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Equilibrium in the jungle

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Abstract

This bachelor thesis firstly introduces the jungle model, in which economic transactions are driven by coercion. The jungle is closely related to the model of exchange economy. The differences between both setups are discussed and consumption sets are introduced as additional constraint of agents in exchange economy. Following is the essential part of the thesis, discussing effects of these sets on welfare properties and competitive equilibria. The results suggest that consumption sets extend the set of Pareto efficient allocations in exchange economy. Analysis of competitive equilibria suggests interesting results about existence of non-efficient competitive equilibrium allocations.

Abstrakt

Tato bakalářská práce nejprve představuje model džungle, ve které jsou ekonomické transakce řízeny nátlakem. Džungle úzce souvisí s modelem směnné ekonomiky. Rozdíly mezi oběmi situacemi jsou diskutovány, spotřební množiny jsou přidány jako další omezení agentů v směnné ekonomice. Následuje stěžejní část práce, diskuze efektů spotřebních množin na blahobyt a kompetitivní rovnováhy. Výsledky ukazují, že spotřební množiny rozšíří množinu Pareto efektivních alokací u směnné ekonomiky. Analýza kompetitivních rovnováh ukazuje zajímavé výsledky ohledně existence neefektivních alokací, které ale jsou kompetitivními rovnováhami.

Keywords

Power, exchange economy, competitive equilibrium, jungle equilibrium, hierarchical organizations involuntary exchange, consumption set

Klíčová slova

Moc, ekonomika směny, kompetitivní rovnováha, rovnováha v džungli, hierarchické organizace, nedobrovolná směna, spotřební množina

Range of thesis: 67 374 characters

Declaration of Authorship

1. The author hereby declares that he compiled this thesis independently, using only the listed resources and literature.

2. The author hereby declares that all the sources and literature used have been properly cited.

3. The author hereby declares that the thesis has not been used to obtain a different or the same degree.

Prague 22.12.2015 Jiří Havlena

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Contents

Chapter 1: Introduction

In the typical exchange economy, agents participate in trading goods when mutually beneficial. But transactions carried by market are not the only way how to allocate resources in economy. It is quite common in economic activities that agents use coercion to seize resources held by weaker agents. In this thesis, we compare the model of jungle economy introduced by Piccione and Rubinstein (2004) with the typical exchange economy. In the jungle, transactions are governed by coercion and stronger agent can seize resources from weaker agents.

We start the analysis by introducing both models while stating the key differences. The effects of eliminating these differences are discussed as we try to make the two models as similar as possible, such that the only remaining difference is the system of allocation. To achieve this setting, we introduce consumption sets as additional constraints of agents in the exchange economy. We discuss how does the set of Pareto efficient allocations change with presence of these bounds. While the set of Pareto efficient allocation is straightforward in a typical exchange economy, with adding consumption sets the problem becomes more complicated. Somewhat counterintuitively, adding additional constraints to agents in the exchange economy extends the set of Pareto efficient allocations. Competitive equilibria in the exchange economy with consumption sets are also discussed, but this problem appears to be difficult and beyond the scope of this thesis. Therefore we focus on pointing out the issues that arise rather than the full analysis which is carried with welfare properties only.

In the contrary to other models of human activities that involve power as a method of obtaining resources (Skaperdas, 1992), power in the jungle model has rather strong notion. There are no transaction costs of taking goods away from weaker agent and, in addition, stronger agent can obtain all the goods from all weaker agents, who have no possibility to resist. Analogically, there are no transaction costs in the exchange economy.

Another possibility for analysis would be to change the notion of power in the jungle model. Obvious choice would be contest functions, where agents invest resources (weapons) to compete against other agents. Instead of production, output of conflict function can reasonably be thought to be win or lose of a conflict (Garfinkel, Skaperdas 2007). How inputs of weapons translate into probabilities of wins or loses is referred to in the literature as "the technology of conflict". A wide class of technologies that has been examined take the following form in the case of two players

$$p_i(G_1, G_2) = \frac{f(G_i)}{f(G_1) + f(G_2)},$$

where G_1, G_2 denotes the choice of weapons for both agents, $f(.)$ is a non-negative, increasing function, and $p_i(G_1, G_2)$ stands for the probability of winning of the party *i*. This case can easily be generalized for multiple agents. However, there is an issue with changing notion of power in the jungle in a way that fighting for resources is costly.

Then, to keep the exchange and the jungle model similar, we have to talk about the costs of enforcing property rights in the exchange economy, which are also non-trivial.

The structure of the thesis is as follows. Chapter 2 is a review of the exchange economy model. Chapter 3 presents the jungle model by Piccione and Rubinstein. These two chapters are in fact literature review. Chapter 4 compares the differences and discusses welfare effects of adding consumption sets to exchange economy, as well as discusses competitive equilibria. Chapter 5 introduces the model of division of tasks in a firm derived from the exchange and jungle economies. Chapter 6 provides summary and conclusions.

Chapter 2: Exchange economy

The goal of this chapter is to present the exchange economy model. This model is a benchmark for all the extensions and modifications in following chapters. Exchange economy is a simple model with no production opportunities. The agents start with initial stocks, endowments of goods. Goods were already obtained in an unspecified way and the only remaining goal is to redistribute them among the agents by mutual trade and consume. We start by defining a simple model of two consumers and two products, which can be extended into multiple dimensions. We discuss the welfare properties of goods allocation and then we turn to the role of market and existence of competitive equilibria. This chapter is based on (Mas Colel *et al,* 1995), (Varian, Repcheck, 2010) and (Serrano, Feldman, 2011).

Two consumers and two goods

Two consumers and two goods is the simplest setting, because if there were less consumers or less goods, then simply no trade could happen. In addition, this situation has a clear graphical illustration in a diagram Edgeworth box, which is suitable for demonstration of the fundamentals. To begin, consider two agents $1,2$ and two products x,y . As already mentioned earlier, we omit the production, and therefore consider the total amounts of goods x,y , noted by capitals, X,Y , as fixed. These amounts are endowed to the agents, so Agent 1 starts with initial endowment (x_1^0, y_1^0) and Agent 2 with (x_2^0, y_2^0) . The lower index refers to the agent and the upper refers to endowment being initial. These initial endowments sum to the total amount of good,

$$x_1^0 + x_2^0 = X \text{ and } y_1^0 + y_2^0 = Y \, . \tag{0.1}$$

Note that if after some trading sequence agents end up with different allocations (x_1, y_1) , (x_2, y_2) then the sum still needs to equal the total

$$x_1 + x_2 = X \text{ and } y_1 + y_2 = Y \, . \tag{0.2}$$

Agent i's preferences among products are represented by utility function depending only on own bundle, i.e.

$$u_i = u(x_i, y_i) \, .$$

These utility function satisfy the monotonicity, smoothness and convexity properties and will generally differ between agents, which makes mutual trade likely.

This problem of two agents and two products can be graphically represented in a diagram called Edgeworth box. The difference of this diagram from other diagrams used in microeconomics is that it has two origins, bottom left corner for agent one and upper right for agent two. It is a rectangle with length X and height Y and any point inside the diagram is some distribution of *X*, *Y* satisfying (0.2). The favorable property is that it shows four quantities in a two dimensional picture.

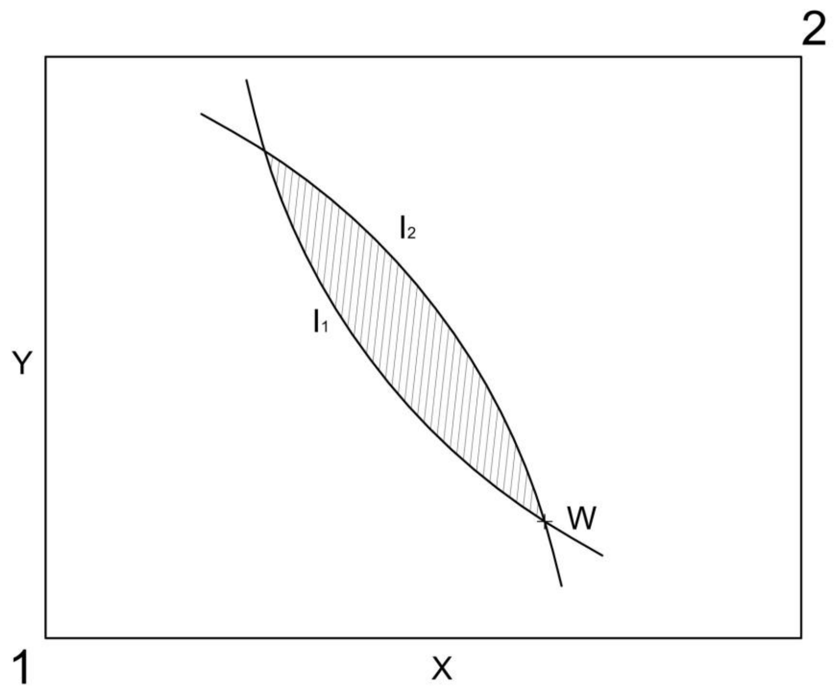

Figure 1

Figure 1 shows the simplest Edgeworth box. *W* is the initial endowment, I_1, I_2 are the indifference curves going through *W*. Remind that the shape of I_2 is due to the fact that origin of agent two is the upper right corner. The hatched area is where both agents can be better-off then in the initial endowment, and therefore, into which area the redistribution will lead.

Feasible allocations

Before we actually get to talk about welfare properties, discussion about feasibility of allocations needs to be done. In the previous section, it was mentioned that every point of the Edgeworth box (including the borders) is some allocation that satisfies the condition (1.2). In addition, these points are also the only feasible allocation in this part of analysis.

Consider an allocation $x_1 + x_2 > X$ or $y_1 + y_2 > Y$. This allocation, shown in Figure 2 as E_1, E_2 (E stand for excess) is *non-feasible* and therefore cannot be Pareto efficient. Obviously excess of only one good also leads to non-feasibility. The counterpart $x_1 + x_2 < X$ or $y_1 + y_2 < Y$ is also not suitable for the welfare analysis. Such allocation is not specifically depicted in Figure 2, but can be easily imagined as $L_1 = E_2$, $L_2 = E_1$ (L stands for lack). Because of monotonicity of preferences, these allocation do not require extra attention, because they will never be the best possible among a set of feasible allocation. For example, allocation where both agents receive half of both unused resources is preferred by both agents.

