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DIPLOMOVÁ PRÁCE

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Souvislost AdS/CFT korespondence a ambientní konstrukce invariantních operátorů

Ústav teoretické fyziky

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Abstrakt: Hlavním tématem diplomové práce je popis rovnic, které definují částečně nehmotné pole na anti de Sitterově prostoročasu a jejich souvislosti s invariantními operátory na ambientním (plochém) prostoru, který má dimenzi o jedna větší. Částečně nehmotným polím v AdS/CFT korespondenci odpovídají pole v konformní teorii pole splňující jistou konformně invariantní diferenciální rovnici. Částečně nehmotná pole mají odlišný počet stupňů volnosti než hmotná nebo nehmotná pole a podobně jako nehmotná pole mají jejich pohybové rovnice kalibrační invarianci. V práci jsou popsány rovnice, které homogenní pole v ambientním prostoru s danou kalibrační invariancí splňují. Omezením polí a operátorů z ambientního prostoru na anti de Sitterův prostoročas získáme pohybové rovnice pro částečně nehmotná pole a jejich kalibrační invarianci. Pro porovnání odvodíme tvar kalibračního pole i jiným způsobem, který nevyužívá ambientní konstrukce. V závěrečné části práce přejdeme od $\mathrm{SO}(d,2)$ kovariantního zápisu k souřadnicím na anti de Sitterově prostoročasu a vyjádříme pohybové rovnice částečně nehmotných polí ve standardním tvaru pomocí kovariantních derivací na anti de Sitterově prostoru.
Klíčová slova: AdS/CFT korespondence, částečně nehmotná pole, kalibrační invariance, homogenní pole

Title: AdS/CFT correspondence and ambient construction of invariant operators
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Abstract: The main topic of the presented diploma thesis is a description of equations, which define partially massless fields, and their connection with invariant differential operators on the ambient (flat) space which has one dimension more. Partially massless fields in AdS/CFT framework correspond to fields in conformal field theory which satisfy a certain conformally invariant differential equation. Partially massless fields have different degrees of freedom than massless or massive ones. Likewise massless fields, their field equations have a gauge invariance. We find the equation of motion for homogeneous fields on the ambient space with a gauge invariance. By restriction to anti de Sitter spacetime we obtain the equations of motion for partially massless fields together with its gauge field. We derive the gauge field by a different method, which does not use the ambient construction, for comparison. Finally, we move from the $\mathrm{SO}(d,2)$ covariant form to intrinsic coordinates of anti de Sitter spacetime and we express the equation of motion for partially massless fields in a standard form using covariant derivatives on the anti de Sitter spacetime.
Keywords: AdS/CFT correspondence, partially massless fields, gauge invariance, homogeneous fields

Chapter 1

Introduction

1.1 Motivation

Higher spin fields in the (pseudo)Euclidian space are either strictly massless or massive. However, once the cosmological constant is present, and therefore we find ourselves sitting in the (pseudo)Riemannian space, we discover more subtle structure - new fields are present - *partially massless fields*. The usual gauge invariance for a massless field splits into a set of partial ones, allowing partially massless fields to exist. A massless field of spin s has helicities $\pm s$, while a massive field has all possible $(2s+1)$-helicities $-s, -s+1, ..s$. In contrast, partially massless fields have helicities in the set $-s, -s+1, .. -n-1, n+1, ..s$, for $s-2 \geq n$, that is, $-n, -n+1, ..n$ is removed from the complete set of helicities of the massive field. For such a field is

$$m^2 = \frac{\Upsilon}{3}(s(s-1) - n(n+1)) \tag{1.1}$$

and Υ is defined by the Einstein equation $R_{\mu\nu} = \Upsilon g_{\mu\nu}$. A partially massless field with spin s is described by a symmetric tensor field $\phi_{\mu_1\mu_2..\mu_s}$ with a gauge invariance

$$\delta\phi_{\mu_1\mu_2..\mu_s} = D_{\mu_1}..D_{\mu_{s-n}}\xi_{\mu_{s-n+1}..\mu_s} + .. \tag{1.2}$$

where the $+..$ covers terms obtained by symmetrization and terms with less derivatives. We can speak of partially massless fields for positive or negative value of cosmological constant (and Υ). But we should stress, that while, for $\Upsilon > 0$, the partially massless field has positive mass squared, it is not the case for $\Upsilon < 0$. Although, it gives us rather surprising consequences, we focus on the case $\Upsilon < 0$ and we use the framework of AdS/CFT correspondence.

It is known that a massless field in the bulk of spin s corresponds to a conserved symmetric tensor of a rank s in the boundary. By analogy, a partially massless field $\phi_{\mu_1\mu_2..\mu_s}$ correspond to a partially conserved tensor $L^{\mu_1\mu_2..\mu_s}$. For a coupling $\int L^{\mu_1\mu_2..\mu_s}\phi_{\mu_1\mu_2..\mu_s}$, which should be gauge invariant, we deduce that tensor $L^{\mu_1\mu_2..\mu_s}$ is symmetric in its indices and obeys a conformally invariant equation (obtained using (1.2) and integrating per partes)

$$D_{\mu_1}..D_{\mu_{s-n}}L^{\mu_1\mu_2..\mu_s} + .. = 0 \tag{1.3}$$

and the +.. again refers to symmetrized terms and lower order terms.

Also, for spin $s \geq 1$, there are more field components than degrees of freedom. Thus, we impose divergencelessness and tracelessness of the field to get the right number of field components. These constraints are in fact Bianchi identites and are also called traceless-transverse decomposition. The first nontrivial case, $s = 2$, $n = 0$, has been treated in [6].

1.2 The structure and results

In this section, we shall describe shortly the contents of the thesis. The second chapter is devoted to representation theory. It contains a summary of basic notions needed in the thesis as well as some other less known facts (together with their proofs). In particular, the following lemma is important for understanding of the transversality condition for fields defined on the ambient space. It shows that the transversality condition for symmetric tensor fields reduces tensors with the symmetry group $\mathrm{SO}(d,2)$ of the ambient space to tensors with the symmetry group $\mathrm{SO}(d,1)$ of the anti de Sitter spacetime.

Lemma 11 (The condition of transversality)
Let us have an irreducible representation of $\mathrm{SO}(d,2)$ *on the space* S_0^s *of symmetric traceless tensor fields of spin* s*. When restricting to a Lorentz subgroup* $G_y \sim \mathrm{SO}(d,1)$ *of* $\mathrm{SO}(d,2)$ *(the group that preserves the metric* $g^{\mu\nu}$*) the tensor* $A \in S_0^s$ *can be decomposed into the same rank tensor and the lower rank tensors. The condition of transversality*

$$y^{C_1}A_{(C_1..C_s)}(y) = 0$$

sets lower rank tensors to zero.

The relation between fields on the ambient space and fields on the anti de Sitter spacetime is very simple. On the ambient space, we consider always homogeneous functions (fields). Hence, they are uniquely determined by their values on the anti de Sitter space. When we take gauge transformations of a certain type (given by symmetrization of multiple gradients), we can show that the transversality conditions for a gauge field and for the field itself fix the corresponding homogeneity. This is shown in the following lemma which is still contained in the second chapter with preliminary results.

Lemma 12 *Let* $s > n \geq 0$ *and define* B *by* $B_{(C_1..C_s)} = \partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)}$ *where* $\Lambda \in {}_\beta S^n$*. If* $y^{C_1}B_{(C_1..C_s)} = 0$ *and* $y^{C_1}\Lambda_{(C_1..C_n)} = 0$ *then* $\beta = s - 1$*.*

In representation theory, there is a standard way how to compute value of the Casimir operator on a representation with the highest weight. We shall need the same fact in alternative formulation for modules with the lowest weight. The corresponding theorem (and its proof) is contained in the last section of the second chapter.

Theorem 7 *Let* $D(E_0, \mathbf{s})$ *be the lowest weight irreducible representation of* $\mathfrak{so}(d,2)$ *characterized by* $E_0 > 0$*, the lowest positive eigenvalue of operator* $iJ_{0\infty}$*, and by* $\mathbf{s} = (-s, 0, ..0)$*;*

$s > 0$, the lowest weight of the representation of $\mathfrak{so}(d)$. Then the Casimir operator

$$\langle C_2 \rangle \,:\, D(E_0, \mathbf{s}) \to D(E_0, \mathbf{s})$$

is equal to $[E_0(E_0 - d) + s(s + d - 2)]Id$.

The next chapter is devoted to partially massless fields. Partially massless fields are fields with a particular gauge invariance. They are not present in the flat Minkowski case (where there are only two possibilities - massless and massive fields). On spacetimes with a nontrivial cosmological constant, however, there is a wider variety of possibilities. In the third chapter, we study partially massless fields in the formalism of the ambient space. An advantage of the ambient space setting is that the equation of motion has a particularly simple form. It is shown in the following theorem.

Theorem 8 *Let $A \in {}_\gamma S^s_0$ and let $\delta_{s-n} A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s)_0}$ be its gauge field where $\Lambda \in {}_\beta S^n_0$. If conditions*

$$\begin{aligned} \bar{\partial} \left| A \right\rangle &= 0 \\ \bar{y} \left| A \right\rangle &= 0 \\ \bar{\partial} \left| \Lambda \right\rangle &= 0 \\ \bar{y} \left| \Lambda \right\rangle &= 0 \end{aligned}$$

hold, then the equation of motion (3.11) takes the form

$$y^2 \Delta \left| A \right\rangle = 0$$

In chapter 4, we reduce our fields to the anti de Sitter space, it means that we consider them only in the points of the corresponding hyperboloid. Even here, we still work in ambient space coordinates. We write down the form of the gauge transformation and of the equation of motion on the anti de Sitter space.

Lemma 18 *Let $A \in {}_\gamma S^s_0$ and let $\delta_{(s-n)} A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s)_0}$ be its gauge field where $\Lambda \in {}_\beta S^n_0$. Then, the gauge field can be written in terms of differential operators on AdS as*

$$\delta_{(s-n)} A_{(C_1..C_s)_0} = (\nabla_{(C_1} - n y_{(C_1}) .. (\nabla_{C_{s-n}} - (s-1) y_{C_{s-n}}) \Lambda_{C_{s-n+1}..C_s)_0}$$

Theorem 9 *For $A \in {}_\gamma S^s_0$, a representation of $\mathfrak{so}(d,2)$ characterized by the lowest energy $E_0 > 0$ and by $\mathbf{s} = (-s, 0, ..0)$, the lowest weight of $\mathfrak{so}(d)$ and for $\Lambda \in {}_\beta S^n_0$ let*

$$\delta_{(s-n)} \left| A \right\rangle = (\nabla - ny)(\nabla - (n+1)y)..(\nabla - (s-1)y) \left| \Lambda \right\rangle$$

be the gauge field of $|A\rangle$. If conditions

$$\begin{aligned} \bar{\nabla} \left| A \right\rangle &= 0 \\ \bar{y} \left| A \right\rangle &= 0 \\ \bar{\nabla} \left| \Lambda \right\rangle &= 0 \\ \bar{y} \left| \Lambda \right\rangle &= 0 \end{aligned}$$

hold, then the equation of motion in terms of differential operators on AdS is

$$[\nabla^2 - (n-1)(n-1+d)]\,|A\rangle = 0$$

We also provide an alternative way of deriving the gauge invariance, which is based on generalization of Metsaev's approach, in this chapter.

In the last chapter, we translate everything to local coordinates of anti de Sitter spacetime. Namely, we write down the equation of motion and the gauge field in these coordinates.

Theorem 10 *Assume that* $A_{\mu_1..\mu_s} \in \odot_0^s T\mathcal{H}^{d,1}$ *and let*

$$\delta_{(s-n)} A_{(\mu_1..\mu_s)_0} = D_{(\mu_1}..D_{\mu_{s-n}}\Lambda_{\mu_{s-n+1}..\mu_s)_0}$$

be the gauge field of the field A *where* $\Lambda_{\mu_1..\mu_n} \in \odot_0^n T\mathcal{H}^{d,1}$*. If condition*

$$D^\mu A_{\mu\mu_2..\mu_s} = 0$$
$$D^\mu \Lambda_{\mu\mu_2..\mu_n} = 0$$

is satisfied, then the equation of motion in the coordinates x^μ *reads*

$$[D^2 + s - (n-1)(d+n-1)]A_{(\mu_1..\mu_s)_0} = 0$$

Chapter 2

Facts from the representation theory

We make a brief introduction into the representation theory which is very well treated in [1] and [2]. Also proofs of all propositions presented here can be found there.

2.1 Basic notions

Definiton 1. *Let $\mathbb{F}$ be a field. An algebra $\mathfrak{g}$ is a vector space over $\mathbb{F}$ with a product $[X, Y]$ that is linear in each variable. The algebra is a Lie algebra if the product also satisfies*

$$\begin{aligned} &[X, Y] = -[Y, X] \\ &[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad \text{(Jacobi identity)} \end{aligned}$$

for every $X, Y, Z \in \mathfrak{g}$.
For any algebra $\mathfrak{g}$ we get a linear map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\, \mathfrak{g}$ given by

$$[\mathrm{ad}(X)](Y) = [X, Y]$$

Comment. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\, \mathfrak{g}$ is called adjoint action or adjoint representation (we will see later that it is representation).

Definiton 2. *For a finite-dimensional Lie algebra $\mathfrak{g}$ we define*

$$\mathfrak{g}^0 = \mathfrak{g} \qquad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \qquad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]$$

We say that $\mathfrak{g}$ is solvable if there exists j such that $\mathfrak{g}^j = 0$.

Definiton 3. *A finite-dimensional Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ is nonabelian and $\mathfrak{g}$ has no proper nonzero ideals. A finite-dimensional Lie algebra $\mathfrak{g}$ is semisimple if $\mathfrak{g}$ has no proper nonzero solvable ideals.*

Definiton 4. *A symmetric bilinear form B on $\mathfrak{g}$ defined as*

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\, \mathrm{ad}(Y))$$

is called the Killing form.

Comment. The Killing form is invariant in the following sense

$$B(\mathrm{ad}(X)Y, Z) = -B(Y, \mathrm{ad}(X)Z) \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{g}$. In other words

$$B([X, Y], Z) = B(X, [Y, Z]) \tag{2.2}$$

If M is a smooth manifold, smooth vector fields on M can be defined as derivations of the algebra $C^\infty(M)$ of smooth real-valued functions on M and then the tangent space is formed at each point of M out of the smooth vector fields. Alternatively, the tangent space may be constructed first at each point and a vector field may be then defined as a collection of tangent vectors, one at each point. In either case, let us write T_pM for the tangent space of M at p. If X is a vector field on M, let X_p be the value of X at p, i.e., the corresponding tangent vector in T_pM. If $\Phi : M \to N$ is a smooth map between smooth manifolds, we write $d\Phi_p : T_pM \to T_{\Phi(p)}N$ for the differential of Φ at p.

Definiton 5. *A Lie group is a separable topological group with the structure of a smooth manifold such that multiplication and inversion are smooth operations.*

Let G be a Lie group and let $L_x : G \to G$ be a left translation by x, that is, the diffeomorphism from G to itself given by $L_x(y) = xy$. A vector field X on G is left invariant if $dL_{yx^{-1}}X_x = X_y$ for any $x, y \in G$. It is equivalent to say that X as an operator on smooth real-valued functions commutes with left translations.

If G is a Lie group then the map $X \to X_e$ is an isomorphism of the real vector space of left-invariant vector fields on G onto T_eG and the inverse map is $Xf(x) = X_e(L_{x^{-1}}f)$ where $L_{x^{-1}}f(y) = f(xy)$. Every left-invariant vector field on G is smooth and the bracket of two left-invariant vector fields is left invariant as well.

2.2 Classical groups

In this section we introduce families of groups of linear transformations. First, we denote by $M_n\,\mathbb{C}$ the space of all $n \times n$ complex matrices.

Definiton 6. *Let* $\mathrm{GL}(n, \mathbb{C})$ *be the group of invertible* $n \times n$ *complex matrices.*

Definiton 7. *The special linear group* $\mathrm{SL}(n, \mathbb{C})$ *consists of all matrices* $g \in \mathrm{GL}(n, \mathbb{C})$ *with* $\det(g) = 1$. *We will call it a group of type* A_l *where* $l = n - 1$.

To see the geometric significance of this group we recall that the n-th exterior power $\Lambda^n\,\mathbb{C}^n$ is one dimensional (an element in this space is a complex volume form). The group $\mathrm{GL}(n, \mathbb{C})$ acts on volume forms by multiplication by $\det(g)$, so $\mathrm{SL}(n, \mathbb{C})$ is the group of linear transformations that preserve volume forms.

