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DIPLOMA THESIS

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Spinorial techniques for constructing quasi-local quantities in general relativity

Institute of Theoretical Physics

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I would like to thank my supervisor. Without his help this thesis would not have been possible. Both the author and his supervisor are grateful to Paul Tod and László Szabados for useful discussions and suggestions.

I declare that I carried out this diploma thesis independently, and only with the cited sources, literature and other professional sources and with the help of my supervisor.

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In Prague, 30. 7. 2014 Lukáš Holka

Název práce: Spinorové techniky pro konstrukci kvazi-lokálních veličin v obecné relativitě

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Katedra: Ústav teoretické fyziky

Vedoucí diplomové práce: Mgr. Martin Scholtz, Ph.D.

Abstrakt: V asymptoticky plochých prostoročasech umíme definovat vyhovující globální hmotnost. V předkládané práci spočítáme Bondiho hmotnost prostoročasu obsahujícího interagující skalární a elektromagnetické pole. Pak odvodíme formuli pro úbytek Bondiho hmoty a ukážeme, že je negativně semi-definitní. Získané výsledky jsou odvozeny pomocí spinorových technik, které jsou popsány v první části práce, která také obsahuje krátký přehled některých jiných konstrukcí pro hmotnost v obecné relativitě.

Klíčová slova: asymptotická jednoduchost, Bondiho hmotnost, spinory, kvazi-lokální veličiny, energie v obecné relativitě

Title: Spinorial techniques for constructing quasi-local quantities in general relativity

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Abstract: It is possible to define reasonable global mass for asymptotically flat space-times. In this work we compute the Bondi mass of asymptotically simple space-time that contains interacting scalar and electromagnetic fields. We then obtain the Bondi mass-loss formula and show that it is negatively semi-definite. These results are derived with the help of spinorial techniques which we introduce in the first part of this thesis, which also contains brief review of several other constructions of energy in general relativity.

Keywords: asymptotic simplicity, Bondi mass, spinors, quasi-local quantities, general-relativistic energy

Contents

Introduction

Notions of energy and momentum play a major role in modern physics. They were present already at the times of birth of the Newtonian mechanics and slowly matured as physics evolved into their current form. They appear in Lagrangian formulation of dynamics and fulfill a pivotal function in the Hamiltonian formalism. Because the canonical quantization relies on Hamiltonian formalism, energy and momentum play an important role (at least historically) also in quantum physics. But one of the pillars of modern physics, the general relativity, resists vigorously all the attempts to define some universal notion of energy and momentum. The source of the difficulties lies in the equivalence principle, which requires the existence of coordinates with respect to which the gravitational field vanishes locally. These difficulties may suggest that the notion of energy-momentum in general relativity is ill-defined and should be abandoned as a dead end. On the other hand, there are several reasons why this problem may be worth of investigation. First of all, the situation is not at all hopeless. A lot of progress has been made in this direction and several interesting and promising constructions for energy-momentum have already been proposed. Moreover, the nature of the problem — one of the central notions of physics being seemingly incompatible with a foundational principle of general relativity — makes it attractive, since we may very well learn valuable lessons while researching it. Additionally, because the concept of energy is connected with Hamiltonian dynamics, investigation of general-relativistic energy-momentum may provide us with tools valuable in the search for the theory of quantum gravitation. In fact, the basic obstacle in formulation of the quantum theory of gravitation — the vast gauge freedom of such a theory — is also a consequence of the equivalence principle.

The most promising approaches to the problem of general-relativistic energy-momentum seem to be of non-local nature. There are two very satisfactory global constructions that are applicable in asymptotically simple space-times, the ADM construction where energy-momentum is measured on space-like hypersurfaces and the Bondi construction with asymptotically null hypersufaces. There are also several promising quasi-local (i.e. measured over a finite region) constructions, although none of them is fully satisfactory and universally applicable. In this thesis we are mainly concerned with the Bondi mass of asymptotically simple space-time that contains interacting electromagnetic and scalar fields, but we also briefly review some other constructions.

In this thesis we employ the spinor formalism. Spinors are very interesting concept on their own and have proven to be a valuable tool in general relativity. We dedicate the majority of the introductory part of this thesis to introduce spinors and related concepts.

In the first chapter, following [11] closely, we introduce the spinors on algebraic level and give their geometrical interpretation. We elucidate the relation between two-component spinors and usual tensor algebra and develop Penrose's abstract index notation in detail. In the second chapter we introduce spinor fields in the spacetime (on the working level) and the notion of covariant differentiation of spinors. We explain how the spinor calculus is related to the Newman-Penrose formalism. This formalism can be built up using purely tensorial terms but the use of spinors provides us with the deeper insight into the structure of the Newman-Penrose equations. In the rest of the second chapter we derive the Bianchi identities and the Ricci identities in the spinor form. These equations serve as the equations of gravitational field, i.e. in the spinor formalism they replace the role of the Einstein equations.

The third chapter is devoted to a modification of the Newman-Penrose formalism, the Geroch-Held-Penrose (GHP) formalism. While the former is adapted to a given null tetrad, the latter is adapted to two null directions. In this chapter we show how the systematic use of boost and spin gauge freedom effectively reduces the number of the Newman-Penrose equation.

We have mentioned that there exist several promising suggestions how to construct quasi-local quantities in general relativity. One of the most fruitful and the most inspiring suggestion has been made by Penrose and we call this construction the Penrose mass. It was inspired by the twistor theory developed by the same author and his collaborators. In the thesis we do not want to go into the details of this theory, but we introduce some basic notions which will be necessary later. We write down the twistor equation and define the twistor as its solution, we discuss the existence of the solutions in the flat and curved spacetime.

