* $p \in [1, \infty]$ *and* Ω *Lipschitz. Select a bounded open set* V *such that* $\Omega \subset\subset V$. *Then there exists a bounded linear operator* $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ *such that for each* $u \in W^{1,p}(\Omega)$

(i) $Eu = u$ *almost everywhere in* Ω,
(ii) Eu *has support within* V,
(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(\Omega)}$, *the constant* c *depending only on* p, Ω *and* V.

Next we discuss the possibility of assigning "boundary values" along $\partial\Omega$ to a function $u \in W^{1,p}(\Omega)$, assuming that $\partial\Omega$ is $\mathcal{C}^{0,1}$. Now if $u \in \mathcal{C}(\overline{\Omega})$, then clearly u has values on $\partial\Omega$ in the common sense. The problem is that a typical function $u \in W^{1,p}(\Omega)$ is not in general continuous and, even worse, is only defined almost everywhere in Ω. Since $\partial\Omega$ has d-dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression "u restricted to $\partial\Omega$". The following operator resolves this problem, we call tr$u$ the trace of u on $\partial\Omega$.

Theorem 1.5 (Trace Theorem)**.** *Consider* $p \in [1, \infty]$ *and* Ω *Lipschitz. Then there exists a bounded linear operator* $\mathrm{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that*

(i) $\mathrm{tr}u = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap \mathcal{C}(\bar{\Omega})$,
(ii) $\|\mathrm{tr}u\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$, *for each* $u \in W^{1,p}(\Omega)$, *with the constant* C *depending only on* p *and* Ω.

For simplicity, in what follows we do not write tr$u$ but only u whenever u is a Sobolev function and we consider its restriction to the boundary.

Difference quotient approximations to weak derivatives

We will need to establish the higher regularity of solution which is not known a priori and therefore we will now describe the equivalent characterization of Sobolev function via difference quotients.

Definition 1.9. Assume $u : \Omega \to \mathbb{R}$ is a locally integrable function and $V \subset\subset \Omega$.

(i) The i^{th} difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}$$

for $i = 1, \ldots, d$, $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < \mathrm{dist}(V, \partial\Omega)$.

(ii) $D^h u := (D_1^h u, \ldots, D_d^h u)$.

Theorem 1.6 (Difference quotients and weak derivatives)**.** (i) *Let* $p \in [1, \infty)$ *and* $u \in W^{1,p}(\Omega)$. *Then for each* $V \subset\subset \Omega$

$$\|D^h u\|_{L^p(V)} \leq c\|Du\|_{L^p(\Omega)}$$

for some constant c *and* $0 < |h| < \mathrm{dist}(V, \partial\Omega)$.

(ii) *Assume* $p \in (1,\infty)$, $u \in L^p(V)$, *and there exists a constant* c *such that*

$$\|D^h u\|_{L^p(V)} \leq c$$

for all $0 < |h| < \frac{1}{2}\text{dist}(V, \partial\Omega)$. *Then* $u \in W^{1,p}(V)$ *with* $\|Du\|_{L^p(V)} \leq c$.

1.3 Functions of bounded variation

Functions of bounded variation on Ω, denoted by $BV(\Omega)$, have similar properties to those from the Sobolev space $W^{1,1}(\Omega)$, as we will see in following chapters. For more details concerning BV spaces we refer to Giusti (1984).

Definition 1.10. Let $\Omega \subset \mathbb{R}^d$ be an open set and let $f \in L^1(\Omega)$. Define

$$\int_\Omega |\nabla f| := \sup \left\{ \int_\Omega f \text{div} g \, \mathrm{d}x;\ g \in \mathcal{C}^{0,1}(\Omega, \mathbb{R}^d) \text{ and } |g(x)| \leq 1 \text{ for } x \in \Omega \right\},$$

where $\text{div} g = \sum_{i=1}^d \dfrac{\partial g_i}{\partial x_i}$ is divergence of g.

If $f \in \mathcal{C}^1(\Omega)$, then integration by parts gives

$$\int_\Omega f \text{div} g \, \mathrm{d}x = -\int_\Omega \sum_{i=1}^d \frac{\partial f}{\partial x_i} g_i \, \mathrm{d}x$$

for $g \in \mathcal{C}^{0,1}(\Omega, \mathbb{R}^d)$ and therefore

$$\int_\Omega |\nabla f| = \int_\Omega |Df| \, \mathrm{d}x,$$

where $Df = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_d}\right)$; for $f \in W^{1,1}(\Omega)$ Sobolev space Df consists of weak derivatives of f.

Definition 1.11. A function $f \in L^1(\Omega)$ is said to have bounded variation in Ω if $\int_\Omega |\nabla f| < \infty$. We define $BV(\Omega)$ as the space of all functions in $L^1(\Omega)$ with bounded variation.

If $f \in BV(\Omega)$ and ∇f is the gradient of f in the sense of distributions, then ∇f is a vector valued Radon measure and $\int_\Omega |\nabla f|$ is the total variation of ∇f on Ω. Moreover, according to the Lebesgue decomposition theorem, we can express ∇f as the sum of its regular and singular part, i.e., $\nabla f = (\nabla f)^r + (\nabla f)^s$, where $(\nabla f)^r$ is absolutely continuous with respect to Lebesgue measure μ, and $(\nabla f)^s$ and μ are singular. This is, that $(\nabla f)^s$ is supported on the Lebesgue null set.

Theorem 1.7 (Semicontinuity, see (Giusti, 1984, Theorem 1.9))**.** *Let* $\Omega \subseteq \mathbb{R}^d$ *be an open set and* $\{f_j\}$ *a sequence of functions in* $BV(\Omega)$ *which converge in* $L^1_{loc}(\Omega)$ *to a function* f. *Then*

$$\int_\Omega |\nabla f| \leq \liminf_{j\to\infty} \int_\Omega |\nabla f_j|.$$

Remark 3. Under the norm

$$\|f\|_{BV} = \|f\|_{L^1} + \int_\Omega |\nabla f|,$$

$BV(\Omega)$ is a Banach space. In the Chapter 3 we will use also notation $|\nabla f|(\Omega)$ instead of $\int_\Omega |\nabla f|$.

Symmetric mollifiers

Thanks to Theorem 1.2, we are able to approximate any Sobolev function by smooth ones, i.e., express it as a limit of smooth functions. A very useful tool to do so is mollification, where the sequence of smooth functions is made of the approximated function using positive symmetric mollifiers.

Definition 1.12. A function $\eta(x)$ is called a positive symmetric mollifier if
(i) $\eta(x) \in \mathcal{D}(\mathbb{R}^d)$,
(ii) η is zero outside a compact subset of B_1,
(iii) $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$,
(iv) $\eta(x) \geq 0$ for all x, and
(v) $\eta(x) = \upsilon(|x|)$ for some function $\upsilon : \mathbb{R}^+ \to \mathbb{R}$.

For a function $f \in L^1_{loc}(\mathbb{R}^d)$, given mollifier η and for any $\varepsilon > 0$ we define

$$\eta_\varepsilon(x) = \varepsilon^{-d}\eta\left(\frac{x}{\varepsilon}\right) \text{ and } f_\varepsilon = \eta_\varepsilon * f.$$

Then, using standard properties of mollifiers, we may show that
(a) $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$, $f_\varepsilon \to f$ in $L^1_{loc}(\mathbb{R}^d)$ and if $f \in L^1(\mathbb{R}^d)$, then $f_\varepsilon \to f$ in $L^1(\mathbb{R}^d)$,
(b) $A \leq f(x) \leq B$ for all x, then $A \leq f_\varepsilon(x) \leq B$ for all x,
(c) if $f, g \in L^1(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d} f_\varepsilon g\,\mathrm{d}x = \int_{\mathbb{R}^d} f g_\varepsilon\,\mathrm{d}x$,
(d) if $f \in \mathcal{C}^1(\mathbb{R}^d)$, then $\dfrac{\partial f_\varepsilon}{\partial x_i} = \left(\dfrac{\partial f}{\partial x_i}\right)_\varepsilon$,
(e) $\mathrm{supp} f \subseteq A$, then $\mathrm{supp} f_\varepsilon \subseteq A_\varepsilon = \{x;\, \mathrm{dist}(x, A) \leq \varepsilon\}$.

Theorem 1.8 (see (Giusti, 1984, Proposition 1.15))**.** *Suppose $f \in BV(\Omega)$ and suppose $A \subset\subset \Omega$ is an open set such that $|\nabla f|(\partial A) = 0$. Then, if f_ε are mollified functions described above (where f is extended to be 0 outside Ω if necessary),*

$$\int_A |\nabla f| = \lim_{\varepsilon \to 0} \int_A |\nabla f_\varepsilon|\,\mathrm{d}x.$$

1.4 Tools

In this section, commonly known results will be stated without proofs. We will refer to them in later chapters.

Notation

Let X be a normed function space of functions defined on $\Omega \subset \mathbb{R}^d$ and X^* be its dual. We simplify the notation

$$\langle a, a^* \rangle := \langle a, a^* \rangle_{(X,X^*)}$$

for all $a \in X$, $a^* \in X^*$ to express the duality. Moreover, if X is Hilbert and $(\cdot,\cdot)_X$ denotes the scalar product on X,

$$\langle a, a^* \rangle = (a, a^*)_X = a \cdot_X a^* =: a \cdot a^*$$

for all $a \in X$, $a^* \in X^*$.

In the text we usually use the last notation with the dot. However, we skip writing the index denoting the space because the information, which Hilbert space we work in, will always be obvious.

Also, many times in the thesis we work with a general constant denoted c (or C), which should be understood as a real (sometimes positive or nonnegative) number changing its value throughout the work, mostly even from line to line. Most of the time we do not concern about its value, the only important property of c is that it always remains finite. Sometimes we use notation $c = c(\alpha, \beta, \gamma)$, which means that c depends on α, β, γ whatever those are (e.g., function, point, domain, other constant). However, note that usually more useful information is what c is independent of, which is explicitly written every time when it is needed.

The next thing to remind is that working with function spaces, the function is usually the main point of interest. That means, we also do not write in which point the function is considered, e.g., instead of $\int_{\mathbb{R}} F(\nabla u(x))\,\mathrm{d}x$ we only would write $\int_{\mathbb{R}} F(\nabla u)\,\mathrm{d}x$. This is done to simplify the formulas significantly and also helps the reader to pay attention to what is important.

Last but not least, working with partial derivatives, the following useful notation will be used (not always, only when the readability and therefore also comprehensibility of expressions would be affected): $D_i u := \frac{\partial u}{\partial x_i}$, $D_{ij} u := \frac{\partial^2 u}{\partial x_i \partial x_j}$, etc.

Inequalities

Cauchy inequality. For $a, b \in \mathbb{R}$,

$$2ab \leq a^2 + b^2. \tag{1.7}$$

Young inequality. Let $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $a, b > 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.8}$$

Young inequality with ε. For $a, b > 0$, $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon > 0$

$$ab \leq \varepsilon a^p + c(\varepsilon) b^q, \tag{1.9}$$

for $c(\varepsilon) = (\varepsilon p)^{-q/p} q^{-1}$.

Hölder inequality. For $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(U)$, $v \in L^q(U)$, we have

$$\int_U |uv|\,\mathrm{d}x \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}. \tag{1.10}$$

Minkowski inequality. Assume $p \in [1, \infty]$ and $u, v \in L^p(U)$. Then

$$\|u + v\|_{L^p(U)} \leq \|u\|_{L^p(U)} + \|v\|_{L^p(U)}. \tag{1.11}$$

Cauchy-Schwarz inequality. For $x, y \in \mathbb{R}^n$,

$$|x \cdot y| \leq |x||y|. \tag{1.12}$$

Poincaré inequality. Let $p, q \in [1, \infty)$ and $U \subset \mathbb{R}^d$ be Lipschitz, then there exists a constant c, depending only on U and p, so that for every $u \in W_0^{1,p}(U)$

$$\|u\|_{L^p(U)} \leq c\|\nabla u\|_{L^p(U)}. \tag{1.13}$$

Interpolation inequality. Let $1 \leq p < q < \frac{dr}{d-r}$, $r \leq d$ and $U \subset \mathbb{R}^d$ be Lipschitz. Then there exists $\alpha \in (0,1)$ and C such that for all $u \in W^{1,r}(U)$

$$\|u\|_q \leq C\|u\|_p^{\alpha} \|u\|_{1,r}^{1-\alpha}. \tag{1.14}$$

If $r < d$, the above inequation holds for such α that $\frac{1}{q} = \alpha\frac{1}{p} + (1-\alpha)\frac{d-r}{dr}$ is satisfied. If $d = r$, it holds for every $\alpha < \frac{p}{q}$.
Jensen inequality.[see Perlman (1974)] Let $(\Omega; A; \mu)$ be a measure space such that $\mu(\Omega) = 1$. Let $g : \Omega \to \mathbb{R}^n$ be μ-measurable and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be continuous and convex, then

$$\varphi\left(\int_\Omega g \mathrm{d}\mu\right) = \int_\Omega (\varphi \circ g)\mathrm{d}\mu. \tag{1.15}$$

Theorems

Theorem 1.9 (Lebesgue Dominated Convergence Theorem)**.** *Assume the functions $\{f_n\}_{n=1}^\infty$ are integrable and $f_n \to f$ almost everywhere. Suppose also $|f_n| \leq g$ almost everywhere, for some summable function g. Then*

$$\int_{\mathbb{R}^d} f_n \, \mathrm{d}x \to \int_{\mathbb{R}^d} f \, \mathrm{d}x.$$

Theorem 1.10 (Levi Monotone Convergence Theorem)**.** *Assume $\{a_n\}_{n=1}^\infty$ is a monotone sequence of real numbers. Then this sequence has limit if and only if it is bounded.*

Theorem 1.11 (Vitali Theorem)**.** *[see Dunford and Schwartz (1958)] Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions with finite integrals over a measurable bounded set $\Omega \subset \mathbb{R}^d$. Suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for almost all $x \in \Omega$ and let f be an almost everywhere finite function. Suppose that the following condition is satisfied:
For every $\varepsilon > 0$ there exists $\delta > 0$ with the property: if $B \subset \Omega$, $\mu(B) < \delta$, then

$$\int_B |f_n(x)|\mathrm{d}x < \varepsilon \quad \textit{for all } n \in \mathbb{N}.$$

Then the function f has a finite integral over Ω and

$$\lim_{n\to\infty} \int_\Omega f_n(x)\mathrm{d}x = \int_\Omega f(x)\mathrm{d}x.$$

Theorem 1.12 (Selection principle)**.** *[see Banach (1932)] In a Banach space with a separable predual, any bounded sequence contains a weakly* convergent subsequence.*

Theorem 1.13 (Dunford – Pettis Theorem)**.** *Let* $\Omega \subset \mathbb{R}^d$ *and* $A \subset L^1(\Omega)$ *be a bounded subset. Then the following is equivalent:*

- A *is weakly pre-compact,*
- A *is uniformly equi-integrable, i.e., for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $U \subset \Omega$, $|U| \leq \delta$ *and all* $u \in A$ *there holds*

$$\int_U |u| \, \mathrm{d}x \leq \varepsilon.$$

Remark 4. In the setting of the theorem above, the following is also equivalent:

- A is uniformly equi-integrable,
- there exists a sequence $\{\lambda_i\}$, $0 < \lambda_1 < \lambda_2 < \ldots$, such that for every $i \in \mathbb{N}$ and for all $u \in A$,

$$\int_{\{|u|>\lambda_i\}} |u| \, \mathrm{d}x \leq 4^i.$$

Chapter 2

Algebraic properties of the mappings F and A

Inspired by (1), we will first define following mappings, $F : \mathbb{R}^d \to \mathbb{R}$, $A : \mathbb{R}^d \to \mathbb{R}^d$ and $B : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, as

$$\begin{aligned} F(\eta) &:= \frac{1}{2}\int_0^{|\eta|^2} \frac{\mathrm{d}s}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}}, \\ A_i(\eta) &:= \frac{\partial F}{\partial \eta_i}(\eta) = \frac{\eta_i}{(1+|\eta|^a)^{\frac{1}{a}}}, \\ B_{i,j}(\eta) &:= \frac{\partial A_i}{\partial \eta_j}(\eta) = \frac{\delta_{i,j}}{(1+|\eta|^a)^{\frac{1}{a}}} - \frac{\eta_i\eta_j|\eta|^{a-2}}{(1+|\eta|^a)^{\frac{1}{a}+1}}, \end{aligned} \tag{2.1}$$

for $a \in (0,\infty]$, $i,j = 1,\dots,d$. We will provide several important properties (monotonicity, convexity, etc.) of these mappings below.

Definition 2.1. A function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called strictly monotone provided

$$(f(x) - f(y))(x-y) > 0$$

for all $x, y \in \mathbb{R}^d$, $u \neq v$. Also, f is called Lipschitz continuous provided there exists $L \in [0,\infty)$ such that

$$\left|\frac{\partial f(x)}{\partial x_i}\right| \le L$$

for all $x \in \mathbb{R}^d$ and $i = 1,\dots,d$.

A function $g : \mathbb{R}^d \to \mathbb{R}$ is called strictly convex provided

$$g(\lambda x + (1-\lambda)y) < \lambda g(x) + (1-\lambda)g(y)$$

for all $x, y \in \mathbb{R}^d$ and each $\lambda \in (0,1)$.

Lemma 2.1. Let F and A be mappings defined in (2.1). Then

(i) A is strictly monotone and bounded,

(ii) F is strictly convex,

(iii) there exists $\varepsilon = \varepsilon(a)$, $c = c(a) \in \mathbb{R}$, $\varepsilon < 1$ such that

$$|F(\xi) - |\xi|| \le c(1+|\xi|)^{1-\varepsilon}, \tag{2.2}$$

(iv) for any u, v

$$A(u)(v-u) \leq F(v) - F(u),$$

(v) A is Lipschitz continuous, i.e., there exists $L > 0$ such that for any $\eta \in \mathbb{R}^d$

$$|B| \leq L.$$

Proof. (i): The boundedness is obvious,

$$|A| = \left| \frac{\eta}{(1+|\eta|^a)^{\frac{1}{a}}} \right| \leq 1.$$

To prove that A is strictly monotone, we evaluate for $u \neq v$

$$\begin{aligned}
&\left(\frac{u}{(1+|u|^a)^{\frac{1}{a}}} - \frac{v}{(1+|v|^a)^{\frac{1}{a}}} \right) \cdot (u-v) \\
=&\frac{|u|^2}{(1+|u|^a)^{\frac{1}{a}}} + \frac{|v|^2}{(1+|v|^a)^{\frac{1}{a}}} - \frac{u \cdot v}{(1+|u|^a)^{\frac{1}{a}}} - \frac{u \cdot v}{(1+|v|^a)^{\frac{1}{a}}} \\
=&\frac{|u|}{(1+|u|^a)^{\frac{1}{a}}} \left(|u| - \frac{u \cdot v}{|u|} \right) + \frac{|v|}{(1+|v|^a)^{\frac{1}{a}}} \left(|v| - \frac{u \cdot v}{|v|} \right) = (\#).
\end{aligned}$$

We now distinguish 2 cases:$|u| = |v|$, or $|u| \neq |v|$. In the first case,

$$(\#) = \frac{2|u|}{(1+|u|^a)^{\frac{1}{a}}} \left(|u| - \frac{u \cdot v}{|u|} \right)$$

and this is positive thanks to $0 < |u-v|^2 = 2|u|^2 - 2u \cdot v$. In the second case we use that $u \cdot v \leq |u| \cdot |v|$ and therefore

$$\begin{aligned}
(\#) &\geq \frac{|u|}{(1+|u|^a)^{\frac{1}{a}}} (|u| - |v|) + \frac{|v|}{(1+|v|^a)^{\frac{1}{a}}} (|v| - |u|) \\
&= (|u| - |v|) \left(\frac{|u|}{(1+|u|^a)^{\frac{1}{a}}} - \frac{|v|}{(1+|v|^a)^{\frac{1}{a}}} \right) > 0
\end{aligned}$$

and we have strict monotonicity of A.