The other classical groups are defined by bilinear forms B on $\mathbb{C}^n$. If B is a bilinear form and $g \in \mathrm{GL}(n, \mathbb{C})$, define a bilinear form $g \cdot B$ by

$$g \cdot B(x, y) = B(g^{-1}x, g^{-1}y) \tag{2.3}$$

If B has matrix T then the bilinear form $g \cdot B$ has matrix $(g^t)^{-1}Tg^{-1}$. We say that B is invariant under g if $g \cdot B = B$. This is equivalent to

$$T = g^t T g \tag{2.4}$$

Definiton 8. *Let B be a nondegenerate symmetric bilinear form on $\mathbb{C}^n$. The orthogonal group relative to B is*

$$\mathrm{O}(\mathbb{C}^n, B) = \{g \in \mathrm{GL}(n, \mathbb{C}); B(gx, gy) = B(x, y),\ x, y \in \mathbb{C}^n\} \tag{2.5}$$

To characterize this group in matrix terms, let S be the matrix of the form: $B(x, y) = x^t Sy$. Then S is a symmetric invertible matrix. From (2.4) we have

$$g \in \mathrm{O}(\mathbb{C}^n, B) \Leftrightarrow g^t S g = S \tag{2.6}$$

Since $\det(S) = \det(g^t S g) = \det(g^2)\det(S)$, we see that $\det(g) = \pm 1$.

Definiton 9. *We define the special orthogonal group relative to B as*

$$\mathrm{SO}(\mathbb{C}^n, B) = \{g \in \mathrm{O}(\mathbb{C}^n, B); \det(g) = 1\} \tag{2.7}$$

We call $\mathrm{SO}(\mathbb{C}^{2l}, B)$ a group of type D_l and $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$ a group of type B_l. The reasons for distinguishing between the even and odd cases are connected with the structure of these groups. When the particular choice of B is understood or irrelevant we will use the notation $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$.

Definiton 10. *Let Ω be a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^n$. Then $n = 2l$ must be even. We define the symplectic group relative to Ω as*

$$\mathrm{Sp}(\mathbb{C}^{2l}, \Omega) = \{g \in \mathrm{GL}(2l, \mathbb{C}); \Omega(gx, gy) = \Omega(x, y), x, y \in \mathbb{C}^{2l}\} \tag{2.8}$$

Let $J \in M_{2l}(\mathbb{C})$ be such that $\Omega(x, y) = x^t Jy$. Then J is skew-symmetric and nonsingular and

$$g \in \mathrm{Sp}(\mathbb{C}^{2l}, \Omega) \Leftrightarrow g^t J g = J \tag{2.9}$$

We call $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ a group of type C_l. Again, if the choice of Ω is understood we shall denote this group as $\mathrm{Sp}(l, \mathbb{C})$.

The groups $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{Sp}(l, \mathbb{C})$ are called the classical groups.

2.3 Lie algebras of the Classical groups

We now determine the Lie algebras of the classical groups that were introduced in the previous subsection. We will use the following result.

Lemma 1. *If G is a Lie group and we set $\mathfrak{g} = T_eG$ then $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ with bracket operation and $\mathfrak{g}$ is called the Lie algebra of G. We also use notation $\mathfrak{g} = \mathrm{Lie}(G)$*

Comment. It is convenient to extend the definition of $X \in \mathfrak{g}$ from real-valued functions to complex-valued functions using $Xf = X(\mathrm{Re}\, f) + iX(\mathrm{Im}\, f)$. Then $\mathfrak{g}$ becomes a Lie algebra over $\mathbb{C}$.

Let $G = \mathrm{SL}(n, \mathbb{C})$. Then

$$\mathfrak{sl}(n, \mathbb{C}) = \mathrm{Lie}(\mathrm{SL}(n, \mathbb{C})) = \{A \in M_n\,\mathbb{C}; \mathrm{Tr}(A) = 0\} \tag{2.10}$$

we also present a derivation of the condition (not really a proof rather a motivation). The left invariant vector fields X form a subalgebra $\mathfrak{g}$ of the Lie algebra of all smooth vector fields, and this is just the Lie algebra of G. We define the function $f(g) = \det(g) - 1$ which is smooth and equal zero on G. For $A \in M_n\,\mathbb{C}$ we have

$$X_A f(g) = \frac{d}{dt} \det(g(I + tA))\bigg|_{t=0} = \det(g)\,\mathrm{Tr}(A) \tag{2.11}$$

If $A \in \mathrm{Lie}(G)$, then $X_A f(g) = 0$ which implies that $\mathrm{Tr}(A) = 0$.

The subgroup $\mathrm{SO}(n, \mathbb{C})$ is open in $\mathrm{O}(n, \mathbb{C})$, so it has the same Lie algebra as $O(n, \mathbb{C})$. In the similar way we could verify that

$$\mathfrak{so}(n, \mathbb{C}) = \{a \in M_n\,\mathbb{C}; A^t = -A\} \tag{2.12}$$

The algebra $\mathfrak{so}(n, \mathbb{C})$ is the Lie algebra of skew-symmetric matrices.

The Lie algebra $\mathfrak{sp}(l, \mathbb{C})$ is given by

$$\mathfrak{sp}(l, \mathbb{C}) = \{A \in M_n\,\mathbb{C}; JA + A^t J = 0\} \tag{2.13}$$

Where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and I is the $l \times l$ identity matrix.

Set $s_l = \mathrm{skewdiag}[1, ..1]$, the $l \times l$ matrix with 1 on the skew diagonal and 0 elsewhere. Denote by

$$J_+ = \begin{pmatrix} 0 & s_l \\ s_l & 0 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & s_l \\ -s_l & 0 \end{pmatrix}$$

and define the bilinear forms

$$B(x, y) = (x, J_+ y), \qquad \Omega(x, y) = (x, J_- y) \qquad \text{for } x, y \in \mathbb{C}^{2l} \tag{2.14}$$

The form B is nondegenerate and symmetric and the form Ω is nondegenerate and skew symmetric.

Lemma 2. *Define B as in (2.14). Then the Lie algebra $\mathfrak{so}(\mathbb{C}^{2l}, B)$ of $\mathrm{SO}(\mathbb{C}^{2l}, B)$ consists of all matrices*

$$A = \begin{pmatrix} a & b \\ c & -s_l a^t s_l \end{pmatrix} \tag{2.15}$$

where $a \in \mathfrak{gl}(l, \mathbb{C})$ and b, c are $l \times l$ matrices such that $b^t = -s_l b s_l$ and $c^t = -s_l c s_l$ (b, c are skew symmetric around the skew diagonal).

Lemma 3. *Define Ω by relation (2.14). Then the Lie algebra $\mathfrak{sp}(\mathbb{C}^{2l}, \Omega)$ of $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ consists of all matrices*

$$A = \begin{pmatrix} a & b \\ c & -s_l a^t s_l \end{pmatrix} \tag{2.16}$$

where $a \in \mathfrak{gl}(l, \mathbb{C})$ and b, c are $l \times l$ matrices such that $b^t = s_l b s_l$ and $c^t = s_l c s_l$ (b, c are symmetric around the skew diagonal).

Lemma 4. *Set*

$$S = \begin{pmatrix} 0 & 0 & s_l \\ 0 & 1 & 0 \\ s_l & 0 & 0 \end{pmatrix} \tag{2.17}$$

and define the symmetric bilinear form $B(x, y) = (x, Sy)$. The Lie algebra $\mathfrak{so}(\mathbb{C}^{2l+1}, B)$ of $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$ consists of all matrices

$$A = \begin{pmatrix} a & w & b \\ u & 0 & -w^t s_l \\ c & -s_l u^t & -s_l a^t s_l \end{pmatrix} \tag{2.18}$$

where $a \in \mathfrak{gl}(l, \mathbb{C})$ and b, c are $l \times l$ matrices such that $b^t = -s_l b s_l$ and $c^t = -s_l c s_l$ (b, c are skew symmetric around the skew diagonal), w is a $l \times 1$ matrix and u is a $1 \times l$ matrix.

2.4 Real forms of classical groups

We will introduce the notion of a real form and study the compact real forms of the classical groups. It is closely connected with enlarging or shrinking of the field of scalars for a Lie algebra. Let $\mathbb{K}$ be a field and let $\mathbb{F}$ be an extension field. If U and V are vector spaces over $\mathbb{K}$ then the tensor product $U \otimes_{\mathbb{K}} V$ is characterized up to canonical isomorphism by the universal mapping property that a $\mathbb{K}$ bilinear map L of $U \otimes V$ into a $\mathbb{K}$ vector space W extends uniquely to a $\mathbb{K}$ linear map $\tilde{L} : U \otimes_{\mathbb{K}} V \to W$. The sense in which $\tilde{L}$ is an extension of L is that $\tilde{L}(u \otimes v) = L(u \otimes v)$ for all $u \in U$ and $v \in V$.

With V as a vector space over $\mathbb{K}$, we are especially interested in the special case $V \otimes_{\mathbb{K}} \mathbb{F}$. If $c \in \mathbb{F}$, then multiplication by c, which we denote $\mu(c)$, is $\mathbb{K}$ linear from $\mathbb{F}$ to $\mathbb{F}$. Thus $1 \otimes \mu(c)$ defines a $\mathbb{K}$ linear map of $V \otimes_{\mathbb{K}} \mathbb{F}$ to itself and we define this to be a scalar multiplication by c in $V \otimes_{\mathbb{K}} \mathbb{F}$. With this definition we easily check that $V \otimes_{\mathbb{K}} \mathbb{F}$ becomes a vector space over $\mathbb{F}$. We write $V^{\mathbb{F}}$ for this vector space. If W is a vector space over the extension field $\mathbb{F}$, we can restrict the definition of scalar multiplication to scalars in $\mathbb{K}$, therefore obtaining a vector space over $\mathbb{K}$. This vector space we denote by $W^{\mathbb{K}}$.

Definiton 11. *In the special case $\mathbb{K} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ and V is a real vector space, the complex vector space $V^{\mathbb{C}}$ is called the complexification of V. If W is complex, then $W^{\mathbb{R}}$ is W regarded as a real vector space.*

Comment. The operations $(\cdot)^{\mathbb{C}}$ and $(\cdot)^{\mathbb{R}}$ are not inverse to each other - $(V^{\mathbb{C}})^{\mathbb{R}}$ has twice the real dimension of V and $(W^{\mathbb{R}})^{\mathbb{C}}$ has twice the complex dimension of W. In fact,

$$(V^{\mathbb{C}})^{\mathbb{R}} = V \oplus iV \tag{2.19}$$

as real vector spaces, where V means $V \otimes 1$ in $V \otimes_{\mathbb{R}} \mathbb{C}$ and the i refers to the real linear transformation of multiplication by i. We often abbreviate the relation (2.19) as

$$V^{\mathbb{C}} = V \oplus iV \tag{2.20}$$

Definiton 12. *When a complex vector space W and a real vector space V are related by*

$$W^{\mathbb{R}} = V \oplus iV \tag{2.21}$$

we say that V is a real form of the complex vector space W.

Let us impose Lie algebra structures on these constructions. Similarly we have

Definiton 13. *When a complex Lie algebra $\mathfrak{g}$ and a real Lie algebra $\mathfrak{g}_0$ are related as a vector space over $\mathbb{R}$ by*

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\, \mathfrak{g}_0 \tag{2.22}$$

we say that $\mathfrak{g}_0$ is a real form of the complex Lie algebra $\mathfrak{g}$. Any real Lie algebra is a real form of its complexification.

Example. Let G denote $\mathrm{O}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}); gg^t = 1\}$ and let $p, q \in \mathbb{N}$ be such that $p + q = n$. Set

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \tag{2.23}$$

with I_r the $r \times r$ identity matrix. The corresponding real form

$$\mathrm{O}(p, q) = \{g \in G; I_{p,q}\bar{g}I_{p,q} = g\} \tag{2.24}$$

We will now present a geometric description of $\mathrm{O}(p, q)$. Let

$$\sigma_{p,q}(z_1, ..z_n) = (\bar{z}_1..\bar{z}_p, -\bar{z}_{p+1}, .. - \bar{z}_n) \tag{2.25}$$

Let $e_1, ..e_n$ be the standard basis of $\mathbb{C}^n$. Then

$$f_1 = e_1, ..f_p = e_p, \qquad f_{p+1} = ie_{p+1}, ..f_n = ie_n \tag{2.26}$$

is a basis of the real vector space $V = \{z \in \mathbb{C}^n; \sigma_{p,q}(z) = z\}$. Let $(z, w) = z_1 w_1 + .. + z_n w_n$. Then $(\cdot,\cdot)|_{V\times V}$ is real valued. Relative to the basis $f_1..f_n$ of V over $\mathbb{R}$ the form is given by

$$s_{p,q}(x, y) = x_1 y_1 + .. + x_p y_p - x_{p+1} y_{p+1} - .. - x_n y_n \tag{2.27}$$

Thus $\mathrm{O}(p, q)$ is isomorphic with the group of all $g \in \mathrm{GL}(n, \mathbb{R})$ such that $s_{p,q}(gx, gy) = s_{p,q}(x, y)$ for all $x, y \in \mathbb{R}^n$.
Example. Let $G = \mathrm{SO}(n, \mathbb{C})$ then the corresponding real form is

$$\mathrm{SO}(p, q) = \{g \in \mathrm{SO}(n, \mathbb{C}); I_{p,q} \bar{g} I_{p,q} = g\} \tag{2.28}$$

2.5 Roots, a root space decomposition, Cartan subalgebra

We will see in this section that each of the classical groups has an abelian subalgebra $\mathfrak{h}$ such that analysis of $\mathrm{ad}\,\mathfrak{h}$ leads to a rather complete understanding of the bracket law in the full Lie algebra. We will give the analysis of $\mathrm{ad}\,\mathfrak{h}$ in each example.

Definiton 14. *A torus is a group $T \subseteq \mathrm{GL}(n, \mathbb{C})$ isomorphic to $\mathbb{C}^\times .. \mathbb{C}^\times$ (l factors) where the integer l is called the rank of T and $\mathbb{C}^\times = \mathbb{C} - \{0\}$. If $G \subseteq \mathrm{GL}(n, \mathbb{C})$, then a torus $H \subset G$ is maximal if it is not contained in any larger torus in G.*

Example. We compute a maximal torus H for some groups G

(1) When $G = \mathrm{SL}(l + 1, \mathbb{C})$, type A_l, then

$$H = \{\mathrm{diag}[x_1, ..x_l, (x_1 \cdot .. \cdot x_l)^{-1}]; x_i \in \mathbb{C}^\times\} \tag{2.29}$$

By our computation in (2.10) we have

$$\mathrm{Lie}(H) = \{\mathrm{diag}[a_1, ..a_{l+1}]; a_i \in \mathbb{C}, \sum a_i = 0\} \tag{2.30}$$

(2) When $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, type C_l or $G = \mathrm{SO}(\mathbb{C}^{2l}, B)$, type D_l, then

$$H = \{\mathrm{diag}[x_1, ..x_l, x_l^{-1}, ..x_1^{-1}]; x_i \in \mathbb{C}^\times\} \tag{2.31}$$

By our computation in (2.12) and (2.13) we have

$$\mathrm{Lie}(H) = \{\mathrm{diag}[a_1, ..a_l, -a_l, .. - a_1]; a_i \in \mathbb{C}\} \tag{2.32}$$

(3) When $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$, type B_l, then

$$H = \{\mathrm{diag}[x_1, ..x_l, 1, x_l^{-1}, ..x_1^{-1}]; x_i \in \mathbb{C}^\times\} \tag{2.33}$$

By our computation in (2.12) we have

$$\mathrm{Lie}(H) = \{\mathrm{diag}[a_1, ..a_l, 0, -a_l, .. - a_1]; a_i \in \mathbb{C}\} \tag{2.34}$$

From now on, G will denote a connected classical group of rank l and $\mathfrak{g} = \mathrm{Lie}(G)$. Thus G is $\mathrm{GL}(l, \mathbb{C})$, $\mathrm{SL}(l+1, \mathbb{C})$, $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$, $\mathrm{SO}(\mathbb{C}^{2l}, B)$ or $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$. The subgroup H of diagonal matrices in G is a maximal torus of rank l and we denote its Lie algebra by $\mathfrak{h}$. Let $x_1, ..x_l$ be the coordinate functions on H as before. Fix a basis for a dual space $\mathfrak{h}^*$ as follows:

(1) Let $G = \mathrm{GL}(l, \mathbb{C})$. Define the linear functional ε_i on $\mathfrak{h}$ by $\varepsilon_i(A) = a_i$ for $A = \mathrm{diag}[a_1, ..a_l] \in \mathfrak{h}$. Then $\{\varepsilon_1, ..\varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.

(2) Let $G = \mathrm{SL}(l+1, \mathbb{C})$. In this case $\mathfrak{h}$ consists of all diagonal matrices of trace zero. Define ε_i as in (1) as a linear functional on the space of diagonal matrices for $i = 1, ..l+1$. The restriction of ε_i to $\mathfrak{h}$ is then an element of $\mathfrak{h}^*$. We will continue to denote this functional by ε_i. The elements of $\mathfrak{h}^*$ can be written uniquely as

$$\sum_{i=1}^{l+1} \lambda_i \varepsilon_i, \qquad \text{with } \lambda_i \in \mathbb{C} \text{ and } \sum_{i=1}^{l+1} \lambda_i = 0 \tag{2.35}$$

The functionals

$$\varepsilon_i - \frac{1}{l+1}(\varepsilon_1 + .. + \varepsilon_{l+1}) \qquad \text{for } i = 1, ..l \tag{2.36}$$

give a basis for $\mathfrak{h}^*$.

