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DIPLOMOVÁ PRÁCE

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Oceňování kreditních derivátů

Katedra pravděpodobnosti a matematické statistiky
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Děkuji svému vedoucímu RNDr. Petru Fraňkovi, Ph.D. za zajímavé téma a za konzultace.

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V Praze dne 11.04 2006 Marek Promer

Contents

Chapter 1. Credit derivatives 5
1. Terminology 5
2. Credit default swaps 8
3. Total return swap 10
4. Credit-linked note 12
5. Banks and Credit Derivatives 14

Chapter 2. Valuation of credit derivatives 17
1. The model of Das and Sundaram 20
2. The Hull-White Model 30
3. The model of Schönbucher 42
4. Conclusion 46

Chapter 3. Enclosures 48

Bibliography 53

Název práce: Oceňování kreditních derivátů
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Abstrakt: Úvěrové riziko hraje při oceňování finančních nástrojů důležitou roli. Ve snaze pojistit se vůči možným problémům plynoucím z tohoto rizika byly vyvinuty finanční nástroje zvané úvěrové deriváty. V práci jsou popsány tři typy úvěrových derivátů: "credit default swap", "total return swap" a "credit linked note".
Vzhledem k rozšíření credit default swapu na trzích s úvěrovými deriváty se práce zabývá oceňováním právě tohoto nástroje, pomocí tří typů modelů. Jsou to modely odhadující cenu credit default swapu, kterou se rozumí pravidelná platba vyžadovaná po dobu trvání smlouvy od jedné se zúčastněných stran.
Klíčová slova: *úvěrové deriváty, credit default swap, pravděpodobnost úpadku*

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Abstract: Credit risk plays an important role in the pricing of financial instruments. In effort to avoid the dangers resulting from this risk were developed new financial instruments called credit derivatives. In this work, the main features of three types of credit derivatives are discussed: credit default swap, total return swap and credit linked note. Regarding to the major portion of credit default swap on the credit derivatives market, the work deals with the valuation of this exact instrument with three models for valuing a credit default swap. These models estimate the value of credit default swap, under which the premium required from one participant of the contract is meant.
Keywords: *credit derivatives, credit default swap, default probability*

CHAPTER 1

Credit derivatives

1. Terminology

When a financial transaction is performed, it is also important to take *credit risk* into consideration. Whenever there is a possibility of loss on financial or nonfinancial contract due to counterparty's failure to meet contractual obligations, we speak about credit risk. Among other properties, it can significantly affect the prices of financial instruments, including derivatives.

There is a variety of ways how to specify, describe or measure credit risk. Typically, it is defined as the risk that a contractual counterparty does not honour its payment obligations stated in the contract and thereby causes a financial loss to the creditor. In effort to avoid this potential problem and to make possible better credit risk management, new financial instruments have been developed in the past few years. When credit risk itself is the underlying variable of a derivative instrument, this instrument is called *credit derivative*.

Generally, the term *credit derivative* is applied to a very broad class of derivative instruments. In Schönbucher [**9**], their common and characteristic features are :

1. A credit derivative is a derivative security that is primarily used to transfer or manage credit risk and also to earn some income.

2. A credit derivative is a contract that has a payoff which is conditioned on the occurrence of a credit event. If the credit event has occurred, the default payment has to be payed out by one of the counterparties.

Nowadays a plenty of various credit derivatives exist. Some of them are discussed in the next sections.

The important basic key terms of most common credit derivatives are the following :

– *Protection buyer* is the buyer of a credit protection, for example the seller of credit risk. He is sometimes also called creditor or insured counterparty. Protection buyer receives a payoff from counterparty of his credit derivative contract if a default or other exactly specified credit event happens and typically pays a periodic premium for this protection. For example, if a bondholder wants to insure himself against the default or another event affecting the credit quality of the bond issuer, he enters into a credit derivative contract with a third party and buys credit protection. One of the advantages of credit derivatives is that in this case the bond issuer doesn't have to know that the bondholder has hedged his exposure.

– *Protection seller* is the insurer. He sells protection or in other words he buys risk. Generally, protection seller receives a fee for this protection but he has to pay a pre-agreed amount in case of a credit event or a default of the reference entity. Protection seller can, with some types of credit derivative contracts, also participate on markets where he has no access other way.

– *Reference entity/reference credit/obligor* is the party that may cause financial loss to the creditor in the case that it will be unable or unwilling to meet its contractual payment obligations. The bond issuer is often a typical example of the reference entity.

– *Reference obligation/reference credit asset* is the asset or a set of assets that causes credit exposure and is used as the underlying instrument to derive the cash-flow of a credit derivative. A common example of a reference obligation is a bond. If the bondholder is exposed to credit risk, he can enter a credit derivative contract with a third party to gain protection against the credit event of the issuer of the bond.

– *Credit event* is a precisely defined event, which triggers the default payment mechanism. Credit event is defined with respect to the reference entity and the reference obligation. In the following sections, the term *default* will be used as a synonym with the term *credit event.* In Tavakoli [**10**], notice of the default event has to be made upon a publicly available information. It is usually an information that has been published in at least one of the two or more predefined internationally recognized financial news sources. Most of usual definitions of credit events include one or more of the following events:

- bankruptcy: a standard clause relating to insolvency and bankruptcy situations that are applicable to the reference entity. It can be also some more subjective occurrence such as the reference entity's inability to pay its debts or the reference entity's default in any payment of principal or interest.
- failure to pay: the reference entity fails to make payments under the defined reference obligation.
- ratings downgrade below a given threshold: the credit rating of the reference entity or reference obligation drops below a specified rating set.
- specified changes in the credit spread or yield spread of the reference obligation.

- repudiation: the reference entity repudiates a defined reference obligation.
- credit event upon merger: a merger or sale of all reference entity's assets in which the surviving entity is weaker than before the action.

– *Default payments* are payments which have to be made if a credit event happens. There are more possibilities how to fulfill a default payment when a credit event occurs. It can be done for example by physical delivery, cash settlement or other pre-agreed fixed payoff.

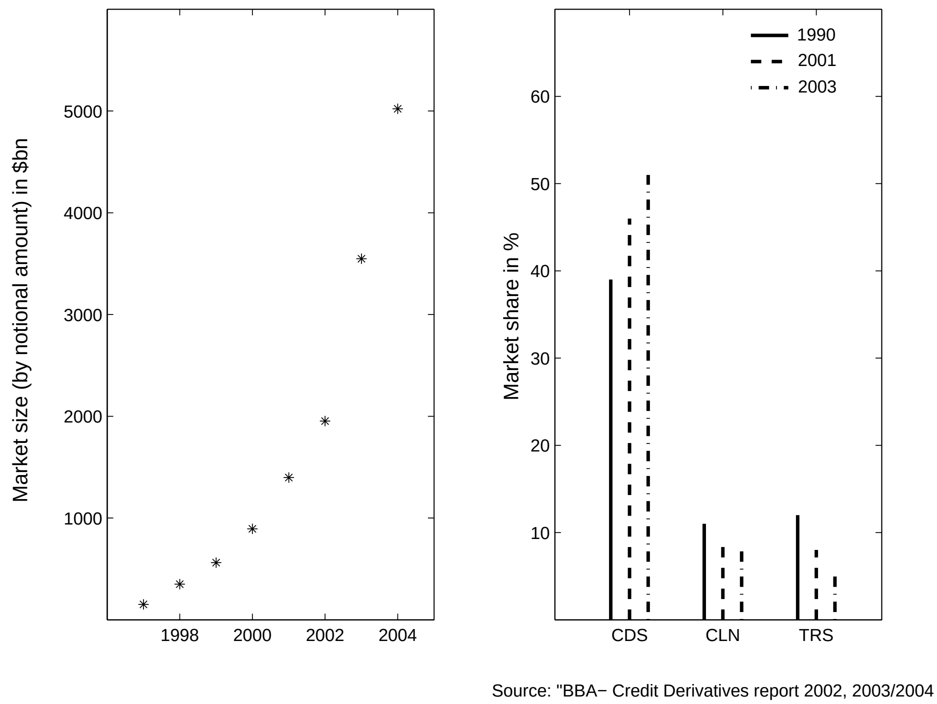

FIGURE 1. *Global market development of credit derivatives (left picture) and global market share of three types of credit derivatives (right picture)*

Credit derivatives have a big influence on the way how banks and other financial institutions manage credit risk. The market of credit derivatives has grown very rapidly since first credit derivatives appeared and is still growing with enormous speed. The British Bankers Association (BBA Credit derivatives report, [2002],[2004]) suggest that the global credit derivatives market increased in size (measured by notional amount outstanding) from around US \$151 billion in 1997 to US \$1 398 billion by the end of 2001. Credit derivatives have a great potential.

There are a lot of possible applications for banks, companies, investors or everyone who wants to deal with unacceptable credit risk connected with financial transactions.

Credit derivatives give an investor the possibility to execute a financial transaction even in the case that the credit risk of the obligor is higher than it is acceptable for him. It is enough for the investor to find a credit derivative counterparty that is willing to take over the credit risk of the obligor.

We can distinguish several types or classes of credit derivatives. Some of the most popular are described in the next sections.

2. Credit default swaps

Credit default swap is a bilateral contract that gives protection to the insurance taking party against the risk of default by the reference entity. This protection is related to the basic feature of this instrument which enables the investor to isolate and transfer the credit risk of the obligor to another counterparty. In a credit default swap the protection buyer agrees to pay regular fee payments to the insuring counterparty until a credit event occurs or the reference obligation matures or the CDS itself defaults. In return the protection seller agrees to pay the default payment in case of reference entity's default. If there is no default, he pays nothing. Insurance premium, default event and also default payment are exactly specified in the contract.

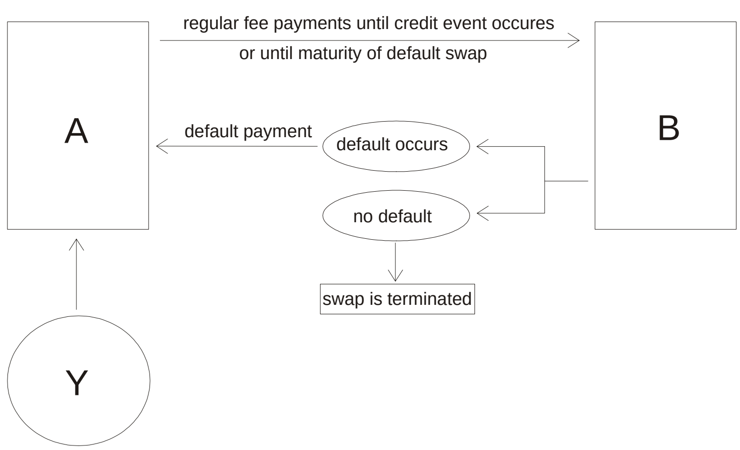

Figure 2. *Credit default swap structure.*

`Example 1.1`: Assume that party A holds one five year coupon bond issued by company Y, with a notional principal of 1 000 SKK. The bondholder wants to hedge his credit exposure of this bond and decides to enter a credit default swap. Party B is willing to enter a five year credit default swap with the protection buyer A. This agreement commits A to pay fee payments of 5 % of notional principal annually. In response B agrees to pay the default payment if a credit event occurs. Suppose that the default payment is the notional amount of the bond minus its recovery value and that the credit event is defined as a credit rating downgrade of company Y. Suppose further that there is a rating downgrade at the end of the third year. Suppose that the recovery value of the bond is 800 SKK and therefore B has to pay 200 SKK to its counterparty. In this case A had to pay $1\,000 \cdot (0.05) = 50$ at the end of the first year, and also at the end of the second and the third year.

Payments at default

Counterparties A and B from `Example 1.1` can agree on a variety of key terms of the default swap. Most used alternatives of payments, when a default occurs, are:

- Physical delivery of the reference asset against repayment at par value. Default protection buyer makes physical delivery of the reference obligation to default protection seller, who pays him the notional amount of the reference asset plus accrued but unpaid interests. When a default occurs between two fee payment dates, the protection buyer still has to pay the fraction of the next fee payment that has accrued until the time of default at which the credit default swap is terminated with no further obligation to either counterparty.
- Notional value minus post-default market value of the reference asset. Default protection buyer receives a payment from protection seller in an amount equal to notional value of the reference obligation minus its market value some specified days after default. Again, protection buyer has to pay to protection seller the accrued payment amount up to the date of credit event. Protection seller gets no other additional premium and the contract is terminated with no further obligation to either counterparty.
- A pre-agreed fixed payoff. This might be set at any percents of the notional amount. Protection buyer and protection seller agree in advance on a fair default payment.