From a broader point of view, many quasi-local constructions that have been suggested can be understood as a 2-surface integral of the quantity called the Nester-Witten form. A unified formalism for description of these construction has been given by Szabados. Following closely his papers [18, 19] and the review paper [20], we present the spinorial analysis of geometry of spacelike 2-surfaces, in particular topological 2-spheres. In usual 3+1 decomposition of general relativity it is customary to introduce the intrinsic covariant derivative induced on the 3-hypersurfaces foliating the spacetime. An interesting feature of the analysis of spacelike 2-surfaces is the observation that this "canonical" intrinsic derivative is too rigid to encompass information which should be relevant for the construction of quasi-local quantities. Indeed: we might expect that the quasi-local energy will be given as the 2-surface integral of the curvature tensor, where the curvature tensor is derived from the intrinsic covariant derivative. However, the Gauss-Bonnet theorem asserts that such integral is always proportional to a single characteristic of the surface: the Euler characteristic, which, for orientable compact surfaces, reduces to $2-2g$ where g is the genus of the surface. This simplified deduction illustrates that the usual intrinsic derivative does not contain all relevant information.

Nevertheless, there is another connection which can be reasonably called intrinsic to a 2-surface: the Sen connection. It turns out that this connection has much richer structure which we reveal in the chapter 5. We introduce the notion of 2-surface spinors and present the decomposition of the Sen derivative into its irreducible parts. Surprisingly enough, one irreducible part of the Sen operator is the Witten operator (which played an important role in Witten's proof of the positivity of mass) and the second irreducible part is the twistor operator. This shows that although the twistor equation has been imposed by Penrose for different reason, it actually appears naturally in the description of intrinsic geometry of 2-surfaces.

In chapter 6 we finally define the Penrose mass. Original Penrose's construction was motivated by the linearized gravity, but in the thesis we present contemporary standard derivation of the Penrose mass [20, 7]. This derivation starts from constructing quasi-local quantities in the Minkowski spacetime and continues by "guessing" an appropriate analogy in the curved spacetime. We arrive at the expression for the Penrose quasi-local charges in the Newman-Penrose formalism. Unfortunately, even the Penrose mass suffers from several drawbacks. Although we do not discuss them in the thesis, it is worth to mention that the twistorial construction works only for the so-called non-contorted surfaces. Roughly speaking, these are the surfaces which can be embedded in the (conformal) Minkowski spacetime, preserving their extrinsic and intrinsic curvature. If such embedding is possible, the twistorial norm can be shown to be constant over the 2-surface which is necessary in order to extract the components of the energy-momentum and the angular momentum from the object known as the kinematical twistor. If the 2-surface cannot be embedded this way, the twistorial norm is not constant over the surface and the construction fails. Such surfaces are called contorted and the Penrose mass cannot be defined for them.

There have been several attempts to overcome these difficulties. For the sake of our thesis we have chosen two important and interesting examples. In the chapter 7 we discuss the notion of global energy in asymptotically flat spacetime and introduce the ADM and the Bondi mass (see above). A natural quasi-localization of the Bondi mass is the Ludvigsen-Vickers construction, which we briefly describe. A more sophisticated suggestion (based on the sheaf-theoretical arguments) have been given by Dougan and Mason. We do not go into details but mention important properties of the Dougan-Mason mass. This concludes the theoretical introduction.

Goals and results

In this thesis we do not aim to solve the long standing problem of appropriate definition of quasi-local mass in general relativity. As we explain in the introduction to paper attached to the thesis, our goal is to make a first step to construct the Penrose mass for spacetimes with the scalar field sources. Although there are many particular examples of Penrose's mass (mainly due to Paul Tod), the examples with scalar fields are missing. However, because of lack of exact solutions with scalar or electromagnetic and scalar fields, it is necessary to construct the Penrose mass indirectly, e.g. by the 3+1 or 2+2 decomposition. An important criterion of correctness of particular construction is whether this construction in the limit of large spheres gives correct Bondi or ADM mass (which are defined unambiguously). Hence, in this thesis we have solved a simpler problem: we have calculated the Bondi mass for the spacetimes with interacting scalar and electromagnetic fields. Details of this construction are explained in the second part of the thesis and in the paper attached.

The main result of this thesis is the expression for the Bondi mass

$$M_B = \frac{1}{2\sqrt{\pi}} \oint \left(\Psi_2^0 + \sigma^{(0)}\dot{\overline{\sigma}}^{(0)} + \frac{1}{6}\frac{\partial}{\partial u}\left(\phi^0\overline{\phi}^0\right) \right) \mathrm{d}\hat{\mathcal{S}} \,.$$

In the formula above, Ψ_2^0 is the leading term in corresponding component of the Weyl spinor, σ^0 is the asymptotic shear of Newman and Penrose, ϕ^0 is the radiative part of the scalar field and u stands for (retarded) time. An interesting feature is that the scalar field itself contributes to the Bondi mass, since this is not the case with purely EM field.

The Bondi mass-loss formula measures how the Bondi mass of the spacetime changes when the gravitating system inside the spacetime produces gravitational or another radiation. For "reasonable" matter, i.e. matter satisfying the null energy condition, the Bondi mass-loss formula is negatively semidefinite, which means that the Bondi mass is either constant or decreasing function of time u. For example, at is was shown in [2], the conformally invariant scalar field does not obey this condition and hence corresponding mass-loss formula is indefinite. In this thesis we have shown that interacting electromagnetic and scalar fields do not suffer from this defect. In paper we have shown that the mass-loss formula for the case of our interest reads

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \oint \left[\dot{\sigma}^{(0)}\dot{\overline{\sigma}}^{(0)} + \phi_2^0\overline{\phi}_2^0 + \dot{\phi}^0\dot{\overline{\phi}}^0 + ie\,A_2^0\left(\phi^0\dot{\overline{\phi}}^0 - \dot{\phi}^0\overline{\phi}^0\right) + e^2 A_2^0 A_2^0 \phi^0\overline{\phi}^0 \right] \mathrm{d}\hat{\mathcal{S}} \,,$$

where ϕ_2^0 is the radiative component of EM field, e is the charge of scalar field and A_2^0 is the Newman-Penrose component of the 4-potential. This formula can be brought into form

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \oint \left[\dot{\sigma}^{(0)}\dot{\overline{\sigma}}^{(0)} + \phi_2^0\overline{\phi}_2^0 + \left(\mathcal{D}_u\phi^{(0)}\right)\left(\mathcal{D}_u\overline{\phi}^{(0)}\right) \right] \mathrm{d}\hat{\mathcal{S}} \,,$$

where $\mathcal{D}_u$ is the gauge-covariant derivative with respect to time. This expression is manifestly gauge invariant and negative semi-definite. Our results reduce to previously known expressions when one of the fields (or both) is missing.