(3) Let G be $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$ or $\mathrm{SO}(\mathbb{C}^{2l}, B)$. Define the linear functionals ε_i on $\mathfrak{h}$ as $\varepsilon_i(A) = a_i$ for $A = \mathrm{diag}[a_1, ..a_l, -a_l, .. - a_1] \in \mathfrak{h}$ and $i = 1, ..l$. Then $\{\varepsilon_1, ..\varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.

(4) Let G be $\mathrm{SO}(\mathbb{C}^{2l+1}, B)$. Define the linear functionals ε_i on $\mathfrak{h}$ by $\varepsilon_i(A) = a_i$ for $A = \mathrm{diag}[a_1, ..a_l, 0, -a_l, .. - a_1] \in \mathfrak{h}$ and $i = 1, ..l$. Then $\{\varepsilon_1, ..\varepsilon_l\}$ is a basis for $\mathfrak{h}^*$.

We now study the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ (i.e. $\mathrm{ad}(\mathfrak{h})\, \mathfrak{g}$).

Definiton 15. *For $\alpha \in \mathfrak{h}^*$ let*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; [A, X] = \alpha(A)X \textit{ for all} A \in \mathfrak{h}\} \tag{2.37}$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$ then α is called a root and $\mathfrak{g}_\alpha$ is called a root space. If α is a root then a nonzero element of $\mathfrak{g}_\alpha$ is called a root vector for α. We call the set Φ of roots the root system of $\mathfrak{g}$. Its definition is dependent on the choice of the maximal torus, so we write $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ when we want to make this choice explicit.

We will calculate the roots and root vectors for each type of classical group.

(1) Let $G = \mathrm{GL}(l, \mathbb{C})$ and let E_{ij} for $1 \leq i, j \leq l$ be the usual elementary matrix that takes the basis vector e_j to e_i. For $A = \mathrm{diag}[a_1, ..a_l] \in \mathfrak{h}$ we have

$$[A, E_{ij}] = (a_i - a_j)E_{ij} = (\varepsilon_i - \varepsilon_j)(A)E_{ij} \tag{2.38}$$

Since the set $\{E_{ij}; 1 \leq i, j \leq l\}$ is a basis of $\mathfrak{g} = M_l\, \mathbb{C}$, the roots are

$$\{\varepsilon_i - \varepsilon_j; 1 \leq i, j \leq l, i \neq j\} \tag{2.39}$$

each with multiplicity one. The root space $\mathfrak{g}_\lambda = \mathbb{C}\, E_{ij}$ for $\lambda = \varepsilon_i - \varepsilon_j$.

(2) Let $G = \mathrm{SL}(l+1, \mathbb{C})$. Just as in the previous case, we calculate that the roots are

$$\{\varepsilon_i - \varepsilon_j; 1 \leq i, j \leq l+1, i \neq j\} \tag{2.40}$$

with multiplicity one. The root space $\mathfrak{g}_\lambda = \mathbb{C}\, E_{ij}$ for $\lambda = \varepsilon_i - \varepsilon_j$.

(3) Let $G = \mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$. Label the basis for $\mathbb{C}^{2l}$ as $e_{\pm 1}, ..e_{\pm l}$ with $e_{-i} = e_{2l+1-i}$. Let E_{ij} be the matrix that takes the basis vector e_j to e_i where i and j range over $\pm 1, .. \pm l$. Set $X_{\varepsilon_i - \varepsilon_j} = E_{ij} - E_{-j,-i}$ for $1 \leq i, j \leq l, i \neq j$. Then $X_{\varepsilon_i - \varepsilon_j} \in \mathfrak{g}$ and

$$[A, X_{\varepsilon_i - \varepsilon_j}] = (\varepsilon_i - \varepsilon_j)(A) X_{\varepsilon_i - \varepsilon_j} \tag{2.41}$$

for $A \in \mathfrak{h}$. Hence $\varepsilon_i - \varepsilon_j$ is a root. The roots are associated with embedding $\mathfrak{gl}(l, \mathbb{C}) \to \mathfrak{g}$ given by

$$Y \to \begin{pmatrix} Y & 0 \\ 0 & -s_l Y^t s_l \end{pmatrix}, \qquad Y \in \mathfrak{gl}(l, \mathbb{C}) \tag{2.42}$$

with the matrix $s_l = \mathrm{skewdiag}[1, ..1]$, the $l \times l$ matrix with 1 on the skew diagoal and 0 elsewhere. Set

$$X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} - E_{j,-i} \qquad X_{-\varepsilon_i - \varepsilon_j} = E_{-i,j} + E_{-j,i} \tag{2.43}$$

for $1 \leq i < j \leq l$ and set $X_{2\varepsilon_i} = E_{i,-i}$ for $1 \leq i \leq l$. These matrices are in $\mathfrak{g}$ and

$$[A, X_{\pm(\varepsilon_i + \varepsilon_j)}] = \pm(\varepsilon_i + \varepsilon_j)(A) X_{\pm(\varepsilon_i + \varepsilon_j)} \tag{2.44}$$

for $A \in \mathfrak{h}$. Hence, $\pm(\varepsilon_i + \varepsilon_j)$ are roots for $1 \leq i \leq j \leq l$. From Lemma**3** we see that

$$\{X_{\pm(\varepsilon_i - \varepsilon_j)}, X_{\pm(\varepsilon_i + \varepsilon_j)}; 1 \leq i < j \leq l\} \cup \{X_{\pm 2\varepsilon_i}; 1 \leq i \leq l\} \tag{2.45}$$

is a basis for $\mathfrak{g}$ mod $\mathfrak{h}$. This shows that the roots are

$$\pm(\varepsilon_i - \varepsilon_j) \qquad \text{and} \qquad \pm(\varepsilon_i + \varepsilon_j) \qquad \text{for } 1 \leq i < j \leq l \tag{2.46}$$

$$\pm 2\varepsilon_i \qquad \text{for } 1 \leq i \leq l \tag{2.47}$$

each with multiplicity one.

(4) Let $G = \mathrm{SO}(\mathbb{C}^{2l}, B)$. Label the basis for $\mathbb{C}^{2l}$ and define $X_{\varepsilon_i - \varepsilon_j}$ as in the case of $\mathrm{Sp}(\mathbb{C}^{2l}, \Omega)$. Then $X_{\varepsilon_i - \varepsilon_j} \in \mathfrak{g}$ and (2.41) holds for $A \in \mathfrak{h}$. Hence $\varepsilon_i - \varepsilon_j$ is a root. These roots arise from the same embedding $\mathfrak{gl}(l, \mathbb{C}) \to \mathfrak{g}$ as in the symplectic case. Set

$$X_{\varepsilon_i + \varepsilon_j} = E_{i,-j} - E_{j,-i} \qquad X_{-\varepsilon_i - \varepsilon_j} = E_{-i,j} + E_{-j,i} \tag{2.48}$$

for $1 \leq i < j \leq l$. Then

$$[A, X_{\pm(\varepsilon_i + \varepsilon_j)}] = \pm(\varepsilon_i + \varepsilon_j)(A) X_{\pm(\varepsilon_i + \varepsilon_j)} \tag{2.49}$$

for $A \in \mathfrak{h}$. Thus, $\pm(\varepsilon_i + \varepsilon_j)$ are roots. From **lemma 2** we see that

$$\{X_{\pm(\varepsilon_i-\varepsilon_j)}, X_{\pm(\varepsilon_i+\varepsilon_j)}; 1 \leq i < j \leq l\} \tag{2.50}$$

is a basis for $\mathfrak{g}$ mod $\mathfrak{h}$. This shows that the roots are

$$\pm(\varepsilon_i - \varepsilon_j) \qquad \text{and} \qquad \pm(\varepsilon_i + \varepsilon_j) \qquad \text{for } 1 \leq i < j \leq l \tag{2.51}$$

each with multiplicity one.

(5) Let $G = \mathrm{SO}(\mathbb{C}^{2l+1}, B)$. We embed $\mathrm{SO}(\mathbb{C}^{2l}, B)$ into G as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \tag{2.52}$$

Since $H \subset \mathrm{SO}(\mathbb{C}^{2l}, B) \subset G$ via this embedding, the roots $\pm(\varepsilon_i \pm \varepsilon_j)$ of $\mathrm{ad}(\mathfrak{h})$ on $\mathfrak{so}(\mathbb{C}^{2l}, B)$ also occur for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Label the basis for $\mathbb{C}^{2l+1}$ as

$$e_{-l}, ..e_{-1}, e_0, e_1, ..e_l \tag{2.53}$$

where $e_0 = e_{l+1}$ and $e_{-i} = e_{2l+2-i}$. Then the corresponding root vectors are

$$X_{\varepsilon_i+\varepsilon_j} = E_{i,-j} - E_{j,-i} \qquad X_{-\varepsilon_i-\varepsilon_j} = E_{-i,j} + E_{-j,i} \tag{2.54}$$
$$X_{\varepsilon_i-\varepsilon_j} = E_{ij} - E_{-j,-i} \qquad X_{-\varepsilon_i+\varepsilon_j} = E_{ji} - E_{-i,-j} \tag{2.55}$$

for $1 \leq i < j \leq l$. Define

$$X_{\varepsilon_i} = E_{i0} - E_{0,-i} \qquad X_{-\varepsilon_i} = E_{0i} - E_{-i,0} \tag{2.56}$$

for $1 \leq i \leq l$. Then $X_{\pm\varepsilon_i} \in \mathfrak{g}$ and $[A, X_{\pm\varepsilon_i}] = \pm\varepsilon_i(A)X_{\pm\varepsilon_i}$ for $A \in \mathfrak{h}$. From **lemma 4** we see that $\{X_{\pm\varepsilon_i}; 1 \leq i \leq l\}$ is a basis for $\mathfrak{g}$ mod $\mathfrak{so}(\mathbb{C}^{2l}, B)$. Hence, from the results for $\mathfrak{so}(\mathbb{C}^{2l}, B)$ imply that the roots of $\mathfrak{so}(\mathbb{C}^{2l+1}, B)$ are

$$\pm(\varepsilon_i - \varepsilon_j) \qquad \text{and} \qquad \pm(\varepsilon_i + \varepsilon_j) \qquad \text{for } 1 \leq i < j \leq l \tag{2.57}$$
$$\pm\varepsilon_i \qquad \text{for } 1 \leq i \leq l \tag{2.58}$$

each with multiplicity one.

Theorem 1. *Let G be a classical group and let $H \subset G$ be a maximal torus. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$ and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{h}$ on $\mathfrak{g}$. Set*

$$P(G) = \bigoplus_{i=1}^{l} \mathbb{Z}\,\varepsilon_i$$

(1) *If* $\alpha \in \Phi$ *then* $\alpha \in P(G)$, $\dim \mathfrak{g}_\alpha = 1$ *and*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

(2) *If* $\alpha \in \Phi$ *and* $c\alpha \in \Phi$ *for some* $c \in \mathbb{C}$ *then* $c = \pm 1$.

(3) *The symmetric bilinear form* $(X, Y) = \mathrm{Tr}(XY)$ *on* $\mathfrak{g}$ *is invariant:*

$$([X, Y], Z) = -(Y, [X, Z]) \qquad \textit{for } X, Y, Z \in \mathfrak{g}$$

(4) *Let* $\alpha, \beta \in \Phi$ *and* $\alpha \neq -\beta$. *Then* $(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$ *and* $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

(5) *The form* (X, Y) *on* $\mathfrak{g}$ *is nondegenerate.*

Lemma 5. *If* $\alpha, \beta \in \Phi$ *and* $\alpha + \beta \in \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Corollary. Let $H \subset G \subseteq \mathrm{GL}(n, \mathbb{C})$ be a maximal torus. Denote $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Then $\mathfrak{h}$ is a maximal abelian Lie subalgebra in $\mathfrak{g}$ and it is called a Cartan subalgebra.

Definiton 16. *We call a subset* $\Delta = \{\alpha_1, ..\alpha_l\} \subset \Phi$ *a set of simple roots if every* $\gamma \in \Phi$ *can be written uniquely as*

$$\gamma = n_1\alpha_1 + .. + n_l\alpha_l \qquad \textit{with } n_1, ..n_l \textit{ integers all of the same sign.} \tag{2.59}$$

Comment. The requirement of uniqueness, together with the fact that Φ spans $\mathfrak{h}^*$, implies that Δ is a basis for $\mathfrak{h}^*$.

Definiton 17. *If* Δ *is a set of simple roots, then it partitions* Φ *into two disjoint subsets*

$$\Phi = \Phi^+ \cup (-\Phi^+)$$

where Φ^+ *consists of all roots for which the coefficients* n_i *in* (2.59) *are nonnegative. We call* $\gamma \in \Phi^+$ *a positive root relative to* Δ.

Example. We define the height of a root $\beta = n_1\alpha_1 + .. + n_l\alpha_l$ as $ht(\beta) = n_1 + .. + n_l$.

(1) $G = \mathrm{SO}(2l + 1, \mathbb{C})$: Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq l - 1$ and $\alpha_l = \varepsilon_l$. Take $\Delta = \{\alpha_1, ..\alpha_l\}$. For $1 \leq i < j \leq l$, we can write

$$\varepsilon_i - \varepsilon_j = \alpha_i + .. + \alpha_{j-1} \tag{2.60}$$

$$\varepsilon_i + \varepsilon_j = \alpha_i + .. + \alpha_{j-1} + 2\alpha_j + .. + 2\alpha_l \tag{2.61}$$

And for $1 \leq i \leq l$ we have

$$\varepsilon_i = \alpha_i + .. + \alpha_l \tag{2.62}$$

These formulas show that Δ is a set of simple roots. The associated set of positive roots is

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j; 1 \leq i < j \leq l\} \cup \{\varepsilon_i; 1 \leq i \leq l\}$$

The highest root is

$$\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$$

(2) $G = \mathrm{SO}(2l, \mathbb{C})$ with $l \geq 3$: Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq l-1$ and $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$. Take $\Delta = \{\alpha_1, ..\alpha_l\}$. For $1 \leq i < j \leq l$, we can write

$$\varepsilon_i - \varepsilon_j = \alpha_i + .. + \alpha_{j-1} \tag{2.63}$$

And for $1 \leq i \leq l-1$ we have

$$\varepsilon_i + \varepsilon_{l-1} = \alpha_i + .. + \alpha_l \qquad \varepsilon_i + \varepsilon_l = \alpha_i + .. + \alpha_{l-2} + \alpha_l \tag{2.64}$$

And for $1 \leq i < j \leq l-2$ we have

$$\varepsilon_i + \varepsilon_j = \alpha_i + .. + \alpha_{j-1} + 2\alpha_j + .. + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l \tag{2.65}$$

These formulas show that Δ is a set of simple roots. The associated set of positive roots is

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j; 1 \leq i < j \leq l\}$$

The highest root is

$$\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$$

Definiton 18. *A representation of a group G over field $\mathbb{F}$ is a pair (ρ, V) where V is a vector space over field $\mathbb{F}$ and*

$$\rho : G \to \mathrm{GL}(V)$$

is a group homomorphism from G to the automorphism group of V, that is, $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$.

Definiton 19. *We say that the representation is regular if $\dim V < \infty$ and the functions on G*

$$g \to \langle \rho(g)v, v^* \rangle$$

which we call matrix coefficients of ρ, are regular for all $v \in V$ and $v^* \in V^*$.

Definiton 20. *If (ρ, V) is a regular representation and $W \subset V$ is a linear subspace, then we say that W is G-invariant if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. In this case we obtain a representation σ of G on W by restriction of ρ.*

Definiton 21. *We say that a representation (ρ, V) with $V \neq \{0\}$ is reducible if there is a G-invariant subspace $W \subset V$ such that $W \neq \{0\}$ and $w \neq V$. A representation that is not reducible is called irreducible.*

Definiton 22. *A representation of Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism*

$$\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$$

from $\mathfrak{g}$ to the Lie algebra of endomorphisms on a vector space V.

Lemma 6. *If $\Phi : G \to H$ is a homomorphism of Lie groups, and $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H respectively, then the induced map $\Phi_* : \mathfrak{g} \to \mathfrak{h}$ on tangent spaces is a Lie algebra homomorphism.*

Corollary. In particular, a representation of Lie group

$$\Phi : G \to \mathrm{GL}(V)$$

determines a Lie algebra homomorphism

$$\Phi_* : \mathfrak{g} \to \mathfrak{gl}(V)$$

from $\mathfrak{g}$ to the Lie algebra of the general linear group $\mathrm{GL}(V)$, that is a representation of a Lie algebra $\mathfrak{g}$.

Theorem 2. *Every representation of a finite-dimensional Lie algebra lifts to a unique representation of the associated simply connected Lie group, so that the representations of the simply connected Lie groups are in one-to-one correspondence with the representations of their Lie algebras.*

Definiton 23. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Let (φ, V) be a representation of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$ we define*

$$V_\lambda = \{v \in V; \varphi(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\} \tag{2.66}$$

If $V_\lambda \neq 0$, then V_λ is called a weight space, λ is a weight, and members of the weight space are called weight vectors. We define the height of the weight λ in the same way as for roots. Thus, we have a partial ordering of the weights.