(ii): Defining the function $g(\lambda)$ in the way that

$$g(\lambda) := \lambda F(v) + (1-\lambda) F(u) - F(\lambda v + (1-\lambda) u),$$

justification of strict convexity of F becomes equal to proving that $g(\lambda) > 0$ for all $\lambda \in (0,1)$. Find the derivation of g,

$$\begin{aligned}
g'(\lambda) &= F(v) - F(u) - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{1}{2} \int_0^{|\lambda v + (1-\lambda) u|^2} \frac{\mathrm{d}s}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}} \\
&= F(v) - F(u) - \frac{(\lambda v + (1-\lambda) u, v-u)}{(1+|\lambda v + (1-\lambda) u|^a)^{\frac{1}{a}}},
\end{aligned}$$

and denote $h := \lambda v + (1-\lambda)u$ to simplify expressions. Finding the sign of the second derivative will tell us about the behaviour of g. In the next we use that $F(v)$, $F(u)$ are independent of λ and that $\frac{\partial |h|^2}{\partial \lambda} = 2(h, v-u)$,

$$\begin{aligned} g''(\lambda) &= -\frac{|v-u|^2}{(1+|h|^a)^{\frac{1}{a}}} + \frac{1}{a}\frac{(h,v-u)}{(1+|h|^a)^{\frac{1}{a}+1}}\frac{\mathrm{d}}{\mathrm{d}\lambda}|h|^a \\ &= -\frac{|v-u|^2}{(1+|h|^a)^{\frac{1}{a}}} + \frac{1}{a}\frac{(h,v-u)}{(1+|h|^a)^{\frac{1}{a}+1}}\frac{\mathrm{d}}{\mathrm{d}\lambda}(|h|^2)^{\frac{a}{2}} \\ &\le -\frac{|v-u|^2}{(1+|h|^a)^{\frac{1}{a}}} + \frac{(h,v-u)^2}{(1+|h|^a)^{\frac{1}{a}+1}}|h|^{a-2} \\ &\le -\frac{|v-u|^2}{(1+|h|^a)^{\frac{1}{a}}} + \frac{|h|^a}{(1+|h|^a)^{\frac{1}{a}+1}}|v-u|^2 \\ &= \frac{|v-u|^2}{(1+|h|^a)^{\frac{1}{a}+1}}(-1-|h|^a+|h|^a) < 0, \end{aligned}$$

Note that $g(0) = g(1) = 0$ and from $g''(\lambda) < 0$ for all $u \neq v$ we see that function g is strictly concave which implies that g is positive on $(0,1)$ and therefore F is strictly convex.

(iii): Note that for small ξ the inequality (2.2) holds. We also remark that constant c is changing its value throughout the proof, however always remains finite. First of all, we express $|\xi|$ in the integral form and deduct it from $F(\xi)$. We denote by $(*) := |F(\xi) - |\xi||$ and will refer to it in the following,

$$(*) = \left|\frac{1}{2}\int_0^{|\xi|^2} \frac{1}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}} - \frac{1}{\sqrt{s}}\,\mathrm{d}s\right| = \left|\frac{1}{2}\int_0^{|\xi|^2} \frac{1-\left(\frac{1+|s|^{\frac{a}{2}}}{|s|^{\frac{a}{2}}}\right)^{\frac{1}{a}}}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}s\right|.$$

Since $\left(\frac{1+|s|^{\frac{a}{2}}}{|s|^{\frac{a}{2}}}\right)^{\frac{1}{a}} > 1$, removing the absolute value will change the sign in the numerator,

$$(*) \le \frac{1}{2}\int_0^{|\xi|^2} \frac{\left(\frac{1+|s|^{\frac{a}{2}}}{|s|^{\frac{a}{2}}}\right)^{\frac{1}{a}} - 1}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}s = \frac{1}{2}\int_0^{|\xi|^2} \frac{\mathrm{e}^{\frac{1}{a}\ln\left(1+\frac{1}{|s|^{\frac{a}{2}}}\right)} - 1}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}s.$$

Thanks to the estimate $\ln(1+x) \le x$ we may simplify the power of e, used as $\frac{1}{a}\ln\left(1+\frac{1}{|s|^{\frac{a}{2}}}\right) \le \frac{1}{a|s|^{\frac{a}{2}}}$. As observed in the beginning, the problem may occur when $\xi \to \infty$. Therefore we split the integral in two pieces for $|\xi|^2$ lower than some number. Let us now deduce this number. In the next we will use that $(\mathrm{e}^x - 1) \le 2x$ for $|x| \le 1$, with $\frac{1}{a|s|^{\frac{a}{2}}}$ representing x. Denote $t := \left(\frac{1}{a}\right)^{\frac{2}{a}}$, then $t \le |s|$ is equivalent to $\frac{1}{a|s|^{\frac{a}{2}}} \le 1$. In this value we split the integral - the first one integrating over [0,t] is bounded by constant c, and the second one over [t, $|\xi|^2$],

$$\begin{aligned} (*) &\le c + \frac{1}{2}\int_t^{|\xi|^2} \frac{\mathrm{e}^{\frac{1}{a|s|^{\frac{a}{2}}}} - 1}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}s \le c + \int_t^{|\xi|^2} \frac{1}{a|s|^{\frac{a}{2}}(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}s \\ &\le c + c\int_t^{|\xi|^2} \frac{1}{a|s|^{\frac{a}{2}+\frac{1}{2}}}\,\mathrm{d}s. \end{aligned}$$

Now, we distinguish two situations, $a = 1$ and $a \neq 1$. Firstly, let $a = 1$. Then

$$(*) \leq c + \int_t^{|\xi|^2} \frac{1}{|s|}\,\mathrm{d}s \leq c + \ln|\xi|^2 \leq c + 4\ln|\xi|^{\frac{1}{2}} \leq c + 4\ln(1+|\xi|)^{\frac{1}{2}} \leq c(1+|\xi|)^{\frac{1}{2}}.$$

Secondly, let $a \neq 1$. In this case

$$(*) \leq c + \frac{2}{a(1-a)} \left[|s|^{\frac{1-a}{2}}\right]_t^{|\xi|^2} \leq c(1+|\xi|)^{1-a}.$$

Setting $\varepsilon = \min\{a, \frac{1}{2}\}$ the proof is complete.

(iv): Hereby we use the fact that functional F is convex, i.e., for any $\lambda \in (0,1)$,

$$\frac{1}{\lambda}\left(F(\lambda v + (1-\lambda)u) - F(u)\right) \leq F(v) - F(u). \tag{2.3}$$

Evaluating the term on the left hand side

$$\frac{1}{\lambda}\left(F(\lambda v + (1-\lambda)u) - F(u)\right) = \frac{1}{\lambda}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} F(u + \tau\lambda(v-u))\,\mathrm{d}\tau$$
$$= \int_0^1 A(u + \tau\lambda(v-u)) \cdot (v-u)\,\mathrm{d}\tau \to A(u)(v-u),$$

as $\lambda \to 0_+$. We are allowed to do so since A is bounded and therefore $|\int_0^1 A(u + \tau\lambda(v-u)) \cdot (v-u)\,\mathrm{d}\tau| \leq c|v-u|$, the last expression is integrable and we are allowed to use the Lebesgue Theorem (Theorem 1.9) to switch the limit $\lambda \to 0_+$ and integral over $(0,1)$. Use of this convergence in (2.3) completes the proof.

(v): This is an easy result, since

$$|B| = \max_{i,j=1,\dots,d} |B_{i,j}(\eta)| = \max_{i,j=1,\dots,d} \left|\frac{1}{(1+|\eta|^a)^{\frac{1}{a}}}\right| + \left|\frac{\eta_i}{|\eta|}\frac{\eta_j}{|\eta|}\frac{|\eta|^a}{(1+|\eta|^a)^{\frac{1}{a}+1}}\right| \leq 2.$$

Setting $L = 2$, we have just proved the Lipschitz continuity of A. □

Remark 5. In the work we will use also other, however weaker, estimate, following directly from (2.2). There exists constant $c = c(a) \in \mathbb{R}$, such that

$$F(\xi) \leq c(1+|\xi|)$$

for all $\xi \in \mathbb{R}^d$.

Remark 6. Similarly, the proof of (v) can be modified as

$$|B| \leq \max_{i,j=1,\dots,d} \left|\frac{1}{1+|\eta|}\right| + \left|\frac{1}{1+|\eta|}\right| \leq \frac{2}{1+|\eta|}.$$

Chapter 3

Possible notions of solution and relationships between them

3.1 The problem and notions of solution

Similarly as before, we consider Lipschitz domain $\Omega \subset \mathbb{R}^d$. We will deal with four notions of solution to our task and will show the context of how they are related. The problem is stated as

$$\begin{aligned} -\mathrm{div}\frac{\nabla u}{(1+|\nabla u|^a)^{\frac{1}{a}}} &= 0 \qquad \text{in } \Omega \\ u &= u_D \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Throughout the chapter, we consider F and A given by (2.1).

Definition 3.1. We say that $u \in W^{1,1}(\Omega)$, $u = u_D$ on $\partial\Omega$ is the weak solution to problem (3.1), if

$$\int_\Omega \frac{\nabla u}{(1+|\nabla u|^a)^{\frac{1}{a}}} \nabla\varphi \,\mathrm{d}x = 0 \tag{3.2}$$

for all $\varphi \in W_0^{1,1}(\Omega)$.

Definition 3.2. We say that $u \in W^{1,1}(\Omega)$, $u = u_D$ on $\partial\Omega$ is the solution to (3.1), if it minimizes the following

$$\int_\Omega F(\nabla u)\,\mathrm{d}x \le \int_\Omega F(\nabla v)\,\mathrm{d}x \tag{3.3}$$

for all $v \in W^{1,1}(\Omega)$, $v = u_D$ on $\partial\Omega$.

Definition 3.3. We say that $u \in W^{1,1}(\Omega)$ is the solution to (3.1) with generalized boundary condition, if it minimizes

$$\int_\Omega F(\nabla u)\,\mathrm{d}x + \int_{\partial\Omega} |u - u_D|\,\mathrm{d}S \le \int_\Omega F(\nabla v)\,\mathrm{d}x + \int_{\partial\Omega} |v - u_D|\,\mathrm{d}S \tag{3.4}$$

for all $v \in W^{1,1}(\Omega)$.

Finally, we define the notion of solution in $BV(\Omega)$. For $f \in BV(\Omega)$, ∇f can be understood as Radon measure with decomposition $\nabla f = (\nabla f)^r + (\nabla f)^s$ as described in Chapter 1.

Definition 3.4. Let $\Omega_0 \supset\supset \Omega$. We say that $u \in BV(\Omega_0)$, $u = u_D$ on $\Omega_0 \setminus \overline{\Omega}$ is the minimizer of (3.1) in the space $BV(\Omega)$, if for all $v \in BV(\Omega_0)$, $v = u_D$ on $\Omega_0 \setminus \overline{\Omega}$ it holds that

$$\int_\Omega F((\nabla u)^r)\,\mathrm{d}x + |(\nabla u)^s|(\overline{\Omega}) \leq \int_\Omega F((\nabla v)^r)\,\mathrm{d}x + |(\nabla v)^s|(\overline{\Omega}). \tag{3.5}$$

We can describe the situation symbolically as

$$(3.2) \Leftrightarrow (3.3) \Rightarrow (3.4) \Rightarrow (3.5).$$

The first equivalence should be read as: if u satisfies (3.2), i.e., solves 3.1 in the sense of Definition 3.1, then it is equivalent to the situation that u satisfies (3.3), i.e., solves 3.1 in the sense of Definition 3.2. The following two implications should be treated in the same way. Throughout the chapter we will provide these proofs. First of all, Theorem 3.1 will show equivalence between the problem of finding the weak solution and finding the minimizer of the functional. Theorem 3.2 says that if the minimizer with boundary condition is provided then we can easily get the more generally defined minimizer. Finally, the minimizer in the space $BV(\Omega)$ will be derived in Theorem 3.3.

3.2 The analogy between notions of solution

Theorem 3.1. *Let $u_D \in W^{1,1}(\Omega)$ be given. The function $u \in W^{1,1}(\Omega)$ is the unique weak solution to problem* (3.1) *(i.e.,* (3.2) *holds for u) if and only if it is the minimizer of* (3.3)*.*

Proof. "$\Rightarrow$": Since $v = u = u_D$ on $\partial\Omega$, the function $\varphi := v - u$ satisfies $\varphi \in W_0^{1,1}(\Omega)$. Note that Lemma 2.1, (iv) says

$$A(\nabla u)(\nabla v - \nabla u) \leq F(\nabla v) - F(\nabla u).$$

Integration over Ω preserves the inequality and using assumptions

$$0 = \int_\Omega A(\nabla u) \cdot \nabla\varphi\,\mathrm{d}x \leq \int_\Omega F(\nabla v) - F(\nabla u)\,\mathrm{d}x,$$

so u really minimizes (3.3).

"$\Leftarrow$": Let $v = u + t\varphi$, $t \in \mathbb{R}^+$, $\varphi \in W_0^{1,1}(\Omega)$. Inequality (3.3) is equivalent to $0 \leq \int_\Omega F(\nabla u + t\nabla\varphi) - F(\nabla u)\,\mathrm{d}x$. Then

$$\begin{aligned} 0 &\leq \int_\Omega \frac{F(\nabla u + t\nabla\varphi) - F(\nabla u)}{t}\,\mathrm{d}x = \frac{1}{t}\int_\Omega\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} F(\nabla u + t\tau\nabla\varphi)\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int_\Omega\int_0^1 \sum_{i=1}^d \frac{\partial F}{\partial\eta_i}(\nabla u + t\tau\nabla\varphi) \cdot (\nabla\varphi)_i\,\mathrm{d}\tau\,\mathrm{d}x, \end{aligned}$$

where using $A_i = \dfrac{\partial F}{\partial \eta_i}$ and reverting to original integral over Ω we obtain

$$0 \leq \int_\Omega\int_0^1 A(\nabla u + t\tau\nabla\varphi) \cdot \nabla\varphi\,\mathrm{d}\tau\,\mathrm{d}x.$$

We know that A is bounded, therefore $|\int_0^1 A(\nabla u(x) + t\tau\nabla\varphi) \cdot \nabla\varphi(x)\,\mathrm{d}\tau| \leq c|\nabla\varphi(x)|$, the last expression is integrable and we are allowed to use the Lebesgue Theorem (Theorem 1.9) to switch the limit $t \to 0_+$ and integral over $(0,1)$ to get $0 \leq \int_\Omega A(\nabla u) \cdot \nabla\varphi\,\mathrm{d}x$. The function φ was chosen arbitrarily, so it holds for $-\varphi$, too. Therefore

$$\int_\Omega A(\nabla u) \cdot \nabla\varphi\,\mathrm{d}x = 0.$$

Since $A(\nabla u) = \dfrac{\nabla u}{(1 + |\nabla u|^a)^{\frac{1}{a}}}$, this proves the first implication.

Note that since F is strictly convex (see Theorem 2.1, (ii)) the minimum is obtained and is unique. Working with gradients, this uniqueness holds up to an additive constant. However, the assumption $u = u_D$ on $\partial\Omega$ fixes u and does not allow any other solutions. □

Theorem 3.2. *Let $u_D \in W^{1,1}(\Omega)$ be given. If $u \in W^{1,1}(\Omega)$, $u = u_D$ on $\partial\Omega$ minimizes* (3.3)*, then this u is also the unique solution to* (3.4)*.*

Proof. Assuming (3.3), it is not difficult to observe that thanks to the boundary condition the following inequality holds,

$$\int_\Omega F(\nabla u)\,\mathrm{d}x + \int_{\partial\Omega} |u - u_D| \leq \int_\Omega F(\nabla(u_D + v))\,\mathrm{d}x \text{ for all } v \in W_0^{1,1}(\Omega). \quad (3.6)$$

In the next, we will prove that (3.6) implies (3.4). Let $\Omega_0 \subset\subset \Omega$ be smooth and $d^0(x) := \operatorname{dist}(x, \partial\Omega_0)$ be the distance function for $x \in \Omega_0$. We note that $d^0(x)$ is at least $\mathcal{C}^2$ near $\partial\Omega_0$ and that $-\nabla d^0 = \mathbf{n}$ on $\partial\Omega_0$. For $\varepsilon > 0$ we define

$$d^0_\varepsilon(x) := \begin{cases} \min\left(1, \dfrac{d^0(x)}{\varepsilon}\right) & \text{for } x \in \Omega_0 \\ 0 & \text{outside } \Omega_0. \end{cases}$$

For $\Omega_{0,\varepsilon} := \{x \in \Omega_0;\ \operatorname{dist}(x, \partial\Omega_0) < \varepsilon\}$, it holds that $d^0_\varepsilon \equiv 1$ in $\Omega_0 \setminus \Omega_{0,\varepsilon}$. If for arbitrary $w \in W^{1,1}(\Omega)$ we set $v^\varepsilon := d^0_\varepsilon(w - u_D)$, one can easily check that $v^\varepsilon \in W_0^{1,1}(\Omega)$ for any $\varepsilon > 0$ and therefore can be used in the right hand side of (3.6). For estimating the term on the right hand side we need to note that

$$|\nabla(u_D + v^\varepsilon)| = |(1 - d^0_\varepsilon)\nabla u_D + d^0_\varepsilon \nabla w + (w - u_D)\nabla d^0_\varepsilon|, \quad (3.7)$$

$$\nabla(u_D + v^\varepsilon) \to \chi_{\Omega\setminus\overline{\Omega}_0}\nabla u_D + \chi_{\Omega_0}\nabla w \text{ a.e. in } \Omega \text{ and} \quad (3.8)$$

$$||u_D + v^\varepsilon||_{1,1} \leq c + \frac{c}{\varepsilon}\int_{\Omega_{0,\varepsilon}} |w - u_D|\,\mathrm{d}x \leq c, \quad (3.9)$$

where in the first inequality in (3.9) we use the fact that both $u_D, w \in W^{1,1}(\Omega)$, also $d^0_\varepsilon \in [0,1]$ and that $d^0_\varepsilon(x)$ is at least $\mathcal{C}^2$ near $\partial\Omega_0$. The second one follows from the fact that Ω_0 is smooth and can be justified in what follows.

Add $\pm \liminf_{\varepsilon\to 0_+} \int_\Omega |\nabla(u_D + v^\varepsilon)|\,\mathrm{d}x$ to the right hand side of (3.6) to see that the first limit is finite thanks to the pointwise convergence in (3.8), the sublinear growth proven in (ii) of Lemma 2.1, and finally the Vitali Theorem (Theorem 1.11). This limit will be forgotten after inequality in the third line, and the second one is expressed using (3.7) and (3.8),

$$\begin{aligned}
&\liminf_{\varepsilon\to 0_+}\int_\Omega F(\nabla(u_D+v^\varepsilon))\,\mathrm{d}x\\
&=\liminf_{\varepsilon\to 0_+}\int_\Omega F(\nabla(u_D+v^\varepsilon))-|\nabla(u_D+v^\varepsilon)|\,\mathrm{d}x+\liminf_{\varepsilon\to 0_+}\int_\Omega|\nabla(u_D+v^\varepsilon)|\,\mathrm{d}x\\
&\le\int_{\Omega_0}F(\nabla w)\,\mathrm{d}x+\int_{\Omega\setminus\Omega_0}F(\nabla u_D)\,\mathrm{d}x+\liminf_{\varepsilon\to 0_+}\int_{\Omega_{0,\varepsilon}}|(u_D-w)\nabla d^0_\varepsilon|\,\mathrm{d}x.
\end{aligned}\tag{3.10}$$

In the next we use that ∇d^0 is Lipschitz in Ω_0, it has the same direction as ∇d^0_ε and $|\nabla d^0|=1$ on $\partial\Omega_0$, then we add $\pm|u_D-w|\nabla d^0_\varepsilon\cdot\nabla d^0$. The inequality that follows will be explained below,

$$\begin{aligned}
|(u_D-w)\nabla d^0_\varepsilon| &= |u_D-w|\nabla d^0_\varepsilon\cdot\left(\frac{\nabla d^0}{|\nabla d^0|}-\nabla d^0\right)+|u_D-w|\nabla d^0_\varepsilon\cdot\nabla d^0\\
&\le\ c|u_D-w|-|u_D-w|\nabla(1-d^0_\varepsilon)\cdot\nabla d^0,
\end{aligned}\tag{3.11}$$

because $\nabla d^0_\varepsilon=-\nabla(1-d^0_\varepsilon)$ and

$$\begin{aligned}
\nabla d^0_\varepsilon\cdot\left(\frac{\nabla d^0}{|\nabla d^0|}-\nabla d^0\right)&\le|\nabla d^0_\varepsilon|\,|\nabla d^0|\left|\frac{1}{|\nabla d^0|}-1\right|=|\nabla d^0_\varepsilon|\,\big|1-|\nabla d^0|\big|\\
&\le c\varepsilon|\nabla d^0_\varepsilon|=c|\nabla d^0|=c.
\end{aligned}$$

In the next we continue estimating the last element in (3.10), firstly by (3.11) and secondly integrating by parts. In both steps we use the fact that the first integrands on the second, third and fourth line are integrable (d^0 is $\mathcal{C}^2$, $(u_D-w)\in W^{1,1}(\Omega)$ implies that also $|u_D-w|\in W^{1,1}(\Omega)$) and Lebesgue integral is absolutely continuous, i.e., the limit of integral over $\Omega_{0,\varepsilon}$ is 0 as $|\Omega_{0,\varepsilon}|\to 0$ (which is as $\varepsilon\to 0_+$), and finally the fact that $-\nabla d^0=\mathbf{n}$ on $\partial\Omega_0$ and $\mathbf{n}\cdot\mathbf{n}=|\mathbf{n}|^2=1$,

$$\begin{aligned}
&\liminf_{\varepsilon\to 0_+}\int_{\Omega_{0,\varepsilon}}|(u_D-w)\nabla d^0_\varepsilon|\,\mathrm{d}x\\
&\le c\liminf_{\varepsilon\to 0_+}\int_{\Omega_{0,\varepsilon}}|u_D-w|\,\mathrm{d}x+\liminf_{\varepsilon\to 0_+}\int_{\Omega_{0,\varepsilon}}-|u_D-w|\nabla(1-d^0_\varepsilon)\cdot\nabla d^0\,\mathrm{d}x\\
&=\liminf_{\varepsilon\to 0_+}\int_{\Omega_{0,\varepsilon}}(1-d^0_\varepsilon)\nabla|u_D-w|\cdot\nabla d^0\,\mathrm{d}x\\
&\quad+\liminf_{\varepsilon\to 0_+}\int_{\Omega_{0,\varepsilon}}|u_D-w|(1-d^0_\varepsilon)\cdot\Delta d^0\,\mathrm{d}x-\int_{\partial\Omega_0}|u_D-w|\nabla d^0\cdot\mathbf{n}\,\mathrm{d}S\\
&=\int_{\partial\Omega_0}|u_D-w|\,\mathrm{d}S.
\end{aligned}$$

Using this inequality in (3.10) and the result in (3.6), what we obtain is that for all $w\in W^{1,1}(\Omega)$ and all smooth $\Omega_0\subset\subset\Omega$ there holds

$$\begin{aligned}
&\int_\Omega F(\nabla u)\,\mathrm{d}x+\int_{\partial\Omega}|u-u_D|\,\mathrm{d}S\\
&\le\int_{\Omega_0}F(\nabla w)\,\mathrm{d}x+\int_{\Omega\setminus\overline{\Omega}_0}F(\nabla u_D)\,\mathrm{d}x+\int_{\partial\Omega_0}|w-u_D|\,\mathrm{d}S,
\end{aligned}$$

letting $\Omega_0 \nearrow \Omega$ and using continuity with respect to domain we finally get (3.4).

