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Behavior of one-dimensional integral operators on function spaces

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Abstract: In this manuscript we study the action of one-dimensional integral operators on rearrangement-invariant Banach function spaces. Our principal goal is to characterize optimal target and optimal domain spaces corresponding to given spaces within the category of rearrangement-invariant Banach function spaces as well as to establish pointwise estimates of the non-increasing rearrangement of a given operator applied on a given function. We apply these general results to proving optimality relations between special rearrangement-invariant spaces. We pay special attention to the Laplace transform, which is a pivotal example of the operators in question.

Keywords: non-increasing rearrangement, K-functional, interpolation, the Laplace transform

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Contents

Introduction

We investigate one-dimensional integral operators of the form

$$S_a f(t) = \int_0^\infty a(st) f(s) \, ds,$$

defined for every suitable function f, locally integrable on $(0, \infty)$ and for an a-priori fixed function a satisfying certain appropriate conditions. The function a is throughout considered to be non-negative, non-increasing, bounded and not identically equal to zero on $(0, \infty)$. Occasionally, further assumptions on a will be imposed.

The investigation is motivated by the study of operators such as the Laplace transform which is a pivotal example of S_a, being at the same time one of the most important integral operators with a wide range of applications throughout analysis and other parts of mathematics.

The principal characteristics of integral operators is how they act on function spaces. Although the Lebesgue spaces L^p, where $p \in [1, \infty]$, play a primary role in many areas of mathematical analysis, there exist other classes of Banach spaces of measurable functions that are also of interest. Some of the well-known larger classes than Lebesgue spaces, such as for instance Orlicz spaces or Lorentz spaces, are of essential importance. The class of the so-called rearrangement-invariant Banach function spaces, which had been built in the first half of the 20th century, mostly through the efforts of Young, Orlicz, Hardy, Littlewood, Luxemburg, Lorentz, Zaanen and many others, provides a very reasonable and at the same time a fairly general environment of function spaces. In particular, it constitutes a common roof for all the classes of function spaces mentioned so far, and many more.

Our aim in this thesis is to investigate the action of the operator S_a on rearrangement-invariant Banach function spaces with a particular focus on the optimality of such results. In particular, our principal goal is to characterize the optimal rearrangement-invariant partner target space when a domain space, also rearrangement-invariant, is given, and also to characterize the optimal domain space for a given target. The approach is based on a combination of the techniques that have been developed in connection with investigation of optimal function spaces in Sobolev embeddings (cf. e.g. [4, 5, 8]) with real interpolation theory and weighted inequalities.

The thesis is structured as follows. In Chapter 1 we collect all the necessary preliminary material such as rearrangement-invariant Banach function spaces, certain function spaces which we will work with throughout the text and the K-functional. We also quote the most important principles of this theory including the Hardy lemma, the Hardy–Littlewood inequality, and the Holmstedt theorem.

We study the spaces containing functions defined on a general totally σ-finite measure spaces. We do not insert proofs of the known results, referring the reader for example to the book Bennett and Sharpley ([2]). In Chapter 2 we characterize the boundedness of the operator S_a on Banach function spaces. We also present the optimality results and also show the usage of the interpolation theory on establishing the pointwise estimate of a non-increasing rearrangement of S_a applied on a given function. Finally, in Chapter 3 we will apply the obtained result on a special case of the Laplace transform and we will study the action of this operator on the Lorentz and the classical Lorentz spaces.

Chapter 1

Preliminaries

1.1 Basic properties of rearrangement-invariant spaces

In this section we collect basic definitions and ingredients from the theory of rearrangement-invariant spaces and from the theory of the K-method of real interpolation. We also fix notation and quote known basic results which will be needed throughout the text. The proofs and further details can be found in [2].

We will denote by (R,μ) a totally σ-finite measure space, by $M_0(R,\mu)$ the set of all μ-measurable and a.e. finite functions on R, by $M^+(R,\mu)$ the set of all μ-measurable functions on R whose values lie in $[0,\infty]$ and by $M_0^+(R,\mu)$ the subset of $M_0(R,\mu)$ involving only non-negative functions.

Definition 1.1.1. A mapping $\rho : M^+(R,\mu) \to [0,\infty]$ is called a *Banach function norm* or just a *function norm* if, for all $f, g, f_n, (n = 1, 2, ...)$, in $M^+(R,\mu)$, for all constants $a \geq 0$, and for all μ-measurable subsets E of R, the following properties hold:

- (P1): $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\quad \rho(af) = a\rho(f); \quad \rho(f+g) \leq \rho(f) + \rho(g);$
- (P2): $0 \leq g \leq f$ μ-a.e. $\Rightarrow \rho(g) \leq \rho(f);$
- (P3): $0 \leq f_n \nearrow f$ μ-a.e. $\Rightarrow \rho(f_n) \nearrow \rho(f);$
- (P4): $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty;$
- (P5): $\mu(E) < \infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$ for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

Definition 1.1.2. Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in $M(R,\mu)$ for which $\rho(|f|) < \infty$ is called a *Banach function space.* For each $f \in X$, define

$$\|f\|_X = \rho(|f|).$$

An example of Banach function spaces are Lebesgue spaces $L^p = L^p(R,\mu)$.

Remark 1.1.3. We note that $\|f\|_X$ is defined for every $f \in M(R,\mu)$ but $f \in X$ if and only if $\|f\|_X < \infty$.

Now we present the *non-increasing rearrangement* of a given function and its certain properties.

Definition 1.1.4. The *distribution function* μ_f of a function f in $M_0(R,\mu)$ is given by

$$\mu_f(\lambda) = \mu\{x \in (0,\infty) : |f(x)| > \lambda\}, \ \lambda \in (0,\infty).$$

Definition 1.1.5. Two functions $f \in M_0(R,\mu)$ and $g \in M_0(S,\nu)$ are said to be *equimeasurable* if they have the same distribution function, that is, if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.

Definition 1.1.6. Suppose that f belongs to $M_0(R,\mu)$. The *non-increasing rearrangement* of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \ t \in (0,\infty).$$

Definition 1.1.7. Let ρ be a function norm over a totally σ-finite measure space (R,μ). Then ρ is said to be *rearrangement-invariant* if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions f and g in $M_0^+(R,\mu)$. In that case, the Banach function space $X = X(\rho)$ generated by ρ is said to be a *rearrangement-invariant space.*

The next proposition gives us certain important properties of the non-increasing rearrangement.

Proposition 1.1.8. *Suppose f, g and f_n, $(n = 1, 2, \ldots)$, belong to $M_0(R,\mu)$ and let a be any scalar. The non-increasing rearrangement f^* is a non-negative, non-increasing, right-continuous function on $[0,\infty)$. Furthermore,*

$$|g| \leq |f| \text{ a.e. } \Rightarrow g^* \leq f^*, \tag{1.1.1}$$

$$(af)^* = |a| f^*, \tag{1.1.2}$$

$$(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2), \tag{1.1.3}$$

$$|f_n| \nearrow |f| \text{ a.e. } \Rightarrow f_n^* \nearrow f^*. \tag{1.1.4}$$

Next we shall present a function that often serves as an alternative for the non-increasing rearrangement in appropriate situations and is sometimes easier to work with.

Definition 1.1.9. Let f belong to $M_0(R,\mu)$. Then f^{**} will denote the *maximal function* of f^* defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \ \text{t} > 0.$$

Theorem 1.1.10. *Let f, g belong to $M_0(R,\mu)$, let $t > 0$. Then*

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t).$$

We shall now present the definition and certain properties of the *associate spaces.*

Definition 1.1.11. If ρ is a function norm, its *associate norm* ρ' is defined on $M^+(R,\mu)$ by

$$\rho'(g) = \sup\left\{\int_R fgd\mu : f \in M^+(R,\mu), \rho(f) \le 1\right\}.$$

Remark 1.1.12. In Definition 1.1.11, the expression $\int_R fgd\mu$ can be replaced with $\int_R |fg|\, d\mu$ or $\int_R f^* g^* d\mu$.

Theorem 1.1.13. *Let ρ be a function norm. Then the associate norm ρ' is itself a function norm.*

Definition 1.1.14. Let ρ be a function norm and let $X = X(\rho)$ be the Banach function space determined by ρ. Let ρ' be the associate norm of ρ. The Banach function space $X(\rho')$ determined by ρ' is called the *associate space* of X and is denoted by X'.

We recall that the sum $X + Y$ of rearrangement-invariant function spaces X and Y over the same measure space (R,μ) is defined by

$$X + Y = \{f \in M(R,\mu) : f = g + h,\ g \in X,\ h \in Y\}$$

and normed by

$$\|f\|_{X+Y} = \inf_{f=g+h}(\|g\|_X + \|h\|_Y),$$

where the infimum is extended over all such decompositions of the function f. The intersection $X \cap Y$ of such spaces is normed by

$$\|f\|_{X\cap Y} = \max\{\|f\|_X, \|f\|_Y\}.$$

Remark 1.1.15. The basic examples of associate spaces are:

- if $1 \le p \le \infty$ then $(L^p)' = L^{p'}$, where $p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty \\ \infty & \text{if } p = 1 \end{cases}$
- $(L^1 \cap L^\infty)' = L^1 + L^\infty$.

We shall now quote several theorems which play an important role in the theory of Banach function spaces.

Theorem 1.1.16 (Hölder's inequality). *Let X be a Banach function space with the associate space X'. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_R |fg|\, d\mu \le \|f\|_X \, \|g\|_{X'}.$$

Theorem 1.1.17 (**Hardy's lemma**). *Let ξ_1 and ξ_2 be non-negative measurable functions on $(0,\infty)$ and suppose*

$$\int_0^t \xi_1(s)\,ds \le \int_0^t \xi_2(s)\,ds$$

for all $t>0$. Let η be any non-negative non-increasing function on $(0,\infty)$. Then

$$\int_0^\infty \xi_1(s)\,\eta(s)\,ds \le \int_0^\infty \xi_2(s)\,\eta(s)\,ds.$$

Theorem 1.1.18 (**Hardy's inequality**). *Let $\psi \ge 0$ on $(0,\infty)$, $-\infty<\lambda<1$ and $1\le q\le\infty$, then*

$$\left\{\int_0^\infty \left(t^\lambda \frac{1}{t}\int_0^t \psi(s)\,ds\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}} \le \frac{1}{1-\lambda}\left\{\int_0^\infty \left(t^\lambda \psi(t)\right)^q \frac{dt}{t}\right\}^{\frac{1}{q}}.$$

We shall now state a result from [4] on a weighted inequality for a kernel operator.

Theorem 1.1.19. *Let v,w be weights, that is, positive mesurable locally integrable functions on $(0,\infty)$. Let $\phi:(0,\infty)\times(0,\infty)\to(0,\infty)$ and let*

$$S_\phi f(x) = \int_0^\infty \phi(x,y)\,f(y)\,dy;\ \ \Phi(x,r) = \int_0^r \phi(x,y)\,dy.$$

If $1\le q\le p<\infty$ and there exists a positive C such that for every $r>0$ we have

$$\left(\int_0^r w\right)^{\frac{1}{p}} \le C\left(\int_0^\infty \Phi(x,r)^q\, v(x)dx\right)^{\frac{1}{q}},$$

then for every f non-negative and non-increasing on $(0,\infty)$ we get

$$\left(\int_0^\infty f^p w\right)^{\frac{1}{p}} \le C\left(\int_0^\infty (S_\phi f)^q\, v\right)^{\frac{1}{q}}.$$

We shall now introduce the concept of Lorentz spaces, which will be needed throughout the text.