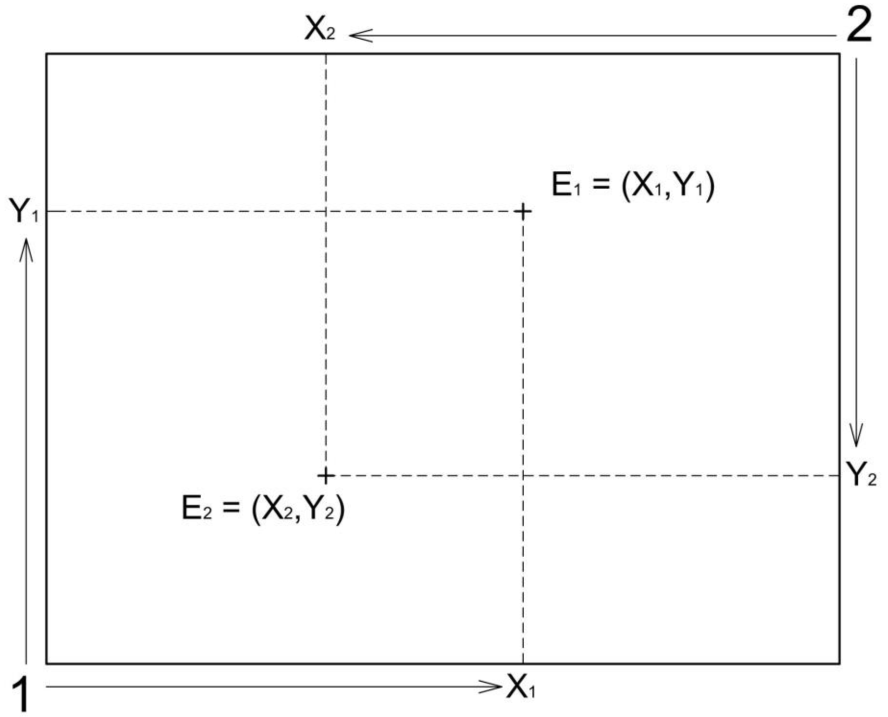

Figure 2

Therefore *feasible allocation* is an allocation that satisfies these properties:

(i) $x_1 + x_2 = X$, $y_1 + y_2 = Y$

(ii) $x_1, x_2, y_1, y_2 \geq 0$

For the following discussion about Pareto efficiency, we only consider feasible allocations.

Pareto efficiency

In the Figure 1, we described the area where both agents are better-off then in the initial endowment *W*. We say that all the points in the hatched area *Pareto dominate W*. Point *A Pareto dominates B* if all agents like *A* at least as much as *B* and one or more of them prefers *A* to *B*. If *A Pareto dominates* a point *B* then we consider a move from *A* to *B* a *Pareto move.* We call a feasible allocation *not Pareto efficient* if there exists a *Pareto move* from it to another feasible allocation. Finally, allocation is *Pareto efficient (Pareto* *optimal)* if there does not exist a *Pareto move* from it, in other words, there is no other allocation which all agents like as much as this and at least one strictly prefers.

Looking back at the Edgeworth box in Figure 1, we see that every point in the hatched area Pareto dominates the point *W*. Due to the shape and properties of the utility functions, if indifferent curves cross at a point inside Edgeworth box, this point cannot be Pareto efficient. There exists an area of points that Pareto dominate it. Therefore the only Pareto efficient points in the diagram are those where the indifferent curves of two agents are tangent, expect the situations that agents do not want to consume one of the goods. But these situation, which lead to the corner solutions, will be left apart for now. In other words, the slopes of the indifferent curves need to be equal, so marginal rates of substitution need to be equal for both agent and therefore

$$\frac{\frac{\partial u_1(x_1, y_1)}{\partial x_1}}{\frac{\partial u_1(x_1, y_1)}{\partial y_1}} = \frac{MU_1^x}{MU_1^y} = \frac{MU_2^x}{MU_2^y} = \frac{\frac{\partial u_2(x_2, y_2)}{\partial x_2}}{\frac{\partial u_2(x_2, y_2)}{\partial y_2}} \tag{0.3}$$

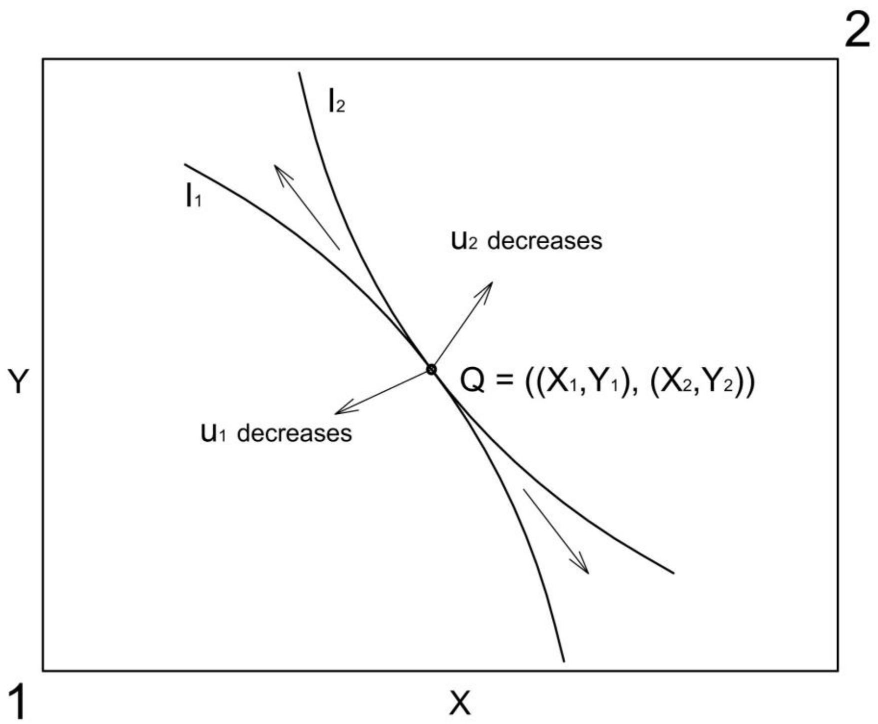

Figure 3

Figure 3 shows one of the Pareto efficient solutions as well as changes in utilities of moving elsewhere from it. If we move to the area between both indifference curves, both agents are worse of. The set of such points that satisfy (0.3) is called *contract curve*. The name is due to the fact, that this is a set of points that might be an outcome of a possible trading contract. Where exactly on the contract curve will agents end depends on their initial endowments and possibly on their bargaining power. But it is clear that all the points along the contract curve are Pareto efficient. Considering some initial endowment W , then a subset of the contract curve where neither is worse-off than in W is called the *core*. It is worth mentioning that Pareto efficiency tells us nothing about equalities of distributions, as both points where one of agents consumes all the goods (upper right and bottom left corner of the Edgeworth box) are clearly Pareto efficient.

Competitive equilibrium

In the initial endowment point, all the goods are consumed, but this allocation is likely not to be Pareto efficient. Then the trading can be introduced as follows. Suppose that an exogenous price vector p_x, p_y is presented to both the agents, who take this as fixed and given. Then each agent faces a budgetary constraint

$$p_x x_i + p_y y_i = p_x x_i^0 + p_y y_i^0 \tag{0.4}$$

and his goal is to maximize $u_i = u(x_1, y_1)$ subject to the constraint (0.4). Now all agents present their optimal consumption bundle and the relative price $\frac{p_x}{p_y}$ is adjusted. Then a new price vector is announced, the agents present their consumption bundles, until, for some price vector p_x, p_y , these bundles equal the given totals of all goods. [1]This is a *competitive equilibrium* (also called market or Walrasian equilibrium) as each consumer chooses his most preferred bundle given price vector and initial endowment and all choices are compatible in a way that supply equals demand for each good.

The first welfare theorem presents the relation between competitive equilibria and Pareto efficiency.

The first welfare theorem

Suppose there are markets and market prices for all goods, all people are competitive price takers and each person's utility only depends on his or her own consumption bundle. Then any competitive equilibrium allocation is Pareto optimal, and lies in the core.

At any competitive equilibrium allocation, there is no alternative allocation that benefits one of agents without hurting the other one. This is an important results, as it states that relying on competitive markets will achieve Pareto optimality. The shortcoming is that it states nothing about equalities, which is discussed in the second welfare theorem, but stating that is not necessary for this thesis.

[1] This Walrassian auctioneering proces is more complex, but the detailed explanation is not necessary for us.

Chapter 3: Jungle model

In this part of the thesis, the original model by Piccione and Rubinstein (2004), denoted as (P&R) model in the following text, will be introduced. The transactions in this model are governed by coercion and bilateral agreement is no longer a necessity for transaction to be realized. Note that by transactions in this model we not only take redistributive acts of goods between agents, but also takings from the aggregate bundle. In order to preserve the clarity of the text the same sequencing as in the original paper will be followed.

The jungle

Consider a set of agents $I = \{1,....,N\}$ and a set of commodities $1,....,K$. An aggregate bundle $w = \{w_{1,}.....,w_k\}$ is available to be distributed among the agents.[2] The agents are characterized by a preference relation $\{\prec=\}$ over the set of consumption bundles R_+^k and by a convex consumption set $X^i \subseteq R$. The consumption preferences of each agent are strictly monotone and continuous. The set X^i is defined as agent i's ability to consume. We assume that X^i is compact and convex and satisfies free disposal assumption, ie. $x^i \in X^i, y \in R_+^k$ and $y \le x^i$ implies that $y \in X^i$.

The power notion is fairly simple and known to all, defined by the relative power of the agents. The strength relation is denoted by *S*. We assume that *S* is a linear (ordering, irreflexive, asymmetric, complete and transitive) and without loss of generality, that $1S2, 2S3,...,(N-1)SN$, while the notion iSj simply means that agent i is stronger than agent j. The stronger agent can use her power to seize all the goods from any weaker agent. Finally, the full notion of jungle is $<\{\ge^i\}_{i\in I}, \{X^i\}_{i\in I}, w, S>$.

As a benchmark for this model, recall the notion of the exchange economy to see the difference ie. $<\{\ge^i\}_{i\in I}, \{X^i\}_{i\in I}, w, \{w_i\}_{i\in I}>$. Here w_i stands for the initial endowment of agent *i* and the initial endowments sum up to the total amount of goods available, ie.

$$\sum_{i=1}^{N} w_i = w$$

.

[2]This bundle is not held by anyone

Remarks

Let's now take a look at the few issues that could come to readers mind while reading the above. First three issues and the text after about limited consumption are mentioned in original paper as well, but they are necessary to clearly understand the nature of the problem

(i) The model does not deal with the source of power, neither does it provide incentives to increase one's power. The power relation is an exogenous ranking. Similarly in exchange economy, the distribution of initial endowments is exogenous.

(ii) There are no transaction costs due to the transfer of resources from one agent to another due to the complete information availability about the power sequence of the agents. Weaker agent being forced to part with his goods to a stronger attacker has no incentive to resist, as she knows the outcome of possible conflict. The uncertainty about the strength order would disrupt this assumption. [3]

(iii) Coalition formation is not allowed in the original P&R model.

(iv) The strict monotonicity of the individual preferences is a key to achieve the desirable properties of equilibria mentioned in the following section.

An important difference from the exchange economy is the presence of bound imposed on individual consumption. Naturally, there are physical bounds on individual's ability to consume, but there is certainly a point to be made about unbound desire to acquire wealth. But the motives of this desire are to influence the collective allocation or gain social status, while in this model we focus on obtaining commodities to accomplish basic needs.

There is another possibility to interpret these bounds as individual's ability to protect his bundle. While this interpretation is satisfactory for some of analysis, it might undermine the efficiency of jungle equilibrium. The presence of the bounds is necessary for the analysis as the absence would cause all the goods to be trivially appropriated by the strongest agent.

Jungle equilibrium

Let us start by defining a few key terms necessary for future analysis. The *feasible allocation* is a vector of non-negative bundles

$z = \{z_0.....z_n\}$ such that $z_0 \in R_+^k$ and $z_i \in X_i, i = 1,....,n$ while $\sum_{i=0}^{N} z_i = w$.

[3] As mentioned in the concluding remarks by Ariel Rubinstein, societies (for example tribal) create rituals to determine the relative power to avoid the costs of possible conflict.

The bundle z_0 stands to the goods not allocated to any agent. It is a result of allocation process if not all goods are distributed. The feasible allocation simply means that all the resources are divided into individual's bundles, none of these bundles excess the individual's consumption bounds and the sum of all bundles and the non-allocated goods equals aggregate bundle. *Efficient allocation* means that no agent can be strictly better-off without making at least one agent worse-off.