Insurance premium

The insurance premium depends on several factors. The first one is the credit quality of the insurance seller and the reference obligation. Agencies such as Moody's or Standard & Poor's are the most famous rating providers in this area. The influence of rating on insurance premium is that a higher rated protection seller demands a higher premium for protection than a lower rated protection seller.

Another important factor which has an effect on the premium amount is the correlation of the protection seller with the reference asset. With a higher correlation level the possibility of default of protection seller is also higher.

Trading with credit default swaps was facilitated by standard documentation produced by the International Swaps and Derivatives Association (ISDA) in 1998.

3. Total return swap

Total return swap is a contract in which the participating parties agree on a specified cash flow exchange. Protection buyer, usually the owner of a reference asset, pays the total return that arises from this reference asset to insuring counterparty. He is called the total return payer. He exchanges the total return on a reference asset for payments from the total return receiver. These payments are usually floating rate payments of LIBOR or another money market rate plus a pre-agreed spread. During the life of the swap, the receiver has to pay any price depreciation of the reference asset. If the reference asset increases in value, the receiver receives an amount from the total return payer which reflects this price appreciation. In a total return swap there are also payments conditioned on the occurrence of a credit event. If there is a default on the reference obligation, the swap is usually terminated and the receiver has to pay the default payment. This payment is typically equal to the notional value of the reference asset minus its recovery value. The key terms of the swap are specified in the contract.

`Example 1.2:` Assume that party A holds a five year coupon bond issued by company Y (notional value 1 000 SKK, coupon payments 6%). He wants to hedge his exposure from this bond and decides to enter a total return swap with party B. On coupon payment dates, the payer pays the coupons earned on the bond. In return he receives a floating interest at LIBOR plus 50 bp on the notional amount of 1 000 SKK. If the bond increases in value, the payer is required to pay also this difference. If the value of the bond falls, the receiver is required to pay this difference. If there is a default on the bond during the life of the swap, the swap is terminated and the receiver pays an amount equal to notional value of the bond and receives its market value after default.

Some advantages of the total return swap

- If the reference asset is unavailable in the market or the receiver is forbidden to purchase the asset, a total return swap may be the only way in which he can invest to the asset and gain the desired amount. This means that the total return receiver can get access to previously unreachable markets.
- The total return swap contract can be signed without the knowledge of the reference entity
- Total return swaps are governed by ISDA rules. The exchange of payments is also executed under the terms of ISDA.
- Total return swap transfers credit risk and also market risk of the reference asset to the total return receiver.

Payments at default

Payments exchanges conditioned on default has to be pre-agreed in terms of the total return swap. There is a variety of ways how it can be done. The receiver can deliver the bond in exchange for its par value. Another possibility for the payer is to keep the asset and pay its recovery value to receiver in return for its notional value. Counterparties can further agree that the payer keeps the reference asset and receives an amount equal to notional principal minus its market value some specified days after default.

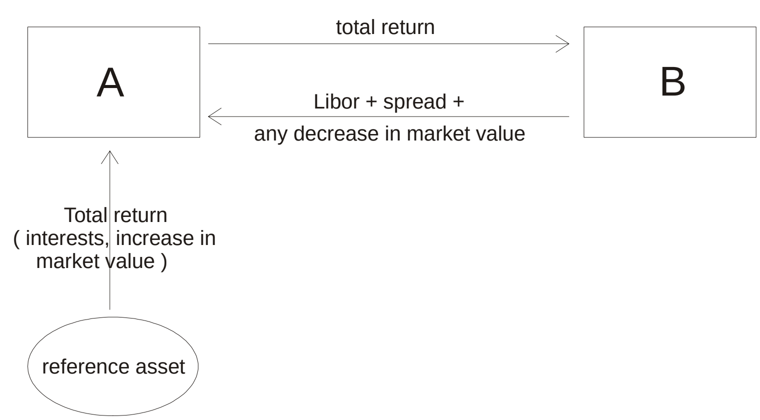

FIGURE 3. *Total return swap structure*

4. Credit-linked note

Credit-linked note is an instrument allowing the issuer of the note to transfer a specific credit risk of a reference entity or a reference asset to *credit investors*. It can be issued by the owner of the reference credit itself, but it is often issued by another party. This party can be a *Special Purpose Vehicle* or a *Trust*, which enters a credit derivative contract with the protection buyer and sells the issued note to the credit investor. The issuer therefore has the hedge money in advance. The cashflows in a credit linked note are conditioned on the occurrence of a credit event. Credit investor receives fixed or floating periodic coupon from the issuer during the life of the note and the par value of the note at maturity, unless a credit event occurs, in which case he receives the recovery value of the note. He has an exposure to the issuer and also to the reference credit. The stream of cashflows is exactly specified in the contract.

Characteristics and advantages of credit linked notes:

- Correlation of the credit investor with the reference asset and the credit quality of the investor are irrelevant, because the issuer gets the hedge money up front.
- Investors can participate through credit linked notes in credit derivative markets where they have no access other way.
- Credit investor sells credit protection in exchange for higher yield on the note. The issuer of credit linked note, on the other hand, buys credit protection. There is a close relationship between the credit default swap and the credit linked note.

`Example 1.3:` Assume that party A holds a five year coupon bond issued by company Y (notional value 1 000 SKK, coupon 8%). It wants to hedge its exposure from this bond and decides to issue a credit linked note. Party B is willing to buy this note. This contract commits party A to pay regular fee payments of 5% of the notional value. If the credit event does not occur, the investor receives at maturity the whole notional amount of the note. If there is a credit event, he receives the recovery value. Suppose that the credit event is defined as a rating downgrade of the reference entity Y. Suppose further that there is a rating downgrade at the end of the second year. The recovery value of the bond after credit event is 900 SKK. Party B delivers the note to its issuer in exchange for its recovery value 900 SKK. Party A paid 50 SKK at the end of the first year and also 50 SKK at the end of the second year. The contract is terminated after the default event. The structure of this credit linked note is outlined in Figure 4.

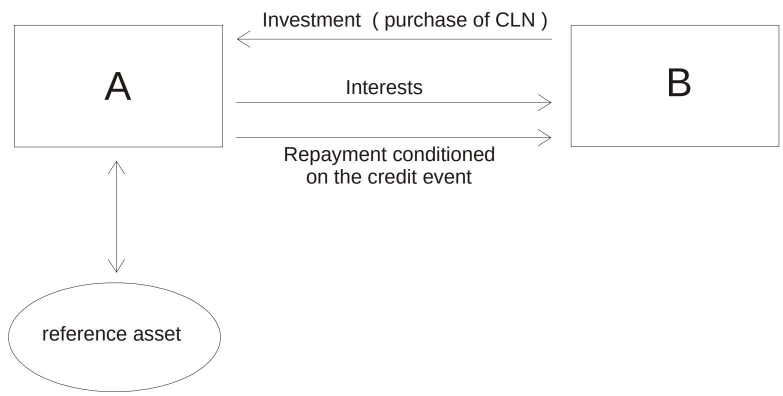

Figure 4. *Simplified credit linked note structure (the issuer of the note is the protection buyer itself)*

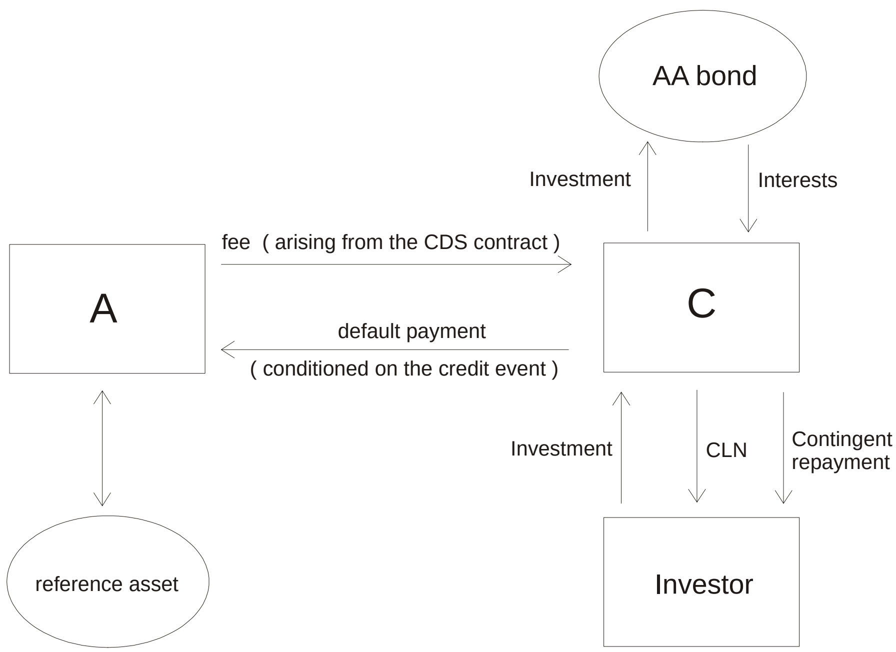

Figure 5. *Credit linked note structure*

`Example 1.4.:` Assume that party A holds a five year coupon bond issued by company Y (notional value 1 000 SKK, coupon 8%). It wants to hedge its exposure from this bond and enters a credit default swap with a special purpose company C. Party C issues a credit linked note, sells it to investor and uses the money to buy a high rated bond. As a result the exposure of party A to the reference credit is passed to the investor. The investor receives a regular fee from the issuer of the note until its maturity (when he receives also the par value of the note) or until a credit event happens. As with the credit default swap, there is a number of credit events that can be included in the contract. After a credit event, the default swap needs to be settled first. This involves a payment equal to the par value of the reference obligation (1 000 SKK) minus its post-default market value. If the issuer of the note is a special purpose company then it will need to liquidate its collateral (a high rated bond in this example) in order to be able to make a payment to the protection buyer A. Once this payment is made, any residual cash is passed to the investor as a redemption amount. The note is then terminated. The structure of this contract is outlined in Figure 5.

5. Banks and Credit Derivatives

Capital Requirements

In 1988, the Basel Committee on Banking Supervision, which is a committee of central banks from the major industrialized countries that meet regularly in the Bank for International Settlements (BIS) in Basel, introduced its Basel I Capital Accord. This Accord requires international banks to hold a certain part of their capital for protection against unexpected losses that may arise from risky assets.

This amount of capital is defined as 8 % of *risk-weighed assets* obtained by multiplying the outstanding amount (net of provisions) with the pre-defined BIS weight. There are four values of BIS risk weights. Claims on OECD governments and central banks have zero percent BIS risk weight. Claims on banks from OECD countries have 20 percent BIS risk weight. Mortgages are weighted by using 50 percent BIS risk weight. Claims on other counterparties (e.g. corporates) have 100 percent BIS risk weight.

If an asset has a 20 percent BIS risk weight, the bank has to hold 20 percent of 8 percent, that means 1.6 percent of the value of the claim as a minimum capital requirement. In this case, an exposure of 1 million SKK is equivalent to risk weighted asset of 200 000 SKK and to minimum capital requirement of 16 000 SKK. This can be shown with a simple calculation:

$$\begin{aligned} Risk\ Weighted\ Asset &= Exposure * Risk\ Weight \\ &= 1\,000\,000 * 0.2 = 200\,000 \end{aligned}$$

$$\begin{aligned} Capital\ Requirement &= Risk\ Weighted\ Asset * 0.08 \\ &= 200\,000 * 0.08 = 16\,000 \end{aligned}$$

This capital adequacy system does not differentiate the actual riskiness of the assets. Consider, for example, the case of an AAA rated bank with a funding cost of LIBOR minus 25 bp which purchases an AA rated asset with a notional value of 1 million Sk and a coupon of LIBOR plus 30 bp. The BIS risk weight of the asset is 100 percent. It means that the net income of the bank from this transaction is 5 500 SKK and the minimum capital requirement is 80 000 SKK, what results into a return on regulatory capital of 6.875 percent. The investment of the same parameters to an OECD bank (no matter its rating) would have 20 percent risk weight and the minimum capital requirement would be 16 000 SKK. The return on regulatory capital would be 34.375 percent.

By using the credit derivatives the banks may manage their return on regulatory capital. Suppose the bank enters a credit default swap contract with an A rated OECD bank with a 20 percent BIS risk weight. The AAA rated bank, the protection buyer, pays 20 bp annually as an insurance premium. As a result, its net income will be 3 500 SKK, but the minimum capital requirement is reduced to 16 000 SKK what represents a return on regulatory capital of 21.875 percent.