Definiton 24. *The coroots $\{H_1, ..H_l\}$ may be identified with the roots using the trace form inner product on $\mathfrak{h}$. That is,*

$$\alpha_i(H) = \langle H_i, H\rangle \qquad \text{for all } H \in \mathfrak{h} \tag{2.67}$$

Definiton 25. *Define the dominant weights for $\mathfrak{g}$ to be*

$$P_{++}(\mathfrak{g}) = \{\mu \in P(\mathfrak{g}); \mu(H_i) \geq 0 \text{ for } i = 1, ..l\} \tag{2.68}$$

Theorem 3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots. If (φ, V) is a representation of $\mathfrak{g}$, then*

(1) $\varphi(\mathfrak{h})$ acts diagonably on V, so V is the direct sum of all the weight spaces.

(2) Roots and weights are related by $\varphi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$.

Theorem 4. *(Theorem of the highest weight) Suppose (φ, V) is an irreducible finite-dimensional representation of $\mathfrak{g}$. Then V has a unique highest weight λ such that $\mu \prec \lambda$ for all other weights μ of V. One has $\lambda \in P_{++}(\mathfrak{g})$ and $\dim V_\lambda = 1$. A nonzero vector $v_0 V_\lambda$ is called a highest weight vector of V. If U is another irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ, then $U \simeq V$.*

2.6 Casimir operator

Lemma 7. *Let G be a classical group whose Lie algebra $\mathfrak{g}$ is simple. Let (φ, V) be a finite-dimensional representation of $\mathfrak{g}$ such that $\varphi(\mathfrak{g})V \neq 0$.*

(1) Define a bilinear form B_φ on $\mathfrak{g}$ by

$$B_\varphi(X, Y) = \mathrm{Tr}(\varphi(X)\varphi(Y)), \qquad \text{for } X, Y \in \mathfrak{g}$$

Then B_φ is nondegenerate and invariant under $\mathfrak{g}$.

(2) Let $\{X_i\}$ be any basis for $\mathfrak{g}$ and let $\{Y_i\}$ be the basis for $\mathfrak{g}$ such that $B_\varphi(X_i, Y_j) = \delta_{ij}$. Then the linear transformation

$$C_\varphi = \sum_i \varphi(X_i)\varphi(Y_i) \tag{2.69}$$

is independent of the choice of basis $\{X_i\}$.

(3) The operator C_φ commutes with $\varphi(X)$ for all $X \in \mathfrak{g}$.

(4) $\mathrm{Tr}(C_\varphi) = \dim \mathfrak{g}$. In particular, $C_\varphi \neq 0$

As a first application, we take $\varphi = \mathrm{ad}$ to be the adjoint representation. For any Lie algebra $\mathfrak{g}$, the bilinear form $B_{\mathrm{ad}}(X, Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ is called the Killing form of $\mathfrak{g}$.
Corollary. The Killing form on $\mathfrak{g}$ is nondegenerate.

Definiton 26. *Let B be the trace form in the defining representation of $\mathfrak{g}$. Let X_i be a basis for $\mathfrak{g}$ and let X^i be the B-dual basis. Define*

$$C = \sum X_i X^i \tag{2.70}$$

We call C the (universal) Casimir operator associated with form B. If φ is a representation of $\mathfrak{g}$ then $\varphi(C)$ is the operator denoted by C_φ in the lemma above. The operator C does not depend on the choice of basis and dual basis, and it commutes with all generators of $\mathfrak{g}$.

We will write

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \tag{2.71}$$

and it is easy to see that $\rho \in P_{++}$.

Theorem 5. *Let (φ, V) be the representation of a semisimple Lie algebra $\mathfrak{g}$ with the lowest weight $-\lambda$. The Casimir operator $C : V \to V$ is equal to $(\lambda, \lambda + 2\rho)Id$.*

Proof. We have $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$. Let E_α be a generator of $\mathfrak{g}_\alpha$. We know that if $(\mathfrak{g}_\beta, \mathfrak{g}_\gamma) \neq 0$, then $\beta + \gamma = 0$. We can normalize E_α so that $(E_\alpha, E_{-\alpha}) = 1$. We choose a basis X_i of $\mathfrak{h}$

and let X^i be its dual basis. Then $\{H_i\}$ and $\{E_\alpha\}$ form a basis of $\mathfrak{g}$ and $\{H^i\}$ and $\{E_{-\alpha}\}$ form a dual basis. So the Casimir element is (recall the **definition 26**)

$$C = \sum_i H_i H^i + \sum_{\alpha \in \Phi} E_\alpha E_{-\alpha} \tag{2.72}$$

Thus,

$$\begin{aligned}
Cv_{-\lambda} &= \sum_i H_i H^i v_{-\lambda} + \sum_{\alpha\in\Phi} E_\alpha E_{-\alpha} v_{-\lambda} \\
&= \sum_i (-\lambda(H_i))(-\lambda(H^i))v_{-\lambda} + \sum_{\alpha\in\Phi} E_\alpha E_{-\alpha} v_{-\lambda} \\
&= (-\lambda, -\lambda)v_{-\lambda} + \sum_{\alpha\in(-\Phi^+)} [E_{-\alpha}E_\alpha + H_\alpha]v_{-\lambda} \qquad \text{where } H_\alpha = [E_\alpha, E_{-\alpha}] \in \mathfrak{h} \\
&= (-\lambda, -\lambda)v_{-\lambda} + -\lambda\left(\sum_{\alpha\in(-\Phi^+)} H_\alpha\right) v_{-\lambda} \\
&= (-\lambda, -\lambda)v_{-\lambda} + (-\lambda, -2\rho)v_{-\lambda} \\
&= (\lambda, \lambda + 2\rho)v_{-\lambda}
\end{aligned} \tag{2.73}$$

♠

Comment. This works for the highest weight and the highest weight vector as well, of course. But we will apply this theorem to the lowest weight case, that is why we modified it slightly.

2.7 Restriction to a subgroup

Theorem 6. *An irreducible representation of* $\mathrm{SO}(2\nu+1, \mathbb{C})$ *determined by the highest weight* $m = (m_1, m_2, ..m_\nu)$ *with integer components restricted to the subgroup* $G_0 \sim \mathrm{SO}(2\nu, \mathbb{C})$ *contains all irreducible representations of* G_0 *with the highest weights* $q = (q_1, q_2, .q_\nu)$ *for which the following conditions are satisfied*

$$m_1 \geq q_1 \geq m_2 \geq q_2 \geq .. \geq m_\nu \geq q_\nu \geq -m_\nu \tag{2.74}$$

The components q_i *are simultaneously all integers. Every irreducible representation occurs with multiplicity one.*

Similarly, the restriction of the irreducible representations of $\mathrm{SO}(2\nu, \mathbb{C})$ *determined by the highest weight* $m = (m_1, m_2, ..m_\nu)$ *with integer components contains all irreducible representations of the subgroup* $G_0 \sim \mathrm{SO}(2\nu-1, \mathbb{C})$ *with the highest weight* $p = (p_1, p_2, ..p_{\nu-1})$ *for which*

$$m_1 \geq p_1 \geq m_2 \geq p_2 .. \geq m_{\nu-1} \geq p_{\nu-1} \geq |m_\nu| \tag{2.75}$$

The components p_i are simultaneously all integers. Every irreducible representation occurs with multiplicity one.

Comment. Clearly, all statements hold for real forms of $\mathrm{SO}(n,\mathbb{C})$ and in particular for orthogonal real groups $\mathrm{SO}(n)$ and $\mathrm{SO}(p,q), p+q=n$.

Definiton 27. *Let $t>0$. By dilatation map we mean*

$$\begin{aligned} \delta_t \ : \ \mathbb{R}^{d,2} \rightarrow \mathbb{R}^{d,2} \\ (y^0, y^1, ..y^\infty) \rightarrow (ty^0, ty^1, ..ty^\infty) \end{aligned} \tag{2.76}$$

The vector field tangent to the dilatations is denoted by T.

Lemma 8. *The generator of the dilatations T can be expressed in the basis of the tangent space as follows*

$$T = y^A \partial_A \tag{2.77}$$

Proof. For any $f \in C^\infty(\mathbb{R}^{d,2})$ and $y \in \mathbb{R}^{d,2}$ we have that

$$\begin{aligned} Tf(y) &= \frac{d}{dt}(\delta_t^* f(y))\Big|_{t=0} = \frac{d}{dt}(f(ty))\Big|_{t=0} \\ &= y^A \partial_A f(y) \end{aligned} \tag{2.78}$$

Therefore, we can write $T = y^A \partial_A$. T is also called the Euler operator. ♠

By AdS spacetime we mean a $(d+1)$-dimensional hypersurface $\mathcal{H}^{d,1}$

$$\eta_{AB} y^A y^B = -1 \tag{2.79}$$

in $(d+2)$-dimensional pseudo-Euclidian space with metric η_{AB}. The differentiation gives

$$\eta_{AB} y^A dy^B = 0 \tag{2.80}$$

and the components of 1-forms dy^B can be expressed using intrinsic coordinates $(x^0, x^1, ..x^d)$ of AdS as $dy^B = (\frac{\partial}{\partial x^0} y^B, .. \frac{\partial}{\partial x^d} y^B)$. It is enough to realize that the matrix $(\frac{\partial}{\partial x^i} y^B)_{i=0,1..d}^{B=0,1..\infty}$ has a rank $d+1$ because it is parametrization of $\mathcal{H}^{d,1}$. We use $T_y \mathcal{H}^{d,1}$ to denote the tangent space of the AdS.

Lemma 9. *Take $y = (y^0, y^1, ..y^\infty)$ and $T_y \mathbb{R}^{d,2}$ a tangent space at y. Let $T = y^C \partial_C$ be the generator of translations. Then,*

$$T \perp T_y \mathcal{H}^{d,1} \tag{2.81}$$

Proof. From the equation (2.80) we see that $(y^C \partial_C y^A)(\frac{\partial}{\partial x^i} y_A) = 0$ for every i and vectors $\frac{\partial}{\partial x^i}$ form the basis of $T_y \mathcal{H}^{d,1}$. ♠

Corollary. We can write for any point $y \in \mathbb{R}^{d,2}$ the tangent space as an orthogonal decomposition

$$T_y \mathbb{R}^{d,2} = \langle T \rangle \oplus T_y \mathcal{H}^{d,1}$$

where $\langle T\rangle$ is a subspace generated by T. Likewise, the cotangent space

$$T_y^* \mathbb{R}^{d,2} = \langle T\rangle \oplus T_y^* \mathcal{H}^{d,1}$$

Lemma 10. *Take $y \in T_y^* \mathbb{R}^{d,2}$. Then there is an orthogonal basis of $T_y^*\mathcal{H}^{d,1}$*

$$\{\phi^{(0)}, \phi^{(1)}, ..\phi^{(d)}\},\ \phi^{(i)} \in T_y^*\mathcal{H}^{d,1}$$

such that

$$\{y, \phi^{(0)}, \phi^{(1)}, ..\phi^{(d)}\}$$

form the basis of $T_y^* \mathbb{R}^{d,2}$.
Then, the components of the metric tensor η_{AB} at a point y can be written as

$$\eta_{AB}(y) = -y_A y_B - \phi_A^{(0)}\phi_B^{(0)} + \sum_{i=1}^{d} \phi_A^{(i)}\phi_B^{(i)} \tag{2.82}$$

Proof. The previous corollary on the orthogonal decomposition gives the claim we only have to ensure the right signature. ♠

Definiton 28. *For every point $y \in \mathbb{R}^{d,2}$ we define the group $G_y \sim \mathrm{SO}(d,1)$ which preserves the metric $g^{\mu\nu}$ induced from η^{AB} by restricting to AdS.*

Let us have an irreducible representation of $\mathrm{SO}(d,2)$ on the space S_0^s of symmetric traceless tensor fields of spin s. In any point $y \in \mathbb{R}^{d,2}$ and for $A \in S_0^s$ we have a representation of $\mathrm{SO}(d,2)$ on the values of field A determined by its highest weight $(s, 0, ..0)$. When restricting to a Lorentz subgroup $G_y \sim \mathrm{SO}(d,1)$ of $\mathrm{SO}(d,2)$ the tensor $A \in S_0^s$ can be decomposed into the same rank tensor and the lower rank tensors. The transversality condition sets the lower rank tensors to zero. To support this statement we present

Lemma 11. *The condition of transversality*

$$y^{C_1} A_{(C_1..C_s)}(y) = 0 \tag{2.83}$$

sets lower rank tensors to zero in the above mentioned decomposition.

Proof. From the definition, $S_{0y}^s = \odot_0^s T_y \mathbb{R}^{d,2}$. Because $A_{C_1..C_s}(y)$ is the representation of $\mathrm{SO}(d,2)$ with the highest weight $(s, 0..0)$ and we want to restrict this representation to a representation of a subgroup $G_y \sim \mathrm{SO}(d,1)$ we can use **theorem 27** to write $A_{C_1..C_s}(y)$ as a sum of $B^{(i)}(y) \in S_0^i$, the irreducible representation of $G_y \sim \mathrm{SO}(d,1)$ with the highest weight $(i, 0, ..0)$ where $s \geq i \geq 0$, which are complemented to a right number of indices by a tensor on which G_y acts trivially. While tensors $B^{(i)}(y)$ are traceless, tensor powers of $\langle y\rangle$ are not. Thus,

$$\begin{aligned} A_{(C_1C_2..C_s)}(y) &= B^{(s)}_{(C_1C_2..C_s)}(y) + y_{(C_1} B^{(s-1)}_{C_2..C_s)}(y) \\ &\quad + [y_{(C_1}y_{C_2} + \frac{1}{d+2}\eta_{(C_1C_2}]B^{(s-2)}_{C_3..C_s)}(y) + .. + y_{(C_1}y_{C_2}..y_{C_s)_0}B^{(0)}(y) \end{aligned} \tag{2.84}$$

We use **lemma 10** to decompose η_{AB}. The transversality condition for $A \in S_0^s$ reads

$$\begin{aligned} 0 &= \eta^{AC_1} y_A A_{(C_1C_2..C_s)}(y) \\ &= \eta^{AC_1} y_A [B^{(s)}_{(C_1C_2..C_s)}(y) + y_{(C_1} B^{(s-1)}_{C_2..C_s)}(y) \\ &+ [\frac{d+1}{d+2} y_{(C_1} y_{C_2} + \frac{1}{d+2} \sum_i \phi^{(i)}_{C_1} \phi^{(i)}_{C_2}] B^{(s-2)}_{C_3..C_s)}(y) + .. + y_{(C_1} y_{C_2} .. y_{C_s)_0} B^{(0)}(y)] \end{aligned} \tag{2.85}$$

First, $y^{C_1} B^{(s)}_{(C_1C_2..C_s)}(y) = 0$ because the decomposition of the tangent space is orthogonal. We don't need to write all terms but it is clear that there will be term equal (up to a non zero constant) to

$$y^{C_1} y_{(C_1} .. y_{C_s)} B^{(0)}(y)$$

There will be no other terms from $\odot^s \langle T \rangle$, therefore, $B^{(0)}(y) = 0$. We can finish the proof by induction on i. Assume that $B^{(i)}(y) = 0$ for every $i < n \leq s - 1$. The term that equals (again, up to a non zero constant) to

$$y^{C_1} y_{(C_1} .. y_{C_{s-n}} B^{(n)}_{C_{s-n+1}..C_s}$$

is the only term laying in $(\odot^{s-n} \langle T \rangle) \odot (\odot^n T^*_y \mathcal{H}^{d,1})$. Hence, $B^{(n)}(y) = 0$ is zero. The tensor $A \in S^s_{0y}$ that satisfies the transversality condition is then

$$A_{(C_1C_2..C_s)}(y) = B^{(s)}_{(C_1C_2..C_s)}(y) \tag{2.86}$$

♠

Definiton 29. *The field $A \in S^s$ is homogeneous of order γ, denoting as $A \in {}_\gamma S^s$, when*

$$(y^X \partial_X) A_{(C_1..C_s)} = \gamma A_{(C_1..C_s)} \tag{2.87}$$

The following lemma computes the right homogeneity of the gauge field.

