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MASTER THESIS

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Stochastic Integrals

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: Tato práce je zaměřena na rozšíření klasického Itôova integrálu $(I)\int_0^T X\ \mathrm{d}B$ na přímce. Rozšíříme Itôův integrál tak, abychom byli schopni integrovat i procesy, které nejsou adaptované. Taktéž představíme integraci vzhledem k frakcionálnímu Brownovu pohybu $B^H,\ 0 < H < 1$, což také pokrývá Itôův integrál, neboť standardní Brownův pohyb (Wienerův proces) B se shoduje s $B^{\frac{1}{2}}$. Navíc, jak známo, Itôův integrál je definován pomocí L^2 procedur za použití Itôovy izometrie, což znamená, že nemůže být definován po trajektoriích. Naproti tomu představíme také stochastické integrály, které jsou definované po trajektoriích a porovnáme je. V poslední kapitole ukážeme použití Kurzweilova integrálu pro stochastickou integraci.

Klíčová slova: Stochastická integrace, Malliavinův počet, Skorochodův integrál, Integrace po trajektoriích, Kurzweilův integrál

Title: Stochastic Integrals

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Abstract: In this Thesis we extend the classic theory of the Itô stochastic integral $(I)\int_0^T X\ \mathrm{d}B$ on real line. We extend the Itô integral so that we can handle anticipating (non adapted) processes. We also introduce the integration with respect to the fractional Brownian motion $B^H,\ 0 < H < 1$ which also covers the Itô integral, because the standard Brownian motion B coincides with $B^{\frac{1}{2}}$. Moreover it is well-known that the basic Itô integral is defined via L^2 procedures using Itô isometry which means that it cannot be defined pathwise. Contrary we introduce some concepts of pathwise stochastic integrals and compare them. In the last chapter we show the usage of the concept of generalized Perron (Kurzweil) integral for the stochastic integration.

Keywords: Stochastic integration, Malliavin calculus, Skorohod integral, Pathwise integration, Kurzweil integral

Contents

Chapter 0

Introduction

We start with a brief history of stochastic integration. Theory of stochastic integration is closely related to the Brownian motion so let us begin with its origins. The earliest attempts to define the Brownian motion mathematically were done by three authors independently: T. N. Thiele, L. Bachelier and A. Einstein. Thiele developed a model studying time series in 1880. Bachelier created a model of Brownian motion while studying the dynamics of Paris stock market in 1900. Einstein wanted to model the behaviour of small particles in a liquid in 1905. Einstein proposed the model of a stochastic process with continuous paths and independent stationary Gaussian increments. The models of Thiele and Bacheliere were not particularly influential but the Einstein's was. However, Einstein was unable to show the existence of such stochastic process. The existence was shown later in 1923 by N. Wiener. After the existence, other important properties of the Brownian motion were proven, such as infinite variation and finite non zero quadratic variation. In 1944 K. Itô introduced his first paper on stochastic integration where the integrand was adapted stochastic process and integrator was the Brownian motion. Later other important results were proven such as Itô formula in 1951, the Doob-Meyer decompostion, conception of stochastic differential equations driven by the Brownian motion, Black-Scholes' application of stochastic calculus to finance, change of time conception etc. A considerable part of this Thesis is devoted to the fractional Brownian motion which was first introduced in 1940 by Kolmogorov who called it Wiener Helix. The name fractional Brownian motion was introduced by Mandelbrot and Van Ness in 1968. It is an useful and widely used model for diffusion processes with correlated increments. Therefore it has many applications e.g. in financial mathematics. During the second half of the 20th century until now many mathematicians studied and developed various conceptions of stochastic integration. They wanted to handle anticipating processes and extend the set of integrators, use different approaches of approximation etc. Many of the conceptions of stochastic integrals are studied here.

In Chapter 1 we introduce Skorohod stochastic integral defined as an adjoint operator to the Malliavin derivative. Chapter 2 is devoted to pathwise integrals. In the third chapter we introduce and compare other "less usual" conceptions of integration. Chapter 4 contains a small summary of results comparing the types of integrals we introduce in the first three chapters and in Chapter 5 we introduce the concept of generalized Perron (Kurzweil) integral and present our own result in Theorem 40 and apply the conception of Kurzweil integral to the stochastic calculus with respect to the fractional Brownian motion.

We expect the reader to know the basic theory of measure and Lebesgue-Stieltjes integration, probability theory, theory of martingales and stopping times as introduced in e.g. Karatzas and Shreve (1998, Chapter 1), existence and basic properties of standard Brownian motion on real line (Karatzas and Shreve (1998, Chapter 2)) and construction of Itô stochastic integral with respect to a semimartingale via certain limit of approximating sums:

$$(I)\int_0^T Y \,\mathrm{d}Z = \lim_{n\to\infty}\sum_{k=0}^{2^n-1} Y_{kT/2^n}(Z_{(k+1)T/2^n} - Z_{kT/2^n}.) \tag{0.1}$$

For detailed procedure see Karatzas and Shreve (1998, Chapter 3) or my Bachelor Thesis: Filip Lacina: Stochastická integrace, Prague, 2013, supervisor: Prof. RNDr. Josef Štěpán, DrSc. Note that the Itô integral admits a continuous version. In the whole Thesis we always assume that we work with the continuous version.

Chapter 1

Skorohod type integrals

We follow the approach in Nualart (2006, Chapter 1). Let us fix a complete probability space $(\Omega, \mathcal{F}, P)$ assuming it is rich enough. We also use a fixed space $L^2(X, \mathcal{B}, \mu)$, where μ is σ-finite nonatomic measure.

1.1 Wiener space and Wiener chaos decomposition

Let H be a real Hilbert space equipped with scalar product $\langle \cdot, \cdot \rangle_H$ and norm induced by the scalar product $\| \cdot \|$.

Definition 1. *A stochastic process $W = (W(h), h \in H)$ defined on $(\Omega, \mathcal{F}, P)$ is called isonormal Gaussian process if W is a centered Gaussian family such that $E(W(g)W(h)) = \langle g, h \rangle_H$.*

Remark. The mapping $h \to W(h)$ is linear and provides a linear isometry of H onto a subspace of $L^2(\Omega, \mathcal{F}, P)$ containing all centered Gaussian variables. Note that $W(h)$ exists by Kolmogorov theorem (see Nualart (2006, p. 4)).

Now let us introduce the definition and basic properties of Hermite polynomials.

Definition 2. *Let*

$$H_n(x) = \frac{-1^n}{n!} \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}}{\mathrm{d}x^n} \left(\mathrm{e}^{\frac{-x^2}{2}} \right), \ n \geq 1.$$

H_n is called the n-th Hermite polynomial.

Lemma 1.

- *Hermite polynomials are the coefficients of Taylor expansion of the function $F(x, \cdot)$ where $F(x,t) = \exp\left(tx - \frac{t^2}{2}\right)$,*
- $H_n'(x) = H_{n-1}(x), \ n \geq 1,$
- $H_{n+1} = xH_n(x) - H_{n-1}(x), \ n \geq 1,$
- $H_n(x) = (-1)^n H_n(x), \ n \geq 1.$

Proof. For the proof see Nualart (2006, p. 4, 5). ❑

Let us now introduce following lemma.

Lemma 2. *Let X, Y be two centered variables with joint Gaussian distribution and unit variance. Then for $n, m \geq 1$ we have*

$$E(H_n(x)H_m(x)) = \begin{cases} 0 & \text{if } n \neq m. \\ \frac{1}{n!}E(XY)^n & \text{if } n = m. \end{cases}$$

Proof. For proof see Nualart (2006, p. 5). ❑

Now we fix the σ-algebra generated by $W = (W(h), h \in H)$ and call it $\mathcal{G}$. The following results strictly depend on the fact that all the randomness is generated by W.

Lemma 3. *The set $(\mathrm{e}^{W(h)}, h \in H)$ is total in $L^2(\Omega, \mathcal{G}, P)$ which means that closure of its linear span equals $L^2(\Omega, \mathcal{G}, P)$.*

Proof. Sketch of the proof: Let $X \in L^2(\Omega, \mathcal{G}, P)$ such that $EX\mathrm{e}^{W(h)} = 0, h \in H$. Then from linearity of the mapping $h \to W(h)$ it follows that

$$E\left(X \exp\left(\sum_{i=1}^{m} t_i W(h_i)\right)\right) = 0.$$

It means that the Laplace transform of measure

$$\mu(B) = EX\mathbf{1}_B(W(h1), \ldots, W(h_m))$$

is identically zero which implies that $X = 0$.
For detailed proof see Nualart (2006, p. 6) ❑

Theorem 4. *The space $L^2(\Omega, \mathcal{G}, P)$ can be orthogonally decomposed:*

$$L^2(\Omega, \mathcal{G}, P) = \oplus_{n=1}^{\infty} \mathcal{H}_n, \tag{1.1}$$

where $\mathcal{H}_n$ denotes the subspace of $L^2(\Omega, \mathcal{G}, P)$ generated by random variables $H_n(W(h)), h \in H, \|h\|_H = 1, n \geq 1$ and $\mathcal{H}_0$ denotes the set of constants. We call $\mathcal{H}_n$ the n-th Wiener chaos.

Proof. The fact that $L^2(\Omega, \mathcal{G}, P)$ can be really decomposed into $\oplus_{n=1}^{\infty} \mathcal{H}_n$ follows from Lemma 3. The orthogonality follows from Lemma 2. ❑

1.2 Multiple Wiener-Itô integrals

In this section we assume that the underlying Hilbert space H has the form

$$H = L^2(X, \mathcal{B}, \mu), \tag{1.2}$$

where X is a Polish space, $\mathcal{B}$ denotes the Borel sets on X and μ is a nonatomic σ-finite nonnegative measure.

Definition 3. *Let $\{W = W(A), A \in \mathcal{B}, \mu(A) < \infty\}$ be a family of centered Gaussian random variables such that $(EW(A)W(B)) = \mu(A \cap B)$. W is then called white noise with underlying measure μ (or based on μ).*

Let us proceed to define the multiple Wiener-Itô integral. Set $\mathcal{B}_0 = \{A \in \mathcal{B}, \mu(B) < \infty\}$ and fix $m \geq 1$.

Definition 4. *Let*

$$f(t_1, \dots, t_m) = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots,i_m} \mathbf{1}_{A_{i_1}\times\dots\times A_{i_m}}(t_1, \dots, t_m), \tag{1.3}$$

where $A_1, \dots, A_n \in \mathcal{B}_0$ are pairwise disjoint and $a_{i_1,\dots,i_m} = 0$ if any two of the indices are equal. Such function is called step function and the set of step function is denoted as $\mathcal{E}_m$.

Definition 5. *Let be f be a function of the form of the form (1.3), then we define the Multiple Wiener-Itô integral of f with respect to W as*

$$(WI) \int_{X^m} f(t_1, \dots, t_m) \, \mathrm{d}(W(t_1), \dots, W(t_m)) =$$

$$= \sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots,i_m} W(A_{i_1}) \times \dots \times W(A_{i_m}).$$

Sometimes for shorter notation we use $WI_m(f)$ instead of $(WI) \int_{X^m} f(t_1, \dots, t_m) \, \mathrm{d}(W(t_1), \dots, W(t_m))$.

Now let us list a few basic properties of such defined integral.

Lemma 5.

- *$WI_m : \mathcal{E}_m L^2(\Omega, \mathcal{G}, P)$ is linear.*
- *$WI_m(f) = WI_m(\tilde{f})$, where $\tilde{f}$ denotes the symmetrization of f:*

  $$\tilde{f}(t_1, \dots, t_m) = \frac{1}{m!} \sum_{\pi} f(t_{\pi(1)}, \dots, t_{\pi(m)}),$$

  where π runs over all permutations of the set $\{1, \dots, m\}$.
- $$E(WI_m(f)WI_q(g)) = \begin{cases} 0 & \text{if } m \neq q. \\ m! \left\langle \tilde{f}, \tilde{g} \right\rangle_{L^2(X^m)} & \text{if } m = q. \end{cases}$$

Proof. (Nualart, 2006, p. 9). ❑

Now we want to extend the Wiener-Itô integral to all elements of $L^2(X^m)$. In order to do that we need to show that $\mathcal{E}_m$ is dense in $L^2(X^m)$. μ is nonatomic and therefore every indicator function $\mathbf{1}_{B_{i_1}\times\cdots\times B_{i_m}}$ can be approximated by a sequence of step functions from $\mathcal{E}_m$. And so $\mathcal{E}_m$ is dense in the set of all indicator functions which is of course dense in the set of all $L^2(X^m)$ functions and so the density is proven. Moreover the last property in Lemma 5 shows that WI_m can be extended to a linear and continuous operator from $L^2(X^m)$ to $L^2(\Omega, \mathcal{G}, P)$ as we can let $f = g$ in the third item of the previous Lemma and obtain the estimate of $E(WI_m(f))^2$ (see (Nualart, 2006, p. 10)). Note that the multiple Wiener-Itô integral cannot be defined pathwise.

Relation between Hermite polynomials and multiple Wiener-Itô integrals

We show two theorems which shows the relation between Hermite polynomials and the concept of Wiener-Itô stochastic integrals.

Set $H = L^2([0,T], \mathcal{B}, \lambda)$, where λ denotes the Lebesgue measure, equipped with the standard scalar product. The isonormal Gaussian process is then a Gaussian family of centered random variables such that

$$E(W(\mathbf{1}(0,t_1])W(\mathbf{1}_{(0,t_2]})) = \left\langle \mathbf{1}_{(0,t_1]}, \mathbf{1}_{(0,t_2]} \right\rangle_{L^2([0,T],\mathcal{B},\lambda)} = \int_{[0,T]} \mathbf{1}_{(0,t_1]}(t)\mathbf{1}_{(0,t_2]}(t)\ \mathrm{d}t = \\ = t_1 \wedge t_2.$$

We see that after, as usual, taking continuous version, $\tilde{W}_t$ defined as $W(\mathbf{1}(0,t])$, $t \in [0,T]$ coincides with the standard Brownian motion $B = \{B_t, t \in [0,T]\}$.

Theorem 6. *Let $H_m(x)$ be the m-th Hermite polynomial and $h \in H = L^2(X)$, $\|h\|_H = 1$. Then*

$$m!H_m(W(h)) = (WI)\int_{X^m} h(t_1)\cdot h(t_2)\cdots h(t_m)\ \mathrm{d}(B_{t_1},\ldots,B_{t_m}). \tag{1.4}$$

Consequently WI_m maps $L^2(X)$ onto the m-th Wiener chaos.

Proof. For the proof see Nualart (2006, p. 13). ❑

Theorem 7. *Let $F \in L^2(\Omega, \mathcal{G}, P)$. Then F can be decomposed as follows:*

$$F = \sum_{n=0}^{\infty} WI_n(f_n), \tag{1.5}$$

where $WI_0(f) = E(f)$. By Lemma 1 $f_n \in L^2(X^n)$ can be without loss of generality taken symmetric and then the decomposition is unique.

Proof. (Nualart, 2006, p. 13). ❑

Relation between Wiener-Itô integral and classic Itô integral

In the following theorem we show the link between the concept of multiple Wiener-Itô integral and the classical Itô integral recalled in Chapter 0.

Theorem 8. *Let f_m be a real symmetric function in $L^2(X^m)$ and let $W(h) = (WI)\int_X h_s \, \mathrm{d}B_s, h \in L^2(X)$ as defined in the previous section. Then the multiple Wiener-Itô integral with respect to W coincides with the iterated Itô integral. It means that when we assume $X = \mathbb{R}^+$, then for $0 \le t_1 \le \cdots \le t_m$:*

$$WI_m(f_m) = m!(I)\int_0^\infty (I)\int_0^{t_m} \ldots (I)\int_0^{t_2} f_m(t_1, \ldots, t_m) \, \mathrm{d}B_{t_1} \ldots \mathrm{d}B_{t_m}. \quad (1.6)$$

Proof. The proof follows simply from the fact that the theorem clearly holds for step functions and in general case is treated by the approximation argument. (Nualart, 2006, p. 23).