Consider now the case that the AAA rated bank with a funding cost of LIBOR minus 15 bp purchases the BBB rated asset with a 100 percent BIS risk weight, a notional value of 1 million SKK and a coupon of LIBOR plus 60 bp. The return on capital from this transaction is 9.375 percent. In effort to increase this return, the bank can enter a total return swap contract with an A rated OECD bank, which is willing to pay LIBOR plus 15 bp. As a result, the BIS risk weight of the asset can be reduced to 20 percent. According to Tavakoli [**10**], the bank can choose between the risk weight of its counterparty and the insured asset. It means that the AAA rated bank has increased its return on capital from 9.375 percent to 18.75 percent.

The A rated bank could purchase the same asset with a higher funding cost than the AAA bank, with a funding cost of LIBOR plus 30 bp. Therefore, it has also improved its position with this transaction, because it has increased its net spread from 0.35 percent to 0.50 percent.

The imperfection of the Basel I system, however, led further to a so called capital arbitrage when banks entered the TRS transactions with other banks and assumed for example all credit risk of the corporate counterparty but only with 20 percent risk weight. On the other hand banks that got rid of some of their credit risk by using credit derivatives were still "punished" with 100 risk weight.

The Basel I system is soon to be replaced by Basel II version which is more sensitive to actual riskiness of assets, however one can assume that banks will sooner or later find their way how to "beat the system" again. It is probable that credit derivatives will again play an important role in these efforts.

Concentrated loan portfolios

Apart from regulatory capital management credit derivatives can be useful in portfolio credit management, e.g. for banks which loan portfolios are concentrated in a specific industrial or geographical area.

Consider a hypothetical bank, Hungarian Agricultural Bank, which lends mostly to farmers. This bank is exposed to a higher credit risk. It can happen some year that these farmers will have a failure of crop (for example as a consequence of bad weather conditions) or they will be unable to sell enough of their commodities. If it is so, they will not have enough money to pay the loan payments.

The credit risk of this bank can be reduced by swapping the payments from some of the bank's assets for payments of an institution which is from a different industrial or geographical area. This way the bank may diversify its portfolio. A total return swap represents a useful instrument for this exchange.

The bank has also some other possibilities to diversify its credit risk. One of them is to sell some loans and purchase others, but the administrative costs of the loan sale transaction can be higher than for the swap transaction. Further, a loan sale requires the records of the borrowers to be transferred to the new owner of the loan.

From the example above is implicit that credit derivatives are widely applicable in the financial practice. For their successful usage is essential to set the "fair" value, that means in the case of credit default swap for swap premium. We are going to discuss this problematic in the following chapter, in which there are introduced three models. The model of Das and Sundaram [**4**], the Hull-White model [**7**] and the model of Schönbucher [**9**]. As it can be seen also from Figure 1 above, credit default swap has the major portion in the market of credit derivatives. So we will deal only with CDS, though derived methods can be transferred on the other types of derivatives.

CHAPTER 2

Valuation of credit derivatives

The theoretical present value of an n-year risk free bond (e.g. issued by the US government) with a fixed coupon rate c paid annually can be calculated as the present value of future cashflows related to the bond, as shown in [**3**]:

$$PV = \sum_{t=1}^{n-1} \frac{C}{(1+i)^t} + \frac{F+C}{(1+i)^n} \ ,$$

where F is the notional amount of the bond, $C = Fc$ the coupon payment and i the risk free interest rate which is assumed constant for n years in this simple case. Here i corresponds to a yield to maturity related to risk free investment with maturity n years.

More generally, the equation for the value of a risk-free bond can be written as:

$$PV = \sum_{j=1}^{n} \frac{C_{t_j}}{\prod_{k=0}^{j-1}(1+f_{t_k})} \ ,$$

or

$$PV = \sum_{j=1}^{n} \frac{C_{t_j}}{(1+r_{t_j})^{t_j}} \ ,$$

where t_j are times of cash flows related to the bond, C_{t_j}, $j = 1, ..., n$, are the cashflows that characterize the bond, f_{t_k} are the forward interest rates for the time intervals $(t_k, t_{k+1}]$ and r_{t_j} are the annual interest rates for the maturity at t_j.

The bondholder of this bond is exposed to the *market risk*, i.e. a risk of depreciation of a value of the bond caused by increased market interest rates. Apart of the market risk, the bondholder of a risky (corporate) bond is exposed to the *credit risk* of the issuer, i.e. to the risk that the issuer will not repay his debt in full.

NOTE 1: Except these main types of risks, market risk and credit risk, there exist also other forms of risks. One of them is for example the operational risk. ◇

According to general rules of investment, investors should be compensated for the risk they are bearing with a higher yield on financial instruments. The valuation formula therefore changes as follows:

$$PV = \sum_{t=1}^{n-1} \frac{C}{(1+i+s)^t} + \frac{F}{(1+i+s)^n} \ ,$$

where i is considered to be the risk free yield to maturity and s is the spread over the risk free yield.

Similarly the second formula changes as follows:

$$PV = \sum_{j=1}^{n} \frac{C_{t_j}}{\prod_{k=0}^{j-1}(1+f_{t_k}+s_{t_k})} \ ,$$

or

$$PV = \sum_{j=1}^{n} \frac{C_{t_j}}{(1+r_{t_j}+s_{t_j})^{t_j}} \ .$$

Hence the spread s plays a crucial role in pricing the risky assets and may be used to quantify the credit risk (up to a certain extent – in practise, the spread reflects also the worse liquidity of more risky assets), i.e. the probability of the counterparty default.

The earliest models of credit risk assumed that default is determined with the firm's value. If the value of a firm's assets when a bond matures is less than the bond's face value, the firm can not completely pay its debt to the bondholder even by liquidating all of its assets. Default occurs when the value of the firm decrease bellow a certain threshold. These models are called the structural models. One of the earliest structural models was developed by Merton [**8**] and is described in the following section.

The structural model of Merton for pricing corporate debt

In the model of Merton, it is assumed that default can occure only at maturity of the debt.

Consider a firm that has issued n shares and that has no other stock. Consider further that the firm issued a T-year zero coupon bond with face value K. The payment to the bondholder at maturity of the bond depends on the market value of the firm. If the value of the firm V is higher than the face value of the bond, the bondholder receives an amount equal the face value of the bond, but if it is lower, he gets the market value of the firm. The price of the bond at maturity can therefore be written as

$$P(T) = min(K, V(T)) \ ,$$

or

$$P(T) = K - max(K - V(T), 0) \ .$$

The term $max(K - V(T), 0)$ is equal to a payoff of an European put option on a non-dividend paying stock, where K corresponds to the strike price and V to the stock price. Merton's model gives the value of the firm's equity at time T as $max(V(T) - K, 0)$, what is a European call option on the value of the firm's assets. The market value of the firm is assumed to follow a stochastic process:

$$dV_t = \mu_v V_t + \sigma_v V_t dW_t \ ,$$

where μ_v is the drift of firm's assets, σ_v is their volatility and W_t is a Wiener process.

The equity value of the firm today is given by the Black-Scholes formula as:

$$E_0 = V_0 \Phi(d_1) - K \mathrm{e}^{-rT} \Phi(d_2) \ ,$$

where

$$d_1 = \frac{\ln(\frac{V_0}{K}) + (r + \frac{1}{2}\sigma_v^2 T)}{\sigma_v \sqrt{T}} \ ,$$

$$d_2 = d_1 - \sigma_v \sqrt{T} \ ,$$

and Φ denotes a cdf of the normal distribution with mean 0 and standard deviation 1. The value of the debt today is $V_0 - E_0$. The value of firm's equity E_0 can be observed, and the volatility of equity σ_e can be estimated from Itô's lemma (See [**7**], 11.6).

One of the main drawbacks of structural models is that parameters related to the firm's value V_0, σ_v are usually unobservable. Another disadvantage of structural models is that they can not incorporate other credit events, e.g. credit rating changes. These limitations make it necessary to look at other classes of models for valuing defaultable securities and securities that are subject to credit risk.

As a result, the models for credit risk have advanced to models that do not require any parameter related to the value of the firm. These models are known as reduced form models. They can be subdivided into several subgroups depending on how they model the default risk and the recovery risk. They include also some intuitive models and models based upon transitions between credit ratings.

1. The model of Das and Sundaram

The model developed by Das and Sundaram [**4**] belongs to reduced form models and enables calculating the premium in a credit default swap. The advantage of this model compared to other models is that it requires a relatively few input parameters.

The notation used in this section is:

(t,T)	... time period
h	... interval between payment dates
$f(t,t_j)$	... forward interest rate determined at time t for the time period (t_j, t_j+h) It is equivalent with the notation $f(t,t_j,t_j+h)$ - forward interest rate over the period (t_j,t_j+h) as seen at time t
$r(t)$	... spot interest rate, $r(t) = f(t,t)$
$s(t,t_j)$	... forward credit spread for the time period (t_j,t_j+h) as seen at time t
X_1, X_2	... random variables taking on values +1 or -1 with equal probabilities under the probability measure P
$p(t,t_j)h$	... probability of default in the time interval (t_j,t_j+h) as seen at time t $p(t,t_j)$ is denoted as default intensity
$R(t)$	... recovery rate in the event of default
$B(t,T)$	... price at time t of the risky bond with notional amount 1 and maturing at time T
$G(t,T)$	... price at time t of the riskless bond with notional amount 1 and maturing at time T
ρ	... coefficient of correlation between forward rates and forward spreads
$\Lambda(t)$	... cumulative default probability at time t

Assumptions used in the model are:

AD_1 : $f(t,T)$ exists for all T.

AD_2 : $s(t,T)$ exists for all T.

AD_3 : There exists a risk neutral measure Q under which the prices of securities are martingales.

AD_4 : The bivariate process $[X_1, X_2]$ takes values $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ with probabilities $\left\{\frac{1+\rho}{4}, \frac{1-\rho}{4}, \frac{1-\rho}{4}, \frac{1+\rho}{4}\right\}$.

AD_5 : Claim made by bondholders in the event of default is the market value of the bond.

AD_6 : There is no counterparty default of the protection seller.

The term structure of forward rates is assumed to follow the process:

$$f(t+h,T) = f(t,T) + \alpha(t,T,f(t,T))h + \sigma(t,T,f(t,T))X_1\sqrt{h} \qquad \forall\ T\ ,$$

where α is the drift and σ the volatility of forward rates. Both α and σ are functions of current time, future forward time and level of the forward rates. The notation can be also $\alpha(T)$ and $\sigma(T)$ when it is clear for which time and level they refer to.

The forward spreads are assumed to follow the process:

$$s(t+h,T) = s(t,T) + \beta(t,T,s(t,T))h + \delta(t,T,s(t,T))X_2\sqrt{h} \qquad \forall\ T\ ,$$

where β is the drift and δ the volatility of forward spreads.

The valuation can be divided into a few steps. First, the drifts of forward rates have to be calculated. Second, the drifts of forward spreads are determined. Finally, an algorithm for calculating the credit default swap premium is constructed.

Deriving the drifts of forward rates

The price of the riskless bond is given as:

$$G(t,T) = \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1} f(t,ih)h\right)\ . \tag{1}$$

In (1), a continuous compounding is used. This compounding is used also in the rest of this work.

