Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

Miroslav Šimko

Interakce stlačitelné tekutiny a obtékaných teles

Matematický ústav Univerzity Karlovy

Vedoucí bakalářské práce: Prof. RNDr. Miloslav Feistauer, DrSc., Katedra Numerické matematiky

Studijní program: Matematické a počítačové modelování ve fyzice a technice

2006

Rád by som poďakoval vedúcemu práce Prof. RNDr. Miloslavovi Feistauerovi, DrSc. za odborné vedenie, ochotu a rady pri tvorbe práce.
Ďakujem tiež Mgr. Janovi Zemenovi za prečítanie anglického textu a pomoc s gramatikou.

Prehlasujem, že som svoju bakalársku prácu napísal samostatne a výhradne s použitím citovaných prameňov. Súhlasím s požičiavaním práce a jej zverejňovaním.

V Prahe dňa 11.8.2006 Miroslav Šimko

Miroslav Šimko

Contents

Název práce: Interakce stlačitelné tekutiny a obtékaných teles
Autor: Miroslav Šimko
Katedra (ústav): Katedra numerické matematiky
Vedoucí bakalářské práce: Prof. RNDr. Miloslav Feistauer, DrSc.
e-mail vedoucího: feist@karlin.mff.cuni.cz

Abstrakt: Obsahom tejto práce je odvodenie matematického popisu interakcie dvojrozmerného prúdenia stlačitelnej vázkej tekutiny a vibrujúceho leteckého profilu. Uvažujeme pevný letecký profil s dvoma stupňami volnosti, ktorý môže rotovať okolo elastickej osi a oscilovať vo vertikálnom smere. V práci sú odvodené Navier-Stokesove rovnice pre stlačitelnú Newtonovskú tekutinu popisujúce nami uvažované prúdenie, dalej matematická formulácia úlohy a jej formulácia v Arbitrary Lagrangian-Eulerian (ALE) popise. Nasleduje slabá formulácia úlohy a vytvorenie numerickej schémy pre výpočet použitím metódy konečných prvkov.

Klíčová slova: stlačitelné prúdenie, ALE popis, slabá formulácia úlohy, metóda konečných prvkov

Title: Interaction of compressible fluid and moving bodies
Author: Miroslav Šimko
Department: Department of Numerical Mathematics
Supervisor: Prof. RNDr. Miloslav Feistauer, DrSc.
Supervisor's e-mail address: feist@karlin.mff.cuni.cz

Abstract: In the present work we derive mathematical description of interaction of a two-dimensional compressible viscous flow and a vibrating airfoil. We considere a solid airfoil with two degrees of freedom, which can rotate around the elastic axis and oscillate in the vertical direction. Derivation of the Navier-Stokes equations for compressible flow is included as well as mathematical formulation of considered problem and its description in Arbitrary Lagrangian-Eulerian (ALE) formulation. Weak formulation of the problem is presented and we derive the formulae for numerical computation using the finite element method.

Keywords: compressible flow, ALE formulation, weak formulation, finite element method

Chapter 1

Mathematical concepts and notation

Throughout this article, we will use the following notation:

The space under consideration will be an euclidean point space $\mathscr{E}$ over a vector space $\mathscr{V}$.

We will use the term **tensor** as a synonym for a linear mapping from $\mathscr{V}$ into $\mathscr{V}$. Let us denote the following sets of tensors:

$$
\begin{aligned}
\mathrm{Lin} &= \text{set of all tensors},\\
\mathrm{Lin}^+ &= \text{set of all tensors } S \text{ with } \det S > 0,\\
\mathrm{Sym} &= \text{set of all symetric tensors},\\
\mathrm{Skw} &= \text{set of all skew (antisymetric) tensors},\\
\mathrm{Psym} &= \text{set of all symetric, positive definite tensors},\\
\mathrm{Orth} &= \text{set of all orthogonal tensors},\\
\mathrm{Orth}^+ &= \text{set of all rotations}.
\end{aligned}
$$

We will use the term **body** $\mathscr{B}$ to describe a regular region in $\mathscr{E}$. We refer to $\mathscr{B}$ as a **reference configuration**. Points $\mathbf{p} \in \mathscr{B}$ are called **material points**.

By the **deformation** of $\mathscr{B}$ we mean a smooth, one-to-one mapping f which maps $\mathscr{B}$ onto a closed region in $\mathscr{E}$, and which satisfies $\det \nabla f > 0$. The tensor $F(\mathbf{p}) = \nabla f(\mathbf{p})$ is called the **deformation gradient** and belongs to Lin^+.

Let $\mathscr{B}$ be a body. A **motion** of $\mathscr{B}$ is a class C^3 function

$$x : \mathscr{B} \times \mathbb{R} \to \mathscr{E}$$

with $x(\cdot,t)$, for each fixed t, a deformation of $\mathscr{B}$. We refer to

$$\mathbf{x} = x(\mathbf{p},t)$$

as the **place** occupied by a material point $\mathbf{p}$ at a time t and we write $\mathscr{B}_t = x(\mathscr{B},t)$ for the region of space occupied by the body at t. We define the **trajectory** of the body as a set

$$\mathscr{T} = \{(\mathbf{x},t) | \mathbf{x} \in \mathscr{B}_t, t \in \mathbb{R}\}.$$

At each t, $x(\cdot,t)$ has an inverse

$$p(\cdot,t) : \mathscr{B}_t \longrightarrow \mathscr{B}$$

such that

$$x(p(\mathbf{x},t),t) = \mathbf{x}, \qquad p(x(\mathbf{p},t),t) = \mathbf{p}.$$

Given $(\mathbf{x},t) \in \mathscr{T}$,

$$\mathbf{p} = p(\mathbf{x},t)$$

is the material point that occupies a place $\mathbf{x}$ at a time t. The map

$$p : \mathscr{T} \longrightarrow \mathscr{B}$$

is called the **reference map** of the motion.