❑

1.3 Malliavin derivative

In this section we define the Malliavin derivative operator and we mention its basic properties. Recall that $W = W(h), h \in H$ is an isonormal Gaussian process associated with a Hilbert space H. Also recall that we assume that W is defined on a complete probability space $(\Omega, \mathcal{G}, P)$, where $\mathcal{G}$ is generated by W.

Notation:

- $C_p^\infty(\mathbb{R}^n)$ denotes the set of all real functions on $\mathbb{R}^n$ which are infinitely continuously differentiable and all its partial derivatives have at most polynomial growth,
- $C_b^\infty(\mathbb{R}^n)$ denotes the set of real functions which are infinitely continuously differentiable, bounded and all its partial derivatives are also bounded,
- $C_0^\infty(\mathbb{R}^n)$ denotes the set of real functions which are infinitely continuously differentiable and have a compact support.

Definition 6. *Let a random variable F have the form*

$$F = f(W(h_1), \ldots, W(h_n)), \quad (1.7)$$

where $f \in C_p^\infty(\mathbb{R}^n), h_1, \ldots, h_n \in H$ and $n \ge 1$. Such F is called smooth random variable. The set of all smooth random variables is denoted $\mathcal{S}$. $\mathcal{S}_b$ and $\mathcal{S}_0$ denotes the class of smooth random variables of the form (1.7), where $f \in C_b^\infty(\mathbb{R}^n)$ and $f \in C_0^\infty(\mathbb{R}^n)$ respectively. Finally let $\mathcal{P}$ denote the class of smooth random variables of the form (1.7) such that f is a polynomial.

Remark. Clearly $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}_0 \subset \mathcal{S}_b \subset \mathcal{S}$. It is also obvious that $\mathcal{S}_0$ is dense in $L^2(\Omega)$ and due to the density of Hermite polynomials, which was shown in Section 1.1, $\mathcal{P}$ is also dense in $L^2(\Omega)$.

Now we define the derivative operator (Malliavin derivative) for a smooth random variable.

Definition 7. *Let F be a smooth random variable of the form* (1.7)*. The Malliavin derivative DF is the H-valued random variable given by*

$$DF = \sum_{i_1,\dots,i_m=1}^{n} \partial_i f(W(h_i),\dots,W(h_n))h_i \tag{1.8}$$

The next lemma represents one of the most important property of the Malliavin derivative.

Lemma 9. *Let F be a smooth random variable and $h \in H$. Then it holds that*

$$E(\langle DF, h\rangle_H) = E(FW(h)). \tag{1.9}$$

Remark. The above lemma is called "Integration by parts lemma". However, the name is not intuitive because the integration by parts formulae are usually of the form

$$G' \cdot H \sim G \cdot H',$$

but here we have

$$E(\langle DF, DW(h)\rangle_H) = E(FW(h))$$

which means both derivatives are on one side of the equation. But the proof of the lemma justifies the name.

Proof. Sketch of the proof: We proceed as in Nualart (2006, p. 26). We can normalize (1.9) and therefore assume without loss of generality that $\|h\| = 1$. We can also assume that F can be written as $F = f(W(e_1),\dots,W(e_n))$, where $f \in C_p^\infty(\mathbb{R}^n)$ and $h = e_1$ and $e_1,\dots,e_n$ are orthonormal elements of H. Let $\phi(x)$ denote the density of standard normal distribution on $\mathbb{R}^n$. Then

$$E(\langle DF, DW(h)\rangle_H) = \int_{\mathbb{R}^n} \partial_1 f(x)\phi(x)\ \mathrm{d}x = \int_{\mathbb{R}^n} f(x)\phi(x)x_1\ \mathrm{d}x =$$

$$= E(FW(e_1)) = E(FW(h)).$$

❑

The following result is a direct consequence of the previous lemma.

Corollary. Let F and G be two smooth random variables and $h \in H$. Then

$$E(G\langle DF, h\rangle_H) = E(-F\langle DG, h\rangle_H) + FGW(h).$$

Proof. The proof is a direct consequence of Lemma 9 when we apply it to FG. ❑

Now we want to extend the domain of the derivative operator. To do it, we need the following fact. As we can see, the previous lemma play a crucial role in the proof.

Proposition 10. *The Malliavin derivative is closable operator from $L^2(\Omega)$ to $L^2(\Omega; H)$.*

Proof. As in Nualart (2006, p. 26) we show that if $F_n, n \geq 1$ is a sequence of smooth random variables such that $F_n \to 0$ in $L^p(\Omega)$ and the sequence $DF_n \to \xi$ in $L^p(\Omega; H)$ then $\xi = 0$. Indeed, we can take $h \in H$ and a smooth random variable $F \in \mathcal{S}_b$ such that $FW(h)$ is bounded. Then by Lemma 9 it holds that:

$$E(\langle \xi, h\rangle_H F) = \lim_{n\to\infty} E(\langle DF_n, h\rangle_H F) =$$
$$= \lim_{n\to\infty} E(-F_n \langle DF, h\rangle_H) + E(F_n F W(h)) = 0$$

because F_n goes to zero and $\langle DF, h\rangle_H$ and $FW(h)$ are bounded. And hence $\xi = 0$.

❑

Now we are finally able to extend the operator D.

Definition 8. *Let $p \geq 1$. $\mathbb{D}^{1,p}$ denotes the closure of $\mathcal{S}$ with respect of the norm of the graph of D in $L^p(\Omega)$:*

$$\|F\|_{1,p} = (E(|F|^p) + E(\|DF\|_H^p))^{\frac{1}{p}}. \tag{1.10}$$

Definition 9. *If we take $p = 2$ in previous definition then $\mathbb{D}^{1,2}$ is a Hilbert space with scalar product*

$$\langle F, G, \rangle_{1,2} = E(FG) + E(\langle DF, DG\rangle_H).$$

The next proposition describes the connection between the Malliavin derivative and the concept of Wiener chaos expansion.

Proposition 11. *Let $F \in L^2(\Omega)$ with Wiener chaos representation $F = \sum_{n=0}^{\infty} K_n(F)$. Then $F \in \mathbb{D}^{1,2}$ if and only if*

$$E(\|DF\|_H^2) = \sum_{n=1}^{\infty} n\|K_n(F)\|_2^2 < \infty. \tag{1.11}$$

In that case we have for all $n \geq 1$ that $DK_n(F) = K_{n-1}(DF)$.

Proof. The proof is very technical and can be found in Nualart (2006, p. 28).

❑

Now we assume again the case that $H = L^2(X, \mathcal{B}, \mu)$ which is of high importance. Recall that μ is a σ-finite nonatomic measure. In this case, the Malliavin derivative of a random variable $F \in \mathbb{D}^{1,2}$ is a random process denoted as $\{D_t F, t \in X\}$.

Last but not least let us mention one very important lemma about the operator D in this case.

Lemma 12. *Let $A \in \mathcal{B}$ and $F \in \mathbb{D}^{1,2}$. If F is $\mathcal{F}_A$-measurable, then $D_t F = 0$ almost surely on $A^c \times \Omega$.*

Proof. For the proof see Nualart (2006, p. 34).

❑

1.4 The divergence operator

Now we define the Divergence Operator which we, in the case $H = L^2(X, \mathcal{B}, \mu)$, call the Skorohod integral.

Definition 10. *Let δ denote the adjoint operator of D. Then the Skorohod integral $\delta : L^2(\Omega; H)$ to $L^2(\Omega)$ such that:*

1. *The domain of δ, called $Dom(\delta)$, is the set of $u \in L^2(\Omega; H)$ which satisfy*

$$|E(\langle DF, u\rangle_H)| \leq c\|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$ and c is a constant depending on u. $\|F\|_2$ denotes the standard norm of $L^2(\Omega)$.

2. *For $u \in Dom\delta$ it holds that $\delta(u) \in L^2(\Omega)$ characterized by*

$$EF\delta(u) = E(\langle DF, u\rangle_H), F \in \mathbb{D}^{1,2}.$$

The operator δ is then the adjoint operator of D.

Remark. We write $(Sk) \int_X u \, \mathrm{d}W$ instead of $\delta(u)$. In case that $X = [0, T]$ we write $(Sk) \int_{[0,T]} u_t \, \mathrm{d}W_t$ and not $(Sk) \int_0^T u_t \, \mathrm{d}W_t$ because of the fact that u is Skorohod integrable does not imply that $u\mathbf{1}_{[0,t]}, t \in [0, T]$ is also Skorohod integrable. In the case of Skorohod integral 0 and T should not be considered as bounds of the integral.

Remark. The δ operator is closed because it is an adjoint of a densely defined operator. Moreover we can easily check by setting $F = 1$ in the above definition that $E\delta(u) = 0$.

Now the problem is that $Dom(\delta)$ is quite a complicated structure. So we want to have a subspace of $Dom(\delta)$ which is easily describable yet large enough to be useful. This is the point in the following proposition.

Proposition 13. *It holds that $\mathbb{D}^{1,2} \subset Dom(\delta)$.*

Proof. See Nualart (2006, p. 37, 38). ❑

The following result is quite useful when comparing different types of integrals.

Proposition 14. *Still assuming $H = L^2(X, \mathcal{B}, \mu)$ let $A \in \mathcal{B}$ and $F \in \mathbb{D}^{1,2}$. Moreover let $u \in L^2(\Omega; H)$ such that $u\mathbf{1}_A$ is in the domain of δ and $Fu\mathbf{1}_A \in L^2(\Omega; H)$. Then $Fu\mathbf{1}_A$ belongs to $Dom(\delta)$ and it holds that*

$$(Sk) \int_X Fu\mathbf{1}_A \, \mathrm{d}W = F(Sk) \int_X u\mathbf{1}_A \, \mathrm{d}W - \int_A D_t F u_t \, \mathrm{d}\mu_t \tag{1.12}$$

if the right side is square integrable.

Proof. For the proof see Nualart (2006, p. 40).

❑

The next proposition shows us the link between the Skorohod integral and the Wiener chaos expansion.

Proposition 15. *Let $u \in L^2(\Omega \times X)$. Then, as stated in Nualart (2006, p. 40), u has Wiener chaos expansion*

$$u(t) = \sum_{n=0}^{\infty} WI_n(\tilde{k}_n(\cdot, t)), \tag{1.13}$$

where for each $n \geq 1, k_n \in L^2(X^{n+1})$ is a symmetric function in the first n variables. It also holds that $u \in Dom(\delta)$ if and only if the sum

$$\sum_{n=0}^{\infty} WI_{n+1}(\tilde{k}_n)$$

converges in $L^2(\Omega)$. In that case

$$\delta(u) = \sum_{n=0}^{\infty} WI_{n+1}(\tilde{k}_n). \tag{1.14}$$

Proof. The proof is provided in Nualart (2006, p. 41).

❑

Definition 11. *The space $\mathbb{D}^{1,2}(L^2(X))$ is in the sequel denoted as $\mathbb{L}^{1,2}$.*

Remark. The previous proposition results in

$$E(\delta^2(u)) = \sum_{n=0}^{\infty} (n+1)! \|\tilde{k}_n\|^2_{L^2(X^{n+1})}$$

and hence the set of Skorohod integrable processes can be characterized as the set of u such that

$$\sum_{n=0}^{\infty} (n+1)! \|\tilde{k}_n\|^2_{L^2(X^{n+1})} < \infty \tag{1.15}$$

and

$$E(\delta(u)\delta(v)) = \int_X E(u_t v_t) \, \mathrm{d}\mu_t + \int_X \int_X E(D_s u_t D_t v_s) \, \mathrm{d}\mu_s \mathrm{d}\mu_t \tag{1.16}$$

whenever u, v belong to $\mathbb{L}^{1,2}$.

Proof. For proof see Nualart (2006, p. 42, 43).

❑

Relation between Skorohod and Itô integral

Now we explain why can be δ considered as a stochastic integral and we show its link to Itô stochastic integral. First let us note that when we take random process u of the form

$$u = \sum_{j=1}^{n} F_j h_j,$$

where F_j are smooth random variables and $h_j \in H$, then by Lemma 9 we can see that $u \in Dom(\delta)$ and aslo that:

$$\delta(u) = \sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H \tag{1.17}$$

(cf. Nualart (2006, p. 37)). Indeed, we can verify the definition of the δ operator. Let us take arbitrary $F \in \mathbb{D}^{1,2}$ and compute

$$EF\delta(u) = EF\left(\sum_{j=1}^{n} F_j W(h_j) - \sum_{j=1}^{n} \langle DF_j, h_j \rangle_H\right) =$$

$$= E\sum_{j=1}^{n} FF_j W(h_j) - \sum_{j=1}^{n} F \langle DF_j, h_j \rangle_H =$$

$$= E\sum_{j=1}^{n} FF_j W(h_j) - \sum_{j=1}^{n} FF_j W(h_j) + E\sum_{j=1}^{n} F_j \langle DF, h_j \rangle_H =$$

$$= E\left\langle DF, \sum_{j=1}^{n} F_j h_j \right\rangle_H = \langle DF, u \rangle_H .$$

The forth equality follows from the corollary of Lemma 9. The fact that the δ operator is defined uniquely completes the proof.

In the case of $H = L^2([0,T], \mathcal{B}, \lambda)$ we can rewrite (1.17) as Nualart (2006, p. 43)

$$(Sk)\int_X u_t \, \mathrm{d}B_t = \sum_{j=1}^{n} (Sk)\int_X F_j h_j(t) \, \mathrm{d}B_t - \sum_{j=1}^{n} \int_X D_t F_j h_j(t) \, \mathrm{d}\mu_t. \tag{1.18}$$

Now we consider stochastic basis $(\Omega, \mathcal{G}, \mathcal{G}_t, P)$ where $\mathcal{G}$ is σ-algebra generated by W and $\mathcal{G}_t$ is a filtration. Set $h_i = \mathbf{1}_{(t_i, t_{i+1}]}(t)$ where $(0 = t_1 < \cdots < t_n = T)$ is a partition of $[0,T]$ and F_i such that F_i is smooth and $\mathcal{G}_{t_i}$-measurable. We constructed an adapted elementary process on $[0,T]$. When we recall that $W(\mathbf{1}_{(t_i, t_{i+1}]}) = W(t_{i+1}) - W(t_i)$, Skorohod integral of u equals

$$(Sk)\int_{[0,T]} u_t \, \mathrm{d}B_t = \sum_{j=1}^{n} F_j(B(t_{j+1}) - B(t_j)) - \sum_{j=1}^{n} \int_X D_t F_j \mathbf{1}_{(t_j, t_{j+1}]}(t). \tag{1.19}$$

F_j is $\mathcal{G}_{t_j}$-measurable so as was shown in Lemma 12 $D_t F_j = 0$ for $t \geq t_j$ but for $t < t_j \mathbf{1}_{(t_j, t_{j+1}](t)}$ equals zero so the second sum in (1.19) equals zero. Therefore we have

$$(Sk)\int_{[0,T]} u_t \, \mathrm{d}B_t = \sum_{j=1}^{n} F_j(W(t_{j+1}) - W(t_j)). \tag{1.20}$$

Finally we see that for smooth adapted elementary random process u we have

$$(Sk)\int_{[0,T]} u_t \, \mathrm{d}B_t = (I)\int_0^T u_t \, \mathrm{d}B_t. \tag{1.21}$$

This result can be of course extended by means of approximation of all $L^2(\Omega)$ random variables by smooth random variables to all $\mathcal{G}_t$-adapted L^2 random processes. This approach really works because δ is closed.

1.5 Malliavin calculus with respect to fractional Brownian motion

This section is devoted to non-pathwise integration with respect to fractional Brownian motion.

Definition and basic properties of fractional Brownian motion

Definition 12. *Let $0 < H < 1$ and let $B^H = \{B^H_t, t \geq 0\}$ be a Gaussian process with zero mean and covariance function*

$$R_H(t,s) = E(B^H_t B^H_s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1.22}$$

Then B^H is called fractional Brownian motion (fBm) with Hurst parameter H.