Part I

Theoretical introduction

1. Spinors in General Relativity

In this chapter we will briefly[1] introduce (2-)spinor formalism which proved to be a fruitful alternative to the commonly used tensor calculus. The formalism is in a sense tailored for a special case of four-dimesional manifold endowed with the Lorentzian metric. As a result, it lacks some of a generality of the tensor formalism, which can be readily applied to manifold of any dimension. But its functionality in the case of four-dimensional manifold compensates more than enough for this deficiency. Many formulae of general relativity simplify considerably when approached from spinorial viewpoint and some hidden structures reveal themselves. In fact, spinor calculus may appear to be more fundamental than the tensorial one, but even if that is not the case, it still constitutes a very useful representation for certain problems in physics.

1.1 Matrix representation of four-vector

We start by examining a general Hermitian matrix $\mathbf{A}$ of dimension 2×2. We say that a matrix is Hermitian, if it equals complex conjugate of its transposition, i.e. $\mathbf{A_{ij}} = \overline{\mathbf{A}}_{\mathbf{ji}}$. Therefore its diagonal elements must be real and the remaining two elements must be related by complex conjugation. Thus a general Hermitian matrix may be written in the form

$$\mathbf{A} = \begin{pmatrix} \alpha+\beta & \gamma+\mathrm{i}\delta \\ \gamma-\mathrm{i}\delta & \alpha-\beta \end{pmatrix}, \tag{1.1}$$

with α, β, γ and δ arbitrary real numbers. Now consider its determinant

$$\det(\mathbf{A}) = \alpha^2 - \beta^2 - \gamma^2 - \delta^2. \tag{1.2}$$

This formula formally resembles the expression for the Lorentz norm[2] of a vector, $s^2 = T^2 - X^2 - Y^2 - Z^2$. Hence we see, that it is possible to assign a Hermitian matrix

$$\begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} \tag{1.3}$$

to a four-vector of components T, X, Y, Z, and that the determinant of the matrix is equal to its length. Moreover, this correspondence is clearly one-to-one.

Let us now restrict our attention to the special case of a null vector, so that matrix (1.3) has zero determinant. Since its columns are linearly dependent, we can factorize the matrix as a direct product of two two-vectors

$$\begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}, \tag{1.4}$$

with all the components of the two-vectors being complex. Because the matrix is Hermitian, we have

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \overline{v}_1 \\ \overline{v}_2 \end{pmatrix} \begin{pmatrix} \overline{u}_1 & \overline{u}_2 \end{pmatrix}, \tag{1.5}$$

[1] We refer an interested reader to the more complete work [11] from which the present thesis took a lot of inspiration.

[2] In this thesis we use $(+---)$ signature.

which yields

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \kappa \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}, \tag{1.6}$$

with κ being real. Thus we see that it is possible to decompose the matrix (1.3) as

$$\begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix}. \tag{1.7}$$

This decomposion is unique up to the phase transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto e^{\mathrm{i}\phi} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{1.8}$$

where ϕ is real. Complex numbers ξ and η may be regarded as the components of a *spin-vector*, an object central to this chapter.

1.2 Stereographic projection

In the previous section we have shown how an entirely new object, a spin-vector, can be extracted from a null four-vector. We will spend the rest of this chapter studying this object, mainly from geometrical point of view. However, we do not give a thorough treatise on the object, as our aim is simply to help to build an intuition for a concept of spin-vector.

So, what is the meaning behind these mysterious numbers ξ and η? It turns out that the answer is not as remote as it may seem. A concept essential to the tensor calculus is that of a direction[3], and we will find that encoding an information on direction[4] in a suitable way, we arrive at just those two numbers. The key to the gate that stands in our way is stereographic projection.

In general, the stereographic projection is a mapping that projects a sphere onto a plane. We are interested in a particular case of such a projection. Consider the three-dimensional Euclidean space $\mathbb{E}^3$ and choose Cartesian coordinates x, y, z with origin at a point O. Consider a unit sphere at origin representing the space of directions from O. Let $P(X, Y, Z)$ be an arbitrary point on the sphere. ($X^2 + Y^2 + Z^2 = 1$.) We find the projection of the point P by drawing a line from the north pole $\mathrm{N}(0, 0, 1)$ through the point P. The point p where this line intersects the plane $z = 0$ is the stereographic projection of the point P from the north pole (see Figure 1.1).

The projection maps the southern hemisphere onto the unit disk in the xy-plane. In particular, it projects the south pole into the origin. The image of northern hemisphere covers the rest of the plane with the projection of the north pole undefined. Note, however, that the closer to the north pole the point P lies, the further from the origin it projects. Therefore, we can associate the projection of the north pole with the infinity.

Our next step is to find the relation between coordinates of the point $P = (X, Y, Z)$ and the coordinates of its image $p = (x_p, y_p, 0)$ under the stereographic projection. Because the line passing through the points p and P also intersects the z-axis, $X/Y = x_p/y_p$ does hold. Therefore, it remains to find how $x_p^2 + y_p^2$ depends on the coordinates of the point P. The situation — as seen in the plane determined by the point p and the z-axis — is shown in the figure 1.2. Two similar right-angled triangles are outlined in the picture. One with the hypotenuse $\mathrm{N}P$, the other

[3]The other one of similar importance is a concept of magnitude.