A **material field** is a function with domain $\mathscr{B} \times \mathbb{R}$; a **spatial field** is a function with domain $\mathscr{T}$. We can transform a material field into a spatial field, and vice versa. We define the **spatial description** Φ_s of a material field $(\mathbf{p},t) \to \Phi(\mathbf{p},t)$ by

$$\Phi_s(\mathbf{x},t) = \Phi(p(\mathbf{x},t),t),$$

and the **material description** Ω_m of a spatial field $(\mathbf{x},t) \to \Omega(\mathbf{x},t)$ by

$$\Omega_m(\mathbf{p},t) = \Omega(x(\mathbf{p},t),t).$$

Given a material field Φ we write

$$\dot{\Phi}(\mathbf{p},t) = \frac{\partial}{\partial t}\Phi(\mathbf{p},t)$$

for a derivative with respect to time t holding the material point $\mathbf{p}$ fixed, and

$$\nabla\Phi(\mathbf{p},t) = \nabla_p\Phi(\mathbf{p},t)$$

for a gradient with respect to $\mathbf{p}$ holding t fixed.

Similarly, given a spatial field Ω we write

$$\Omega'(\mathbf{x}, t) = \frac{\partial}{\partial t}\,\Omega(\mathbf{x}, t)$$

for the derivative with respect to time t holding the place $\mathbf{x}$ fixed, and

$$\operatorname{grad}\Omega(\mathbf{x}, t) = \nabla_x\,\Omega(\mathbf{x}, t)$$

for the gradient with respect to $\mathbf{x}$ holding t fixed.

We define the **material time derivative** $\dot{\Omega}$ of a spatial field Ω by

$$\dot{\Omega} = ((\Omega_m)^{\cdot})_s;$$

that is,

$$\dot{\Omega}(\mathbf{x}, t) = \frac{\partial}{\partial t}\,\Omega(x(\mathbf{p}, t), t)\big|_{\mathbf{p}=p(\mathbf{x},t)}.$$

Further, we define the **spatial divergence** div to be a divergence operation for a spatial field, so that grad is the underlying gradient. Thus, for a spatial vector field v, we have

$$\operatorname{div} v(\mathbf{x}, t) = \operatorname{tr}\operatorname{grad} v(\mathbf{x}, t).$$

We call

$$\dot{x}(\mathbf{p}, t) = \frac{\partial}{\partial t}\,x(\mathbf{p}, t)$$

the **velocity** of the material point $\mathbf{p}$, and $v : \mathcal{T} \to \mathcal{V}$ defined by

$$v(\mathbf{x}, t) = \dot{x}(p(\mathbf{x}, t), t)$$

the **spatial description of velocity**. The spatial field

$$L = \operatorname{grad} v$$

is called the **velocity gradient**. We write

$$L = D + W,$$

where D and W, respectively, denote the symmetric and skew parts of L.

Using the concept of velocity gradient previously defined, one can show that $\dot{F} = L_m F$ for the material time derivative of a deformation gradient F.

By the **system of forces** for $\mathscr{B}$ during a motion (with trajectory $\mathscr{T}$), we mean a pair (s, b) of functions

$$s : \mathscr{N} \times \mathscr{T} \to \mathscr{V}, \qquad b : \mathscr{T} \to \mathscr{V},$$

where $\mathscr{N}$ is the set of all unit vectors from $\mathscr{V}$[1].

By Cauchy's Theorem[2], there exists a spatial tensor field T (called the **Cauchy stress**) such that

- $s(n) = Tn$ for each unit vector n,
- T is symmetric,
- T satisfies the **equation of motion**

$$\operatorname{div} T + b = \rho \dot{v},$$

where ρ is the density in motion.

By the **dynamical process** we mean a pair (x, T) with

- x motion,
- T symmetric tensor field on trajectory $\mathscr{T}$ of x,
- $T(\mathbf{x}, t)$ smooth function of $\mathbf{x}$ on $\mathscr{B}_t$.

A **material body** is a body $\mathscr{B}$ together with a family $\mathscr{C}$ of dynamical processes. $\mathscr{C}$ is called the **constitutive class** of the body.

Let x and x^* be motions of $\mathscr{B}$. We say, that x and x^* are *related by a change in observer*, if

$$x^*(\mathbf{p}, t) = q(t) + Q(t)[x(\mathbf{p}, t) - o] \tag{1.1}$$

for every material point $\mathbf{p}$ and time t, where $q(t)$ is a point of the space and $Q(t)$ is a rotation.

Letting

$$L = \operatorname{grad} v, \qquad L^* = \operatorname{grad} v^*,$$

where

$$v = (\dot{x})_s, \qquad v^* = (\dot{x}^*)_s.$$

[1] More precise definition is in Gurtin [gu, p. 99].
[2] See Gurtin [gu, p. 101].

we obtain

$$L^* = QLQ^T + \dot{Q}Q^T, \qquad D^* = QDQ^T,$$

where D and D^*, respectively, are symmetric parts of L and L^*. Thus, we have $\operatorname{tr} L^* = \operatorname{tr} L$.

We say that two dynamical processes (x, T) and (x^*, T^*) **are related by a change in observer** if there exist C^3 functions

$$q : \mathbb{R} \to \mathscr{E}, \qquad Q : \mathbb{R} \to \mathrm{Orth}^+$$

such that

- (1.1) holds for all $\mathbf{p} \in \mathscr{B}$ and $t \in \mathbb{R}$,
- $T^* = QTQ^T$ in trajectory of x.

We say that **a response of a material body is independent of the observer** provided its constitutive class $\mathscr{C}$ has the following property: if a process (x, T) belongs to $\mathscr{C}$, so does every dynamical process related to (x, T) by a change in observer.

Chapter 2

Governing equations

In this work, we will solve a flow of compressible Newtonian fluid, which is a material, for which the Cauchy stress is defined by the constitutive equation of the form

$$T = -\pi I + C[L], \tag{2.1}$$

where C is a linear function of the velocity gradient

$$L = \operatorname{grad} v.$$

As considered in Gurtin [gu, p. 147], Newtonian fluid means *incompressible* Newtonian fluid. The Navier-Stokes equations are derived with the assumption $\operatorname{tr} L = 0$, which means incompressibility. In our case, we need to consider compressibility effects and cannot neglect the term $\operatorname{tr} L = \operatorname{div} v$. We will use the name Newtonian fluid for *compressible* Newtonian fluid. In order to simplify the constitutive equation, we define the **extra stress** T_0 by

$$T_0 = T + \pi I = T - \frac{1}{3}(\operatorname{tr} T)I.$$

Then the constitutive equation (2.1) takes the simple form

$$T_0 = C[L]. \tag{2.2}$$

In view of the previous, we consider **Newtonian fluid** a compressible material body consistent with the following constitutive equation: there exists a linear *response function*

$$C : \mathrm{Lin} \to \mathrm{Sym}$$

such that the constitutive class $\mathscr{C}$ is a set of all dynamical processes (x, T) which obey the constitutive equation (2.2).