Lemma 12. *Let $s > n \geq 0$ and define B by $B_{(C_1..C_s)} = \partial_{(C_1} .. \partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s)}$ where $\Lambda \in {}_\beta S^n$. If $y^{C_1} B_{(C_1..C_s)} = 0$ and $y^{C_1} \Lambda_{(C_1..C_n)} = 0$ then $\beta = s - 1$.*

Proof. First, from the condition $y^{C_1} B_{(C_1..C_s)} = 0$:

$$\begin{aligned} 0 &= y^{C_1} [\partial_{(C_1} .. \partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s)}] \\ &= y^{C_1} [\frac{1}{s!} \sum_\pi \partial_{(C_{\pi(1)}} .. \partial_{C_{\pi(s-n)}} \Lambda_{C_{\pi(s-n+1)}..C_{\pi(s)})}] \end{aligned} \tag{2.88}$$

where π runs through all permutations on s indices. We know that Λ is symmetric. Also, partial derivatives commute because we are in the flat space. If $\pi^{-1}(1) \in \{1, 2, ..(s-n)\}$ we can always commute derivatives and use the fact that derivative decreases the homogeneity by one. Then we have

$$y^{C_1}\partial_{C_1}\frac{1}{(s-1)!}\sum_{\tilde{\pi}}[\partial_{C_{\tilde{\pi}(2)}}..\partial_{C_{\tilde{\pi}(s-n+1)}}\Lambda_{C_{\tilde{\pi}(s-n+1)}..C_{\tilde{\pi}(s)}}] =$$
$$= (\beta-(s-n-1))[\partial_{(C_2}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)}] \quad (2.89)$$

and $\tilde{\pi}$ is permutation on indices $\{2,..s\}$.
If this is not the case and $\pi^{-1}(1)\in\{(s-n+1),..s\}$ we use transversality of Λ

$$y^{C_1}\Lambda_{(C_1..C_n)} = 0$$

to prove

$$y^{C_1}\frac{1}{(s-1)!}\sum_{\tilde{\pi}}[\partial_{C_{\tilde{\pi}(2)}}..\partial_{C_{\tilde{\pi}(s-n+1)}}\Lambda_{..C_1..)}] = (-s+n)[\partial_{(C_2}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)}] \quad (2.90)$$

Together,

$$\begin{aligned}
0 &= y^{C_1}[\partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)}] \\
&= \frac{1}{s!}[(s-n)(s-1)!\,y^{C_1}\partial_{C_1}\frac{1}{(s-1)!}\sum_{\tilde{\pi}}[\partial_{C_{\tilde{\pi}(2)}}..\partial_{C_{\tilde{\pi}(s-n+1)}}\Lambda_{C_{\tilde{\pi}(s-n+1)}..C_{\tilde{\pi}(s)}}] \\
&+ n(s-1)!y^{C_1}\frac{1}{(s-1)!}\sum_{\tilde{\pi}}[\partial_{C_{\tilde{\pi}(2)}}..\partial_{C_{\tilde{\pi}(s-n+1)}}\Lambda_{..C_1..)}] \\
&= \frac{1}{s}[(s-n)(\beta-(s-n-1))+n(-s+n)]\partial_{(C_2}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)} \\
&= \frac{s-n}{s}(\beta-s+1)\partial_{(C_2}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)} \qquad (2.91)
\end{aligned}$$

♠

Corollary. If $\delta_{(s-n)}A_{(C_1..C_s)} = \partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)}$ is the gauge field of $A\in {}_{\gamma}S^s$ for $\Lambda\in {}_{\beta}S^n$ and the transversality conditions

$$y^{C_1}A_{(C_1..C_s)} = 0 \quad (2.92)$$
$$y^{C_1}\Lambda_{(C_1..C_n)} = 0 \quad (2.93)$$

then the right homogeneity of the field A is

$$\gamma = s-1-(s-n) = n-1 \quad (2.94)$$

2.8 Realization of $\mathfrak{so}(d,2)$ on the space of symmetric tensors

We start with the group $\mathrm{O}(n)$. The generators X_{ij} of $\mathrm{O}(n)$ can be expressed in terms of the generators E_{ij} of the group $\mathrm{GL}(n)$ as follows:

$$X_{ij} = E_{ij} - E_{ji} \tag{2.95}$$

where, of course, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Also, the generators of $\mathrm{GL}(n)$ satisfy the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{jk} \tag{2.96}$$

The commutation relations for generators X_{ij} can be obtained from those of E_{ik}:

$$[X_{ij}, X_{kl}] = \delta_{jk}X_{il} + \delta_{il}X_{jk} - \delta_{ik}X_{jl} - \delta_{jl}X_{ik} \tag{2.97}$$

Let $J_{AB} \in \mathfrak{so}(d,2)$, $A, B = 0, 1..d, \infty$; $A < B$, be a generator of the group $\mathrm{SO}(d,2)$ where the metric is $(d+2)\times(d+2)$ matrix $\eta_{AB} = \mathrm{diag}(-1,+1,..+1,-1)$. We can carry out the previous computation in the similar way and we find out that

$$(J_{AB})^C_D = \eta_{AD}\eta^C_B - \eta^C_A\eta_{BD} \tag{2.98}$$

and (2.97) is modified into

$$[J_{AB}, J_{CD}] = \eta_{BC}J_{AD} + \eta_{AD}J_{BC} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC} \tag{2.99}$$

Then $g = e^{tJ_{AB}}$ is an element of $\mathrm{SO}(d,2)$.

Lemma 13. *Let $g \in \mathrm{SO}(d,2)$, $y \in \mathbb{R}^{d,2}$, and let $A \in \odot^s T_y\,\mathbb{R}^{d,2}$. Then, the representation π of $\mathrm{SO}(d,2)$ on the space of symmetric tensors of order s is given by*

$$[\pi(g)A](y) = (g^{-1})^{C'_1}_{C_1}..(g^{-1})^{C'_s}_{C_s}A_{C'_1..C'_s}(g^{-1}y) \tag{2.100}$$

We can derive the action of $\mathfrak{so}(d,2)$ on the symmetric tensors by differentiation along the parameter t where we take $g = e^{tJ_{AB}}$.

$$\begin{aligned}
&\frac{d}{dt}\,[\pi(e^{tJ_{AB}})A](y)\Big|_{t=0} = \\
&\quad= \frac{d}{dt}\,(e^{-tJ_{AB}})^{C'_1}_{C_1}..(e^{-tJ_{AB}})^{C'_s}_{C_s}A_{C'_1..C'_s}((e^{-tJ_{AB}})y)\Big|_{t=0} \\
&\quad= [((-J_{AB}e^{-tJ_{AB}})^{C'_1}_{C_1}..(e^{-tJ_{AB}})^{C'_s}_{C_s}) + ..((e^{-tJ_{AB}})^{C'_1}_{C_1}..(-J_{AB}e^{-tJ_{AB}})^{C'_s}_{C_s})]A_{C'_1..C'_s}((e^{-tJ_{AB}})y) \\
&\qquad+(e^{-tJ_{AB}})^{C'_1}_{C_1}..(e^{-tJ_{AB}})^{C'_s}_{C_s}\frac{d}{dt}A_{C'_1..C'_s}((e^{-tJ_{AB}})y)\Big|_{t=0} = \\
&\quad= [(-J_{AB})^{C'_1}_{C_1}A_{C'_1C_2..C_s}(y) + .. + (-J_{AB})^{C'_s}_{C_s}A_{C_1..C_{s-1}C'_s}(y)] - (y_A\partial_B - y_B\partial_A)A_{C_1..C_s}(y) \\
&\quad= [(-\eta_{AC_1}\eta^{C'_1}_B + \eta^{C'_1}_A\eta_{BC_1})A_{C'_1C_2..C_s}(y) + .. + (-\eta_{AC_s}\eta^{C'_s}_B + \eta^{C'_s}_A\eta_{BC_s})A_{C_1..C_{s-1}C'_s}(y) \\
&\qquad+ [-y_A\partial_B + y_B\partial_A]A_{C_1..C_s}(y) \\
&\quad= [-\eta_{AC_1}A_{BC_2..C_s}(y) + \eta_{BC_1}A_{AC_2..C_s}(y) + .. - \eta_{AC_s}A_{C_1..C_{s-1}B}(y) + \eta_{BC_s}A_{C_1..C_{s-1}A}(y) \\
&\qquad+ [-y_A\partial_B + y_B\partial_A]A_{C_1..C_s}(y)
\end{aligned} \tag{2.101}$$

where we used formula

$$
\begin{aligned}
&(e^{-tJ_{AB}})^{C_1'}_{C_1}..(e^{-tJ_{AB}})^{C_s'}_{C_s}\frac{d}{dt}A_{C_1'..C_s'}((e^{-tJ_{AB}})y)\Big|_{t=0} = \\
&= (e^{-tJ_{AB}})^{C_1'}_{C_1}..(e^{-tJ_{AB}})^{C_s'}_{C_s}\frac{d}{dt}(y'^X)\partial'_X A_{C_1'..C_s'}(y')\Big|_{t=0} \\
&= (-J_{AB}y)^X \partial_X A_{C_1..C_s}(y) \\
&= ((-\eta_{AD}\eta^X_B + \eta^X_A\eta_{BD})y^D)\partial_X A_{C_1..C_s}(y) \\
&= (-y_A\partial_B + y_B\partial_A)A_{C_1..C_s}(y)
\end{aligned}
\tag{2.102}
$$

with obvious substitution $y' = (e^{-tJ_{AB}})y$.

2.9 Irreducible representations of $\mathfrak{so}(d,2)$

The lowest weight irreducible representations of $\mathfrak{so}(d,2)$ denoted as $D(E_0,\mathbf{s})$ are defined by $E_0 > 0$, the lowest positive eigenvalue of operator $iJ_{0\infty}$, and by $\mathbf{s} = (-s,0,..0)$; $s > 0$, the lowest weight of the representation of $\mathfrak{so}(d)$. Since the representations in question have, by definition, the energy bounded from below, they contain the vacuum $|\Phi(E_0,\mathbf{s})\rangle$. Thus, $|\Phi(E_0,\mathbf{s})\rangle$ is the eigenvector of $iJ^{\infty 0}$ with eigenvalue E_0 and also a representation of $\mathfrak{so}(d)$ with the lowest weight $\mathbf{s}$. The second order Casimir operator C_2 of $\mathfrak{so}(d,2)$ can be expressed in terms of generators

$$
C_2 = \eta^{AD}\eta^{BC}J_{AB}J_{CD};\ A < B \tag{2.103}
$$

For convenience, we change our coordinates $(y^0,y^I,y^\infty) \to (z,y^I,\bar{z})$ where $z = \frac{1}{\sqrt{2}}(y^0 + iy^\infty)$, $\bar{z} = z^*$. Here and below $I,K,J,L = 1,..d$. Nonzero elements of transformed metric are $\eta_{z\bar{z}} = -1$ and $\eta_{II} = 1$. The energy operator in new coordinates is $J_{z\bar{z}}$, in other words:

$$
J_{z\bar{z}}\,|\Phi(E_n,\mathbf{s})\rangle = E_n\,|\Phi(E_n,\mathbf{s})\rangle \tag{2.104}
$$

It is easy to prove that operators J_{zI} decrease the energy

$$
\begin{aligned}
J_{z\bar{z}}J_{zI}\,|\Phi(E_n,\mathbf{s})\rangle &= ([J_{z\bar{z}},J_{zI}] + J_{zI}J_{z\bar{z}})\,|\Phi(E_n,\mathbf{s})\rangle \\
&= (\eta_{\bar{z}z}J_{zI} + J_{zI}E_n)\,|\Phi(E_n,\mathbf{s})\rangle \\
&= (E_n - 1)J_{zI}\,|\Phi(E_n,\mathbf{s})\rangle
\end{aligned}
\tag{2.105}
$$

Since E_0 is the lowest energy we immediately get

$$
J_{zI}\,|\Phi(E_0,\mathbf{s})\rangle = 0 \tag{2.106}
$$

Likewise, $J_{\bar{z}I}$ increases the energy and the representation $D(E_0,\mathbf{s})$ can be built by acting with these operators on vacuum $|\Phi(E_0,\mathbf{s})\rangle$:

$$
D(E_0,\mathbf{s}) = \bigoplus_{n=0}^{\infty} J_{\bar{z}I_1}..J_{\bar{z}I_n}\,|\Phi(E_0,\mathbf{s})\rangle \tag{2.107}
$$

Now, we are ready to compute the value of the second order Casimir operator $\langle C_2 \rangle$ of $D(E_0, \mathbf{s})$

Theorem 7. *Let $D(E_0, \mathbf{s})$ be the lowest weight irreducible representation of $\mathfrak{so}(d, 2)$ characterized by $E_0 > 0$, the lowest positive eigenvalue of operator $iJ_{0\infty}$, and by $\mathbf{s} = (-s, 0, ..0)$; $s > 0$, the lowest weight of the representation of $\mathfrak{so}(d)$. Then the Casimir operator*

$$\langle C_2 \rangle \,:\, D(E_0, \mathbf{s}) \to D(E_0, \mathbf{s})$$

is equal to $[E_0(E_0 - d) + s(s + d - 2)]Id$.

Proof. Let us first compute the Casimir operator on the vacuum state $|\Phi(E_0, \mathbf{s})\rangle$

$$\begin{aligned}
\langle C_2 \rangle |\Phi(E_0, \mathbf{s})\rangle &= [\eta^{z\bar{z}}\eta^{IK} J_{zI} J_{K\bar{z}} + \eta^{z\bar{z}}\eta^{\bar{z}z} J_{z\bar{z}} J_{z\bar{z}} \\
&\quad + \frac{1}{2}\eta^{IL}\eta^{JK} J_{IJ} J_{KL} + \eta^{IK}\eta^{\bar{z}z} J_{I\bar{z}} J_{zK}] \,|\Phi(E_0, \mathbf{s})\rangle \\
&= [-\eta^{z\bar{z}}\eta^{IK}[J_{zI}, J_{\bar{z}K}] + (J_{z\bar{z}})^2 + C_2(\mathfrak{so}(d)) + 2\eta^{IK}\eta^{\bar{z}z} J_{I\bar{z}} J_{zK}] \,|\Phi(E_0, \mathbf{s})\rangle \\
&= [\eta^{IK}(\eta_{I\bar{z}} J_{zK} + \eta_{zK} J_{I\bar{z}} - \eta_{z\bar{z}} J_{IK} - \eta_{IK} J_{z\bar{z}}) + E_0^2 + C_2(\mathfrak{so}(d))] \,|\Phi(E_0, \mathbf{s})\rangle \\
&= [-dE_0 + E_0^2 + C_2(\mathfrak{so}(d))] \,|\Phi(E_0, \mathbf{s})\rangle \\
&= [E_0(E_0 - d) + s(s + d - 2)] \,|\Phi(E_0, \mathbf{s})\rangle
\end{aligned} \tag{2.108}$$

where we used formula for the second Casimir operator of $\mathfrak{so}(d)$ for totally symmetric representation with the lowest weight $\mathbf{s} = (-s, 0, ..0)$. Recalling **theorem 5** we can easily compute its value

$$C_2(\mathfrak{so}(d)) = (\mathbf{s}, \mathbf{s} + 2\rho) = s(s + d - 2)$$

Moreover, from **lemma 7**, part (3), we know that the Casimir operator commutes with $J_{\bar{z}I_i}$. Thus, our computation holds also for $|\Phi(E_n, \mathbf{s})\rangle = J_{\bar{z}I_1}..J_{\bar{z}I_n} |\Phi(E_0, \mathbf{s})\rangle \in D(E_0, \mathbf{s})$ where $n \in \mathbb{N}$ is arbitrary. The linearity completes the proof for any $|\Phi\rangle \in D(E_0, \mathbf{s})$. ♠

Chapter 3

Partially massless fields on the ambient space $R^{d,2}$

3.1 The description of partially massless fields

We will briefly describe our notation here. We denote the set of all symmetric fields of spin s which are traceless and homogeneous of degree β as ${}_{\beta}S_0^s$ where, of course, $S^s = \odot^s T\,\mathbb{R}^{d,2}$. Thus, for any point $y \in \mathbb{R}^{d,2}$ we have $S_y^s = \odot^s T_y^*\,\mathbb{R}^{d,2}$. All fields in consideration are defined everywhere except the origin (but we will usually omit this). It is clear that such a field is given for example by its values on the hyperplane $y^\infty = 1$. Coordinates on the ambient space are $(y^0, y^1, ..y^d, y^\infty)$ and the metric $\eta^{AB} = \mathrm{diag}(-1, 1, ..1, -1)$. We use capital letters for indices on the ambient space and their values cover all the coordinate axis (of course) - for example $A = 0, 1, ..n, \infty$. We find useful to identify symmetric tensors with vectors in the Fock space. For the $A \in S^s$ with components $A_{(C_1..C_s)}$ the corresponding vector is:

$$|A\rangle = a^{C_1}..a^{C_s} A_{C_1..C_s} |0\rangle \tag{3.1}$$

and a^C ($\bar{a}^C$) are the creation (annihilation) operators which satisfy the commutation relations $[\bar{a}^C, a^D] = \eta^{CD}$ and for annihilation operators also $\bar{a}^C |0\rangle = 0$ holds. Since the creation operators commute we have a nonzero contribution only for symmetric tensors or reading it the other way around, we don't have to stress the symmetrization of the field A. The isomorphism between symmetric tensors and vectors in the Fock space is clear and we will interchange these two descriptions a lot.