[4]Actually hand in hand with information on magnitude.

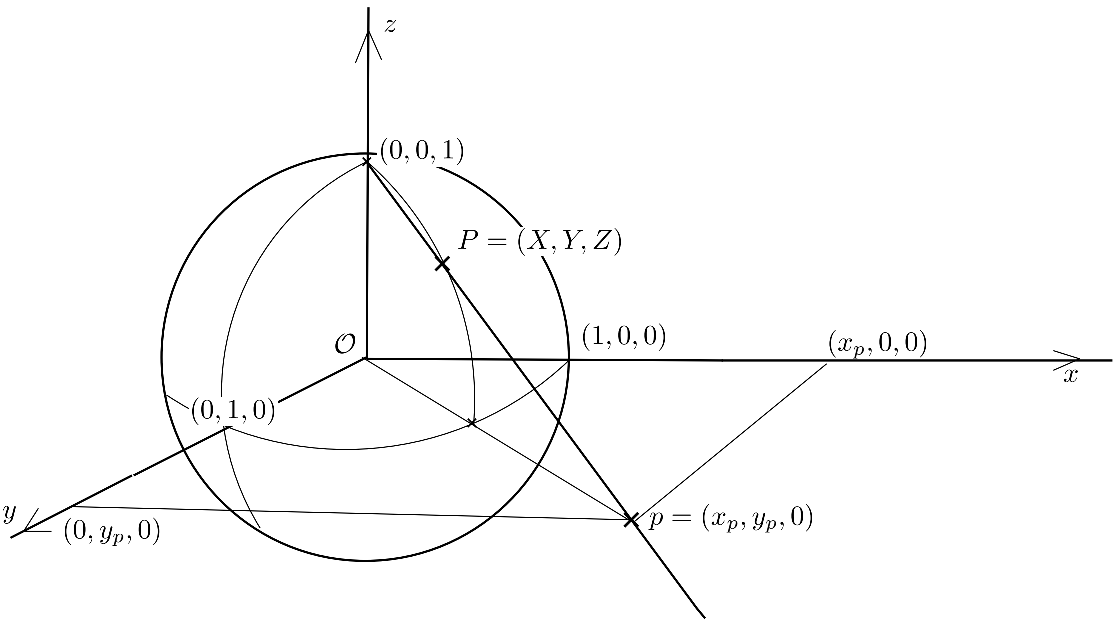

Figure 1.1: Stereographic projection from the north pole. It maps the point P on the unit sphere to the point p in the xy-plane.

one with the hypotenuse Np. The ratio of their lenghts is $\frac{1-Z}{1}$ and so $\frac{\sqrt{X^2+Y^2}}{\sqrt{x_p^2+y_p^2}} = \frac{1-Z}{1}$. Putting everything together, we get

$$\begin{aligned} x_p =& \sqrt{x_p^2+y_p^2}\frac{X}{\sqrt{X^2+Y^2}} = \frac{X}{1-Z}, \\ y_p =& \sqrt{x_p^2+y_p^2}\frac{Y}{\sqrt{X^2+Y^2}} = \frac{Y}{1-Z}. \end{aligned} \tag{1.9}$$

Employing spherical coordinates on the unit sphere,

$$\begin{aligned} X =& \sin\theta\,\cos\phi\,, \\ Y =& \sin\theta\,\sin\phi\,, \\ Z =& \cos\theta\,, \end{aligned}$$

equations (1.9) acquire the form

$$\begin{aligned} x_p =& \frac{X}{1-Z} = \frac{\sin\theta\cos\phi}{1-\cos\theta} = \cos\phi\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2})} = \cos\phi\cot\frac{\theta}{2}\,, \\ y_p =& \frac{Y}{1-Z} = \frac{\sin\theta\sin\phi}{1-\cos\theta} = \sin\phi\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2})} = \sin\phi\cot\frac{\theta}{2}\,. \end{aligned} \tag{1.10}$$

Now, with the infinity regarded as the image of the north pole under the stereographic projection, we have effectively constructed a one-to-one map between the unit sphere and the xy-plane including infinity. We can use this map to define new coordinates on the sphere with the coordinates of the point P being x and y coordinates of its stereographic projection.

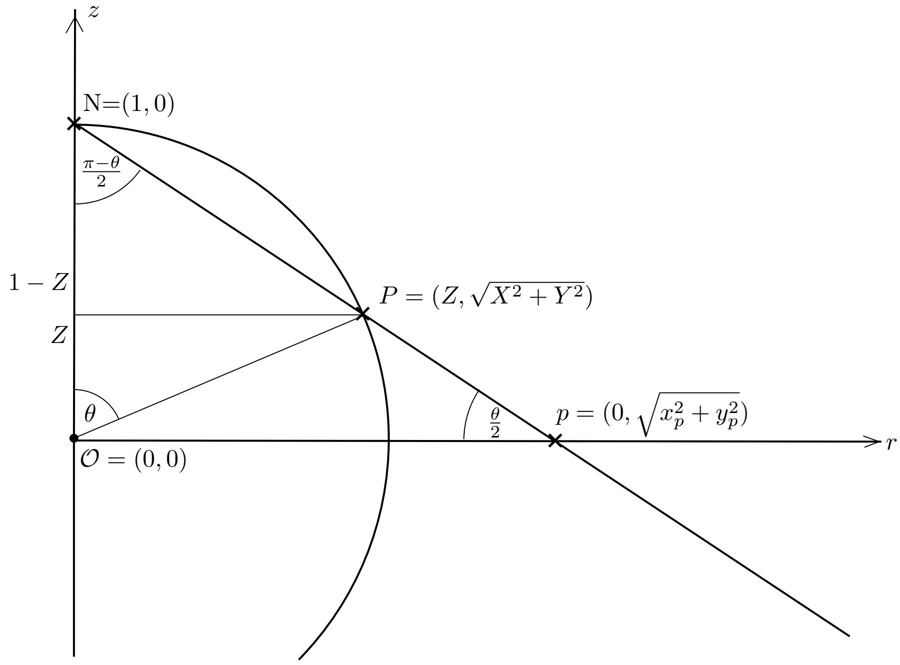

Figure 1.2: Stereographic projection from the north pole — as seen in the half-plane N$\mathcal{O}P$ — using cylindrical coordinates (z, r).