Remark. A new problem with notation arises. In literature the Hurst index of the fractional Brownian motion is usually denoted H but the underlying Hilbert space is also usually denoted H. To avoid misunderstanding we denote the underlying Hilbert space $\mathscr{H}$.

Lemma 16. *Let B^H be fractional Brownian motion with Hurst parameter H. Then it holds that*

1. *For any $a > 0$ it is true that $\{a^{-H}B_{at}\}$ has the same distribution as B^H.*
2. *For any s, t it holds that $E(|B^H_s - B^H_t|^2) = |t-s|^{2H}$ and so fBm has stationary increments.*
3. *FBm admits a continuous version and the continuous fBm B^H has trajectories which are Hölder of order $H - \varepsilon$ for every $\varepsilon > 0$.*
4. *$B^{1/2}$ after taking continuous version coincides with standard Brownian motion.*
5. *For $H \neq \frac{1}{2}$ fBm is not a semimartingale.*

Proof. The proofs of the parts of this lemma are not difficult and can be found in Nualart (2006, p. 274 - 275).

❑

Definition 13. *Let* $u = \{u_t, t \in [0,T]\}$ *be a stochastic process and take a partition* $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. *Define*

$$V_p\left(u, \mathscr{P}\right) = \sum_{i=1}^{n} |u_{t_k} - u_{t_{k-1}}|^p.$$

The strong p-variation of u over $[0,T]$ *is defined as*

$$\mathcal{V}_p(u, [0,T]) = \sup_{\mathscr{P}} V_p\left(u, \mathscr{P}\right), \tag{1.23}$$

where $\mathscr{P}$ *denotes a finite partition of* $[0,T]$. *Moreover the index of p-variation of a stochastic process is defined as*

$$I(u, [0,T]) = \inf\{p > 0; \mathcal{V}_p(u, [0,T]) < \infty\}.$$

We also define the weak p-variation of u over $[0,T]$ *as*

$$\mathscr{V}_p(u, [0,T]) = \lim_{n\to\infty} \sum_{i=1}^{n} \left| u\left(\frac{Ti}{n}\right) - u\left(\frac{T(i-1)}{n}\right)\right|^p, \tag{1.24}$$

where the limit is taken in probability.

We refer to Øksendal (2008, p. 13) that

$$I(B^H, [0,T]) = \frac{1}{H}.$$

Moreover it is stated there that $\mathscr{V}_p(B^H, [0,T]) = 0$ when $pH > 1$ and $\mathscr{V}_p(B^H, [0,T]) = \infty$ if $pH < 1$.

As we know from the construction of classic Itô stochastic integral, the weak quadratic variation of the integrator plays an essential role. Let us have a closer look at the quadratic variation of fBm. As we can see in Nualart (2006, p. 275), there are three distinct cases:

- $H < 1/2$: In this case we choose $p > 2$ which satisfies $pH < 1$ and we see that the weak p-variation is infinite and hence the weak quadratic variation is also infinite.
- $H = 1/2$: The weak quadratic variation $\mathscr{V}_2(B^H, [0,T]) = T$.
- $H > 1/2$: Set p such that $\frac{1}{H} < p < 2$. We see that the weak p-variation is zero and so the weak quadratic variation is also zero. But if we choose $1 < p < \frac{1}{H}$ we can see that the total variation (weak 1-variation) is infinite.

Now set $X = [0,T]$. Let $B^H = \{B_t^H, t \in [0,T]\}$ be a fBm. Let $\mathcal{E}$ denote the set of all step functions on $[0,T]$. The underlying Hilbert space $\mathscr{H}$ is now defined as the completion of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \right\rangle_{\mathscr{H}} = R_H(t,s). \tag{1.25}$$

Now we can extend $\mathbf{1}_{[0,T]} \to B_t^H$ to an isometry between $\mathscr{H}$ and the Gausssian space $\mathcal{H}_1$ defined in Theorem 4. After that we see that $\left\{ B^H(\phi), \phi \in \mathscr{H} \right\}$ is a isonormal Gaussian process associated with $\mathscr{H}$ as defined in Definition 1. $B^H(\phi)$ may be understood as the integral if the deterministic element $\phi \in \mathscr{H}$ with respect to B^H.

Case $H > \frac{1}{2}$

We can easily check that

$$R_H(t,s) = c_H \int_0^t \int_0^s |u-v|^{2H-2} \, \mathrm{d}v\mathrm{d}u,$$

where $c_H = H(2H-1)$. And so for any two step funtions ϕ, ψ from $\mathcal{E}$ it holds that

$$\langle \phi, \psi \rangle_{\mathscr{H}} = c_H \int_0^T \int_0^T |u-v|^{2H-2} \phi_u \psi_v \, \mathrm{d}v\mathrm{d}u. \tag{1.26}$$

Now have a look at a square integrable kernel

$$K_H(t,s) = \alpha_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}. \tag{1.27}$$

It can be shown (see Øksendal (2008, p. 24)) that

$$\int_0^{s \wedge t} K_H(t,u) K_H(s,u) \, \mathrm{d}u = R_H(t,s). \tag{1.28}$$

Now let us introduce the operator K_H^*

$$(K_H^* \phi)(s) = \int_s^T \phi(t) \frac{\partial K_H}{\partial t}(t,s) \, \mathrm{d}t. \tag{1.29}$$

It holds that

$$(K_H^* \mathbf{1}_{[0,t]})(s) = K_H(t,s) \mathbf{1}_{[0,t]}(s)$$

(cf. Øksendal (2008, p. 30)) and moreover

$$\left\langle K_H^* \mathbf{1}_{[0,t]}, K_H^* \mathbf{1}_{[0,s]} \right\rangle_{L^2([0,T])} = R_H(t,s) = \left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathscr{H}}. \tag{1.30}$$

Therefore we can see that the operator K_H^* provides (after the standard extension) an isometry between $\mathscr{H}$ and $L^2([0,T])$. As a consequence of (1.30) it holds that

$$B_t^H = \int_0^t K_H(t,s) \, \mathrm{d}B_s, \tag{1.31}$$

where B_s denotes the standard Brownian motion. The above equality holds in law and also pathwise with

$$B_t = B^H((K_H^*)^{-1}(\mathbf{1}_{[0,T]}))$$

(see Nualart (2006, p. 279)). It follows that Øksendal (2008, p. 32)

$$B^H(\phi) = \int_0^T (K_H^*\phi)(s)\, \mathrm{d}B_s,\ \phi \in \mathcal{H}. \tag{1.32}$$

Now let us define an important subspace of $\mathscr{H}$. $\left|\mathscr{H}\right|$ denotes the set of all measurable functions which satisfy

$$\|\phi\|^2_{\left|\mathscr{H}\right|} = c_H \int_0^T \int_0^T |\phi_u|\, |\phi_v|\, |u-v|^{2H-2}\, \mathrm{d}u\mathrm{d}v < \infty. \tag{1.33}$$

Therefore the space $\left|\mathscr{H}\right|$ is continuously embedded into $\mathscr{H}$.

Case $H < \frac{1}{2}$

The approach issimilar and so we need to find a square integrable kernel which satisfies (1.28). The problem is solved in the following equation. In Øksendal (2008, p. 35) we can see that in case $H < \frac{1}{2}$ it holds that the kernel

$$K_H(t,s) = \alpha_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\, \mathrm{d}u \right] \tag{1.34}$$

satisfies (1.28). Let us now introduce the operator K_H^* for the case $H < \frac{1}{2}$.

$$(K_H^*\phi)(s) = K_H(T,s)\phi(s) + \int_s^T (\phi(t) - \phi(s)) \frac{\partial K_H}{\partial t}(t,s)\, \mathrm{d}s. \tag{1.35}$$

Again as in the case $H > \frac{1}{2}$, as we can see in Øksendal (2008, p. 37), it holds that standard Brownian motion B can be expressed as

$$B_t^H = \int_0^t K_H(t,s)\, \mathrm{d}B_s.$$

In this case it holds that

$$\mathscr{H} \hookleftarrow \left|\mathscr{H}\right| \hookleftarrow C^\varepsilon([0,T]), \varepsilon > \frac{1}{2} - H,$$

where $C^\varepsilon([0,T])$ denotes the set of Hölder continuous functions of order ε (cf. Nualart (2006, p. 287)).

Remark. Note that in the case $H = \frac{1}{2}$ of course $\mathscr{H} = L^2([0,T])$.

1.5.1 Derivative and Divergence operator with respect to fractional Brownian motion

Now we investigate the relation between the operators D and δ with respect to fBm and standard Brownian motion. Let us denote D^B and δ^B the derivative and divergence operators with respect to the standard Brownian motion and D and δ those with respect to fractional Brownian motion. The following two results from Nualart (2006, p. 288) are called transfer principles.

Regarding derivative, the smooth random variables are now of the form

$$F = f(B^H(h_1), \dots, B^H(h_n))$$

and the Malliavin derivative with respect to B^H is defined as

$$DF = \sum_{i=1}^{n} \partial_i f(B^H(h_i), \dots, B^H(h_n))h_i.$$

Regarding the divergence operator the following two results hold.

Proposition 17. *It holds that*

$$K_H^* DF = D^B F, \tag{1.36}$$

whenever $F \in \mathbb{D}^{1,2}$.

Proposition 18.

$$Dom(\delta) = (K_H^*)^{-1}(Dom(\delta^B)) \tag{1.37}$$

and for any $u \in Dom(\delta)$ *this equality holds*

$$\delta(u) = \delta^B(K_H^* u). \tag{1.38}$$

Both transfer principles follow from the fact that

$$B^H(\phi) = \int_0^t K_H(t, s) \, \mathrm{d}B_s.$$

Maximal inequalities

In this section we state a couple of maximal inequalities for the divergence integral in case of the fractional Brownian motion.

Theorem 19. *Let* $H > \frac{1}{2}$ *and* u *be a stochastic process in* $\mathbb{D}^{1,2}\left(\left|\mathscr{H}\right|\right)$. *Then we have that*

$$E(|\delta(u)|^p) \le C_{H,p} \left(\|Eu\|^p_{|\mathscr{H}|} + E\left(\|D(u)\|^p_{|\mathscr{H}|\otimes|\mathscr{H}|} \right) \right), \tag{1.39}$$

where $C_{H,p}$ *is a constant depending on* H *and* p *and*

$$\|\phi\|_{|\mathscr{H}|\otimes|\mathscr{H}|} = c_H^2 \int_{[0,T]^4} |\phi_{r,\theta}|\phi_{u,\eta}|r-u|^{2H-2}|\theta-\eta|^{2H-2} \, \mathrm{d}r\mathrm{d}u\mathrm{d}\theta\mathrm{d}\eta.$$

Proof. For the proof see Nualart (2003, p. 19). ❑

Theorem 20. *Let $H > \frac{1}{2}$ and u be a stochastic process in $\mathbb{D}^{1,2}\left(\left|\mathscr{H}\right|\right)$ which satisfies*

$$\|u\|_{p,1} = \left[\int_0^T E(|u_s|^p)\ \mathrm{d}s + E\left(\int_0^T \left(\int_0^T |D_r u_s|^{\frac{1}{H}}\ \mathrm{d}r\right)^{pH}\ \mathrm{d}s\right)\right]^{\frac{1}{p}} < \infty.$$

Then the maximal inequality

$$E\left(\sup_{t\in[0,T]}\left|(Sk)\int_0^t u_s\ \mathrm{d}B_s^H\right|\right) < C\|u\|_{p,1}^p \tag{1.40}$$

holds. C denotes a constant depending on p, H and T. $(Sk)\int_0^t u_s\ \mathrm{d}B_s^H$ denotes $(Sk)\int_{[0,T]} u_s \mathbf{1}_{[0,t]}(s)\ \mathrm{d}B_s^H$.

Proof. For the proof we refer to Nualart (2003, p. 19.). ❑

Remark. The above theorem implicitly says that under its assumptions $u_s\mathbf{1}_{[0,t]}$ is Skorohod integrable which is not always true.

Now we want to define an analogy of the Wiener-Itô integrals with respect to fBm and not only standard Brownian motion as above. To do it we have to choose a different approach because the model of white noise is not useful here because the fBm has correlated increments over disjoint intervals and hence the condition $EW(A)W(B) = \mu(A \cap B)$ is too restrictive. To define the multiple integrals for an arbitrary isonormal Gaussian process we need the following definitions.

Definition 14. *Let W be an arbitrary isonormal Gaussian process. For a smooth random variable F defined in (1.7) and a positive integer m we define the m-th Malliavin derivative as the $\mathscr{H}^{\otimes m}$-valued random variable satisfying*

$$D^m F = \sum_{i=1}^n \frac{\partial^m}{\partial_{x_{i_1}} \dots \partial_{x_{i_n}}} f(W(h_i), \dots, W(h_n)) h_{i_1} \otimes \dots \otimes h_{i_n}, \tag{1.41}$$

where $\otimes$ denotes the tensor product:

$$f \otimes g(h_1, h_2) = f(h_1) \cdot g(h_2).$$

Remark. The m-th Malliavin derivative has similar properties as the classic Malliavin derivative as closability etc., cf. Nourdin and Peccati (2012, p. 26).

Definition 15. *Let m be a positive integer. We denote by $Dom(\delta^m)$ the subspace of $L^2\left(\Omega; \mathscr{H}^{\otimes m}\right)$ containing elements u satisfying for every smooth random variable F*

$$|E\left\langle D^m F, u\right\rangle_{\mathscr{H}^{\otimes m}}| \le c\|F\|_2,$$

where c is a constant depending on u.

For an $u \in Dom(\delta^m)$ the m-th divergence operator δ^m is then the unique element of $L^2(\Omega)$ defined by

$$EF\delta^m = E \langle D^m F, u \rangle_{\mathscr{H}^{\otimes m}} \tag{1.42}$$

for any smooth random variable F.

Remark. Similarly in the above remark, even the m-th divergence operator has similar properties as the classic operator δ. For more details see Nourdin and Peccati (2012, p. 30).

Now let us consider the set $\mathcal{E}$ of step functions on $[0,T]$. The underlying Hilbert space $\mathscr{H}$ is now the completion of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathscr{H}} = R_H(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

Consider an isonormal Gaussian process W on such $\mathscr{H}$. From the construction we can see that the process $\{W(\mathbf{1}_{[0,t]}), t \in [0,T]\}$ is a family of centered Gaussian variables with the same covariance structure as the fBm B^H. Hence those two processes, after taking continuous version, coincide. Therefore $W(h), h \in \mathscr{H}$ can be considered as a stochastic integral of a deterministic function h with respect to the fractional Brownian motion B^H.

Now denote by $\mathcal{E}_m$ the set of all step functions on $\mathscr{H}^{\otimes m}$ of the form

$$f(t_1, \dots, t_m) = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots,i_m} \mathbf{1}_{A_{i_1} \times \dots \times A_{i_m}}(t_1, \dots, t_m), \tag{1.43}$$

where A_i's are pairwise disjoint Borel sets on $[0,T]$. For such function f we define the multiple integral in a similar way as in Definition 5:

$$I_m f = \sum_{i_1,\dots,i_m=1}^{n} a_{i_1,\dots,i_m} W(\mathbf{1}_{A_{i_1}}) \times \dots \times W(\mathbf{1}_{A_{i_m}}). \tag{1.44}$$

Now we use the fact that for all $m \geq 1$ the set $\mathcal{E}_m$ is dense in $\mathscr{H}^{\otimes m}$ (see Nourdin and Peccati (2012, p. 34)) an extend the I_m operator to the whole $\mathscr{H}^{\otimes m}$. This approach is really correct due to the fact that for a step function f of the form (1.43) it holds that

$$I_m f = \delta^m f$$

(cf. Nourdin and Peccati (2012, p. 34)). Now we can use the closability of δ^m and really extend the operator I_m to $\mathscr{H}^{\otimes m}$. Therefore we have the complete analogy of the Wiener-Itô integral for the fractional Brownian motion B^H.