In the following theorem, we will show that the response is determined by *two constants*.

Theorem[1] A necessary and sufficient condition for the response of a Newtonian fluid to be independent of the observer is that its response function C has the form

$$C[L] = 2\mu D + \lambda(\operatorname{tr} L)I \tag{2.3}$$

for every $L \in \text{Lin}$, where

$$D = \frac{1}{2}(L + L^T).$$

The scalar constants μ and λ are called the first and the second **viscosity coefficients** of the fluid.

Proof. Our proof will copy the one from Gurtin, so that we include the compressibility.

(Sufficiency) Assume that (2.3) holds. Let (x, T) belong to the constitutive class $\mathscr{C}$ of the fluid. Then

$$T_0 = 2\mu D + \lambda(\operatorname{tr} L)I.$$

Let (x^*, T^*) be related to (x, T) by a change in observer. Then

$$T^* = QTQ^T, \qquad D^* = QDQ^T,$$

and

$$\operatorname{tr} T^* = \operatorname{tr}(QTQ^T) = \operatorname{tr} T.$$

Therefore

$$\begin{aligned} T_0^* &= T^* - \frac{1}{3}(\operatorname{tr} T^*)I = QTQ^T - \frac{1}{3}(\operatorname{tr} T)QQ^T = QT_0Q^T \\ &= Q(2\mu D)Q^T + \lambda \operatorname{tr}(QLQ^T)I = 2\mu D^* + \lambda(\operatorname{tr} L^*)I, \end{aligned}$$

because

$$L^* = QLQ^T + \dot{Q}Q^T, \qquad \operatorname{tr} L^* = \operatorname{tr}(QLQ^T) = \operatorname{tr} L,$$

since $\dot{Q}Q^T \in \text{Skw}$.

Thus $(x^*, T^*) \in \mathscr{C}$ and the response is independent of the observer.

[1] Cf. Gurtin [gu, p. 149].

The proof of necessity is faciliated by the following Lemma and Representation Theorem:

Lemma. Let $L \in \text{Lin}$ be a constant tensor. Then there exists a motion x with velocity gradient

$$\text{grad}\, v = L. \tag{2.4}$$

Proof. Take

$$F(t) = e^{Lt}$$

so that F is a unique solution of

$$\dot{F} = LF, \qquad F(0) = I. \tag{2.5}$$

Thus

$$x(\mathbf{p}, t) = \mathbf{q} + F(t)[\mathbf{p} - \mathbf{q}]$$

defines a motion with the deformation gradient F. Further, (2.4) follows from $(2.5)_1$, since $\dot{F} = (\text{grad}\, v)_m F$ and $L = L_m$. □

Representation Theorem for Isotropic Tensor Functions.[2] A linear function

$$G : Sym \to Sym$$

is isotropic if and only if there exist scalars μ and λ such that

$$G(A) = 2\mu A + \lambda(\text{tr}\, A)I \tag{2.6}$$

for every $A \in \text{Sym}$.

We now return to the proof of the previous theorem. To establish the *necessity* of (2.3) we assume that

$$\text{the response is independent of the observer.} \tag{2.7}$$

Let $L \in \text{Lin}$ be arbitrary, let x be the motion constructed in the previous lemma, and let $T = T_0 = C[L]$ be the constant field defined by (2.2). Then, clearly, $(x, T) \in \mathscr{C}$. Let (x^*, T^*) be related to (x, T) by a change in observer. Then by (2.7), $(x^*, T^*) \in \mathscr{C}$ and

$$T_0^* = C[L^*]. \tag{2.8}$$

But

$$T_0^* = QT_0Q^T, \qquad L^* = QLQ^T + \dot{Q}Q^T;$$

[2]Cf. Gurtin [gu, p. 235].

hence (2.8) yields

$$QT_0Q^T = C[QLQ^T + \dot{Q}Q^T],$$

and we conclude from (2.2) and (2.4) that

$$QC[L]Q^T = C[QLQ^T + \dot{Q}Q^T]. \qquad (2.9)$$

Clearly, this relation holds for every $L \in \mathrm{Lin}$ (the definition scope of C) and every C^3 function $Q : \mathbb{R} \to \mathrm{Orth}^+$. Fix L and take

$$Q(t) = e^{-Wt},$$

where

$$W = \frac{1}{2}(L - L^T).$$

Then $Q(t)$ is rotation, since W is skew, and

$$Q(0) = I, \qquad \dot{Q}(0) = -W.$$

Using this function Q in (2.9) at $t = 0$ yields

$$C[L] = C[L - W] = C[D],$$

where

$$D = \frac{1}{2}(L + L^T).$$

Thus C is completely determined by its restriction to Sym. Next, let Q be a constant function with values in Orth^+. Then (2.9) with $L = D$ ($D \in \mathrm{Sym}$) implies that

$$QC[D]Q^T = C[QDQ^T].$$

Since this relation must hold for every $D \in \mathrm{Sym}$ and every $Q \in \mathrm{Orth}^+$, the restriction of C to Sym is isotropic: we therefore conclude from the representation (2.6) that

$$C[D] = 2\mu D + \lambda(\mathrm{tr}\, L)I$$

for all $D \in \mathrm{Sym}$.