We can also regard the xy-plane as the representation of the Argand plane of complex numbers (including infinity), stereographic projection then induces one-dimensional complex coordinate $\zeta = x + \mathrm{i}y$ on the sphere. From (1.9) and (1.10) we get

$$\zeta = \frac{X + \mathrm{i}Y}{1 - Z} = e^{\mathrm{i}\phi} \cot \frac{\theta}{2}. \tag{1.11}$$

In what follows we will also need the inverse relations. To solve (1.11) for the Cartesian coordinates we start with Z (recall that $X^2 + Y^2 + Z^2 = 1$):

$$\begin{aligned} \zeta\bar{\zeta} =& \frac{X^2 + Y^2}{(1 - Z)^2} = \frac{1 + Z}{1 - Z}, \\ Z =& \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1}. \end{aligned} \tag{1.12}$$

Now we can easily obtain

$$X = \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \quad Y = \frac{\zeta - \bar{\zeta}}{\mathrm{i}(1 + \zeta\bar{\zeta})}, \quad Z = \frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}. \tag{1.13}$$

Note however that, strictly speaking, ζ is not true coordinate on the whole sphere, because the coordinate of the north pole is not finite. Recall, that it is in general impossible to cover the sphere by single coordinate chart. We could, of course, define coordinates using two different stereographic projections, e. g. one projecting from the north pole and the second one projecting from the south pole. But there are other options on how to specify points on the sphere, rather than just covering it by coordinate patches. For example, we could say that the point (x, y, z) of $\mathbb{R}^3$ represents a direction in $\mathbb{R}^3$ and the point P of the sphere emerges as an intersection of this direction with the sphere. This way any point $q(x, y, z)$ specifies an equivalence class

$$\{q' : x_{q'}/x = y_{q'}/y = z_{q'}/z > 0\},$$

which does in turn specify a point on the sphere.

Let us adopt a similar approach: Our projection maps points of the sphere onto the points of the xy-plane, mapping the P of the sphere to the p of the plane. Coordinates of the point $P = (X, Y, Z)$ are then given by single complex number $\zeta = x_p + iy_p$. This complex number can be written as a ratio $\zeta = \frac{\beta}{\alpha}$ for some $\alpha, \beta \in \mathbb{C}$. Numbers α and β will be called *homogeneous coordinates* of the point ζ. Obviously, these coordinates are non-unique, because any other coordinates $\kappa\alpha$ and $\kappa\beta$ determine the same ζ for any non-zero complex κ. (Thus the name homogeneous.) Numbers α and β are basically ξ and η of (1.7). But to see this, and to actually make the identification meaningful, we need to switch to Minkowski space $\mathbb{M}$. To do so, we need to find out a way how to translate our construction to $\mathbb{M}$ so that it will respect the Lorentz symmetry. For example, if we tried to identify the space $\mathbb{R}^3$ of the outlined procedure with the hyperplane $T = 0$ (where T stands for a time coordinate), such a construction would clearly be not Lorentz invariant, because simultaneity is not an absolute concept in special relativity (i.e. hyperplanes $T = \text{const}$ are not Lorentz invariant). To overcome this problem, we regard ζ as a coordinate on the light cone. Consider the intersection of the light cone with the hyperplane $T = 1$. In Euclidean space $T = 1$, this intersection is an unit sphere and each point on the sphere may represent a null direction. In the spirit of (1.13) we can write

$$X = \frac{\zeta + \overline{\zeta}}{1 + \zeta\overline{\zeta}}, \quad Y = \frac{\zeta - \overline{\zeta}}{\mathrm{i}(1 + \zeta\overline{\zeta})}, \quad Z = \frac{\zeta\overline{\zeta} - 1}{1 + \zeta\overline{\zeta}}, \quad T = 1. \tag{1.14}$$

Substituting β/α for ζ, these equations acquire the form

$$X = \frac{\beta\overline{\alpha} + \overline{\beta}\alpha}{\alpha\overline{\alpha} + \beta\overline{\beta}}, \quad Y = \frac{\beta\overline{\alpha} - \overline{\beta}\alpha}{\mathrm{i}(\alpha\overline{\alpha} + \beta\overline{\beta})}, \quad Z = \frac{\beta\overline{\beta} - \alpha\overline{\alpha}}{\alpha\overline{\alpha} + \beta\overline{\beta}}, \quad T = 1. \tag{1.15}$$

We have already noted that relations (1.15) are unchanged under transformation $\alpha, \beta \mapsto \kappa\alpha, \kappa\beta$. Thus we have two redundant degrees of freedom. We can harness this ambiguity and use one degree of freedom to encode the information on the extent of vector. To do so, we need to extend our description on the whole null cone, rather than just its intersection with the hyperplane $T = 1$. This can be accomplished by simply multiplying equations (1.15) by some convenient real function of α and β (same function for each of the equations). We choose this function to be $\alpha\overline{\alpha} + \beta\overline{\beta}$, as the form of the equations suggests, and add a factor $\frac{1}{\sqrt{2}}$ for later convenience. Thus we get

$$X = \frac{1}{\sqrt{2}}\left(\beta\overline{\alpha} + \overline{\beta}\alpha\right), \quad Y = \frac{1}{\mathrm{i}\sqrt{2}}\left(\beta\overline{\alpha} - \overline{\beta}\alpha\right), \quad Z = \frac{1}{\sqrt{2}}\left(\beta\overline{\beta} - \alpha\overline{\alpha}\right), \quad T = \frac{1}{\sqrt{2}}\left(\beta\overline{\beta} + \alpha\overline{\alpha}\right). \tag{1.16}$$

Substituting now from (1.16) for the components of four-vector in (1.3), we finally see that α, β of (1.16) are essentially[5] the same as η, ξ of (1.7).