Chapter 2

Pathwise integrals

This chapter is devoted to pathwise stochastic integrals. At first we need some preliminaries and then we define various types of pathwise stochastic integrals, compare them and apply them to fBm. All the conceptions have in common that they try to overcome problems with infinite total variation of the integrator.

2.1 Preliminaries

Let us now define a few terms which is very useful in the sequel.

Definition 16. *Let $f \in L^1$ and $\alpha > 0$ and fix an interval (a, b). We define the left- and right-sided Liouville integrals as*

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) \, \mathrm{d}y, \tag{2.1}$$

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) \, \mathrm{d}y, \tag{2.2}$$

respectively for λ-almost all $x \in (a, b)$. Γ denotes the Gamma function.

We refer to Zähle (1998, p. 337) that for both forward and backward Liouville integrals we have that

$$I^{\alpha}(I^{\beta} f) = I^{\alpha+\beta} f. \tag{2.3}$$

The equation (2.3) is called the composition formula. As we defined Liouville integrals now we define the inverse operation, Liouville derivative.

Definition 17. *Let $0 < \alpha < 1$. The left- and right-sided Liouville derivatives of order α are defined as*

$$D_{a+}^{\alpha} f(x) = \mathbf{1}_{(a,b)}(x) \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} \, \mathrm{d}y, \tag{2.4}$$

$$D_{b-}^{\alpha} f(x) = \mathbf{1}_{(a,b)}(x) \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b \frac{f(y)}{(x-y)^{\alpha}} \, \mathrm{d}y. \tag{2.5}$$

The composition formula for both types of Liouville derivatives also holds (see Zähle (1998, p. 339)) and it reads

$$D^{\alpha}(D^{\beta} f) = D^{\alpha+\beta} f. \tag{2.6}$$

Now we mention some basic but very important properties which may clarify the names "derivative" and "integral".

Lemma 21. *For both right- and left- sided derivatives and integrals and for a suitable f we have that*

$$I^\alpha(D^\alpha f) = f,$$
$$D^\alpha(I^\alpha f) = f.$$

The second equation holds for any $f \in L^1$, the first one for any function which can be interpreted as I^α-integral of an L^1 function. Moreover

$$\lim_{\alpha\to 1} D^\alpha f(x) = f'(x), f \in C^1$$

$$\lim_{\alpha\to 0} D^\alpha g(x) = g(x).$$

For $g \in L^1$ we have

$$\lim_{\alpha\to 0} I^\alpha_{a+} g(x) = g(x+),$$
$$\lim_{\alpha\to 0} I^\alpha_{b-} g(x) = g(x-),$$

where $g(x+)$ denotes $\lim_{\varepsilon\to 0+} g(x+\varepsilon)$ and similarly $g(x-) = \lim_{\varepsilon\to 0+} g(x-\varepsilon)$, provided that those limits exist.

Proof. The proof is mostly straight calculation. For more details see Zähle (1998, p. 341).

❑

We use the following notation.

Definition 18. *For functions f, g we define*

$$f_{a+}(x) = \mathbf{1}_{(a,b)}(x)(f(x) - f(a+)), \tag{2.7}$$
$$g_{b-}(x) = \mathbf{1}_{(a,b)}(x)(g(x) - f(b-)), \tag{2.8}$$

provided the limits exist.

2.2 Riemann-Stieltjes integral

In this short section we show a theorem which allows us to define a pathwise integral with respect to fractional Brownian motion in case we have Hölder functions.

Definition 19. *Let $f : [a,b] \to \mathbb{R}$ be a function. We say that f is Hölder continuous of order α if there exist nonnegative real constants α, C such that for all $t_1, t_2 \in [a,b]$*

$$|f(t_1) - f(t_2)| \le C|t_1 - t_2|^\alpha$$

holds. The set of all Hölder continuous functions of order α on $[a,b]$ is denoted $C^\alpha([a,b])$.

Definition 20. *Let $\mathscr{K}$ be a finite sequence of numbers such that*

$$\mathscr{K} = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}.$$

Moreover let

$$a = \alpha_0 < \alpha_1 < \dots < \alpha_k < b$$

and

$$\alpha_{j-1} \le \tau_j \le \alpha_j, \;\; j = 1, 2, \dots, k,$$

then we call $\mathscr{K}$ Kurzweil partition of the finite interval $[a, b]$.

Definition 21. *Let $\mathscr{K}$ be a Kurzweil partition of the interval $[0, T]$ as defined above and u, v be two stochastic processes. We define (pathwise) the Riemann-Stieltjes integral sum as*

$$RS\left(u, v, \mathscr{K}, [0, T]\right) = \sum_{j=1}^{k} u(\tau_j)(v(\alpha_j) - v(\alpha_{j-1})).$$

We say that the Riemann-Stieltjes integral of u with respect to v exists and equals $I \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\gamma > 0$ such that for all Kurzweil partitions $\mathscr{K}_\gamma$ satisfying $\max_{j=1,\dots,n}\{\alpha_j - \alpha_{j-1}\} < \gamma$

$$\left| RS\left(u, v, \mathscr{K}_\gamma, [0, T]\right) \right| < \varepsilon \tag{2.9}$$

holds. The Riemann-Stieltjes integral I wil be denoted

$$(RS)\int_0^T u_t \; \mathrm{d}v_t. \tag{2.10}$$

Theorem 22. *Let $\{u_t, t \in [0, T]\}, \{v_t, t \in [0, T]\}$ be random processes defined on $(\Omega, \mathcal{F}, P)$ with Hölder sample paths such that $u \in C^\nu([0, T]), v \in C^\upsilon([0, T])$ and $\nu, \upsilon > 0$ and $\nu + \upsilon > 1$. Then the Riemann-Stieltjes integral*

$$(RS)\int_0^T u_t \; \mathrm{d}v_t \tag{2.11}$$

exists (pathwise).

Proof. Let $\mathscr{P}_\Delta$ denote the set of all partitions $\mathscr{P} = \{0 = t_1 < t_2 < \dots < t_n = T\}$ of the interval $[0, T]$ which satisfy $\sup_i (t_{i+1} - t_i) < \Delta$. For arbitrary partition $\mathscr{P}$ and a random process u we define the approximating step function

$$\hat{u}_{\mathscr{P}} = \sum_{i=1}^{\infty} u(t_i)\mathbf{1}_{(t_i, t_{i+1}]}.$$

Obviously for continuous functions it holds that

$$\|\hat{u}_{\mathscr{P}} - u\|_{L^\infty([0,T])} \overset{\Delta \to 0}{\longrightarrow} 0, \; \mathscr{P} \in \mathscr{P}_\Delta$$

uniformly in Δ. It is well known that in the case of continuous function it is sufficient to investigate the behaviour of Riemann-Stieltjes sums. Let us calculate

$$\sup_{\mathscr{P}_\Delta} \left| \sum_{i=1}^{n} u(t_i^*)(v(t_{i+1}) - v(t_i)) - \sum_{i=1}^{n} u(t_i)(v(t_{i+1}) - v(t_i)) \right| \leq$$

$$\leq \sup_{\mathscr{P}_\Delta} \sum_{i=1}^{n} |u(t_i^*) - u(t_i)| \cdot |v(t_{i+1}) - v(t_i)| \leq$$

$$\leq C(\nu)C(\upsilon) \sup_{\mathscr{P}_\Delta} \sum_{i=1}^{n} (t_{i+1} - t_i)^{\nu+\upsilon} \leq C(\nu)C(\upsilon)T\Delta^{\nu+\upsilon-1} \overset{\Delta\to 0}{\longrightarrow} 0,$$

where $C(\nu), C(\upsilon)$ denote the Hölder constants of u and v and t_i^* is an arbitrary point in $(t_i, t_{i+1}]$. And therefore the existence of Riemann-Stieltjes integral is proven.

❑

Now we apply this theorem to fractional Brownian motion. Recall that the trajectories of fBm with Hurst parameter H are Hölder of order $H - \varepsilon$ for every $\varepsilon > 0$. So when we take a stochastic process u which has Hölder trajectiories of order greater than $1 - H$ we can define the stochastic integral

$$(RS) \int_0^T u_t \, \mathrm{d}B_t^H. \tag{2.12}$$

2.3 Fractional integral

In this section we use another approach in order to define a pathwise integral. Recall the Definition 18.

Definition 22. *Let f, g be two functions then we define*

$$(Fr) \int_a^b f(x) \, \mathrm{d}g(x) = (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) \, \mathrm{d}x + f(a+)(g(b-) - g(a+)), \tag{2.13}$$

whenever $f_{a+} \in I_{a+}^\alpha(L^p), g_{b-} \in I_{b-}^{1-\alpha}(L^q)$ such that $\frac{1}{p} + \frac{1}{q} \leq 1, 0 \leq \alpha \leq 1$. Here the spaces $I_{a+}^\alpha(L^p)$ and $I_{b-}^{1-\alpha}(L^q)$ denote the sets of function which can be represented as an I_{a+}^α-integral of a function from L^p, respectively as an $I_{b-}^{1-\alpha}$-integral of a function from L^q. For $p > 1$ it holds that $I_{a+}^\alpha(L^p) \subset L^q$, where $1/q = 1/p - \alpha$ (see Zähle (1998, p. 338)).

Remark. We refer to Zähle (1998, p. 338) that the definition is correct, which means independent of the choice of α.

As we can see the fractional integral defined in (2.13) is directed because of the choice of left- and right- sided derivatives of f and g. Similarly we could define the integral

$$(F) \int_a^b \mathrm{d}g(x) f(x) = (-1)^{-\alpha} \int_a^b D_{b-}^\alpha f_{b-}(x) D_{a+}^{1-\alpha} g_{a+} \, \mathrm{d}x + f(b-)(g(b-) - g(a+)) \tag{2.14}$$

for f, g such that $f_{b-} \in I_{b-}^{\alpha'}\left(L^{p'}\right), g_{a+} \in I_{a+}^{1-\alpha'}\left(L^{q'}\right)$ for some $\frac{1}{p'} + \frac{1}{q'} \leq 1$, $0 \leq \alpha' \leq 1$.

Proposition 23. *If the functions f, g satisfy both the conditions for* (2.13) *and* (2.14) *then the fractional integrals defined in* (2.13) *and* (2.14) *coincide.*

Proof. The proof can be found in Zähle (1998, p. 346). ❑

Link between the Riemann-Stieltjes integral and fractional integral

It is a natural question how the Riemann-Stieltjes and fractional integral behave when they both exist. The following result solves this question for Hölder continuous functions.

Theorem 24. *Assuming the situation in Theorem 22 we already know that the Riemann-Stieltjes integral exists. The fractional integral also exists and agrees the Riemann-Stieltjes integral which means*

$$(RS)\int_0^T u_t \, \mathrm{d}v_t = (Fr)\int_0^T u_t \, \mathrm{d}v_t. \tag{2.15}$$

Proof. The proof can be seen in Zähle (1998, p. 349). ❑

Application of fractional integral to stochsatic calculus

Now we want to apply the concept of fractional integral to stochastic calculus. Again we move to our fixed interval $[0, T]$. We refer to Zähle (1998, p. 354) that the integral

$$(Fr)\int_0^T u(s) \, \mathrm{d}B_s^H$$

exists almost surely for any measurable random function u on $[0, T]$ which satisfies
$u_{0+} \in I_{0+}^{\alpha}(L^1([0, T]))$, where $\alpha > 1 - H$. That means no requirement on adaptedness is needed.

At first we show how the classic Itô integral and fractional integral are related.

Theorem 25. *Let B be a standard Brownian motion and u a random process adapted to its filtration. Assume that $u \in I_{0+}^{\alpha}(L^2([0, T]))$ for some $\alpha > \frac{1}{2}$. Then we have*

$$(I)\int_0^T u(t) \, \mathrm{d}B_t = (Fr)\int_0^T u(t) \, \mathrm{d}B_t \tag{2.16}$$

almost surely. Recall that, as was stated in the Chapter 0, we assume the continuous version of the Itô integral.

Proof. It can be seen in Zähle (1998, p. 355) that if we have a continuously differentiable process u, then we can use integration by parts formula and obtain that both sides of (2.16) equal to

$$-\int_0^T u_t' B_t \, \mathrm{d}t + u_T B_T.$$

The general case is solved by means of approximation.

❑

According to Lemma 21 for a suitable function g it holds that

$$\lim_{\alpha \to 0} D^\alpha g(x) = g(x).$$

When we want to establish a similar formula for the Liouville integral

$$\lim_{\alpha \to 0} I^\alpha g(x) = g(x),$$

the situation is slightly more complicated but the following result shows us that an "integral version" of this formula holds.

Proposition 26. *Let u be an adapted stochastic process which satisfies $u \in I_{0+}^\alpha(L^2)$ for some $\alpha < \frac{1}{2}$ almost surely. Then*

$$P - \lim_{\varepsilon \searrow 0} \int_0^T I_{0+}^\varepsilon u_t \, \mathrm{d}B_t = (I) \int_0^T u_t \, \mathrm{d}B_t. \tag{2.17}$$

Proof. We refer to Zähle (1998, p. 356) for the proof.

❑

Relation between Skorohod and fractional integral

Let us investigate the link between the Skorohod and the fractional integral in case the integrator is the standard Brownian motion B.

Definition 23. *Let $0 < \alpha < 1$. Then we define the class $I_{0+}^\alpha(n, L^2)$ of functions from $L^2([0,T]^{n+1})$ which are symmetric in the first n arguments and can be interpreted as I_{0+}^α-integral with respect to the last variable of an $L^2([0,T]^{n+1})$ function.*

Theorem 27. *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process with representation* (1.13)*, where for every $n \geq 1$ it is true that $\tilde{k}_n \in I_{0+}^\alpha(n, L^2)$ for some $\alpha > \frac{1}{2}$, then the following equality holds almost surely*

$$(Fr) \int_0^T u_t \, \mathrm{d}B_t = \delta(u) + \sum_{n=1}^\infty n \int_0^T W I_{n-1}(\tilde{k}_n(\cdot, t, t)) \, \mathrm{d}t, \tag{2.18}$$

where δ denotes the divergence operator with respect to the standard Brownian motion.

Proof. The results follows from the fact that (see Zähle (1998, p. 360)) for processes u with such representation the right side is well determined and from the fact that

$$(Fr)\int_0^T WI_n(\tilde{k}_n(\cdot,t))\ \mathrm{d}B_t = WI_{n+1}(\tilde{k}_n) + n\int_0^T WI_{n-1}(\tilde{k}_n(\cdot,t,t))\ \mathrm{d}t$$

and the equation (1.14) completes the proof.

❑

The above equality suggests the following definition of a new integral.

Definition 24. *Let* $u = \{u_t, t \in [0,T]\}$ *be a stochastic process with representation* (1.13)*. We define the anticipating integral of* u *with respect to the standard Brownian motion* B *as*

$$(A)\int_0^T u_t\ \mathrm{d}B_t = \delta(u) + \sum_{n=1}^{\infty} n\int_0^T WI_{n-1}(\tilde{k}_n(\cdot,t,t))\ \mathrm{d}t. \qquad (2.19)$$

Chapter 3

Other types of integrals

In this chapter we see other types of stochastic integrals with conception which differs from those in the first two chapters.

3.1 Stratonovich integral

In this section we investigate the concept of so called Stratonovich integral. Let us introduce an approximation.

Definition 25. *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process such that $\int_0^T |u_t| \, \mathrm{d}t < \infty$ a.s. and let $\mathscr{P}$ be partition of the interval $[0,T]$. We define the approximating family of processes $\hat{u}^{\mathscr{P}}$ as*

$$\hat{u}^{\mathscr{P}}(t) = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} u_s \, \mathrm{d}s \right) \mathbf{1}_{(t_i, t_{i+1}]}(t). \tag{3.1}$$

Remark. In the sequel we need the convergence of the approximating family $\hat{u}^{\mathscr{P}}$. The family is indexed by set of partitions and hence forms a net. Indeed, it is not a sequence because the set of partitions is not linearly ordered. When we write that something converges as $|\mathscr{P} \to 0|$ we understand it that the convergence holds for any sequence of partitions whose norm tends to zero.