A *jungle equilibrium* is a feasible allocation such that no agent can be better-off by combining his bound with the free-disposal resources or with bounds held by weaker agent. Formally, it is a feasible allocation that there are no such agents i and j , iSj and a bundle $y_i \in X_i$, such that $y^i \le z^i + z^o$ or $y^i \le z^i + z^j$ while $y^i \succ z^i$.

Proposition 1

The jungle equilibrium exists.

Proof:[4]

The construction goes as follows. Let z_1 be the best bundle of Agent 1 in the set $x^1 \in X^1, x^1 \le w$. Define by induction z_i as agent i's best bundle in the set

$z_i = \{x_i \le X^i, x_i \le w - \sum_{j=1}^{i-1} z_j\}$ and $z_0 = w - \sum_{j=1}^{n} z_j$. $\hat{z} = (z_0, z_1 ... z_n)$ is a jungle equilibrium.

The construction process goes as follows. Starting by the strongest agent, agents take their optimal bundle from the not-yet allocated part of the aggregate bundle. That means Agent 1 has access to whole w, Agent 2 selects his bundle from $w - z_1$ and so on. After the weakest Agent n takes his bundle, the remaining goods are placed into z_o .

The process of acquiring goods in the jungle equilibrium proof is referred to in the literature as *Serial dictatorship* (Abdulkadiroğlu, Sönmez, 1998). It means that there exists some exogenous ordering of agents and then the agents take their top choices in this ordering and these choices are removed from the bundle. Note that the generating process does not need to generate a unique jungle equilibrium, neither does the proposition claim the unambiguity. Consider a simple example of a jungle

$N = 2$, $K = 2$, consumption sets $x_i^1 + x_i^2 \le 3$, $w = \{3,3\}$, $u_i = x_i^1 + x_i^2$. (0.5)

Recall the definition of equilibrium, stating that no agent can be better off combining his bound with the bound of any weaker agent or with the free disposal, which is irrelevant here. There is infinite number of equilibria, in fact, any allocation with $z_o = 0$ is a jungle equilibrium in this set. Neither has the proposition claimed anything about efficiency. In the example case, the jungle equilibrium is clearly efficient, however, just a slight modification most likely leads to inefficiency.

Consider a slightly modified example with Agent 1 being the stronger agent:

$N = 2$, $K = 2$, $x_i^1 + x_i^2 \le 3$ $w = \{3,3\}$ $u_1 = x_1^1 + x_1^2$ $u_2 = x_2^1 + (x_2^2)^\wedge 2$ (0.6)

[4] Taken from P&R

Here, as we can see, the existence of equilibrium remains exactly the same as in the above example. However, only the distribution $X^1 = (3,0)$ and $X^2 = (0,3)$ is efficient as in any other equilibrium Agent 1 can ''trade'' commodity 1 for commodity 2 of Agent 2 in 1-for-1 ratio and increase his utility, while his own remains exactly the same.

The next section shows the sufficient conditions to achieve our desired properties, unambiguity and efficiency.

Jungle smoothness and unambiguity of equilibrium

In this section we present two propositions from the P&R model that guarantee uniqueness and efficiency of jungle equilibrium, given that jungle smoothness assumptions are satisfied. The proofs of these propositions can be found in the P&R paper. We say the jungle as smooth if is satisfies the following two conditions for each agent *i*:

(i) the preferences are represented by a strictly quasiconcave and continuously differentiable utility function $u_i : R_+^K -> R$ and with $\nabla u_i >> 0$

(ii) there exists a quasiconvex and differentiable function g^i , such that $X^i = \{x^i \in R_+^K \,|\, g^i(x^i) \le 0\}$ (at the points on the boundary the gradients are defined as limits)

Recall that both the utility functions in (0.5) and the utility function of Agent 1 in (0.6) do not satisfy (i).

Proposition 2

If the jungle is smooth, $\hat{z}$ (equilibrium allocation from proposition 1) is the unique jungle equilibrium.

Proposition 3

If the jungle is smooth, $\hat{z}$ is efficient.

In this section, we introduced the jungle model by P&R, where transaction of goods is driven by coercion. The model was formally defined and existence of jungle equilibrium was showed. Smoothness property adds additional properties of equilibria given that smoothness conditions are satisfied.

Chapter 4: Consumption sets in Exchange economy

Differences of jungle vs exchange

This chapter can be characterized as comparison of exchange economy with respect to the jungle model. We discuss the differences between the two models presented in previous chapters, and, while modifying several assumptions, introduce consumption sets as another constraint of agents in exchange economy. We focus mainly on analyzing welfare properties and finding competitive equilibria.

In Section 4 of their work P&R analyze the differences between the jungle and the exchange model, recalling the difference being symmetric property rights in the exchange model being replaced by the asymmetric property rights in the jungle model. Therefore bilateral participation constraint characterizes transactions in the exchange economy, whereas order of the stronger agent characterizes transactions in the jungle economy. This difference is clearly the most obvious contrast, but for their ongoing analysis of efficiencies and welfare theorems, they do not discuss any additional differences. Both models differ in at least two additional features, initial endowments and bounds on individual consumptions.

Initial endowments in jungle economy

The initial endowments of the goods in the exchange model are a necessity for existence of the model, as if all goods were part of an aggregate bundle not owned by anyone, no trade would happen. The P&R model in contrast presents an aggregate bundle not owned by anyone, but this can be replaced by the initial endowments relatively easily.

With the strict convexity of preferences, the set of jungle equilibria is invariant to the distribution of initial endowments including all goods being help by Agent 0, aggregate bundle. Without strict convexity of preferences, these initial endowment may become relevant. This idea of introducing initial endowment bundles to the jungle model was already covered by Houba *et al.* (2014)

They find that under the initial assumptions of the P&R model, initial endowments are irrelevant for welfare maximization. The construction of the equilibrium goes in a very similar way as in Chapter 3 of this thesis; the only difference is that stronger agents obtain their bundles through a sequence of bilateral takings from weaker agents, while each of these takings improves taker's utility. The initial bundles only determine from whom the stronger agents will take.

The endowments only become relevant if the assumptions of strong monotonicity and strict convexity of preferences are relaxed to monotonicity and convexity. Recall the definition of the jungle equilibrium from P&R to see the difference: a *jungle equilibrium* is a feasible allocation such that no agent can be better-off by combining his bound with the free-disposal resources or with bound held by a weaker agent. To see the argument, suppose that one of stronger agents has Leontief preferences $U_i = \min\{x_i, y_i\}$ over the goods, and two weaker agents hold each one of a pair of shoes. Then the stronger agent cannot increase his utility directly by taking from any of them, and therefore, the jungle is in equilibrium, if we suppose that weaker agents cannot take from each other. He could, however, increase his utility by taking from both of them simultaneously.

As relaxing the initial assumptions of P&R model is not a purpose of this thesis, we will not go deeper into this problem.

Consumption bounds in exchange economy

The second major difference between the models and the main focus of this chapter will be the consumption sets, present in the jungle model and initially absent in the exchange model, interpreted as the bounds of individual's ability to consume. The question we try to deal with is how the existence of consumption sets C_1, C_2 affects the set of Pareto efficient allocations in exchange economy. The result seems to be somewhat counterintuitive, as adding additional constraint seems to extend the set of Pareto efficient allocations.

It is worth mentioning what happens if the bounds on individual consumption are removed from the P&R model, despite the result being trivial. Because of the absolute ordering of power, all goods will be acquired by the strongest agent, so in the Edgeworth box for two players, the result is a corner solution.

The consequences of adding consumption bounds into the exchange economy is ambiguous. In that case agents face two restrictions in their optimization problem, budget constraint given by the initial endowment and the price vector and the consumption set unrelated to it. We will discuss possible combinations of these bounds in the Edgeworth box and focus mainly on the welfare properties and competitive equilibria. For the initial analysis we will focus on the bounds of consumption sets and the bounds of initial endowment. Price vectors will be discussed later.

For the ongoing analysis we will stick to the problem of two players and two goods because of the advantages of graphical representations in the Edgeworth box. The consumption sets will be noted

$C_i, i = 1,...,n$ [5]

where the index stands for the agent whose consumption set we refer to. We will still stick to the assumptions about consumption sets from the jungle model, being compact, convex and satisfying free disposal. Because of these assumptions, there are only three possible interactions of two such sets in the Edgeworth box.

Figure 4 shows all three possibilities. We take consumption set of agent 1 as fixed and show three possibilities by modifying consumption set of agent 2. We see, that

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[5] We need to change the notion used in chapter 3, because the X stands for the total amount of good x in economy

intersection is either an empty set, a single point or a set of points bounded by both consumption sets.

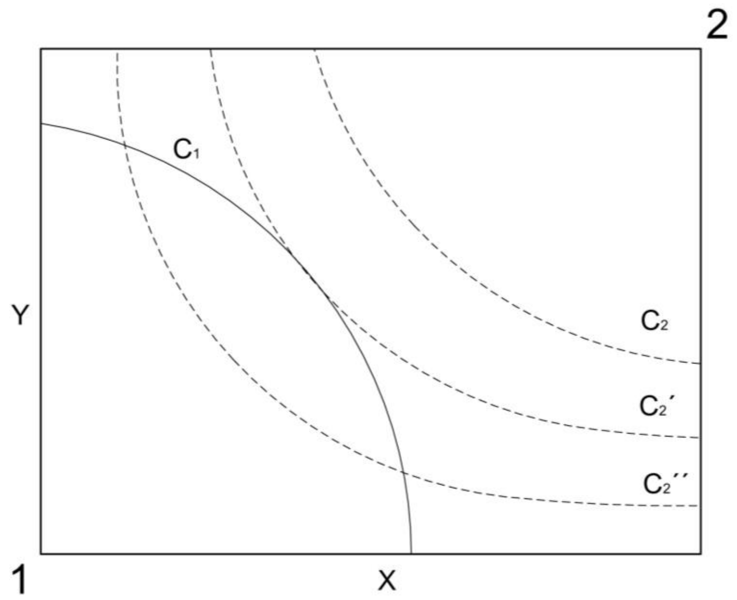

Figure 4

In order to cover the cases where there is no intersection or a single-point intersection of the consumption sets in the Edgeworth box, we need to clearly state the difference between the feasible allocation and consumption. Definition of *Feasible allocation* remains the same as it is always represented by a single point in the Edgeworth box. Define a set of feasible allocation $F_i(x_i, y_i)$ satisfying

(1) $x_1 + x_2 = X$, $y_1 + y_2 = Y$

(2) $x_1, x_2, y_1, y_2 \geq 0$ (0.7)

(3) $x_i \in [0, X]$, $y_i \in [0, Y]$

Now if a feasible allocation is also a part of individual's consumption set, no modification is needed. But if it exceeds the consumption sets, the agents need to modify their consumption to move the allocation points into their consumption sets. They have to decrease the consumption of some goods. To do this, each individual agents maximizes his utility such that his bounds are within his consumption set and the "initial endowment" given by the allocation.Formally, consider a feasible allocation (x_i, y_i) and derive a consumption x_i^C, y_i^C ;

$\{x_i^C, y_i^C : 0 \leq x_i^C \leq x_i; 0 \leq y_i^C \leq y_i\}$:

If $(x_i, y_i) \in C_i$ => $(x_i^C, y_i^C) = (x_i, y_i)$

If $(x_i, y_i) \notin C_i$ => $(x_i^C, y_i^C) = \arg\max u_i$ such that $(x_i^C, y_i^C) \in \{C_i \cap F_i(x_i, y_i)\}$ (0.8)

This type of bounded optimization problems with inequality constraints can be rigorously treated using so called Karush-Kuhn-Tucker (KKT) theorem (Krogstad,

,2012). However, as we present the results for two agents only, we will use an intuitive solution that can be justified on the visualization provided by the Edgeworth box.