As for the uniqueness, assume that u and v are two minimizers of (3.4). From the strict convexity of F and the convexity of the absolute value it follows that $\nabla(u - v) = 0$ in Ω, what directly implies that $u = v + c$ where $c \in \mathbb{R}$. Thus, we have got uniqueness up to an additive constant. Therefore, subsequently, (3.4) reduces to the identity

$$\int_{\partial\Omega} |u - u_D| \, \mathrm{d}S = \int_{\partial\Omega} |v - u_D| \, \mathrm{d}S.$$

However, $u = u_D$ on $\partial\Omega$ and therefore necessarily $v = u_D$ on $\partial\Omega$ and that finishes the proof. □

Theorem 3.3. *For given $u_D \in W^{1,1}(\Omega)$, let $u \in W^{1,1}(\Omega)$ be a function that solves* (3.4) *for all $v \in W^{1,1}(\Omega)$. Then u is also solution to* (3.5).

Proof. This proof is a little longer than others and for the purpose of easier orientation we will split it into few steps. First of all, the function $u \in W^{1,1}(\Omega)$ is extended to $\Omega_0 \supset\supset \Omega$ in the way that $u \in BV(\Omega_0)$. In the second step we express what the seminorm $|\nabla u|(\Omega_0)$ is equal to. In the Step 3, equality between the left hand sides of (3.4) and (3.5) will be proven. In the next part, taking arbitrary $v \in BV(\Omega_0)$ we construct such a function $\tilde{v}^\varepsilon$, that again $\tilde{v}^\varepsilon \in W^{1,1}(\Omega)$ and also $\tilde{v}^\varepsilon \in BV(\Omega_0)$. For this $\tilde{v}^\varepsilon$, (3.4) holds. Finally, in the Step 5, we justify the limit as $\varepsilon \to 0$ and by obtaining the right hand side of (3.5) the proof will be finished.

Step 1: Since BV spaces allow the function to jump, a problem can occur on the boundary. The domain needs to be extended to capture the behaviour there. Let $\Omega_0 \supset\supset \Omega$. The Extension Theorem (Theorem 1.4) guarantees that u_D can be extended to $W^{1,1}(\Omega_0)$. We claim that $\tilde{u}$ defined as

$$\begin{aligned} \tilde{u} &:= u \quad && \text{in } \Omega \\ \tilde{u} &:= u_D \quad && \text{on } \Omega_0 \setminus \overline{\Omega} \end{aligned}$$

is in the space $BV(\Omega_0)$. Indeed, it is easy to observe that $\tilde{u} \in L^1(\Omega_0)$, since $u \in L^1(\Omega)$, $u_D \in L^1(\Omega_0 \setminus \overline{\Omega})$ and $\partial\Omega$ is Lebesgue null set. Let us now take a look at what $\nabla\tilde{u}$ looks like on Ω_0. Its i-th component, in notation $D_i\tilde{u} = \dfrac{\partial \tilde{u}}{\partial x_i}$, is

$$\left\langle \frac{\tilde{u}}{x_i}, \varphi \right\rangle = -\int_{\Omega_0} \tilde{u} \cdot \frac{\varphi}{x_i} \, \mathrm{d}x = -\int_{\Omega} u \cdot \frac{\varphi}{x_i} \, \mathrm{d}x - \int_{\Omega_0 \setminus \Omega} u_D \cdot \frac{\varphi}{x_i} \, \mathrm{d}x.$$

Using integration by parts and the fact that if $\mathbf{n}$ is the unit outward vector for Ω, then $-\mathbf{n}$ is the unit outward vector for $\Omega_0 \setminus \Omega$,

$$\left\langle \frac{\tilde{u}}{x_i}, \varphi \right\rangle = \int_{\Omega} \frac{u}{x_i} \varphi \, \mathrm{d}x + \int_{\Omega_0 \setminus \overline{\Omega}} \frac{u_D}{x_i} \varphi \, \mathrm{d}x - \int_{\partial\Omega} (u - u_D)\varphi \mathbf{n}_i \, \mathrm{d}S \tag{3.12}$$

holds for all test functions $\varphi \in \mathcal{D}(\Omega_0)$. Let the seminorm of $\nabla\tilde{u}$ in $BV(\Omega_0)$ be denoted by $|\nabla\tilde{u}|(\Omega_0)$. Sum (3.12) through $i = 1, \ldots d$ and take the supremum over all $\varphi \in \mathcal{C}_0^{0,1}(\Omega_0)$, $||\varphi|| \leq 1$ to obtain the definition of the seminorm. Here we claim that the supremum is acquired and is equal to the following expression, which is finite. This claim is proven in the Step 2.

$$
\begin{aligned}
|\nabla\tilde{u}|(\Omega_0) &= \sup_{\substack{\varphi\in\mathcal{C}_0^{0,1}(\Omega_0),\\ ||\varphi||\leq 1}} \int_\Omega \nabla u\cdot\varphi\,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}} \nabla u_D\cdot\varphi\,\mathrm{d}x - \int_{\partial\Omega}(u-u_D)\varphi\cdot\mathbf{n}\,\mathrm{d}S \\
&= \int_\Omega |\nabla u|\,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}} |\nabla u_D|\,\mathrm{d}x + \int_{\partial\Omega}|u-u_D|\,\mathrm{d}S < \infty
\end{aligned} \tag{3.13}
$$

and that is why $\tilde{u}\in BV(\Omega_0)$ and its norm is $\|\tilde{u}\|_{BV(\Omega_0)} = \|\tilde{u}\|_{L^1(\Omega_0)} + |\nabla\tilde{u}|(\Omega_0)$ (according to Remark 3). Also the decomposition of $\nabla\tilde{u}$ into regular and singular part can be seen at this moment,

$$
\begin{aligned}
(\nabla\tilde{u})^r &= \nabla u\chi_\Omega + \nabla u_D\chi_{\Omega_0\setminus\overline{\Omega}} \\
(\nabla\tilde{u})^s &= -(u-u_D)\mathbf{n}H_{n-1}(\partial\Omega).
\end{aligned} \tag{3.14}
$$

This is because $u\in W^{1,1}(\Omega)$ and $u_D\in W^{1,1}(\Omega_0)$, so the first two integrals are absolutely continuous with respect to Lebesgue measure. The jump may occur on the boundary so is supported on the Lebesgue null set, therefore it is included in the singular part. From now on, we will use notation u instead of $\tilde{u}$ referring to $u\in W^{1,1}(\Omega)$ and concurrently $u\in BV(\Omega_0)$.

Step 2: In this step we prove that supremum in (3.13) is really obtained by $\int_\Omega|\nabla u|\,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}}|\nabla u_D|\,\mathrm{d}x + \int_{\partial\Omega}|u-u_D|\,\mathrm{d}S$. The inequality "$\leq$" is obvious if taking $\varphi := -\mathbf{n}$. The opposite inequality is not that trivial.

First of all, denote $D_\delta := \{x;\ \mathrm{dist}(x,\partial\Omega) < 2\delta\}$. Next, consider

$$
\begin{aligned}
&S_1 := \{\varphi\in\mathcal{D}(\Omega);\ |\varphi|\leq 1,\ \varphi(x)=0 \text{ on } D_\delta\}, \\
&S_2 := \left\{\varphi\in\mathcal{D}(\Omega_0\setminus\overline{\Omega});\ |\varphi|\leq 1,\ \varphi(x)=0 \text{ on } D_\delta\right\}, \\
&\tau_\delta\in\mathcal{D}(\Omega_0),\ \tau_\delta(x)=1 \text{ for } x\in\partial\Omega,\ \tau_\delta(x)=0 \text{ if } \mathrm{dist}(x,\partial\Omega)>\delta, \\
&\varphi_3^{\delta,\alpha} := \frac{\nabla d}{|\nabla d|}\frac{(u-u_D)}{\alpha+|u-u_D|}\tau_\delta \text{ and } \varphi_3^{\delta,\alpha,\gamma} := \varphi_3^{\delta,\alpha} * \rho_\gamma,
\end{aligned}
$$

where $\alpha,\gamma,\delta>0$, $\gamma<\delta$, ρ_γ is symmetric mollifier and $d(x) := \mathrm{dist}(x,\partial\Omega)$ inside Ω, $d(x) := 0$ on $\Omega_0\setminus\Omega$. Now, taking

$$
\begin{aligned}
&\varphi_1^n\in S_1, \varphi_1^n \nearrow \frac{\nabla u}{|\nabla u|} \quad \text{in } \Omega\setminus D_\delta \text{ as } n\to\infty, \text{ and} \\
&\varphi_2^n\in S_2, \varphi_2^n \nearrow \frac{\nabla u_D}{|\nabla u_D|} \quad \text{in } \left(\Omega_0\setminus\overline{\Omega}\right)\setminus D_\delta \text{ as } n\to\infty
\end{aligned} \tag{3.15}
$$

we define $\varphi^n := \varphi_1^n + \varphi_2^n + \varphi_3^{\delta,\alpha,\gamma}$.

It surely holds that $\varphi^n\leq 1$, because each of $|\varphi_1^n|, |\varphi_2^n|, |\varphi_3^{\delta,\alpha,\gamma}|\leq 1$ and thanks to the way they are defined, their supports either do not intersect or only one function is non-zero in the point of intersection. Moreover, $\varphi^n\in\mathcal{C}_0^{0,1}(\Omega_0)$ and therefore

$$
\begin{aligned}
|\nabla u|(\Omega_0) &\geq \int_\Omega \nabla u\cdot\varphi^n\,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}} \nabla u_D\cdot\varphi^n\,\mathrm{d}x - \int_{\partial\Omega}(u-u_D)\varphi^n\cdot\mathbf{n}\,\mathrm{d}S \\
&= \int_\Omega \nabla u\cdot\varphi_1^n + \nabla u\cdot\varphi_3^{\delta,\alpha,\gamma}\,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}} \nabla u_D\cdot\varphi_2^n + \nabla u_D\cdot\varphi_3^{\delta,\alpha,\gamma}\,\mathrm{d}x \\
&\quad - \int_{\partial\Omega}(u-u_D)\varphi_3^{\delta,\alpha,\gamma}\cdot\mathbf{n}\,\mathrm{d}S =: (*).
\end{aligned}
$$

Now, for mollified function we have $\varphi_3^{\delta,\alpha,\gamma} \to \varphi_3^{\delta,\alpha}$ in $W^{1,1}(\Omega_0)$ and then also $\varphi_3^{\delta,\alpha,\gamma} \to \varphi_3^{\delta,\alpha}$ in $L^1(\partial\Omega)$ and this is that $\varphi_3^{\delta,\alpha,\gamma} \to \varphi_3^{\delta,\alpha}$ almost everywhere on $\partial\Omega$. Note that we write φ instead of $\mathrm{tr}\varphi$ on $\partial\Omega$ as the use of this convention was mentioned after the Trace Theorem (Theorem 1.5) stated in Chapter 1. Using this and convergence of φ_1^n and φ_2^n in (3.15) we can go with $n \to \infty$ and $\gamma \to 0$ to get that

$$(*) \to \int_{\Omega\setminus D_\delta} |\nabla u| \,\mathrm{d}x + \int_{(\Omega_0\setminus\overline{\Omega})\setminus D_\delta} |\nabla u_D| \,\mathrm{d}x + \int_\Omega \nabla u \cdot \varphi_3^{\delta,\alpha} \,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}} \nabla u_D \cdot \varphi_3^{\delta,\alpha} \,\mathrm{d}x - \int_{\partial\Omega} (u - u_D)\varphi_3^{\delta,\alpha} \cdot \mathbf{n} \,\mathrm{d}S.$$

Taking the limit $\delta \to 0$, the set D_δ is eliminated and also the domain of $\varphi_3^{\delta,\alpha}$ shrinks to $\partial\Omega$, because only here $\tau_\delta \neq 0$. Indeed, $\tau_\delta = 1$ on $\partial\Omega$ for any δ and therefore

$$(*) \to \int_\Omega |\nabla u| \,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}} |\nabla u_D| \,\mathrm{d}x + \int_{\partial\Omega} \frac{|u-u_D|^2}{\alpha + |u - u_D|} \frac{(-\nabla d)}{|\nabla d|} \cdot \mathbf{n} \,\mathrm{d}S.$$

Finally, note that $-\nabla d = \mathbf{n}$ and nothing can stop us from using the limit $\alpha \to 0$ to get the desired expression,

$$(*) \to \int_\Omega |\nabla u| \,\mathrm{d}x + \int_{\Omega_0\setminus\overline{\Omega}} |\nabla u_D| \,\mathrm{d}x + \int_{\partial\Omega} |u - u_D| \,\mathrm{d}S.$$

Step 3: We will point out that left hand sides of (3.4) and (3.5) are equal, i.e., that

$$\int_\Omega F((\nabla u)^r) \,\mathrm{d}x + |(\nabla u)^s|(\overline{\Omega}) = \int_\Omega F(\nabla u) \,\mathrm{d}x + \int_{\partial\Omega} |u - u_D| \,\mathrm{d}S. \tag{3.16}$$

This follows directly from (3.14). The equality $\int_\Omega F(\nabla u) \,\mathrm{d}x = \int_\Omega F((\nabla u)^r) \,\mathrm{d}x$ holds as ∇u has no singular part inside Ω. Using also the information from the Step 2,

$$|(\nabla u)^s|(\overline{\Omega}) = \sup_{\varphi\in C_0^{0,1}(\Omega_0),||\varphi||\leq 1} - \int_{\partial\Omega} (u - u_D)\varphi \cdot \mathbf{n} \,\mathrm{d}S = \int_{\partial\Omega} |u - u_D| \,\mathrm{d}S$$

and the equation (3.16) is clear.

Step 4: In this step we provide a proper regularization of v after which we are able to prove that (3.4) truly implies (3.5). We mollify $v \in BV(\Omega_0)$, $v = u_D$ on $\Omega_0 \setminus \overline{\Omega}$ to get $v^\varepsilon \in W^{1,1}(\Omega_0)$, $v^\varepsilon := v * \rho_\varepsilon$. Similarly as in the first step, define

$$\tilde{v}^\varepsilon := v^\varepsilon \quad \text{in } \Omega$$
$$\tilde{v}^\varepsilon := u_D \quad \text{on } \Omega_0 \setminus \overline{\Omega}.$$

We remind that both functions, $\tilde{v}^\varepsilon$ and u are in $W^{1,1}(\Omega)$ and also in $BV(\Omega_0)$. Therefore we are allowed to apply it to (3.4)

$$\begin{aligned}\int_\Omega F((\nabla u)^r) \,\mathrm{d}x + |(\nabla u)^s|(\overline{\Omega}) &= \int_\Omega F(\nabla u) \,\mathrm{d}x + \int_{\partial\Omega} |u - u_D| \,\mathrm{d}S \\ &\leq \int_\Omega F(\nabla v^\varepsilon) \,\mathrm{d}x + \int_{\partial\Omega} |v^\varepsilon - u_D| \,\mathrm{d}S.\end{aligned} \tag{3.17}$$

Step 5: Our last aim is to justify the limit $\varepsilon \to 0$. Firstly, using the same procedure as in the Step 2 we find out that

$$|\nabla \tilde{v}^\varepsilon|(\Omega_0) = \sup_{\varphi \in C_0^{0,1}, ||\varphi|| \leq 1} \int_\Omega \nabla v^\varepsilon \cdot \varphi \, \mathrm{d}x + \int_{\Omega_0 \setminus \Omega} \nabla u_D \cdot \varphi \, \mathrm{d}x - \int_{\partial\Omega} (v^\varepsilon - u_D)\varphi \cdot \mathbf{n} \, \mathrm{d}S$$
$$= \int_\Omega |\nabla v^\varepsilon| \, \mathrm{d}x + \int_{\Omega_0 \setminus \Omega} |\nabla u_D| \, \mathrm{d}x + \int_{\partial\Omega} |v^\varepsilon - u_D| \, \mathrm{d}S.$$

Without loss of generality, we assume that $\nabla u_D \equiv 0$ on $\Omega_0 \setminus E$ for some $\Omega \subset\subset E \subset\subset \Omega_0$. Applying the Theorem 1.8 we compute

$$\int_\Omega |\nabla v^\varepsilon| \, \mathrm{d}x + \int_{E \setminus \Omega} |\nabla u_D| \, \mathrm{d}x + \int_{\partial\Omega} |v^\varepsilon - u_D| \, \mathrm{d}S = |\nabla \tilde{v}^\varepsilon|(E) \to |\nabla \tilde{v}|(E), \quad (3.18)$$

where $\nabla \tilde{v}$ is the limit of mollifiers $\nabla \tilde{v}^\varepsilon$ as $\varepsilon \to 0$. It is useful to remind that

$$|\nabla \tilde{v}|(E) - |\nabla u_D|(E \setminus \Omega) = |\nabla \tilde{v}|(\overline{\Omega}), \quad (3.19)$$

because $|\nabla \tilde{v}|(E) = |\nabla \tilde{v}|(E \setminus \overline{\Omega}) + |\nabla \tilde{v}|(\overline{\Omega})$ and $|\nabla \tilde{v}|(E \setminus \overline{\Omega}) = |\nabla u_D|(E \setminus \Omega)$.

Note the limit property

$$\nabla v^\varepsilon \to (\nabla v)^r$$

pointwisely in Ω. This is because $\nabla v^\varepsilon = ((\nabla v)^r)^\varepsilon + ((\nabla v)^s)^\varepsilon$ and $((\nabla v)^r)^\varepsilon \to (\nabla v)^r$ in $L^1(\Omega)$, while $((\nabla v)^s)^\varepsilon \rightharpoonup^* (\nabla v)^s$ in $\mathcal{M}$ which is supported on a Lebesgue null set.

Sublinear growth

$$F(\nabla v^\varepsilon) - |\nabla v^\varepsilon| \leq c(1 + |\nabla v^\varepsilon|)^{1-\beta}$$

for $\beta \in (0, 1)$, proven in (iii) of Lemma 2.1, allows us to use the Hölder inequality with conjugates $\frac{1}{1-\beta}$ and $\frac{1}{\beta}$ and show that for $U \subset \Omega$,

$$\int_U c(1 + |\nabla v^\varepsilon|)^{1-\beta} \, \mathrm{d}x \leq c \left(\int_U 1 + |\nabla v^\varepsilon| \, \mathrm{d}x \right)^{1-\beta} |U|^\beta \leq c|U|^\beta.$$

Now, for any $\overline{\varepsilon} > 0$ we can find $\overline{\delta}$ such that $|U| < \overline{\delta}$ gives $c|U|^\beta < \overline{\varepsilon}$. Really, this is true for $\overline{\delta} < \left(\frac{\overline{\varepsilon}}{c}\right)^{\frac{1}{\beta}}$ and thus it follows that

$$F(\nabla v^\varepsilon) - |\nabla v^\varepsilon| \to F((\nabla v)^r) - |(\nabla v)^r| \quad (3.20)$$

using the Vitali Theorem (Theorem 1.11).