Lemma 28. *The family $u^{\mathscr{P}}$ converges to u in the norm of the space $L^2([0,T] \times \Omega)$ as $|\mathscr{P}| \to 0$. This convergence holds also in $\mathbb{L}^{1,2}$ if $u \in \mathbb{L}^{1,2}$. Recall that $\mathbb{L}^{1,2}$ denotes the space $\mathbb{D}^{1,2}(L^2([0,T]))$, where $\mathbb{D}^{1,2}$ is defined in Definition 9.*

Proof. The proof can be found in Nualart (2006, p. 171). ❑

Now let us define the partial sum for the oncoming Stratonovich integral.

Definition 26. *Let u be a stochastic process with approximation as in (3.1). Then we define the partial sum $S^{\mathscr{P}}$ as follows*

$$S^{\mathscr{P}}(u) = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} u_s \, \mathrm{d}s \right) (B(t_{i+1}) - B(t_i)). \tag{3.2}$$

Now we are ready to define the Stratonovich integral.

Definition 27. *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process such that $\int_0^T |u_t|\ \mathrm{d}t < \infty$ a.s. We say that u is Stratonovich integrable when the family $S^{\mathscr{P}}(u)$ converges in probability as $|\mathscr{P}| \to 0$. The limit is then called the Stratonovich integral and is denoted*

$$(St) \int_0^T u_t\ \mathrm{d}B_t. \tag{3.3}$$

The set of all Stratonovich integrable functions is rather complicated. It is not sufficient for a process to be in $\mathbb{L}^{1,2}$ to be Stratonovich integrable. We follow the approach in Nualart (2006, p. 173) to establish a reasonable class of Stratonovich integrable functions. Now let u be a stochastic process in $\mathbb{L}^{1,2}$ and $1 \leq p \leq 2$. We denote by D^+u (resp. D^-u) the element of $L^p([0,T] \times \Omega)$ satisfying

$$\lim_{n\to\infty} \int_0^T \sup_{s<t\leq(s+\frac{1}{n})\wedge T} E(|D_s u_t - (D^+u)_s|^p)\ \mathrm{d}s = 0, \tag{3.4}$$

resp.

$$\lim_{n\to\infty} \int_0^T \sup_{s-\frac{1}{n}\vee 0\leq t<s} E(|D_s u_t - (D^-u)_s|^p)\ \mathrm{d}s = 0, \tag{3.5}$$

where D denotes the Malliavin derivative with respect to the standard Brownian motion.

We denote by $\mathbb{L}^{1,2}_{p+}$ (resp. $\mathbb{L}^{1,2}_{p-}$) the class of processes in $\mathbb{L}^{1,2}$ such that (3.4) (resp. (3.5)) holds. We define $\mathbb{L}^{1,2}_{p}$ as $\mathbb{L}^{1,2}_{p+} \cap \mathbb{L}^{1,2}_{p-}$. For $u \in \mathbb{L}^{1,2}_{p}$ we set

$$(\nabla u)_t = (D^+u)_t + (D^-u)_t. \tag{3.6}$$

As we can see in Nualart (2006, p. 173), if the mapping $(s,t) \to D_s u_t$ is continuous from the neighbourhood of the diagonal $N_\varepsilon = \{s,t : |s-t| < \varepsilon\}$ into $L^p(\Omega)$, then $u \in \mathbb{L}^{1,2}_{p}$ and $D^+u = D^-u = Du$. It holds that processes from the space $\mathbb{L}^{1,2}_{1}$ are Stratonovich integrable.

Relation between Stratonovich and Skorohod integral

The following result shows us how are the concepts of Stratonovich and Skorohod integral related in case of $u \in \mathbb{L}^{1,2}_{1}$.

Theorem 29. *Let u be a measurable stochastic process in $\in \mathbb{L}^{1,2}_{1}$, then both Stratonovich and Skorohod integral exist and it holds that*

$$(St) \int_0^T u_t\ \mathrm{d}B_t = (Sk) \int_0^T u_t\ \mathrm{d}B_t + \frac{1}{2} \int_0^T (\nabla u)_t\ \mathrm{d}t. \tag{3.7}$$

Proof. The proof can be found in Nualart (2006, p. 175). ❑

Remark.

- If the mapping $(s,t) \to D_s u_t$ is continuous from the neighborhood of the diagonal $N_\varepsilon = \{s,t : |s-t| < \varepsilon\}$ into $L^p(\Omega)$ then (3.7) has the form

$$(St)\int_0^T u_t \, \mathrm{d}B_t = (Sk)\int_0^T u_t \, \mathrm{d}B_t + \int_0^T D_t u_t.$$

- If u is a continuous semimartingale, then we have

$$(St)\int_0^T u_t \, \mathrm{d}B_t = (Sk)\int_0^T u_t \, \mathrm{d}B_t + \frac{1}{2}\langle u, B\rangle_t, \tag{3.8}$$

where $\langle u, B\rangle_t$ denotes the covariation between u and the standard Brownian motion B.

Remark. The relation (3.7) and suggest another approach how the Stratonovich integral can be defined. If we take a stochastic process u of the form (1.13) and take into account Proposition 11 and (1.14) we could define the Stratonovich integral as

$$\sum_{n=0}^{\infty}\left(WI_{n+1}(\tilde{k}_n) + \frac{n}{2}\int_0^T \left(WI_{n-1}(\tilde{k}_n(\cdot,t,t-)) + WI_{n-1}(\tilde{k}_n(\cdot,t,t+))\right) \mathrm{d}t\right) \tag{3.9}$$

whenever the sum on the right side converges in the mean square.

3.2 L^2-integral

In this section we show the possibility of integration using the Fourier coefficients. We have the same setup as at the beginning of Chapter 1. We have a space $(X, \mathcal{B}, \mu)$ and a Gaussian measure
$W = \{W(A), A \in \mathcal{B}, \mu(A) < \infty\}$. We also consider the Hilbert space
$H = L^2(X, \mathcal{B}, \mu)$. We can fix a complete orthonormal system $\{e_i, i \geq 1\}$ in H. We can pathwise compute the (random) Fourier coefficients of $u \in L^2(X \times \Omega)$ as

$$u(t) = \sum_{i=1}^{\infty} \langle u, e_i\rangle_H \, e_i(t) \tag{3.10}$$

and define the L^2-integral as

$$(L^2)\int_X u \, \mathrm{d}W = \sum_{i=1}^{\infty} \langle u, e_i\rangle_H \, W(e_i), \tag{3.11}$$

provided that the sum converges in probability and the result does not depend on the choice of the complete orthonormal system.

Now we return to the case when $X = [0,T]$ and the process W as an isonormal Gaussian process coincides with the standard Brownian motion. In this case we have the L^2 integral with respect to the Brownian motion:

$$(L^2)\int_0^T u_t \, \mathrm{d}B_t.$$

Link between L^2-integral and Stratonovich integral

The following result shows us, that the concept of L^2 integral is closely related to Stratonovich integral.

Theorem 30. *Let u be measurable stochastic process such that $\int_0^T u_t^2 \, \mathrm{d}t < \infty$ a.s. Then both L^2 integral and Stratonovich integral exist and*

$$(St) \int_0^T u_t \, \mathrm{d}B_t = (L^2) \int_0^T u_t \, \mathrm{d}B_t. \tag{3.12}$$

Proof. For proof see Nualart (2006, p. 176).

❑

3.3 Three integrals theorem

Let us define a special class of functions called Slobodetsky-type space.

Definition 28. *Let us denote by $\mathbb{W}_{2,+}^{\alpha}$ the space of random processes u which satisfy*

1.

$$E(u(0+)^2) < \infty.$$

2.

$$E \int_0^T \frac{(u(t) - u(0+))^2}{t^{2\alpha}} \, \mathrm{d}t < \infty.$$

3.

$$E \int_0^T \int_0^T \frac{(u(t) - u(s))^2}{|t - a|^{2\alpha+1}} \, \mathrm{d}s\mathrm{d}t < \infty,$$

where $0 < \alpha < 1$.

Remark. Note that it is not hard to check the first two properties in the above definition imply that

$$E \int_0^T u(t)^2 \, \mathrm{d}t < \infty.$$

The following result shows us the behaviour of three stochastic integrals on the above defined Slobodeckij-type space.

Theorem 31. *Let $u \in \mathbb{W}_{2,+}^{\alpha}$ for an $\alpha > \frac{1}{2}$. Then the fractional integral, Stratonovich integral and anticipating integral exist and*

$$(A) \int_0^T u_t \, \mathrm{d}B_t = (St) \int_0^T u_t \, \mathrm{d}B_t = (Fr) \int_0^T u_t \, \mathrm{d}B_t \tag{3.13}$$

holds.

Proof. For the proof see Zähle (1998, p. 365).

❑

3.4 Symmetric, forward and backward integral

Let us return to the fractional Brownian motion.

Definition 29. *Let $0 < H < 1$. Moreover let $u = \{u_t, t \in [0, T]\}$ be a stochastic process with integrable sample paths. Provided the limits on the right side exist in probability, we define the symmetric integral of u with respect to B^H as*

$$(Sy) \int_0^T u_t \, \mathrm{d}B_t^H = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^T u_t (B_{t+\varepsilon}^H - B_{t-\varepsilon}^H) \, \mathrm{d}t, \tag{3.14}$$

forward integral as

$$(Fo) \int_0^T u_t \, \mathrm{d}B_t^H = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^T u_t (B_{t+\varepsilon}^H - B_t^H) \, \mathrm{d}t \tag{3.15}$$

and backward integral as

$$(Ba) \int_0^T u_t \, \mathrm{d}B_t^H = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^T u_t (B_t^H - B_{t-\varepsilon}^H) \, \mathrm{d}t, \tag{3.16}$$

whenever the limits exist P-a.s.

We need the following definition to establish the relation between the symmetric and the forward integral.

Definition 30. *Let u, v be two continuous stochastic processes. Their extended covariation is defined as the limit*

$$[u, v]_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t (u_{s+\varepsilon} - u_s)(v_{s+\varepsilon} - v_s) \, \mathrm{d}s \tag{3.17}$$

if the limit exists in uniform convergence in probability.

As it is written in Øksendal (2008, p. 125) we claim the following result:

Proposition 32. *Let u, v be two continuous stochastic processes, then it holds that*

$$(Sy) \int_0^t u_s \, \mathrm{d}v_s = (Fo) \int_0^t u_s \, \mathrm{d}v_s + [u, v]_t.$$

Link between forward and fractional integral

In this subsection we show the relation between the concept of symmetric and fractional integral. To do this we need the following definition.

Definition 31. *Let u be a stochastic process. We define (taken from* **?***, p. 125) the extended forward integral as*

$$(EFo) \int_0^t u_s \, \mathrm{d}B_s^H = \lim_{\varepsilon \to 0} \frac{1}{\Gamma(\varepsilon)} \int_0^T v^{\varepsilon - 1} \int_0^t u_s \frac{B_{s+v}^H - B_s^H}{s} \, \mathrm{d}s \mathrm{d}v, \tag{3.18}$$

provided that the limit on right side exists in uniform convergence in probability.

Remark. As we can see in Øksendal (2008, p. 126) this definition really extends the definition of the forward integral.

We refer to Øksendal (2008, p. 286) that the equality (2.17) in the deterministic case holds for functions f, g which satisfy $f_{0+} \in I^{\alpha}_{0+}(L^p([0,T])), g_{T-} \in I^{1-\alpha}_{T-}(L^q([0,T]))$, $g(0+)$ exists, $1/p+1/q \leq 1,\ 0 \leq \alpha \leq 1, f \in C^{\alpha-1/p}$ and $\alpha p > 1$. It means

$$(Fr)\int_0^T f(t)\ \mathrm{d}g(t) = \lim_{\varepsilon \searrow 0}\int_0^T I^{\varepsilon}_{0+}f(t)\ \mathrm{d}g(t).$$

When we take into account the formula (see Øksendal (2008, p. 286))

$$\int_0^T I^{\alpha}_{a+}f(t)g(t)\ \mathrm{d}t = \int_0^T f(t)I^{\alpha}_{T-}g(t)\ \mathrm{d}t,$$

the fractional integral $\int_0^T I^{\varepsilon}_{0+}f(t)\ \mathrm{d}g(t)$ now equals

$$\frac{1}{\Gamma(\varepsilon)}\int_0^T u^{\varepsilon-1}\int_0^T f(s)\frac{g_{T-}(s+u)-g_{T-}(s)}{u}\ \mathrm{d}s\mathrm{d}u,$$

when the sum of degrees of fractional differentiability is higher or equals $1-\varepsilon$. And so we can improve the definition of fractional integral and define the extended fractional integral for deterministic functions as

$$(Efr)\int_0^T f(t)\ \mathrm{d}g(t) = \lim_{\varepsilon\to 0}\frac{1}{\Gamma(\varepsilon)}\int_0^T u^{\varepsilon-1}\int_0^T f(s)\frac{g_{T-}(s+u)-g_{T-}(s)}{u}\ \mathrm{d}s\mathrm{d}u, \tag{3.19}$$

whenever the right side is determined. When we want to apply this formula to random processes we take the limit on the right side as limit in uniform convergence in probability. So at the end we can see how the formulae (3.18) and (3.19) are essentially similar.

3.5 Relation between symmetric and Skorohod integral

First let us recall the definition of $|\mathcal{H}|$ (1.33) and the space $\mathbb{D}^{1,2}$ introduced in Definition 9 and the notation D^H of the Malliavin derivative with respect to the fractional Brownian motion with Hurst parameter H. As usual let us start with the case $H > \frac{1}{2}$.

Theorem 33. *Let $H > \frac{1}{2}$ and let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in $\mathbb{D}^{1,2}\left(\left|\mathscr{H}\right|\right)$. Suppose that*

$$\int_0^T\int_0^T |D^H_s u_t| \cdot |t-s|^{2H-2}\ \mathrm{d}s\mathrm{d}t < \infty a.s., \tag{3.20}$$

Then the Skorohod integral and the extended symmetric integral exist and we have

$$(Sy)\int_0^T u_t \, \mathrm{d}B_t^H = (Sk)\int_{[0,T]} u_t \, \mathrm{d}B_t^H + H(2H-1) + \int_0^T \int_0^T D_s^H u_t |t-s|^{2H-2} \, \mathrm{d}s\mathrm{d}t. \tag{3.21}$$

Moreover the symmetric, forward and backward integrals coincide and sufficient condition for (3.20) *is that for some* $p > 1/(2H-1)$ *it holds that*

$$\int_0^T \left(\int_0^T \left| D_s^H u_t \right|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \, \mathrm{d}s < \infty. \tag{3.22}$$

Proof. The proof can be found in Øksendal (2008, p. 130). ❑

The following theorem shows us a sufficient condition for existence of the symmetric integral.

Theorem 34. *Let* $H > \frac{1}{2}$ *and let* u *be an adapted stochastic process which is continuous in the norm of* $\mathbb{D}^{1,2}\left(\mathscr{H}\right)$, *which means for* $s, t \in [0,T]$ $s \to t$ *implies* $u(s) \to u(t)$ *in* $\| \cdot \|_{\mathbb{D}^{1,2}\left(\mathscr{H}\right)}$, *and*

$$\lim_{n\to\infty} \int_0^T \sup_{s,s' \in (r, r+1/n)\cap[0,T]} E\left[\left| D_r^H u_s - D_r^H u(s') \right|^2 \right] \, \mathrm{d}r = 0,$$

then

$$\lim_{|\mathscr{P}|\to 0} \sum_{i=1}^n u_{t_i}(B_{t_{i+1}}^H - B_{t_i}^H) = (Sy)\int_0^T u_t \, \mathrm{d}B_t^H, \tag{3.23}$$

where we used the notation for partitions as in the proof of Theorem 22. The convergence holds in $L^2(\mathbb{P}^H)$, *where* $\mathbb{P}^H$ *denotes the law of* B^H *and hence the convergence holds also almost surely.*

Proof. For the proof see Øksendal (2008, p. 131) ❑

Now we move to the case $H < \frac{1}{2}$. This case is again more complicated than the previous one but we establish conditions under which the symmetric integral exists and we show its relation to the divergence operator. To do that we need the following definition.