Definition 1.1.20. Let $0<p,q\le\infty$. Then a *Lorentz space* $L^{p,q}=L^{p,q}(0,\infty)$ is the collection of all measurable functions f on $(0,\infty)$ such that

$$\|f\|_{p,q} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} < \infty.$$

Proposition 1.1.21. *The Lorentz space $L^{p,p}$, $(0<p\le\infty)$, coincides with the Lebesgue space L^p, and for $f\in L^p$*

$$\|f\|_{p,p} = \|f\|_p.$$

Theorem 1.1.22. *Suppose $1\le q\le p<\infty$ or $p=q=\infty$. Then $(L^{p,q}, \|\cdot\|_{p,q})$ is a rearrangement-invariant Banach function space.*

Theorem 1.1.23. *Let (R, μ) be a measure space and suppose $1 < p < \infty$, $1 \leq q \leq \infty$ (or $p = q = 1$ or $p = q = \infty$). Then the associate space of $L^{p,q}(R, \mu)$ is the Lorentz space $L^{p',q'}(R, \mu)$.*

We will also introduce the concept of the so-called *endpoint spaces*, which will be also needed later in the text.

Definition 1.1.24. Let $a \in (0, \infty]$. A non-decreasing function $\varphi : [0, a) \longmapsto [0, \infty)$ is called *quasi-concave* on $[0, \infty)$ if

$$\varphi(t) = 0 \Leftrightarrow t = 0$$

and

$$\frac{t}{\varphi(t)} \text{ is non-decreasing on } (0, a).$$

Now we will define two types of Marcinkiewicz endpoint spaces.

Definition 1.1.25. Let φ be a quasi-concave function on $[0, \mu(R))$. We then denote by $\|\cdot\|_{M_\varphi}$ the functional defined by

$$\|g\|_{M_\varphi} := \sup_{t \in (0, \mu(R))} \varphi(t) g^{**}(t), \; g \in M_0(R, \mu),$$

and by M_φ the collection

$$M_\varphi := \left\{ g \in M_0(R, \mu); \|g\|_{M_\varphi} < \infty \right\}.$$

Definition 1.1.26. Let φ be a quasi-concave function on $[0, \mu(R))$. We then denote by $\|\cdot\|_{m_\varphi}$ the functional defined by

$$\|g\|_{m_\varphi} := \sup_{t \in (0, \mu(R))} \varphi(t) g^{*}(t), \; g \in M_0(R, \mu),$$

and by m_φ the collection

$$m_\varphi := \left\{ g \in m_0(R, \mu); \|g\|_{m_\varphi} < \infty \right\}.$$

Remark 1.1.27. We immediately see that $M_\varphi(R, \mu) \hookrightarrow m_\varphi(R, \mu)$.

An important proposition follows.

Proposition 1.1.28. *The functional $\|\cdot\|_{M_\varphi}$ is a Banach function norm and the corresponding Marcinkiewicz endpoint space M_φ is a rearrangement-invariant Banach function space. On the other hand the functional $\|\cdot\|_{m_\varphi}$ is not necessarily a norm on $M_0(R, \mu)$.*

Example 1.1.29. We present some examples of the spaces M_φ and m_φ:

- Let $\varphi(t) = t$. Then $M_\varphi = L^1$ and $m_\varphi = L^{1,\infty}$, which is known not to be equivalent to a norm.
- Let $\varphi(t) = t^{\frac{1}{p}}$ for $p > 1$. Then $M_\varphi = m_\varphi = L^{p,\infty}$.
- Let $\varphi(t) = \chi_{(0,\infty)}$. Then $M_\varphi = m_\varphi = L^\infty$.

1.2 Facts from the theory of K-method

We shall now present some basic ingredients of the theory of the K-method of real interpolation.

Definition 1.2.1. A pair (X_0, X_1) of Banach spaces X_0 and X_1 is called a *compatible couple* if there exists some Hausdorff topological vector space into which each of X_0 and X_1 is continuously embedded.

Now we define the key notion of the K-functional.

Definition 1.2.2. Let (X_0, X_1) be a compatible couple of Banach spaces. The *K-functional* is defined for each f from $X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) = \inf\left\{\|f_0\|_{X_0} + t\,\|f_1\|_{X_1} : f = f_0 + f_1\right\},$$

where the infimum extends over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

In the case when no misunderstanding can appear we will denote $K(f, t) = K(f, t; X_0, X_1)$. The next proposition is a useful tool for working with K-functionals.

Proposition 1.2.3. *For each f in $X_0 + X_1$, the K-functional $K(f, t; X_0, X_1)$ is a non-negative concave function of $t > 0$, and*

$$t^{-1} K(f, t; X_0, X_1) = K\left(f, t^{-1}; X_1, X_0\right).$$

The next theorem shows that the value of the K-functional for the spaces L^1 and L^∞ can be expressed exactly. We shall show later that for certain different pairs of spaces at least upper and lower estimates can be obtained.

Theorem 1.2.4. *Let (R, μ) be a totally σ-finite measure space. Then, for each f in $(L^1 + L^\infty)(R, \mu)$ and $t > 0$ it holds*

$$K\left(f, t; L^1, L^\infty\right) = \int_0^t f^*(s)\, ds.$$

Definition 1.2.5. Let (X_0, X_1) be a compatible couple and suppose $0 < \theta < 1$, $1 \le q < \infty$ or $0 \le \theta \le 1$, $q = \infty$. The space $(X_0, X_1)_{\theta,q}$ consists of all f in $X_0 + X_1$ for which the functional

$$\|f\|_{\theta,q} = \left\| t^{-\theta - \frac{1}{q}} K(f, t) \right\|_q$$

is finite.

The following theorem can be obtained directly from definitions but is very useful in practice.

Theorem 1.2.6. *Let $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ where (X_0, X_1) and (Y_0, Y_1) are compatible couples. Then*

$$K(Tf, t; Y_0, Y_1) \le M_0 K\left(f, t\frac{M_1}{M_0}; X_0, X_1\right)$$

for all f in $X_0 + X_1$, all $t > 0$ and some constants M_0 and M_1.

Let us fix some notation now. Let (X_0, X_1) be a compatible couple and consider two interpolation spaces

$$\overline{X}_{\theta_0} = (X_0, X_1)_{\theta_0, q_0} \text{ and } \overline{X}_{\theta_1} = (X_0, X_1)_{\theta_1, q_1}$$

where $0 < \theta_0 < \theta_1 < 1$ and $1 \leq q_0, q_1 \leq \infty$. Then $\left(\overline{X}_{\theta_0}, \overline{X}_{\theta_1}\right)$ is itself a compatible couple. We shall write $\overline{K}(f,t) = K\left(f, t; \overline{X}_{\theta_0}, \overline{X}_{\theta_1}\right)$.

The next theorem is one of the most important in the theory of the K-functional. The symbol $A \approx B$ denotes that there exists a positive constant C independent of appropriate quantities such that $C^{-1}A \leq B \leq CA$.

Theorem 1.2.7 (**T. Holmstedt**). *Let (X_0, X_1) be a compatible couple and suppose $0 < \theta_0 < \theta_1 < 1$ and $1 \leq q_0, q_1 \leq \infty$, Let $\delta = \theta_1 - \theta_0$. Then*

$$\overline{K}\left(f, t^{\delta}\right) \approx \left\{ \int_0^t \left(s^{-\theta_0} K(f,s)\right)^{q_0} \frac{ds}{s} \right\}^{\frac{1}{q_0}} + t^{\delta} \left\{ \int_t^{\infty} \left(s^{-\theta_1} K(f,s)\right)^{q_1} \frac{ds}{s} \right\}^{\frac{1}{q_1}},$$

for all f in $\overline{X}_{\theta_0} + \overline{X}_{\theta_1}$ and all $t > 0$. If q_0 or q_1 is infinite, the corresponding integral is replaced by the supremum in the usual way.

The following result corresponds to the extreme case $\theta_0 = 0$ and $\theta_1 = 1$ of the previous theorem considering $X_0 = L^1$, $X_1 = L^{\infty}$.

Corollary 1.2.8. *Suppose $0 < \theta < 1$, $1 \leq q \leq \infty$, and let $X_{\theta,q} = (L^1, L^{\infty})_{\theta,q}$. Then*

$$K\left(f, t^{\theta}; L^1, X_{\theta,q}\right) \approx t^{\theta} \left\{ \int_t^{\infty} \left(s^{-\theta} K\left(f, s; L^1, L^{\infty}\right)\right)^q \frac{ds}{s} \right\}^{\frac{1}{q}}$$

and

$$K\left(f, t^{1-\theta}; X_{\theta,q}, L^{\infty}\right) \approx \left\{ \int_0^t \left(s^{-\theta} K\left(f, s; L^1, L^{\infty}\right)\right)^q \frac{ds}{s} \right\}^{\frac{1}{q}}$$

with obvious modification if $q = \infty$.

Chapter 2

The operator S_a

In our preceding work [6] we described various properties of the Laplace transform operator, such as boundedness between certain spaces or a pointwise estimate of the non-increasing rearrangement of the Laplace transform of a given function. Here we shall extend this theory to a fairly more general class of integral operators.

Given an appropriate function a on $(0, \infty)$, we define operator S_a as

$$S_a f(t) := \int_0^\infty a(st) f(s)\, ds,$$

for suitable functions f. An important example of such an operator is the above-mentioned Laplace transform which corresponds to $a(x) = e^{-x}$.

2.1 The action of one-dimensional integral operators on function spaces

In this section we will examine properties of the operator S_a. We will start with a precise definition of such operator and then we shall study its boundedness between function spaces.

Definition 2.1.1. Let X be a rearrangement-invariant space over $(0, \infty)$. Let a be a non-negative measurable bounded and non-increasing function on $(0, \infty)$ such that $a \not\equiv 0$ and a belongs to X'. Then we can define the operator S_a for every function f from $L^1 + X$ by

$$S_a f(t) = \int_0^\infty a(st) f(s)\, ds \text{ for } t \in (0, \infty).$$

We note that the integral in the definition of S_a is well defined for any $f \in M^+(0, \infty)$. Moreover, it is convergent for functions from $L^1 + X$ (this follows from the Hölder inequality).

We start with the basic boundedness result which holds in the case when a is essentially bounded.

Theorem 2.1.2. *Let $a \in L^\infty(0, \infty)$. Then the operator S_a is well defined on $L^1(0, \infty)$, and, moreover,*

$$S_a : L^1(0, \infty) \to L^\infty(0, \infty).$$

Proof. We have

$$\begin{aligned}\|S_a f\|_\infty &= \inf\{\alpha \ge 0 : |S_a f| \le \alpha \text{ a.e. on } (0,\infty)\}\\ &\le \inf\left\{\alpha \ge 0 : K\int_0^\infty |f(s)|\,ds \le \alpha \text{ a.e. on } (0,\infty)\right\}\\ &= K\|f\|_{L^1},\end{aligned}$$

where $K = \|a\|_\infty$, proving both the assertions. □

Our next goal is to describe the boundedness of S_a between spaces in the most general way possible. To this end, we shall take advantage of the *dilation operator*.

Given $t \in (0,\infty)$, let us define the operator E_t at $g \in M(0,\infty)$ by

$$(E_t g)(s) := g\left(\frac{s}{t}\right) \text{ for } s \in (0,\infty).$$

Definition 2.1.3. Given a rearrangement invariant space X, we define the set H_X by

$$H_X = \left\{g \in M(0,\infty) ; \|g\|_{H_X} < \infty\right\},$$

where

$$\|g\|_{H_X} = \sup_{t\in(0,\infty)} \frac{t}{\|E_t\|_{X\to X}} g^*(t).$$

Remark 2.1.4. The function $\|\cdot\|_{H_X}$ from the previous definition is a quasinorm for any choice of X, but it is not always a function norm. We will add more details in the Proposition 2.1.7.

The next theorem gives us a useful property of the function $\|E_t\|_{X\to X}$.