Note 2: A rate r_m with compounding of m times per annum can be converted into a continuously compounded rate r_c as:

$$r_c = m\ln\left(1+\frac{r_m}{m}\right)$$

Deriving of the relationship between r_m and r_c can be found in [**7**]. ◇

Further, an auxiliary variable for an accumulation account is defined, which will be useful for deriving the equation for the drifts of forward rates. Its definition is as follows:

$$D(t) = \exp\left(\sum_{i=0}^{\frac{t}{h}-1} r(ih)h\right)\ . \tag{2}$$

Note 3: Definition of a martingale. Define $(\Omega,\mathcal{A},Q)$ a probability space and $(\mathcal{F}_i)$ filtration on this space. A process $(U_t, t\geq 0)$ is a martingale with respect to the probability measure Q and a filtration $(\mathcal{F}_i)$ if

$$0\leq t\leq s\ \Rightarrow\ E_Q[U_s|\mathcal{F}_t] = U_t \text{ a.s.}$$ ◇

According to Heat Jarrow and Morton [**6**], all discounted asset prices are martingales under the Q measure and it holds that:

$$\frac{G(t,T)}{D(t)} = \mathrm{E}_Q\left\{\frac{G(t+h,\,T)}{D(t+h)}\right\}$$

$$\Rightarrow 1 = \mathrm{E}_Q\left\{\frac{G(t+h,\,T)}{G(t,\,T)}\frac{D(t)}{D(t+h)}\right\}\ . \tag{3}$$

The components of (3) can be obtained with simple calculations. The expression $G(t+h,T)/G(t,T)$ can be written as:

$$\begin{aligned}\frac{G(t+h,T)}{G(t,T)} &= \exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} f(t+h,ih)h\right)\exp\left(\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1} f(t,ih)h\right)\\ &= \exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(f(t+h,ih)-f(t,ih))h+f(t,t)h\right)\ .\end{aligned}$$

For $D(t+h)/D(t)$ it is:

$$\begin{aligned}\frac{D(t+h)}{D(t)} &= \exp\left(\sum_{i=0}^{\frac{t}{h}} r(ih)h\right)\exp\left(-\sum_{i=0}^{\frac{t}{h}-1} r(ih)h\right)\\ &= \exp(r(t)h)=\exp(f(t,t)h)\ .\end{aligned}$$

The expression in (3) can therefore be written as:

$$\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(f(t+h,ih)-f(t,ih))h+f(t,t)h\right)\exp(f(t,t)h)^{-1}\right\}\ .$$

Using the term structure of forward rates, this is equal to:

$$\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(f(t,ih)+\alpha(ih)h+\sigma(ih)X_1\sqrt{h}-f(t,ih))h\right)\right\}\ ,$$

what can be further rewritten as:

$$\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(\alpha(ih)h+\sigma(ih)X_1\sqrt{h})h\right)\right\}\ ,$$

or equivalently:

$$\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\alpha(ih)h^2\right)\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\sigma(ih)X_1\sqrt{h}h\right)\right\} .$$

Finally, the equation in (3) is equivalent to:

$$1=\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\alpha(ih)h^2\right)\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\sigma(ih)X_1\sqrt{h}h\right)\right\} .$$

The last relation enables to get the recursive equation for the risk-neutral drift terms $\alpha(ih)$:

$$\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\alpha(ih)=\frac{1}{h^2}\ln\left[\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\sigma(ih)X_1\sqrt{h}h\right)\right\}\right] .$$

The drifts of forward spreads are calculated with analogy.

Deriving the drifts of forward spreads $s(t,T)$

The price at time t of a defaultable bond maturing at time T is given by:

$$B(t,T)=\exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}(f(t,ih)+s(t,ih))h\right)$$

To derive the drift of the particular forward spread, it is helpful to define another accumulation account:

$$D^*(t)=\exp\left(\sum_{i=0}^{\frac{t}{h}-1}(f(ih,ih)+s(ih,ih))h\right) .$$

Similarly as in (3), for the price of risky bond holds that:

$$\frac{B(t,T)}{D^*(t)}=\mathrm{E}_Q\left\{\frac{B(t+h,T)}{D^*(t+h)}\right\}$$

$$\Rightarrow 1=\mathrm{E}_Q\left\{\frac{B(t+h,\,T)}{B(t,\,T)}\frac{D^*(t)}{D^*(t+h)}\right\} . \tag{4}$$

To obtain the recursive expression for the drifts of forward spreads, it is enough to express the components of (4).

For $B(t+h,T)/B(t,T)$ it is:

$$\frac{B(t+h,T)}{B(t,T)} = \exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(f(t+h,ih)+s(t+h,ih))h\right)\times$$
$$\times\exp\left(\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}(f(t,ih)+s(t,ih))h\right) \ ,$$

and also

$$\frac{B(t+h,T)}{B(t,T)} = \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}(f(t+h,ih)+s(t+h,ih)-f(t,ih)+s(t,ih))h\right)\times$$
$$\times\exp(f(t,t)h+s(t,t)h) \ .$$

Similarly, for $D^*(t+h)/D^*(t)$ we get:

$$\frac{D^*(t+h)}{D^*(t)} = \exp(f(t,t)h+s(t,t)h) \ .$$

The equation (4) can now be written as:

$$\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(f(t+h,ih)+s(t+h,ih)-f(t,ih)+s(t,ih))h\right)\right\} = 1 \ ,$$

or

$$\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(\alpha(ih)h+\beta(ih)h+\sigma(ih)X_1+\delta(ih)X_2\sqrt{h})h\right)\right\} = 1 \ .$$

As $\alpha(ih)$ has already been computed, $\beta(ih)$ can be solved recursively from the equation:

$$\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(\alpha(ih)+\beta(ih)) = \frac{1}{h^2}\ln\left[\mathrm{E}_Q\left\{\exp\left(-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(\sigma(ih)X_1+\delta(ih)X_2\sqrt{h}h\right)\right\}\right] \ .$$

Decomposition of the forward spread

The price of a risky bond with maturity after one period with notional amount of one is:

$$B(t,t+h) = \exp(-(f(t,t)+s(t,t))h) \ .$$

According to Das and Sundaram [**4**], the price of this bond is also equal:

$$B(t,t+h) = \exp(-f(t,t)h)[p(t,t)hR(t) + 1 - p(t,t)h] \ , \tag{5}$$

where $p(t,t)hR(t)$ is the probability of default times the recovery amount, and $(1 - p(t,t)h)1$ is the probability of no default times the market value of the underlying zero coupon bond at maturity.

NOTE 4: It is known that $\exp(-x) \simeq 1 - x$, for $x \to 0$. Since $p(t,t)$ is decreasing in h as h goes to zero, this property can be applied on the expression in (5):
$$\exp(-p(t,t)h[1-R(t)]) \simeq p(t,t)hR(t) + 1 - p(t,t)h \ , \ h \to 0 \ .$$ ◇

According to the previous note, (5) can be written:

$$B(t,t+h) = \exp(-f(t,t)h - p(t,t)h(1-R(t)) \ ,$$

what implies that $s(t,t) = p(t,t)(1-R(t))$. Therefore, in general we can write:

$$\begin{aligned}
B(t,T) &= \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}[f(t,ih)+s(t,ih)]\,h\right) \\
&= \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}[f(t,ih)+p(t,ih)(1-R(ih))]\,h\right) \\
&= \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-2}[f(t,ih)+p(t,ih)(1-R(ih))]\,h\right) \times \\
&\quad \times \exp\left(-\sum_{i=\frac{T}{h}-1}^{\frac{T}{h}-1}[f(t,ih)+p(t,ih)(1-R(ih))]\,h\right) 1 \\
&= \exp\left(-\sum_{i=\frac{t}{h}}^{\frac{T}{h}-2}[f(t,ih)+p(t,ih)(1-R(ih))]\,h\right) B(T-h,T) \ .
\end{aligned}$$

The model is based on the idea, that the spreads reflect the cost of default and are functions of the probability of default and the recovery rate in the event of default.

The one period forward spread $s(t,t)$ contains information about default probabilities and recovery rates over one period. As $s(t,t) = p(t,t)[1-R(t)]$, an additional equation is required that allows to calculate the two components $p(t,t),\ [1-R(t)]$.

In [**4**], this is solved with the adoption of the following assumption about the default probability:

AD_7 : The default probabilities are defined using a logit equation:

$$p(t,t) = \frac{\exp(a + bf(t,t) + cs(t,t))}{1 + \exp(a + bf(t,t) + cs(t,t))} \ .$$

If we have the value of $p(t,t)$, the value of the recovery rate can be calculated from:

$$R(t) = 1 - \frac{s(t,t)}{p(t,t)} \ .$$

To ensure that $R(t) \in [0,1]$, it is necessary to assume that $p(t,t) \geq s(t,t), \quad \forall\ t$.

The implementation approach is according to the assumption AD_4 a tree model. Consider a tree with four branches emanating from each node. It is illustrated in Figure 1 bellow. At each node we calculate the forward rates, spreads, default probabilities and recovery rates. This leads to the notation: $f(t,t,\varpi),\ s(t,t,\varpi)$, $p(t,t,\varpi),\ \phi(t,\varpi)$, where ϖ represents the random choice of one of the four branches of the lattice at each node at time t.

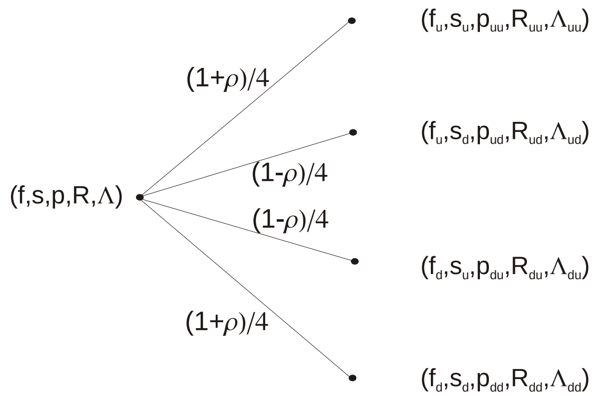

FIGURE 1. *One step in the tree.*

To value a credit default swap, it is necessary to define the cumulative probabilities of default. This is done with the recursive equation:

$$\Lambda(t+h,\varpi) = \Lambda(t) + (1-\Lambda(t))p(t,t,\varpi)h, \quad \Lambda(0) = 0 \ ,$$

where $p(t,t)h$ denotes the probabilities of default over one period.

Credit default swap

Assume that the payoff in the event of default is the loss on default:

$$1 - R(t) \ .$$

To calculate the credit default swap premium s^*, it is defined the probability of default conditional on no prior default as:

$$(1 - \Lambda(t))p(t, t) \ ,$$

where $p(t, t)$ is calculated as given in assumption AD_7. The cashflow to the credit default swap in this model is the loss on default times these probabilities of default:

$$(1 - \Lambda(t))p(t, t) \times (1 - R(t)) \ ,$$

what can be further written as:

$$(1 - \Lambda(t))p(t, t)\frac{s(t, t)}{p(t, t)} = (1 - \Lambda(t))s(t, t) \ . \tag{6}$$

For simplicity, consider a two year credit default swap, where the reference obligation is a zero coupon bond with notional amount of 1$. The tree looks then like shown in Figure 2. Das and Sundaram [**4**] assume that it is simpler to calculate the credit default swap premium required from protection buyer as a single up-front payment rather than a stream of payments. The value given in (6) is calculated for nodes that correspond to the time of maturity and also for each other node of the tree. These values are then discounted back appropriately. For Figure 2 this means:

$$s^*_{uuuu}(t_2) = (1 - \Lambda(t_2))s_{uuuu}(t_2, t_2) \ ,$$

$$s^*_{uu}(t_1) = (1 - \Lambda(t_1))s_{uu}(t_1, t_1) + \exp(-f(t_1, t_1))\times \tag{7}$$

$$\times((1 + \rho)/4 \cdot s^*_{uuuu} + (1 - \rho)/4 \cdot s^*_{uuud} + (1 - \rho)/4 \cdot s^*_{uudu} + (1 + \rho)/4 \cdot s^*_{uudd}) \ ,$$

and similarly for other nodes, $s^*_{ud}(t_1)$,$s^*_{du}(t_1)$ and $s^*_{dd}(t_1)$. The credit default swap premium $s^*(t_0)$ is calculated as in (7), but with discounting of values from nodes corresponding to time t_1:

$$s^*(t_0) = (1 - \Lambda(t_0))s(t_0, t_0) + \exp(-f(t_0, t_0))\times$$

$$\times((1 + \rho)/4 \cdot s^*_{uu} + (1 - \rho)/4 \cdot s^*_{ud} + (1 - \rho)/4 \cdot s^*_{du} + (1 + \rho)/4 \cdot s^*_{dd}) \ .$$

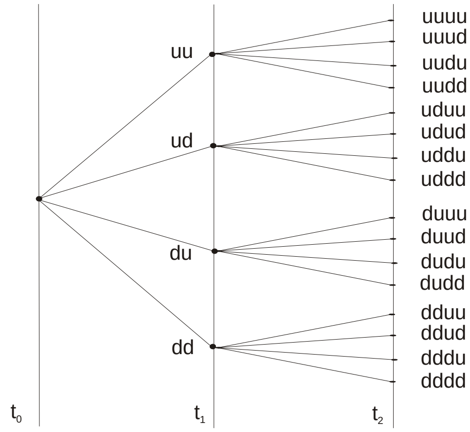

FIGURE 2. *Example of a two-step tree*

Consider now a 5 year credit default swap, where the reference obligation is a zero-coupon bond with notional amount of 1\$. Assume that hypothetical data shown in Figure 3 are used to calculate the cds premium.