Therefore, using this notation, we can rewrite the formula (2.101) (identifying $\pi(J_{AB})$ and J_{AB})

$$J_{AB} |A\rangle = -[(a_A\bar{a}_B - a_B\bar{a}_A) + (y_A\partial_B - y_B\partial_A)] |A\rangle \tag{3.2}$$

If we set

$$M_{AB} = a_A\bar{a}_B - a_B\bar{a}_A \qquad\qquad L_{AB} = y_A\partial_B - y_B\partial_A \tag{3.3}$$

we have a decomposition

$$J_{AB} = -(M_{AB} + L_{AB}) \tag{3.4}$$

3.2 The field equation

As an easy application of the **theorem 5** we get that, for an irreducible representation of SO$(d,2)$ realized on ${}_\gamma S^s_0$,

$$(C_2 - t)\,|A\rangle = 0 \tag{3.5}$$

where $t \in \mathbb{R}$ is constant given by the lowest weight. It is true for the lowest weight vector but since the Casimir operator commutes with all generators, it is true for all $|A\rangle \in {}_\gamma S^s_0$. This does not imply the irreducibility but it is a necessary condition.

Comment. We often abbreviate the gauge transformation as

$$\delta_{(s-n)} A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s)_0} \tag{3.6}$$

and we call this field a gauge field. The gauge field $\delta_{(s-n)}A$ satisfies the equation (3.5) with same parameter t as the field A.

We write for short $y = y_A a^A$, $\bar{y} = y_A \bar{a}^A$ and in the same manner $\partial = \partial_A a^A$, $\bar{\partial} = \partial_A \bar{a}^A$.

To further reduce the space of solutions of the equation (3.5) we impose the following subsidiary conditions

$$\bar{\partial}\,|A\rangle = 0 \tag{3.7}$$

$$\bar{y}\,|A\rangle = 0 \tag{3.8}$$

The condition (3.7) is an analogy of the divergencelessness of the electromagnetic field. The equation (3.8) is the already discussed transversality condition. We proved in **lemma 11** that this condition sets lower rank tensors to zero when reducing to SO$(d,1)$.

We introduce another piece of our notation which is used subsequently $y^2 = \eta_{AB} y^A y^B$ and $\Delta = \eta^{AB}\partial_A\partial_B$ (the Laplace operator).

Lemma 14. *Let $A \in {}_\gamma S^s_0$ and furthermore, assume that conditions* (3.7) *and* (3.8) *hold. Then*

$$C_2\,|A\rangle = \left[-y^2\Delta + \gamma(d+\gamma) + s(s+d-2)\right]|A\rangle \tag{3.9}$$

Proof. Since $A \in {}_\gamma S^s_0$, we have the following useful relations

$$\eta_{AB}\bar{a}^A\bar{a}^B\,|A\rangle = 0 \qquad \text{(Tracelessness)}$$

$$\eta_{AB}a^A\bar{a}^B\,|A\rangle = s\,|A\rangle \qquad \text{(Spin } s\text{)}$$

$$y^A\partial_A\,|A\rangle = \gamma\,|A\rangle \qquad (\gamma\text{-homogeneity})$$

We apply the Casimir operator (2.103) on $|A\rangle$ (instead of the condition that $A > B$ we take $\frac{1}{2}$ of both possibilities). Then we plug formulae (3.3) into the decomposition (3.4).

$$\begin{aligned}
C_2\,|A\rangle &= \frac{1}{2}\eta^{AD}\eta^{BC}(L_{AB}L_{CD} + M_{AB}L_{CD} + L_{AB}M_{CD} + M_{AB}M_{CD})\,|A\rangle \\
&= \frac{1}{2}\eta^{AD}\eta^{BC}[(y_A\partial_B - y_B\partial_A)(y_C\partial_D - y_D\partial_C) + (a_A\bar{a}_B - a_B\bar{a}_A)(y_C\partial_D - y_D\partial_C) \\
&\quad + (y_A\partial_B - y_B\partial_A)(a_C\bar{a}_D - a_D\bar{a}_C) + (a_A\bar{a}_B - a_B\bar{a}_A)(a_C\bar{a}_D - a_D\bar{a}_C)]\,|A\rangle \\
&= \frac{1}{2}\eta^{AD}\eta^{BC}[2(y_A(\eta_{BC} + y_C\partial_B)\partial_D - y_A(y_D\partial_B + \eta_{BD})\partial_C) \\
&\quad + 4((\partial_B y_A - \eta_{AB})a_C\bar{a}_D - y_A\partial_B a_D\bar{a}_C) \\
&\quad + 2(a_A(a_C\bar{a}_B + \eta_{BC})\bar{a}_D - a_A(a_D\bar{a}_B + \eta_{BD})\bar{a}_C)]\,|A\rangle \\
&= [(\gamma(d+2+\gamma-1) - y^2\Delta - \gamma) - 2s + (s(s-1+d+2) - s)]\,|A\rangle \\
&= [-y^2\Delta + \gamma(d+\gamma) + s(s+d-2)]\,|A\rangle
\end{aligned} \tag{3.10}$$

♠

We want the space of solutions of the equation (3.5) to be an irreducible representation of $\mathfrak{so}(d,2)$ characterized by the lowest energy $E_0 > 0$ and by $\mathbf{s} = (-s, 0, ..0)$, the lowest weight of $\mathfrak{so}(d)$. In **theorem 7** we have computed the value of the Casimir operator in that case and we will use it in the equation of motion (3.5) as the value of the parameter t.

Lemma 15. *Let $A \in {}_{\gamma}S_0^s$ be a representation of $\mathfrak{so}(d,2)$ characterized by the lowest energy $E_0 > 0$ and by $\mathbf{s} = (-s, 0, ..0)$, the lowest weight of $\mathfrak{so}(d)$. Take*

$$\delta_{(s-n)}A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)_0}$$

as the gauge field of A where $\Lambda \in {}_{\beta}S_0^n$. If $|A\rangle$ and $|\Lambda\rangle$ satisfy the transversality condition (3.8) and if $|A\rangle$ also satisfies the condition (3.7), then the equation of motion has the form

$$(y^2\Delta - (n-1)(d+n-1) + E_0(E_0-d))\,|A\rangle = 0 \tag{3.11}$$

Proof. The equation (3.5) with the right value of the parameter t computed in the **theorem 7** looks like (where we also utilize the previous **lemma**)

$$\begin{aligned}
(-y^2\Delta + \gamma(d+\gamma) + s(d+s-2) - E_0(E_0-d) - s(s+d-2))\,|A\rangle &= 0 \\
(y^2\Delta - \gamma(d+\gamma) + E_0(E_0-d))\,|A\rangle &= 0
\end{aligned} \tag{3.12}$$

Due to (2.94) we can write the field equation on the ambient space as

$$(y^2\Delta - (n-1)(d+n-1) + E_0(E_0-d))\,|A\rangle = 0 \tag{3.13}$$

♠

Now, we will compute the value of E_0.

Lemma 16. *Let* $A \in {}_{\gamma}S_0^s$ *and take* $\delta_{(s-n)}A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)_0}$ *as its gauge field where* $\Lambda \in {}_{\beta}S_0^n$*, as always* $s > n$*. If* $|A\rangle$ *and* $|\Lambda\rangle$ *satisfy the divergencelessness condition* (3.7) *then* $E_0^1 = d+n-1$ *and* $E_0^2 = -n+1$ *are two possible values of* E_0*.*

Proof. If $|A\rangle$ satisfies the divergencelessness condition, then the same is true for its gauge field

$$\begin{aligned} 0 = \bar{\partial}\delta_{(s-n)}\,|A\rangle &= \bar{a}^A\partial_A(a^{C_1}..a^{C_s}\partial_{C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s})\,|0\rangle \\ &= (s-n)\Delta(a^{C_2}..a^{C_s}\partial_{C_2}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s})\,|0\rangle \end{aligned}$$

We derive it one more time

$$\begin{aligned} 0 &= \partial[\bar{\partial}\delta_{(s-n)}]\,|A\rangle \\ &= \partial_A a^A(s-n)\Delta(a^{C_2}..a^{C_s}\partial_{C_2}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s})\,|0\rangle \\ &= (s-n)\Delta\delta_{(s-n)}\,|A\rangle \end{aligned} \tag{3.14}$$

The equation (3.11) holds for the gauge field $\delta_{(s-n)}\,|A\rangle$ too. And if we apply (3.14) we are left with

$$\begin{aligned} (-(n-1)(d+n-1) + E_0(E_0-d))\delta_{s-n}\,|A\rangle = 0 \\ (n-1)(d+n-1) = E_0(E_0-d)) \end{aligned} \tag{3.15}$$

The last equation has solutions $E_0^1 = d+n-1$ and $E_0^2 = -n+1$. ♠

Comment. Whereas we have a condition $E_0 > 0$, thereby we drop the second solution (it is positive only for $n = 0$, that is, for scalars).
Comment. Looking at the equation of motion (3.11) we see that there are special discrete values of the energy E_0 ($E_0^1 = d+n-1$ and $E_0^2 = -n+1$) such that the equation takes the form $y^2\Delta\,|A\rangle = 0$. This is only another way to obtain values of energy E_0.

Theorem 8. *Let* $A \in {}_{\gamma}S_0^s$ *and let* $\delta_{(s-n)}A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)_0}$ *be its gauge field where* $\Lambda \in {}_{\beta}S_0^n$*. If conditions*

$$\bar{\partial}\,|A\rangle = 0 \tag{3.7}$$
$$\bar{y}\,|A\rangle = 0 \tag{3.8}$$
$$\bar{\partial}\,|\Lambda\rangle = 0 \qquad \text{(Divergencelessness for } \Lambda\text{)}$$
$$\bar{y}\,|\Lambda\rangle = 0 \qquad \text{(Transversality for } \Lambda\text{)}$$

hold, then the equation of motion (3.11) *takes the form*

$$y^2\Delta\,|A\rangle = 0 \tag{3.16}$$

Proof. We have already proved it in previous lemmas. ♠

Chapter 4

Partially massless fields in (d+1)-dimensional AdS

4.1 Massless fields in (d+1)-dimensional AdS

In contradiction with the name of the chapter we start with massless fields. By AdS spacetime we mean a (d+1)-dimensional spacetime $\mathcal{H}^{d,1}$ defined earlier by

$$\eta_{AB} y^A y^B = -1 \tag{2.79}$$

in $(d+2)$-dimensional pseudo-Euclidian space with metric η_{AB}. We use η^{AB} and η_{AB} to raise and lower indices as usual and we will continue to describe everything in SO$(d,2)$ covariant form.

We first present differential operators defined on the hypersurface (2.79):

$$\nabla_A = (\eta_A^B + y_A y^B)\partial_B \tag{4.1}$$

The tangent derivative ∇_A satisfies relations

$$[\nabla_A, y^B] = [(\eta_A^C + y_A y^C)\partial_C, y^B] = (\eta_A^C + y_A y^C)\partial_C y^B - y^B(\eta_A^C + y_A y^C)\partial_C = \eta_A^B + y_A y^B \tag{4.2}$$

$$y^A \nabla_A = y^A(\eta_A^B + y_A y^B)\partial_B = (y^B - y^B)\partial_B = 0 \tag{4.3}$$

$$\nabla_A y^A = (\eta_A^B + y_A y^B)\partial^B y^A = d+1 \tag{4.4}$$

From the relations (3.3), we also see

$$\begin{aligned} L_{AB} = y_A\partial_B - y_B\partial_A &= y_A(\eta_B^C + y_B y^C)\partial_C - y_B(\eta_A^C + y_A y^C)\partial_C \\ &= y_A\nabla_B - y_B\nabla_A \end{aligned} \tag{4.5}$$

and, to be complete,

$$
\begin{aligned}
[\nabla_A, \nabla_B] &= [(\eta_A^C + y_A y^C)\partial_C, (\eta_B^D + y_B y^D)\partial_D] \\
&= (\eta_A^C + y_A y^C)\partial_C(\eta_B^D + y_B y^D)\partial_D - (\eta_B^D + y_B y^D)\partial_D(\eta_A^C + y_A y^C)\partial_C \\
&= y_B \nabla_A - y_A \nabla_B \\
&= -L_{AB}
\end{aligned} \tag{4.6}
$$

We prepared our ground to compute the second order Casimir operator in terms of differential operators ∇_A. Again, we abbreviate $\nabla = \nabla_A a^A$ and $\bar{\nabla} = \nabla_A \bar{a}^A$.

Lemma 17. *Let $A \in S_0^s$ and assume that*

$$\bar{\nabla}\,|A\rangle = 0 \tag{3.7'}$$

$$\bar{y}\,|A\rangle = 0 \tag{3.8}$$

Denote $\nabla^2 = \eta^{AB}\nabla_A\nabla_B$, then

$$C_2\,|A\rangle = [\nabla^2 + s(s+d-2)]\,|A\rangle \tag{4.7}$$

Proof. The proof is nearly a complete analogy of the proof in **lemma 14** but we still carry out all the computation

$$
\begin{aligned}
C_2\,|A\rangle &= \frac{1}{2}\eta^{AD}\eta^{BC}(L_{AB}L_{CD} + M_{AB}L_{CD} + L_{AB}M_{CD} + M_{AB}M_{CD})\,|A\rangle \\
&= \frac{1}{2}\eta^{AD}\eta^{BC}[(y_A\nabla_B - y_B\nabla_A)(y_C\nabla_D - y_D\nabla_C) + (a_A\bar{a}_B - a_B\bar{a}_A)(y_C\nabla_D - y_D\nabla_C) \\
&\quad + (y_A\nabla_B - y_B\nabla_A)(a_C\bar{a}_D - a_D\bar{a}_C) + (a_A\bar{a}_B - a_B\bar{a}_A)(a_C\bar{a}_D - a_D\bar{a}_C)]\,|A\rangle \\
&= \frac{1}{2}\eta^{AD}\eta^{BC}2[(y_A\nabla_B y_C\nabla_D - y_A(\eta_{BD} + y_B y_D + y_D\nabla_B)\nabla_C) \\
&\quad + 2(a_A(\nabla_D y_C - \eta_{CD} - y_C y_D)\bar{a}_B - a_A y_D\bar{a}_B\nabla_C) \\
&\quad + (a_A(a_C\bar{a}_B + \eta_{BC})\bar{a}_D - a_A(a_D\bar{a}_B + \eta_{BD})\bar{a}_C)]\,|A\rangle \\
&= [\eta^{BC}\nabla_B\nabla_C - 2s + ((s-1+d+2)s - s)]\,|A\rangle \\
&= [\nabla^2 + s(s+d-2)]\,|A\rangle
\end{aligned} \tag{4.8}
$$

♠

Comment. It is obvious that if $\bar{y}\,|A\rangle = 0$ holds then $\bar{\nabla}\,|A\rangle = 0$ is equivalent to $\bar{\partial}\,|A\rangle = 0$

The equation of motion (3.5) with the value of the parameter t taken from **theorem 7** on AdS is then

$$
\begin{aligned}
[-\nabla^2 - s(s+d-2) + E_0(E_0-d) + s(s+d-2)]\,|A\rangle &= 0 \\
[\nabla^2 - E_0(E_0-d)]\,|A\rangle &= 0
\end{aligned} \tag{4.9}
$$

For the $E_0^1 = d+(s-1)-1$, that is, if $A \in {}_{s-2}S_0^s$ and $\Lambda \in {}_{s-1}S_0^{s-1}$, the equation of motion for a massless field on the AdS looks like

$$[\nabla^2 - (s-2)(d+s-2)]\,|A\rangle = 0 \tag{4.10}$$

with the subsidiary conditions on a transversality and divergencelessness

$$\begin{aligned} \bar{y}\,|A\rangle &= 0 \\ \bar{\nabla}\,|A\rangle &= 0 \end{aligned}$$

and a gauge field

$$\begin{aligned} \delta_{(1)}\,|A\rangle &= \partial\,|\Lambda\rangle \\ &= (\nabla - yy^A\partial_A)\,|\Lambda\rangle \\ &= (\nabla - (s-1)y)\,|\Lambda\rangle \end{aligned} \tag{4.11}$$

All our results are in agreement with Metsaev's results [4], [5] for the totally symmetric field (except a slightly different dimension notation). But we take all the computations one step further and generalize them for the case of partially massless fields.