□

By (2.3) the constitutive equation (2.1) takes the form

$$T = -\pi I + 2\mu D + \lambda(\operatorname{tr} L)I. \tag{2.10}$$

We consider the equation of motion

$$\rho[v' + (\operatorname{grad} v)v] = \operatorname{div} T + b.$$

and substitute (2.10) for T. We have,

$$2 \operatorname{div} D = \operatorname{div}(\operatorname{grad} v + \operatorname{grad} v^T) = \Delta v + \operatorname{grad} \operatorname{div} v$$

and

$$\operatorname{div}(\operatorname{tr} L)I = \operatorname{grad}(\operatorname{tr} L) = \operatorname{grad} \operatorname{div} v,$$

where $\Delta = \operatorname{div} \operatorname{grad}$ is the spatial Laplacian. Thus the equation of motion reduces to

$$\rho[v' + (\operatorname{grad} v)v] = \mu\Delta v + (\lambda + \mu) \operatorname{grad} \operatorname{div} v - \operatorname{grad} \pi + b. \tag{2.11}$$

These (vector) relations are the **Navier-Stokes equations**; given μ, λ and b they constitute a nonlinear system of partial differential equations for the velocity v, density ρ and pressure π. We supplement these equations by the continuity equation

$$\rho' + \operatorname{div}(\rho v) = 0. \tag{2.12}$$

Now, we have four equations for five unknowns, so we have to add one more equation to the system. We will consider barotropic flow[3], where the pressure is a known function of the density

$$\pi = \hat{\pi}(\rho). \tag{2.13}$$

[3]Cf. Feistauer et al. [fe, p. 33].

Chapter 3

Formulation of the problem

In what follows, we shall be concerned with a two-dimensional model describing an interaction of a viscous, compressible fluid with an airfoil. The airfoil is considered to be a rigid body with two degrees of freedom - its vertical and torsion vibrations (see Figure 3.1). Equations describing the airfoil motion will be presented later.

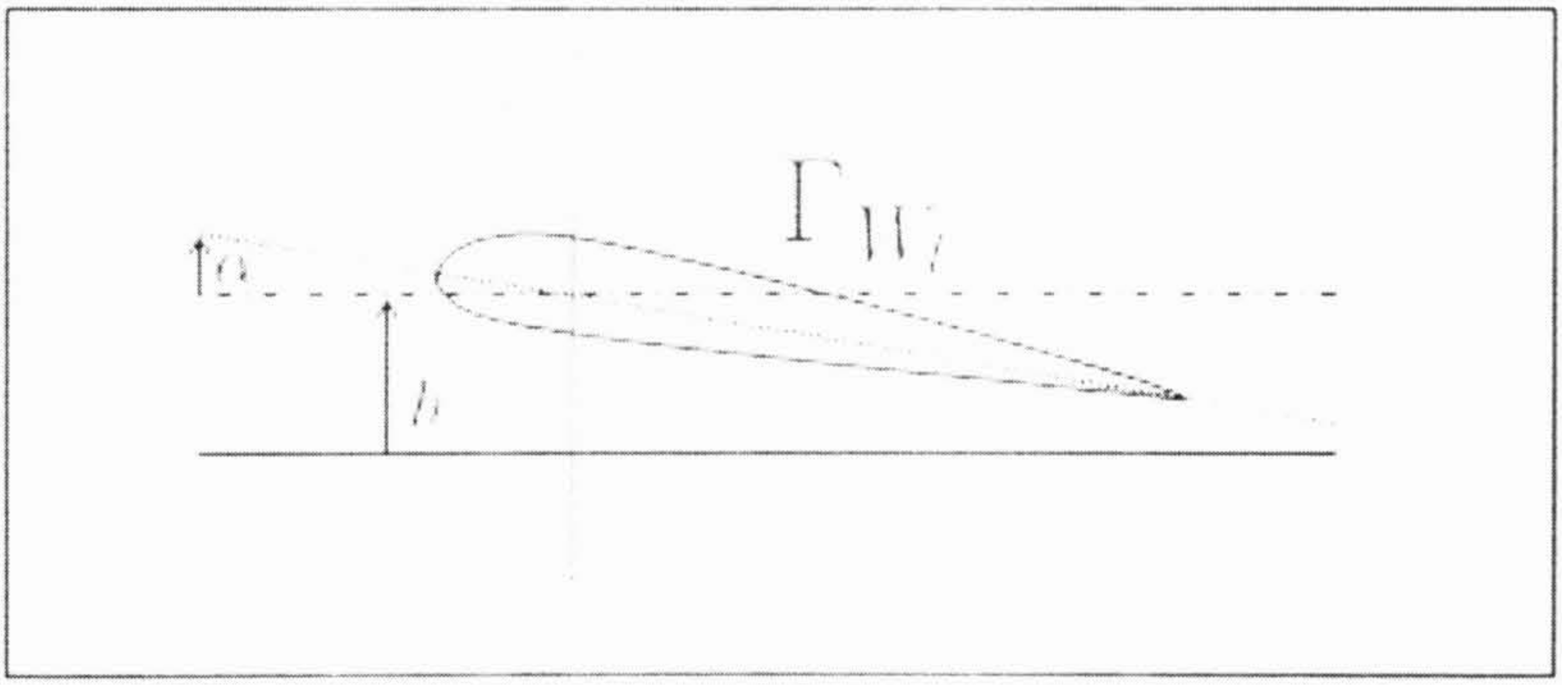

Figure 3.1: Airfoil model

This problem has a time-dependent boundary (moving airfoil) and therefore, a time-dependent computational domain (see Figure 3.2).