Definition 32. *Recall the space* $\mathcal{E}$ *of all step functions on* $[0,T]$. *We equip it with the following seminorm*

$$\|\phi\|_{K_H}^2 = \int_0^T \phi_s^2 K_H(T,s)^2 \, \mathrm{d}s + \int_0^T \left[\int_0^T |\phi_t - \phi_s|(t-s)^{H-\frac{3}{2}} \, \mathrm{d}t \right]^2 \, \mathrm{d}s,$$

the operator K_H *was defined in Chapter 1. The completion of* $\mathcal{E}$ *with respect to* $\|\phi\|_{K_H}$ *is denoted* $\mathscr{H}_{K_H}$.

Remark. As we can see in Øksendal (2008, p. 128) the space $\mathscr{H}_{K_H}$ is continuously embedded in $\mathscr{H}$.

Now we are ready to investigate the existence of symmetric integral and its relation to Skorohod integral from a slightly different point of view than in Theorem 33.

Theorem 35. *Let $H < \frac{1}{2}$ and $u = \{u_t, t \in [0,T]\}$ be a random process in $\mathbb{D}^{1,2}(\mathscr{H}_{K_H})$. Assume that the trace defined as*

$$Tr\ D^H u = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T \left\langle D^H u_s, \mathbf{1}_{[s-\varepsilon, s+\varepsilon]} \right\rangle_H$$

exists as limit in probability and moreover

$$E\left[\int_0^T u_s^2 (s^{2H-1} + (T-s)^{2H-1})\ \mathrm{d}s\right] < \infty$$

and

$$E\left[\int_0^T \int_0^T (D_r^H u_s)^2 (s^{2H-1} + (T-s)^{2H-1})\ \mathrm{d}s\mathrm{d}r\right] < \infty.$$

Then both symmetric and Skorohod integral exist and we have that

$$(Sy)\int_0^T u_t\ \mathrm{d}B_t^H = (Sk)\int_{[0,T]} u_t\ \mathrm{d}B_t^H + Tr\ D^H u. \tag{3.24}$$

Proof. The theorem was taken from Øksendal (2008, p. 130) and the refference for the proof is also there.

❑

Chapter 4

Summary

This chapter provides a summary of definitions and relation formulae between the concepts of stochastic integrals we have already introduced. Here we omit some technical details which were mentioned in the chapters before.

4.1 Definitions

Multiple Wiener-Itô integral:
For a step function f of the form

$$f(t_1, \dots, t_m) = \sum_{i_1, \dots, i_m = 1}^{n} a_{i_1, \dots, i_m} \mathbf{1}_{A_{i_1} \times \dots \times A_{i_m}}(t_1, \dots, t_m)$$

we define in Definition 5 (and extend thereafter) the multiple Wiener-Itô integral as

$$(WI) \int_{X^m} f(t_1, \dots, t_m) \; \mathrm{d}(W(t_1), \dots, W(t_m)) = \sum_{i_1, \dots, i_m = 1}^{n} a_{i_1, \dots, i_m} W(A_{i_1}) \times \dots \times W(A_{i_m}).$$

For a general f the integral is defined via approximation by step functions as limit of the multiple Wiener-Itô integrals of the approximating sequence.

Skorohod integral:
For a random process u such that for any $F \in \mathbb{D}^{1,2}$ it holds that

$$|E(\langle DF, u \rangle_H)| \leq c \|F\|_2$$

the Skorohod integral $\delta(u)$ is defined in Definition 10 by the relation

$$EF\delta(u) = E(\langle DF, u \rangle_H),$$

which must hold for all $F \in \mathbb{D}^{1,2}$.

Riemann-Stieltjes integral
Let $\{u_t, t \in [0, T]\}, \{v_t, t \in [0, T]\}$ be two random processes such that

$u \in C^{\varepsilon}, v \in C^{v}$, $\varepsilon, v > 0$ and $\varepsilon + v > 1$. Then, as proved in Theorem 22, the Riemann-Stieltjes integral of u with respect to v defined in Definition 21 exists.

Fractional integral

For two functions f, g such that $f_{a+} \in I^{\alpha}_{a+}(L^p), g_{b-} \in I^{1-\alpha}_{b-}(L^q)$, where $\frac{1}{p} + \frac{1}{q} \leq 1, 0 \leq \alpha \leq 1$ we define the fractional integral in Definition 2.13 as

$$(Fr) \int_a^b f(x)\,\mathrm{d}g(x) = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x)\,\mathrm{d}x + f(a+)(g(b-) - g(a+)).$$

Anticipating integral

The anticipating integral is defined in (2.18) for a random process of the form

$$u(t) = \sum_{n=0}^{\infty} WI_n(\tilde{k}_n(\cdot, t))$$

as

$$(A) \int_0^T u_t\,\mathrm{d}B_t = WI_{n+1}(k_n) + n \int_0^T WI_{n-1}(k_n(\cdot, t, t))\,\mathrm{d}t,$$

whenever the sum on the right side converges in the mean square.

Stratonovich integral

Let u be a random process with integrable trajectories. The Stratonovich integral of u with respect to B is defined in Definition 27 as the limit in probability (if it exists) of

$$S^{\mathscr{P}}(u) = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} u_s\,\mathrm{d}s \right) (B(t_{i+1}) - B(t_i))$$

as $|\mathscr{P}|$ goes to zero.

L^2-integral

Let $u \in L^2(X \times \Omega)$, then we define the L^2-integral in (3.11) as

$$(L^2) \int_X u\,\mathrm{d}W = \sum_{i=1}^{\infty} \langle u, e_i \rangle_H W(e_i),$$

whenever the sum converges in probability and the result does not depend on the choice of the orthonormal system $\{e_i, i \geq 1\}$.

Symmetric, forward and backward integrals

For a random process u with integrable trajectiories and $0 < H < 1$ we define (provided the right sides converge in probability) in Definition 29 the symmetric integral of u with respect to fBm with Hurst parameter H as

$$(Sy)\int_0^T u_t \, \mathrm{d}B_t^H = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int u_t (B_{t+\varepsilon}^H - B_{t-\varepsilon}^H) \, \mathrm{d}t,$$

the forward integral as

$$(Fo)\int_0^T u_t \, \mathrm{d}B_t^H = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int u_t (B_{t+\varepsilon}^H - B_t^H) \, \mathrm{d}t$$

and the backward integral as

$$(Ba)\int_0^T u_t \, \mathrm{d}B_t^H = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int u_t (B_t^H - B_{t-\varepsilon}^H) \, \mathrm{d}t.$$

4.2 Relation formulae

Multiple Wiener-Itô integral and iterated Itô integral
Let f_m be a real symmetric function in $L^2(X^m)$ and let $W(h) = (WI)\int_X h_s \, \mathrm{d}W_s, h \in L^2(X)$, then, as stated in Theorem 8, it holds that

$$WI_m(f_m) = m!(I)\int_0^\infty (I)\int_0^{t_m} \dots (I)\int_0^{t_2} f_m(t_1, \dots, t_m) \, \mathrm{d}W_{t_1} \dots \mathrm{d}W_{t_m}.$$

Skorohod integral and Itô integral
For an adapted process u, as we show in (1.21), it holds that

$$(Sk)\int_{[0,T]} u_t \, \mathrm{d}B_t = (I)\int_0^T u_t \, \mathrm{d}B_t,$$

provided u is both Skorohod as well as Itô integrable.

Riemann-Stieltjes integral and fsractional integral
Let $\{u_t, t \in [0,T]\}, \{v_t, t \in [0,T]\}$ be two random processes such that $u \in C^\varepsilon, v \in C^\upsilon$, $\varepsilon, \upsilon > 0$ and $\varepsilon + \upsilon > 1$, then, as we show in Theorem 24, we have that

$$(RS)\int_0^T u_t \, \mathrm{d}v_t = (Fr)\int_0^T u_t \, \mathrm{d}v_t.$$

Itô integral and fractional integral
Let u be an adapted stochastic process which satisfies $u \in I_{0+}^\alpha(L^2)$ for some $\alpha < \frac{1}{2}$ almost surely. We show in Theorem 25 that

$$P - \lim_{\varepsilon \searrow 0} \int_0^T I_{0+}^\varepsilon u_t \, \mathrm{d}B_t = (I)\int_0^T u_t \, \mathrm{d}B_t.$$

Skorohod integral and fractional integral

Let u be a random process which can be interpreted as (1.13), where for every $n \geq 1$ it holds that $k_n \in I^{\alpha}_{0+}(n, L^2)$ for an $\alpha > \frac{1}{2}$. Then Theorem 27 suggests that

$$(Fr) \int_0^T u_t \, \mathrm{d}B_t = \delta(u) + \sum_{n=1}^{\infty} n \int_0^T W I_{n-1}(k_n(\cdot, t, t)) \, \mathrm{d}t.$$

Skorohod integral and Stratonovich integral

Let u be a measurable stochastic process in $\in \mathbb{L}_1^{1,2}$, then we state in Theorem 29 that

$$(St) \int_0^T u_t \, \mathrm{d}B_t = (Sk) \int_0^T u_t \, \mathrm{d}B_t + \frac{1}{2} \int_0^T (\nabla u)_t \, \mathrm{d}t.$$

Stratonovich integral and L^2-integral

Let u be measurable stochastic process such that $\int_0^T u_t^2 \, \mathrm{d}t < \infty$ a.s. Then as we can see in Theorem 30 it is true that

$$(St) \int_0^T u_t \, \mathrm{d}B_t = (L^2) \int_0^T u_t \, \mathrm{d}B_t.$$

Anticipating integral, Skorohod integral and fractional integral

Let $u \in \mathbb{W}^{\alpha}_{2,+}$ for an $\alpha > \frac{1}{2}$. Then according to Theorem 31 it holds that

$$(A) \int_0^T u_t \, \mathrm{d}B_t = (St) \int_0^T u_t \, \mathrm{d}B_t = (Fr) \int_0^T u_t \, \mathrm{d}B_t.$$

Symmetric integral and forward integral

Let u, v be two continuous (locally bounded) stochastic processes, then according to Proposition 32 it holds that

$$(Sy) \int_0^t u_s \, \mathrm{d}v_s = (Fo) \int_0^t u_s \, \mathrm{d}v_s + [u, v]_t.$$

Symmetric integral and Skorohod integral

Let $H > \frac{1}{2}$ and $u = \{u_t, t \in [0, T]\}$ be a stochastic process in $\mathbb{D}^{1,2}\left(\left|\mathscr{H}\right|\right)$. Suppose that

$$\int_0^T \int_0^T |D_s^H u_t| \cdot |t - s|^{2H-2} \, \mathrm{d}s\mathrm{d}t < \infty, a.s.$$

Then as we can see in Theorem 33 we have that

$$(Sy) \int_0^T u_t \, \mathrm{d}B_t^H = (Sk) \int_{[0,T]} u_t \, \mathrm{d}B_t^H + H(2H-1) + \int_0^T \int_0^T D_s^H u_t |t-s|^{2H-2} \, \mathrm{d}s\mathrm{d}t.$$

Symmetric integral and Skorohod integral II

Let $H < \frac{1}{2}$ and let $u = \{u_t, t \in [0,T]\}$ be a random process in $\mathbb{D}^{1,2}\left(McallH_{K_H}\right)$. Assume that the trace defined as

$$Tr\ D^H u = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T \left\langle D^H u_s, \mathbf{1}_{[s-\varepsilon, s+\varepsilon]} \right\rangle_H$$

exists as limit in probability and moreover

$$E\left[\int_0^T u_s^2 (s^{2H-1} + (T-s)^{2H-1})\ \mathrm{d}s\right] < \infty$$

and

$$E\left[\int_0^T \int_0^T (D_r^H u_s)^2 (s^{2H-1} + (T-s)^{2H-1})\ \mathrm{d}s\mathrm{d}r\right] < \infty.$$

Then Theorem 35 says that

$$(Sy) \int_0^T u_t\ \mathrm{d}B_t^H = (Sk) \int_{[0,T]} u_t\ \mathrm{d}B_t^H + Tr\ D^H u.$$

Chapter 5

Kurzweil integral

The idea of Kurzweil integration is to use non-uniform meshes. Recal the definition of Kurzweil partition Definition 20. We start with the conception of weak Kurzweil-Stieltjes integral as introduced in Toh and Chew (2012). The word "weak" suggest that the result is not a pathwise integral but a limit taken in L^2.

Definition 33. *Let $[a,b]$ be a finite interval and $\mathscr{K}$ be its Kurzweil partition. Any strictly positive function γ on $[a,b]$ is called gauge. Given a gauge γ on $[a,b]$, the Kurzweil partition $\mathscr{K}$ is called γ-fine if*

$$[\alpha_{j-1}, \alpha_j] \subset [\tau_j - \gamma(\tau_j), \tau_j + \gamma(\tau_j)],\ j = 1, 2, \ldots, k.$$

The set of all γ-fine Kurzweil partitions of $[a,b]$ is denoted $\mathscr{K}(\gamma)$. A Kurzweil partition $\mathscr{K}$ is called belated if

$$\mathscr{K} = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

i.e. the tag τ_j always coincides with the left point of the interval $[\alpha_{j-1}, \alpha_j]$.

In the sequel we need the following Cousin lemma.

Lemma 36. *Let $[a,b]$ be a finite interval and let γ be a gauge on $[a,b]$. Then the set of γ-fine partitions in nonempty.*

Proof. For the proof we refer to Schwabik (1985, p. 7).

❑

We see that given a gauge γ the set $\mathscr{K}(\gamma)$ of partitions of $[a,b]$ is nonempty. However we refer to Toh and Chew (2003, p. 135) that for a particular gauge γ belated γ-fine Kurzweil partition does not always exist. This fact suggests us to introduce the following concept.

Definition 34. *Let λ denote the one-dimensional Lebesgue measure and let γ be a gauge on $[a,b]$. A finite collection $\mathscr{B}$ of intervals*
$\{(\alpha_j, \beta_j] : j = 1, 2, 3, \ldots k\}$ is called a γ-fine partial belated Kurzweil partition if

1. *$(\alpha_j, \beta_j]$ are left-open subintervals of $[a,b]$,*

2. *each $[\alpha_j, \beta_j]$ is γ-fine belated, i.e. $[\alpha_j, \beta_j] \subset [\alpha_j, \alpha_j + \gamma(\alpha_j))$.*

Moreover, given a positive real number ξ we say that a partial belated Kurzweil partition $\mathscr{B}$ fails to cover $[a, b]$ by at most λ-measure ξ if

$$\lambda([a, b]) - \sum_{j=1}^{k} \lambda([\alpha_i, \beta_i]) < \xi.$$

The concept of partial belated Kurzweil partition is very useful because given a gauge γ there always exist a γ-fine partial belated Kurzweil (cf. Toh and Chew (2003, p. 135)).

Remark. The Lebesgue measure was chosen in Definition 34 due to the fact that we want to build an integral with respect to standard Brownian motion. Hence the Lebesgue measure plays the role of the measure induced by quadratic variation of the integrator. If we wanted to integrate with respect a general continuous semimartingale M, we would have to replace λ by the appropriate measure induced by the quadratic variation of M in the following construcion.

5.1 Weak Kurzweil integral

Now we are ready to define the weak Kurzweil integral.