Theorem 2.1.5. *Let X be a rearrangement invariant space. Then for all $t \ge 0$ the function $\|E_t\|_{X\to X}$ is quasi-concave.*

Proof. We will first show that $\|E_t\|_{X\to X}$ is non-decreasing. We will use the fact that for all $y \ge 0$ it is

$$(E_t g)^*(y) = g^*\left(\frac{y}{t}\right),$$

which we get from the chain

$$\begin{aligned}(E_t g)^*_t(y) &= \inf\{\lambda : \mu_{E_t g}(\lambda) \le y\} = \inf\{\lambda : \mu\{x \in (0,\infty) : |(E_t g)(x)| > \lambda\} \le y\}\\ &= \inf\left\{\lambda : \mu\left\{x \in (0,\infty) : \left|g\left(\frac{x}{t}\right)\right| > \lambda\right\} \le y\right\}\\ &=^{z=\frac{x}{t}} \inf\{\lambda : \mu\{zt \in (0,\infty) : |g(z)| > \lambda\} \le y\}\\ &= \inf\{\lambda : t\mu\{z \in (0,\infty) : |g(z)| > \lambda\} \le y\}\\ &= \inf\left\{\lambda : \mu\{x \in (0,\infty) : |g(x)| > \lambda\} \le \frac{y}{t}\right\}\\ &= g^*\left(\frac{y}{t}\right).\end{aligned}$$

By using Remark 1.1.12 we get

$$\|E_t\|_{X\to X} = \sup_{g\neq 0} \frac{\|E_t g\|_X}{\|g\|_X} = \sup_{g\neq 0} \sup_{\|h\|_{X'}\leq 1} \frac{\int_0^\infty (E_t g)^* (s)\, h^* (s)\, ds}{\|g\|_X}$$
$$= \sup_{g\neq 0} \sup_{\|h\|_{X'}\leq 1} \frac{\int_0^\infty g^* \left(\frac{s}{t}\right) h^* (s)\, ds}{\|g\|_X},$$

which is increasing in t thanks to the monotonicity of g^*.

Let us now write $\|E_t\|$ instead of $\|E_t\|_{X\to X}$. We would like to show that the function $\frac{\|E_t\|}{t}$ is non-increasing. Again by using Remark 1.1.12 we have

$$\frac{\|E_t\|}{t} = \sup_{\|g\|_X=1} \frac{1}{t} \|E_t g\|_X = \sup_{\|g\|_X=1} \sup_{\|h\|_{X'}=1} \frac{1}{t} \int_0^\infty g^* \left(\frac{s}{t}\right) h^* (s)\, ds$$
$$=^{y=\frac{s}{t}} \sup_{\|g\|_X=1} \sup_{\|h\|_{X'}=1} \int_0^\infty g^* (y)\, h^* (ty)\, dy,$$

which is non-increasing in t due to the monotonicity of h^*. $\square$

The next theorem shows a boundedness result for the operator S_a in the case when a belongs to a general rearrangement-invariant space. For convenience, we shall assume that a belongs to the associate space of a given rearrangement-invariant space.

Theorem 2.1.6. *Let X be a rearrangement-invariant space and let $a \in X'$, non-increasing, non-negative. Then*

$$S_a : X \to H_X.$$

Proof. Let us denote $K = \|a\|_{X'}$. Then we get on using the change of variables and Hölder's inequality (Theorem 1.1.16) that

$$tS_a f(t) = t \int_0^\infty a(ts)\, f(s)\, ds$$
$$=^{y=ts} \int_0^\infty a(y)\, f\left(\frac{y}{t}\right) dy$$
$$\leq \|a\|_{X'} \left\| f\left(\frac{\cdot}{t}\right)\right\|_X$$
$$= \|a\|_{X'} \|E_t f\|_X$$
$$\leq K \|E_t\|_{X\to X} \|f\|_X,$$

which implies

$$\sup_{t\in(0,\infty)} \frac{t}{\|E_t\|} (S_a f)(t) \leq K \|f\|_X.$$

Let $f \geq 0$. Then $S_a f = (S_a f)^*$ since a is non-increasing. We thus get

$$S_a : X \to H_X.$$

Now let $f \in X$ (not necessarily non-negative). It is enough to show that

$$\sup_{t\in(0,\infty)} \frac{t}{\|E_t\|} (S_a f)^* (t) \leq \sup_{t\in(0,\infty)} \frac{t}{\|E_t\|} (|S_a f|)(t).$$

Let us denote $\varphi(t) := \frac{t}{\|E_t\|}$ and $h(t) := |S_a f(t)|$. Then

$$\sup_{t\in(0,\infty)} \varphi(t)\, h(t) \leq \sup_{t\in(0,\infty)} \varphi(t) \sup_{s\in(t,\infty)} h(s) = \sup_{s\in(0,\infty)} h(s) \sup_{t\in(0,s)} \varphi(t)$$

$$= \sup_{s\in(0,\infty)} h(s)\, \varphi(s)\,,$$

because according to Theorem 2.1.5, φ is increasing. Therefore

$$\sup_{t\in(0,\infty)} \varphi(t)\, h(t) = \sup_{t\in(0,\infty)} \varphi(t) \sup_{s\in(t,\infty)} h(s)\,.$$

Let us denote $\bar{h}(t) := \sup_{s\in(t,\infty)} h(s)$. Then $\bar{h} \geq h$, therefore also $\left(\bar{h}\right)^* \geq h^*$. Moreover, $\left(\bar{h}\right)^* = \bar{h}$ because $\bar{h}$ is non-increasing. Consequently,

$$\sup_{t\in(0,\infty)} \varphi(t)\, h(t) = \sup_{t\in(0,\infty)} \varphi(t)\, \bar{h}(t) \geq \sup_{t\in(0,\infty)} \varphi(t)\, h^*(t)\,.$$

□

We will now show that under certain conditions the quantity $\|\cdot\|_{H_X}$ has properties of a Banach function norm. As we have already observed, this is not true for any rearrangement-invariant space X.

Proposition 2.1.7. *Let us assume that φ is quasi-concave and that there exists $c \geq 0$ such that for all $t \geq 0$*

$$\int_0^t \frac{ds}{\varphi(s)} \leq \frac{ct}{\varphi(t)}.$$

Then $M_\varphi = m_\varphi$ and therefore also m_φ is a Banach function space.

Proof. We get $M_\varphi \subseteq m_\varphi$ immediately from the fact that $g^{**} \geq g^*$. Let us assume now that $g \in m_\varphi$ and denote $K := \sup_{t\in(0,\infty)} \varphi(s)\, g^*(s)$. Then, for all $s \geq 0$, one has $g^*(s) \leq \frac{K}{\varphi(s)}$, whence

$$g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)\, ds \leq \frac{1}{t}\int_0^t \frac{K}{\varphi(s)} ds \leq \frac{Kc}{\varphi(t)},$$

which implies

$$\sup_{t\in(0,\infty)} \varphi(t)\, g^{**}(t) \leq Kc.$$

Thus $m_\varphi \subseteq M_\varphi$. Altogether, $m_\varphi = M_\varphi$. □

Corollary 2.1.8. *Let X be a rearrangement-invariant space and let us assume that there exists a constant c such that for all $t \geq 0$ the inequality*

$$\int_0^t \frac{\|E_s\|_{X\to X}}{s} ds \leq c\, \|E_t\|_{X\to X}$$

holds. Then H_X is a Banach function space.

Proof. This is an immediate consequence of Proposition 2.1.7 applied on $\varphi(t) := \frac{t}{\|E_t\|}$. □

Examples of such spaces were already mentioned in Examples 1.1.29.

We shall now present a useful simple observation.

Proposition 2.1.9. *For a rearrangement-invariant space X and $t \geq 0$ we have*

$$\|E_t\|_{X' \to X'} = t \left\| E_{\frac{1}{t}} \right\|_{X \to X}.$$

Proof.

$$\begin{aligned}
\|E_t\|_{X' \to X'} &= \sup_{\|g\|_{X'}=1} \|E_t g\|_{X'} = \sup_{\|g\|_{X'}=1} \sup_{\|h\|_X=1} \int_0^\infty E_t g(s) h(s) \, ds \\
&= \sup_{\|h\|_X=1} \sup_{\|g\|_{X'}=1} \int_0^\infty g\left(\frac{s}{t}\right) h(s) \, ds \\
&= \sup_{\|h\|_X=1} \sup_{\|g\|_{X'}=1} t \int_0^\infty g(s) h(ts) \, ds = t \sup_{\|h\|_X=1} \left\| E_{\frac{1}{t}} h \right\|_X \\
&= t \left\| E_{\frac{1}{t}} \right\|_{X \to X}.
\end{aligned}$$

□

The exact value of $\|E_t\|_{X \to X}$ is in general very difficult or even impossible to compute. However, we will show that for example for the two-parameter Lorentz space $X = L^{p,q}$ this can be done.

Example 2.1.10. For $L^{p,q}(0,\infty)$, where $1 \leq p, q \leq \infty$, and for all $t > 0$ it holds

$$\|E_t\|_{L^{p,q} \to L^{p,q}} = t^{\frac{1}{p}},$$

and therefore for $a \in L^{p',q'}$, non-increasing and non-negative, we have

$$S_a : L^{p,q} \to H_{L^{p,q}},$$

and $H_{L^{p,q}} = L^{p',\infty}$.

Proof. We observe that, by a change of variables

$$\begin{aligned}
\|E_t\|_{L^{p,q} \to L^{p,q}} &= \sup_{\|g\|_{L^{p,q}}=1} \|E_t g\|_{L^{p,q}} = \sup_{\|g\|_{L^{p,q}}=1} \left\| y^{\frac{1}{p}-\frac{1}{q}} (E_t g)^*(y) \right\|_{L^q(0,\infty)} \\
&= \sup_{\|g\|_{L^{p,q}}=1} \left(\int_0^\infty \left(y^{\frac{1}{p}-\frac{1}{q}} \right)^q \left(g^*\left(\frac{y}{t}\right) \right)^q dy \right)^{\frac{1}{q}} \\
&=^{z=\frac{y}{t}} \sup_{\|g\|_{L^{p,q}}=1} \left(\int_0^\infty \left((zt)^{\frac{1}{p}-\frac{1}{q}} \right)^q (g^*(z))^q \, t dz \right)^{\frac{1}{q}} \\
&= \left(t^{\frac{q}{p}-1} t \right)^{\frac{1}{q}} \sup_{\|g\|_{L^{p,q}}=1} \left(\int_0^\infty \left(z^{\frac{1}{p}-\frac{1}{q}} \right)^q (g^*(z))^q \, dz \right)^{\frac{1}{q}} \\
&= t^{\frac{1}{p}} \sup_{\|g\|_{L^{p,q}}=1} \|g\|_{L^{p,q}} = t^{\frac{1}{p}}.
\end{aligned}$$

Thus, by the definition of the norm in H_X, we get

$$\|g\|_{H_{L^{p,q}}} = \sup_{t \in (0,\infty)} \frac{t}{t^{\frac{1}{p}}} g^*(t) = \sup_{t \in (0,\infty)} t^{\frac{1}{p'}} g^*(t) = \|g\|_{p',\infty}.$$

□

Remark 2.1.11. The Example 2.1.10 works also for the extreme case $p=\infty$ even though $L^{p',\infty}=L^{1,\infty}$ which is not a Banach function space and it is determined by a quasinorm.