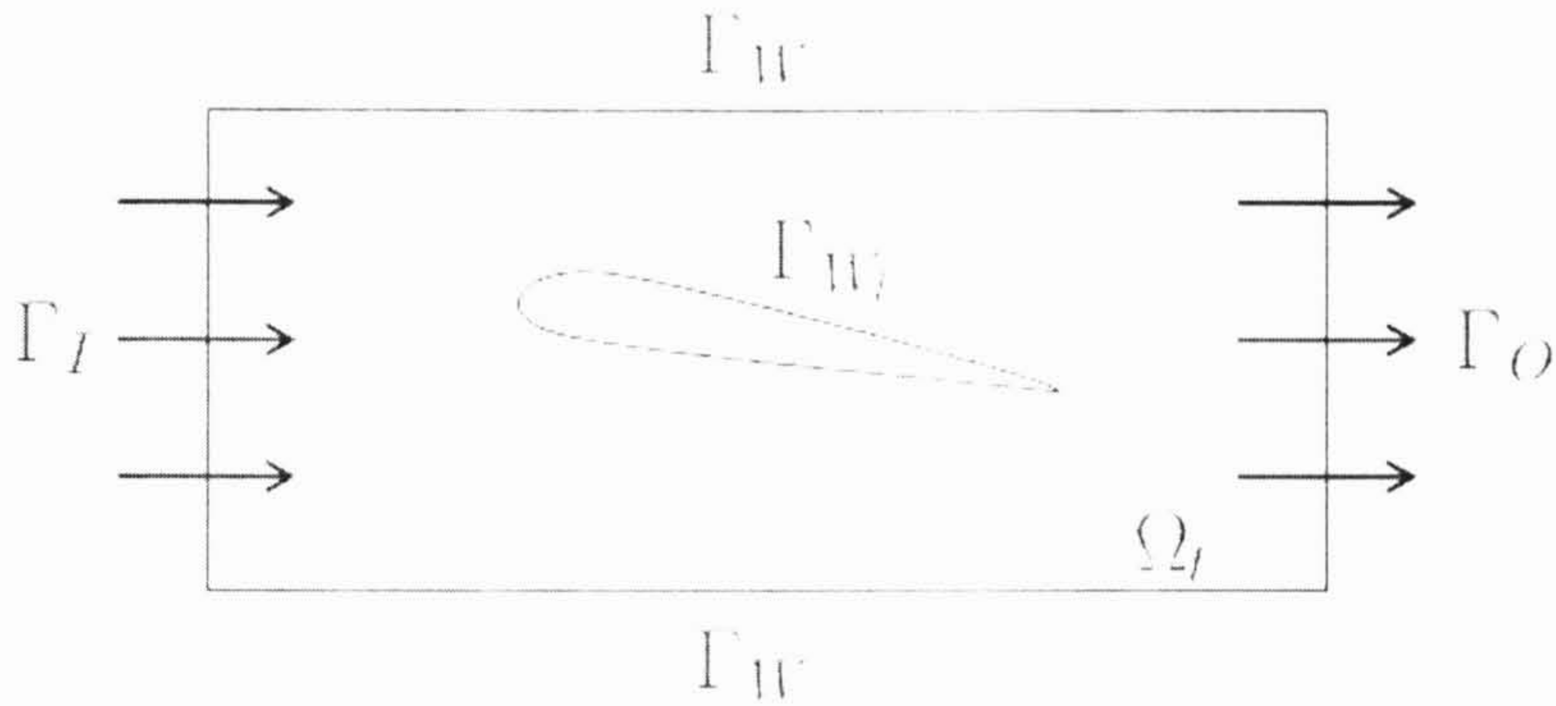

Figure 3.2: Problem setting

3.1 Input data of our problem

We consider the problem in the domain

$$\widetilde{\Omega} := \bigcup_{t\in[0,T]} \Omega_t \times \{t\}.$$

We split the domain boundary into four parts. Three of them are time independent, whereas the part representing the moving airfoil depends on time:

$$\begin{array}{ll} \Gamma_I := \Gamma_I \times [0,T] & \text{inlet,} \\ \Gamma_O := \Gamma_O \times [0,T] & \text{outlet,} \\ \Gamma_W := \Gamma_W \times [0,T] & \text{virtual flow wall,} \\ \Sigma := \bigcup_{t\in[0,T]} \Gamma_{W_t} \times \{t\} & \text{airfoil.} \end{array}$$

In the domain $\widetilde{\Omega}$, we consider the Navier-Stokes equations, the continuity equation and the condition of barotropic flow:

$$\begin{aligned} \rho[v' + (\operatorname{grad} v)v] &= \mu\Delta v + (\lambda+\mu)\operatorname{grad}\operatorname{div} v - \operatorname{grad}\pi + b \qquad && \text{in } \widetilde{\Omega}, \\ \rho' + \operatorname{div}(\rho v) &= 0 && \text{in } \widetilde{\Omega}, \quad (3.1) \\ \pi &= \widehat{\pi}(\rho) && \text{in } \widetilde{\Omega}. \end{aligned}$$

Boundary conditions for the time independent part of the boundary:

$$\begin{aligned} v &= v_D && \text{on } \Gamma_I \cup \Gamma_W, \\ -(\pi - \pi_{ref})n + \mu(\operatorname{grad} v)n + (\lambda + \mu)(\operatorname{div} v)n &= 0 && \text{on } \Gamma_O, \\ \rho &= \rho_D && \text{on } \Gamma_I. \end{aligned} \tag{3.2}$$

Initial conditions:

$$\begin{aligned} v(x, 0) &= v_0(x) && \text{in } \Omega_0, \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Omega_0. \end{aligned} \tag{3.3}$$

We also need to prescribe boundary conditions on the part Σ of the boundary and initial conditions on Ω_0. This will be discussed further.

The fact that the domain occupied by the fluid depends on time causes difficulties. In order to overcome them, we can use Arbitrary Lagrangian-Eulerian (ALE) formulation for the mathematical description of the problem with a moving boundary.

3.2 Equations of airfoil motion

In our case, the airfoil can perform its vertical and torsion vibrations. These vibrations are described by two degrees of freedom: airfoil deflection angle α and vertical displacement h. The evolution of these values for small angles of deflection is described by the following differential equations[1]

$$\begin{aligned} m\ddot{h} + D_{hh}\dot{h} + D_{h\alpha}\dot{\alpha} + S_\alpha\ddot{\alpha} + k_{hh}h &= -L_2, \\ I_\alpha\ddot{\alpha} + D_{\alpha h}\dot{h} + D_{\alpha\alpha}\dot{\alpha} + S_\alpha\ddot{h} + k_{\alpha\alpha}\alpha &= M. \end{aligned} \tag{3.4}$$

Here we use the following notation:

$m = \int_{\Pi_t} \rho \, dx$	weight of airfoil,
$S_\alpha = \int_{\Pi_t} x\rho \, dx$	static momentum,
$I_\alpha = \int_{\Pi_t} x^2\rho \, dx$	momentum of inertia,
$L_2 = -\int_{\Gamma_{W_t}} \sum_{j=1}^2 T_{2j} n_j \, dS$	aerodynamic lift,
$M = -\int_{\Gamma_{W_t}} \sum_{i,j=1}^2 T_{ij} n_j r_i^{ort}) \, dS$	aerodynamic momentum,

where Π_t is the area of the airfoil, $\Gamma_{W_t} = \partial\Pi_t$, T is the stress tensor obtained from (2.10), $r_1^{ort} = -(x_2 - x_{EA2})$ and $r_2^{ort} = x_1 - x_{EA1}$. Further,

k_{hh}	vertical stiffness,
$k_{\alpha\alpha}$	torsion stiffness,
$D_{hh}, D_{h\alpha}, D_{\alpha h}, D_{\alpha\alpha}$	components of viscous damping

[1] See Růžička [ru, p. 17].