Definition 35. *Let us fix interval $[0, T]$. Moreover, let $B = \{B_t, t \in T\}$ be the standard Brownian motion and let $u = \{u_t, t \in [0, T]\}$ be a stochastic process adapted to the filtration generated by B. We say that u is weakly Kurzweil integrable over $[0, T]$ with respect to B to a random variable $A \in L^2(\Omega)$ if for any $\varepsilon > 0$ there exist a gauge γ on $[0, T]$ and a positive number ξ such that for any γ-fine partial belated partition $\mathscr{B} = \{(\alpha_j, \beta_j] : j = 1, 2, \ldots, k\}$ of $[0, T]$ which fails to cover $[0, T]$ by at most ξ*

$$\left(\sum_{j=1}^{k} u_{\alpha_j} (B_{\beta_j} - B_{\alpha_j}) - A \right)^2 < \varepsilon$$

holds. A is then called the weak Kurzweil integral of u with respect to B and is denoted

$$(WK) \int_0^T u_t \, \mathrm{d}B_t.$$

Now we show the link between weak Kurzweil integral and classic Itôintegral.

Theorem 37. *Let u be a stochastic process adapted to the filtration generated by B which satisfies*

$$E\left(\int_0^T u_t^2 \, \mathrm{d}t \right) < \infty.$$

Then both classic Itô integral and weak Kurzweil integral exist and it holds that

$$(I) \int_0^T u_t \, \mathrm{d}B_t = (WK) \int_0^T u_t \, \mathrm{d}B_t.$$

Proof. For the proof of a stronger result see Toh and Chew (2003, p. 145). ❑

Hence we see that the weak Kurzweil integral encompasses the classic Itô integral in the case of $E\left(\int_0^T u_t \, \mathrm{d}t\right)^2 < \infty$. In Toh and Chew (2003) is studied even the case of integration with respect to a local semimartingale and even then the Weak Kurzweil integral, defined for a local semimartingale as an integrator, exists and coincides with the classic Itô integral provided the Itô integral exists. However, we want to use the fractional Brownian motion as the integrator and even the more general construction in Toh and Chew (2003) heavily relies on the fact that the integrator is at least local semimartingale so it is not useful for our purpose. To be able to construct a Kurzweil stochastic integral with respect to the fractional Brownian motion we need to define the strong (pathwise) Kurzweil integral.

5.2 Strong Kurzweil integral

In this chapter we introduce the concept of strong (pathwise) Kurzweil (generalized Perron) integral according to Schwabik (1985). Firstly we mention some preliminaries and definitions. After that we apply this concept to the stochastic case via building the Kurzweil integral pathwise for a proper set of integrands, where the integrator is the fractional Brownian motion.

First we need to define a special set which plays an important role in the concept of the Kurzweil integral.

Definition 36. *Let $\mathcal{S} = \mathcal{S}([a,b])$ denote the system of all sets $S \subset \mathbb{R}^2$ such that there exists a gauge γ so that*

$$\{(\tau, t) \in \mathbb{R}^2; \tau \in [a,b], t \in [\tau - \gamma(\tau), \tau + \gamma(\tau)] \cap [a,b]\} \subset S.$$

Definition 37. *Let f be a real function of two variables defined on $S \in \mathcal{S}$. Let γ be the gauge corresponding to S then for every Kurzweil partition $\mathscr{K} = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{k-1}, \tau_k, \alpha_k\}$ we define the Kurzweil integral sum as*

$$s\left(f, \mathscr{K}\right) = \sum_{j=1}^{k} (f(\tau_j, \alpha_j) - f(\tau_j, \alpha_{j-1})). \tag{5.1}$$

Now we are ready to define the strong Kurzweil integral.

Definition 38. *Let f be a function on $S \in \mathcal{S}$. Then f is called Kurzweil integrable over $[a,b]$ if there exists a number I such that for every $\varepsilon > 0$ there is a gauge γ so that for every γ-fine Kurzweil partition $\mathscr{K}$ it holds that*

$$\left| s\left(f, \mathscr{K}\right) - I \right| < \varepsilon.$$

Such I, if it exists, is called the Kurzweil integral of f over $[a,b]$ and is denoted

$$(K) \int_a^b f(\tau, t) \, \mathrm{d}(\tau, t). \tag{5.2}$$

The set of all Kurzweil integrable functions f over $[a,b]$ is denoted by $\mathcal{K}([a,b])$.

The following remark shows us the connection of the Riemann, respective Riemann-Stieltjes integral and the Kurzweil integral.

Remark. If the function $f(\tau, t)$ is of the form $g(\tau) \cdot t$ then for $\tau \in [a, b], \alpha_1, \alpha_2 \in [a, b]$ it holds that $f(\tau, \alpha_2) - f(\tau, \alpha_1) = g(\tau)(\alpha_2 - \alpha_1)$. For a Kurzweil partition $\mathscr{K} = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ of $[a, b]$ the Kurzweil integral sum $s\left(f, \mathscr{K}\right)$ coincides with the usual Riemann integral sum $\sum_{j=1}^{k} g(\tau_j)(\alpha_j - \alpha_{j-1})$. Similarly if the function f is of the form $f(\tau, t) = g(\tau) \cdot h(t)$, then the Kurzweil integral sum coincides with the Riemann-Stieltjes integral sum $\sum_{j=1}^{k} g(\tau_j)(h(\alpha_j) - h(\alpha_{j-1}))$. If the function f is of the form $f(\tau, t) = g(\tau) \cdot h(t)$ we usually write $(K)\int_a^b g(t)\ \mathrm{d}h(t)$ instead of $(K)\int_a^b f(\tau, t)\ \mathrm{d}(\tau, t)$. In that case we sometimes call the integral Kurzweil-Stieltjes.

The definition of Kurzweil integral requires the the existence of a γ-fine Kurzweil partition for a given gauge γ. As was mentioned in the beginning of this chapter, the set $\mathscr{K}(\gamma)$ is nonempty and hence the definition of the Kurzweil integral is indeed not trivial.

Lemma 38. *As can be seen in Schwabik (1985, p. 7., 9), the Kurzweil integral is linear. More precisely, for a real constant α and two Kurzweil integrable functions f, g it holds that*

$$(K)\int_a^b \alpha f(\tau, t)\ \mathrm{d}(\tau, t) = \alpha\ \ (K)\int_a^b f(\tau, t)\ \mathrm{d}(\tau, t),$$

$$(K)\int_a^b f(\tau, t)\ \mathrm{d}(\tau, t) + g(\tau, t)(\tau, t) = (K)\int_a^b f(\tau, t)\ \mathrm{d}(\tau, t) + (K)\int_a^b g(\tau, t)\ \mathrm{d}(\tau, t).$$

Moreover, as it is usual for integral it is additive which means if a function f is Kurzweil integrable over $[a, c]$ as well as $[c, b]$, then the integral of f over $[a, b]$ exists and equals the sum of the integrals over $[a, c]$ and $[c, b]$.

The following lemma shows an interesting property which is not intuitive for an integral, namely that the indefinite Kurzweil integral is not continuous in general.

Lemma 39. *The indefinite Kurzweil integral*

$$(K)\int_a^s f(\tau, t)\ \mathrm{d}(\tau, t),\ \ a \le s \le b,$$

as a function of s, is continuous at a point $c \in [a, b]$ if and only if $f(c, t)$, as function of t, is continuous at the point c.

Proof. The proof can be found in Schwabik (1985, p. 14).

❑

Remark. The Kurzweil integral has more, for integral usual, properties such as monotonicity, change of limit and integral formula, change of variable formula, dominated convergence, per partes formula etc. For detailed survey see Schwabik (1985, Chpater 1). A considerable part of the Chapter 1 there is devoted to the case when the integrated function $f(\tau, t)$ is of the form $g(\tau) \cdot h(t)$ and $h(t)$ is of bounded variation. In stochastic calculus this case is not particularly useful because the fBm of course is not of finite variation.

Now we are ready to apply the concept of the Kurzweil integral to stochastic calculus. Let us return to our in previous chapters fixed interval $[0, T]$ and $(\Omega, \mathcal{F}, P)$. We can set a random process u of two variables to have the form $u(\tau, t) = v(\tau)B^H(t)$, where $v = v(\tau, \omega), \omega \in \Omega$ is now random and then pathwise define (if it exists) the stochastic Kurzweil integral (almost surely) of u with respect to the fractional Brownian motion

$$(K)\int_0^T v(\tau)B^H(t)\ \mathrm{d}(\tau, t) \tag{5.3}$$

and use the notation

$$(K)\int_0^T v(\tau)\ \mathrm{d}B^H(\tau).$$

However, although the Kurzweil integral is a very powerful tool which, in certain situations, even generalizes the Perron integral (see Schwabik (1985, p. 5)), taking into account that fBm is not of finite strong 1-variation, it is rather complicated to establish conditions on v which imply the existence of the integral. It can be shown (cf. Tvrdý (2012, p. 146)) that if v has sample paths with finite 1-variation then the integral $(SK)\int_0^T v_t\ \mathrm{d}B_t^H$ exists a.s. as B^H has continuous sample paths. However, this result is not particularly useful because the assumption of finite strong 1-variation is too restrictive for stochastic calculus. It can be shown (see Tvrdý (2012, p. 105)) that if for a fixed function g the integral $(SK)\int_0^T f(t)\ \mathrm{d}g(t)$ exists for every function f then g has finite strong 1-variation over $[a, b]$. Hence we are not able to integrate all continuous functions with respect to the fractional Brownian motion.

From the construction it is clear that the strong Kurzweil-Stieltjes integral is a very powerful instrument and it generalizes the Riemann-Stieltjes integral defined in Definition 21 as the approximating sums coincide but the Riemann-Stieltjes integral assumes only uniform meshes. So one might think that the Kurzweil integral could exist even for pair of Hölder functions where the sum of their Hölder orders is less than one. However the answer is negative as we show in the following theorem which is our own result.

Theorem 40. *Let ν, υ be two nonnegative numbers such that $\nu + \upsilon < 1$ then there exist two real functions on $[0, 1]$ such that f if Hölder of order ν and g if Hölder of order υ such that the integral $(SK)\int_0^1 f(t)\ \mathrm{d}g(t)$ does not exist.*

Proof. The proof consists of three steps: construct two funcions f and g, show they are Hölder of order ν and υ respectively and show that the integral $(SK)\int_0^1 f(t)\ \mathrm{d}g(t)$ does not exist.
Let us fix the number ν. We use the mathematical induction to construct a function f as a limit of sequence of functions $(f_n, n \geq 1)$ which are continuous and piecewise affine. For a give n the function f_n has "break points" $\left(\frac{k}{2^n}, k = 1, 2, \ldots, 2^n - 1\right)$. Set $f_0 \equiv 0$. Assume that we already have $(f_i, i = 1, 2, \ldots, n)$. We construct the function f_{n+1} as follows:
$f_{n+1}(x) := f_n(x), x \in \left\{\frac{k}{2^n}, k = 1, 2, \ldots, 2^n\right\}$. Let Z_n denote the set $\left\{\frac{k}{2^n}, k = 1, 2, \ldots, 2^n\right\}$. Now we define f_{n+1} on $Z_{n+1} \setminus Z_n \left\{\frac{k}{2^{n+1}}, k = 1, 2, \ldots, 2^{n+1}\right\}$ where k is odd. Set $x_k := \frac{k}{2^{n+1}}, x_k^+ := \frac{k+1}{2^{n+1}}$ and

$x_k^- := \frac{k-1}{2^{n+1}}$. There are two different cases:

$$f_{n+1}(x_k) := \begin{cases} \frac{f_n(x_k^+)+f_n(x_k^-)}{2} & \text{if } |f_n(x_k^+)+f_n(x_k^-)| \geq 2\left(\frac{1}{2^{n+1}}\right)^\nu, \\ \max\left\{f_n(x_k^+), f_n(x_k^-),\right\} + \left(\frac{1}{2^{n+1}}\right)^\nu & \text{otherwise.} \end{cases}$$

Now we claim that

1. there really exist a function f such that $f_n \rightrightarrows f$ on $[0,1]$.
2. For all $n \geq 1$, for all $x \in Z_n$ it holds that $f(x) = f_n(x)$.
3. For all $n \geq 1$, for all $x, y \in Z_n$ which are "neighbours" in Z_n (means $|x-y| = \frac{1}{2^n}$) we have that for all $n \geq 1$ it holds that $|f(x) - f(y)| \geq \left(\frac{1}{2^n}\right)^\nu$ and $|f(x) - f(y)| \leq 3\left(\frac{1}{2^n}\right)^\nu$.
4. f if Hölder continuous of order ν.

Proof of 1.: It is clear that function f_{n+1} differs from f_n on $[x_k^-, x_k^+]$ only if $|f_n(x_k^+) + f_n(x_k^-)| < 2\left(\frac{1}{2^{n+1}}\right)^\nu$. However, for all k it holds that the deviation does not exceed $3 \cdot \left(\frac{1}{2^{n+1}}\right)^\nu$ on $[x_k^-, x_k^+]$. Hence $\|f_{n+1} - f_n\|_\infty \leq 3 \cdot \left(\frac{1}{2^{n+1}}\right)^\nu$ for all $n \geq 1$. Clearly $\sum_{n=1}^\infty 3 \cdot \left(\frac{1}{2^{n+1}}\right)^\nu = \sum_{n=1}^\infty 3 \cdot \left(\frac{1}{2^\nu}\right)^{n+1} < \infty$ hence there exists a function f such that $\sum_{n=1}^\infty (f_{n+1} - f_n) \rightrightarrows f$ as $n \to \infty$ on $[0,1]$ and so $f_n \rightrightarrows f$ as $n \to \infty$ on $[0,1]$.
Proof of 2.: Immediate consequence of the construction of $\{f_n\}_{n=0}^\infty$.
Proof of 3.: The first inequality follows directly from the construction. The second
inequality is proven by means of the mathematical induction. For $i = 0$ the inequality $|f(x) - f(y)| \leq 3\left(\frac{1}{2^i}\right)^\nu$ holds. Now assume the inequality holds for $i = 0, 1, 2, \ldots, n-1$ and we want to show that it holds for $i = n$. Let us fix the two points $x, y \in Z_n$ so that $|x - y| = \frac{1}{2^n}$. Clearly either $x \in Z_{n-1}$ or $y \in Z_{n-1}$. Without loss of generality assume that $y \in Z_{n-1}$ hence $x \in Z_n \setminus Z_{n-1}$. Also without loss of generality assume that $y = x^-$, i.e. $y < x$. The value of $f(x) = f_n(x)$ was fixed during the construction of f_n when we were working with the triplet $x^- < x < x^+$. Let A denote $|f_n(x) - f_{n-1}(x^+)| + |f_{n-1}(x^+) - f_{n-1}(x^-)|$. If $|f_n(x^+) + f_n(x^-)| < 2\left(\frac{1}{2^{n+1}}\right)^\nu$ then $f_n(x^-) = f_{n-1}(x^-)$ because $x \in Z_{n-1}$ and it holds that

$$|f(x) - f(y)| = |f(x) - f(x^-)| = |f_n(x) - f_{n-1}(x^-) \leq$$

$$\leq \begin{cases} \left(\frac{1}{2^n}\right) & \text{if } f_{n-1}(x^-) = \max\{f_{n-1}(x^-), f_{n-1}(x^+)\}, \\ A & \text{if } f_{n-1}(x^+) = \max\{f_{n-1}(x^-), f_{n-1}(x^+)\}. \end{cases}$$