At the end, adding $\pm \int_\Omega |\nabla v^\varepsilon| \, \mathrm{d}x$ and $\pm \int_{E \setminus \Omega} |\nabla u_D| \, \mathrm{d}x$ to the last line in (3.17) and using (3.18) and (3.20) we get the first inequality; (3.19) explains the following step and decomposition (3.14) finishes the estimate,

$$\int_\Omega F(\nabla v^\varepsilon) - |\nabla v^\varepsilon| + |\nabla v^\varepsilon| \, \mathrm{d}x + \int_{E \setminus \Omega} |\nabla u_D| - |\nabla u_D| \, \mathrm{d}x + \int_{\partial\Omega} |v^\varepsilon - u_D| \, \mathrm{d}S$$
$$\to \int_\Omega F((\nabla v)^r) - |(\nabla v)^r| \, \mathrm{d}x + \int_E |\nabla \tilde{v}| \, \mathrm{d}x - \int_{E \setminus \Omega} |\nabla u_D| \, \mathrm{d}x$$
$$= \int_\Omega F((\nabla \tilde{v})^r) - |(\nabla \tilde{v})^r| \, \mathrm{d}x + \int_{\overline{\Omega}} |\nabla \tilde{v}| \, \mathrm{d}x$$
$$= \int_\Omega F((\nabla \tilde{v})^r) \, \mathrm{d}x + \int_{\overline{\Omega}} |(\nabla \tilde{v})^s| \, \mathrm{d}x.$$

This information together with (3.17) completes the proof. □

Chapter 4

The existence of solution

In this part we shall obtain the main result of this thesis. We still consider F and A defined in Chapter 2, where also their algebraic properties are shown. The work below is inspired by the results achieved in Chapter 3 and the question is, whether we are able to prove the existence of the solution in the sense of Definitions 3.1 and 3.2 (which are equivalent). Unfortunately, this solution does not exist in general, as shown in the counterexample in Chapter 5. Therefore, we focus our attention on the existence of solutions related to Definitions 3.3 or 3.4. We were inspired by articles Bulíček et al. (2014); Bulíček et al. (2015), in which authors pay attention to the existence of solution to (3.1) in the sense of Definition 3.2, i.e., up to the boundary for special data and nonconvex domains, and by Bildhauer and Fuchs (1999, 2002) where the situation related to Definition 3.3 and the interior regularity are studied.

As shown in Chapter 3, the solution in the sense of Definition 3.4 is weaker notion than the one of Definition 3.1. The first result, based on the lower semicontinuity of F, says that for all $a \in (0, \infty)$ and reasonable data there always exists a solution according to Definition 3.4. However, the most important result is that for $a \in (0, 2]$ and reasonable data there actually exists a solution in the sense of Definition 3.3.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $u_D \in W^{1,1}(\Omega)$ and $a \in (0, \infty)$. Then there exists a solution u to problem* (3.1) *in the sense of Definition 3.4. In addition, $u \in L^\infty_{loc}(\Omega)$. Moreover, if $a \in (0, 2]$, u is also a solution to* (3.1) *in the sense of Definition 3.3.*

The rest of the chapter is devoted to the proof of this theorem, which means that also all data will be considered as those used in the statement of the Theorem 4.1. In the section 4.1, the approximative problem is set and it is also shown that the approximative solutions u^ε converge to the one in BV spaces from Definition 3.4. Uniform estimates for the second derivative of u^ε inside Ω are shown in 4.2, and L^∞-estimates for u^ε in 4.3. That implies the L^∞-estimate for the limit u as well and leaves only the last part of the Theorem 4.1 unproven. Finally, section 4.4 supplies us with the proof of uniform equi-integrability of $\{\nabla u^\varepsilon\}$ and the final limit $\varepsilon \to 0_+$ is provided in 4.5 to get the solution u in $W^{1,1}(\Omega)$, according to Definition 3.3.

4.1 Approximative problem

For $\varepsilon > 0$, consider the new functional

$$F_\varepsilon(\nabla u) := \varepsilon|\nabla u|^2 + F(\nabla u).$$

Lemma 4.1. Let $u_D^\varepsilon \in W^{1,2}(\Omega)$ be given. There exists unique $u^\varepsilon \in W^{1,2}(\Omega)$, $u^\varepsilon = u_D^\varepsilon$ on $\partial\Omega$ such that

$$\int_\Omega F_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x \leq \int_\Omega F_\varepsilon(\nabla v)\,\mathrm{d}x \tag{4.1}$$

for all $v \in W^{1,2}(\Omega)$, $v = u_D^\varepsilon$ on $\partial\Omega$.

Proof. Firstly, note the following estimates, which are direct consequences of definition of F_ε and Remark 5:

$$F_\varepsilon(\nabla u) \geq \varepsilon|\nabla u|^2, \tag{4.2}$$
$$F_\varepsilon(\nabla u) \leq c(1 + |\nabla u| + \varepsilon|\nabla u|^2). \tag{4.3}$$

Denote

$$I := \inf_{v\in W^{1,2}(\Omega), v=u_D^\varepsilon \text{ on } \partial\Omega} \int_\Omega F_\varepsilon(\nabla v)\,\mathrm{d}x.$$

Especially, $I \leq \int_\Omega F_\varepsilon(\nabla u_D^\varepsilon)\,\mathrm{d}x$. From the definition of infimum we know that there exists a sequence $\{u^n\}_{n\in\mathbb{N}} \in W^{1,2}(\Omega)$, $u_n = u_D$ on $\partial\Omega$ for all $n \in \mathbb{N}$ such that

$$I = \lim_{n\to\infty} \int_\Omega F_\varepsilon(\nabla u^n)\,\mathrm{d}x.$$

Thanks to that and (4.2) it holds that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\varepsilon \int_\Omega |\nabla u^n|^2\,\mathrm{d}x \leq \int_\Omega F_\varepsilon(\nabla u^n)\,\mathrm{d}x \leq 1 + \int_\Omega F_\varepsilon(\nabla u_D^\varepsilon)\,\mathrm{d}x,$$

and this is bounded. Indeed, consider (4.3) and the fact that $u_D^\varepsilon \in W^{1,2}(\Omega)$. Integrating over a bounded set does not harm the boundedness and therefore $\|u^n\|_{1,2} \leq c(\varepsilon)$. Due to the reflexivity of $W^{1,2}(\Omega)$ there exists $u^\varepsilon \in W^{1,2}(\Omega)$, $u^\varepsilon = u_D^\varepsilon$ on $\partial\Omega$ such that for a subsequence that we do not relabel

$$u^n \rightharpoonup u^\varepsilon \text{ in } W^{1,2}(\Omega).$$

Both functions $|\cdot|^2$ and $F(\cdot)$ are convex, then also $F_\varepsilon(\cdot)$ is convex. In other words, the tangent line is below the graph of F_ε and therefore

$$F_\varepsilon(\nabla u^n) - F_\varepsilon(\nabla u^\varepsilon) \geq \frac{\partial F_\varepsilon(\nabla u^\varepsilon)}{\partial \eta}(\nabla u^n - \nabla u^\varepsilon) = (2\varepsilon\nabla u^\varepsilon + A(\nabla u^\varepsilon))(\nabla u^n - \nabla u^\varepsilon),$$

where last equation is trivial from the definitions of F_ε and A. Integrating over Ω and passing to limes inferior gives

$$\liminf_{n\to\infty} \int_\Omega F_\varepsilon(\nabla u^n) - F_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x \geq \liminf_{n\to\infty} \int_\Omega (2\varepsilon\nabla u^\varepsilon + A(\nabla u^\varepsilon))(\nabla u^n - \nabla u^\varepsilon)\,\mathrm{d}x.$$

As for the right hand side, $(2\varepsilon\nabla u^\varepsilon + A(\nabla u^\varepsilon))$ is bounded in $L^2(\Omega)$ and $(\nabla u^n - \nabla u^\varepsilon) \rightharpoonup 0$ in $L^2(\Omega)$, so the integral goes to 0. Therefore

$$\int_\Omega F_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega F_\varepsilon(\nabla u^n)\,\mathrm{d}x = I,$$

thus we proved (4.1) for all $v \in W^{1,2}(\Omega)$, $v = u_D^\varepsilon$ on $\partial\Omega$. □

In the following lemma other representation of (4.1) is introduced. Also, its results are crucial in the proof of the first part of Theorem 4.1.

Lemma 4.2. Let $u_D^\varepsilon \in W^{1,2}(\Omega)$ be given. For $u^\varepsilon \in W^{1,2}(\Omega)$, $u^\varepsilon = u_D^\varepsilon$ on $\partial\Omega$, the relation (4.1) holds for all $v \in W^{1,2}(\Omega)$, $v = u_D^\varepsilon$ on $\partial\Omega$ if and only if

$$2\varepsilon \int_\Omega \nabla u^\varepsilon \cdot \nabla\varphi\,\mathrm{d}x + \int_\Omega A(\nabla u^\varepsilon)\cdot\nabla\varphi\,\mathrm{d}x = 0 \tag{4.4}$$

for all $\varphi \in W_0^{1,2}(\Omega)$. Further, assume that $u_D^\varepsilon \to u_D$ in $W^{1,1}(\Omega)$ and that

$$\varepsilon\|\nabla u_D^\varepsilon\|_2^2 \le K \tag{4.5}$$

for some $K = K(\varepsilon) \in \mathbb{R}$, $K(\varepsilon) \to 0_+$ as $\varepsilon \to 0_+$. Then there exists a constant $c = c(K(\varepsilon), \|u_D^\varepsilon\|_{1,1}, \Omega, a) \in \mathbb{R}$ such that

$$\|u^\varepsilon\|_{1,1} + \varepsilon\|\nabla u^\varepsilon\|_2^2 \le c. \tag{4.6}$$

Moreover, there exists a subsequence of u^ε (for which we do not change the notation) such that

$$u^\varepsilon \rightharpoonup^* u \text{ in } BV(\Omega), \tag{4.7}$$

where u solves (3.5).

Proof. Step 1: The equivalence (4.1) ⇔ (4.4) has a very similar proof to that of Theorem 3.1.
"⇒": Similarly as in the proof of Theorem 3.1, after substitution $v := u^\varepsilon + t\varphi$ for $\varphi \in W_0^{1,2}(\Omega)$ it holds that

$$\begin{aligned} 0 &\le \int_\Omega \frac{F(\nabla u^\varepsilon + t\nabla\varphi) - F(\nabla u^\varepsilon)}{t}\,\mathrm{d}x + \varepsilon\int_\Omega \frac{2t\nabla u^\varepsilon\cdot\nabla\varphi + t^2|\nabla\varphi|^2}{t}\,\mathrm{d}x \\ &= \int_\Omega\int_0^1 \sum_{i=1}^d \frac{\partial F}{\partial\eta_i}(\nabla u^\varepsilon + t\tau\nabla\varphi)\cdot(\nabla\varphi)_i\,\mathrm{d}\tau\,\mathrm{d}x + \varepsilon\int_\Omega \nabla\varphi\cdot(2\nabla u^\varepsilon + t\nabla\varphi)\,\mathrm{d}x. \end{aligned}$$

Therefore

$$0 \le \int_\Omega\int_0^1 A(\nabla u^\varepsilon + t\tau\nabla\varphi)\cdot\nabla\varphi\,\mathrm{d}\tau + \varepsilon\nabla\varphi\cdot(2\nabla u^\varepsilon + t\nabla\varphi)\,\mathrm{d}x,$$

Using the Lebesgue Theorem (Theorem 1.9), limit $t \to 0_+$ and the fact that it holds for $-\varphi$, too, we obtain the equation (4.4).

"⇐": For an arbitrary $v \in W^{1,2}(\Omega)$, $v = u_D^\varepsilon$ on $\partial\Omega$ consider $\varphi := v - u^\varepsilon$. The result of Lemma 2.1, (iv) says that

$$A(\nabla u^\varepsilon)(\nabla v - \nabla u^\varepsilon) \le F(\nabla v) - F(\nabla u^\varepsilon).$$

Applying this inequality in (4.4), we get

$$0 \leq \int_\Omega F(\nabla v) - F(\nabla u^\varepsilon)\,\mathrm{d}x + 2\varepsilon \int_\Omega \nabla u^\varepsilon \cdot (\nabla v - \nabla u^\varepsilon)\,\mathrm{d}x. \tag{4.8}$$

Next, we use the Cauchy inequality for vectors $2a \cdot b \leq |a|^2 + |b|^2$, where ∇u^ε stands for a and ∇v stands for b. Then

$$\varepsilon \int_\Omega 2\nabla u^\varepsilon \cdot \nabla v - 2|\nabla u^\varepsilon|^2\,\mathrm{d}x \leq \varepsilon \int_\Omega |\nabla v|^2 - |\nabla u^\varepsilon|^2\,\mathrm{d}x \tag{4.9}$$

and, finally, moving the terms containing ∇u^ε to the left hand side in (4.8), we get

$$\int_\Omega \varepsilon|\nabla u^\varepsilon|^2 + F(\nabla u^\varepsilon)\,\mathrm{d}x \leq \int_\Omega \varepsilon|\nabla v|^2 + F(\nabla v)\,\mathrm{d}x,$$

which is, according to the definition of F_ε, exactly (4.1). Therefore this step is complete.

Step 2: In the next, we will obtain (4.6), assuming (4.4) and (4.5). First of all, note that $u^\varepsilon - u^\varepsilon_D \in W^{1,2}_0(\Omega)$ and therefore substitution $\varphi := u^\varepsilon - u^\varepsilon_D$ in (4.4) is correct,

$$2\varepsilon \int_\Omega \nabla u^\varepsilon \cdot (\nabla u^\varepsilon - \nabla u^\varepsilon_D)\,\mathrm{d}x + \int_\Omega A(\nabla u^\varepsilon) \cdot (\nabla u^\varepsilon - \nabla u^\varepsilon_D)\,\mathrm{d}x = 0.$$

Next, we use estimate (4.9) and (iv) from Lemma 2.1, although a little modified - opposite sign will change the direction of inequations,

$$\varepsilon \int_\Omega |\nabla u^\varepsilon|^2 - |\nabla u^\varepsilon_D|^2\,\mathrm{d}x \leq 2\varepsilon \int_\Omega |\nabla u^\varepsilon|^2 - \nabla u^\varepsilon \cdot \nabla u^\varepsilon_D\,\mathrm{d}x,$$
$$F(\nabla u^\varepsilon) - F(\nabla u^\varepsilon_D) \leq A(\nabla u^\varepsilon)(\nabla u^\varepsilon - \nabla u^\varepsilon_D).$$

When applied, we get

$$\varepsilon \int_\Omega |\nabla u^\varepsilon|^2 - |\nabla u^\varepsilon_D|^2\,\mathrm{d}x + \int_\Omega F(\nabla u^\varepsilon) - F(\nabla u^\varepsilon_D)\,\mathrm{d}x \leq 0.$$

Finally, estimate from Remark 5 says that $F(\xi) \leq c(a)(1 + |\xi|)$. Therefore, after we move the terms that do not contain u to the right hand side, for a proper constant c it holds that

$$\varepsilon \int_\Omega |\nabla u^\varepsilon|^2\,\mathrm{d}x + \int_\Omega |\nabla u^\varepsilon|\,\mathrm{d}x \leq c\left(1 + \int_\Omega |\nabla u^\varepsilon_D|\,\mathrm{d}x + \varepsilon \int_\Omega |\nabla u^\varepsilon_D|^2\,\mathrm{d}x\right) \leq c(a, K), \tag{4.10}$$

where for the boundedness we used the assumption $u^\varepsilon_D \in W^{1,1}(\Omega)$ and (4.5). To get the last estimate we only use the triangle and Poincaré inequalities, respectively, and that $u^\varepsilon, u^\varepsilon_D \in W^{1,1}(\Omega)$,

$$\begin{aligned} \|u^\varepsilon\|_{1,1} &\leq \|u^\varepsilon - u^\varepsilon_D\|_{1,1} + \|u^\varepsilon_D\|_{1,1} \leq c + \|u^\varepsilon - u^\varepsilon_D\|_{1,1} \\ &\leq c + c\|\nabla(u^\varepsilon - u^\varepsilon_D)\|_1 \leq c(1 + \|\nabla u^\varepsilon\|_1 + \|\nabla u^\varepsilon_D\|_1) \leq c. \end{aligned} \tag{4.11}$$

Then (4.6) holds, considering (4.10) and (4.11).

Step 3: The last thing to be proved is the existence of a weakly* convergent subsequence in $BV(\Omega)$. Using (4.11), there exists u such that

$$u^\varepsilon \to u \text{ in } L^1(\Omega). \tag{4.12}$$

Especially, $\|\nabla u^\varepsilon\|_1 \leq c$. Also $(\mathcal{C}(\overline{\Omega}))^* = \mathcal{M}(\Omega) \subset L^1(\Omega)$ and according to the Selection principle (Theorem 1.12) there exists a weakly* convergent subsequence of ∇u^ε such that

$$[\nabla u^\varepsilon]_i \rightharpoonup^* D_i \text{ in } \mathcal{M}(\Omega), \tag{4.13}$$

where $\mathcal{M}(\Omega)$ is the space of Radon measures over Ω and $D = \nabla u$ is the vector of distributional derivatives. Indeed, for an arbitrary $\varphi \in \mathcal{D}(\Omega)$,

$$\langle D_i, \varphi \rangle = \lim_{\varepsilon \to 0_+} \int_\Omega \frac{\partial u^\varepsilon}{\partial x_i} \varphi \, \mathrm{d}x = - \lim_{\varepsilon \to 0_+} \int_\Omega u^\varepsilon \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = - \int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x.$$

The last equality holds thanks to (4.12). Then, (4.12) and (4.13) gives (4.7).

The following estimate holds for any $v \in \mathcal{D}(\Omega)$,

$$\begin{aligned} \int_\Omega F_\varepsilon(\nabla u^\varepsilon) \, \mathrm{d}x &\leq \int_\Omega F_\varepsilon(\nabla u_D^\varepsilon + \nabla v) \, \mathrm{d}x \\ &= \int_\Omega \varepsilon \|\nabla u_D^\varepsilon + \nabla v\|_2^2 + F(\nabla u_D^\varepsilon + \nabla v) \, \mathrm{d}x \\ &\to \int_\Omega F(\nabla u_D + \nabla v) \, \mathrm{d}x \text{ as } \varepsilon \to 0_+, \end{aligned} \tag{4.14}$$

where the inequality holds thanks to (4.1), the equality follows from the definition of F_ε, for the first convergence we use (4.5) and the fact that v does not depend on ε. For $F(\nabla u_D^\varepsilon + \nabla v) \to F(\nabla u_D + \nabla v)$ we follow the same procedure as in the proof of Theorem 3.3, where we used the Vitali Theorem (Theorem 1.11) to obtain (3.20). Indeed, $\nabla u_D + \nabla v \in L^1(\Omega)$ and therefore $(\nabla u_D + \nabla v)^r = \nabla u_D + \nabla v$ and

$$\begin{aligned} &\int_\Omega F(\nabla u_D^\varepsilon + \nabla v) \, \mathrm{d}x \\ &= \int_\Omega F(\nabla u_D^\varepsilon + \nabla v) - |\nabla u_D^\varepsilon + \nabla v| \, \mathrm{d}x + \int_\Omega |\nabla u_D^\varepsilon + \nabla v| \, \mathrm{d}x \\ &\to \int_\Omega F((\nabla u_D + \nabla v)^r) - |(\nabla u_D + \nabla v)^r| \, \mathrm{d}x + \int_\Omega |(\nabla u_D + \nabla v)^r| \, \mathrm{d}x \\ &= \int_\Omega F(\nabla u_D + \nabla v) \, \mathrm{d}x. \end{aligned}$$

From (4.14), after passing to supremum, we get that

$$\limsup_{\varepsilon \to 0_+} \int_\Omega F_\varepsilon(\nabla u^\varepsilon) \, \mathrm{d}x \leq \int_\Omega F(\nabla u_D + \nabla v) \, \mathrm{d}x. \tag{4.15}$$

Moreover, define $w := u_D + v$. Then $w \in W^{1,1}(\Omega)$, $w = u_D$ on $\partial\Omega$ and

$$\int_\Omega F(\nabla u_D + \nabla v) \, \mathrm{d}x = \int_\Omega F((\nabla w)^r) \, \mathrm{d}x + |(\nabla w)^s|(\overline{\Omega}). \tag{4.16}$$

In the next, we will prove that

$$\int_\Omega F((\nabla u)^r)\,\mathrm{d}x + |(\nabla u)^s|(\overline{\Omega}) \le \limsup_{\varepsilon\to 0_+} \int_\Omega F_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x \tag{4.17}$$

and that, together with (4.15) and (4.16), means that u fulfills the condition to be the minimizer in the $BV(\Omega)$.