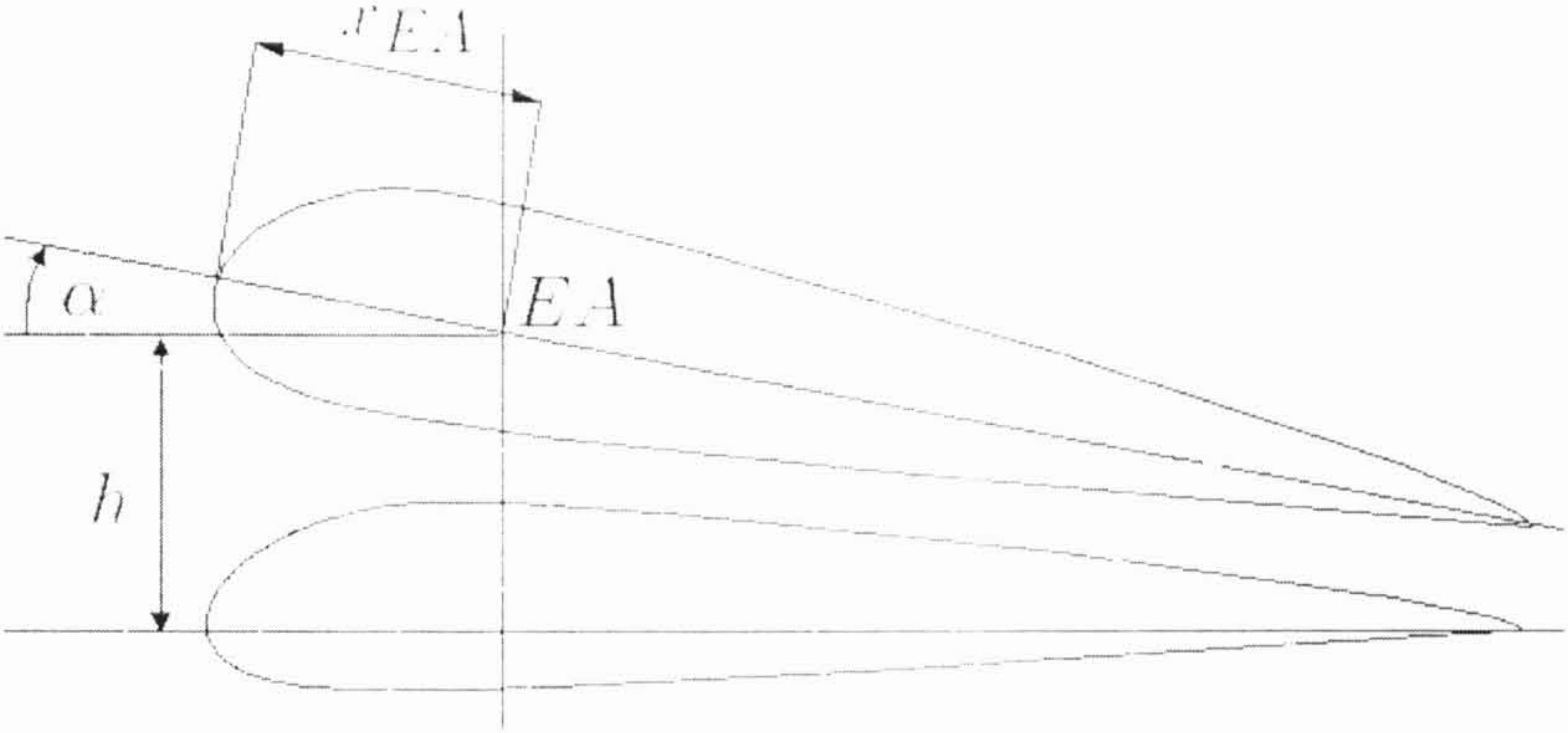

Figure 3.3: Airfoil vibrations

are given (constant) parameters.

Equations (3.4) are supplemented with these initial conditions

$$\alpha(0) = \alpha_0, \quad \dot{\alpha}(0) = \alpha_1,$$
$$h(0) = h_0, \quad \dot{h}(0) = h_1.$$

3.3 ALE formulation

We consider the Navier-Stokes equations in a moving domain $\widetilde{\Omega} = \Omega_t \times [0, T]$ (which means $\bigcup_{t\in[0,T]} \Omega_t \times \{t\}$ - the so-called non-cylindrical domain).
In order to simulate a fluid flow on a moving domain, we employ the *Arbitrary Lagrangian-Eulerian* (ALE) method[2].

Let Ω_0 be the original domain and Ω_t be the computational domain at a (later) time t. We introduce the ALE mapping

$$\mathscr{A}_t : \Omega_0 \to \Omega_t$$
$$X \mapsto y = y(X, t) = \mathscr{A}_t(X),$$

which maps the original domain Ω_0 onto the computational domain Ω_t, such that $\mathscr{A}_t$ is continuous and bijective on Ω_0.

[2]See, e.g. Quarteroni [qa, p. 37]; Sváček [sv, p. 6].

We define the *domain velocity* field at points X of the original domain at each time level t

$$\tilde{w}(X,t) = \frac{\partial}{\partial t} y(X,t) = \frac{\partial}{\partial t} \mathscr{A}_t(X),$$

which, in spatial coordinates has the form

$$w = \tilde{w} \circ A_t^{-1}, \qquad \text{i.e.} \quad w(y,t) = \tilde{w}(\mathscr{A}_t^{-1}(y),t).$$

For a function $f : \widetilde{\Omega} \to \mathbb{R}$, we define the *ALE derivative* of f as

$$\frac{D^{\mathscr{A}}}{Dt} f(y,t) = \frac{\partial}{\partial t} \tilde{f}(X,t),$$

where $\tilde{f} = f \circ \mathscr{A}_t$ and $X = \mathscr{A}_t^{-1}(y)$.