Best possible consumption with unlimited resources

In the previous section we defined the consumption sets for both agents. Now let us assume that one of the agents has unlimited access to all resources in economy, as if he were a stronger agent in the jungle model, so his initial endowment is (X, Y), all goods in the economy. He faces the bound of his consumption set C_i. We did not state explicitly that this full-access allocation is not a part of consumption set, however, because of the free-disposal assumption, if it were a part of it for both agents, these consumption sets would not affect the result.[6] Both agents could simply consume all the goods in economy in this case if they were the stronger agent.

Now define BP_i (best possible, or bliss point) as a consumption of agent i such that his initial endowment is (X, Y), all goods in the economy. In other words, if one of the agents is a jungle leader, he chooses to consume $BP_i = (x_i^{BP}, y_i^{BP})$. By strict monotonicity of preferences, BP_i has to be located on the borderline of C_i.

The main reason why we defined BPs is to discuss their interaction in two player case. The intuition is that based on location of BP's we see which good is in supply shortage and which is sufficient.

Figure 5 shows four possible interactions of BPs of two agents. We take some consumption set C_1, specify BP_1 and take these as fixed. Now we divide the Edgeworth box into four areas by two straight lines parallel with the axie and going through BP_1. If BP_2 is any point in *a*, the upper right region, supply of both goods is sufficient, because $x_1^{BP} + x_2^{BP} \leq X$ and $y_1^{BP} + y_2^{BP} \leq Y$. We call this situation *no conflict* of bliss points.

[6] Recall the free disposal assumption of consumption sets

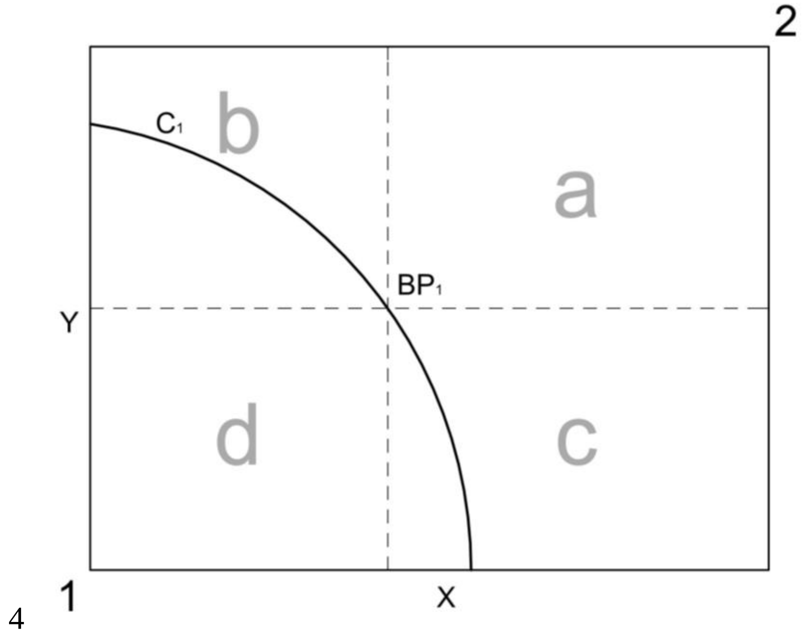

Figure 5

If BP_2 lies in the opposite quarter *d*, the situation is exactly opposite, $x_1^{BP} + x_2^{BP} \geq X$ $y_1^{BP} + y_2^{BP} \geq Y$ and we call it *full conflict*. The two last regions are similar, as we have a conflict over one of the resources, *x* in region *b* and *y* in region *c*. We call this situation *partial conflict*.

Not all interactions of bliss points are possible with various interactions of consumption sets. What can be said immediately is that non-empty intersect of consumption sets is a necessity for possibility of full conflict.[7] Table 1 combines figure 10 and figure 11 and shows possible settings of bliss points given consumption sets.

Consumption sets	**Bliss point possibilities**
No intersect	No conflict, partial conflict
Tangent	No conflict, partial conflict
Set-of points intersect	No conflict, partial conflict, full conflict

Table 1

[7] In a very rare case that both agents would have the same BP, being exactly the point of tangency of consumption set, it is also a no-conflict case. In fact, any case where both agents share their BP is a non-conflict case. More discussion about this in the part about full conflict.

Pareto efficiency

For the Pareto efficiency analysis, we divide the cases by the possibilities of bliss point conflict. When this division is no longer sufficient, we add the mutual setting of the consumption sets.

Pareto efficiency with no conflict of BP's

The division into multiple cases by consumption sets is not necessary for this easiest case of no conflict. We know, that because $x_1^{BP} + x_2^{BP} \leq X$ and $y_1^{BP} + y_2^{BP} \leq Y$, both agents can simultaneously obtain their most preferred bundle and there are still some undivided goods left. Figure 11 shows that BP_1 is bottom-left from BP_2. Let us draw a rectangle, sides parallel with axie and with two these BPs as corners. This rectangle gives a region of allocations, such that $x_1 \geq x_1^{BP}, x_2 \geq x_2^{BP}, y_1 \geq y_1^{BP}, y_2 \geq y_2^{BP}$. The consumption derived from any point in this rectangle is $BP_i = (x_i^{BP}, y_i^{BP})$ for both agents. All these allocations are clearly Pareto efficient as none of agents has any possibility to increase his utility whatsoever.

Figure 6 illustrates the case with no conflict of BP's and empty intersect of consumption sets. The set of Pareto efficient allocations is labeled *a*. No other points in the Edgeworth box are Pareto efficient.

The consumptions derived from any point in *a* are the *BP* consumptions for both agents. The hatched area is where one of the agents can benefit from gift-giving while other agent does not lose utility by decreasing his consumption. In the left part of the hatched area, Agent 2 parts with goods and Agent 1 receives goods and in the right section vice versa.

In corners, labeled *b* and *c*, one agent has excess possession of one good and lack of the other and the other vice versa, therefore there is an opportunity for trade. Take some initial endowment *W* in the interior of *b*. Then the consumption derived from *W* are the projections to the consumption sets. For every initial endowment in interior of *b*, derived consumption for Agent 1 lies at the part of C_1 between *M* and BP_1 . For Agent 2, it lies on the part of C_2 between *N* and BP_2 . In region c, situation is very similar and can easily be imagines. Both agents will always accept a trade that takes them to the interior of *a*.

Note that if we take different configuration of consumption sets than no intersection and take BP configuration as no conflict, the result will still look very similar to this one. The set of Pareto efficient allocations will always be outside of both agents consumption set and will be constructed identically. Therefore other cases do not need special attention.

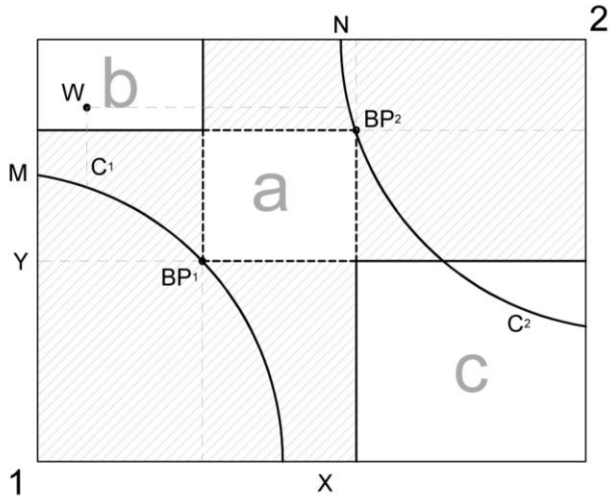

Figure 6

Pareto efficiency with partial conflict of BP's

For the analysis of this setup, we can merge the tangent and no-intersection cases. The result is very similar in both. The case of set-of-points interest is difficult and will be discussed afterwards. Consider a partial conflict case, such that $x_1^{BP} + x_2^{BP} \leq X$, $y_1^{BP} + y_2^{BP} > Y$. The intuition is that there should exist a set of points, such that both agents are saturated in consumption of x and at least one of them is not saturated in consumption of y. Because of the saturation in x for both agents, there is no opportunity for trade. Figure 7 illustrates this case with tangent consumption sets.

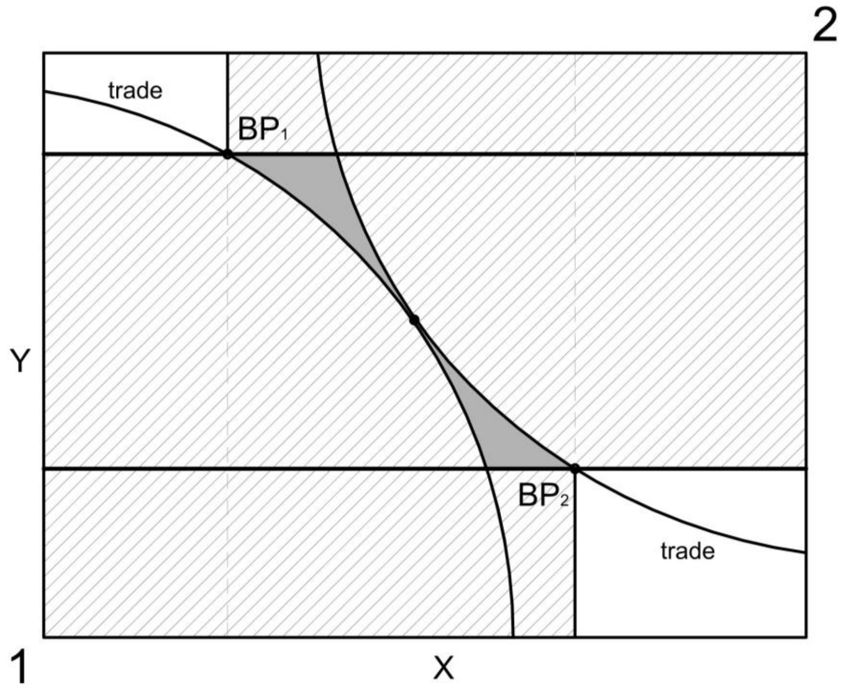

Figure 7

We start by identifying and removing the non-efficient feasible allocations and get to the efficient ones. For corner areas we use same argument as in previous case. In top-left corner Agent 1 is saturated in *x* and lacks *y*, while Agent 2 is saturated in *y* and lacks *x*. The bottom-right corner is exactly opposite. Therefore these are non-efficient because there is an opportunity for trade that makes both agents better off. Now suppose one of agents has more resources than he needs to cover his BP consumption. Formally, there exists agent *i*, such that $x_i > x_i^{BP}, y_i > y_i^{BP}$. These points are the upper-right section from BP_1 and bottom-left section from BP_2. Neither of these is Pareto efficient, as gift-giving increases utility of non-saturated agent while other agent's utility does not change.

What we now have left is a horizontal stripe bounded by two bold lines. In all these allocations, expect for the bold lines, both agents lack some of good y to reach their bliss point consumption. But the points in the interior of either of consumption sets are also not efficient. Suppose an allocation in the interior of C_1 . Agent 2 is already saturated in consumption of x , while Agent 1 is not. Therefore gift-giving of x with no change of y is a Pareto move. In the interior of C_2 situation is exactly opposite. Neither of these points is Pareto efficient.