To prove (4.17), we first notice that from the definition of F_ε it trivially follows that

$$\limsup_{\varepsilon\to 0_+} \int_\Omega F_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}x \ge \limsup_{\varepsilon\to 0_+} \int_\Omega F(\nabla u^\varepsilon)\,\mathrm{d}x.$$

Next, using the continuity of the Extension Theorem (Theorem 1.4), we can find $\Omega_0 \supset\supset \Omega$ and a sequence $Eu_D^\varepsilon \in W_0^{1,1}(\Omega_0)$ such that $Eu_D^\varepsilon \to Eu_D$ strongly in $W_0^{1,1}(\Omega_0)$. Moreover, extending u^ε by Eu_D^ε outside Ω, we also see that

$$u^\varepsilon \rightharpoonup^* u \quad \text{in } BV(\Omega_0),$$

where $u = Eu_D$ outside Ω. However, from this extension directly follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0_+} \int_\Omega F(\nabla u^\varepsilon)\,\mathrm{d}x &= \limsup_{\varepsilon\to 0_+} \int_{\Omega_0} F(\nabla u^\varepsilon)\,\mathrm{d}x - \limsup_{\varepsilon\to 0_+} \int_{\Omega_0\setminus\Omega} F(\nabla Eu_D^\varepsilon)\,\mathrm{d}x \\ &= \limsup_{\varepsilon\to 0_+} \int_{\Omega_0} F(\nabla u^\varepsilon)\,\mathrm{d}x - \int_{\Omega_0\setminus\Omega} F(\nabla Eu_D)\,\mathrm{d}x,\end{aligned} \tag{4.18}$$

where for the second equality, we used the strong convergence of Eu_D^ε, the continuity and the linear growth of F and the Lebesgue Dominated Convergence Theorem (Theorem 1.9).

Next, for any fixed δ, we recall the definition of the mollifier η_δ (Definition 1.12) and we also denote the mollified limit function u and its distributional derivative ∇u by

$$\begin{aligned} u_\delta &:= \eta_\delta * u, \\ (\nabla u)_\delta &:= \eta_\delta * \nabla u = \eta_\delta * (\nabla u)^r + \eta_\delta * (\nabla u)^s =: (\nabla u)^r_\delta + (\nabla u)^s_\delta. \end{aligned}$$

Due to the mollification and the weak* convergence of u^ε we see that also

$$u^\varepsilon_\delta := \eta_\delta * u^\varepsilon \to u_\delta \quad \text{in } \mathcal{D}(\mathbb{R}^d). \tag{4.19}$$

If we use the fact that $\nabla u^\varepsilon = 0$ outside Ω_0 and that $F(0) = 0$; the fact that $\int_{\mathbb{R}^d} \eta_\delta(x-y)\,\mathrm{d}x = 1$; the Fubini Theorem; the convexity of F and therefrom excused Jensen inequality (1.15); and finally the definition of mollifiers, we obtain the following estimate

$$\begin{aligned}\int_{\Omega_0} F(\nabla u^\varepsilon)\,\mathrm{d}x &= \int_{\mathbb{R}^d} F(\nabla u^\varepsilon)\,\mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(\nabla u^\varepsilon(x))\eta_\delta(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} F(\nabla u^\varepsilon(x))\eta_\delta(x-y)\,\mathrm{d}x\right)\mathrm{d}y \\ &\ge \int_{\mathbb{R}^d} F\left(\int_{\mathbb{R}^d} \nabla u^\varepsilon(x)\eta_\delta(x-y)\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} F((\nabla u^\varepsilon)_\delta)\,\mathrm{d}x.\end{aligned} \tag{4.20}$$

Hence, using (4.19), we see that for any $\delta > 0$ there holds

$$\limsup_{\varepsilon\to 0_+} \int_{\Omega_0} F(\nabla u^\varepsilon)\,\mathrm{d}x \geq \int_{\mathbb{R}^d} F((\nabla u)_\delta)\,\mathrm{d}x. \tag{4.21}$$

Finally, we let $\delta \to 0_+$ and almost step by step we follow the procedure of the proof of Theorem 3.3 in the preceding chapter. It follows from the properties of the mollification that

$$\begin{aligned}
&\|(\nabla u)_\delta\|_{L^1(\mathbb{R}^d)} \leq C, \\
&\qquad (\nabla u)_\delta \to (\nabla u)^r \qquad \text{almost everywhere in } \mathbb{R}^d, \\
&\qquad\quad u_\delta \rightharpoonup^* u \qquad\quad \text{in } BV(\mathbb{R}^d).
\end{aligned}$$

Consequently, using the Vitali Theorem (Theorem 1.11) and the property (2.2) of F, we see that

$$\begin{aligned}
\lim_{\delta\to 0_+} \int_{\mathbb{R}^d} F((\nabla u)_\delta) - |(\nabla u)_\delta|\,\mathrm{d}x &= \int_{\Omega_0} F((\nabla u)^r) - |(\nabla u)^r|\,\mathrm{d}x \\
= \int_{\Omega_0\setminus\Omega} F(\nabla E u_D) - |\nabla E u_D|\,\mathrm{d}x &+ \int_{\Omega} F((\nabla u)^r) - |(\nabla u)^r|\,\mathrm{d}x.
\end{aligned} \tag{4.22}$$

On the other hand, since the singular part $(\nabla u)^s$ is supported in $\overline{\Omega}$ and consequently $|\nabla u|(\partial\Omega_0) = 0$, we can use Theorem 1.8 to conclude

$$\begin{aligned}
\lim_{\delta\to 0_+} \int_{\mathbb{R}^d} |(\nabla u)_\delta|\,\mathrm{d}x &= |\nabla u|(\Omega_0) \\
&= \int_{\Omega_0\setminus\Omega} |\nabla E u_D|\,\mathrm{d}x + \int_{\Omega} |(\nabla u)^r|\,\mathrm{d}x + |(\nabla u)^s|(\overline{\Omega}).
\end{aligned} \tag{4.23}$$

Hence, substituting (4.22) and (4.23) into (4.21) and using the resulting inequality in (4.18), we deduce (4.17), which finishes the proof.

□

Basically, Lemma 4.2 finishes the proof of the first part of Theorem 4.1. It is not difficult to show that for any $u_D \in W^{1,1}(\Omega)$ one can always find a sequence of smooth functions u_D^ε such that the condition (4.5) holds with $K(\varepsilon) \to 0_+$ and, in addition, $u_D^\varepsilon \to u_D$ in $W^{1,1}(\Omega)$. This is an easy consequence of the approximation theorem (Theorem 1.2), possibly after choosing a subsequence of approximating functions and thereafter its relabelling.

Therefore, in what follows we focus on proving the second part of the Theorem 4.1. In the lemmas below we use common assumptions, so we state them here together and denote by $(\mathcal{A})$:

Assumption $(\mathcal{A})$:

- let $u_D^\varepsilon \to u_D$ in $W^{1,1}(\Omega)$ and assume that $\varepsilon\|\nabla u_D^\varepsilon\|_2^2 \leq K$ for some $K = K(\varepsilon) \in \mathbb{R}$, $K(\varepsilon) \to 0_+$ as $\varepsilon \to 0_+$,
- let $u^\varepsilon \in W^{1,2}(\Omega)$, $u^\varepsilon = u_D^\varepsilon$ on $\partial\Omega$ be such that (4.4) holds for all $\varphi \in W_0^{1,2}(\Omega)$.

4.2 Uniform interior $W^{2,2}$-regularity

In the section we provide estimates proving the uniform $W^{2,2}$-regularity inside Ω and from now on, u will appear instead of u^ε, as well as u_D instead of u_D^ε. This is used only to avoid the formulas to seem too complicated, however we keep this dependence on ε in minds. In the following lemma we show that u, the solution to (4.4), behaves better inside Ω as it belongs to $W^{2,2}_{loc}(\Omega)$.

Lemma 4.3. Assume $(\mathcal{A})$. Then $u \in W^{2,2}_{loc}(\Omega)$.

Proof. Let $\Omega_0 \subset\subset \Omega$ and let $\operatorname{dist}(x, \partial\Omega) > \delta$ for any $x \in \Omega_0$. This condition guarantees that $(x + he_i) \in \Omega$, whenever $h \le \delta$. Therefore, in this setting, (4.4) implies

$$\begin{aligned} 2\varepsilon \int_{\Omega_0} &(\nabla u(x+he_i) - \nabla u(x)) \cdot \nabla v(x)\,\mathrm{d}x \\ &+ \int_{\Omega_0} (A(\nabla u(x+he_i)) - A(\nabla u(x))) \cdot \nabla v(x)\,\mathrm{d}x = 0 \end{aligned}$$

for any $v \in W_0^{1,2}(\Omega_0)$. Consider

$$v(x) := (u(x+he_i) - u(x)) \cdot \eta^2(x)$$

for $\eta \in \mathcal{D}(\Omega_0)$, $\eta \ge 0$ and use it in previous. What we get is, after we keep the "nice" terms on the left hand side and the rest we move to the right hand side,

$$\begin{aligned} (*) :=& 2\varepsilon \int_{\Omega_0} |\nabla u(x+he_i) - \nabla u(x)|^2 \eta^2(x)\,\mathrm{d}x \\ &+ \int_{\Omega_0} [A(\nabla u(x+he_i)) - A(\nabla u(x))] \cdot (\nabla u(x+he_i) - \nabla u(x))\eta^2(x)\,\mathrm{d}x \\ =& -4\varepsilon \int_{\Omega_0} [(\nabla u(x+he_i) - \nabla u(x)) \cdot \eta(x)]\,[(u(x+he_i) - u(x)) \cdot \nabla\eta]\,\mathrm{d}x \\ &- 2\int_{\Omega_0} [(A(\nabla u(x+he_i)) - A(\nabla u(x))) \cdot \eta(x)]\,[(u(x+he_i) - u(x)) \cdot \nabla\eta]\,\mathrm{d}x. \end{aligned}$$

Moreover, A is Lipschitz continuous (with L denoting the Lipschitz constant) and therefore the right hand side can be simplified,

$$(*) \le c(\varepsilon, L) \int_{\Omega_0} [(\nabla u(x+he_i) - \nabla u(x)) \cdot \eta(x)]\,[(u(x+he_i) - u(x)) \cdot \nabla\eta]\,\mathrm{d}x.$$

In the following step we use the Young inequality (the version with ε, note that the ε in the statement of the inequality is general and different from ε we work with now, the assignment of parameters is explained in the next sentence). In detail, $[(\nabla u(x+he_i) - \nabla u(x)) \cdot \eta(x)]$ will stand for a, $[(u(x+he_i) - u(x)) \cdot \nabla\eta]$ for b and $c(\varepsilon, L)$ for ε. Furthermore, in the next computation we identify c with any (finite) modification of $c(\varepsilon, L)$, and similarly C with $C(\varepsilon, L, \|\nabla\eta(x)\|_\infty)$. Then

$$(*) \le \varepsilon \int_{\Omega_0} |\nabla u(x+he_i) - \nabla u(x)|^2 \eta^2(x)\mathrm{d}x + C \int_{\Omega_0} |u(x+he_i) - u(x)|^2 \mathrm{d}x.$$

Subtracting ($\varepsilon \int_{\Omega_0} |\nabla u(x+he_i) - \nabla u(x)|^2 \eta^2(x)\, \mathrm{d}x$) from both sides of the inequality and then dividing it by h^2 we obtain

$$\varepsilon \int_{\Omega_0} \frac{|\nabla u(x+he_i) - \nabla u(x)|^2 \eta^2(x)}{h^2} \mathrm{d}x \leq C \int_{\Omega_0} \frac{|u(x+he_i) - u(x)|^2}{h^2} \mathrm{d}x$$

Note that the last fraction is $\|D^h u\|^2_{L^2(\Omega_0)}$ from the first chapter and therefore using Theorem 1.6,

$$\varepsilon \int_{\Omega_0} \frac{|\nabla u(x+he_i) - \nabla u(x)|^2 \eta^2(x)}{h^2} \mathrm{d}x \leq C \|u\|^2_{1,2}.$$

This estimate holds for arbitrary η and thus $\nabla u \in W^{1,2}_{loc}(\Omega_0)$. However, note that this estimate is not uniform with respect to ε. At the end, we use the limit $\delta \to 0_+$, which means that $\Omega_0 \to \Omega$ and therefore $u \in W^{2,2}_{loc}(\Omega)$. □

In the next lemma we recall the notation of partial derivatives introduced in Chapter 1.

Lemma 4.4. Assume ($\mathcal{A}$). Then there exists a constant $c = c(K, \eta)$ such that

$$(*) := \sum_{i,k=1}^{d} \int_{\Omega} 2\varepsilon \, |\nabla^2 u|^2 \, \eta^2 + D_k A_i(\nabla u) \, D_{ik} u \, \eta^2 \, \mathrm{d}x \leq c.$$

Proof. Integrate (4.4) by parts,

$$-2\varepsilon \int_{\Omega} \Delta u \cdot \varphi \, \mathrm{d}x - \int_{\Omega} \mathrm{div} A(\nabla u) \cdot \varphi \, \mathrm{d}x = 0$$

and use the fact that $|\mathrm{div} A(\nabla u)| \leq \left|\frac{\partial A(\nabla u)}{\partial \nabla u}\right| |\nabla^2 u|$, where $\left|\frac{\partial A(\nabla u)}{\partial \nabla u}\right|$ is bounded from Lemma 2.1, (v) and $|\nabla^2 u| = |\nabla(\nabla u)|$ is in $L^2_{loc}(\Omega)$,

$$2\varepsilon \Delta u + \mathrm{div} A(\nabla u) \equiv 0 \text{ almost everywhere in } \Omega. \tag{4.24}$$

Multiplying (4.24) by $(\mathrm{div}(\nabla u \; \eta^2))$ for $\eta \in \mathcal{D}(\Omega)$ and integrating the equation over Ω,

$$\int_{\Omega} 2\varepsilon \; \Delta u \; \mathrm{div}(\nabla u \; \eta^2) + \mathrm{div} A(\nabla u) \mathrm{div}(\nabla u \; \eta^2) \, \mathrm{d}x = 0.$$

If we rewrite it by coordinates and integrate by parts, what we get is (using Einstein's summation convention)

$$\int_{\Omega} 2\varepsilon \; D_{ki} u \; D_i(D_k u \; \eta^2) + D_k A_i(\nabla u) \; D_i(D_k u \; \eta^2) \, \mathrm{d}x = 0.$$

Applying the product rule for derivatives and moving the terms containing $D_i \eta$ on the right hand side we get the $(*)$ defined in the statement of the lemma,

$$(*) = \int_{\Omega} 2\varepsilon \; |\nabla^2 u|^2 \; \eta^2 + D_k A_i(\nabla u) \; D_{ik} u \; \eta^2 \, \mathrm{d}x \tag{4.25}$$

$$= -4 \int_{\Omega} \varepsilon \; D_{ik} u \; D_k u \; \eta \; D_i \eta + D_k A_i(\nabla u) \; D_k u \; \eta \; D_i \eta \, \mathrm{d}x. \tag{4.26}$$

We remind that $A_i(\xi) = \dfrac{\partial F(\xi)}{\partial \xi_i}$ and that $B_{ij}(\xi) = \dfrac{\partial^2 F(\xi)}{\partial \xi_i \partial \xi_j}$. Now, using the chain rule, for fixed k, i the following holds

$$D_k A_i(\nabla u) = \sum_{j=1}^{d} B_{ij}(\nabla u) D_{jk} u.$$

Moreover, since F is strictly convex, then B is positive definite matrix, i.e., elliptic - there exists $c > 0$ such that $B_{ij}(\xi)\zeta_i\zeta_j \geq c(\xi)|\zeta|^2$ for any $\xi, \zeta \in \mathbb{R}^d$. Applied to (4.25), we get

$$(*) = \int_\Omega 2\varepsilon |\nabla^2 u|^2 \eta^2 \,\mathrm{d}x + \int_\Omega B_{ij}(\nabla u)\ D_{jk}u\ D_{ik}u\ \eta^2 \,\mathrm{d}x,$$

and to (4.26), using also Young inequality (1.8),

$$\begin{aligned}(*) &\leq \int_\Omega \varepsilon |\nabla^2 u|\ \eta\ 4|\nabla u||\nabla \eta| \,\mathrm{d}x - 4\int_\Omega B_{ij}(\nabla u)\ D_{jk}u\ D_k u\ \eta\ D_i\eta \,\mathrm{d}x \\ &\leq \int_\Omega \frac{\varepsilon}{2} |\nabla^2 u|^2 \eta^2 + 8\varepsilon |\nabla u|^2 |\nabla \eta|^2 \,\mathrm{d}x - 4\int_\Omega B_{ij}(\nabla u)\ D_{jk}u\ \eta\ D_k u\ D_i\eta \,\mathrm{d}x.\end{aligned}$$

What is more, B is also symmetric and therefore defines the scalar product and the Cauchy-Schwarz inequality holds,

$$|B_{ij}(\xi)a_i b_j| \leq (B_{ij}(\xi)a_i a_j)^{\frac{1}{2}}\ (B_{ij}(\xi)b_i b_j)^{\frac{1}{2}}.$$

Applied to our situation, $(D_{ik}u\ \eta)$ stands for a_i and $(D_k u\ D_i\eta)$ for b_i. Moreover, $\int_\Omega \frac{\varepsilon}{2} |\nabla^2 u|^2 \eta^2 \,\mathrm{d}x$ can be subtracted from both (4.25) and (4.26). Then

$$\begin{aligned}(*) &\leq c + \int_\Omega \sum_{k=1}^{d} (B_{ij}(\nabla u)\ D_{ik}u\ D_{jk}u\ \eta^2)^{\frac{1}{2}}\ (16 B_{ij}(\nabla u)(D_k u)^2 D_i\eta\ D_j\eta)^{\frac{1}{2}} \,\mathrm{d}x \\ &\leq c + \frac{1}{2}\int_\Omega \sum_{k=1}^{d} B_{ij}(\nabla u)\ D_{ik}u\ D_{jk}u\ \eta^2 + 4\sum_{k=1}^{d} B_{ij}(\nabla u)(D_k u)^2 D_i\eta\ D_j\eta \,\mathrm{d}x,\end{aligned}$$

thanks to the Young inequality. Now $(\frac{1}{2}\int_\Omega \sum_{k=1}^{d} B_{ij}(\nabla u)\ D_{ik}u\ D_{jk}u\ \eta^2 \,\mathrm{d}x)$ can be again subtracted from (4.25) and (4.26). Therefore

$$(*) \leq c + c\int_\Omega |B(\nabla u)||\nabla u|^2 |\nabla \eta|^2 \,\mathrm{d}x,$$

and Remark 6 says that $|B(\nabla u)| \leq \frac{2}{1+|\nabla u|}$, then

$$(*) \leq c + c\int_\Omega |\nabla u||\nabla \eta|^2 \,\mathrm{d}x \leq c(K, \eta).$$

□

4.3 Uniform interior L^∞-estimates

In what follows we shall show that $u \in L^p_{loc}(\Omega)$ for $p \in [1,\infty]$, adapting our calculations on the Moser iteration technique.

Lemma 4.5. Assume $(\mathcal{A})$, then there exists a constant c such that for any $\Omega' \subset\subset \Omega$,

$$\|u\|_{L^\infty(\Omega')} \leq c(\Omega', \|u\|_{1,1}, \varepsilon\|u\|^2_{1,2}).$$

The estimate (4.7) proven in Lemma 4.2 says that $u^\varepsilon \rightharpoonup^* u$ in $BV(\Omega)$ as $\varepsilon \to 0_+$. Thanks to that, the result of Lemma 4.5 implies that $u^\varepsilon \in L^\infty_{loc}(\Omega)$ and also $u \in L^\infty_{loc}(\Omega)$ and that proves the part of Theorem 4.1 stating the L^∞-estimate.