Note that $|f_n(x) - f_{n-1}(x^+)| + |f_{n-1}(x^+) - f_{n-1}(x^-)| < 3 \cdot \left(\frac{1}{2^n}\right)^\nu$. On the other hand, if $|f_n(x^+) + f_n(x^-)| \geq 2\left(\frac{1}{2^{n+1}}\right)^\nu$ then from the base case follows that

$$|f(x^+) - f(x^-)| \leq 3 \cdot \left(\frac{1}{2^{n-1}}\right)^\nu = 6\frac{1}{2}\frac{1}{2^{n-1}} \overset{*}{<} 6 \cdot \frac{1}{2^\nu}\left(\frac{1}{2^{n-1}}\right)^\nu = 6 \cdot \left(\frac{1}{2^n}\right)^\nu,$$

where the inequality $\overset{*}{<}$ follows from the fact that $\nu \in (0,1)$ hence $2^\nu \in (1,2)$. Recall that in this case $|f(x) - f(x^-)| = \frac{1}{2}|f(x^+) - f(x^-)|$ hence the inductive

step is proven.
Proof of 4.: Let us fix arbitrary two points $a, b \in [0,1]$, $a < b$ and let n be the smallest index such that $[a,b] \cap Z_n$ contains at least two points (i.e. $[a,b] \cap Z_{n-1}$ contains at most one point). Then we have that $\frac{1}{2^n} \leq b - a < 4 \cdot \frac{1}{2^n}$. From this inequality, construction of f_n and 3. it follows that

$$|f_n(b) - f_n(a)| < 4 \cdot 3 \cdot \left(\frac{1}{2^n}\right)^{\nu} = 12 \cdot 2^{-\nu n}. \tag{5.4}$$

Moreover, (recall proof of 1.)

$$|f_n(b) - f(b)| \leq \|f_n - f\|_\infty \leq \sum k = n^\infty \|f_k - f_{k+1}\|_\infty \leq$$

$$\leq \sum_{k=n}^{\infty} \left(\frac{1}{2^{k+1}}\right)^{\nu} = \cdots = \frac{3}{2^\nu - 1} \cdot 2^{-\nu n}.$$

Similarly $f_n(a) - f(a) \leq \frac{3}{2^\nu - 1} \cdot 2^{-\nu n}$ and therefore we have

$$|f(b) - f(a)| \leq |f_n(b) - f(b)| + |f_n(b) - f_n(a)| + |f_n(a) - f(a)| \leq$$

$$\leq \frac{3}{2^\nu - 1} \cdot 2^{-\nu n} + 12 \cdot 2^{-\nu n} + \frac{3}{2^\nu - 1} \cdot 2^{-\nu n} = \left(12 + \frac{6}{2^\nu - 1}\right) \cdot 2^{-\nu n} \leq$$

$$\left(12 + \frac{6}{2^\nu - 1}\right) \cdot (b - a)^\nu.$$

Hence f is indeed Hölder continuous of order ν. Similarly we construct the second function g which is Hölder continuous of order υ. Now it remains to show that the integral $(SK) \int_0^1 f(t)\, \mathrm{d}g(t)$ does not exist. Let $\mathscr{K} = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ be a Kurzweil partition of the interval $[0,1]$. In this proof we use the notation $\mathscr{K} = (\boldsymbol{\alpha}, \boldsymbol{\tau})$. $\boldsymbol{\alpha}$ is called set of points and $\boldsymbol{\tau}$ is called set of tags. We also introduce the notation of the Kurzweil integral sum for this case

$$S(f, g, \boldsymbol{\alpha}, \boldsymbol{\tau}) = \sum_{j=1}^{k} f(\tau_i)(g(\alpha_i) - g(\alpha_{i-1})).$$

Indeed, it can be easily checked that the sum s in (5.1) coincides with the sum S. To prove that the integral $(SK) \int_0^1 f(t)\, \mathrm{d}g(t)$ does not exist it is sufficient to show that for every gauge γ there exist two γ-fine Kurzweil partitions $(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1), (\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)$ of interval $[0,1]$ such that

$$|S(f, g, \boldsymbol{\alpha}_1, \boldsymbol{\tau}_1) - S(f, g, \boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)| > 1. \tag{5.5}$$

Let γ be an arbitrary gauge on $[0,1]$. Define $M_n = \{x \in [0,1] : \gamma(x) > \frac{1}{2^n}$. Then $[0,1] = \bigcup_{n=1}^{\infty}$ and hence, according to the Baire Theorem, there exists $n_0 \geq 1$ such that M_{n_0} is not nowhere dense, i.e. there exists an interval $I := [a,b]$ such that $I \subseteq \overline{M_n}$. The sequence $(2^n(b-a) - 2)(2^{-n-1})^{\nu+\upsilon}$ converges to infinity as $n \to \infty$ because $\nu + \upsilon < 1$. Therefore we can fix n so that $n > n_0$ and for every $m \geq n :\ (2^m(b-a) - 2)(2^{-m-1})^{\nu+\upsilon} > 1$ and $\frac{1}{2^m} < \frac{b-a}{3}$. Moreover, fix an arbitrary $\varepsilon > 0$ such that $\left(\frac{1}{2^n}\right)^{\upsilon} - 2\varepsilon > \left(\frac{1}{2^{n+1}}\right)^{\upsilon}$ and $\left(\frac{1}{2^n}\right)^{\nu} - 2\varepsilon > \left(\frac{1}{2^{n+1}}\right)^{\nu}$. The functions f

and g are continuous on $[0,1]$ hence they are uniformly continuous so we can fix an $\eta > 0$ such that $\eta < \frac{1}{2^n}$ and for all $x, y \in [0,1]$ it holds that

$$|x-y| < \eta \Rightarrow |f(x) - f(y)| < \varepsilon$$

and also

$$|x-y| < \eta \Rightarrow |g(x) - g(y)| < \varepsilon.$$

Let $\tilde{D}$ denote $Z_n \cap I = Z_n \cap [a,b]$. As we are assuming $\frac{1}{2^n} < \frac{b-a}{3}$ we have that $\tilde{D}$ contains at least two points. We use the notation $\tilde{D} = \{d_0 < d_1 < \cdots < d_N\}$. Set $\tilde{P} = \{p_0 < p_1 < \cdots < p_N\}$ so that

$$\forall_{i \in \{1,2,\ldots,N} : p_i \in H_{n_0} \cap I \cap (d_i - \eta, d_i + \eta).$$

We have that

$$\forall_{i \in \{1,2,\ldots,N-1\}} |f(p_i) - f(p_{i+1})| \geq |f(d_i) - f(d_{i+1})| - |f(d_i) - f(p_{i+1})| -$$

$$-|f(d_{i+1}) - f(p_{i+1})| > \left(\frac{1}{2^n}\right)^{\nu} - \varepsilon - \varepsilon > \left(\frac{1}{2^{n+1}}\right)^{\nu}.$$

Similarly it holds that

$$\forall_{i \in \{1,2,\ldots,N-1\}} |g(p_i) - g(p_{i+1})| > \left(\frac{1}{2^{n+1}}\right)^{\upsilon}.$$

Now define two partitions $(\tilde{\boldsymbol{\alpha}}_1, \tilde{\boldsymbol{\tau}}_1), (\tilde{\boldsymbol{\alpha}}_2, \tilde{\boldsymbol{\tau}}_2)$ as follows:
$(\tilde{\boldsymbol{\alpha}}_1, \tilde{\boldsymbol{\tau}}_1) := (\tilde{P}, \boldsymbol{T})$ and $(\tilde{\boldsymbol{\alpha}}_2, \tilde{\boldsymbol{\tau}}_2) := (\tilde{P}, \boldsymbol{t})$, where $\boldsymbol{T} = \{T_0, T_1, \ldots, T_{N-1}\}$ and $\boldsymbol{t} = \{t_0, t_1, \ldots, t_{N-1}\}$ such that

$$T_i = \begin{cases} p_i & \text{if } (f(p_{i+1}) - f(p_i)) \cdot (g(p_{i+1} - g(p_i))) < 0, \\ p_{i+1} & \text{if } f(p_{i+1}) - f(p_i) \cdot g(p_{i+1} - g(p_i)) \geq 0 \end{cases}$$

and

$$t_i = \begin{cases} p_{i+1} & \text{if } (f(p_{i+1}) - f(p_i)) \cdot (g(p_{i+1} - g(p_i))) < 0, \\ p_i & \text{if } (f(p_{i+1}) - f(p_i)) \cdot (g(p_{i+1} - g(p_i))) \geq 0. \end{cases}$$

Observe that both $(\tilde{\boldsymbol{\alpha}}_1, \tilde{\boldsymbol{\tau}}_1)$ and $(\tilde{\boldsymbol{\alpha}}_2, \tilde{\boldsymbol{\tau}}_2)$ are γ-fine partitions of $[p_0, p_n]$. Indeed, the tags are from $\tilde{P}$ which means the border points of the intervals which means they are elements of the set M_{n_0}. Hence for all $i = 1, 2, \ldots, N$ it holds that

$$\gamma(T_i) > \frac{1}{2^{n_0}} \overset{**}{\geq} \frac{1}{2^n} + \frac{1}{2^n} > (d_{i+1} - d_i) + 2\eta > p_{i+1} - p_i,$$

where the inequality $\overset{**}{\geq}$ holds because of the fact that $n > n_0$. Similarly

$$\gamma(t_i) > p_{i+1} - p_i.$$

Now we can arbitrarily extend the partitions $(\tilde{\boldsymbol{\alpha}}_1, \tilde{\boldsymbol{\tau}}_1), (\tilde{\boldsymbol{\alpha}}_2, \tilde{\boldsymbol{\tau}}_2)$ to be γ-fine by adding points and tags (same for both partitions) and obtain two partitions $(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1)$ and $(\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)$ which are γ-fine on $[0,1]$. Note that only tags on $[\min_{\tilde{P}}, \max_{\tilde{P}}]$ are different. Finally let us compute

$$|S(f, g, \boldsymbol{\alpha}_1, \boldsymbol{\tau}_1) - S(f, g, \boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)| = |S(f, g, \tilde{P}, \boldsymbol{T}) - S(f, g, \tilde{P}, \boldsymbol{T}) =$$

$$\left|\sum_{i=1}^{N-1} f(T_i)(g(p_{i+1}) - g(p_i)) - \sum_{i=1}^{N-1} f(t_i)(g(p_{i+1}) - g(p_i))\right| =$$

$$= \left|\sum_{i=1}^{N-1} (f(T_i) - f(t_i))(g(p_{i+1}) - g(p_i))\right|.$$

Let $E = \{i : (f(p_{i+1}) - f(p_i)) \cdot (g(p_{i+1}) - g(p_i)) < 0\}$ and let
$F = \{i : f(p_{i+1}) - f(p_i) \cdot g(p_{i+1} - g(p_i)) \geq 0\}$. We continue our computation

$$\left|\sum_{i=1}^{N-1} (f(T_i) - f(t_i))(g(p_{i+1}) - g(p_i))\right| =$$

$$= \left|\sum_{i \in E} (f(p_i) - f(p_{i+1})) \cdot (g(p_{i+1}) - g(p_i))\right| + \left|\sum_{i \in F} (f(p_{i+1}) - f(p_i)) \cdot (g(p_{i+1}) - g(p_i))\right|.$$

Note that all summands are greater or equal to zero due to the choice of tags. Therefore

$$\left|\sum_{i \in E} (f(p_i) - f(p_{i+1})) \cdot (g(p_{i+1}) - g(p_i))\right| + \left|\sum_{i \in F} (f(p_{i+1}) - f(p_i)) \cdot (g(p_{i+1}) - g(p_i))\right| =$$

$$\sum_{i=0}^{N-1} |f(p_{i+1}) - f(p_i)| \cdot |g(p_{i+1}) - g(p_i)| > \sum_{i=0}^{N-1} \left(2^{-n-1}\right)^{\nu} \left(2^{-n-1}\right)^{\upsilon} =$$

$$= N \left(2^{-n-1}\right)^{\nu+\upsilon}.$$

Recall that N is defined as the number of intervals in $\tilde{D}$. $\tilde{D}$ divides the interval $[a, b]$ into parts of length 2^{-n}. Hence it is clear that $N \geq \frac{(b-a)}{2^{-n}} - 2 = 2^n(b - a) - 2$ therefore

$$N \left(2^{-n-1}\right)^{\nu+\upsilon} \geq (2^n(b - a) - 2) \left(2^{-n-1}\right)^{\nu+\upsilon} > 1,$$

where the last inequality holds due to the choice of n. Hence we proved that

$$|S(f, g, \boldsymbol{\alpha}_1, \boldsymbol{\tau}_1) - S(f, g, \boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)| > 1.$$

The proof is now completed.

❑

However, the Kurzweil integral can be useful but we need to use variational approach and not Hölder continuity.

Theorem 41. *Let ν, υ be two nonegative numbers such that*

$$\frac{1}{\nu} + \frac{1}{\upsilon} > 1.$$

Let f, g be two real continuous functions on $[a, b]$ such that f has finite strong ν-variation and g has finite strong υ-variation. Then the strong Kurzweil integral $(SK) \int_a^b f(t)\, \mathrm{d}g(t)$ (and therefore $(SK) \int_a^b g(t)\, \mathrm{d}f(t)$ as we can change f and g in the assumtpions) exist.

Proof. In Dudley and Norvaiša (2010, p. 183) there is proven that the assumption on variation of f and g implies the existence of Kolmogorov integral which under the assumption of continuity of f and g imply the existence of the strong Kurzweil-Stieltjes integral.

❑

Remark. We note that (cf. Dudley and Norvaiša (2010, p. 88)) if we take nonnegative numbers ν, υ such that $\frac{1}{\nu}+\frac{1}{\upsilon} \leq 1$ then the integral $(SK)\int_a^b f(t)\,\mathrm{d}g(t)$ does not exist in general, i.e. there exist functions f and g with finite μ- and υ-variation such that the integral $(SK)\int_a^b f(t)\,\mathrm{d}g(t)$ does not exist.

Finally we apply this result to the stohcastic calculus with respect to the fractional Brownian motion. Recall that

$$I(u,[0,T]) = \inf\{p > 0; \mathcal{V}_p(u,[0,T]) < \infty\}$$

and

$$I(B^H,[0,T]) = \frac{1}{H}.$$

We return to our fixed interval $[0,T]$. We take the fractional Brownian motion B^H with $0 < H < 1$ on $[0,T]$. Then the above theorem suggests that the integral $(SK)\int_a^b u_t\,\mathrm{d}B_t^H$ exists pathwise for any continuous stochastic process u whenever u is of finite strong p-variation such that $H + \frac{1}{p} > 1$, i.e. $p < \frac{1}{1-H}$.

Afterword

At the end of this Thesis we note that many of interesting conceptions related to the stochastic integration were not studied here. It is due to the fact that they are rather complex and, as we want to introduce and compare various ways to define a stochastic integral, they are not of primary interest. For example Itô formulas for the particular integrals. In the majority of cases the Itô formulas can be found in appropriate chapters of the referred sources because they are important for investigating the stochastic differential equations which are also not studied in this Thesis, as it is focused on integration. Next important issue which was not studied is the conception of indefinite integral. If there is fixed interval $[0,T]$ it could be useful to define the particular integral not only as $(\cdot)\int_0^T f\ \mathrm{d}g$ (the symbol $(\cdot)$ before the integral means that we have an arbitrary type of integral discussed in this Thesis) as a random variable but as a random process, i.e. as a function of the upper bound: $(\cdot)\int_0^t f\ \mathrm{d}g, t \in [0,T]$. In the case of integrals defined pathwise there is no ptoblem. On the other hand, in the case of integrals not defined pathwise, e.g. the Skorohod integral, some difficulties appear. The indefinite integral is in the case of non-pathwise integrals usually defined as

$$(\cdot)\int_0^T f\mathbf{1}_{[0,t]}\ \mathrm{d}g, t \in [0,T].$$

The problem is that existence of the integral $(\cdot)\int_0^T f\ \mathrm{d}g$ does not in general imply that the integral $(\cdot)\int_0^T f\mathbf{1}_{[0,t]}\ \mathrm{d}g$ also exists. The conditions for the existence of such defined indefinite integral are also usually discussed in the appropriate chapters of the books and articles which were cited.

Moreover there are other conceptions of stochastic integration which were not studied in this Thesis. Namely for example fWIS and WIS integrals defined by means of the Wick product. This conceptions are rather complex and very different from the conceptions which were studied and hence they are beyond the scope of this Thesis. An exhaustive survey of the fWIS and WIS integrals can be found in Øksendal (2008), Chapter 3 and Chapter 4 respectively.

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