Finally, we get to the efficient allocations. That is the area bounded by consumption sets and bold lines. Both agents are saturated in x, but they both lack y. No trade is possible and all these allocations are Pareto efficient. The consumptions derived from these allocations are just the horizontal projections on borderlines of consumption sets.

We only dealt with the part of partial conflict case when there is a lack of good y in economy. The opposite case with a lack of x does not need any extra attention, as it is basically the same problem, only difference being the bold lines being vertical and the result being turned by 90 degrees.

Pareto efficiency with full conflict of BP's

This setup is the most difficult, mainly due to variety of possibilities. Cases we haven't covered yet are the full conflict of BP's and the partial conflict of BP's with a set-of-points intersect. The latter does not bring much new and will be briefly discussed at the end of this part. We start by stating some assumptions that hold for both these cases and then separate them. For simplicity, define D as a set of points, such that they are in the interior of intersection of consumption sets, D^+ as the intersection including borderlines and D^- only the borderlines. Therefore D is open and bounded[8]. In additional, recall the definition of contract curve from section about exchange economy: The contract curve (CC) is a set of points that satisfy $MRS_1 = MRS_2$. Therefore this is a set of points that will be an outcome of possible trading contract in initial exchange economy without consumption sets and therefore these are the Pareto efficient allocations in exchange economy without consumption sets. Where on the contract curve we end depends on initial endowments and bargaining power. The analysis of efficiency is carried same as in previous sections, first we find non-efficient allocations and then get to the efficient. The following proposition states a relation between D and contract curve:

[8] It is an intersection of two non-empty open sets therefore it is open. It is bounded because X, Y are real nonnegative numbers.

Proposition 4

Let $A \in D$ and $A \notin CC$. Then A is not Pareto efficient.

The proposition is quite obvious. D is a region where consumption sets are not bounding for either of agents. One can imagine that without the existence of consumption sets in exchange economy, D would be the whole interior of Edgeworth box. Because D is open[9], some small vicinity of A is also a part of D. But because A does not lie on the contract curve, then utility functions for both agents cross in A and there exist a set of points where utility is greater for both agents. Some of these points will definitely lie in the vicinity of A that is part of D. Therefore A is not efficient.

In next step, we show that allocations in the interior of any of the consumption sets are not efficient either.

Proposition 5

Suppose $B(x, y)$, $B \in C_1, B \notin C_2$ and B does not lie on the borderline of C_1 . Then B is not efficient.

Proof: Compare the location of $B(x_b, y_b)$ and $BP_2(x_{BP}, y_{BP})$. If $x_b \leq x_{BP}$ or $y_b \leq y_{BP}$ and at least one of inequalities is strict then B is not efficient. Reason is that B lies in the interior of C_1. Hence by the strict monotonicity and non-saturation of preferences of agent 1, there is an opportunity for gift-giving, which is a Pareto move. Therefore B is not efficient. Only remaining case, $x_b > x_{BP}$ and $y_b > y_{BP}$ is not possible due to the definition of B. If it holds, B is definitely in the interior of C_2 because of free-disposal assumption about consumption sets which is contradiction to the definition of B.

We show the non-efficiency of allocations in the interior of C_2 similarly by modifying Proposition 2. Now, exactly as in the previous case, suppose all allocations such that one agent has more resources than he needs to cover his BP consumption. Using exactly same reasoning as before, neither of these is Pareto efficient.

[9] In fact, it consists of two open sets, because CC divides D into two parts.

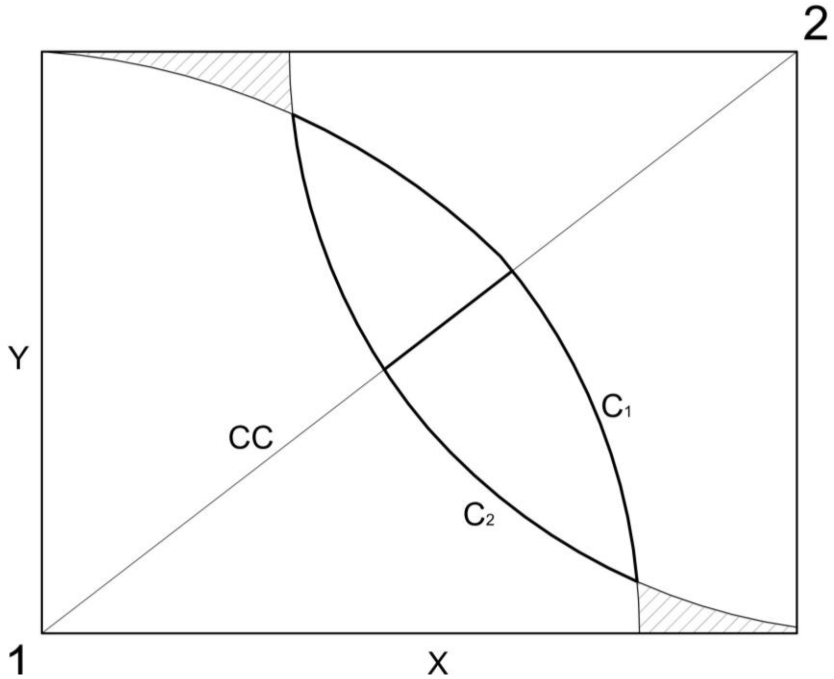

Figure 8

Figure 8 shows the sets of points with partial conflict setting that were identified as inefficient so far, these are the white regions. We see that the only remaining allocations left for possible efficiency are the borderlines of D^+, the part of contract curve in D and the corner allocations not included in anyone's consumption set.

Recall the definition of full conflict of BPs, $x_1^{BP} + x_2^{BP} > X$, $y_1^{BP} + y_2^{BP} > Y$. It can be also said that BP_1 is upper-right from BP_2 in the Edgeworth box. From Figure 5, we see that both BPs need to be located in D^+, and by the monotonicity of preferences both are in D^- . If either of BP's is located outside of D^+, there is no possibility for location of the second BP such that they form a full conflict setting. The implication does not hold the other way, as both BP's can be in D^+ but the setting is only partial conflict.

The big advantage of full conflict is that we can immediately pronounce the allocations not included in anyone's consumption sets as inefficient by using the proposition bellow:

Proposition 6

Consider the borderline of C_i and BP_i located on this borderline. Then every move among the borderline getting closer to the *BP* increases utility of agent *i*.

Discussion about Proposition 6 can be found in Appendix. For any allocation in the corner areas, we just derive the consumptions from that allocation (which reduces consumption of one of the goods for each agent). Then, using Proposition 6, we see that the crossing point of two consumption sets is a feasible allocation reachable by trade which Pareto dominates every allocation in the corner areas. In addition, the consumptions derived from this allocation does not require parting with goods for neither agent.

Only allocations left for possible efficiency are D^- and the part of the contract curve in D. Among these, this part of the contract curve is clearly efficient from the definition of the contract curve as utility functions are tangent here and both BP's are clearly Pareto efficient as there is no opportunity to increase the utility for one of the agents whatsoever. But so far we know little about the remaining parts of consumption set borderlines. We know that neither of these points would be efficient if C_1, C_2 wouldn't exist, as in exchange economy model from Chapter 2. We would simply move away from these allocations to some efficient allocation located on the CC. But with the existence of C_1, C_2, some of these allocations may become efficient. The reason is that the Pareto move would take us outside of D^+. If, however, there exist a Pareto move that takes us to the interior of D^+ or among the borderlines, this allocation is clearly inefficient. Therefore the goal is to find a set of points that Pareto dominate every point of D^- and find intersection of this set with D^+. If this intersection is empty, allocation is efficient, if not, it is inefficient.

The process goes as follows. We take every point in $C_i \cap D^-$ and draw indifference curves in such point. If this allocation does not lie on the CC, then there exists a region such that both agents can be better-off. We compare this region with C_i. If the intersection is empty, it is efficient, if not, there exists a Pareto move into D^+. Using the process, we can simply show why BP's are efficient. From perspective of agent *i*, all allocations preferred to BP_i are outside of D^+.

Figure 9 shows the indifferent curves of both agents going through two allocation on the borderline of C_1, BP_1, F. Allocation F is the intersection of contract curve with C_1. The arrows are the gradients of utility functions in F and BP_1. It is clearly visible that if the gradients of utility functions are not exactly opposite, there exists a set of points that are better for both agents. Only points where the gradients are exactly opposite in Edgeworth lie on the contract curve. In BP_1, the set of points that Pareto dominate, shown as the hatched area, lies outside of C_1. Therefore BP_1 is Pareto efficient.

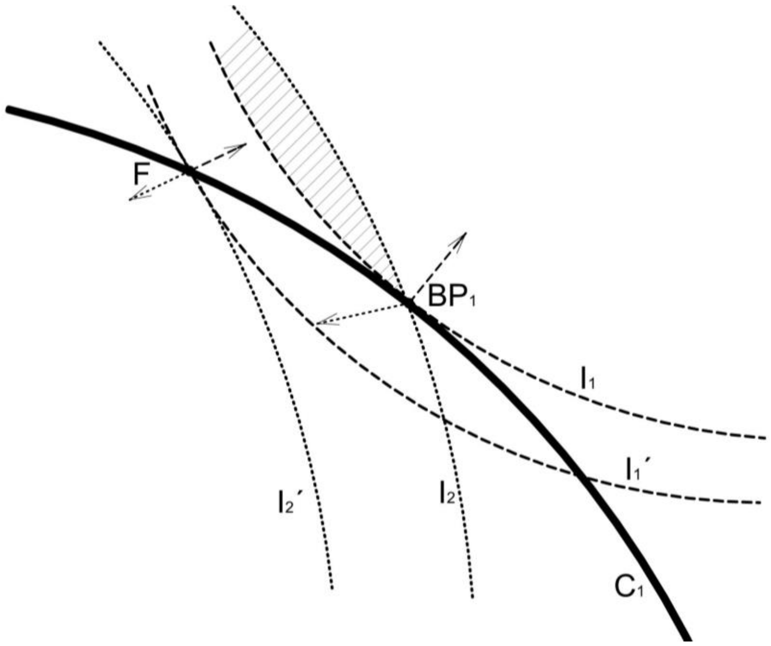

Figure 9

Now we want to check for Pareto dominating allocations for every other allocation on boarderine of C_1 . We start by an allocation $F^{'}$ that lies anywhere down from BP_1 , shown in Figure 10. The set of Pareto dominating allocations is nonempty and the intersection with D^+ is shown as the hatched area. Therefore, none of these allocations is Pareto efficient. In addition, moving among the borderline of C_1 closer to BP_1 increases utility for both agents.