[5]Ignoring discrepancies due to normalization.

1.3 Spin transformations

In this section we will study an important concept of spin transformation. Consider a regular complex matrix

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with numbers α and β not related to α and β of the previous section. A transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{1.17}$$

maps a spin-vector into another spin-vector. Through (1.16) this induces a linear transformation of the null cone. The change in null directions is given by the transformation of ζ

$$\zeta \mapsto \tilde{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \tag{1.18}$$

which is conformal (because holomorphic) and invariant under $\mathbf{A} \mapsto k\mathbf{A}$, $k \neq 0$. Normalizing determinant of $\mathbf{A}$ to unity has thus no influence on how null directions are transformed, altering only the way in which the extent of vectors is affected. We define the *spin-matrix* to be a complex matrix with unit determinant. If $\mathbf{A}$ in transformation (1.17) is a spin-matrix, we refer to it as *spin transformation.*

From (1.3) we can readily infer how X, Y, Z, T transform. We have

$$\begin{aligned} \begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} \mapsto \begin{pmatrix} \tilde{T}+\tilde{Z} & \tilde{X}+\mathrm{i}\tilde{Y} \\ \tilde{X}-\mathrm{i}\tilde{Y} & \tilde{T}-\tilde{Z} \end{pmatrix} = \\ = \sqrt{2} \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} \left(\bar{\tilde{\xi}} \bar{\tilde{\eta}} \right) = \sqrt{2} \left[\mathbf{A} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right] \left[\mathbf{A} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right]^{\dagger} = \\ = \mathbf{A} \begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} \mathbf{A}^{\dagger}, \end{aligned} \tag{1.19}$$

where $\dagger$ denotes Hermitian conjugation[6].

Recall that Lorentz transformations are exactly those linear transformations that preserve the lengths of the vectors. Because the length of a four-vector is equal to the determinant of matrix (1.3), we find from (1.19) that a spin transformation induces a Lorentz transformation of the null cone[7]. If we extend the domain of a spin transformation (1.19) such that it may act on arbitrary Hermitian matrix (1.3), it clearly defines a Lorentz transformation of the whole Minkowski vector space.

Notice that both $\mathbf{A}$ and its negative $-\mathbf{A}$ generate the same Lorentz transformation. It can be shown that every spin transformation corresponds to a unique restricted[8] Lorentz transformation and that to each restricted Lorentz transformation there are exactly two corresponding spin transformation, one being the negative of the other.

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[6]The right hand side of the (1.19) can be written in the form (1.3), because the resulting matrix $\mathbf{AXA}^{\dagger}$ is Hermitian if $\mathbf{A}$ is.

[7]$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

[8]Restricted Lorentz transformation is a Lorentz transformation that does not involve space or time reversal. Consequently, any such transformation lies in the identity component of the Lorentz group, i. e. it can be continuously changed to the identity.

For illustrative purposes we will find explicit form of spin transformation that corresponds to the rotation about the z axis and the one that corresponds to the boost in the z direction.

In the case of rotation by angle ϕ about the z axis, the null directions are also rotated in the same manner. Therefore, in terms of ζ, such a transformation corresponds to

$$\tilde{\zeta} = \zeta e^{\mathrm{i}\phi}. \tag{1.20}$$

Thus the spin-matrix we are looking for has the form

$$\mathbf{A} = \pm \begin{pmatrix} e^{\frac{\mathrm{i}\phi}{2}} & 0 \\ 0 & e^{-\frac{\mathrm{i}\phi}{2}} \end{pmatrix}. \tag{1.21}$$

Notice that this matrix is unitary. Actually, any unitary matrix corresponds to a spatial rotation and vice versa. To see this, observe that trace of the matrix (1.3) equals $\sqrt{2}T$ and that $\mathrm{Tr}(\mathbf{A}\mathbf{X}\mathbf{A}^\dagger) = \mathrm{Tr}(\mathbf{X})$ if and only if $\mathbf{A}$ is unitary.

Under the boost in the z direction the coordinates transform as

$$\tilde{T} = \frac{T + vZ}{\sqrt{1-v^2}}, \quad \tilde{X} = X, \quad \tilde{Y} = Y, \quad \tilde{Z} = \frac{Z + vT}{\sqrt{1-v^2}}, \tag{1.22}$$

where v is the velocity parameter. We rewrite these relations as

$$\tilde{T} + \tilde{Z} = w\,(T + Z), \quad \tilde{X} = X, \quad \tilde{Y} = Y, \quad \tilde{T} - \tilde{Z} = w^{-1}\,(T - Z), \tag{1.23}$$

with $w = \sqrt{\frac{1+v}{1-v}}$. Comparing this result with (1.3) we find out that the corresponding spin-matrix has the form

$$\mathbf{A} = \pm \begin{pmatrix} w^{\frac{1}{2}} & 0 \\ 0 & w^{-\frac{1}{2}} \end{pmatrix}. \tag{1.24}$$

This transformation is particularly simple when expressed in terms of ζ, as all we get is a simple expansion

$$\tilde{\zeta} = w\zeta. \tag{1.25}$$

1.4 Geometric representation of spin-vector

Our aim here is to find a way how to geometrically represent a spin-vector. We have already established a correspondence between a null four-vector and a spin-vector, but the structure of four-vector is not rich enough to capture all the information on spin-vector: a four-vector associated with the spin-vector (ξ, η) is unaffected by transformation $(\xi, \eta) \mapsto e^{\mathrm{i}\phi}(\xi, \eta)$, where ϕ is real. As we will find out, it is possible to constuct a geometric object that determines the associated spin-vector nearly completely, leaving unknown only the overal sign of the spin-vector — an ambiguity that cannot be removed. The object is a *null flag* and we will closely follow [11] in its construction.