Example 2.1.12. Let φ be a quasi-concave function on $(0,\infty)$. For the space m_φ, we have

$$\|E_t\|_{m_\varphi\to m_\varphi}=\sup_{s\in(0,\infty)}\frac{\varphi(st)}{\varphi(s)}.$$

Proof. Note that

$$\begin{aligned}\|E_t\|_{m_\varphi\to m_\varphi}&=\sup_{\|g\|_{m_\varphi}\le 1}\|E_tg\|_{m_\varphi}=\sup_{\|g\|_{m_\varphi}\le 1}\left\|g\left(\frac{\cdot}{t}\right)\right\|_{m_\varphi}\\&=\sup_{\|g\|_{m_\varphi}\le 1}\sup_{s\in(0,\infty)}\varphi(s)\,g^*\left(\frac{s}{t}\right)\\&=\sup_{\left\{g;\sup_{s\in(0,\infty)}\varphi(s)g^*(s)\le 1\right\}}\sup_{s\in(0,\infty)}\varphi(st)\,g^*(s)\\&\le\sup_{s\in(0,\infty)}\frac{\varphi(st)}{\varphi(s)}.\end{aligned}$$

We get the converse inequality immediately from the special case $g^*(s)\,\varphi(s)=1$, more explicitly

$$\|E_t\|_{m_\varphi\to m_\varphi}=\sup_{\left\{g;\sup_{s\in(0,\infty)}\varphi(s)g^*(s)\le 1\right\}}\sup_{s\in(0,\infty)}\varphi(st)\,g^*(s)$$

$$\ge\sup_{s\in(0,\infty)}\frac{\varphi(st)}{\varphi(s)}.$$

□

We can now make a pointwise estimate for a non-increasing rearrangement of S_a applied to a given function. To this end, we will first need a characterization of the K-functional for the couple $(L^{1,\infty},L^\infty)$. Such result is an analogue of Theorem 1.2.4, even though in this case we do not get the exact formula. Instead, we only obtain lower and upper estimates and we will show that the constants in both these estimates are sharp.

Theorem 2.1.13. *For every $f\in M_0(0,\infty)$ and all $t\in(0,\infty)$, one has*

$$\sup_{s\in(0,t)}sf^*(s)\le\inf_{f=g+h}\left(\|g\|_{1,\infty}+t\,\|h\|_\infty\right)\le 2\sup_{s\in(0,t)}sf^*(s)\tag{2.1.1}$$

and the constants in both inequalities are optimal.

Proof. The first inequality has already been proved in [6] but we shall recall the detailed proof here for the sake of completeness.

We fix f and $t>0$ and let $\alpha:=\inf_{f=g+h}\{\|g\|_{L^{1,\infty}}+t\,\|h\|_{L^\infty}\}$. Without loss of generality we can assume that $f\in L^{1,\infty}+L^\infty$, therefore $f=g+h$ where $g\in L^{1,\infty}$ and $h\in L^\infty$.

In the next step we use Proposition 1.1.8, part (1.1.3) and we get

$$f^*(s) \le g^*(s) + h^*(0),$$

therefore

$$\begin{aligned}\sup_{s\in(0,t]} sf^*(s) &\le \sup_{s\in(0,t]} sg^*(s) + \sup_{s\in(0,t]} sh^*(0) \le \sup_{s\in(0,\infty)} sg^*(s) + th^*(0)\\ &= \|g\|_{L^{1,\infty}} + t\|h\|_{L^\infty}.\end{aligned}$$

This holds for all g and h such that $f = g + h$ so it holds for the infimum as well. This completes the proof of the first inequality in (2.1.1). Now we shall turn our attention to the second one. It is enough to show that it holds for one particular pair of functions h_0 and g_0. For a fixed $t > 0$ we put

$$h_0(s) := \min\{|f(s)|, f^*(t)\}$$

and

$$g_0(s) := |f(s)| - h_0(s) = \max\{|f(s)| - f^*(t), 0\}.$$

We need to show that $h_0 \in L^\infty$ and that $g_0 \in L^{1,\infty}$. The former assertion is obvious. Next, observe that

- $h_0^*(s) = \min\{f^*(s), f^*(t)\} = \begin{cases} f^*(t) & \text{for } s \in (0,t) \\ f^*(s) & \text{for } s \in (t,\infty), \end{cases}$

  which follows easily from the fact that

  $$\begin{aligned}\mu_{h_0}(\lambda) &= \mu\{s \in (0,\infty) : \min\{|f(s)|, f^*(t)\} > \lambda\}\\ &= \mu\{s \in (0,\infty) : \min\{f^*(s), f^*(t)\} > \lambda\},\end{aligned}$$

  where μ_{h_0} is a distribution function of the function h_0. Similarly,

- $g_0^* = \max\{f^*(s) - f^*(t), 0\} = (f^*(s) - f^*(t))\,\chi_{(0,1)}(s)$,
  where the argument is similar to the previous, specifically

  $$\begin{aligned}\mu_{g_0}(\lambda) &= \mu\{s \in (0,\infty) : \max\{|f(s)| - f^*(t), 0\} > \lambda\}\\ &= \mu\{s \in (0,\infty) : \max\{f^*(s) - f^*(t), 0\} > \lambda\},\end{aligned}$$

  including the fact that the non-increasing rearrangement satisfies $g_0^*(s) = 0$ for all $s \ge t$.

Without loss of generality, we can assume that

$$\sup_{s\ge 0} sf^*(s) < \infty.$$

Then we have

$$sg_0^*(s) = (sf^*(s) - sf^*(t))\,\chi_{(0,t)}(s) \le sf^*(s) < \infty.$$

Therefore, applying supremum we get

$$\sup_{s\in(0,\infty)} sg_0^*(s) < \infty,$$

thus

$$g_0 \in L^{1,\infty}.$$

We continue with showing that the inequality stated in the theorem holds:

$$\begin{aligned}
\|g_0\|_{1,\infty} &= \sup_{s\in(0,\infty)} sg_0^*(s) = \sup_{s\in(0,\infty)} (sf^*(s) - sf^*(t))\,\chi_{(0,t)}(s) \\
&= \sup_{s\in(0,t)} (sf^*(s) - sf^*(t)) \\
&\le \sup_{s\in(0,t)} sf^*(s),
\end{aligned}$$

and

$$t\,\|h_0\|_\infty = tf^*(t) \le \sup_{s\in(0,t)} sf^*(s).$$

Putting all these estimates together we get

$$\|g_0\|_{1,\infty} + t\,\|h_0\|_\infty \le 2 \sup_{s\in(0,t)} sf^*(s).$$

It remains to show the optimality of the constants in both estimates. The constant 1 in the first inequality is clearly optimal. To see this, it is enough to take $f = \chi_{(0,1)}$ and $t = 1$. Indeed, then the left-hand side of the first inequality in (2.1.1) equals

$$\sup_{s\in(0,1)} s\chi_{(0,1)}(s) = 1,$$

while the quantity on the right hand side equals

$$\inf_{f=g+h} (\|g\|_{L^{1,\infty}} + \|h\|_{L^\infty}),$$

which is obviously less than or equal to the quantity corresponding to the trivial decomposition $f = 0 + f$, that is, $\|\chi_{(0,1)}\|_{L^\infty} = 1$. This shows that one cannot have a smaller constant than 1 on the right-hand side of the first inequality in (2.1.1).

Now let us prove that the constant 2 at the second inequality is sharp, too. We would like to find a function f for which the inequality would turn into an equality for certain $t > 0$.

We put

$$f(x) = \begin{cases} \frac{1}{x} & \text{on } (0,1] \\ 1 & \text{on } (1,\infty). \end{cases}$$

We are going to show that for $t = 1$ it is

$$\inf_{f=g+h} \left(\|g\|_{1,\infty} + \|h\|_\infty \right) = 2.$$

The function f is non-increasing, therefore $f = f^*$.

It also holds that

$$\sup_{s\in(0,t)} f^*(s) = \max\{1, t\}$$

and we verify that as follows. In case $t \leq 1$ we have

$$\sup_{s\in(0,1)} sf^*(s) = \sup_{s\in(0,1)} s\frac{1}{s} = 1 = \max\{1,t\},$$

while in case $t > 1$ we have

$$\begin{aligned}\sup_{s\in(0,t)} sf^*(s) &= \max\left\{\sup_{s\in(0,1)} sf^*(s), \sup_{s\in(1,t)} sf^*(s)\right\} \\ &= \max\left\{1, \sup_{s\in(1,t)} s\right\} \\ &= \max\{1,t\}.\end{aligned}$$

Thus for $t = 1$ we get

$$\sup_{s\in(0,1)} f^*(s) = 1.$$

Now we are going to show that $\|h\|_\infty \geq 1$ and $\|g\|_{1,\infty} \geq 1$ for any decomposition $f = g + h$ where $g \in L^{1,\infty}$ and $h \in L^\infty$.

- If there exists $\gamma < 1$ and h such that

  $$\|h\|_\infty = \gamma < 1$$

  then there exists $c > 0$ such that for all $s > 0$ it is

  $$g^*(s) \geq c,$$

  and that follows from the fact that $f \geq 1$ on $(0,\infty)$ and that $f = g + h$. But then we get $\|g\|_{1,\infty} = \sup_{s\in(0,\infty)} sg^*(s) \geq \sup_{s\in(0,\infty)} sc = \infty$ which is a contradiction.

- If there exists $\gamma < 1$ such that

  $$\|g\|_{1,\infty} = \sup_{s\in(0,\infty)} sg^*(s) = \gamma,$$

  we get that for all $s \geq 0$ it is

  $$sg^*(s) \leq \gamma$$

  therefore

  $$g^*(s) \leq \frac{\gamma}{s} < \frac{1}{s}$$

  which is a contradiction again.

□

Now we are in a position to state the first of our pointwise estimates for the operator S_a. It corresponds to the case when $X = L^\infty$, hence $X' = L^1$.

Theorem 2.1.14. *Let $a \in L^1 \cap L^\infty$ be non-increasing and non-negative on $(0,\infty)$. Then there exists a positive constant C such that for every function $g \in L^1 + L^\infty$ one has*

$$(S_a g)^*(t) \leq C \int_0^{\frac{1}{t}} g^*(s)\,ds \text{ for all } t \geq 0.$$

Proof. We fix $t > 0$. Using Theorem 2.1.2 and Example 2.1.10, we get

$$\inf_{g=g_1+g_2} \left(\|S_a g_1\|_\infty + t \|S_a g_2\|_{1,\infty} \right) \leq \inf_{g=g_1+g_2} \left(C_1 \|g_1\|_1 + t C_2 \|g_2\|_\infty \right).$$

Using Theorem 2.1.13 for the second inequality we get

$$\begin{aligned} \inf_{g=g_1+g_2} \left(\|S_a g_1\|_\infty + t \|S_a g_2\|_{1,\infty} \right) &\geq t \inf_{S_a g = f_1 + f_2} \left(\|f_2\|_{1,\infty} + \frac{1}{t} \|f_1\|_\infty \right) \\ &\geq t \sup_{s \in \left(0, \frac{1}{t}\right)} s \left(S_a g\right)^*(s). \end{aligned}$$

Therefore

$$t \sup_{s \in \left(0, \frac{1}{t}\right)} s \left(S_a g\right)^*(s) \leq \inf_{g=g_1+g_2} \left(C_1 \|g_1\|_1 + t C_2 \|g_2\|_\infty \right),$$

and from Theorem 1.2.4 we get

$$t \sup_{s \in \left(0, \frac{1}{t}\right)} s \left(S_a g\right)^*(s) \leq C \int_0^t g^*(y)\,dy.$$

That implies for $s = \frac{1}{t}$

$$(S_a g)^*(t) \leq C \int_0^{\frac{1}{t}} g^*(y)\,dy.$$

□

In what follows we shall need to know when a functional of the form $f \mapsto \|S_a f^*\|_{X'}$, where X is a given rearrangement-invariant space, has properties of a Banach function norm. For this purpose, the following assertion will be useful.