T	f(0,T)	σ_f	s(0,T)	σ_s
0	0.05	0.017	0.008	0.004
1	0.06	0.014	0.011	0.005
2	0.07	0.012	0.015	0.006
3	0.09	0.009	0.020	0.008
4	0.10	0.008	0.022	0.009

ρ	h	a	b	c
0.25	0.5	-3	8	50

FIGURE 3. *Input parameters for the valuation algorithm.*

The computed credit default swap premium is 1.67% from the swaps notional value. It can be calculated using a recursive programming technique.

1.1. Model implications.

The assumed forward probability of default used in the model does not always correspond with the default probability in reality, because the parameters of the function in AD_7 are determined empirically. The function used for calculating the premium of the CDS above is shown in Figure 4. To demonstrate what influence the parameters have, there is a plot of the same function but with other parameters in enclosure A.

One of the possibilities to solve this problem could be to take historical default probabilities into consideration searching for parameters of the default probability function. The drawback of this idea is that the concrete historical probabilities are not easy to obtain.

Another drawback of this model is that the functions f and s can take negative values.

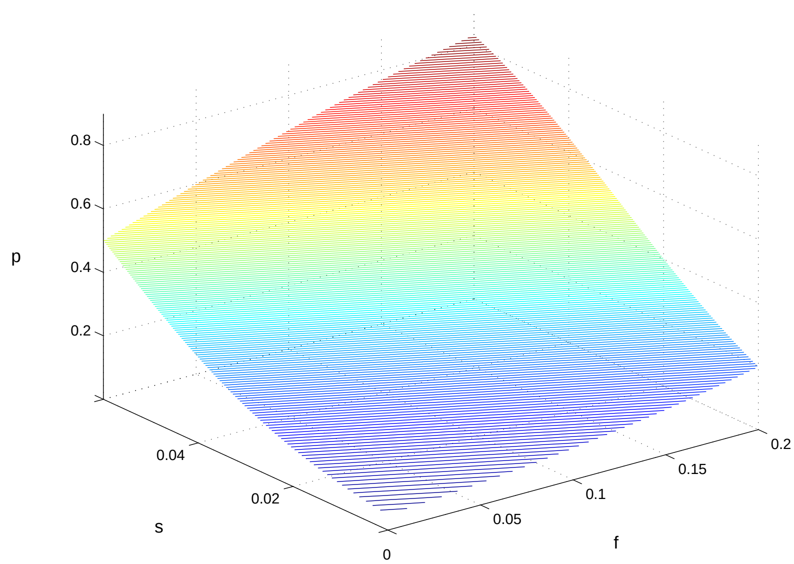

FIGURE 4. *The default probability function*

2. The Hull-White Model

In their work, John Hull and Allan White [**7**] deal with the problematics of valuing a credit default swap. It is a model that belongs to models of the reduced form type and differs from approaches that rely on using a hazard rate.

The assumptions of the model are:

AH_1: There is no counterparty default risk of the protection seller.

AH_2: Probability of default, interest rate, and recovery rates are mutually independent.

AH_3: Claim made by bondholders in the event of default is the face value of the bond plus accrued interest.

AH_4: Expected recovery rate is independent of time and maturity of the reference bond.

AH_5: The risk free rate is given by the treasury rate.

AH_6: Default can happen at any time during the life of the bonds.

AH_7: There is no systematic risk in recovery rates so that real world recovery is equal to risk neutral recovery.

The valuation of credit default swap can be divided into two steps. First, the risk neutral probability of default has to be estimated from bond prices. Secondly, the credit default swap spread is estimated.

NOTE 5: *Real world and risk-neutral world*

According to Hull [**7**], the risk neutral world is a world where it is assumed that:

1. The expected return from all traded securities is the risk-free interest rate and investors require no compensation for risk.

2. Future cashflows can be valued by discounting their expected values at the risk-free interest rate. $\diamond$

2.1. Estimation of default probabilities.

Assume there are N bonds used in the analysis that are issued either by the reference entity or by other company with the same probability of default as the reference entity. The maturity of the ith bond is denoted t_i, with $t_1 < t_2 < \cdots < t_N$.

It is defined a default intensity $q(t)$, such that $q(t)\delta t$ is the unconditional probability of default between times t and $t + \delta t$ as seen from time zero. The assumption about the default intensity used in the model is:

AH_8: The default intensity $q(t)$ is piecewise constant for any time interval $t_{i-1} < t < t_i$.

The variables required for the estimation of default probabilities are:

B_j ... price of a defaultable bond maturing at time t_j
G_j ... price of a default free bond maturing at time t_j which has the same cash flow as the defaultable bond
$F_j(t)$ ... forward price of a default free bond maturing at time t_j for a forward contract maturing at time t, $t < t_j$
It is the forward value of the risk free bond G_j.
$v(t)$ ... present value of 1 monetary unit received at time t with certainty $v(t)$ will be used as a discount factor.
$C_j(t)$ ... claim amount made by bondholder if there is a default at time t of a bond maturing at time t_j.
$R_j(t)$ ... recovery rate on the bond maturing at time t_j if default occurs at time t. According to assumption AH_4, the notation will be: $R_j(t) = \hat{R}$
q_i ... default intensity for period i, ($q(t)$ for time interval $t_{i-1} < t < t_i$)
β_{ij} ... present value of the loss, relative to the value the bond with maturity t_j would have if there was no possibility of default, when default occurs at time t_i.
α_{ij} ... present value of the loss from a default on the jth bond at time t_i

At this point, the situation could be made easier by adopting a new assumption:

AH_9: Default can occur only at discrete times, for example at or immediately before maturity of bonds.

Although this assumption does not correspond with the default arrival in reality, replacing assumption AH_6 with AH_9 is a useful procedure for getting a notion of default probabilities. It is enough to calculate the present value of the loss from a default at time t_i on the bond maturing at time t_j. This loss is given by:

$$\alpha_{ij} = v(t_i)[F_j(t_i) - \hat{R}C_j(t_i)] . \tag{8}$$

The model is based on the idea that the difference in value between a risk-free bond and a defaultable bond, is due entirely to the expected present value of the costs of default.

The present value of the losses on the defaultable bond B_j, is therefore given by:

$$G_j - B_j = \sum_{i=1}^{j} p_i \alpha_{ij} , \tag{9}$$

where p_i is the probability of default at time t_i.

The probability of default p_j at time t_j is then determined inductively as:

$$p_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} p_i \alpha_{ij}}{\alpha_{jj}} \ , \qquad \alpha_{jj} > 0 \ \forall j. \tag{10}$$

The case $\alpha_{jj} = 0$ is trivial and may be omitted.

Now we return to assumption AH_6 and relax assumption AH_9. β_{ij} is then given by:

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(t)[F_j(t) - \hat{R}C_j(t)]dt \ . \tag{11}$$

The price $F_j(t)$ of the forward contract on the j-th bond represents the no default value of the bond at future time t.

With a similar consideration about the relationship between the price difference of risk-free and defaultable bonds and the expected present value of the cost of default, as already mentioned when analyzing assumption AH_9, the present value of losses on the defaultable bond B_j is expressed as:

$$G_j - B_j = \sum_{i=1}^{j} q_i \beta_{ij} \ . \tag{12}$$

By induction, it can be found any one q_j. This gives:

$$q_j = \frac{G_j - B_j - \sum_{i=1}^{j-1} q_i \beta_{ij}}{\beta_{jj}} \ , \qquad \beta_{jj} > 0 \ \forall j. \tag{13}$$

The case $\beta_{jj} = 0$ is trivial and may be omitted.

Calculation of default probabilities

Consider now the hypothetical bonds given in Figure 5 bellow.

Time to Maturity	Coupon (%)	Bond yield (%)
1	6	6.5
2	6	6.6
3	6	6.7
4	6	6.8
5	6	6.9
10	6	7.10

FIGURE 5. *Hypothetical data on bonds issued by a corporation*

Assume that all bonds have the same seniority and notional value and that the risk free zero rate i is flat at 5%. All bonds have coupons paid semiannually. The previous approach can now be applied.

Default probabilities (10) and intensities (13) for bonds given in Figure 5 are shown in Figure 6. Calculations are made under the assumption that the expected recovery rate is 30%. Suppose that it is a recovery rate that was observed from historical recovery rates for BBB rated bonds, as the coupons and spreads over treasury yields in Figure 5 are typical for BBB rated bonds. The forward prices are calculated using the expression:

$$F_j(t_k) = (G_j - I_k)\mathrm{e}^{it_k} \ ,$$

where I_k is the present value of income during the life of the forward contract, i is the risk free rate with continuous compounding and t_k is the time to maturity of the forward contract. The present values of risk free and defaultable bonds are:

$$G_j = \sum_{t=1}^{2\cdot t_j - 1} C_j \mathrm{e}^{-i\cdot\frac{t}{2}} + (Nom_j + C_j)\mathrm{e}^{-i\cdot t_j} \ ,$$

$$B_j = \sum_{t=1}^{2\cdot t_j - 1} C_j \mathrm{e}^{-r_j\cdot\frac{t}{2}} + (Nom_j + C_j)\mathrm{e}^{-r_j\cdot t_j} \ ,$$

where Nom_j represents the notional value of the j-th bond, C_j the coupon on the bond, i the continuously compounded risk free rate and r_j the interest rate for defaultable bonds from Figure 5.

NOTE 6: The integral in (11) can be calculated using a procedure for numerical integration as for example the Simpson's rule:

$$\int_a^b f(x)dx \approx$$

$$\approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \ .$$

Here, $(x_0, x_1, ..., x_n)$ is the division of the interval (a, b) such that $a = x_0, \ b = x_n$ and $x_k - x_{k-1} = constant, \ \forall \ k = 1, ..., n$. In general, the finer the division is, the better the approximation is. More precisely, the error estimate for the Simpson rule depends on the fourth derivative of $f(x)$. Suppose that $|f^{(4)}(x)| \le K$ for some constant K, $\forall \ a \le x \le b$. Then :

$$error \le \frac{K(b-a)^5}{180n^4} \ .$$

◇

The second and fifth column in Figure 6 contains probabilities of default and default intensities (denoted as $p_t^*, \ q^*(t)$) assuming that the claim made by bondholders is equal the no default value of the bond, whereas for probabilities and intensities in the third and sixth column, it is assumed that the claim equals the face value of the bond plus accrued interest (denoted as $p_t,\ q(t)$).

Time (t)	p_t^*	p_t	Time	$q^*(t)$	$q(t)$
1	0.0210	0.0210	(0-1]	0.0207	0.0206
2	0.0235	0.0234	(1-2]	0.0231	0.0230
3	0.0259	0.0258	(2-3]	0.0255	0.0253
4	0.0283	0.0281	(3-4]	0.0279	0.0276
5	0.0307	0.0303	(4-5]	0.0302	0.0297
10	0.1622	0.1596	(5-10]	0.0288	0.0281

FIGURE 6. *Default probabilities and intensities calculated from bonds in* Figure 5.

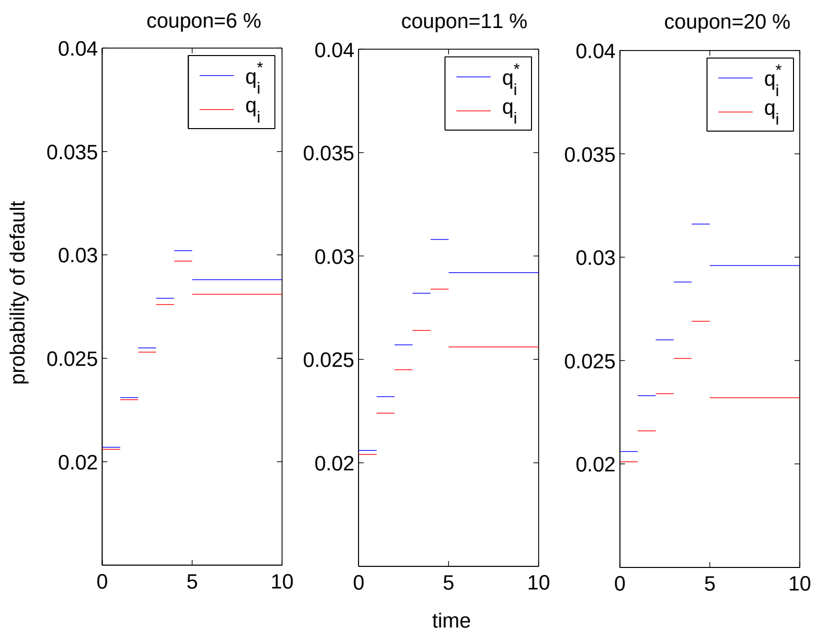

FIGURE 7. *Differences between default probabilities according to the assumption about the claim amount. (red line: claim is equal the face value plus accrued interest, blue line: claim equals the no default value of the bond).*

As it can be seen, the probabilities of default for these two different assumptions have a quite small differences. They would differ if the coupons on bonds would be much higher or much lower than the risk free interest rates, as shown in Figure 7.