We start with considering an abstract two-dimensional sphere that represents null directions. A (future) null direction specifies a point on the sphere, thus it determines a coordinate $\zeta = \frac{\xi}{\eta}$. We will assign one more geometric object to a spin-vector: a real vector $\mathbf{L}$ tangent to the sphere. These two objects — null direction and a vector tangent to the sphere — will together determine components ξ, η up to the overall sign, and will constitute basis for a structure of a null flag. A suitable tangent vector $\mathbf{L}$ needs to meet three criteria:

- Since a vector is real and tangent to the sphere, its form is

$$\mathbf{L} = \lambda \frac{\partial}{\partial \zeta} + \overline{\lambda} \frac{\partial}{\partial \overline{\zeta}}. \tag{1.26}$$

- Because the purpose of our construction is to "geometrize" a spin-vector, it has to be a function of components ξ and η, i.e. $\lambda = \lambda(\xi, \eta)$.
- It needs to be a true geometric vector. Thus its components have to transform correctly under passive spin-transformations[9].

A function $\lambda(\xi, \eta)$ fixes $\mathbf{L}$ for each coordinate system. To make it consistent with the last point, we need to find $\lambda(\xi, \eta)$ that — when regarded as a component of a true vector — is invariant under spin-transformations, i.e. the transformed λ must be the same expression in transformed ξ, η as was the old λ in original ξ, η.

Let us inspect how restrictive our conditions are. Consider coordinate systems $\zeta = \frac{\xi}{\eta}$ and $\tilde{\zeta} = \frac{\tilde{\xi}}{\tilde{\eta}}$ that are mutually related by spin transformation

$$\tilde{\xi} = \alpha\xi + \beta\eta\,, \quad \tilde{\eta} = \gamma\xi + \delta\eta\,, \quad \tilde{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}\,. \tag{1.27}$$

Since $\mathbf{L}$ is vector, we must have

$$\lambda \frac{\partial}{\partial \zeta} + \overline{\lambda} \frac{\partial}{\partial \overline{\zeta}} = \tilde{\lambda} \frac{\partial}{\partial \tilde{\zeta}} + \overline{\tilde{\lambda}} \frac{\partial}{\partial \overline{\tilde{\zeta}}}\,. \tag{1.28}$$

From (1.27) we can find the relation between $\frac{\partial}{\partial \zeta}$ and $\frac{\partial}{\partial \tilde{\zeta}}$. We arrive at

$$\begin{aligned}
\frac{\partial}{\partial \zeta} &= \frac{\partial \tilde{\zeta}}{\partial \zeta}\frac{\partial}{\partial \tilde{\zeta}} = \left[\frac{\partial}{\partial \zeta}\left(\frac{\alpha\zeta + \beta}{\gamma\zeta \ + \delta}\right)\right]\frac{\partial}{\partial \tilde{\zeta}} = \left(\frac{\alpha(\gamma\zeta + \delta)}{(\gamma\zeta + \delta)^2} - \frac{\gamma(\alpha\zeta + \beta)}{(\gamma\zeta + \delta)^2}\right)\frac{\partial}{\partial \tilde{\zeta}} \\
&= \frac{\alpha\delta - \beta\gamma}{(\gamma\zeta + \delta)^2}\frac{\partial}{\partial \tilde{\zeta}} = \frac{1}{(\gamma\zeta + \delta)^2}\frac{\partial}{\partial \tilde{\zeta}} = \frac{\eta^2}{\tilde{\eta}^2}\frac{\partial}{\partial \tilde{\zeta}},
\end{aligned} \tag{1.29}$$

because $\alpha\delta - \beta\gamma = 1$ by the requirement that (1.27) is spin transformation. Substituting (1.29) in (1.28) we have

$$\frac{\eta^2}{\tilde{\eta}^2}\lambda \frac{\partial}{\partial \tilde{\zeta}} + \frac{\overline{\eta}^2}{\overline{\tilde{\eta}}^2}\overline{\lambda} \frac{\partial}{\partial \overline{\tilde{\zeta}}} = \tilde{\lambda} \frac{\partial}{\partial \tilde{\zeta}} + \overline{\tilde{\lambda}} \frac{\partial}{\partial \overline{\tilde{\zeta}}}\,. \tag{1.30}$$

Coefficients standing by $\frac{\partial}{\partial \tilde{\zeta}}$ on both sides of the last equation must be equal (and the same holds true for coefficients standing by $\frac{\partial}{\partial \overline{\tilde{\zeta}}}$). This yields

$$\eta^2 \lambda = \tilde{\eta}^2 \tilde{\lambda}\,. \tag{1.31}$$

Thus λ must be proportional to $\frac{1}{\eta^2}$ and we choose

$$\mathbf{L} = -\frac{1}{\sqrt{2}\eta^2}\frac{\partial}{\partial \zeta} - \frac{1}{\sqrt{2}\overline{\eta}^2}\frac{\partial}{\partial \overline{\zeta}}\,. \tag{1.32}$$

[9]The whole construction relies on the structure of the Minkowski space, therefore we restrict ourselves to Lorentz transformations.