Theorem 2.1.15. *Let a be non-increasing, non-negative on $(0,\infty)$ and $a \not\equiv 0$. Then for a rearrangement-invariant space X such that $\frac{1}{t}\int_0^t a(y)\,dy$ belongs to X' we define the functional F as*

$$F : f \longmapsto \|S_a f^*\|_{X'}$$

for f from $M^+(0,\infty)$. Then F is a Banach function norm.

Proof. We first note that since a is assumed to be non-increasing, one has $a(t) \leq \frac{1}{t}\int_0^t a(s)ds$, whence the assumption $a \in X'$ is automatically satisfied. Also, since f^* is non-negative for every $f \in M(0,\infty)$, the functional F is well defined.

We shall now verify the axioms (P1)-(P5) of a Banach function norm from Definition 1.1.1.

- (P1)

  It is very easy to show that $F(g) = 0 \Leftrightarrow g = 0$ a.e. and that for $\lambda \geq 0$ it is $F(\lambda g) = \lambda F(g)$. Therefore we will only show that $F(f+g) \leq F(f) + F(g)$.

  We fix f and g. Considering the fact from Theorem 1.1.10 we apply Hardy's lemma (Theorem 1.1.17) on

$$\xi_1 := (f+g)^*,$$

$$\xi_2 := f^* + g^*,$$

$$\mu(t) := a(xt) \text{ for } x > 0 \text{ fixed}$$

  and we get

$$\int_0^\infty (f+g)^*(s)\, a(sx)\, ds \leq \int_0^\infty (f^*(s) + g^*(s))\, a(sx)\, ds.$$

  Therefore

$$S_a (f+g)^*(x) \leq S_a f^*(x) + S_a g^*(x) \text{ for every } x > 0.$$

  Thus it immediately follows that

$$\begin{aligned}\|S_a (f+g)^*\|_{X'} &\leq \sup_{\|h\|_X \leq 1} \int_0^\infty h^*(t) \left(S_a f^*(t) + S_a g^*(t)\right)(t)\, dt \\ &= \|S_a f^*\|_{X'} + \|S_a g^*\|_{X'}.\end{aligned}$$

- (P2) is obvious and (P3) follows easily from the Levi theorem.

- (P4)

  Let E be such that $\mu(E) < \infty$. We have

$$\begin{aligned}S_a(\chi_E^*)(x) &= S_a\left(\chi_{(0,\mu(E))}\right)(x) \\ &= \int_0^{\mu(E)} a(xt)\, dt \\ &=^{y=xt} \frac{1}{x} \int_0^{x\mu(E)} a(y)\, dy.\end{aligned}$$

  Therefore

$$\|S_a(\chi_E^*)\|_X = \sup_{\|h\|_X \leq 1} \int_0^\infty |h(x)| \frac{1}{x} \left(\int_0^{x\mu(E)} a(y)\, dy \right) dx,$$

  and the last expression is finite if and only if $\frac{1}{x}\int_0^x a(y)\, dy$ is finite.

- (P5)

  It holds

  1. $\int_E g d\mu \leq \int_0^{\mu(E)} g^*(t)\, dt$

2. $\int_0^{\mu(E)} a(xt)\,dx = \frac{1}{t}\int_0^{t\mu(E)} a(y)\,dy$

3. Therefore $\int_E g d\mu \le \int_0^{\mu(E)} g^*(t)\left(\int_0^{\mu(E)} a(xt)\,dx\right)\frac{t}{\int_0^{t\mu(E)} a(y)dy}dt.$

Now we need the expression $\frac{t}{\int_0^{t\mu(E)} a(y)dy}$ to be bounded on $(0,\mu(E))$. The problem is in the case when t is close to 0. But since a is non-increasing and not identically equal to 0, we have $\liminf_{t\to 0+}\frac{1}{t}\int_0^t a(y)\,dy > 0$.
Let us denote $K_E := \max_{t\in(0,\mu(E))}\frac{t}{\int_0^{t\mu(E)} a(y)dy}$.

Then

$$\begin{aligned}\int_E g d\mu &\le K_E \int_0^{\mu(E)} g^*(t)\left(\int_0^{\mu(E)} a(xt)\,dx\right)dt\\ &= K_E \int_0^{\mu(E)} \chi_E^*(x)\left(\int_0^{\mu(E)} g^*(t)\,a(xt)\,dt\right)dx\\ &\le K_E \int_0^{\infty} \chi_E^*(x)\left(\int_0^{\mu(E)} g^*(t)\,a(xt)\,dt\right)dx,\end{aligned}$$

which is by Hölder's inequality (Theorem 1.1.16) less than or equal to

$$K_E\left\|\chi_E^*\right\|_X\left\|\int_0^{\mu(E)} g^*(t)\,a(xt)\,dt\right\|_{X'} \le K_E\left\|\chi_E^*\right\|_X\left\|S_a g^*\right\|_{X'}.$$

□

One of our main goals is to characterize optimal pairs of rearrangement-invariant Banach function spaces for an operator S_a. We shall first give a precise definition of what the term *optimality* means in this case.

Definition 2.1.16. Given a rearrangement invariant space X, an operator T and some class of function spaces W, we say that Y is the *optimal range space for X with respect to T in W* if the following conditions are satisfied:

1. $Y \in W$;

2. $T : X \to Y$;

3. if there exists $Z \in W$ such that $T : X \to Z$ then $Y \hookrightarrow Z$ ($Y \subset Z$ and there exists $c > 0$ such that, for all $g \in M_0^+(0,\infty)$, $\|g\|_Z \le c\,\|g\|_Y$).

The next theorem is one of our main results. It provides us with a characterization of the optimal rearrangement-invariant range space which corresponds to a given rearrangement-invariant domain space.

Theorem 2.1.17. *Let X be a rearrangement-invariant space such that*

$$\frac{1}{x}\int_0^x a(y)\,dy \in X'.$$

Define the space Y' by fixing the norm $\|g\|_{Y'} := \|S_a g^*\|_{X'}$ as

$$Y' = \left\{g \in M_0^+(0,\infty) : \|g\|_{Y'} < \infty\right\}.$$

Then $\|\cdot\|_{Y'}$ is a rearrangement-invariant norm and the space Y (obtained via $Y = Y''$) is the optimal range space for X with respect to S_a in the class of rearrangement-invariant spaces.

Proof. We will verify the axioms from the Definition 2.1.16. For the operator S_a it holds $S_a : X \to Y \Leftrightarrow S_a : Y' \to X'$ for every pair (X, Y) of rearrangement-invariant spaces. This is a well-known standard observation based on the fact that the operator S_a is self-adjoint with respect to the L^1-pairing, namely, one has, by the Fubini theorem, for each appropriate pair of functions f, g

$$\int_0^\infty (S_a f)g = \int_0^\infty f(S_a g).$$

Thus, we establish the following facts:

1. Y' is rearrangement-invariant because $\|g\|_{Y'} = \|S_a g^*\|_{X'}$ and $\|g^*\|_{Y'} = \|S_a (g^*)^*\|_{X'} = \|S_a g^*\|_{X'}$ therefore also Y is rearrangement-invariant.

2. For a given $g \in M_0^+(0,\infty)$ we have

   $$S_a : X \to Y \Leftrightarrow S_a : Y' \to X' \Leftrightarrow \|S_a g\|_{X'} \le C\|g\|_{Y'}$$

   and we also have

   $$\|S_a g\|_{X'} = \|(S_a g)^*\|_{X'} \le K_a \left\|\int_0^{\frac{1}{t}} g^*(s)\, ds\right\|_{X'} \le K_a \|g\|_{Y'},$$

   which follows from

   $$S_a g(t) = \int_0^\infty a(st)\, g(s) \ge \int_0^{\frac{1}{t}} a(st)\, g(s)$$

   $$\ge \int_0^{\frac{1}{t}} a(1)\, g(s)\, ds \text{ for } a \text{ non-increasing.}$$

   This implies

   $$\left\|\int_0^{\frac{1}{t}} g^*(s)\, ds\right\|_{X'} \le \|S_a g^*\|_{X'} = \|g\|_{Y'}.$$

   Therefore

   $$S_a : Y' \to X'.$$

3. Let Z be a rearrangement invariant space such that $S_a : X \to Z$, then also $S_a : Z' \to X'$ therefore $\|S_a g\|_{X'} \le K\|g\|_{Z'}$. We also know that $\|g\|_{Z'} = \|g^*\|_{Z'}$. Then also $\|S_a g^*\|_{X'} \le K\|g^*\|_{Z'}$ as a special case. Then

   $$\|g\|_{Y'} = \|S_a g^*\|_{X'} \le K\|g^*\|_{Z'} = K\|g\|_{Z'}.$$

   Therefore $Z' \hookrightarrow Y'$ whence $Y \hookrightarrow Z$.

□

Our next result is in some sense dual to the preceding one. We shall characterize the optimal domain space for our operator as well. The definition of the *optimal domain space* is the following.

Definition 2.1.18. Given a rearrangement invariant space Y, an operator T and some class of function spaces W, we say that X is the *optimal domain space for Y with respect to T in W* if the following conditions are satisfied:

1. $X \in W$;
2. $T : X \to Y$;
3. if there exists $Z \in W$ such that $T : Z \to Y$ then $Z \hookrightarrow X$ ($Z \subset X$ and there exists $c > 0$ such that, for all $g \in M_0^+ (0, \infty)$, $\|g\|_X \leq c \|g\|_Z$).

And as the next theorem shows we can also get an optimal domain space for a given rearrangement-invariant space.

Theorem 2.1.19. *Let Y be a rearrangement-invariant space such that*

$$\frac{1}{x} \int_0^x a(y)\, dy \in Y.$$

We define the space X by fixing the norm $\|g\|_X = \|S_a g^*\|_Y$ as

$$X = \left\{ g \in M_0^+ (0, \infty) : \|g\|_X < \infty \right\}.$$

Then $\|\cdot\|_X$ is a rearrangement-invariant norm and X is the optimal domain space for Y with respect to S_a in the class of rearrangement-invariant spaces.

Proof. For a given $g \in M_0^+ (0, \infty)$ we have

$$\|S_a g\|_Y = \|(S_a g)^*\|_Y \leq K_a \left\| \int_0^{\frac{1}{t}} g^*(s)\, ds \right\|_Y \leq K_a \|g\|_X,$$

using the same argument as in the proof of the Theorem 2.1.17. So we obtain that $S_a : X \to Y$. Let Z be such that $S_a : Z \to Y$. Then

$$\|g\|_Z = \|g^*\|_Z \geq c \|S_a g^*\|_Y = c \|f\|_X,$$

therefore $Z \hookrightarrow X$.

□

2.2 Pointwise estimates of the operator S_a

In this section we take a closer look at a special case $X = L^{p,q}(0, \infty)$.

We will proceed by developing the interpolation theory for our operator S_a. We will do a pointwise estimate of the non-increasing rearrangement of our operator S_a applied on a given function.

Theorem 2.2.1. *Let $a \in L^\infty \cap L^{p',q'}$ be a non-increasing, non-negative function on $(0,\infty)$ and $1 \le p, q \le \infty$. Then there exists a positive constant C depending on a, p, q such that for every function $g \in L^1 + L^{p,q}$ one has*

$$(S_a g)^*(y) \le C y^{-\frac{1}{p'}} \left(\int_{\frac{1}{y}}^{\infty} s^{\frac{q}{p}-1} (g^*(s))^q \, ds \right)^{\frac{1}{q}} \text{ for all } y \ge 0.$$

Proof. Let us recall some facts from the basic theory and what we have shown so far.