In the left picture, there are the default probabilities for bonds in Figure 1. In the second and third picture, there are the default probabilities for the same bonds, but with coupons of 11% and 20%.

2.2. Valuation of credit default swap.

For simplicity, Hull and White assume that the par value of the reference obligation is 1$. The variables used in the valuation are the following:

T	... maturity of credit default swap
$u(t)$	... present value of an annuity of 1 per year at payment dates between times zero and t
$v(t)$	... present value of 1$ received at time t
$e(t)$	... present value of the accrual fee payment between times t^* and t where t^* is the last payment date
w	... total payments per year, or fees, paid by the CDS buyer
s	... CDS spread, given by the value of w that makes the value of the CDS contract zero at issuance
π	... risk neutral survival probability or risk neutral probability of no credit event during the life time of the swap
$A(t)$	... accrued interest on the reference obligation as a percent of face value
$p_i(t)$	... risk neutral probability of default at time t

The assumption AH_6 about the default arrival is now expelled again and replaced with the assumption AH_9, that default can happen only at discrete times, at or immediately before maturity of bonds ($0 < t_1, t_2, .., t_n = T$). Then, the value of π is given by:

$$\pi = 1 - \sum_1^n p_i \ .$$

It is calculated as one minus the probability that the credit event will occur by time T.

The CDS is composed of two legs: the fee payment leg and the payment at default leg. The cash flow of a credit default swap is outlined in Figure 8.

The expected present value of the first leg, of payments made by the protection buyer, is:

$$w \sum_1^n p_i[u(t_i) + e(t_i)] + w\pi u(T) \ . \tag{14}$$

The first term in this expression is the expected present value of the fee payments if default occurs at any time between zero and T. The second term is the expected present value of payments if no default occurs.

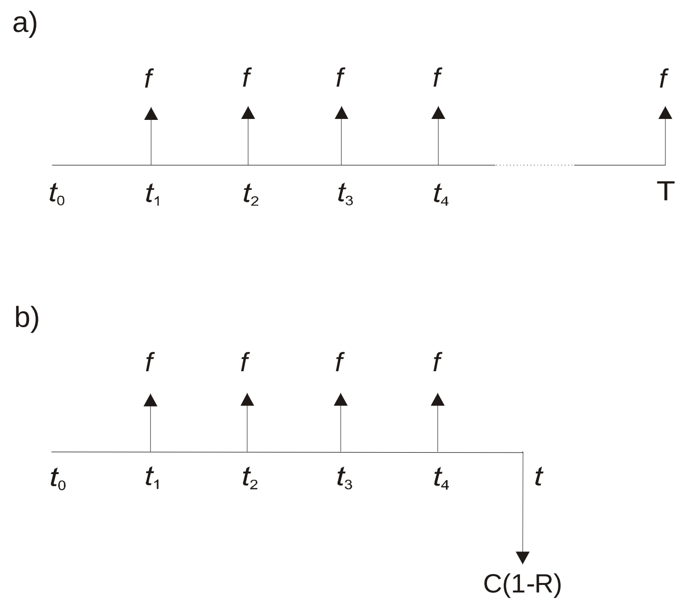

FIGURE 8. *Cash flow of a CDS. Case a): There is no default until maturity T of the CDS; case b): Default happens at time t<T. Recovery value is denoted by R, claim amount by C and the regular fee payments made by protection buyer at payment dates are denoted by f.*

The expected present value of the second leg, of payment made by the protection seller, is:

$$\sum_{1}^{n}[1 - \hat{R} - A(t_i)\hat{R}]p_i v(t_i) \ , \tag{15}$$

when the notional amount Nom equals 1. When it would differ from 1, the expected present value of the second leg could be written as:

$$\sum_{1}^{n}[Nom - \hat{R} - A(t_i)\hat{R}]p_i v(t_i) \ . \tag{16}$$

The term in brackets reflects the assumption on the claim amount. This term for each time t_i is discounted by $v(t_i)$ and weighted by the probability of default at each instant between $t = 0$ and T.

The value of the CDS for the buyer is the value of w such that the values of the two legs defined above are equal:

$$\sum_1^n [1 - \hat{R} - A(t_i)\hat{R}] p_i v(t_i) - w \sum_1^n p_i [u(t_i) + e(t_i)] - w\pi u(T) = 0 \ .$$

The CDS spread under the assumption that default can happen only at discrete times is denoted $\hat{s}^*$, and found as:

$$\hat{s}^* = \frac{\sum_1^n (1 - R[1 + A(t_i)]) p_i v(t_i)}{\sum_1^n p_i [u(t_i) + e(t_i)] + \pi u(T)} \ . \tag{17}$$

After the expression of the credit default swap premium (17) under the assumption about the default arrival at discrete times, the credit default swap spread can be calculated with similar analysis according to assumption AH_6. The value of π is given by:

$$\pi = 1 - \int_0^T q(t) dt \ . \tag{18}$$

The expected present value of the first leg, of payments made by the protection buyer, is:

$$w \int_0^T q(t)[u(t) + e(t)] dt + w\pi u(T) \ . \tag{19}$$

The expected present value of the second leg, of payment made by the protection seller, is:

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}] q(t) v(t) dt \ . \tag{20}$$

As already mentioned, the value of the CDS for the buyer is the value of w such that the values of the two legs defined above are equal:

$$\int_0^T [1 - \hat{R} - A(t)\hat{R}] q(t) v(t) dt - w \int_0^T q(t)[u(t) + e(t)] dt - w\pi u(T) = 0 \ .$$

The CDS spread, s^*, is therefore found as:

$$s^* = \frac{\int_0^T (1 - R[1 + A(t)]) q(t) v(t) dt}{\int_0^T q(t)[u(t) + e(t)] dt + \pi u(T)} \ . \tag{21}$$

Calculation of the CDS premium

It is reasonable to calculate the price of the credit default swap for both assumptions that were made about the default arrival.

First, assume that default can take place at the end of years 1, 2, 3, 4 and 5. Consider a five year credit default swap where payments are made semiannually. Suppose that the reference obligation is a five year bond that pays a coupon semiannually of 9% per year. Assume that the risk free rate is 5% per annum with semiannual compounding, the recovery rate is 30% and that the probabilities of default are as calculated in the third column in Figure 6.

In this case:

$$v(t_i) = 1 \cdot \exp\left(-2 \cdot log\left(1 + \frac{0.05}{2}\right) \cdot t_i\right) \ ,$$

$$u(t_i) = 0.5 \cdot \sum_{k=1}^{2 \cdot t_i} \exp\left(-2 \cdot log\left(1 + \frac{0.05}{2}\right) \cdot \frac{k}{2}\right) \ .$$

The values of $e(t_i)$ and $A(t_i)$ are clear from their definitions. The credit default swap spread, $\hat{s}^*$, is therefore 0.0181 or 181 basis points. It means that payments equal to 0.905% of the CDS notional principal would be required every six months. All variables required for the expression (17) are shown in the following Figure 9.

t_i	p_i	$A(t_i)$	$v(t_i)$	$e(t_i)$	$u(t_i)$
1	0.0210	0.05	0.95181	0	0.9637
2	0.0234	0.05	0.90595	0	1.8810
3	0.0258	0.05	0.86230	0	2.7541
4	0.0281	0.05	0.82075	0	3.5851
5	0.0303	0.05	0.78120	0	4.3760

FIGURE 9. *Variables for CDS spread calculation*

The second case is when default can happen at any time during the life of the five year credit default swap mentioned above. Assume that default intensities are as shown in the last column of Figure 6. Equation (21) can be solved, for example, with a numerical integration. It gives the value of annualized spread s^* for the credit default swap with semiannual payments as 0.018626 or 186.26 basis points. It means that payments equal to 0.9313% of the CDS notional are required every six months.

The probability of default under the assumption AH_9 and the credit default swap spread (17) can be calculated as demonstrated in the segment of a program code that can be found in enclosures and that is written in Matlab.

2.3. Model implications.

Expected recovery rates

To test whether the assumed recovery rate is consistent with observed bond prices, equation (13) can be used to derive an interval in which the bond prices must be found, when expected recovery rates and the yields on bonds maturing at earlier times are specified.

Because default intensities are greater then zero, from equation (13) this means:

$$0 < \frac{G_j - B_j - \sum_{i=1}^{j-1} q_i \beta_{ij}}{\beta_{jj}}$$

$$B_j < G_j - \sum_{i=1}^{j-1} q_i \beta_{ij} \ . \tag{22}$$

The lower bound for bond prices is derived from the assumption that the cumulative probability of default must be less or equal than 1:

$$\sum_{i=1}^{j} q_i (t_{i-1} - t_i) \leq 1$$

From equation (12) this means:

$$\frac{G_j - B_j - \sum_{i=1}^{j-1} q_i \beta_{ij}}{\beta_{ij}} (t_{i-1} - t_i) \leq 1 - \sum_{i=1}^{j-1} q_i (t_{i-1} - t_i)$$

$$B_j \geq G_j - \sum_{i=1}^{j-1} q_i \beta_{ij} - \frac{\beta_{jj}}{(t_{i-1} - t_i)} [1 - \sum_{i=1}^{j-1} q_i (t_{i-1} - t_i)] \ . \tag{23}$$

When for example the expected recovery rate is assumed to be 30%, and there were only the first four rows of the table in Figure 6, the yield of a five year bond with a coupon of 6% has to be according to (22) and (23) between 6.4866% and 7.9%.

The expected recovery rate is a parameter that is not directly observable in the market. In the presented model for valuing credit default swap, the same value of expected recovery rate is used for calculating the default probability and the payoff from this derivative instrument. Hull and White argue that there is an offset. If the estimates of default probabilities increase as a result of increasing expected recovery rates, the payoffs decrease. The impact of the assumed recovery rate will be negligible as long as the assumption lies within reasonable bounds. According to [**7**], it is a range between 0% and 50%. This is illustrated in the following Figure 10, where the spread of the five year CDS is represented according to the expected recovery rate. For calculations, the same parameters were used as for calculation of the CDS spread above.

In some cases, the CDS spread is an increasing function of expected recovery rate. It is, for example, when the yield curves are upward slopping or when the coupon on a T-year bond is less than the T-year par yield. This is illustrated in Figure 11, where the coupon of 6% is replaced with a coupon of 4%.