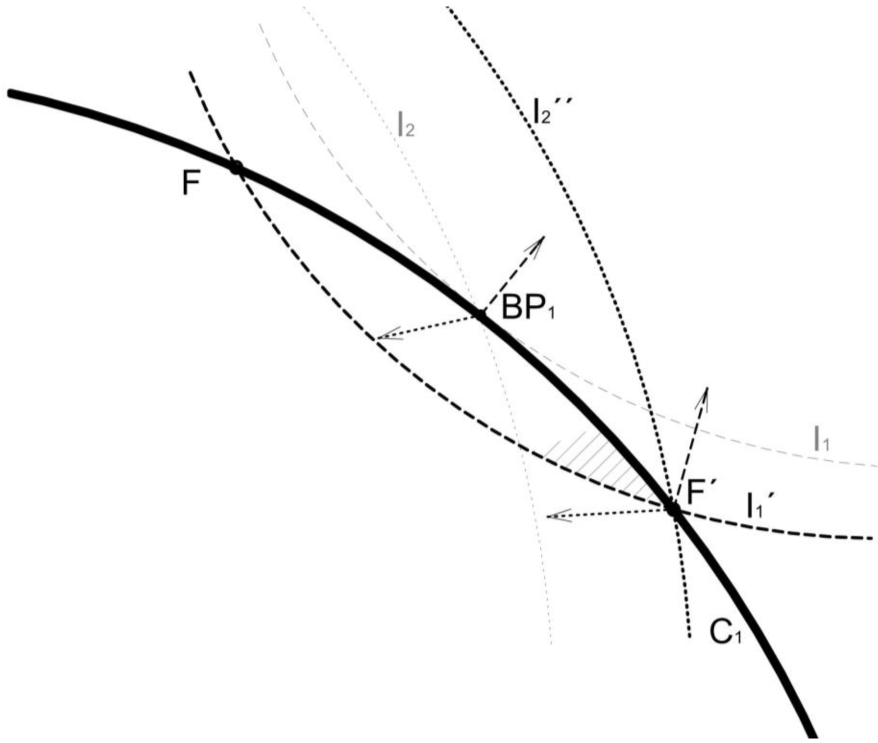

Figure 10

Now let us take any allocation between F and BP_1 labeled $F^{''}$. We see that the region of preferred allocations for both agents lies outside of D^+, therefore any such allocations is efficient. Figure 11 shows such setting

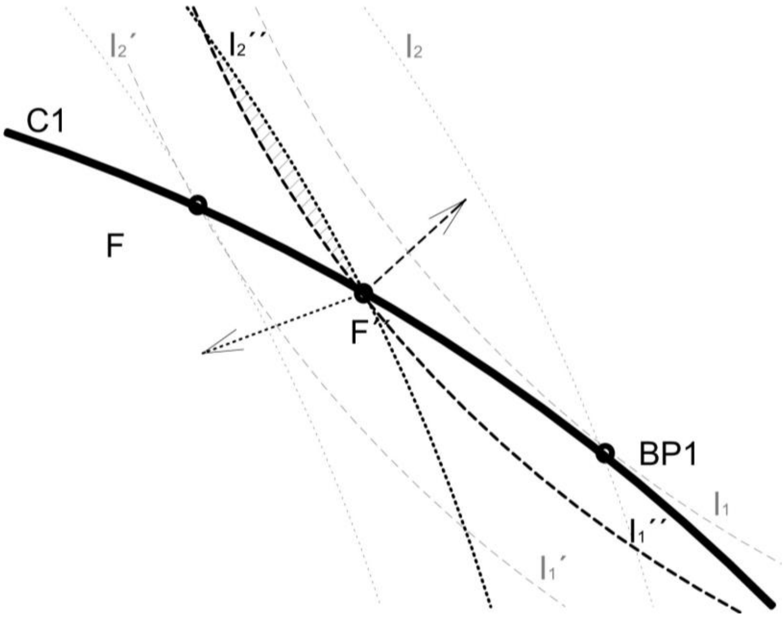

Figure 11

Finally, take any allocation $F^{'''}$ located further from BP_1 than *F*. None of these is efficient either. Such setting is shown in Figure 12. We see, that despite the fact that moving among the borderline of C_1 in any direction is not a Pareto move, a Pareto move to the interior of C_1 can be found. This case is important to demonstrate that for efficiency analysis with this setting, we have to focus not only on the moves among the C_1 borderline, but also on the moves into the interior.

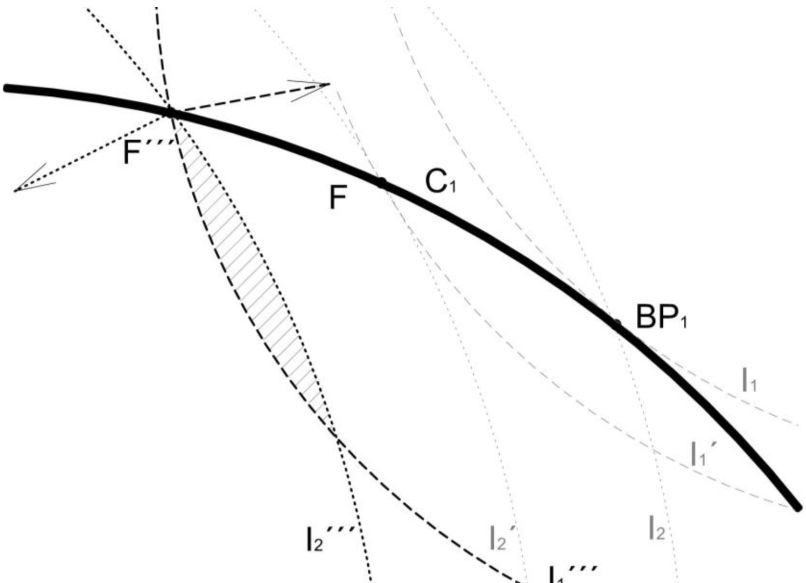

Figure 12

Every other allocation on the borderline is identic case to one of the three cases described above. Every other allocation in the Edgeworth box was dealt with before. For the allocations on the borderline of C_2 we use the same method. So we finally get to the conclusion about the set of efficient allocations in a full conflict setting of BP's. It is the part of contract curve in the interior of D^+ and the allocations on the borderlines of consumption set between BP and intersection of CC with the borderline. Figure 13 shows the set of Pareto efficient allocations as bold lines.

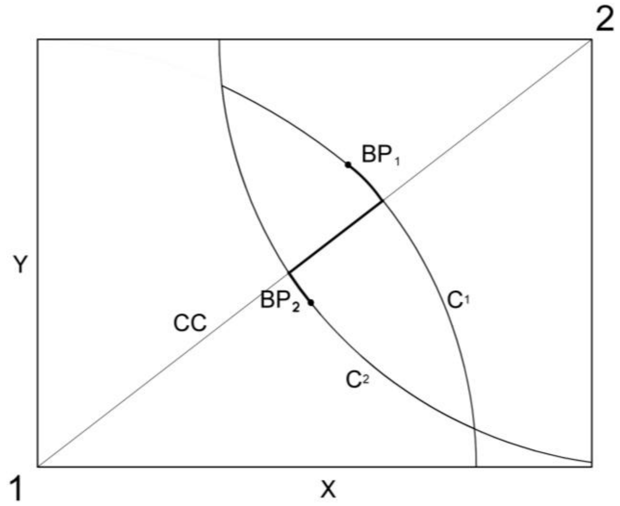

Figure 13

The cases with set-of-points intersect were not covered, however, they do not require much extra attention as no new problems seem to appear in these. These cases are simply the combinations of what was discussed in the previous text. In addition, there are way too many possibilities how two BP's can interact such that they form a partial conflict setting with set-of-points intersect of consumption sets, so discussing each of them separately with a figure for each will take great amount of space with. Let us rather discuss different interesting feature of exchange economy, competitive equilibria.

Competitive equilibria with bounded consumption

Recall the definition, *Competitive equilibrium* for an Edgeworth box is a price vector $p^* = \frac{p_x}{p_y}$ and an feasible allocation x_i^* , such that for $i = 1,2$ $\{ x_i^* \geq x_j^{'}$ for all $x_j^{'} \in B_j(p^*)$ $\}$ where $B_j(p^*)$ is a part of the Edgeworth box bounded by price vector p^* going through some initial endowment x_i^0, y_i^0 which represents allocations available for each agent (Mas Colel *et al*, 1995). In a standard exchange economy model from Chapter 2, for every feasible initial endowment allocation there exists a single price vector p^* [10]that leads us to the competitive equilibrium on the contract curve. Recall that the difference between standard exchange economy and model discussed in this chapter is presence of bounds on individual consumption. Note that for our case of bounded consumption, the definition from competitive equilibrium needs to be slightly modified. From the definition at the very beginning of this part, we say that x_i^* is competitive equilibrium allocation if consumptions derived from allocation x_i^* using (0.8) are the most preferred among the all consumptions derived from $x_j^{'}$ for both agents.

We already showed that these bounds significantly change welfare properties. We will now try to indicate how these bounds modify competitive equilibria. At first, in standard exchange economy from Chapter 2, in any feasible allocation in the interior of Edgeworth box, both agents are not saturated in consumption of either of goods. However, with the consumption sets present, there are some feasible allocations, such that either one of the agents has already saturated his consumption in both goods or both agents are saturated in consumption of one of the goods and both lack the second. In all these areas, trade is not an option.

The price vector $p^* = \frac{p_x}{p_y}$ given some initial endowment is basically a straight line in Edgeworth box going through this endowment. Note that from the perspective of Agent 1, it has to go from the top-left to bottom-right in the Edgeworth box. It basically means that for each Agent, reducing the amount of one good leads to increasing amount of the other good. If the line would be parallel with axis *x*, it would be simply gift giving of good *x* with no change of good *y*. Similarly for a line parallel with *y*. Any line going from the bottom-left to the top-right is simultaneous gift giving of both goods.

Issue that might arise is that the BP of an agent lies in the interior $B_j(p^*)$. In that case, there exists a part of the $B_j(p^*)$ where agent *i* is indifferent. The consumption derived from any such point is BP_i. Therefore there is also a part of the boarder of $B_j(p^*)$ where agent *i* is indifferent. This is something that can never happen in standard exchange economy with non-bounded consumption. Problem with these settings is that a competitive equilibrium allocation does not have to be efficient allocation. Such case will be shown as one of the examples.

Mainly because of the possible satiation of agents, discussing competitive equilibria in the exchange economy with consumptions sets is way more difficult than in exchange economy from Chapter 2. We will not try to describe all cases as there is a huge variety of them, but rather show two settings to illustrate the issues mentioned above

[10] In fact, there is infinite amount of such price vectors p_x, p_y. But if we set one of prices to 1, there is only a single such vector.

Competitive equilibria with no conflict of BP's

We take the setting from Figure 6 and add an initial endowment *w* in the region *c*, the region where Agent 1 has sufficient amount of *x* and lacks *y* and Agent 2 vice versa. Figure 14 illustrates this setting. Take a price vector p_1 that goes through the region *a*, a set of efficient allocations. The whole part of line p_1 in the interior of *a*, allocations $p \cap a$. These are efficient, as consumptions derived from any of these are BP consumptions. Therefore every allocation that lies in $p \cap a$ is a competitive equilibrium for this setting. Then suppose different price vector p_2, such that it goes through BP_1. Then the optimal allocation of Agent 1 given p_2, is only the allocation BP_1, while Agent 2 is indifferent in the whole part of $B_2(p_2)$ that lies in the bottom-left quadrant from BP_2. Allocation BP_1 is one of those he is indifferent between. Therefore BP_1 is the competitive equilibrium for this setting.