For a belonging to $L^\infty \cap L^{p',q'}$, non-increasing, non-negative we have that

$$S_a : L^1 \to L^\infty$$

and

$$S_a : L^{p,q} \to L^{p',\infty}.$$

Therefore from Theorem 1.2.6 we get that

$$K\left(S_a g, t; L^\infty, L^{p',\infty}\right) \le cK\left(g, t; L^1, L^{p,q}\right) \text{ for some constant } c.$$

So now our intention is to look closer at expressions $K\left(S_a g, t; L^\infty, L^{p',\infty}\right)$ and $K\left(g, t; L^1, L^{p,q}\right)$. We will express them without using the K-functional.

1. Case $K\left(g, t; L^1, L^{p,q}\right)$

   We want to use the Holmstedt theorem (Theorem 1.2.7), that is, Corollary 1.2.8 for $X_0 := L^1$, $X_1 := L^\infty$ and $X_{\theta,q} := L^{p,q}$. So we need to find out what is θ.

   For the assumption $X_{\theta,q} = L^{p,q}$ we need the equivalence of the norms of $X_{\theta,q}$ and $L^{p,q}$, in other words we need

   $$\|g\|_{\theta,q} = \|g\|_{L^{p,q}},$$

   and we will also show that $\|g\|_{\theta,q}$ is equivalent to $\left\|t^{-\theta+1-\frac{1}{q}} g^*(t)\right\|_q$, therefore

   $$\theta = 1 - \frac{1}{p}.$$

   This observation can be verified as follows:

   $$\begin{aligned}\|g\|_{\theta,q} &= \left\|t^{-\theta-\frac{1}{q}} K(g, t; X_0, X_1)\right\|_q = \left\|t^{-\theta-\frac{1}{q}} \int_0^t g^*(s)\, ds\right\|_q \\ &= \left\|t^{-\theta+1-\frac{1}{q}} g^{**}(t)\right\|_q \\ &\approx \left\|t^{-\theta+1-\frac{1}{q}} g^*(t)\right\|_q \text{ because:}\end{aligned}$$

   - Using Hardy's inequality (Theorem 1.1.18) for $\lambda = 1 - \theta$ we get

     $$\begin{aligned}\left\|t^{-\theta+1-\frac{1}{q}} g^{**}(t)\right\|_q &= \left(\int_0^\infty \left(t^{-\theta} \int_0^t g^*(s)\, ds \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\le \frac{1}{\theta} \left(\int_0^\infty \left(t^{1-\theta} g^*(t)\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \frac{1}{\theta} \left\|t^{1-\theta-\frac{1}{q}} g^*(t)\right\|_q.\end{aligned}$$

- The converse inequality holds as well, because $g^{**}(t) \geq g^{*}(t)$ for all $t \geq 0$.

So now that we have the equivalence of the norms stated above, we use Corollary 1.2.8 for $\overline{X_0} := L^1$, $\theta := 1 - \frac{1}{p}$ and $X_{\theta,q} := L^{p,q}$ and we get

$$K\left(g, t^{\theta}; L^1, L^{p,q}\right) \approx t^{\theta} \left(\int_t^{\infty} \left(s^{-\theta} \int_0^s g^{*}(y)\, dy \right)^q \frac{ds}{s} \right)^{\frac{1}{q}},$$

therefore

$$K\left(g, t; L^1, L^{p,q}\right) \approx t \left(\int_{t^{p'}}^{\infty} s^{\frac{q}{p}-1} \left(g^{**}(s)\right)^q ds \right)^{\frac{1}{q}}$$

and by Hardy's inequality for $\lambda := \frac{1}{p}$ we also get

$$t \left(\int_{t^{p'}}^{\infty} s^{\frac{q}{p}-1} \left(g^{**}(s)\right)^q ds \right)^{\frac{1}{q}} \approx t \left(\int_{t^{p'}}^{\infty} s^{\frac{q}{p}-1} \left(g^{*}(s)\right)^q ds \right)^{\frac{1}{q}}.$$

2. Case $K\left(S_a g, t; L^{\infty}, L^{p',\infty}\right)$

We will first compute $K\left(g, t^{1-\theta}; L^{p',\infty}, L^{\infty}\right)$ and then by easy modifications we will get the desired expression above.

We set $\left(L^1, L^{\infty}\right)_{\theta,\infty} := L^{p',\infty}$ so we need to find θ again. Of course, in this case the value of θ will be different from its previous occurrence. By the same concept we get

$$\begin{aligned} \|g\|_{\theta,\infty} &= \left\| t^{-\theta} K\left(g, t; L^1, L^{\infty}\right) \right\|_{\infty} \\ &= \sup_{t\in(0,\infty)} t^{-\theta} \int_0^t g^{*}(s)\, ds = \sup_{t\in(0,\infty)} t^{1-\theta} g^{**}(t) \\ &= \|g\|_{L^{\frac{1}{1-\theta}},\infty}, \end{aligned}$$

therefore

$$\frac{1}{1-\theta} = p' \text{ which implies } \theta = \frac{1}{p}.$$

Therefore

$$\begin{aligned} K\left(g, t^{1-\theta}; L^{p',\infty}, L^{\infty}\right) &\approx \sup_{s\in(0,t)} s^{-\theta} \int_0^s g^{*}(y)\, dy \\ &= \sup_{s\in(0,t)} s^{1-\theta} g^{**}(s) = \sup_{s\in(0,t)} s^{\frac{1}{p'}} g^{**}(s) \\ &\approx \sup_{s\in(0,t)} s^{\frac{1}{p'}} g^{*}(s), \end{aligned}$$

which follows from Hardy's inequality for $q = \infty$ and $\lambda = \frac{1}{p'}$.

Then

$$K\left(g, t; L^{p',\infty}, L^{\infty}\right) \approx \sup_{s\in\left(0,t^{p'}\right)} s^{\frac{1}{p'}} g^{*}(s)$$

and by Proposition 1.2.3 we get

$$K\left(g,t;L^{\infty},L^{p',\infty}\right)=tK\left(g,\frac{1}{t};L^{p',\infty},L^{\infty}\right)$$
$$=t\sup_{s\in\left(0,t^{-p'}\right)}s^{\frac{1}{p'}}g^{*}\left(s\right).$$

The desired expression is thus

$$K\left(S_{a}g,t;L^{\infty},L^{p',\infty}\right)\approx t\sup_{s\in\left(0,t^{-p'}\right)}s^{\frac{1}{p'}}\left(S_{a}g\right)^{*}\left(s\right).$$

So now we are ready to make the pointwise estimate. Using Theorem 1.2.6 and the considerations above we get

$$t\sup_{s\in\left(0,t^{-p'}\right)}s^{\frac{1}{p'}}\left(S_{a}g\right)^{*}\left(s\right)\leq Ct\left(\int_{t^{p'}}^{\infty}s^{\frac{q}{p}-1}\left(g^{*}\left(s\right)\right)^{q}ds\right)^{\frac{1}{q}}$$

for some constant C, which for $y=t^{-p'}$ gives

$$\left(S_{a}g\right)^{*}\left(y\right)\leq Cy^{-\frac{1}{p'}}\left(\int_{\frac{1}{y}}^{\infty}s^{\frac{q}{p}-1}\left(g^{*}\left(s\right)\right)^{q}ds\right)^{\frac{1}{q}}.$$

□

Remark 2.2.2. The function $y\longmapsto y^{-\frac{1}{p'}}\left(\int_{\frac{1}{y}}^{\infty}s^{\frac{q}{p}-1}\left(g^{**}\left(s\right)\right)^{q}ds\right)^{\frac{1}{q}}$ is non-increasing.

Proof. It is enough to prove that the function $y\longmapsto y^{\frac{1}{p'}}\left(\int_{y}^{\infty}s^{\frac{q}{p}-1}\left(g^{**}\left(s\right)\right)^{q}ds\right)^{\frac{1}{q}}$ is non-decreasing.

Using the change of variables $z=\frac{s}{y}$ for y fixed we get

$$y^{\frac{1}{p'}}\left(\int_{1}^{\infty}\left(zy\right)^{\frac{q}{p}-1}\left(g^{**}\left(zy\right)\right)^{q}dzy\right)^{\frac{1}{q}}$$
$$=y^{\frac{1}{p'}}\left(\int_{1}^{\infty}\left(zy\right)^{\frac{q}{p}-1}\left(\frac{1}{zy}\int_{0}^{zy}g^{*}\left(t\right)dt\right)^{q}dzy\right)^{\frac{1}{q}}$$
$$=\left(\int_{1}^{\infty}z^{\frac{q}{p}-1}\left(\frac{1}{z}\int_{0}^{zy}g^{*}\left(t\right)dt\right)^{q}\right)^{\frac{1}{q}},$$

which is clearly increasing in y. □

Now we would like to point out that the obtained pointwise estimate from Theorem 2.2.1 works as a special case of the Laplace transform. This gives an alternative proof of an inequality that we have already proved in [6] (however with a different constant).

Let $X=L^{1}$, $a\left(t\right):=\exp^{-t}$, $t\geq0$. Then $S_{a}=L$, where L is the Laplace transform defined on every function $f\in\left(L^{1}+L^{\infty}\right)\left(0,\infty\right)$ by

$$(Lf)(x)=\int_{0}^{\infty}f(s)e^{-xs}\,ds,\quad x\in(0,\infty).$$

Proposition 2.2.3. *For every $t > 0$ and every measurable function g on $(0,\infty)$, we have*

$$(Lg)^*(t) \leq C \int_0^{1/t} g^*(s)\, ds.$$

Proof. In the case of the Laplace transform we had

$$L : L^1 \to L^\infty$$

$$L : L^\infty \to L^{1,\infty}$$

therefore $p' := 1$ and $q' := 1$ which implies $p = q = \infty$. Plugging those in the inequality from the Theorem 2.2.1 (we are using the obtained expression with g^{**} instead of g^* which we showed are equivalent)

$$(S_a g)^*(y) \leq C y^{-\frac{1}{p'}} \left(\int_{\frac{1}{y}}^{\infty} s^{\frac{q}{p}-1} \left(g^{**}(s)\right)^q ds \right)^{\frac{1}{q}},$$

we get

$$\begin{aligned}(S_a g)^*(y) &\leq C \frac{1}{y} \sup_{s \in \left(\frac{1}{y},\infty\right)} g^{**}(s) \\ &= C \frac{1}{y} \sup_{s \in \left(\frac{1}{y},\infty\right)} \frac{1}{s} \int_0^s g^*(t)\, dt \\ &= C \frac{1}{y} y \int_0^{\frac{1}{y}} g^*(t)\, dt.\end{aligned}$$

□

The last theorem of this chapter shows that the obtained pointwise estimate for the non-increasing rearrangement of the operator S_a applied on a given function is in fact a Banach function norm.