Counterparty risk

In this model it was assumed that there is no risk of the counterparty of the derivative contract. This is not really true in reality. As it can be found in [**7**], there is a way how could this assumption AH_1 be relaxed. For this purpose, some new variables have to be defined:

$\theta(t)\Delta t$ ... the probability of default by reference entity between times t and $t + \Delta t$ and no earlier default by counterparty

$\phi(t)\Delta t$ ... the probability of default by counterparty between times t and $t + \Delta t$ and no earlier default by reference entity

The credit default swap is then calculated as:

$$s^* = \frac{\int_0^T (1 - R[1 + A(t)])\theta(t)v(t)dt}{\int_0^T [\theta(t)u(t) + \theta(t)e(t) + \phi(t)u(t)]dt + \pi u(T)} \; .$$

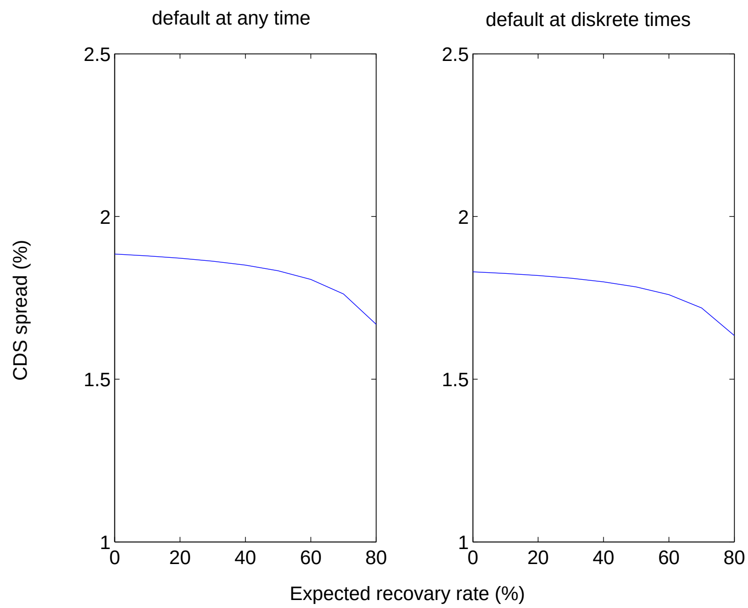

FIGURE 10. *CDS spread sensitivity on expected recovery rate.*

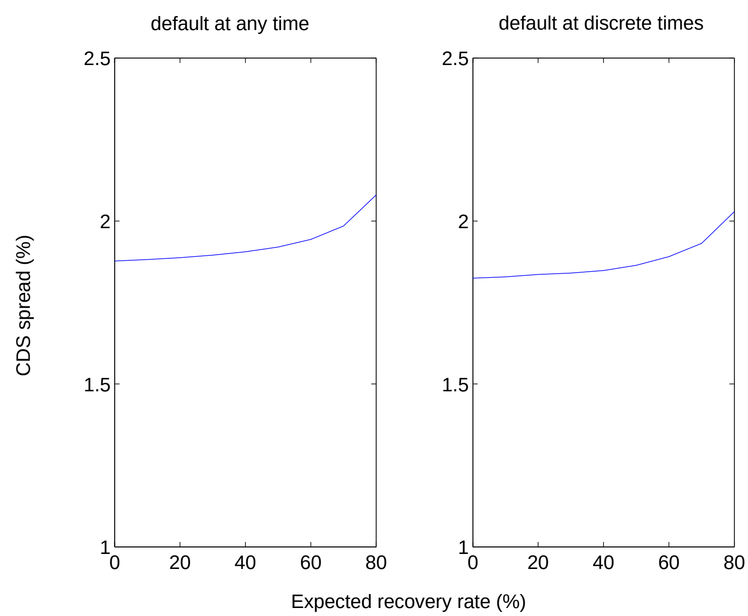

FIGURE 11. *CDS spread as an increasing function of expected recovery rate.*

3. The model of Schönbucher

The model developed by Schönbucher [**9**] is presented to demonstrate the basic ideas of valuing a single credit default swap with a reduced form model, the so cold bond based pricing model. As it can be found in [**9**], the problem of pricing a CDS is reduced to the pricing of a set of basic building block securities.

Notation:

$G(t,T)$	... price at time t of a default free zero-coupon bond with notional amount of 1\$ and maturing at time T
$B(t,T)$	... price at time t of a defaultable zero-coupon bond maturing at time T
$P(t,T)$	... survival probability between times t and T as seen at time t
$P^{\mathrm{def}}(t,T)$	... implied default probability over (t,T)
$P(t,t_1,t_2)$	... conditional survival probability over (t_1,t_2) as seen at time t
$P^{\mathrm{def}}(t,t_1,t_2)$	... the conditional probability of default over (t_1,t_2)
$H(t,t_1,t_2)$	... discrete implied hazard rate over (t_1,t_2) as seen at time t
$F(t,t_1,t_2)$	... default-free simply compounded forward rate over the period (t_1,t_2) as seen from time t
$\overline{F}(t,t_1,t_2)$	... defaultable simply compounded forward rate over the period (t_1,t_2) as seen from time t

The assumptions of the model:

AS_1: There is no arbitrage opportunity, the bond prices are arbitrage-free.

AS_2: The prices of all defaultable and default-free zero-coupon bonds of all maturities are known.

AS_3: The default free interest rates are independent of the default time τ.

AS_4: The recovery value R is independent of the time of default τ.

AS_5: There is given a probability space (Ω, F, Q) under the risk-neutral probability measure, the martingale measure Q.

AS_6: All notionals are normalized to 1.

NOTE 7: To ensure absence of arbitrage it is required that defaultable bonds are always worth less than default-free bonds of the same maturity:

$$0 \leq B(t,T) < G(t,T) \quad \forall t < T \qquad \diamond$$

Under the measure Q, the price of the default free bond is in [**9**] given by the expected value of the discounted expected payoff:

$$G(t,T) = \mathrm{E}\left[\exp^{-\int_t^T r(s)\,ds} \cdot 1\right] \ ,$$

where $r(s)$ is the instantaneous short rate.

NOTE 8: The instantaneous short rate at time t is the rate that applies to an infinitesimally short period of time at time t, as can be found also in [**7**]. $\diamond$

With analogy, the price of a defaultable zero coupon bond with a notional amount of 1 is:

$$B(t,T) = \mathrm{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds} \cdot I(T)\right] , \tag{24}$$

where $I(t)$ in (24) is a survival indicator function:

$$I(t) = \begin{cases} 1, & if \quad \tau > t \\ 0, & if \quad \tau \le t \end{cases}$$

and $I(T)$ is the payoff at maturity that depends on the occurrence of default, τ represents the time of default.

According to assumption AS_3, the defaultable bond price from (24) can be written as:

$$\begin{aligned} B(t,T) &= \mathrm{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds} \cdot I(T)\right] = \mathrm{E}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\right] \mathrm{E}[I(T)] \\ &= G(t,T)\mathrm{E}[I(T)] \ . \end{aligned}$$

For building the model, following definitions are adopted:

$$P(t,T) = \frac{G(t,T)}{B(t,T)} , \qquad P^{\mathrm{def}} = 1 - P(t,T) \ , \qquad \forall\ 0 \le t \le T$$

$$P(t,t_1,t_2) = \frac{P(t,t_1)}{P(t,t_2)} , \qquad P^{\mathrm{def}}(t,t_1,t_2) = (1 - P(t,t_1,t_2)) \ , \qquad \forall\ t \le t_1 < t_2$$

$$F(t,t_1,t_2) = \frac{G(t,t_1)/G(t,t_2) - 1}{t_2 - t_1} , \qquad \forall\ t \le t_1 < t_2$$

$$\overline{F}(t,t_1,t_2) = \frac{B(t,t_1)/B(t,t_2) - 1}{t_2 - t_1} , \qquad \forall\ t \le t_1 < t_2$$

$$H(t,t_1,t_2) = \frac{1}{\Delta t}\left(\frac{P(t,t_1)}{P(t,t_2)} - 1\right) = \frac{1}{\Delta t}\frac{P^{\mathrm{def}}(t,t_1,t_2)}{P(t,t_1,t_2)} , \qquad \forall\ t \le t_1 < t_2 \ .$$

The implied survival probability is initially at one, $P(t,t) = 1$. It is non-negative, decreasing in T and increasing in t.

The value at time t of a payoff 1 that is paid at time $t_1 + \Delta t$ if and only if a default happens in the time interval $(t_1, t_1 + \Delta t)$ is:

$$u(t,t_1,t_1+\Delta t) = \mathrm{E}[\beta(t,t_1+\mathit{\Delta} t)(I(t_1) - I(t_1+\mathit{\Delta} t)) \mid \mathit{default\ in}\ [t_1,t_1+\mathit{\Delta} t]] \ .$$

Here, β represents the discount factor, $I(t_1)$ the survival indicator function and Δt is a time unit. It can be seen from this equation that the variable

$$(I(t_1) - I(t_1 + \Delta t))$$

is 1 if default occurs in the time interval $(t_1, t_1 + \Delta t)$ and is 0 other way.

The value of $u(t, t_1, t_1 + \Delta t)$ can now be written as:

$$\begin{aligned} u(t, t_1, t_1 + \Delta t) &= \mathrm{E}\left[\mathrm{e}^{\int_{t_1}^{t_1+\Delta t} r(s)\,ds} \cdot I(t_1) - \mathrm{e}^{\int_{t_1}^{t_1+\Delta t} r(s)\,ds} \cdot I(t_1 + \Delta t)\right] \\ &= \mathrm{E}\left[\mathrm{e}^{\int_{t_1}^{t_1+\Delta t} r(s)\,ds}\right] \mathrm{E}\left[I(t_1)\right] - \mathrm{E}\left[\mathrm{e}^{\int_{t_1}^{t_1+\Delta t} r(s)\,ds}\right] \mathrm{E}\left[I(t_1 + \Delta t)\right] \\ &= G(t, t_1 + \Delta t)\mathrm{E}\left[I(t_1)\right] - G(t, t_1 + \Delta t)\mathrm{E}\left[I(t_1 + \Delta t)\right] \ , \end{aligned}$$

what can be further rewritten:

$$\begin{aligned} u(t, t_1, t_1 + \Delta t) &= G(t, t_1 + \Delta t)P(t, t_1) - B(t, t_1 + \Delta t) \\ &= B(t, t_1 + \Delta t)\left(\frac{P(t, t_1)}{P(t, t_1 + \Delta t)} - 1\right) \\ &= \Delta t \ B(t, t_1 + \Delta t)H(t, t_1, t_1 + \Delta t) \ . \end{aligned}$$

Building blocks

Suppose that there are K coupon payment days for bonds. These dates are denoted by $0 = T_0, T_1, ..., T_K$. The distance between two dates is denoted by $\delta_k = T_{k+1} - T_k, \ \forall \ 0 \leq k \leq K$.

The value of a credit default swap is find with the pricing building blocks that are in [**9**] defined as follows:

The price of a default-free zero coupon bond:

$$G(0, T_k) = \prod_{i=1}^{k} \frac{1}{1 + \delta_{i-1}F(0, T_{i-1}, T_i)} \ .$$

The price of a defaultable zero-coupon bond with zero recovery:

$$B(0, T_k) = G(0, T_k)P(0, T_k) = \prod_{i=1}^{k} \frac{1}{1 + \delta_{i-1}H(0, T_{i-1}, T_i)} \ .$$

The value of 1 at T_{k+1} if default occurred in $[T_k, T_{k+1}]$:

$$u(0, T_k, T_{k+1}) = \delta_k H(0, T_k, T_{k+1})B(0, T_k) \ .$$

Credit default swap

Assume that there are N payments made by the credit default swap buyer and the days of these payments are indexed with k_n, $n = 1, ..., N$. The credit default swap payment dates are denoted by T_{k_n}, $n = 1, 2..., N$. The distance between these payment dates is $\delta'_n = T_{k_{n+1}} - T_{k_n}$.

Assume that the credit default swap buyer makes a payment of $s^* \cdot \delta'_{n-1}$ at T_{k_n} if there is no default until T_{k_n}, where s^* is the credit default swap rate.

The value of payments made by the protection buyer is:

$$s^* \sum_{n=1}^{N} \delta'_{n-1} B(0, T_{k_n}) \ . \tag{25}$$

If default occurs in $[T_k, T_{k+1}]$, the credit default swap buyer receives a payment of $(1 - R)$ at T_k. The value of these payments is:

$$(1 - R) \sum_{k=1}^{K} u(0, T_{k-1}, T_k) = (1 - R) \sum_{k=1}^{K} \delta_{k-1} H(0, T_k, T_{k+1}) B(0, T_k) \ . \tag{26}$$

The market credit default swap spread s^* is chosen so that these payment streams has the same value. Therefore, combining (25) and (26), the market credit default swap rate can be obtained as:

$$s^* = (1 - R) \frac{\sum\limits_{k=1}^{K} \delta_{k-1} H(0, T_k, T_{k+1}) B(0, T_k)}{\sum\limits_{n=1}^{N} \delta'_{n-1} B(0, T_{k_n})} \ . \tag{27}$$

Consider now a five year credit default swap, where the reference obligation is a zero coupon bond with the same maturity date. Assume that $\delta_k = \delta'_k$. Consider further the forward rates as are those in Figure 12. The calculated cds premium s^* is then 0.94% annually.

t_j	$F(0, t_j, t_{j+1})$	$\overline{F}(0, t_j, t_{j+1})$
0	0.05	0.058
1	0.06	0.071
2	0.07	0.085
3	0.09	0.110
4	0.10	0.122

FIGURE 12. *Variables for CDS premium calculation*

3.1. Model implications.

The approach presented above can be according to [**9**] further improved, for example by adopting assumptions about the time of default and the probability of default.