4.2 Partially massless fields on $\mathcal{H}^{d,1}$

We present the results on (d+1)-dimensional AdS in the following

Lemma 18. *Let $A \in {}_{\gamma}S_0^s$ and let $\delta_{(s-n)}A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)_0}$ be its gauge field where $\Lambda \in {}_{\beta}S_0^n$. Then, the gauge field can be written in terms of differential operators on AdS as*

$$\delta_{(s-n)}A_{(C_1..C_s)_0} = (\nabla_{(C_1} - ny_{(C_1}).. (\nabla_{C_{s-n}} - (s-1)y_{C_{s-n}})\Lambda_{C_{s-n+1}..C_s)_0} \tag{4.12}$$

Proof. The relation (4.1) can be written as $\partial_A = \nabla_A - y_A T$ where T is the Euler operator. Using **lemma 12** we obtain the correct form of the gauge field. ♠

Theorem 9. *For $A \in {}_{\gamma}S_0^s$, a representation of $\mathfrak{so}(d,2)$ characterized by the lowest energy $E_0 > 0$ and by $\mathbf{s} = (-s, 0, ..0)$, the lowest weight of $\mathfrak{so}(d)$ and for $\Lambda \in {}_{\beta}S_0^n$ let*

$$\delta_{(s-n)}\,|A\rangle = (\nabla - ny)(\nabla - (n+1)y)..(\nabla - (s-1)y)\,|\Lambda\rangle$$

be the gauge field of $|A\rangle$. If conditions

$$\begin{aligned} \bar{\nabla}\,|A\rangle &= 0 && (3.7') \\ \bar{y}\,|A\rangle &= 0 && (3.8) \\ \bar{\nabla}\,|\Lambda\rangle &= 0 && \text{(Divergencelessness for } \Lambda) \\ \bar{y}\,|\Lambda\rangle &= 0 && \text{(Transversality for } \Lambda) \end{aligned}$$

hold, then the equation of motion (3.5) *in terms of differential operators on AdS is*

$$[\nabla^2 - (n-1)(n-1+d)] |A\rangle = 0 \tag{4.13}$$

Proof. We already have all the necessary ingredients. We just plug in the right energy value $E_0 = d + n - 1$ computed in **lemma 16** into the equation (4.9). ♠

4.3 Metsaev's approach - generalization

We will introduce Metsaev's approach of deriving the gauge field which we will generalize for a partially massless fields later on. We ought to say that Metsaev treats all representations of SO$(d, 2)$ while we stick to a totally symmetric representation. We assume that we have a field equation and we are looking for its gauge invariance.

The most general gauge transformations of the massless field would look like

$$\delta_{(1)} |A\rangle = \alpha\nabla |\Lambda\rangle + \beta y |R\rangle \tag{4.14}$$

where the gauge parameter fields $\Lambda, R \in S^{s-1}$ and $\alpha, \beta \in \mathbb{R}$. We also impose conditions

$$\begin{aligned} \bar{\nabla} |A\rangle &= 0 \\ \bar{y} |A\rangle &= 0 \\ \bar{\nabla} |\Lambda\rangle &= 0 \\ \bar{y} |\Lambda\rangle &= 0 \end{aligned}$$

We are left to compute the coefficients of the linear combination but first, we prepare our ground. For short $a^+ = \eta_{AB} a^A a^B$ and $a^- = \eta_{AB} \bar{a}^A \bar{a}^B$.

$$\begin{aligned} [\bar{y}, y] &= \bar{a}^A y_A a^B y_B - a^B y_B \bar{a}^A y_A = \eta^{AB} y_A y_B = -1 && (4.15) \\ [\bar{y}, \nabla] &= \bar{a}^A y_A a^B \nabla_B - a^B \nabla_B \bar{a}^A y_A = a^B \bar{a}^A (\nabla_B y_A - \eta_{AB} - y_A y_B) - a^B \nabla_B \bar{a}^A y_A \\ &= -\eta_{AB} a^A a^B - y\bar{y} && (4.16) \\ [\bar{y}, a^+] &= \bar{a}^A y_A \eta_{BC} a^B a^C - \eta_{BC} a^B a^C \bar{a}^A y_A = 2 y_A a^A = 2y && (4.17) \end{aligned}$$

$$\begin{aligned} [\bar{\nabla}, y] &= \bar{a}^A \nabla_A a^B y_B - a^B y_B \bar{a}^A \nabla_A = (d+1) - a^B \bar{a}^A (-\eta_{AB} - y_A y_B) \\ &= (d+1) + \eta_{AB} a^A a^B + y\bar{y} && (4.18) \\ [\bar{\nabla}, \nabla] &= \bar{a}^A \nabla_A a^B \nabla_B - a^B \nabla_B \bar{a}^A \nabla_A = \nabla^2 + a^B \bar{a}^A (-y_A \nabla_B + y_B \nabla_A) \\ &= \nabla^2 - \nabla\bar{y} + \eta_{AB} a^A a^B + y\bar{y} + y\bar{\nabla} && (4.19) \\ [\bar{\nabla}, a^+] &= \bar{a}^A \nabla_A \eta_{BC} a^B a^C - \eta_{BC} a^B a^C \bar{a}^A \nabla_A = 2\nabla_A a^A = 2\nabla && (4.20) \end{aligned}$$

$$[a^-, y] = \eta_{AC}\bar{a}^A\bar{a}^C a^B y_B - a^B y_B \eta_{AC}\bar{a}^A\bar{a}^C = 2\bar{a}^A y_A = 2\bar{y} \tag{4.21}$$

$$[a^-, \nabla] = \eta_{AC}\bar{a}^A\bar{a}^C a^B \nabla_B - a^B \nabla_B \eta_{AC}\bar{a}^A\bar{a}^C = 2\bar{a}^A \nabla_A = 2\bar{\nabla} \tag{4.22}$$

$$\begin{aligned}[a^-, a^+] &= \eta_{AC}\bar{a}^A\bar{a}^C \eta_{BD} a^B a^D - \eta_{BD} a^B a^D \eta_{AC}\bar{a}^A\bar{a}^C \\ &= \eta_{AC}\eta_{BD}\bar{a}^A(\eta^{BC} + a^B\bar{a}^C)a^D - \eta_{BD}a^B a^D \eta_{AC}\bar{a}^A\bar{a}^C \\ &= (d+2) + \eta_{AD}a^A\bar{a}^D + \eta_{AC}\eta_{BD}(\eta^{AB} + a^B\bar{a}^A)\bar{a}^C a^D - \eta_{BD}a^B a^D \eta_{AC}\bar{a}^A\bar{a}^C \\ &= 2(d+2) + 2\eta_{AD}a^A\bar{a}^D + \eta_{AC}\eta_{BD}a^B\bar{a}^A\bar{a}^C a^D - \eta_{BD}a^B a^D \eta_{AC}\bar{a}^A\bar{a}^C \\ &= 2(d+2) + 4\eta_{AD}a^A\bar{a}^D \end{aligned} \tag{4.23}$$

$$\begin{aligned}[\nabla^2, \nabla] &= \eta^{AC}\nabla_A\nabla_C a^B\nabla_B - a^B\nabla_B\eta^{AC}\nabla_A\nabla_C \\ &= \eta^{AC}\nabla_A a^B(\nabla_B\nabla_C - y_C\nabla_B + y_B\nabla_C) - a^B\nabla_B\eta^{AC}\nabla_A\nabla_C \\ &= -(d+1)\nabla + y\nabla^2 + \nabla + \eta^{AC}a^B(-y_A\nabla_B + y_B\nabla_A)\nabla_C \\ &= -(d-1)\nabla + 2y\nabla^2 \end{aligned} \tag{4.24}$$

$$\begin{aligned}[\nabla^2, y] &= \eta^{AC}\nabla_A\nabla_C a^B y_B - a^B y_B \eta^{AC}\nabla_A\nabla_C \\ &= \eta^{AC}a^B\nabla_A(y_B\nabla_C + \eta_{BC} + y_B y_C) - a^B y_B\eta^{AC}\nabla_A\nabla_C \\ &= \nabla + (d+1)y + \eta^{AC}a^B(\eta_{AB} + y_A y_B)\nabla_C \\ &= 2\nabla + (d+1)y \end{aligned} \tag{4.25}$$

Previous results greatly simplify our further computation. We begin with the requirement on a transversality

$$\begin{aligned}\bar{y}\delta_{(1)}|A\rangle &= \bar{y}\,(\alpha\,\nabla|\Lambda\rangle + \beta\,y\,|\Lambda\rangle) \\ &= [\alpha(-y\bar{y} - (s-1) + \nabla\bar{y}) + \beta(-1 + y\bar{y})]\,|\Lambda\rangle \\ &= [-(s-1)\alpha - \beta]\,|\Lambda\rangle = 0 \end{aligned} \tag{4.26}$$

Therefore,

$$\beta = -(s-1)\alpha \tag{4.27}$$

and the gauge field has the form

$$\delta_{(1)}|A\rangle = \nabla|\Lambda\rangle - (s-1)y\,|\Lambda\rangle \tag{4.28}$$

which is also the Metsaev's result. We verify the tracelessness

$$\begin{aligned}a^-\delta_{(1)}|A\rangle &= a^-\,(\nabla|\Lambda\rangle - (s-1)y\,|\Lambda\rangle) \\ &= 2(\bar{\nabla} - (s-1)\bar{y})\,|\Lambda\rangle = 0 \end{aligned} \tag{4.29}$$

And from the divergencelessness

$$\begin{aligned}\bar{\nabla}\delta_{(1)}|A\rangle &= \bar{\nabla}\left(\nabla|\Lambda\rangle - (s-1)y|\Lambda\rangle\right) \\ &= (\nabla^2 + (s-1) - (s-1)(d+1+s-1))|\Lambda\rangle \\ &= (\nabla^2 - (s-1)(d+1+s-2))|\Lambda\rangle = 0 \end{aligned} \tag{4.30}$$

we derived the equation of motion for the gauge parametr field (see also [4]). Before further generalization we show more complicated example of a gauge field for partially massless field. Next step is to take a gauge parameter field $|\Lambda\rangle \in S^{s-2}$. Obviously, (4.14) will look like

$$\delta_{(2)}|A\rangle = (\alpha\,\nabla\nabla + \beta\,\nabla y + \gamma\,y\nabla + \delta\,yy)|\Lambda\rangle \tag{4.31}$$

We continue in a same manner as before.

$$\begin{aligned}\bar{y}\delta_{(2)}|A\rangle &= \bar{y}(\alpha\,\nabla\nabla + \beta\,\nabla y + \gamma\,y\nabla + \delta\,yy)|\Lambda\rangle \\ &= [\alpha(-y\bar{y} - (s-1) + \nabla\bar{y})\nabla + \beta(-y\bar{y} - (s-1) + \nabla\bar{y})y \\ &\quad + \gamma(-1 + y\bar{y})\nabla + \delta(-1 + y\bar{y})y]|\Lambda\rangle \\ &= [\alpha(y(s-2) - \nabla(s-2) - (s-1)\nabla) + \beta(y - y(s-1) - \nabla) \\ &\quad + \gamma(-y(s-2) - \nabla) + \delta(-2y)|\Lambda\rangle = 0 \end{aligned} \tag{4.32}$$

We can split this equation into two which are coefficients in front of $y|\Lambda\rangle$ and $\nabla|\Lambda\rangle$

$$(s-2)\alpha - (s-2)\beta - (s-2)\gamma - 2\delta = 0 \tag{4.33}$$

$$(-2s+3)\alpha - \beta - \gamma = 0 \tag{4.34}$$

We apply the tracelessness condition

$$\begin{aligned}a^-\delta_{(2)}|A\rangle &= \bar{y}(\alpha\,\nabla\nabla + \beta\,\nabla y + \gamma\,y\nabla + \delta\,yy)|\Lambda\rangle \\ &= [\alpha(2\bar{\nabla} + \nabla a^-)\nabla + \beta(2\bar{\nabla} + \nabla a^-)y \\ &\quad + \gamma(2\bar{y} + ya^-)\nabla + \delta(2\bar{y} + ya^-)y]|\Lambda\rangle \\ &= 2[\alpha(\nabla^2 + (s-2)) + \beta(d+1+s-2) + \gamma(-s+2) - \delta]|\Lambda\rangle = 0 \end{aligned} \tag{4.35}$$

And finally, divergencelessness

$$\begin{aligned}\bar{\nabla}\delta_{(2)}|A\rangle &= \bar{y}(\alpha\,\nabla\nabla + \beta\,\nabla y + \gamma\,y\nabla + \delta\,yy)|\Lambda\rangle \\ &= [\alpha(\nabla^2 - \nabla\bar{y} + y\bar{y} + (s-1) + (y+\nabla)\bar{\nabla})\nabla \\ &\quad + \beta(\nabla^2 - \nabla\bar{y} + y\bar{y} + (s-1) + (y+\nabla)\bar{\nabla})y \\ &\quad + \gamma(d+1+(s-1) + y\bar{y} + y\bar{\nabla})\nabla \\ &\quad + \delta(d+1+(s-1) + y\bar{y} + y\bar{\nabla})y]|\Lambda\rangle \\ &= [\alpha(\nabla(2-d+3(s-2)+2\nabla^2) + 3y\nabla^2) \\ &\quad + \beta(\nabla(d+4+s-2) + y(2(d+1) + 2(s-2) + \nabla^2)) \\ &\quad + \gamma(\nabla(d+2+s-2) + y\nabla^2) + \delta 2y(d+1+s-2)]|\Lambda\rangle = 0 \end{aligned} \tag{4.36}$$

We can do the same trick as before and split it into two equations

$$[3\nabla^2\alpha + (2d+2s-2+\nabla^2)\beta + \nabla^2\gamma + 2(d+s-1)\delta]\,|\Lambda\rangle = 0 \tag{4.37}$$
$$[(2-d+3(s-2)+2\nabla^2)\alpha + (d+s+2)\beta + (d+s)\gamma]\,|\Lambda\rangle = 0 \tag{4.38}$$

Solving the system of equations (4.33),(4.34),(4.35),(4.37),(4.38) gives the coefficients of linear combination. From (4.34)

$$\beta + \gamma = (-2s+3)\alpha \tag{4.39}$$

(4.33)+(4.39):

$$(s-2)\alpha - (s-2)(-2s+3)\alpha - 2\delta = 0$$
$$\delta = (s-1)(s-2)\alpha \tag{4.40}$$

We use (4.40) and (4.39) to eliminate γ and δ from the system. (4.35):

$$[(\nabla^2+s-2)\alpha + (d+1+2(s-2))\beta - (s-2)(-2s+3)\alpha - (s-1)(s-2)\alpha]\,|\Lambda\rangle = 0$$
$$[(\nabla^2+(s-1)(s-2))\alpha + (d+1+2(s-2))\beta]\,|\Lambda\rangle = 0 \tag{4.41}$$

(4.37):

$$2[(-(s-3)\nabla^2 + (d+s-1)(s-1)(s-2))\alpha + (d+s-1)\beta]\,|\Lambda\rangle = 0 \tag{4.42}$$

(4.38):

$$[(2-d+3(s-2)+2\nabla^2+(d+s)(-2s+3))\alpha + 2\beta]\,|\Lambda\rangle = 0$$
$$[(\nabla^2-(s-1)(d+s-2))\alpha + \beta]\,|\Lambda\rangle = 0 \tag{4.43}$$

(4.41)-(4.43):

$$[((s-1)(s-2)+(s-1)(d+s-2))\alpha + (d+1+2(s-2)-1)\beta]\,|\Lambda\rangle = 0$$
$$(d+2s-4)[(s-1)\alpha+\beta] = 0$$
$$\beta = -(s-1)\alpha \tag{4.44}$$

For a choice of $\alpha = 1$ equation (4.35) becomes

$$[\nabla^2 - (s-1)(d+1+s-2)]\,|\Lambda\rangle = 0 \tag{4.45}$$

which is precisely the same as (4.30), that is, equation of motion for gauge parameter field. Eventually, we can write down the gauge field

$$\delta_{(2)}\,|A\rangle = (\nabla\nabla - (s-1)\nabla y - (s-2)y\nabla + (s-1)(s-2)yy)\,|\Lambda\rangle \tag{4.46}$$

Comment. We would have to treat the situation for $\Lambda \in S^0$, that is, for a scalar parameter field, individually. But we don't find it necessary now.

If we would like to generalize this approach we would run into serious difficulties. Yet, it is enough to realize that the general form of the gauge field can be also written in more convenient form

$$\delta_{(m)} |A\rangle = (\nabla + \beta_m y)(\nabla + \beta_{m-1} y)..(\nabla + \beta_1 y) |\Lambda\rangle \tag{4.47}$$

for $A \in S^s$, $\Lambda \in S^n$ and $m = s - n$. Below $a^0 = \eta_{AB} a^A \bar{a}^B$. As usual, we apply the transversality condition

$$\begin{aligned} \bar{y}\delta_{(m)} |A\rangle &= \bar{y}(\nabla + \beta_m y)(\nabla + \beta_{m-1} y)..(\nabla + \beta_1 y) |\Lambda\rangle \\ &= [-a^0 - \beta_m + (\nabla + (\beta_m - 1)y)\bar{y}](\nabla + \beta_{m-1} y)..(\nabla + \beta_1 y) |\Lambda\rangle \end{aligned} \tag{4.48}$$

We have to treat the case $n = 0$ separately. Here and below we assume $n \geq 1$. If we look at this expression as a polynomial in y and ∇ all the coefficients must be zero. The coefficient of the $\nabla..\nabla$ ($(m-1)$-times):

$$(-s + 1 - \beta_m) + (-s + 2 - \beta_{m-1}) + .. + (-s + m - \beta_1) = 0 \tag{4.49}$$

The coefficient of the $y\nabla..\nabla$:

$$\begin{aligned} 0 &= (-s + 1 - \beta_m)\beta_{m-1} + (\beta_m - 1)[(-s + 2 - \beta_{m-1}) + .. + (-s + m - \beta_1)] \\ &= (-s + 1 - \beta_m)\beta_{m-1} + (\beta_m - 1)(s - 1 + \beta_m) \\ &= (s - 1 + \beta_m)(\beta_m - 1 - \beta_{m-1}) \end{aligned} \tag{4.50}$$

The choice $\beta_m = -(s-1)$ is in disagreement with the previous result (4.46). Therefore, $\beta_m = \beta_{m-1} + 1$, similarly the others, and from the equation (4.49)

$$\beta_i = -(s - i) \tag{4.51}$$

And the general gauge field takes the form

$$\delta_{(s-n)} |A\rangle = (\nabla - ny)(\nabla - (n+1)y)..(\nabla - (s-1)y) |\Lambda\rangle \tag{4.52}$$

which is in an agreement with our previous computation (4.12)

Chapter 5

Intrinsic coordinates of AdS

We formulated our results in SO$(d, 2)$ covariant form so far. Now, we will present a formulation in terms of intrinsic coordinates of AdS.