Theorem 2.2.4. *For a rearrangement-invariant space X such that*

- $\chi_{(1,\infty)}(t) \frac{1}{t} \in X$
- $t^{-\frac{1}{p'}} \chi_{(1,\infty)}(t) \in X,$

and for any $p, q \geq 1$ we define the space $Z_{X,p,q}$ as

$$\|g\|_{Z_{X,p,q}} = \left\| t^{-\frac{1}{p'}} \left(\int_{\frac{1}{t}}^{\infty} s^{\frac{q}{p}-1} \left(g^{**}(s)\right)^q ds \right)^{\frac{1}{q}} \right\|_X .$$

Then $\|\cdot\|_{Z_{X,p,q}}$ is a Banach function norm and also

$$S_a : X' \to Z'_{X,p,q} \text{ for } a \in L^{p',q'}.$$

Proof. To prove the first part we will verify the (P1)-(P5) from the Definition 1.1.1

- (P1) is trivial where the triangle inequality follows easily from the Minkowski inequality.
- (P2) and (P3) are also obvious.
- (P4) Without loss of generality let us assume that $E = (0,1)$ and let Z_X denote $Z_{X,p,q}$. Then

$$\begin{aligned}\left\|\chi_{(0,1)}\right\|_{Z_X} &= \left\| t^{-\frac{1}{p'}} \left(\int_{\frac{1}{t}}^{\infty} s^{\frac{q}{p}-1} \left(\min\left\{1, \frac{1}{s}\right\}\right)^q ds \right)^{\frac{1}{q}} \right\|_X \\ &= \left\| \chi_{(0,1)}(t)\, t^{-\frac{1}{p'}} \left(\int_{\frac{1}{t}}^{\infty} s^{\frac{q}{p}-1} s^{-q} ds \right)^{\frac{1}{q}} \right\|_X + \\ &+ \left\| \chi_{(1,\infty)}(t)\, t^{-\frac{1}{p'}} \left(\int_{\frac{1}{t}}^{1} s^{\frac{q}{p}-1} ds + \int_1^{\infty} s^{\frac{q}{p}-1-q} ds \right)^{\frac{1}{q}} \right\|_X .\end{aligned}$$

Let us denote the sum of the two norms from above as (1) + (2). Then

$$\begin{aligned}(1) &= \left\| \chi_{(0,1)}(t)\, t^{-\frac{1}{p'}} \left(\left[s^{\frac{q}{p}-q} \right]_{\frac{1}{t}}^{\infty} \right)^{\frac{1}{q}} \right\|_X = \left\| \chi_{(0,1)} t^{-\frac{1}{p'}} \left(\frac{1}{t}\right)^{\frac{1}{p}-1} \right\|_X \\ &= \left\| \chi_{(0,1)}(t)\, t^{-\frac{1}{p'}} \left(\frac{1}{t}\right)^{-\frac{1}{p'}} \right\|_X = \left\| \chi_{(0,1)}(t) \right\|_X ,\end{aligned}$$

which is finite. We continue

$$\begin{aligned}(2) &= \left\| \chi_{(1,\infty)}(t)\, t^{-\frac{1}{p'}} \left(\left[s^{\frac{q}{p}} \right]_{\frac{1}{t}}^{1} + \left[s^{\frac{q}{p}-q} \right]_{1}^{\infty} \right)^{\frac{1}{q}} \right\|_X = \left\| \chi_{(1,\infty)}(t)\, t^{-\frac{1}{p'}} \left(t^{-\frac{q}{p}} \right)^{\frac{1}{q}} \right\|_X \\ &= \left\| \chi_{(1,\infty)} t^{-\frac{1}{p'}} t^{-\frac{1}{p}} \right\|_X = \left\| \chi_{(1,\infty)}(t) \frac{1}{t} \right\|_X ,\end{aligned}$$

and that is finite according to the assumption.
- (P5) Without loss of generality we can assume $|E| = 1$. It also holds $\int_E g d\mu \le \int_0^{\mu(E)} g^*(t)\, dt$ so it is enough to show that $\int_0^1 g^*(s)\, ds \le A_E \|g\|_{Z_X}$. We have

$$\begin{aligned}\|g\|_{Z_X} &\ge \left\| \chi_{(1,\infty)}(t)\, t^{-\frac{1}{p'}} \left(\int_{\frac{1}{t}}^{\infty} s^{\frac{q}{p}-1} \left(\frac{1}{s} \int_0^s g^*(y)\, dy \right)^q ds \right)^{\frac{1}{q}} \right\|_X \\ &\ge \int_0^1 g^*(y)\, dy \left\| \chi_{(1,\infty)}(t)\, t^{-\frac{1}{p'}} \left(\int_1^{\infty} s^{\frac{q}{p}-1} s^{-q} ds \right)^{\frac{1}{q}} \right\|_X .\end{aligned}$$

Let us denote $A := \left(\int_1^{\infty} s^{\frac{q}{p}-1} s^{-q} ds \right)^{\frac{1}{q}} = \left(\int_1^{\infty} s^{-\frac{q}{p'}-1} ds \right)^{\frac{1}{q}}$ which is finite. Then

$$\|g\|_{Z_X} \ge A \int_0^1 g^*(y)\, dy \left\| t^{-\frac{1}{p'}} \chi_{(1,\infty)}(t) \right\|_X ,$$

where the last expression is finite from the assumption.

We get the second part by using Fubini theorem and the Hölder's inequality as follows

$$\begin{aligned}
\|S_a g\|_{Z'_X} &= \|(S_a g)^*\|_{Z'_X} \leq \|S_a g^*\|_{Z'_X} = \sup_{\|h\|_Z \leq 1} \int_0^\infty (S_a g^*)^* (s)\, h^* (s) \\
&= \sup_{\|h\|_Z \leq 1} \int_0^\infty (S_a h^*)^* (s)\, g^* (s)\, ds \\
&\leq C \sup_{\|h\|_Z \leq 1} \int_0^\infty g^* (s)\, s^{-\frac{1}{p'}} \left(\int_{\frac{1}{s}}^\infty y^{\frac{q}{p}-1} \left(h^{**} (y)\right)^q dy \right)^{\frac{1}{q}} ds \\
&\leq C \|g^*\|_{X'} \sup_{\|h\|_Z \leq 1} \left\| s^{-\frac{1}{p'}} \left(\int_{\frac{1}{s}}^\infty y^{\frac{q}{p}-1} \left(h^{**} (y)\right)^q dy \right)^{\frac{1}{q}} \right\|_X \\
&= C \|g^*\|_{X'} \sup_{\|h\|_Z \leq 1} \|h\|_Z = C \|g^*\|_{X'}
\end{aligned}$$

□

Chapter 3

Results and examples for the Laplace transform

In this chapter we shall take a closer look at the Laplace transform, which is a special case of our operator S_a obtained by taking $a(x) := e^{-x}$. We recall that the Laplace transform L is defined on every function $f \in (L^1 + L^\infty)(0,\infty)$ by

$$(Lf)(x) = \int_0^\infty f(t)e^{-xt}\,dt, \quad x \in (0,\infty).$$

3.1 The action of the Laplace transform on two-parameter Lorentz spaces

In Theorem 2.1.17 we obtained a result showing optimality between rearrangement-invariant spaces with respect to the Laplace transform, and we already know the optimality for the case of L^p spaces for $1 < p < \infty$. Now we will focus on more general examples, namely one the spaces $L^{p,q}$ for $1 < p, q < \infty$ and the spaces $\Lambda^p(w)$ for $1 < p < \infty$, where w is a weight function which has certain specific properties that will be stated later.

Theorem 3.1.1. *Assume that* $1 < p, q < \infty$. *Then*

$$L : L^{p,q} \to L^{p',q}.$$

Moreover, the range space is optimal.

Proof. We know that for $1 < p, q < \infty$ it is $(L^{p,q})' = L^{p',q'}$, therefore $\left(L^{p',q}\right)' = L^{p,q'}$. To get that $L^{p',q}$ is the optimal range space for $L^{p,q}$ we need to show that $\|g\|_{L^{p,q'}} \approx \|Lg^*\|_{L^{p',q'}}$.

1. We shall prove that $\|Lg^*\|_{L^{p',q'}} \le K\|g\|_{L^{p,q'}}$ for some constant K.

   It holds

   $$\|Lg^*\|_{L^{p',q'}} = \left\|t^{\frac{1}{p'}-\frac{1}{q'}}(Lg^*)^*(t)\right\|_{L^{q'}} = \left\|t^{\frac{1}{p'}-\frac{1}{q'}}(Lg^*)(t)\right\|_{L^{q'}}$$

   because $g^* \ge 0$.

We continue

$$\begin{aligned}\left\| t^{\frac{1}{p'}-\frac{1}{q'}} \left(Lg^*\right)(t) \right\|_{L^{q'}} &= \left(\int_0^\infty t^{\left(\frac{1}{p'}-\frac{1}{q'}\right)q'} \left(Lg^*(t)\right)^{q'} dt \right)^{\frac{1}{q'}} \\ &= \left(\int_0^\infty t^{\frac{q'}{p'}-1} \left(Lg^*(t)\right)^{q'} dt \right)^{\frac{1}{q'}} \\ &\le \left(\int_0^\infty t^{\frac{q'}{p'}-1} \left(\int_0^{\frac{1}{t}} g^*(s)\, ds \right)^{q'} dt \right)^{\frac{1}{q'}},\end{aligned}$$

which follows from Theorem 2.2.3.

Now we substitute $\frac{1}{t}$ for y and get

$$\begin{aligned}\left(\int_0^\infty y^{1-\frac{q'}{p'}} \left(\int_0^y g^*(s)\, ds \right)^{q'} \frac{1}{y^2} dy \right)^{\frac{1}{q'}} &= \left(\int_0^\infty \left(y^{\frac{1}{p}} \frac{1}{y} \int_0^y g^*(s)\, ds \right)^{q'} \frac{1}{y} dy \right)^{\frac{1}{q'}} \\ &\le \frac{1}{1-\frac{1}{p}} \left(\int_0^\infty \left(y^{\frac{1}{p}} g^*(y) \right)^{q'} \frac{1}{y} dy \right)^{\frac{1}{q'}}\end{aligned}$$

from Theorem 1.1.18 setting $\lambda := \frac{1}{p}$.

Finally we get

$$\begin{aligned}\frac{1}{1-\frac{1}{p}} \left(\int_0^\infty \left(y^{\frac{1}{p}} g^*(y) \right)^{q'} \frac{1}{y} dy \right)^{\frac{1}{q'}} &= p' \left(\int_0^\infty y^{\frac{q'}{p}-1} \left(g^*(y)\right)^{q'} dy \right)^{\frac{1}{q'}} \\ &= p' \left\| y^{\frac{1}{p}-\frac{1}{q'}} g^*(y) \right\|_{L^{q'}} \\ &= p' \left\| g \right\|_{L^{p,q'}}.\end{aligned}$$

2. Now we want to prove the converse inequality. We are going to use Theorem 1.1.19 and we set

   - $\phi(x,t) := e^{-xt}$
   - $f := g^*$
   - $w := t^{\frac{q'}{p}-1}$
   - $v := t^{\frac{q'}{p'}-1}$
   - $\tilde{p} = \tilde{q} := q'$

     Therefore:
   - $S_\phi f(x) = \int_0^\infty e^{-xy} g^*(y)\, dy$
   - $\Phi(x,r) = \int_0^r e^{-xy} dy$

   The assumption of Theorem 1.1.19 is that for all $r > 0$ it holds

$$\left(\int_0^r t^{\frac{q'}{p}-1} \right)^{\frac{1}{q'}} \le C \left(\int_0^\infty \left(\int_0^r e^{-ty} dy \right)^{q'} t^{\frac{q'}{p'}-1} dt \right)^{\frac{1}{q'}}.$$

Now we have to verify this assumption.

The left-hand side equals

$$\left(\left[\frac{t^{\frac{q'}{p}}}{\frac{q'}{p}}\right]_0^r\right)^{\frac{1}{q'}} = \left(\frac{r^{\frac{q'}{p}}}{\frac{q'}{p}}\right)^{\frac{1}{q'}} = r^{\frac{1}{p}}\left(\frac{p}{q'}\right)^{\frac{1}{q'}}.$$

We shall consider the right-hand side. We set

$$F(r) := \left(\int_0^\infty \left(\int_0^r e^{-ty}dy\right)^{q'} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}} = \left(\int_0^\infty \left(\frac{1-e^{-tr}}{t}\right)^{q'} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}}.$$

Then we have

$$\begin{aligned}\frac{F(r)}{r} &= \left(\int_0^\infty \left(\frac{1-e^{-tr}}{tr}\right)^{q'} t^{\frac{q'}{p'}-1}dt\right)^{\frac{1}{q'}}\\ &= \left(\int_0^\infty \left(\frac{1-e^{-y}}{y}\right)^{q'} \frac{y^{\frac{q'}{p'}-1}}{r^{\frac{q'}{p'}-1}}\frac{dy}{r}\right)^{\frac{1}{q'}}\\ &= F(1)\, r^{-\frac{1}{p'}},\end{aligned}$$

therefore

$$F(r) = F(1)\, r^{-\frac{1}{p'}+1} = F(1)\, r^{\frac{1}{p}}.$$

By comparing both sides we get

$$r^{\frac{1}{p}}\left(\frac{p}{q'}\right)^{\frac{1}{q'}} \le CF(1)\, r^{\frac{1}{p}}$$

for some C.