Proof. In (4.4), set

$$\varphi := |T_M(u)|^{p-1}u \cdot \tau^2$$

for $T_M(u) := \operatorname{sgn} u \min\{M, |u|\}$, $M \in \mathbb{R}$ fixed, $p \in \mathbb{R}$, $p > 1$ fixed and $\tau \in \mathcal{D}(\Omega)$. To verify that this choice of φ indeed is in $W^{1,2}_0(\Omega)$, note that $T_M(u)$ is bounded by M, $M, p \in \mathbb{R}$ are fixed, $u \in W^{1,2}(\Omega)$ and τ is compactly supported in Ω. Now (4.4) is

$$2\varepsilon \int_\Omega \nabla u \cdot \nabla(|T_M(u)|^{p-1}u \cdot \tau^2)\,\mathrm{d}x + \int_\Omega A(\nabla u) \cdot \nabla(|T_M(u)|^{p-1}u \cdot \tau^2)\,\mathrm{d}x = 0$$

and applying the gradient on the product $(|T_M(u)|^{p-1}u) \cdot \tau^2$ we get

$$\begin{aligned} &2\varepsilon \int_\Omega \nabla u \cdot \nabla(|T_M(u)|^{p-1}u)\tau^2\,\mathrm{d}x + \int_\Omega A(\nabla u) \cdot \nabla(|T_M(u)|^{p-1}u)\tau^2\,\mathrm{d}x \\ =& -2\varepsilon \int_\Omega \nabla u \cdot |T_M(u)|^{p-1}u\nabla\tau^2\,\mathrm{d}x - \int_\Omega A(\nabla u) \cdot |T_M(u)|^{p-1}u\nabla\tau^2\,\mathrm{d}x. \end{aligned} \tag{4.27}$$

The element $\nabla u \cdot \nabla(|T_M(u)|^{p-1}u)$, written by coordinates and using that $u \cdot u = u^2 = |u|^2$, gives

$$\begin{aligned} \sum_{i=1}^d D_i u\, D_i(|T_M(u)|^{p-1}u) &= \sum_{i=1}^d D_i u\, D_i u\,|T_M(u)|^{p-1} + D_i u\, u\, D_i|T_M(u)|^{p-1} \\ &= |\nabla u|^2|T_M(u)|^{p-1} + \sum_{i=1}^d \frac{1}{2} D_i|u|^2\, D_i|T_M(u)|^{p-1} \\ &= |\nabla u|^2|T_M(u)|^{p-1} + (p-1)|T_M(u)|^{p-2}T'_M(u)|\nabla u|^2|u| =: (*). \end{aligned} \tag{4.28}$$

Note that if $|u| < M$ then $T_M(u) = |u|$ and $T'_M(u) = 1$. Moreover, the second element of $(*)$ disappears otherwise, because $T_M(u) = M$ and $T'_M(u) = 0$. Anyway, $T'_M(u) = |T'_M(u)|^2$ and therefore

$$(*) \geq p|T'_M(u)|^2|\nabla u|^2|T_M(u)|^{p-1} = p|\nabla T_M(u)|^2|T_M(u)|^{p-1}, \tag{4.29}$$

but also

$$(*) \geq \frac{p}{2}|\nabla T_M(u)|^2|T_M(u)|^{p-1} + \frac{1}{2}|\nabla u|^2|T_M(u)|^{p-1}. \tag{4.30}$$

In the following computation we will use both estimates. Thanks to the way in which $A(\nabla u)$ is defined one can see that the use of (4.29) is correct for the second element on the left hand side of (4.27). However, for the first one the use of (4.30) will be more suitable as we will see later. The right hand side of (4.27) will be estimated by its absolute value, using also that $|\nabla\tau^2| \leq 2|\nabla\tau||\tau|$. After all, what we get is

$$\begin{aligned}
&\varepsilon \int_\Omega |\nabla u|^2 |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x + \varepsilon p \int_\Omega |\nabla T_M(u)|^2 |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x \\
&+p \int_\Omega \frac{1}{(1+|\nabla u|^a)^{\frac{1}{a}}} |\nabla T_M(u)|^2 |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x \\
\leq& 4\varepsilon \int_\Omega |\nabla u||T_M(u)|^{p-1}|u||\nabla\tau||\tau| \,\mathrm{d}x + 2\int_\Omega |A(\nabla u)||T_M(u)|^{p-1}|u||\nabla\tau||\tau| \,\mathrm{d}x.
\end{aligned} \tag{4.31}$$

We rewrite the first element on the right hand side of (4.31) in the way that use of the Young inequality will be convenient and what is more, part of it will be annulled with the first term on the left hand side later,

$$\begin{aligned}
&4\varepsilon \int_\Omega |\nabla u||T_M(u)|^{p-1}|u||\nabla\tau||\tau| \,\mathrm{d}x \\
=& \int_\Omega \left(2\varepsilon|\nabla u|^2 |T_M(u)|^{p-1}\tau^2\right)^{\frac{1}{2}} \left(2^{\frac{3}{2}}\varepsilon^{\frac{1}{2}} |T_M(u)|^{\frac{p-1}{2}}|u||\nabla\tau|\right) \mathrm{d}x \\
\leq& \varepsilon \int_\Omega |\nabla u|^2 |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x + 4\varepsilon \int_\Omega |T_M(u)|^{p-1}|u|^2|\nabla\tau|^2 \,\mathrm{d}x.
\end{aligned} \tag{4.32}$$

When estimating the third element on the left hand side of (4.31) from below (note that $|\nabla T_M(u)|^2 = |T_M'(u)|^2|\nabla u|^2$), we need to differ whether $|\nabla u|$ is greater than, equal to or lower than 1. Anyway,

$$\frac{1}{2^{\frac{1}{a}}}|\nabla u| - 1 \leq \frac{|\nabla u|^2}{(1+|\nabla u|^a)^{\frac{1}{a}}},$$

and finally, using this, (4.32) and the fact that always $|A(\nabla u)| \leq 1$, we estimate (4.31) as

$$\begin{aligned}
&\varepsilon p \int_\Omega |\nabla T_M(u)|^2 |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x + \frac{p}{2^{\frac{1}{a}}} \int_\Omega |\nabla u||T_M'(u)|^2 |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x \\
\leq& C\Big(\varepsilon \int_\Omega |T_M(u)|^{p-1}|u|^2|\nabla\tau|^2 \,\mathrm{d}x + \int_\Omega |T_M(u)|^{p-1}|u||\nabla\tau|^2 \,\mathrm{d}x \\
&+p \int_\Omega |T_M'(u)|^2 |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x\Big).
\end{aligned} \tag{4.33}$$

Note that

$$\begin{aligned}
\left|\nabla |T_M(u)|^{\frac{p+1}{2}}\right|^2 &= \frac{(p+1)^2}{4}|T_M(u)|^{p-1}|\nabla T_M(u)|^2, \quad \text{and} \\
|\nabla |T_M(u)|^p| &= p|T_M(u)|^{p-1}|T_M'(u)||\nabla u| = p|T_M(u)|^{p-1}|T_M'(u)|^2|\nabla u|.
\end{aligned}$$

This, applied to the left hand side of (4.33), gives

$$\begin{aligned}
&\frac{4\varepsilon p}{(p+1)^2} \int_\Omega \left|\nabla |T_M(u)|^{\frac{p+1}{2}}\right|^2 \tau^2 \,\mathrm{d}x + \frac{1}{2^{\frac{1}{a}}} \int_\Omega |\nabla |T_M(u)|^p|\,\tau^2 \,\mathrm{d}x \\
\leq& C\left(\int_\Omega (\varepsilon|u|^2 + |u|)|T_M(u)|^{p-1}|\nabla\tau|^2 \,\mathrm{d}x + p\int_\Omega |T_M(u)|^{p-1}\tau^2 \,\mathrm{d}x\right).
\end{aligned} \tag{4.34}$$

In order to use the Sobolev embedding we need to move the smooth function τ into the gradient. To do so, this estimate will be used

$$\begin{aligned}\left|\nabla\left(|T_M(u)|^{\frac{p+1}{2}}\tau\right)\right|^2 &= \left|\nabla|T_M(u)|^{\frac{p+1}{2}}\tau + |T_M(u)|^{\frac{p+1}{2}}\nabla\tau\right|^2\\ &\leq 2\left|\nabla|T_M(u)|^{\frac{p+1}{2}}\tau\right|^2 + 2\left||T_M(u)|^{\frac{p+1}{2}}\nabla\tau\right|^2,\end{aligned}$$

in the following way

$$\left|\nabla\left(|T_M(u)|^{\frac{p+1}{2}}\tau\right)\right|^2 - 2|T_M(u)|^{p+1}|\nabla\tau|^2 \leq 2\left|\nabla|T_M(u)|^{\frac{p+1}{2}}\right|^2\tau^2.$$

It will be used to estimate the left hand side of (4.34) from below, together with

$$|\nabla(|T_M(u)|^p\tau^2)| - |T_M(u)|^p|\nabla\tau|^2 \leq |\nabla|T_M(u)|^p|\tau^2,$$

where the same procedure was used for the second element. However, we only keep better elements on the left hand side and those worse, which contain $|\nabla\tau|^2$, go to the right. Moreover, C denotes a constant that is independent of p, ε and τ and also, from now on, only the power of p will be taken into consideration,

$$\begin{aligned}&\frac{\varepsilon}{p}\int_\Omega\left|\nabla\left(|T_M(u)|^{\frac{p+1}{2}}\tau\right)\right|^2\,\mathrm{d}x + \int_\Omega|\nabla(|T_M(u)|^p\tau^2)|\,\mathrm{d}x\\ \leq& C\bigg(\int_\Omega\Big((\varepsilon|u|^2+|u|)|T_M(u)|^{p-1} + |T_M(u)|^p + \frac{\varepsilon}{p}|T_M(u)|^{p+1}\Big)|\nabla\tau|^2\,\mathrm{d}x \qquad (4.35)\\ &+p\int_\Omega|T_M(u)|^{p-1}\tau^2\,\mathrm{d}x\bigg).\end{aligned}$$

Note that $(|T_M(u)|^{\frac{p+1}{2}}\tau)$ and $(|T_M(u)|^p\tau^2)$ are supported on a bounded set, thanks to the fact that $\tau\in\mathcal{D}(\Omega)$ and Ω is bounded. Sobolev embeddings give

$$\begin{aligned}\left\||T_M(u)|^{\frac{p+1}{2}}\tau\right\|^2_{\frac{2d}{d-1}} \leq c\left\||T_M(u)|^{\frac{p+1}{2}}\tau\right\|^2_{\frac{2d}{d-2}} \leq c\left\|\nabla(|T_M(u)|^{\frac{p+1}{2}}\tau)\right\|^2_2 \qquad \text{and}\\ \left\||T_M(u)|^p\tau^2\right\|_{\frac{d}{d-1}} \leq c\left\|\nabla(|T_M(u)|^p\tau^2)\right\|_1,\end{aligned}$$

and therefore (4.35) can be finally rewritten in the form

$$\begin{aligned}&\frac{\varepsilon}{p}\left\||T_M(u)|^{\frac{p+1}{2}}\tau\right\|^2_{\frac{2d}{d-1}} + \left\||T_M(u)|^p\tau^2\right\|_{\frac{d}{d-1}}\\ \leq& C\bigg(\int_\Omega\Big((\varepsilon|u|^2+|u|)|T_M(u)|^{p-1} + |T_M(u)|^p + \frac{\varepsilon}{p}|T_M(u)|^{p+1}\Big)|\nabla\tau|^2\,\mathrm{d}x \qquad (4.36)\\ &+p\int_\Omega|T_M(u)|^{p-1}\tau^2\,\mathrm{d}x\bigg)\end{aligned}$$

and this holds for all $M\in\mathbb{R}$ and $p\in\mathbb{R}$, $p>1$. The left hand side can be formulated as

$$\begin{aligned}\left\||T_M(u)|^{\frac{p+1}{2}}\tau\right\|^2_{\frac{2d}{d-1}} &= \left(\int_\Omega|T_M(u)|^{\frac{p+1}{2}\frac{2d}{d-1}}\tilde{\tau}\mathrm{d}x\right)^{\frac{d-1}{2d}\frac{2(p+1)}{p+1}} = \|T_M(u)\tilde{\tau}\|^{p+1}_{(p+1)\frac{d}{d-1}},\\ \||T_M(u)|^p\tau\|_{\frac{d}{d-1}} &= \|T_M(u)\tilde{\tau}\|^p_{\frac{dp}{d-1}},\end{aligned}$$

where $\tilde{\tau} = \tau^{\frac{2d}{d-1}}$ in the first case and $\tilde{\tau} = \tau^{\frac{d}{d-1}}$ in the second, but the important thing is that it remains smooth. Since τ was chosen arbitrarily, we keep writing τ instead of $\tilde{\tau}$ in what follows. Moreover, for $u \in W^{1,2}(\Omega)$ the Sobolev embedding gives

$$\|u\|_{\frac{d}{d-1}} \leq c\|u\|_{1,2} \leq c.$$

We also have that $\varepsilon\|\nabla u\|_2^2 \leq c$ and $u = u_D$ on $\partial\Omega$, therefore using the assumptions the following expression is bounded,

$$\varepsilon\|u\|^2_{\frac{2d-1}{d-1}} \leq c\varepsilon\|u\|^2_{1,2} \leq c\varepsilon(\|u-u_D\|^2_{1,2}+\|u_D\|^2_{1,2}) \leq c\varepsilon(\|\nabla u\|_2^2+\|\nabla u_D\|_2^2+\|u_D\|^2_{1,2}).$$

Set $p = \frac{d}{d-1}$ and we want to let $M \to \infty$. Then $|T_M(u)| \to |u|$ and we get that for any $\tau \in \mathcal{D}(\Omega)$,

$$\begin{aligned} &\frac{\varepsilon(d-1)}{d}\, \||u|\tau\|^{\frac{2d-1}{d-1}}_{\frac{(2d-1)d}{(d-1)^2}} + \left\||u|\tau^2\right\|^{\frac{d}{d-1}}_{\frac{d}{d-1}} \\ \leq &C\bigg(\int_\Omega\Big(\frac{\varepsilon(d-1)}{d}|u|^{\frac{2d-1}{d-1}} + |u|^{\frac{d}{d-1}}\Big)|\nabla\tau|^2\,\mathrm{d}x + \frac{d}{d-1}\int_\Omega |u|^{\frac{1}{d-1}}\tau^2 \mathrm{d}x\bigg) \leq C \end{aligned} \tag{4.37}$$

and the boundedness is clear using the estimates above. Therefore the limit step with M was used correctly. It also holds that

$$\||u|\tau\|^{\frac{2d-1}{d-1}}_{\frac{d^2}{(d-1)^2}} \leq \||u|\tau\|^{\frac{2d-1}{d-1}}_{\frac{(2d-1)d}{(d-1)^2}}$$

and

$$\|u\|_{\frac{d^2+(d-1)^2}{(d-1)^2}} = \|u\|_{\frac{2d^2-2d+1}{(d-1)^2}} \leq c\|u\|_{\frac{2d^2-d}{(d-1)^2}},$$

thus $u \in L_{loc}^{\frac{d^2}{(d-1)^2}}$. Therefore, coming back to (4.36), setting $p = \frac{d^2}{(d-1)^2}$ and letting $M \to \infty$, the right hand side will become

$$C\bigg(\int_\Omega\Big(\frac{\varepsilon(d-1)^2}{d^2}|u|^{\frac{d^2+(d-1)^2}{(d-1)^2}} + |u|^{\frac{d^2}{(d-1)^2}}\Big)|\nabla\tau|^2\,\mathrm{d}x + \frac{d^2}{(d-1)^2}\int_\Omega |u|^{\frac{d^2-(d-1)^2}{(d-1)^2}}\tau^2 \mathrm{d}x\bigg)$$

and is bounded using the same argument of embeddings as above. Then the left hand side is bounded as well and implies that $u \in L_{loc}^{\frac{d^3}{(d-1)^3}}$. Repeating the same procedure for $p = \frac{d^i}{(d-1)^i}$, $i \in \mathbb{N}$ we always get the boundedness of the right hand side thanks to the estimate gained in the previous step. Therefore $u \in L^p_{loc}(\Omega)$, that means, for any $\Omega' \subset\subset \Omega$,

$$\|u\|_{L^p(\Omega')} \leq C(p, \Omega', \|u\|_{1,1}, \varepsilon\|u\|^2_{1,2}) \tag{4.38}$$

for all $p \in (1, \infty)$. Also, it means that for any $p \in (1, \infty)$ we can go with the limit $M \to \infty$ in (4.36) to get the relation

$$\begin{aligned} &\frac{\varepsilon}{p}\, \||u|\tau\|^{p+1}_{(p+1)\frac{d}{d-1}} + \left\||u|\tau^2\right\|^{p}_{\frac{dp}{d-1}} \\ \leq &C\bigg(\int_\Omega\Big(\frac{\varepsilon}{p}|u|^{p+1} + |u|^p\Big)|\nabla\tau|^2\,\mathrm{d}x + p\int_\Omega |u|^{p-1}\tau^2 \mathrm{d}x\bigg). \end{aligned} \tag{4.39}$$

With the help of Moser iteration technique we will reach that $u \in L^p_{loc}$ for all $p \in (1, \infty]$. That is, the constant in (4.38) will be independent of p.

Let $x \in \Omega$, $\rho > 0$ and $R > 0$ be given, such that $\rho < R < \operatorname{dist}(x, \partial\Omega)$. Denote B_R to be a ball of a radius R, centered at x. Let $\{R_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that

$$R_0 = R, \quad R_k := \rho + \frac{R-\rho}{2^k} \quad \text{and} \quad R_k \searrow \rho \text{ as } k \to \infty,$$

fulfilling $0 < |R_{k-1} - R_k| = \dfrac{R-\rho}{2^k} << 1$. Note that for each k we can choose τ in the way that

$$\tau = 1 \text{ on } B_{R_k},\ \tau = 0 \text{ outside } B_{R_{k-1}} \text{ and } |\nabla \tau| \leq \frac{c}{R_{k-1} - R_k}.$$

Our aim is to bound the left hand side of (4.39), so in the case that for some k $|u|\tau \leq 1$, we do not need the right hand side at all. If $|u|\tau > 1$, note that the last element on the right hand side of (4.39) is small in comparison with those which contain $|\nabla\tau|^2$. Really, if $|u|\tau > 1$, then necessarily $|u| > 1$, $|u|^{p-1} < |u|^p$ and $p << \frac{c}{|R_{k-1}-R_k|^2}$, together with $\tau \leq 1$ we get that

$$p \int_\Omega |u|^{p-1}\tau^2 \, \mathrm{d}x < \int_\Omega |u|^p |\nabla\tau|^2 \, \mathrm{d}x,$$

and therefore we consider it being absorbed by the second element since now. Finally, by decreasing the left hand side and using the introduced notation we get from (4.39) that

$$\|u\|^p_{L^{\frac{dp}{d-1}}(B_{R_k})} \leq \frac{C}{|R_{k-1} - R_k|^2} \left(\frac{\varepsilon}{p} \|u\|^{p+1}_{L^{p+1}(B_{R_{k-1}})} + \|u\|^p_{L^p(B_{R_{k-1}})} \right). \tag{4.40}$$

The ball B_{R_k} is used on the left hand side because it is the largest possible estimate, since $\tau = 1$ on B_{R_k} and $\tau < 1$ everywhere else. On the other hand, $|\nabla\tau|^2 \neq 0$ on $B_{R_{k-1}} \setminus B_{R_k}$ and that is why we need to stay at $B_{R_{k-1}}$ on the right hand side. Moreover, thanks to the $u \in L^p_{loc}(\Omega)$ property for all $p \in (1, \infty)$ and that $|u| > 1$, we can increase the element on the right hand side of (4.40) by

$$\|u\|^p_{L^p(B_{R_{k-1}})} \leq c\|u\|^p_{L^{p+1}(B_{R_{k-1}})} \leq c\|u\|^{p+1}_{L^{p+1}(B_{R_{k-1}})}$$

and raise the (4.40) to $\frac{1}{p}$. Note that for finite p, the element $p^{\frac{1}{p}}$ is a finite number. For $p \to \infty$, $p^{\frac{1}{p}} \to 1$. Therefore this term will be included in C. For ε being fixed yet small, we can do the same. After all, the final inequation we are about to iterate is

$$\|u\|_{L^{\frac{dp}{d-1}}(B_{R_k})} \leq C^{\frac{1}{p}} 2^{\frac{2k}{p}} \|u\|^{\frac{p+1}{p}}_{L^{p+1}(B_{R_{k-1}})}. \tag{4.41}$$

For $0 < \alpha < \frac{1}{d-1}$, consider

$$q := \left(\frac{d}{d-1} - \alpha \right) > 1 \quad \text{and} \quad p := \frac{d-1}{d} q^{k_0},$$

for k_0 big enough to hold that $1 < \alpha q^{k_0-1}\frac{d-1}{d}$. Then

$$p+1 \le q^{k_0-1} < q^{k_0} = p\frac{d}{d-1}, \tag{4.42}$$

because $q > 1$ and

$$p+1 = q^{k_0-1}\left(\frac{d}{d-1} - \alpha\right)\frac{d-1}{d} + 1 = q^{k_0-1} - \alpha q^{k_0-1}\frac{d-1}{d} + 1 < q^{k_0-1}$$