Using the chain rule for derivative, we obtain

$$\begin{aligned} \frac{D^{\mathscr{A}}}{Dt} f(y,t) &= \frac{\partial}{\partial t} f(\mathscr{A}_t(X),t) \\ &= \frac{\partial}{\partial t} f(y,t) + \operatorname{grad} f(y,t) \cdot \frac{\partial}{\partial t} \mathscr{A}_t(X)|_{X=\mathscr{A}_t^{-1}(y)} \\ &= \frac{\partial}{\partial t} f(y,t) + \operatorname{grad} f(y,t) \cdot w(y,t). \end{aligned}$$

Using ALE derivative, we can rewrite the Navier-Stokes equations in the form

$$\begin{aligned} \rho[\tfrac{D^{\mathscr{A}}}{Dt} v + (\operatorname{grad} v)(v - w)] &= \mu \Delta v + (\lambda + \mu) \operatorname{grad} \operatorname{div} v - \operatorname{grad} \pi + b \\ \tfrac{D^{\mathscr{A}}}{Dt} \rho + \operatorname{div}(\rho v) - \operatorname{grad} \rho \cdot w &= 0 \\ \pi &= \widehat{\pi}(\rho) \end{aligned} \tag{3.5}$$

where all equations are considered on the domain $\widetilde{\Omega}$. Note, that the continuity equation can be written in the form

$$\frac{D^{\mathscr{A}}}{Dt} \rho + \rho \operatorname{div}(v) + \operatorname{grad} \rho \cdot (v - w) = 0. \tag{3.6}$$

Boundary and initial conditions remain the same as before.

3.4 Weak formulation

First, we define the spaces of test functions. Let $q \in Q = L^2(\Omega_t)$ and $u \in V = \{u \in H^1(\Omega_t)^2 : u|_{\Gamma_D} = 0\}$, where $\Gamma_D = \Gamma_I \cup \Gamma_W \cup \Gamma_{W_t}$ is the part of the boundary, where we prescribe the Dirichlet condition.

Multiplying equation $(3.5)_1$ with any $u \in V$, integrating over Ω_t and using Green's theorem, we obtain

$$\int_{\Omega_t} \rho \frac{D^{\mathcal{A}}}{Dt} v \cdot u \, dx + \int_{\Omega_t} \rho(\operatorname{grad} v)(v - w) \cdot u \, dx =$$
$$- \mu \int_{\Omega_t} \operatorname{grad} v \cdot \operatorname{grad} u \, dx - (\lambda + \mu) \int_{\Omega_t} \operatorname{div} v \operatorname{div} u \, dx$$
$$+ \int_{\Omega_t} \pi \operatorname{div} u \, dx + \int_{\Omega_t} b \cdot u \, dx$$
$$+ \int_{\Gamma_O} [-\pi + \mu (\operatorname{grad} v) + (\lambda + \mu)(\operatorname{div} v)] \, u \cdot n \, dS$$

Same proceeding with (3.6), in which we use any $q \in Q$, yields

$$\int_{\Omega_t} \frac{D^{\mathcal{A}}}{Dt} \rho \, q \, dx + \int_{\Omega_t} \rho \operatorname{div} v \, q \, dx + \int_{\Omega_t} \operatorname{grad} \rho \cdot (v - w) \, q \, dx = 0$$

For simplicity, we define the following forms[3]:

$$\begin{aligned}
a(v, u) &= \mu \, (\operatorname{grad} v, \operatorname{grad} u) + (\lambda + \mu)(\operatorname{div} v, \operatorname{div} u), \\
b(u, q) &= (\operatorname{div} u, q), \\
\alpha(v, \rho, q) &= (v \cdot \operatorname{grad} \rho, q), \\
d(\rho, w, v, u) &= (\rho(\operatorname{grad} v) w, u), \\
c(\rho, v, q) &= (\rho \operatorname{div} v, q).
\end{aligned}$$

Then, we can rewrite the previous equations in the form

$$\begin{aligned}
&(\rho \frac{D^{\mathcal{A}}}{Dt} v, u) + d(\rho, v - w, v, u) + a(v, u) \\
&\qquad = b(u, \pi) + (b, u) + \int_{\Gamma_O} \pi_{ref} \, u \cdot n \, dS, \qquad (3.7) \\
&(\frac{D^{\mathcal{A}}}{Dt} \rho, q) + c(\rho, v, q) + \alpha(v - w, \rho, q) = 0,
\end{aligned}$$

where we put the so-called soft boundary condition $(3.2)_2$.

---

[3]Denotation from Feistauer et al. [fe, p. 368].

3.5 Boundary conditions

We assume that for each $t \in [0, T]$ there exists $v^* \in H^1(\Omega_t)^2$, such that

$$\begin{aligned} v^*(x,t) &= v_D(x,t), \quad x \in \Gamma_I \cup \Gamma_W \\ v^*(x,t) &= w(x,t), \quad x \in \Gamma_{W_t} \end{aligned}$$

(in the sense of traces). Then the *weak formulation* reads:

- Find v, such that $v - v^* \in V$; $\rho \in Q$
- equation $(3.7)_1$ is satisfied $\forall u \in V$.

The boundary condition for the density ρ prescribed on inlet Γ_I is formulated in the so-called weak integral sense[4]

$$\left(\frac{D^{\mathscr{A}}}{Dt}\rho, q\right) + c(\rho, v, q) + a(v - w, \rho, q) - \gamma \int_{\Gamma_I} \rho v_D \cdot nq \, dS =$$

$$- \gamma \int_{\Gamma_I} \rho_D v_D \cdot nq \, dS \qquad \forall q \in Q,$$

where γ is a suitable parameter.