Consider a counting process:

$$N(t) = \sum_i \mathbf{1}_{\{\tau_i \le t\}} \ ,$$

where

$$\mathbf{1}_{\{\tau_i \le t\}} = \begin{cases} 1, & if \quad \tau_i \le t \\ 0, & if \quad \tau_i > t \end{cases}$$

The time of default τ can be the time of the first jump of N:

$$\tau = inf\{t \in \mathrm{R}_+ \,|\, N(t) > 0\} \ .$$

The survival probabilities can then be given by:

$$P(0,T) = \mathbf{P}[N(t) = 0] \ .$$

The recovery rate is also an important parameter in the credit default swap pricing formula (27). It is not directly observable in the market. Similarly as in the Hull-White model, it is important to look at historical recovery estimates when choosing the recovery rate for valuation of the credit default swap.

4. Conclusion

This work was devoted to introducing the principles of credit derivatives and to pricing of the most used one – the credit default swap. Structures of the three basic types of credit derivatives – namely credit default swap, total return swap and credit linked note – were discussed in detail in the first chapter together with their usage in credit risk management.

The next chapter was then focused on setting the correct risk premium in the case of the credit default swap, probably the most important credit derivative structure. Three different approaches from the category of reduced-form models were discussed in this regard. All three models were implemented in a mathematic software.

Although based on similar variables and assumptions the introduced approaches use quite different techniques to achieve the premium of the credit default swap. None of the approaches has been widely accepted as a correct approach by the market yet (in a way similar to, for example, the Black-Scholes model in the case of options). We may assume that it will be the complexity and feasibility of the particular solution which will determine the winning approach in the future.

The model of Das and Sundaram is a quite intuitive numerical solution that is easy to apply and use. On the other way, however, it employs several simplifications that are not fully compliant with the practise. The model of Hull and White is mathematically more elaborate. Unfortunately it requires sufficient number of quoted bond prices for its calibration. In practise, especially in the environment of emerging markets, this may be a practical problem. The same note applies to the model of Schönbucher which assumes the knowledge of several distributions that are hard to observe in practise.

There are lot of issues that could not be discussed in this work due to a space and time limitations:

- The sensitivity of the introduced methods regarding the input parameters. This might be achieved e.g. by a numerical study.
- The pricing methodology should be extended to other types of credit derivatives. Probably this may be achieved by decomposing these derivatives into simpler structures and using the methodology derived for credit default swap.
- All discussed models may be further improved by relaxing some of the unreal assumptions (especially the assumption of a constant and non-stochastic recovery rate).

CHAPTER 3

Enclosures

A *The model of Das and Sundaram*

In this section, an example of an algorithm for calculating the default swap premium is presented. The program code can be modified to calculate also with other types of bond than zero coupon bonds.

```
function[premium]=crval(level, f, fsig, s, ssig, cumdef, d);

rho=0.25; h=0.5;
a=-4; b=10; c=70; n=d;
puu=(1+rho)/4; pdd=puu;
pud=(1-rho)/4; pdu=pud;

if level==(n-1)               % time of maturity
    premium=(1-cumdef)*s(1)
elseif level<(n-1)
    m=length(f)-1
    for i=2:length(f)
        fuu(i-1)=f(i); suu(i-1)=s(i);
        fsigma(i-1)=fsig(i); ssigma(i-1)=ssig(i);
    end ;
    fud=fuu;fdu=fuu;fdd=fuu; sud=suu;sdu=suu;sdd=suu;
    alpha=zeros(1,m); beta=zeros(1,m);
    for j=1:m
        if j==1             % calculation of the drifts of forward rates and spreads
            alpha(j)=log(0.5*(exp(-fsigma(j)*h*sqrt(h))
                                +exp(fsigma(j)*h*sqrt(h))))/h^2;
            beta(j)=log(puu*exp((-fsigma(j)-ssigma(j))*h*sqrt(h))
                       +pud*exp((-fsigma(j)+ssigma(j))*h*sqrt(h))
                       +pdu*exp((fsigma(j)-ssigma(j))*h*sqrt(h))
                       +pdd*exp((fsigma(j)+ssigma(j))*h*sqrt(h)))/h^2
                     -alpha(j);
           elseif j>1
               pom1=0; poma=0; pomb1=0; pomb2=0; pomb3=0;
               pomb4=0; poma2=0; pomb=0;
               for i=1:j
                   pom1=pom1+fsigma(i);
                   pomb1=pomb1+(-fsigma(j)-ssigma(j))*h*sqrt(h);
                   pomb2=pomb2+(-fsigma(j)+ssigma(j))*h*sqrt(h);
                   pomb3=pomb3+(fsigma(j)-ssigma(j))*h*sqrt(h);
                   pomb4=pomb4+(fsigma(j)+ssigma(j))*h*sqrt(h);
               end ;
               for i=1:(j-1)
```

```
                poma=poma+alpha(i);
                pomb=pomb+beta(i);
            end ;
            alpha(j)=log(0.5*(exp(-pom1*h*sqrt(h))
                              +exp(pom1*h*sqrt(h))))/h^2-poma;
            for i=1:j
                poma2=poma2+alpha(i);
            end ;
            beta(j)=log(puu*exp(pomb1)+pud*exp(pomb2)
                        +pdu*exp(pomb3)+pdd*exp(pomb4))/h^2
                    -poma2-pomb;
        end ;
    end;
fuu=fuu+alpha*h+fsigma*sqrt(h); fud=fud+alpha*h+fsigma*sqrt(h);
fdu=fdu+alpha*h-fsigma*sqrt(h); fdd=fdd+alpha*h-fsigma*sqrt(h);
suu=suu+beta*h+ssigma*sqrt(h); sud=sud+beta*h-ssigma*sqrt(h);
sdu=sdu+beta*h+ssigma*sqrt(h); sdd=sdd+beta*h-ssigma*sqrt(h);
cumd=cumdef+(1-cumdef)*exp(a+b*f(1)+c*s(1))/(1+exp(a+b*f(1)+c*s(1))); % cumulative
                                                                      % probability of
                                                                      % default
premium=(1-cumdef)*s(1)+exp(-(f(1))*h)*
        *(puu*crval(level+1, fuu, fsigma, suu, ssigma, cumd)
          +pud*crval(level+1,fud, fsigma, sud, ssigma, cumd)
          +pdu*crval(level+1, fdu, fsigma, sdu, ssigma, cumd)
          +pdd*crval(level+1, fdd, fsigma, sdd, ssigma, cumd))   % recursion
end;
```

To demonstrate the influence of parameters used for calculating the default probability function in the Das Sundaram model, a plot of this function with other parameters is shown in the following figure.

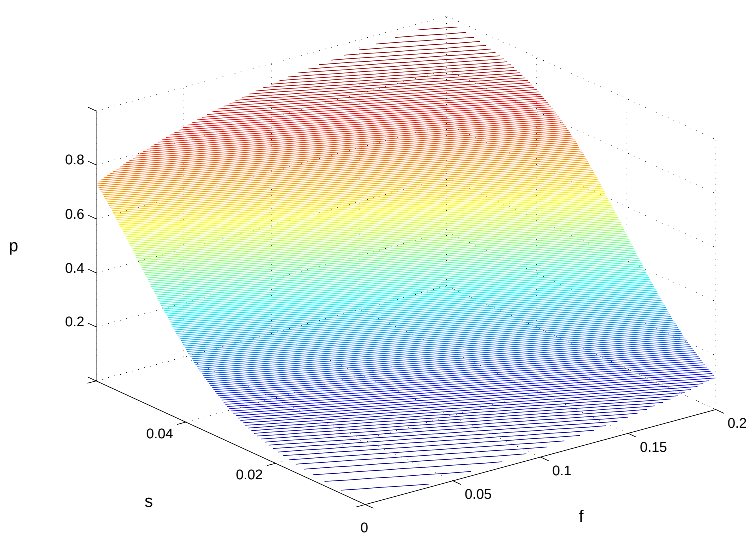

FIGURE 1. *Example of a default probability function (a=-5,b=15,c=100)*

If the forward rates are assumed flat, as shown in Figure 2 below, the credit default swap premium increases. Concretely to 2.42%.

T	f(0,T)	σ_f	s(0,t)	σ_s
0	0.05	0.017	0.015	0.005
1	0.05	0.014	0.016	0.006
2	0.05	0.012	0.017	0.007
3	0.05	0.011	0.018	0.008
4	0.05	0.010	0.019	0.009

FIGURE 2. *Input parameters, where the forward curve is flat.*

B *The model of Hull - White*

In this section, an example of an algorithm for calculating the default probabilities and credit default swap premium from the Hull White model is presented. The procedure is written for a five year credit default swap, but can be modified to calculate the spread of a CDS of any maturity. The input parameters are as those in Figure 7:

risk free zero rates, defaultable bond yields, coupons on bonds, recovery rate, CDS notional, coupon of the reference obligation and vector of maturities.

```
function[premium]=value(b,r,c,R,nom,cr,t)

n=length(b); B=zeros(size(n)); G=zeros(size(n));
alfa=zeros(n); p=zeros(size(n)); F=zeros(n); v=zeros(size(n));
vs=zeros(size(5)); us=zeros(size(5)); ps=zeros(size(5));

for m=1:n
    kupony=0
    kuponyr=0
    for k=0.5:0.5:(t(m)-0.5)
        kupony=kupony+c*exp(-(2*log(1+b(m)/2))*k)
        kuponyr=kuponyr+c*exp(-(2*log(1+r(m)/2))*k)
    end
    G(m)=kupony+(nom+c)*exp(-(2*log(1+b(m)/2))*t(m))
    B(m)=kuponyr+(nom+c)*exp(-(2*log(1+r(m)/2))*t(m))
    v(m)=exp(-(2*log(1+b(m)/2))*t(m))
    I(m)=kupony
    for i=1:m
        F(m,i)=(G(m)-I(i))*exp((2*log(1+b(m)/2))*t(i))
    end
end
for i=1:n
    for m=i:n
        alfa(i,m)=v(i)*(F(m,i)-R*(nom+c))
    end
```

```
end
p(1)=(G(1)-B(1))/alfa(1,1);
for m=2:n
    sucet=0
    for i=1:(m-1)
        sucet=sucet+(p(i)*alfa(i,m))
    end
    p(m)=(G(m)-B(m)-sucet)/alfa(m,m)           % default probabilities
end

citatel2=0; menov1=0;
for k=1:5                           % time to maturity of the CDS is 5 years
    vs(k)=1*exp(-(2*log(1+rs(k)/2))*k)
    ps(k)=p(k)
    pom=0
    for j=0.5:0.5:k
        pom=pom+0.5*exp(-(2*log(1+rs(k)/2))*j)
    end
    us(k)=pom
    citatel2=citatel2+ps(k)*vs(k)
    menov1=menov1+us(k)*ps(k)
end
citatel1=(nom-R-(cs/2)*R)
citatel=citatel1*citatel2
survive=1-sum(ps)
menov2=survive*us(5)
menovatel=menov1+menov2
premium=citatel/menovatel                    % cds premium
```

C *The model of Schönbucher*

The credit default swap premium given in (27) can be calculated using the following program code written in Matlab:

```
function[premium]=schon(f,fc,recovery)

bc=zeros(1,length(f));
br=zeros(1,length(f));
h=zeros(1,length(f));
bcn=zeros(1,length(f)+1);
brn=zeros(1,length(f)+1);
bcn(1)=1;brn(1)=1;
for i=2:length(f)+1           %prices of default free bonds
    pom1=1;
    for j=1:(i-1)
        pom1=pom1*(1/(1+f(j)));
    end
    bcn(i)=pom1;
end
for i=2:length(f)+1           %prices of defaultable bonds
    h(i-1)=bcn(i)/bcn(i-1)*(fc(i-1)-f(i-1))
    pom=1;
    for j=1:(i-1)
```

```
      pom=pom*(1/(1+h(j)));
    end
    brn(i)=bcn(i)*pom;
end
citatel=0;menovatel=0;
for i=1:length(f)
    citatel=citatel+h(i)*brn(i+1);
    menovatel=menovatel+brn(i+1);
end
premium=(1-recovery)*citatel/menovatel      %premium of cds
```

When the forward rates are assumed to be flat at 5%, as shown in Figure 3 below, the CDS premium increases to 1.12%.

t_j	$F(0, t_j, t_{j+1})$	$\overline{F}(0, t_j, t_{j+1})$
0	0.05	0.065
1	0.05	0.066
2	0.05	0.067
3	0.05	0.068
4	0.05	0.069

FIGURE 3. *Variables for CDS premium calculation*

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