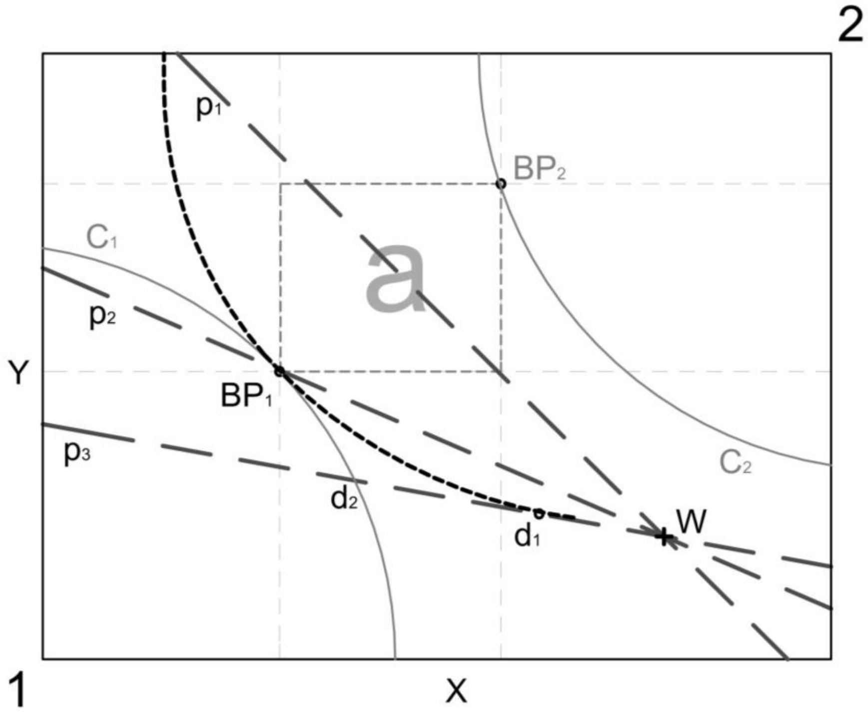

Figure 14

Finally, suppose a price vector p_3 such that Agent 1 can no longer achieve his BP allocation. In contrast, the region where Agent 2 is indifferent expanded. Then agent 1 maximizes his utility given the $B_1(p_3)$ constraint. His optimal allocation is labeled d_1, but note that this is not part of his consumption set. He has to move to a corner solution d_2 .Because d_2 lies in a region where Agent 2 is indifferent, d_2 is a competitive equilibrium for p_3 setting. Note, however, that if it would be out of the indifference region of Agent 2, there would be no competitive equilibrium for this price vector. This case can easily happen if we stretched out the consumption set of Agent 2 and afterwards moved his BP to the left. Also, note that d_2 is definitely not an efficient allocation, as a transfer to BP_1 is a Pareto move. Every other price vector going through *w* is similar case to one the three cases described above.

Competitive equilibria with full conflict of BP's

We take the setting from Figure 8 and an initial endowment *w* in the same region as in previous example. Figure 15 illustrates this setting. At first, take the price vectors $p_{1.1}, p_{1.2}$ going through BP's. These two cases, and the price vectors steeper than $p_{1.1}$ and flatter than $p_{1.2}$ are similar to previous case. One of the agents can cover his BP consumption and there exists a set where he is indifferent. However, as these indifferent sets are significantly smaller than in no-conflict case, no competitive equilibria are likely for these price vectors.

One competitive equilibrium that can exist is the one that emerges from Walrasian auctioneering process described in Chapter 2. The computational process of getting Walrasian equilibrium allocation given initial endowments and utility functions is described in Chapter 5. If this allocation lies in D^+, it is available for both agents. Then consumption sets are not bounding constraint for neither of agents and this allocation is still a competitive equilibrium even with the existence of consumption sets. Price vector p_{ce} is a competitive equilibrium price vector for this setting.

For other price vectors in this setting, intuition is that none of these will form a competitive equilibrium. We know that if the consumption sets weren't present, none of these would form a competitive equilibrium. As the consumption sets are another boundary for agents, intuitively their presence should not cause some of these to form an equilibrium.

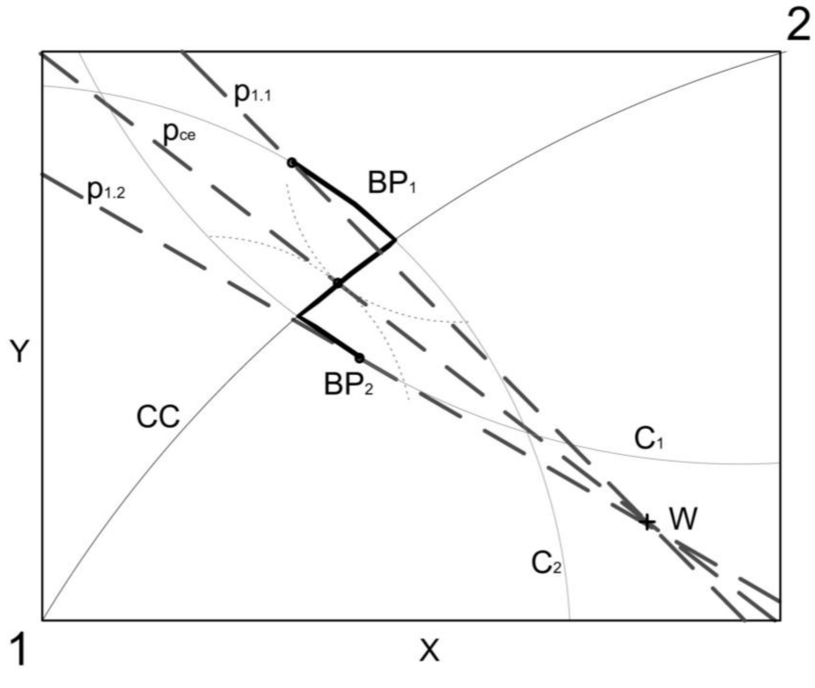

Figure 15

Conclusion of chapter 4

In this chapter, the differences between exchange economy and jungle model were discussed and afterwards bounds on individual consumption to the exchange economy were added as another constraint of agents. Somewhat counterintuitively, the set of Pareto efficient allocations extends with adding this additional constraint. Although author is well aware that the problem is a bit artificial, the analysis of the welfare properties is definitely interesting. The set of Pareto efficient allocations is mainly impacted by mutual arrangement of best possible allocations if one of agents had access to all resources in economy and mutual arrangement of consumption sets. The set of Pareto efficient allocations expands as the conflict over goods decreases. Analyzing competitive equilibria in the exchange model with consumption sets is a difficult problem, because of variety of possible settings and satiation of agents. Satiation causes agents to be indifferent between multiple allocations, which is something that never happens in competitive equilibrium analysis in standard exchange economy.

Chapter 5: The firm model

In this section, we introduce a model for division of tasks in firm, derived from the exchange economy. The problem is in some sense dual to the exchange economy. In contrast to consuming goods which generates utility, performing task for a firm generates disutility. Every agent starts with an endowment of tasks and can "trade" this work for another work with different agent. Then we introduce a "jungle firm" model, such that the employees choose their tasks in a serial dictatorship process described in Chapter 3. They face some bound on the minimal amount of tasks they have to do. Motivation is to show that the problem of division of tasks in firm is equivalent to the problem of division goods in exchange or jungle economy.

The model, two agents

Define:

$s_1 = x_2 = X - x_1$

$t_1 = y_2 = X - y_1$

$s_2 = x_1 = X - x_2$

$t_2 = y_1 = Y - y_2$

$s_{1,} s_2, t_1, t_2$ are quantities of two task for both agents and *X,Y* denote the total amount of work in firm.

From exchange economy utility function u, we derive new utility function v

$u_1(x_1, y_2) = u_1(X - x_2, Y - y_2) = v_1(X - s_1, Y - t_1)$

Function v is decreasing in both s_1, t_1, therefore the preferred allocation for agent *i* is $s_i = t_i = 0$. In this allocation, *i* has no tasks and everything is done by second agent. Maximization problem for Agent 1 in this setting is shown in Figure 16, where the hatched area is the set of admissible allocations.

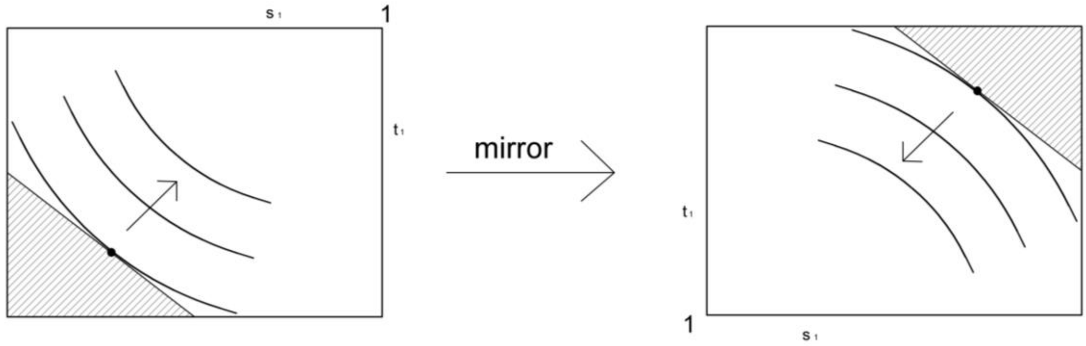

Figure 16

Both agents start with some initial endowment $(s_1^0, t_1^0), (s_2^0, t_2^0) \geq 0$, and equalities of amounts before and after trading process need to hold.

$s_1^0 + s_2^0 = X$

$t_1^0 + t_2^0 = Y$

$s_1 + s_2 = X$

$t_1 + t_2 = Y$

In jungle firm model, agents face a constraint of minimal amount of work they have to do for the firm such that they won't get fired (equivalent of the consumption sets in jungle model).

Two agents, two resources, four settings

Simple example will now be presented, comparing all 4 possibilities, involuntary or voluntary exchange of tasks or commodities. We stick to the same utility functions in all cases and only modify the allocation system and subject of transactions.

Example, voluntary and commodities

Consider two commodities x, y and two agents $1, 2$ with corresponding utility functions

$u_1 = x_1 y_1$,

$u_2 = x_2 y_2$.

The initial endowments $x_1^0, x_2^0, y_1^0, y_2^0 \geq 0$, $x_1^0 + x_2^0 = X$, $y_1^0 + y_2^0 = Y$, are given as the parameters of the model.

Let P_x, P_y be prices of x, y and define price ratio $p = P_x / P_y$. In optimal consumption sets x^*, y^* there must hold $MRS_1 = MRS_2 = p$ while $MRS_i = \frac{\delta u_i}{\delta x_i} / \frac{\delta u_i}{\delta y_i}$.

Start by defining budget lines and dividing by P_y for simplicity [11]

$x_1^0 P_x + y_1^0 P_y = P_x x_1^* + P_y y_1^* \quad / P_y$,

$x_1^0 p + y_1^0 = p x_1^* + y_1^*$,

and similarly for second agent

$x_2^0 P_x + y_2^0 P_y = P_x x_2^* + P_y y_2^* \quad / P_y$,

$x_2^0 p + y_2^0 = p x_2^* + y_2^*$.

Now let's focus on utility functions, because in equilibrium $MRS_1 = MRS_2 = p$ needs to hold.

$MRS_1 = \frac{\delta u_1}{\delta x_1} / \frac{\delta u_1}{\delta y_1}$, therefore $y_1^* / x_1^* = p$ and $y_1^* = p x_1^*$.

$MRS_2 = \frac{\delta u_2}{\delta x_2} / \frac{\delta u_2}{\delta y_2}$, therefore $y_2^* / x_2^* = p$ and $y_2^* = p x_2^*$.

Now we can simply plug these values into the budget lines to get a system of 3 equations with 3 variables x_1^*, x_2^*, p

$x_1 + x_2 = X$,

$x_1^0 p + y_1^0 = p x_1^* + p x_1^*$,

$x_2^0 p + y_2^0 = p x_2^* + p x_2^*$.