For $k \ge k_0$, set

$$p_k := q^k \quad \text{and} \quad a_k := \|u\|_{L^{p_k}(B_{R_k})}.$$

It holds that $p_k \to \infty$ as $k \to \infty$, also $\frac{p+1}{p} = \frac{p_{k_0} + \frac{d}{d-1}}{p_{k_0}} = 1 + \frac{d}{(d-1)p_{k_0}}$ and (4.41) turns into

$$a_{k_0} \le C^{\frac{d}{p_{k_0}(d-1)}} 2^{\frac{2k_0 d}{p_{k_0}(d-1)}} a_{k_0-1}^{1+\frac{d}{p_{k_0}(d-1)}}. \tag{4.43}$$

We can observe that using the same procedure we are able to obtain (4.43) for any $k > k_0$, and then iterate

$$\begin{aligned} a_k \le & C^{\frac{d}{p_k(d-1)}} 2^{\frac{2kd}{p_k(d-1)}} \left(C^{\frac{d}{p_{k-1}(d-1)}} 2^{\frac{2(k-1)d}{p_{k-1}(d-1)}} a_{k-2}^{1+\frac{d}{p_{k-1}(d-1)}} \right)^{1+\frac{d}{p_k(d-1)}} \\ \le & C^{\frac{d}{p_k(d-1)} + \sum_{j=k_0+1}^{k-1} \frac{d}{p_j(d-1)} \prod_{i=j+1}^{k}\left(1+\frac{d}{p_i(d-1)}\right)} 2^{\frac{2kd}{p_k(d-1)} + \sum_{j=k_0+1}^{k-1} \frac{2jd}{p_j(d-1)} \prod_{i=j+1}^{k}\left(1+\frac{d}{p_i(d-1)}\right)} \\ & a_{k_0}^{\prod_{i=k_0+1}^{k}\left(1+\frac{d}{p_i(d-1)}\right)}. \end{aligned}$$

The product $\prod_{i=k_0+1}^{k}\left(1+\frac{d}{p_i(d-1)}\right)$ is finite if and only if its logarithm is finite, also using the upper bound for logarithm, $\ln y \le y-1$, the fact that $\frac{1}{q} < 1$ and therefrom resulting power series convergence we get that

$$\begin{aligned} &\ln \prod_{i=k_0+1}^{k}\left(1+\frac{d}{p_i(d-1)}\right) = \sum_{i=k_0+1}^{k} \ln\left(1+\frac{d}{q^i(d-1)}\right) \le \sum_{i=k_0+1}^{k} \frac{d}{q^i(d-1)} \\ =&\frac{d}{q^{k_0+1}(d-1)} \sum_{i=0}^{\infty} \frac{1}{q^i} = \frac{d}{q^{k_0}(d-1)(q-1)} < \frac{1}{(d+1)(q-1)} < c. \end{aligned}$$

Obviously, this proves that $\prod_{i=k_0+2}^{k}\left(1+\frac{d}{p_i(d-1)}\right)$ is bounded, as well. Note that we can use the result for what comes next, too, together with the root test as the criterion for convergence of $\sum_{j=k_0+1}^{\infty}\frac{2jd}{q^j(d-1)}$, used as

$$\lim_{j\to\infty} \sqrt[j]{\frac{2jd}{q^j(d-1)}} = \frac{1}{q} < 1,$$

which gives that (we use c for constant resulting from the convergence and C remains being the multiplication constant)

$$\begin{aligned} a_k &\le C^{c\sum_{j=k_0+1}^{k}\frac{d}{p_j(d-1)}} 2^{c\sum_{j=k_0+1}^{k}\frac{2jd}{p_j(d-1)}} a_{k_0}^{c} \\ &\le C^{c\sum_{j=k_0+1}^{\infty}\frac{d}{p_j(d-1)}} 2^{c\sum_{j=k_0+1}^{\infty}\frac{2jd}{p_j(d-1)}} a_{k_0}^{c} \le C, \end{aligned}$$

and C is independent of k, therefore letting $k \to \infty$ it holds that $p_k \to \infty$, also $R_k \to \rho$ and we get that $u \in L^\infty_{loc}(\Omega)$, i.e.,

$$\|u\|_{L^\infty(B_\rho)} \leq C(R, \rho, \|u\|_{1,1}, \varepsilon\|u\|^2_{1,2}).$$

□

4.4 Uniform equi-integrability of $|\nabla u^\varepsilon|$

To finish the proof of Theorem 4.1, we would like to show that for $a \in (0, 2]$, the sequence of approximative solutions converges weakly in $W^{1,1}_{loc}(\Omega)$. The limit would be the solution u to (3.1) in the sense of Definition 3.3. According to Dunford-Pettis Theorem (Theorem 1.13), the weakly pre-compactness of $\{u^\varepsilon\}$ is equivalent to its uniform equi-integrability.

The two following lemmas have very similar proofs. The difference between them is whether we consider $a \in (0, 2)$ or $a \in (0, 2]$. Note that the first result is stronger, however we pay the price of allowing $a = 2$.

Lemma 4.6. Assume $(\mathcal{A})$ and $a \in (0, 2)$, then there exists a constant δ, $0 < \delta <<$ 1, and a constant $c = c(\delta, \tau)$ independent of ε such that

$$\int_\Omega \frac{|\nabla u|^2}{(1 + |\nabla u|^a)^{\frac{1}{a}}} (1 + |\nabla u|^2)^\delta \tau \,\mathrm{d}x \leq c$$

for any $\tau \in \mathcal{D}(\Omega)$.

Proof. In (4.4), set

$$\varphi := u \cdot (1 + |\nabla u|^2)^\delta \tau$$

for $\tau \in \mathcal{D}(\Omega)$ and $0 < \delta << 1$. Now, after we use the product rule, move the $\nabla \left[(1 + |\nabla u|^2)^\delta \tau\right]$ part on the right hand side, use the product rule again and estimate it by its absolute value, using also that $|A(\nabla u)| = \left|\frac{\nabla u}{(1 + |\nabla u|^a)^{\frac{1}{a}}}\right| \leq 1$, we get

$$\begin{aligned}
&2\varepsilon \int_\Omega |\nabla u|^2 (1 + |\nabla u|^2)^\delta \tau \,\mathrm{d}x + \int_\Omega \frac{|\nabla u|^2}{(1 + |\nabla u|^a)^{\frac{1}{a}}} (1 + |\nabla u|^2)^\delta \tau \,\mathrm{d}x \\
\leq& 4\delta\varepsilon \int_\Omega |\nabla u|\, |u| \left|1 + |\nabla u|^2\right|^{\delta-1} |\nabla u|\, |\nabla^2 u| |\tau| \mathrm{d}x + 2\varepsilon \int_\Omega |\nabla u|\, |u| \left|1 + |\nabla u|^2\right|^\delta |\nabla \tau| \mathrm{d}x \\
&+ 2\delta \int_\Omega |u| \left|1 + |\nabla u|^2\right|^{\delta-1} |\nabla u|\, |\nabla^2 u| |\tau| \,\mathrm{d}x + \int_\Omega |u| \left|1 + |\nabla u|^2\right|^\delta |\nabla \tau| \,\mathrm{d}x.
\end{aligned} \tag{4.44}$$

Firstly, remember the results from the text above, particularly that

$$\|u\|_{L^\infty(\Omega')} + \varepsilon \int_\Omega |\nabla u|^2 \mathrm{d}x + \varepsilon \int_{\Omega'} |\nabla^2 u|^2 \mathrm{d}x \leq c,$$

$$\int_{\Omega'} \frac{|\nabla^2 u|^2}{(1 + |\nabla u|)^{1+a}} \leq \int_{\Omega'} |B(\nabla u)| |\nabla^2 u|^2 \leq c,$$

for $\Omega' \subset\subset \Omega$. The sum on the right hand side of (4.44) is finite, which results from the following analysis

$$I_1 := 4\delta \int_\Omega |u|\ \varepsilon^{\frac{1}{2}}|\nabla u|\ \varepsilon^{\frac{1}{2}}|\nabla^2 u||\tau|\ |\nabla u|\left|1+|\nabla u|^2\right|^{\delta-1} \mathrm{d}x \leq c,$$

because $|u| \in L^\infty_{loc}(\Omega)$, $\varepsilon^{\frac{1}{2}}|\nabla u| \in L^2(\Omega)$, $\varepsilon^{\frac{1}{2}}|\nabla^2 u||\tau| \in L^2(\Omega)$ from the results above and $|\nabla u|\,|1+|\nabla u|^2|^{\delta-1} \in L^\infty(\Omega)$, because the powers of $|\nabla u|$ are negative, $1+2\delta-2 \leq 0$. Similarly,

$$I_2 := 2\int_\Omega |u|\ \varepsilon^{\frac{1}{2}}|\nabla u|\ \varepsilon^{\frac{1}{2}}|\nabla u|\ |\nabla\tau|\ \frac{\left|1+|\nabla u|^2\right|^\delta}{|\nabla u|}\mathrm{d}x \leq c,$$

since $|u| \in L^\infty_{loc}(\Omega)$, $\varepsilon^{\frac{1}{2}}|\nabla u| \in L^2(\Omega)$, $|\nabla\tau| \in L^\infty(\Omega)$ and $\frac{\left|1+|\nabla u|^2\right|^\delta}{|\nabla u|} \in L^\infty(\Omega)$, which is basically the same as before. As for the third one,

$$I_3 := 2\delta\int_\Omega |u| \left(\frac{|\nabla^2 u|^2\tau^2}{(1+|\nabla u|)^{1+a}}\right)^{\frac{1}{2}} (1+|\nabla u|)^{\frac{1+a}{2}}\ |\nabla u|\left|1+|\nabla u|^2\right|^{\delta-1} \mathrm{d}x \leq c,$$

because $|u| \in L^\infty_{loc}(\Omega)$, $\left(\frac{|\nabla^2 u|^2\tau^2}{(1+|\nabla u|)^{1+a}}\right)^{\frac{1}{2}} \in L^2(\Omega)$ and for the last element to fulfill that $(1+|\nabla u|)^{\frac{1+a}{2}}\ |\nabla u|\,|1+|\nabla u|^2|^{\delta-1} \in L^2(\Omega)$, we have the condition for powers of $|\nabla u|$. It is true, if it holds that

$$2\left(\frac{1+a}{2}+1+2\delta-2\right) \leq 1,$$

which is satisfied when $a \leq 2-4\delta$. For δ being small it means that $a < 2$.

Finally,

$$I_4 := \int_\Omega |u|\ \left|1+|\nabla u|^2\right|^\delta |\nabla\tau|\,\mathrm{d}x \leq c,$$

because $|u| \in L^\infty_{loc}(\Omega)$, $|\nabla\tau| \in L^\infty(\Omega)$ and $|1+|\nabla u|^2|^\delta \in L^1(\Omega)$, since $2\delta \leq 1$. To sum it up, we just got that

$$\int_\Omega \frac{|\nabla u|^2}{(1+|\nabla u|^a)^{\frac{1}{a}}}(1+|\nabla u|^2)^\delta\tau\,\mathrm{d}x \leq I_1+I_2+I_3+I_4 \leq c$$

with c being independent of ε. Then the easy calculation of powers, $2-1+2\delta = 1+2\delta$, puts $|\nabla u|$ in the space $L^{1+2\delta}_{loc}(\Omega)$ for $a < 2$ and δ small and positive. □

Lemma 4.7. Assume $(\mathcal{A})$ and $a \in (0,2]$, then there exists a constant $c = c(\tau)$ independent of ε such that

$$\int_\Omega |\nabla u| \ln(1+|\nabla u|^2)\tau\,\mathrm{d}x \leq c$$

for any $\tau \in \mathcal{D}(\Omega)$.

Proof. In (4.4), set

$$\varphi := u \cdot \ln(1 + |\nabla u|^2)\tau$$

for $\tau \in \mathcal{D}(\Omega)$, similarly as before we get the left hand side as

$$2\varepsilon \int_\Omega |\nabla u|^2 \ln(1 + |\nabla u|^2)\tau \,\mathrm{d}x + \int_\Omega \frac{|\nabla u|^2}{(1 + |\nabla u|^a)^{\frac{1}{a}}} \ln(1 + |\nabla u|^2)\tau \,\mathrm{d}x,$$

and the partial integrals on the right hand side as (using that $(\ln(1 + |\nabla u|^2))' = 2\frac{|\nabla u|\ |\nabla^2 u|}{1+|\nabla u|^2}$, and that $\ln y \leq y - 1$),

$$\begin{aligned}
\tilde{I}_1 &= 4 \int_\Omega |u|\ \varepsilon^{\frac{1}{2}} |\nabla^2 u||\tau|\ \varepsilon^{\frac{1}{2}} |\nabla u|\ \frac{|\nabla u|}{1 + |\nabla u|^2} \mathrm{d}x, \\
\tilde{I}_2 &= 2 \int_\Omega |u|\ \varepsilon^{\frac{1}{2}} |\nabla u|\ \varepsilon^{\frac{1}{2}} \ln(1 + |\nabla u|^2)|\nabla\tau| \mathrm{d}x, \\
\tilde{I}_3 &= 2 \int_\Omega |u| \left(\frac{|\nabla^2 u|^2 \tau^2}{(1 + |\nabla u|)^{1+a}} \right)^{\frac{1}{2}} (1 + |\nabla u|)^{\frac{1+a}{2}} \frac{|\nabla u|}{1 + |\nabla u|^2} \,\mathrm{d}x, \\
\tilde{I}_4 &= \int_\Omega |u|\ \ln(1 + |\nabla u|^2)|\nabla\tau| \,\mathrm{d}x.
\end{aligned}$$

The procedure is very similar to the one above. In fact, $\tilde{I}_1$ is estimated in exactly the same way. In $\tilde{I}_2$ and $\tilde{I}_4$ we use the estimate

$$\ln(1 + |\nabla u|^2) \leq \ln(1 + |\nabla u|)^2 \leq 2(1 + |\nabla u|) - 1 \leq c|\nabla u|,$$

and then the calculation corresponds with I_2, I_4. The difference between $\tilde{I}_3$ and I_3 is in the powers of $|\nabla u|$, and therefore in the condition on a,

$$2\left(\frac{1+a}{2} + 1 - 2\right) \leq 1 \qquad \Rightarrow \qquad a \leq 2.$$

On the left hand side we take into consideration only the powers of $|\nabla u|$ again to get that

$$\int_\Omega |\nabla u| \ln(1 + |\nabla u|^2)\tau \,\mathrm{d}x \leq \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 \leq c,$$

and c depends on $\Omega' \subset\subset \Omega$, however, the most important fact is that it is independent of ε. □

With the help of the estimate above the uniform equi-integrability of $\{\nabla u^\varepsilon\}$ can be proven. To do so, the equivalent statement from Remark 4 will be used. Let $0 < \lambda_1 < \lambda_2 < ...$ and $U_i \subset\subset \Omega$ be sets on which $|\nabla u^\varepsilon|^2 > \lambda_i$ for $i \in \mathbb{N}$. Then for every i,

$$\begin{aligned}
\int_{U_i} |\nabla u^\varepsilon| \,\mathrm{d}x &= \int_{U_i} |\nabla u^\varepsilon| \frac{\ln(1 + |\nabla u^\varepsilon|^2)}{\ln(1 + |\nabla u^\varepsilon|^2)} \,\mathrm{d}x \\
&\leq \frac{1}{\ln(1 + \lambda_i)} \int_{U_i} |\nabla u^\varepsilon| \ln(1 + |\nabla u^\varepsilon|^2) \,\mathrm{d}x \leq \frac{c}{\ln(1 + \lambda_i)}.
\end{aligned}$$

This proves the uniform equi-integrability, because for every $i \in \mathbb{N}$ we can find λ_i such that $c(\mathrm{e}^{4^i} - 1) < \lambda_i$. Then $\frac{c}{\ln(1+\lambda_i)} < \frac{1}{4^i}$ and the condition from Remark 4 is satisfied.

Applying the Dunford-Pettis Theorem (Theorem 1.13), it is equivalent to the existence of weakly convergent subsequence of $\{\nabla u^\varepsilon\}$ in $L^1_{loc}(\Omega)$. From Lemma 4.2 we know that $u^\varepsilon \rightharpoonup^* u$ in $BV(\Omega)$. Joining these facts we get that the limit has to be u,

$$\nabla u^\varepsilon \rightharpoonup \nabla u \text{ in } L^1_{loc}(\Omega) \quad \Rightarrow \quad u \in W^{1,1}_{loc}(\Omega).$$

Actually, the following estimate says more, for $\Omega' \subset\subset \Omega$, using the weakly lower semicontinuity of the L^1-norm and uniform equi-integrability of ∇u^ε,

$$\int_{\Omega'} |\nabla u| \, \mathrm{d}x \leq \lim_{\varepsilon \to 0_+} \int_{\Omega'} |\nabla u^\varepsilon| \, \mathrm{d}x \leq \lim_{\varepsilon \to 0_+} \int_{\Omega} |\nabla u^\varepsilon| \, \mathrm{d}x \leq c,$$

with c independent of Ω'. Therefore the following holds,

$$\lim_{\Omega' \to \Omega} \int_{\Omega'} |\nabla u| \, \mathrm{d}x = \int_{\Omega} |\nabla u| \, \mathrm{d}x \leq c,$$

where the limit exists from the Levi Theorem (Theorem 1.10). Then $u \in W^{1,1}(\Omega)$.

4.5 Limit $\varepsilon \to 0_+$

As this is the last section of the chapter, the proof of Theorem 4.1 will be finally completed here by limiting with $\varepsilon \to 0_+$. Obviously, at this moment we need to return from the notation u back to u^ε.