The assumption is therefore verified and the desired inequality follows from the statement of the Theorem 1.1.19.

The last thing we need to verify is that $\frac{1-e^{-x}}{x} \in L^{p',q'}(0,\infty)$ in other words $\left\|\frac{1-e^{-x}}{x}\right\|_{L^{p',q'}} < \infty$.

It holds

$$\begin{aligned}\left\|\frac{1-e^{-x}}{x}\right\|_{L^{p',q'}} &= \left\|x^{\frac{1}{p'}-\frac{1}{q'}}\left(\frac{1-e^{-x}}{x}\right)^*\right\|_{L^{q'}} = \left\|x^{\frac{1}{p'}-\frac{1}{q'}}\frac{1-e^{-x}}{x}\right\|_{L^{q'}}\\ &= \left(\int_0^\infty x^{\frac{q'}{p'}-1}\left(\frac{1-e^{-x}}{x}\right)^{q'}dx\right)^{\frac{1}{q'}}.\end{aligned}$$

We sill use the elementary fact that the function $\frac{1-e^{-x}}{x}$ is equivalent to $\min\left\{1, \frac{1}{x}\right\}$ on $(0,\infty)$ (up to absolute positive constants).

- For the interval $(0,1)$ we get

$$\int_0^1 x^{\frac{q'}{p'}-1}dx = \left[\frac{x^{\frac{q'}{p'}}}{\frac{q'}{p'}}\right]_0^1 = \frac{p'}{q'} < \infty.$$

- For the interval $(1,\infty)$ we get

$$\begin{aligned}\int_1^\infty x^{\frac{q'}{p'}-1}\frac{1}{x^{q'}}dx &= \int_1^\infty x^{\frac{q'}{p'}-1-q'}dx = \int_1^\infty x^{q'\left(\frac{1}{p'}-1\right)-1}dx \\ &= \int_1^\infty x^{-\left(1+\frac{q'}{p}\right)}dx.\end{aligned}$$

The last expression is less than infinity if and only if $\frac{q'}{p} > 0$ and that always holds.

□

3.2 The action of the Laplace transform on classical Lorentz spaces

In this section we intend to point out one more example of an optimality result for the Laplace transform. We will start with the definition of the so-called *classical Lorentz spaces* of type Λ and Γ.

Definition 3.2.1. Let $0 < p \leq \infty$ and let w be a weight on (R,μ). The *classical Lorentz space* $\Lambda^p(w)$ is the collection of all functions $f \in M_0(R,\mu)$ such that $\|f\|_{\Lambda^p}(w)$ is finite, where

$$\|f\|_{\Lambda^p}(w) := \begin{cases}\left(\int_0^{\mu(R)} (f^*(t))^p\, w(t)\, dt\right)^{\frac{1}{p}} & \text{if } p<\infty \\ \sup_{0<t<\mu(R)} f^*(t)\, w(t) & \text{if } p=\infty.\end{cases}$$

The space $\Gamma^p(w)$ is the collection of all functions $f \in M_0(R,\mu)$ such that $\|f\|_{\Gamma^p}(w)$ is finite, where

$$\|f\|_{\Gamma^p}(w) := \begin{cases}\left(\int_0^{\mu(R)} (f^{**}(t))^p\, w(t)\, dt\right)^{\frac{1}{p}} & \text{if } p<\infty \\ \sup_{0<t<\mu(R)} f^{**}(t)\, w(t) & \text{if } p=\infty.\end{cases}$$

Theorem 3.2.2. *It holds*

$$L : \Lambda^p(w) \to \Lambda^p(\bar{w})$$

for $0 < p \leq \infty$ *and some weight function* w*, where* $\bar{w}(t) = \tilde{w}\left(\frac{1}{t}\right)t^{p'-2}$ *and* $\tilde{w}(t) = \frac{t^{p'}w(t)}{\left(\int_0^t w(s)ds\right)^{p'}}$*, with the following conditions on* w

- $w \in L^1(0,\infty)$,
- $\tilde{w} \in L^1(0,1)$,

- $\int_1^\infty (1+\ln t)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt < \infty$,
- $\tilde{w}$ *is non-negative.*

Proof. It holds

$$(\Lambda^p(w))' = \Gamma^{p'}(\tilde{w}).$$

- At first we want to find the conditions on w and $\tilde{w}$. We immediately see that $w \in L^1(0,\infty)$. The assumption for Theorem 2.1.15 is that $\min\left\{1, \frac{1}{t}\right\} \in \Gamma^{p'}(\tilde{w})$:

$$\left\| \min\left\{1, \frac{1}{t}\right\} \right\|_{\Gamma^{p'}(\tilde{w})} = \left(\int_0^\infty \left(\int_0^t \min\left\{1, \frac{1}{s}\right\} ds \right)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt \right)^{\frac{1}{p'}}$$

$$= \left(\int_0^1 \left(\int_0^t 1 ds \right)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt + \int_1^\infty \left(\int_0^1 1 ds + \int_1^t \frac{1}{s} ds \right)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt \right)^{\frac{1}{p'}}$$

$$= \left(\int_0^1 \tilde{w}(t)\, dt + \int_1^\infty (1+\ln t)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt \right)^{\frac{1}{p'}}.$$

Therefore $\tilde{w}$ is in $L^1(0,1)$ and is such that

$$\int_1^\infty (1+\ln t)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt < \infty.$$

- Now we want to show that $\|Lg^*\|_{\Gamma^{p'}(\tilde{w})} \approx \|g^*\|_{\Gamma^{p'}(\tilde{w})}$.

$$\|Lg^*\|_{\Gamma^{p'}(\tilde{w})} = \left(\int_0^\infty \left(\int_0^t Lg^*(s)\, ds \right)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt \right)^{\frac{1}{p'}}$$

$$\leq \left(\int_0^\infty \left(\int_0^t \int_0^{\frac{1}{s}} g^*(x)\, dx ds \right)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt \right)^{\frac{1}{p'}},$$

and that follows from the fact that

$$Lg^*(s) = (Lg^*(s))^* \leq \int_0^{\frac{1}{s}} (g^*(s))^*\, ds = \int_0^{\frac{1}{s}} g^*(s)\, ds.$$

This however immediately follows from Theorem 2.2.3.

We continue

$$\left(\int_0^\infty \left(\int_0^t \int_0^{\frac{1}{s}} g^*(x)\, dx ds \right)^{p'} \frac{\tilde{w}(t)}{t^{p'}} dt \right)^{\frac{1}{p'}}$$

$$= \left(\int_0^\infty \left(\frac{1}{t^{\frac{p'-1}{p'}}} \tilde{w}(t)^{\frac{1}{p'}} \int_0^t \int_0^{\frac{1}{s}} g^*(x)\, dx ds \right)^{p'} \frac{dt}{t} \right)^{\frac{1}{p'}}.$$

We next use Hardy's inequality (Theorem 1.1.18). We put

$$\psi := \tilde{w}(t)^{\frac{1}{p'}} \int_0^{\frac{1}{s}} g^*(x)\,dx,$$

$$\lambda := \frac{1}{p'},$$

$$q := p',$$

and get

$$\begin{aligned}
&\left(\int_0^\infty \left(\frac{1}{t^{\frac{p'-1}{p'}}}\tilde{w}(t)^{\frac{1}{p'}} \int_0^t \int_0^{\frac{1}{s}} g^*(x)\,dx ds\right)^{p'} \frac{dt}{t}\right)^{\frac{1}{p'}} \\
&\leq p\left(\int_0^\infty \left(t^{\frac{1}{p'}}\tilde{w}(t)^{\frac{1}{p'}} \int_0^{\frac{1}{t}} g^*(x)\,dx\right)^{p'} \frac{dt}{t}\right)^{\frac{1}{p'}} \\
&= p\left(\int_0^\infty \left(\int_0^{\frac{1}{t}} g^*(x)\,dx\right)^{p'} \tilde{w}(t)\,dt\right)^{\frac{1}{p'}} \\
&=^{y=\frac{1}{t}} p\left(\int_0^\infty \left(\int_0^{y} g^*(x)\,dx\right)^{p'} \tilde{w}\left(\frac{1}{y}\right)\frac{dy}{y^2}\right)^{\frac{1}{p'}} \\
&= p\left(\int_0^\infty \left(\int_0^{y} g^*(x)\,dx\right)^{p'} \frac{\bar{w}(y)}{y^{p'}}dy\right)^{\frac{1}{p'}} \\
&= p\,\|g^*\|_{\Gamma^{p'}(\bar{w})}\,.
\end{aligned}$$

Now we want the converse inequality.
It holds:

$$Lg^*(t) = \int_0^\infty e^{-xt}g^*(x)\,dx \geq \int_0^{\frac{1}{t}} e^{-xt}g^*(x)\,dx \geq \frac{1}{e}\int_0^{\frac{1}{t}} g^*(x)\,dx.$$

Therefore

$$\begin{aligned}
\|Lg^*\|_{\Gamma^{p'}(\tilde{w})} &= \left(\int_0^\infty \left(\int_0^t Lg^*(s)\,ds\right)^{p'} \frac{\tilde{w}(t)}{t^{p'}}dt\right)^{\frac{1}{p'}} \\
&\geq \left(\frac{1}{e}\int_0^\infty \left(\int_0^t \int_0^{\frac{1}{s}} g^*(x)\,dx ds\right)^{p'} \frac{\tilde{w}(t)}{t^{p'}}dt\right)^{\frac{1}{p'}}.
\end{aligned}$$

We finish the proof by the following chain based on an elementary change of variables:

$$
\begin{aligned}
\|Lg^*\|_{\Gamma^{p'}(\tilde{w})} &= \left(\frac{1}{e}\int_0^\infty \left(\int_{\frac{1}{t}}^\infty \int_0^y g^*(x)\,dx\frac{dy}{y^2}\right)^{p'} \frac{\tilde{w}(t)}{t^{p'}}dt\right)^{\frac{1}{p'}} \\
&\geq \left(\frac{1}{e}\int_0^\infty \left(\int_0^{\frac{1}{t}} g^*(x)\,dx \int_{\frac{1}{t}}^\infty \frac{1}{y^2}dy\right)^{p'} \frac{\tilde{w}(t)}{t^{p'}}dt\right)^{\frac{1}{p'}} \\
&= \left(\frac{1}{e}\int_0^\infty \left(\int_0^{\frac{1}{t}} g^*(x)\,dx\right)^{p'} \tilde{w}(t)\,dt\right)^{\frac{1}{p'}} \\
&=^{y=\frac{1}{t}} \left(\frac{1}{e}\int_0^\infty \left(\frac{1}{y}\int_0^y g^*(x)\,dx\right)^{p'} \tilde{w}\left(\frac{1}{y}\right) y^{p'}\frac{dy}{y^2}\right)^{\frac{1}{p'}} \\
&= \left(\frac{1}{e}\int_0^\infty \left(\int_0^y g^*(x)\,dx\right)^{p'} \frac{\bar{w}(y)}{y^{p'}}dy\right)^{\frac{1}{p'}} \\
&= \left(\frac{1}{e}\right)^{\frac{1}{p'}} \|g^*\|_{\Gamma^{p'}(\bar{w})}.
\end{aligned}
$$

The proof is complete. □

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