Expressing x_1^*, x_2^* in terms of p from 2nd and 3rd equation and plugging to first one leads to

$p = \frac{y_1^0 + y_2^0}{2X - x_1^0 - x_2^0} = \frac{Y}{X}$ and consequently $x_1^* = \frac{x_1^0 p + y_1^0}{2p}$, $x_2^* = \frac{x_2^0 p + y_2^0}{2p}$,

$y_1^* = \frac{x_1^0 p + y_1^0}{2}$, $y_2^* = \frac{x_2^0 p + y_2^0}{2}$. Thatsi the end of computation, as we expressed all the unknown variables in terms of parameters known, i.e. initial endowments and their totals. Furthermore, it is clearly visible that $MRS_1 = MRS_2 = p$ holds.

Example, voluntary and tasks

Consider two tasks s, t and two agents $1, 2$ with corresponding utility functions $v_i = (X - s_i)(Y - t_i)$. The initial endowments $s_1^0, s_2^0, t_1^0, t_2^0 \geq 0$ are given as the parameters of the model.

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[11] We set the price of one of the goods as 1; nummeraire good

Let P_s, P_t be prices of s,t and define price ratio $p = P_s / P_t$. In optimal consumption sets s^*, t^* there must stand $MRS_1 = MRS_2 = p$ while $MRS_i = \frac{\delta v_i}{\delta s_i} / \frac{\delta v_i}{\delta t_i}$.

The budget constraints are derived identically as in previous example,

$s_1^0 p + t_1^0 = ps_1^* + t_1^*$,

$s_2^0 p + t_2^0 = ps_2^* + t_2^*$.

In equilibrium consumption, $MRS_1 = MRS_2 = p$ needs to hold.

$MRS_1 = \frac{\delta v_1}{\delta s_1} / \frac{\delta v_1}{\delta t_1}$, therefore $\frac{Y - t_1}{X - s_1} = p$ and $t_1 = Y - p(X - s_1)$.

$MRS_2 = \frac{\delta v_2}{\delta s_2} / \frac{\delta v_2}{\delta t_2}$, therefore $\frac{Y - t_2}{X - s_2} = p$ and $t_2 = Y - p(X - s_2)$.

We plug these into budget constraint to get system of 3 equations with 3 variables

$s_1^* + s_2^* = X$,

$s_1^0 p + t_1^0 = ps_1^* + ps_1^*$.

$s_2^0 p + t_2^0 = ps_2^* + ps_2^*$.

This system is solved identically as before, and we get

$p = \frac{2Y - t_1^* - t_2^*}{X} = \frac{Y}{X}$, because $t_1^* + t_2^* = Y$ and therefore

$s_1^* = \frac{s_1^0 p + t_1^0}{2p}$, $s_2^* = \frac{s_2^0 p + t_2^0}{2p}$, $t_1^* = \frac{s_1^0 p + t_1^0}{2}$, $t_2^* = \frac{s_2^0 p + t_2^0}{2}$.

Example, involuntary and commodities

This case is probably the simplest, solved by Lagrange multiplies. Define again utilities

$u_1 = x_1 y_1$,

$u_2 = x_2 y_2$,

with consumption sets $x_1 + y_1 \le k(X + Y)$ and $x_2 + y_2 \le (1 - k)(X + Y)$, $k \in (0,1)$.

The reason these sets were chosen is due to the similar properties as budget constraint in exchange economy, as all goods are divided which is not a necessity in a jungle model. In addition, if we used an interpretation that consumption sets in jungle represent the ability to preserve wealth, then with increasing k this ability increases. It is reasonable to assume that if Agent 1 is stronger, k>1/2, but this assumption is not necessary.

Suppose that Agent one is stronger. We want to maximize $u_1 = x_1 y_1$ with respect to $x_1 + y_1 \le k(X + Y)$. Note that due to monotonicity of preferences, maximal solution needs to be located on the exact line $x_1 + y_1 = k(X + Y)$.

Using Lagrange multipliers, we get

$y_1^* + k\lambda = 0$,

$x_1^* + k\lambda = 0$,

$x_1^* + y_1^* = k(X+Y)$.

Therefore $x_1^* = y_1^* = \frac{k(X+Y)}{2}$, $x_2^* = X - x_1^*$, $y_2^* = Y - y_1^*$.

The necessary condition $0 \le x_i^* \le X \quad 0 \le y_i^* \le Y$ moves us to the corner solution if the aggregate bound is quite asymmetric, ($X > 3Y$ or $Y > 3X$ for $k = \frac{1}{2}$ for example). In that case, the weaker agent gets a bundle of only one resource

Example, involuntary and tasks

Yet again, take two tasks s,t and two agents $1,2$ with corresponding utility functions $v_i = (X - s_i)(Y - t_i)$ and the minimal effort sets,

$x_1 + y_1 \ge k(X+Y)$, $x_2 + y_2 \ge (1-k)(X+Y)$, $k \in (0,1)$.

Note the difference in equalities from previous case. Let Agent 1 be the stronger agent. He wants to maximize $v_1 = (X - s_1)(Y - t_1)$ with respect to constraint $x_1 + y_1 \ge k(X+Y)$.

Using Lagrange multipliers:

$t_1^* - Y + \lambda = 0$,

$s_1^* - X + \lambda = 0$,

$s_1^* + t_1^* = k(X+Y)$.

Subtracting the second equation from the first leads to $Y - t_1^* = X - s_1^*$ and with the third equation,

$$s_1^* = \frac{k(X+Y) + X - Y}{2} \quad t_1^* = \frac{k(X+Y) + Y - X}{2}, \; s_2^* = X - s_1^* \quad t_2^* = Y - t_1^*.$$

Agent 2 simply gets whatever is left. Same discussion about corner solutions as in previous section needs to be done here. Note the difference from previous case, the preferred tasks are dependent on X,Y in this cases. This is intuitive, as X,Y are now present in the utility functions. Same discussion about corner solutions as in previous example needs be done here, as $0 \le s_i^* \le X \quad 0 \le t_i^* \le Y$.

Discussion of results

Four possible settings for distribution of goods were presented. We see that the differences in optimization problem in case of voluntary allocation systems are negligible. Given initial endowments and the same utility function, the price vector at which resources are traded remains the same. The final consumptions derived seem to be very similar as well.

In the involuntary exchange model, results differ more. The reason seems to be that the total amounts of resources, *X, Y,* are included in the utility function. Therefore the utility depends directly on their values in contrast to voluntary exchange. In addition, discussion about corner solution needs to be added. The corner solution seem to be a problem in case that k is really large in the commodities exchange and k really small in tasks exchange. In addition, asymmetric totals of resources *X*,*Y* increase the probability of corner solution. Suppose totals $X = 5, Y = 1$ and same utility functions as in the example. We have to move to corner solution in this case if $k > \frac{1}{3}$ in the commodity case and if $k < \frac{2}{3}$ in the tasks case. Note, however, that these results are strongly influenced by the way we defined limitations over resources, i.e. consumption set and minimal effort set. These limitation strongly impact the results. Note that no such sets had to be added in voluntary exchange.

Purpose of this chapter was not to analyze the firm setting in general. We rather used an example of very simply defined utility functions and showed the results for this case. The case of voluntary exchange seems very similar both for both resources, while involuntary exchange in the firm model might be an interesting area for future discussion.

Chapter 6: Conclusion

In this thesis, we examined multiple systems for allocation of resources. We started with the standard exchange economy, where agents trade goods while mutually beneficial. Then we discussed the jungle model, driven by coercion, where stronger agents can seize resources from weaker agents. The key differences between the two models were discussed.

We tried to compare the two allocation systems in a situation where models are identical in every other aspect. To achieve this, we had to deal with two major differences - initial endowments in the exchange economy and consumption sets. As adding initial endowments in the case of strictly monotone preferences does not affect the jungle equilibria, we rather focused on the consumption set difference. Removing consumption sets from jungle economy does not make much sense, as it simply causes all the goods to be appropriated by the strongest agent. The result of adding these consumption bounds into exchange economy is more interesting.

Presence of consumption sets as an additional constraint of agents in the exchange economy extends the set of Pareto efficient allocations. The Pareto set mostly depends on two factors, mutual setting of consumption sets and mutual setting of BP allocations, best possible allocations of an agent if he had access to all goods in the economy. The set of Pareto efficient allocations extends as the conflict over BPs decreases. Only in the case of set-of-points intersect of consumption sets and full conflict over BPs, the contract curve (the set of Pareto efficient allocation in a standard exchange economy) is an important aspect for determining the Pareto set. It is due to the fact that there exists a region where consumption sets are not bounding for either agent. In other settings, the Pareto set is determined only by arrangement of BPs and consumption sets.

For the whole welfare analysis, we assumed that for every price vector, consumptions of both agents are positive, instead of non-negative. This means that BPs are always located in the interior of the Edgeworth box. This assumption is related to the shape of consumption sets. Allowing BPs to be located at the borderlines of the Edgeworth box adds another dimension to the problem.

We also discussed how consumption sets modify the competitive equilibria in the exchange economy. This problem appears to be rather difficult. Instead of full analysis, we pointed out some issues that arise. The big difference that arises is that when the price vectors go beyond BPs, there are areas in the Edgeworth box where one of the agents is indifferent. In addition, with the presence of consumption bounds we can get a competitive equilibrium that is not Pareto efficient.

In Chapter 5, we introduced a division of tasks in a firm model derived from the exchange economy. In a very simple setting, we discussed how the comparison of the hierarchical or exchange allocation is equivalent in the tasks or source setting.

The objective of the thesis was the comparison of two standard models – the exchange and jungle economy. During the work on the thesis, a number of possible extensions of the state-of-the-art theory emerged. Our focus was on motivation and intuitive treatment of the emerging problems, using the powerful visualization tool – the Edgeworth box. The author is fully aware, that formal rigorous treatment of the problems depicted would require significantly more space and effort, going beyond the scope of a bachelor thesis.

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Appendix A

Proposition 6

Consider the borderline of C_i and BP_i located on this borderline. Then every move among the borderline getting closer to the *BP* increases utility of agent *i*.

Discussion:

Consider a move among the borderline of C_1 from any allocation $F(x, y)$ located on the borderline, $x_f < x_{BP1}$ to BP_1. Define MRT as the shape of the consumption set, $MRT = -\frac{dy}{dx}$. MRS is the shape of indifference curve, such that Agent 1 maintains his utility level, $\frac{\frac{\partial u_1}{\partial x}}{\frac{\partial u_1}{\partial y}}$.

For a move among the consumption set from *F* to BP, we want to show that this inequality holds:

$$\frac{\partial u_1}{\partial x}\Big|_{C_1} = \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} * \frac{\partial y}{\partial x} > 0$$

$\frac{\partial u_1}{\partial x} > 0, \frac{\partial u_1}{\partial y} > 0$ because of monotonicity of preferences. We divide equation by $\frac{\partial u_1}{\partial y}$:

$$\frac{\frac{\partial u_1}{\partial x}}{\frac{\partial u_1}{\partial y}} + \frac{\partial y}{\partial x} > 0$$

$$MRS + MRT > 0$$

This equation holds for any *F* between *M* and BP_1. Once we consider allocations behind the BP_1, in every allocation of the borderline of C_1, shape of consumption set is greater than shape of indifference curve. Therefore moving among borderline while increasing of *x* decreases the utility of Agent 1. Vice versa, decreasing *x* increases utility. Therefore any move closer to the BP increases utility. The proposition is quite obvious from the shape of indifference curves and tangency of indifference curve in the BP allocation.