To prove that $\nabla u^\varepsilon \overset{loc}{\to} \nabla u$, we will use the Minty trick, i.e., for $\tau \in \mathcal{D}(\Omega)$ we will show that

$$\lim_{\varepsilon \to 0_+} \int_{\Omega} (A(\nabla u^\varepsilon) - A(\nabla u)) \, (\nabla u^\varepsilon - \nabla u) \, \tau \, \mathrm{d}x = 0. \tag{4.45}$$

Firstly, note that the expression is non-negative, since A is monotone. We separate it into simpler terms about which we already know something,

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \to 0_+} \int_{\Omega} (A(\nabla u^\varepsilon) - A(\nabla u))(\nabla u^\varepsilon - \nabla u)\tau \, \mathrm{d}x \\ &= \lim_{\varepsilon \to 0_+} \left(\int_{\Omega} A(\nabla u^\varepsilon) \nabla u^\varepsilon \tau \, \mathrm{d}x - \int_{\Omega} A(\nabla u^\varepsilon) \nabla u \tau \, \mathrm{d}x - \int_{\Omega} A(\nabla u)(\nabla u^\varepsilon - \nabla u)\tau \, \mathrm{d}x \right) \end{aligned}$$

The third element goes to 0, as $A(\nabla u) \in L^\infty(\Omega)$, $\tau \in L^\infty(\Omega)$ and $\nabla u^\varepsilon \rightharpoonup \nabla u$ in $L^1(\Omega)$. In the second term, $\{A(\nabla u^\varepsilon)\}_\varepsilon$ is a bounded sequence and according to the Selection Principle (Theorem 1.12), it has a weakly* convergent subsequence, $A(\nabla u^\varepsilon) \rightharpoonup^* \overline{A}$ in $L^\infty(\Omega)$. As for the first term, we adjust it a little bit,

$$\lim_{\varepsilon \to 0_+} \int_{\Omega} A(\nabla u^\varepsilon) \nabla u^\varepsilon \tau \, \mathrm{d}x = \lim_{\varepsilon \to 0_+} \left(\int_{\Omega} A(\nabla u^\varepsilon) \nabla (u^\varepsilon \tau) \, \mathrm{d}x - \int_{\Omega} A(\nabla u^\varepsilon) u^\varepsilon \nabla \tau \, \mathrm{d}x \right),$$

and again $A(\nabla u^\varepsilon) \rightharpoonup^* \overline{A}$ in $L^\infty(\Omega)$, $u^\varepsilon \to u$ in $L^1(\Omega)$ and $\nabla \tau \in L^\infty(\Omega)$. Note that for u^ε the weak formulation (4.4) holds. Using it with $\varphi := (u^\varepsilon \tau)$ and applying the product rule again,

$$\begin{aligned} \lim_{\varepsilon \to 0_+} \int_{\Omega} A(\nabla u^\varepsilon) \nabla (u^\varepsilon \tau) \, \mathrm{d}x &= - \lim_{\varepsilon \to 0_+} \int_{\Omega} 2\varepsilon \nabla u^\varepsilon \nabla (u^\varepsilon \tau) \, \mathrm{d}x \\ &= - \lim_{\varepsilon \to 0_+} \int_{\Omega} 2\varepsilon |\nabla u^\varepsilon|^2 \tau \, \mathrm{d}x - \lim_{\varepsilon \to 0_+} \int_{\Omega} 2\varepsilon \nabla u^\varepsilon \, u^\varepsilon \nabla \tau \, \mathrm{d}x. \end{aligned}$$

In the first element we may use that $\varepsilon^{\frac{1}{2}}\nabla u^\varepsilon \in L^2(\Omega)$ and $\tau \in L^\infty(\Omega)$, therefore the integral is finite and the limit exists and is even non-negative. Denote this limit as $\overline{B}$. For the second one we can use the same, and will shortly show that $\varepsilon^{\frac{1}{2}}u^\varepsilon \rightharpoonup 0$ in $L^2(\Omega)$ which, together with $\nabla\tau \in L^\infty(\Omega)$, guarantees that this term goes to 0 as $\varepsilon \to 0_+$. So let us take a look at why $\varepsilon\int_\Omega (u^\varepsilon)^2 \to 0$. According to the Interpolation inequality (1.14), there exists $0 < \alpha < 1$, such that

$$\lim_{\varepsilon\to 0_+} \varepsilon^{\frac{1}{2}}\|u^\varepsilon\|_2 \le \lim_{\varepsilon\to 0_+} c\varepsilon^{\frac{\alpha}{2}}\|u^\varepsilon\|_1^\alpha(\varepsilon^{\frac{1}{2}}\|u^\varepsilon\|_{1,2})^{1-\alpha}. \tag{4.46}$$

We know that $u^\varepsilon \to u$ in $L^1(\Omega)$, so the term $\|u^\varepsilon\|_1^\alpha$ is bounded. Moreover, the estimates we have, $\varepsilon\|\nabla u^\varepsilon\|_2^2 \le c$ and $\varepsilon\|u_D^\varepsilon\|_{1,2}^2 \le c$ from Lemma 4.2 implies that $\varepsilon^{\frac{1}{2}}\|u^\varepsilon\|_{1,2} \le c$, because

$$\|u^\varepsilon\|_{1,2} \le c\|\nabla(u^\varepsilon - u_D^\varepsilon)\|_2 + \|u_D^\varepsilon\|_{1,2} \le c\left(\|\nabla u^\varepsilon\|_2 + \|u_D^\varepsilon\|_{1,2}\right),$$

where the Poincaré inequality (1.13) was used on $\|u^\varepsilon - u_D^\varepsilon\|_{1,2}$. However, to use this estimate in (4.27), we have an additional condition on α, implying from the Sobolev embedding. That is,

$$\alpha + \frac{(1-\alpha)(d-2)}{2d} \le \frac{1}{2} \quad \Rightarrow \quad 0 < \alpha \le \frac{2}{d+2}$$

Thus, everything but $\varepsilon^{\frac{\alpha}{2}}$ in (4.46) is bounded and this goes to 0. Finally, putting everything together,

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0_+} \int_\Omega (A(\nabla u^\varepsilon) - A(\nabla u))(\nabla u^\varepsilon - \nabla u)\tau \,\mathrm{d}x \\ &= -\overline{B} - 0 - \int_\Omega \overline{A}\, u\, \nabla\tau \,\mathrm{d}x - \int_\Omega \overline{A}\, \nabla u\, \tau \,\mathrm{d}x - 0 \\ &\le -\int_\Omega \overline{A}\, \nabla(u\tau)\,\mathrm{d}x. \end{aligned} \tag{4.47}$$

At this moment, consider (4.4) again and apply limit $\varepsilon \to 0_+$,

$$0 = \lim_{\varepsilon\to 0_+} 2\varepsilon \int_\Omega \nabla u^\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \lim_{\varepsilon\to 0_+} \int_\Omega A(\nabla u^\varepsilon)\cdot\nabla\varphi\,\mathrm{d}x,$$

and note that in the first element $\nabla u^\varepsilon \rightharpoonup \nabla u$ in $L^1(\Omega)$, therefore the integral is finite and multiplied by 2ε goes in the limit to 0. The second element gives that

$$0 = \int_\Omega \overline{A}\, \nabla\varphi \,\mathrm{d}x$$

for all $\varphi \in W_0^{1,2}(\Omega)$. The term $\overline{A}$ is bounded and the space $W_0^{1,2}(\Omega)$ is dense in $W_0^{1,1}(\Omega)$. Therefore the same relation holds for all $\varphi \in W_0^{1,1}(\Omega)$, as well. Especially, set $\varphi := u\tau$, which is allowed since $u \in W^{1,1}(\Omega)$ and $\tau \in \mathcal{D}(\Omega)$. Then

$$0 = \int_\Omega \overline{A}\, \nabla(u\tau)\,\mathrm{d}x,$$

and this, used in (4.47) gives us the final equality

$$0 = \lim_{\varepsilon\to 0_+} \int_\Omega (A(\nabla u^\varepsilon) - A(\nabla u)) \cdot (\nabla u^\varepsilon - \nabla u)\tau \,\mathrm{d}x.$$

Since the interand is nonnegative (which follows from the fact that A is a monotone operator) we see that

$$(A(\nabla u^{\varepsilon}) - A(\nabla u)) \cdot (\nabla u^{\varepsilon} - \nabla u) \to 0$$

strongly in $L^1(\Omega')$ for any $\Omega' \subset\subset \Omega$. Consequently, since A is strictly monotone operator then necessarily we get the local pointwise convergence

$$\nabla u^{\varepsilon} \overset{loc}{\to} \nabla u.$$

Consequently, we see that $\overline{A} = A(\nabla u)$ almost everywhere in Ω.

Now, we finish the proof of Theorem 4.1 by showing that (3.4) is valid for u. From Lemma 4.1 we know that for every ε, u^{ε} satisfies (4.1). Taking the limit of (4.1) we would get (3.4). The limit of right hand side,

$$\lim_{\varepsilon \to 0_+} \int_{\Omega} F_{\varepsilon}(\nabla v) \, \mathrm{d}x = \int_{\Omega} F(\nabla v) \, \mathrm{d}x,$$

would be shown in the same way as it is already done in the proof of the Lemma 4.2. Therefore we focus on the limit on the left hand side. To finish the proof, it is enough to show that

$$\liminf_{\varepsilon \to 0_+} \int_{\Omega} F_{\varepsilon}(\nabla u^{\varepsilon}) \, \mathrm{d}x \geq \int_{\Omega} F(\nabla u) \, \mathrm{d}x + \int_{\partial\Omega} |u - u_D| \, \mathrm{d}S. \tag{4.48}$$

It is easy to see that

$$F_{\varepsilon}(\nabla u^{\varepsilon}) \geq F(\nabla u^{\varepsilon}) = F(\nabla u^{\varepsilon}) - |\nabla u^{\varepsilon}| + |\nabla u^{\varepsilon}|.$$

With the help of the pointwise convergence, the Vitali Theorem (Theorem 1.11) and the estimate (iii) from Lemma 2.1 we obtain that

$$\begin{aligned}\liminf_{\varepsilon \to 0} \int_{\Omega} F_{\varepsilon}(\nabla u^{\varepsilon}) \, \mathrm{d}x &\geq \lim_{\varepsilon \to 0} \int_{\Omega} F(\nabla u^{\varepsilon}) - |\nabla u^{\varepsilon}| \, \mathrm{d}x + \liminf_{\varepsilon \to 0} \int_{\Omega} |\nabla u^{\varepsilon}| \, \mathrm{d}x \\ &= \int_{\Omega} F(\nabla u) - |\nabla u| \, \mathrm{d}x + \liminf_{\varepsilon \to 0} \int_{\Omega} |\nabla u^{\varepsilon}| \, \mathrm{d}x\end{aligned}$$

After all, what we need to check is

$$\liminf_{\varepsilon \to 0} \int_{\Omega} |\nabla u^{\varepsilon}| \, \mathrm{d}x \geq \int_{\Omega} |\nabla u| \, \mathrm{d}x + \int_{\partial\Omega} |u - u_D| \, \mathrm{d}S. \tag{4.49}$$

Extending u_D^{ε} outside Ω by the Extension Theorem (Theorem 1.4), we also have that $u_D^{\varepsilon} \to u_D$ strongly in $W^{1,1}(\Omega_0)$, where $\Omega \subset\subset \Omega_0$. Finally, we also extend u^{ε} by u_D^{ε} outside Ω. Notice that such extension is possible and that u^{ε} is bunded in $W^{1,1}(\Omega_0)$ independently of ε. Consequently, for arbitrary smooth $\varphi \in \mathcal{D}(\Omega_0; \mathbb{R}^d)$ such that $\|\varphi\|_{\infty} \leq 1$, we have

$$\int_{\Omega} |\nabla u^{\varepsilon}| \, \mathrm{d}x = \int_{\Omega_0} |\nabla u^{\varepsilon}| \, \mathrm{d}x - \int_{\Omega_0 \setminus \Omega} |\nabla u_D^{\varepsilon}| \, \mathrm{d}x \geq \int_{\Omega_0} u^{\varepsilon} \mathrm{div}\varphi \, \mathrm{d}x - \int_{\Omega_0 \setminus \Omega} |\nabla u_D^{\varepsilon}| \, \mathrm{d}x.$$

Hence, using the above proven convergence results, we can easily let $\varepsilon \to 0$ in the above inequality to show that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u^\varepsilon|\,\mathrm{d}x &\geq \int_{\Omega_0} u\ \mathrm{div}\varphi\,\mathrm{d}x - \int_{\Omega_0\setminus\Omega}|\nabla u_D|\,\mathrm{d}x\\ &= \int_\Omega u\ \mathrm{div}\varphi\,\mathrm{d}x + \int_{\Omega_0\setminus\Omega} u_D\mathrm{div}\varphi - |\nabla u_D|\,\mathrm{d}x\end{aligned}$$

Next, since $u \in W^{1,1}(\Omega)$ and $u_D \in W^{1,1}(\Omega_0)$, we may use the integration by parts to get

$$\liminf_{\varepsilon\to 0}\int_\Omega |\nabla u^\varepsilon|\,\mathrm{d}x \geq -\int_\Omega \nabla u\cdot\varphi\,\mathrm{d}x - \int_{\Omega_0\setminus\Omega}\nabla u_D\cdot\varphi + |\nabla u_D|\,\mathrm{d}x + \int_{\partial\Omega}(u-u_D)\varphi\cdot\mathbf{n}\,\mathrm{d}S.$$

Finally, taking the supremum over all possible φ (here we refer to the Step 2 of the proof of Theorem 3.3), we get the desired relation (4.49).

Chapter 5

Counterexample to the existence of weak solution

In this chapter we will show that in some cases the weak solution (the one in the sense of Definition 3.1) to (3.1) does not exist. We will provide a model example (defined in (5.1)) and show that for $a > 1$, not even smooth data would guarantee the existence of weak solutions.

We use standard notation B_r for the ball in $\mathbb{R}^d$ with center in 0 and radius r. Let Ω be the annulus $B_2 \setminus B_1$ and u be the function in Sobolev space $W^{1,1}(\Omega)$. Consider the problem

$$\begin{aligned} -\mathrm{div}\frac{\nabla u}{(1+|\nabla u|^a)^{\frac{1}{a}}} &= 0 \quad \text{in } \Omega \\ u_D(x) &= K \quad \text{on } \partial B_1, \\ u_D(x) &= 0 \quad \text{on } \partial B_2. \end{aligned} \tag{5.1}$$

Lemma 5.1. For $a > 1$, the problem (5.1) has a weak solution according to Definition 3.1 if and only if

$$|K| < \int_1^2 \frac{1}{(z^{ad-a}-1)^{\frac{1}{a}}}\,\mathrm{d}z.$$

If $a \in (0,1]$, then for any $K \in \mathbb{R}$ there exists a weak solution to problem (5.1) in the sense of Definition 3.1.

Proof. First of all, we show that $u \in W^{1,1}(\Omega)$, a weak solution to (5.1), is independent of rotation. For this fact, however mathematically incorrectly, we use notation $u(x) = u(|x|)$. To prove it, we work with the rotation matrix $Q \in \mathbb{R}^{d\times d}$, which is orthogonal, i.e., $QQ^\top = I$.

Let $u \in W^{1,1}(\Omega)$ be the weak solution to (5.1). Let us define

$$v(x) := u(Qx).$$

We can easily see that $v \in W^{1,1}(\Omega)$, because rotation does not change qualitative properties of the function. It also does not change values on the boundary since both B_1 and B_2 are balls. Therefore

$$\begin{aligned} v_D(x) &= K \quad \text{on } \partial B_1, \\ v_D(x) &= 0 \quad \text{on } \partial B_2. \end{aligned}$$

In the following, we would like to verify that if u is a weak solution to (5.1), then also $v(x)$ is a weak solution to (5.1). Notice that the change of variables $y = Q^\top x$ does not change the shape of the domain Ω. Consider the test function $\psi \in \mathcal{D}(\Omega)$ such that $\psi(x) := \varphi(Q^\top x)$ for some $\varphi \in \mathcal{D}(\Omega)$. Then for the i-th component of $\nabla\psi$ it holds that

$$[\nabla\psi(x)]_i = \sum_{j=1}^{d} \left[\nabla\varphi(Q^\top x)\right]_j Q^\top_{ji} = \left[Q\nabla\varphi(Q^\top x)\right]_i$$

By the same procedure we get that $\nabla v(x) = Q^\top \nabla u(Qx)$. In the following calculation we use both these facts, starting with the weak formulation (3.2) for u,

$$\begin{aligned}
\int_\Omega \frac{\nabla u(x)}{(1+|\nabla u(x)|^a)^{\frac{1}{a}}} \cdot \nabla\psi(x)\,\mathrm{d}x = 0\\
\int_\Omega \frac{\nabla u(QQ^\top x)}{(1+|\nabla u(QQ^\top x)|^a)^{\frac{1}{a}}} \cdot \left(Q\nabla\varphi(Q^\top x)\right)\,\mathrm{d}x = 0\\
\int_\Omega \frac{\nabla u(Qy)}{(1+|\nabla u(Qy)|^a)^{\frac{1}{a}}} \cdot \left(Q\nabla\varphi(y)\right)\,\mathrm{d}y = 0\\
\int_\Omega \frac{Q\nabla v(y)}{(1+|\nabla v(y)|^a)^{\frac{1}{a}}} \cdot \left(Q\nabla\varphi(y)\right)\,\mathrm{d}y = 0\\
\int_\Omega \frac{\nabla v(y)}{(1+|\nabla v(y)|^a)^{\frac{1}{a}}} \cdot \nabla\varphi(y)\,\mathrm{d}y = 0.
\end{aligned}$$

The last equation identifies $v(x)$ to be a weak solution to (5.1). It holds that

$$Q\nabla u \cdot Q\nabla v = \sum_{i,j,k=1}^{d} Q_{ij}\frac{\partial u}{\partial x_j} \cdot Q_{ik}\frac{\partial v}{\partial x_k} = \sum_{j,k=1}^{d} \delta_{jk}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} = \nabla u \cdot \nabla v,$$

and therefore Q vanished in denominator in the element $|\nabla v(y)|^a$, as

$$|Q\nabla v(y)|^a = \left(|Q\nabla v(y)|^2\right)^{\frac{a}{2}} = \left(|\nabla v(y)|^2\right)^{\frac{a}{2}} = |\nabla v(y)|^a,$$

and so did both Qs in the last integral. Moreover, in Chapter 3 we stated uniqueness of weak solution and therefore

$$u(x) = v(x) = u(Qx).$$

Since the rotation matrix Q was chosen arbitrarily, this proves that $u(x) = u(|x|)$.

Therefore $\nabla u(|x|) = u'(|x|)\frac{x}{|x|}$, which implies that $|\nabla u|^a = (u')^a$ and at this moment we start solving the problem (5.1),

$$\begin{aligned}
\int_\Omega \frac{\nabla u}{(1+|\nabla u|^a)^{\frac{1}{a}}} \cdot \nabla\varphi\,\mathrm{d}x = 0\\
\int_\Omega \frac{u'(|x|)}{(1+(u')^a)^{\frac{1}{a}}}\frac{x}{|x|} \cdot \nabla\varphi\,\mathrm{d}x = 0
\end{aligned} \tag{5.2}$$

Now, we express the test function φ as a function g defined on $\mathbb{R}$ in the way that $\varphi(x) =: g(|x|)$, $g(1) = g(2) = 0$. Now $\nabla\varphi(x) = g'(|x|)\frac{x}{|x|}$ and therefore $\nabla\varphi(x)\frac{x}{|x|} = g'(|x|)$. Using this in (5.2) and consequently changing to polar coordinates we get that

$$\int_\Omega \frac{u'(|x|)}{(1+(u'(|x|))^a)^{\frac{1}{a}}} g'(|x|)\,\mathrm{d}x = 0$$

$$\int_1^2 \int_{\partial B_r} \frac{u'(r)}{(1+(u'(r))^a)^{\frac{1}{a}}} g'(r)\,\mathrm{d}S\,\mathrm{d}r = 0$$

$$H_d \int_1^2 \frac{r^{d-1}u'(r)}{(1+(u'(r))^a)^{\frac{1}{a}}} g'(r)\,\mathrm{d}r = 0,$$

where H_d is Hausdorff measure of the unit sphere in $\mathbb{R}^d$ and this expression is true for any $g \in \mathcal{D}(1,2)$. Therefore

$$\frac{u'(r)}{(1+(u'(r))^a)^{\frac{1}{a}}} = \frac{c}{r^{d-1}}$$

for all $r \in [1,2]$. Absolute value of the left hand side is always less or equal to one, $r \geq 1$ and therefore $|c| \leq 1$. Next, we raise the equation to the power of a and after few easy steps get that

$$|u'(r)| = \frac{c|}{(r^{ad-a} - |c|^a)^{\frac{1}{a}}}.$$

Using the Cauchy Mean Value Theorem,

$$|u(r)| = \int_r^2 \frac{|c|}{(z^{ad-a} - |c|^a)^{\frac{1}{a}}}\,\mathrm{d}z.$$

Without loss of generality, let c and K be nonnegative, i.e., $c \in [0;1]$ and $K = u(1) \geq 0$. We get that

$$u(1) \leq \int_1^2 \frac{1}{(z^{ad-a} - 1)^{\frac{1}{a}}}\,\mathrm{d}z. \tag{5.3}$$

Our question is, what parameter a we can choose to be able to obtain arbitrary value of K only by an appropriate selection of the constant c in this step. Firstly, note that if $c = 0$, then $u(1) = 0$ and therefore (5.3) holds if and only if $K = 0$. Solving the more interesting case $c \in (0,1]$ we start with the following approximation,

$$K \leq \int_1^2 \frac{1}{(e^{(ad-a)\ln z} - 1)^{\frac{1}{a}}}\,\mathrm{d}z \backsim \int_1^2 \frac{1}{((ad-a)\ln z)^{\frac{1}{a}}}\,\mathrm{d}z \backsim \int_1^2 \frac{1}{(\ln z)^{\frac{1}{a}}}\,\mathrm{d}z$$
$$\backsim \int_1^2 \frac{1}{(z-1)^{\frac{1}{a}}}\,\mathrm{d}z \backsim \int_0^1 \frac{1}{t^{\frac{1}{a}}}\,\mathrm{d}t.$$

From the row properties we know that the value of K is finite if $a > 1$ and K may be infinite only when $a \in (0,1]$. This completes the proof. □

Conclusion

In the work we studied a generalized minimal surface problem. In the first chapter we built the theory, which we consequently used. We tried to include all important definitions and theorems to provide everything a graduate student of mathematics would need to comprehend the following text. In Chapter 2 we stated the main problem using the functional F and studied the properties of F, which were used to prove the theorems in Chapters 3 and 4. These two chapters contain the most work that was done. In the third chapter we defined four different notions of solution and showed how they relate. In the next one, these results were used to prove that although the solution in the weakest sense always exists, to be able to get the stronger notion defined in the previous chapter we need to accept some restrictions on how general the modification of the minimal surface problem can be. Finally, the last chapter contains a counterexample on the existence of the weak solution on non-convex domain, which is the strongest notion of solution we considered.

However, there are still some open problems awaiting to be solved in the future, which we state here and maybe inspire someone to think about them.

1. One of the results of the fourth chapter says that for $a \in (0, 2]$, there exists a solution to (3.1) according to Definition 3.3. However, it is not known yet, whether one can find a satisfying solution also for $a > 2$ and non-smooth data or non-convex domain.

2. Also, the counterexample shows that for $a < 1$, there exists a weak solution even on non-convex domain, however the geometry we used was really specific. Therefore, there remains a conjecture, whether one can prove the existence in the case $a < 1$ even for some more general geometry and more relaxed conditions on data.

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