3.6 Discrete problem

Let $\{\mathscr{T}_h\}_{h \in (0,T)}$ be a regular system of triangulations of the domain $\widetilde{\Omega} = \Omega_t \times \{t\}$. In a time interval $[0, T]$ we construct a partition $t_n = n\tau, n = 0, \ldots, r$ with time step τ. For a function f defined in $\widetilde{\Omega}$, we set

$$\begin{aligned} \frac{D^{\mathscr{A}}}{Dt} f(y_n, t_n) &= \frac{\partial}{\partial t} \tilde{f}(X, t_n) \\ &\approx (\tilde{f}(X, t_n) - \tilde{f}(X, t_{n-1}))/\tau \\ &= (f(y_n, t_n) - f(y_{n-1}, t_{n-1}))/\tau, \end{aligned}$$

where $y_n = \mathscr{A}_{t_n}(X)$.

For simplicity, we will write $f^n = f(y_n, t_n)$ and $d_{\mathscr{A}} f^n = (f^n - f^{n-1})/\tau$.

The approximate solution will be sought at each time level t_n in finite dimensional spaces of finite elements X_h and Q_h.

[4] See Feistauer et al. [fe, p. 373].

We set $Q_h = X_h^{(m)}$, $X_h = [X_h^{(k)}]^2$, $V_h = \{v_h \in [X_h^{(k)}]^2;\ v_h|_{\Gamma_D} = 0\}$, where $X_h^{(p)} = \{v_h \in C(\bar{\Omega}_h);\ v_h|_K \in P^p(K)\ \forall K \in \mathcal{T}_h\}$ and $P^p(K)$ is a set of all polynomials on K of degree $\leq p$.

First, we approximate the spaces V and Q by V_h and Q_h respectively. We use the approximations

$$
\begin{aligned}
v^n &\approx v_h^n \in V_h, \\
\rho^n &\approx \rho_h^n \in Q_h, \\
\frac{D^{\mathcal{A}}}{Dt} v^n &\approx (v^n - v^{n-1})/\tau \approx (v_h^n - v_h^{n-1})/\tau = d_{\mathcal{A}_t} v_h^n, \\
\frac{D^{\mathcal{A}}}{Dt} \rho^n &\approx (\rho^n - \rho^{n-1})/\tau \approx (\rho_h^n - \rho_h^{n-1})/\tau = d_{\mathcal{A}_t} \rho_h^n
\end{aligned}
$$

Moreover, we will use the streamline diffusion test function

$$
q_h + \delta q_{h,\beta} \quad \text{with } q_{h,\beta} = (v_h^{n-1}, \operatorname{grad} q_h)
$$

for suitable constant $\delta > 0$, which will be used instead of q_h to avoid Gibb's phenomenon in the numerical solution[5].

Let $v_h^* \in X_h$ be the approximation of v^*, we can use the approximation

$$
\begin{aligned}
v_h^*(P_i, t) &= v_D(P_i, t) & \forall P_i &\in \Gamma_I \cup \Gamma_W, \\
v_h^*(P_i, t) &= w(P_i, t) & \forall P_i &\in \Gamma_{W_t}, \\
v_h^*(P_i, t) &= 0 & \forall P_i &\in \Omega_t.
\end{aligned}
$$

We obtain the following formulation of the discrete problem:
Find $v_h^n \in X_h$, such that $v_h^n - v_h^{*n} \in V_h$; $\rho_h^n \in Q_h$ and the following equations hold:

$$
\begin{aligned}
&(\rho_h^{n-1}\, d_{\mathcal{A}_t} v_h^n, u_h) + d(\rho_h^{n-1}, v_h^{n-1} - w_h^{n-1}, v_h^n, u_h) + a(v_h^n, u_h) \\
&\qquad = b(u_h, \pi_h^{n-1}) + (b_h^{n-1}, u_h) + \int_{\Gamma_O} \pi_{ref}\, u_h \cdot n\, dS \qquad \forall u_h \in V_h, \\
&(d_{\mathcal{A}_t} \rho_h^n, q_h) + e(\rho_h^{n-1}, v_h^n, q_h + \delta q_{h,\beta}) \\
&\qquad + \alpha(v_h^{n-1} - w_h^{n-1}, \rho_h^n, q_h + \delta q_{h,\beta}) - \gamma \int_{\Gamma_I} \rho_h^n v_D^n \cdot n q_h\, dS \\
&\qquad = -\gamma \int_{\Gamma_I} \rho_D^n v_D^n \cdot n q_h\, dS \qquad \forall q_h \in Q_h, \\
&\pi_h^n = \widehat{\pi}(\rho_h^n).
\end{aligned}
\tag{3.8}
$$

[5]See Feistauer et al. [fe, p. 346]

We can write

$$v_h^n = v_h^{*n} + z_h^n, \quad \text{with } z_h^n \in V_h.$$

Assuming that $u_h^{n-1}, \rho_h^{n-1}, \pi_h^{n-1}, w_h^{n-1}$ are known, using substitution for v_h^n, we get a linear system for parameters determining the unknown functions z_h^n and ρ_h^n.

System (3.8) can be solved in two separate steps. First, we find v_h^n by solving the first equation. Using the result, we can find ρ_h^n by solving the second one.

Chapter 4

Conclusion

In this work, we derived mathematical description of the problem with a moving boundary. We also formulated scheme for numerical experiments, which will be the goal of a following master thesis. Several tasks were not mentioned in this work, i.e. the method of solving ordinary differential equations describing the airfoil motion or a construction of ALE mapping. These, along with some numerical experiments, will be included in master thesis. We will also test whether the explicit scheme given in (3.8) is sufficiently accurate, or there is a need of better (implicit) scheme.

Bibliography

[fe] M. Feistauer et al.:
Mathematical and Computational Methods for Compressible Flow,
Oxford University Press, New York, 2003.

[gu] Morton E. Gurtin:
An Introduction to Continuum Mechanics,
Academic Press, New York, 1981.

[qa] A. Quarteroni:
Simulation of Fluid and Structure Interaction,
European Mathematical Society, Prague, 2001.

[ru] M. Růžička:
Interaction of Incompressible Fluid and Moving Bodies,
Master Thesis, MFF UK, Prague, 2005.

[sv] P. Sváček:
Numerical Solution of Fluid Over a Moving Profile,
MFF UK, Prague, 2005.