Definiton 30. *Let x^μ for $\mu = 0, 1, ..d$ be intrinsic coordinates in AdS spacetime. Define the embedding maps*

$$y^A(x) \,:\, \Omega \subset \mathbb{R}^{d,1} \to \mathcal{H}^{d,1} \subset \mathbb{R}^{d,2} \qquad \text{where } A = 0, 1, ..d, \infty, \tag{5.1}$$

such that the equation (2.79) holds. Let us denote

$$y^C_\mu = \partial_\mu y^C \tag{5.2}$$

tangent vectors in $T_y\mathcal{H}^{d,1}$ induced by coordinates. Let us consider a tensor field $A_{\mu_1..\mu_s} \in \otimes^s T\mathcal{H}^{d,1}$ given by

$$A_{\mu_1..\mu_s}(x) = y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s} A_{C_1..C_s}(y) \tag{5.3}$$

where $A_{C_1..C_s}(y) \in S^s_y$. Induced Riemannian metric tensor $g_{\mu\nu}$ has the form

$$g_{\mu\nu} = (\partial_\mu y^A)(\partial_\nu y_A) \tag{5.4}$$

We will prove other useful relations in the following

Lemma 19. *Let x^μ and y^A be the same as in the definition above. Let also D_μ be the usual covariant derivative.*

$$\eta_{AB} + y_A y_B = \partial^\mu y_A \partial_\mu y_B \tag{5.5}$$

$$D_\mu y^A_\nu = g_{\mu\nu} y^A \tag{5.6}$$

$$y^B_\nu \nabla_B = y^B_\nu \partial_B \tag{5.7}$$

Denote $D^2 = D_\mu D^\mu$. Suppose moreover that $A \in {}_{n-1}S^s_0$ and $\bar{y}\,|A\rangle = 0$. Then,

$$\partial_\nu A_{C_1..C_s} = D_\nu A_{C_1..C_s} \tag{5.8}$$

$$y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s} \nabla^2 A_{C_1..C_s} = (D^2 + s) A_{\mu_1..\mu_s} \tag{5.9}$$

Proof.

(1) The equation (5.5): Derivatives $\{\partial_\mu\}$ form the basis of the tangent space of $T_y\mathcal{H}^{d,1}$. The vector $y^C\partial_C$ completes this basis to a basis of the full tangent space $T_y\,\mathbb{R}^{d,2}$. It is easy to verify using the usual scalar product

$$(y^C\partial_C,\partial_\mu)=(y^C\partial_C y^A)(\partial_\mu y_A)=y^A\partial_\mu y_A=\frac{1}{2}\partial_\mu(y^Ay_A)=0 \tag{5.10}$$

The basis of a dual space is $\{\partial^\mu,-y^C\partial_C\}$. It is obvious that we get a dual basis to $\{\partial_\mu\}$ when we lift indices. The rest follows from (5.10) and

$$(-y^C\partial_C,y^B\partial_B)=(-y^C\partial_C y^A)(y^B\partial_B y_A)=-y^Ay_A=1$$

Therefore, we can write

$$\begin{aligned}\eta_{AB}&=(\partial_\mu y_A)(\partial^\mu y_B)-(y^C\partial_C y_A)(y^D\partial_D y_B)\\ \eta_{AB}+y_Ay_B&=(\partial_\mu y_A)(\partial^\mu y_B)\end{aligned}$$

As a side note we remind that we already gave that proof in **lemma 9** and **lemma 10** (not complete, though).

(2) The equation (5.6): From the definition of the covariant derivative and using the previous equation we have

$$\begin{aligned}D_\mu y^A_\nu&=D_\mu\partial_\nu y^A=\partial_\mu\partial_\nu y^A-\Gamma^\iota_{\mu\nu}\partial_\iota y^A\\&=\partial_\mu\partial_\nu y^A-\frac{1}{2}g^{\iota\kappa}[g_{\kappa\mu,\nu}+g_{\nu\kappa,\mu}-g_{\mu\nu,\kappa}]\partial_\iota y^A\\&=\partial_\mu\partial_\nu y^A-\frac{1}{2}[\partial_\nu((\partial_\kappa y^B)(\partial_\mu y_B))+\partial_\mu((\partial_\nu y^B)(\partial_\kappa y_B))-\partial_\kappa((\partial_\mu y^B)(\partial_\nu y_B))]\partial^\kappa y^A\\&=\partial_\mu\partial_\nu y^A-\frac{1}{2}[2(\partial_\mu\partial_\nu y_B)(\partial_\kappa y^B)]\partial^\kappa y^A\\&=\partial_\mu\partial_\nu y^A-(\partial_\mu\partial_\nu y_B)(\eta^{AB}+y^Ay^B)\\&=\partial_\mu\partial_\nu y^A-[\partial_\mu\partial_\nu y^A+(\partial_\mu\partial_\nu y_B)y^By^A]\\&=-[\frac{1}{2}\partial_\mu\partial_\nu(y_By^B)-(\partial_\mu y^B)(\partial_\nu y_B)]y^A\\&=(\partial_\mu y^B)(\partial_\nu y_B)y^A\\&=g_{\mu\nu}y^A\end{aligned}$$

(3) The equation (5.7): This is in fact a projection of the derivative ∂_B into the tangent space $T_y\mathcal{H}^{d,1}$

$$\begin{aligned}y^B_\nu\nabla_B&=(\partial_\nu y^B)(\partial_B+y_By^A\partial_A)\\&=y^B_\nu\partial_B+(\partial_\nu y^B)y_By^A\partial_A\\&=y^B_\nu\partial_B\end{aligned}$$

(4) The equation (5.8): This also clear. We use only transversality condition.

$$\begin{aligned}
y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}\partial_\nu A_{C_1..C_s} &= \partial_\nu(y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}A_{C_1..C_s}) - \partial_\nu(y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s})A_{C_1..C_s}\\
&= \partial_\nu A_{\mu_1..\mu_s} - \sum_{i=1}^{s}\Gamma^\iota_{\nu\mu_i}A_{\mu_1..\iota..\mu_s}\\
&= D_\nu A_{\mu_1..\mu_s}\\
&= y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}D_\nu A_{C_1..C_s}
\end{aligned}$$

(5) The equation (5.9): This is the most difficult part of the lemma. We will carry out this proof in three steps. Firstly,

$$\begin{aligned}
\partial_\mu\partial_\nu A_{\mu_1..\mu_s} &= \partial_\mu\partial_\nu y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}A_{C_1..C_s}\\
&= \partial_\mu[(\partial_\nu(y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}))A_{C_1..C_s} + y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}\partial_\nu A_{C_1..C_s}]\\
&= \partial_\mu[\sum_{i=1}^{s}(g_{\nu\mu_i}y^{C_i} + \Gamma^\iota_{\nu\mu_i}y^{C_i}_\iota)y^{C_1}_{\mu_1}..\hat{y}^{C_i}_{\mu_i}..y^{C_s}_{\mu_s}A_{C_1..C_s} + y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}\partial_\nu A_{C_1..C_s}]\\
&= \partial_\mu[\sum_{i=1}^{s}\Gamma^\iota_{\nu\mu_i}A_{\mu_1..\iota..\mu_s} + y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}y^B_\nu\partial_B A_{C_1..C_s}]
\end{aligned}$$

We know that $y^B_\mu\nabla_B = D_\mu$ from the formula (5.3). Thanks to the relations (5.7) and (5.8) we can write $y^B_\mu\partial_B A_{(C_1..C_s)_0} = \partial_\mu A_{(C_1..C_s)_0}$. Then, from the computation above

$$\begin{aligned}
\partial_\mu D_\nu A_{\mu_1..\mu_s} &= \partial_\mu[\partial_\nu A_{\mu_1..\mu_s} - \sum_{i=1}^{s}\Gamma^\iota_{\nu\mu_i}A_{\mu_1..\iota..\mu_s}]\\
&= \partial_\mu y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}y^B_\nu\partial_B A_{C_1..C_s}\\
&= \sum_{i=1}^{s}(g_{\mu\mu_i}y^{C_i} + \Gamma^\iota_{\mu\mu_i}y^{C_i}_\iota)y^{C_1}_{\mu_1}..\hat{y}^{C_i}_{\mu_i}..y^{C_s}_{\mu_s}y^B_\nu\partial_B A_{C_1..C_s} + y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}\partial_\mu(y^B_\nu\partial_B A_{C_1..C_s})\\
&= \sum_{i=1}^{s}[g_{\mu\mu_i}y^{C_1}_{\mu_1}..\hat{y}^{C_i}_{\mu_i}..y^{C_s}_{\mu_s}y^B_\nu(\partial_B y^{C_i} - \eta^{C_i}_B)A_{C_1..C_s} + \Gamma^\iota_{\mu\mu_i}y^{C_1}_{\mu_1}..y^{C_i}_\iota..y^{C_s}_{\mu_s}y^B_\nu\partial_B A_{C_1..C_s}]\\
&\quad + y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}[(g_{\mu\nu}y^B + \Gamma^\iota_{\mu\nu}y^B_\iota)\partial_B A_{C_1..C_s} + y^B_\nu y^D_\mu\partial_D\partial_B A_{C_1..C_s}]\\
&= \sum_{i=1}^{s}[-g_{\mu\mu_i}A_{\mu_1..\nu..\mu_s} + \Gamma^\iota_{\mu\mu_i}D_\nu A_{\mu_1..\iota..\mu_s}]\\
&\quad + \Gamma^\iota_{\mu\nu}D_\iota A_{\mu_1..\mu_s} + y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}(g_{\mu\nu}y^B + y^B_\nu y^D_\mu\partial_D)\partial_B A_{C_1..C_s}
\end{aligned}$$

Finally, we proceed

$$
\begin{aligned}
(D^2+s)A_{\mu_1..\mu_s} &= (g^{\mu\nu}D_\mu D_\nu + s)A_{\mu_1..\mu_s} \\
&= g^{\mu\nu}[\partial_\mu D_\nu A_{\mu_1..\mu_s} - (\Gamma^{\iota}_{\mu\nu}D_\iota + \sum_{i=1}^{s}\Gamma^{\iota}_{\mu\mu_i}D_\nu)A_{\mu_1..\iota..\mu_s}] + sA_{\mu_1..\mu_s} \\
&= g^{\mu\nu}[-\sum_{i=1}^{s} g_{\mu\mu_i}A_{\mu_1..\nu..\mu_s} + y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}(g_{\mu\nu}y^B + y^B_\nu y^D_\mu \partial_D)\partial_B A_{C_1..C_s}] \\
&\quad + sA_{\mu_1..\mu_s} \\
&= g^{\mu\nu}[y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}(g_{\mu\nu}y^B + y^B_\nu y^D_\mu \partial_D)\partial_B A_{C_1..C_s}] \\
&= y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}[(d+1)y^B + (\eta^{BD} + y^B y^D)\partial_D]\partial_B A_{C_1..C_s} \\
&= y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}[(d+1)(n-1) + (n-1)(n-2) + \Delta]A_{C_1..C_s} \\
&= y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s}\nabla^2 A_{C_1..C_s}
\end{aligned}
$$

It is worth to mention that we used the transversality condition to eliminate some terms. The homogeneity helped us to reach the final form (we already did this computation when we derived the equation of motion on the $\mathcal{H}^{d,1}$ from the one on the ambient space).

♠

Comment. Homogenous fields $A_{C_1..C_s}$ on the ambient space can be identified with fields $A_{\mu_1..\mu_s}$ on anti de Sitter spacetime $\mathcal{H}^{d,1}$ and we use transversality condition to rule out all lower ranked tensors, of course (likewise for Λ). Bearing this fact in mind and recalling the **theorem 8**
Let $A \in {}_{\gamma}S^s_0$ and let $\delta_{(s-n)}A_{(C_1..C_s)_0} = \partial_{(C_1}..\partial_{C_{s-n}}\Lambda_{C_{s-n+1}..C_s)_0}$ be its gauge field where $\Lambda \in {}_{\beta}S^n_0$. If conditions

$$\bar{\partial}\,|A\rangle = 0 \tag{3.7}$$

$$\bar{y}\,|A\rangle = 0 \tag{3.8}$$

$$\bar{\partial}\,|\Lambda\rangle = 0 \qquad \text{(Divergencelessness for } \Lambda)$$

$$\bar{y}\,|\Lambda\rangle = 0 \qquad \text{(Transversality for } \Lambda)$$

hold, then the equation of motion (3.11) *takes the form*

$$y^2\Delta\,|A\rangle = 0 \tag{5.11}$$

we can state similar theorem for fields on anti de Sitter spacetime.

Theorem 10. *Assume that $A_{\mu_1..\mu_s} \in \odot^s_0 T\mathcal{H}^{d,1}$ and let*

$$\delta_{(s-n)}A_{(\mu_1..\mu_s)_0} = D_{(\mu_1}..D_{\mu_{s-n}}\Lambda_{\mu_{s-n+1}..\mu_s)_0} \tag{5.12}$$

be the gauge field of the field A where $\Lambda_{\mu_1..\mu_n} \in \odot_0^n T\mathcal{H}^{d,1}$. If conditions on divergencelessness

$$D^\mu A_{\mu\mu_2..\mu_s} = 0 \tag{5.13}$$
$$D^\mu \Lambda_{\mu\mu_2..\mu_n} = 0 \tag{5.14}$$

are satisfied, then the equation of motion (4.13) *in the coordinates x^μ reads*

$$[D^2 + s - (n-1)(d+n-1)]A_{(\mu_1..\mu_s)_0} = 0 \tag{5.15}$$

Proof. It follows from the formula

$$A_{\mu_1..\mu_s}(x) = y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s} A_{C_1..C_s}(y)$$

that a symmetric traceless field on the ambient space corresponds to a symmetric traceless field on anti de Sitter spacetime. The symmetry is clear. Tracelessness:

$$\begin{aligned} 0 &= \eta^{C_1C_2} A_{(C_1..C_s)} \\ &= \eta^{C_1C_2} y^{\mu_1}_{C_1}..y^{\mu_s}_{C_s} A_{(\mu_1..\mu_s)} \\ &= y^{\mu_3}_{C_3}..y^{\mu_s}_{C_s}(\partial^{\mu_1} y_{C_1} \partial^{\mu_2} y^{C_2}) A_{(\mu_1..\mu_s)} \\ &= y^{\mu_3}_{C_3}..y^{\mu_s}_{C_s}(g^{\mu_1\mu_2} A_{(\mu_1..\mu_s)}) \end{aligned}$$

The gauge field is

$$\begin{aligned} \delta_{(s-n)} A_{(\mu_1..\mu_s)_0} &= y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s} \delta_{s-n} A_{(C_1..C_s)_0} \\ &= y^{C_1}_{\mu_1}..y^{C_s}_{\mu_s} \partial_{(C_1}..\partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s)_0} \\ &= y^{C_2}_{\mu_2}..y^{C_s}_{\mu_s} D_{\mu_1} \partial_{C_2}..\partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s} \\ &= y^{C_3}_{\mu_3}..y^{C_s}_{\mu_s} (D_{\mu_1} D_{\mu_2} - g_{\mu_1\mu_2} y^{C_2} \partial_{C_2}) \partial_{C_3}..\partial_{C_{s-n}} \Lambda_{C_{s-n+1}..C_s} \\ &= D_{(\mu_1}..D_{\mu_{s-n}} \Lambda_{\mu_{s-n+1}..\mu_s)_0} \end{aligned}$$

where we left out all terms with metric tensor since we know that the result is traceless. The divergencelessness condition

$$\begin{aligned} 0 = D^\mu A_{(\mu\mu_2..\mu_s)_0} &= D^\mu y^C_\mu y^{C_2}_{\mu_2}..y^{C_s}_{\mu_s} A_{(CC_2..C_s)_0} \\ &= y^{C_2}_{\mu_2}..y^{C_s}_{\mu_s} y^C_\mu D^\mu A_{(CC_2..C_s)_0} \\ &= y^{C_2}_{\mu_2}..y^{C_s}_{\mu_s} (\partial^C A_{(CC_2..C_s)_0}) \end{aligned}$$

also corresponds to the divergencelessness condition on the ambient space. Using the previous **lemma** to rewrite the equation of motion, together with the comment before, finishes the proof. ♠

Comment. For $n = s-1$, that is, for massless fields, it reduces to [4] (with slightly different notation $d' = d-1$).

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