Vector $\mathbf{L}$ that we have found lies in the tangent bundle of the abstract two-sphere of null directions. But since we can identify this sphere with an intersection of the null cone and a hyperplane $T = 1$, we can also assign a four-vector of Minkowski vector-space to the vector $\mathbf{L}$. We will denote both the vector in the abstract space of null directions and its image in the Minkowski vector-space by the same symbol $\mathbf{L}$. This four-vector $\mathbf{L}$ is tangent to the sphere lying in the hyperplane $T = 1$, and therefore it is space-like and orthogonal to the null direction given by ζ.

So far we have associated two objects with a spin-vector: a null vector $\mathbf{K}$ given by (1.16) and a space-like vector $\mathbf{L}$ specified by (1.32). Both are fully determined by the spin-vector (ξ, η). Conversely, given vectors $\mathbf{K}$ and $\mathbf{L}$, we can find values of ξ and η up to the sign ambiguity. For a null direction given by $\mathbf{K}$ determines a ratio $\zeta = \frac{\xi}{\eta}$ and from $\mathbf{L}$ we can deduce η^2. Hence we know all of the values ξ^2, ξ/η and η^2.

Notice that while $\mathbf{L}$ of abstract space of null directions is Lorentz invariant, its image in the Minkowski space is not. We can easily see this, once we recall that, by construction, a four-vector $\mathbf{L}$ lies in hyperplane $T = 1$, which is not Lorentz invariant. Hence a need for a concept of null flag arises. Consider a null half-plane Π given by set of vectors

$$a\,\mathbf{K} + b\,\mathbf{L}, \quad a,\, b \in \mathbb{R},\, b \geq 0\,. \tag{1.33}$$

We will call this half-plane a null flag. It is determined by a null vector $\mathbf{K}$ — its *flagpole* — and a space-like vector $\mathbf{L}$ orthogonal to the $\mathbf{K}$. Conversely, a null flag uniquely fixes a null vector $\mathbf{K}$ and determines a space-like vector $\mathbf{L}$ up to the transformation $\mathbf{L} \mapsto k_1\mathbf{L} + k_2\mathbf{K}, k_2 \geq 0$, thus fixing $\mathbf{L}$ of abstract sphere uniquely. Hence, a null flag is capable of representing a spin-vector no less than a pair of vectors $\mathbf{K}$ and $\mathbf{L}$. Moreover, a null flag is invariant under the Lorentz transformations. Indeed, recall that a four-vector $\mathbf{L}$ corresponds to a vector tangent to the abstract sphere of null directions and therefore to two infinitesimally close null directions. Half-plane Π is clearly determined by these two directions.

Let us now return to the phase transformation (1.8):

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto e^{\mathrm{i}\phi} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \phi \in \mathbb{R}.$$

Since the four-vector $\mathbf{K}$ is not altered by this transformation, its effect on null flag is determined by its effect on vector $\mathbf{L}$. So how does the vector $\mathbf{L}$ transform under (1.8)? Consider first a situation with the vector given by (1.32) that lies in an Argand-Gauss plane of complex numbers, with ζ being its standard coordinate. Because multiplication $\zeta \mapsto \exp(\mathrm{i}\theta)\zeta$ results in rotation of the Argand plane (through angle θ), we can see that transformation $(\xi, \eta) \mapsto \exp(\mathrm{i}\phi)(\xi, \eta)$ results in rotation of that vector through an angle 2ϕ.[10] Correspondence between Argand plane and the abstract sphere of null directions is provided by stereographic projection, and the stereographic projection is conformal[11]. Hence $\mathbf{L}$ is also rotated through angle 2ϕ under the phase transformation (1.8).

Now, apply the phase transformation (1.8) and starting from $\phi = 0$ increase ϕ gradually. Once ϕ attains value $\phi = \pi$, the spin-vector is transformed into its negative, while the null flag revolves through the whole circle. It takes one more rotation of the null flag to transform the spin-vector into its original form. Phase transformation (1.8) corresponds to an actual spin transformation: a rotation through an angle 2ϕ about the axis given by vector $\mathbf{K}$, as can be most easily seen from (1.21). Thus we can see that no "ordinary" geometric object is capable of representing spin-vector completely, for a rotation through an angle 2π leaves that object

[10] Components of $\mathbf{L}$ are quadratic in η^{-1}.

[11] Conformality is well known property of stereographic projection, although we have not shown it to be true in this introduction. We refer an interested reader to [11].

unchanged, while the spin-vector is transformed into its negative. We will refer to the quantities that must be rotated through 4π to return to their original state as *spinorial objects*. A well known example of such objects are fermions.

The property that rotation through 2π is not fully equivalent to the identity stems from topology of proper rotations in Euclidean 3-space. Euler proved that any such rotation can be written as a rotation about some fixed axis[12]. Therefore we can represent any proper rotation by a vector in Euclidean 3-space: the direction of the vector will determine the axis of the rotation, while the length of the vector will determine the angle through which the space is rotated. But because rotations through angle θ and through $\theta + 2\pi$ result in the same transformation, we need to identify vectors of the same direction the (oriented) length of which does differ by a multiple of 2π. Hence the set of all proper rotations can be visualized as a closed ball of radius π centered at the origin. And since the rotation corresponding to the vector $\mathbf{v}$ of length π is the same as the one corresponding to the vector $-\mathbf{v}$, points opposite on the boundary must be considered identical.

A closed curve lying in the ball represents a continuous rotation that returns to the original rotation (original orientation). It may cross the boundary and in such a case it returns inside the ball on the opposite side. We can deform the curve by a continuous deformation. The number of points in which the curve crosses the boundary may then change, but they appear or disappear only in pairs (see Figure 1.3). Therefore there are two classes of closed curves in the space of proper rotations: those that cross the boundary in an even number of points and those that have odd number of intersections with the boundary. Any closed curve can be continuously deformed into any other closed curve of the same class, but it is not possible to deform it into a curve of the other class. A rotation through 2π intersects the boundary once and therefore cannot be deformed into the identity rotation, while a rotation through 4π crosses the boundary two times and thus is continuous with the identity.

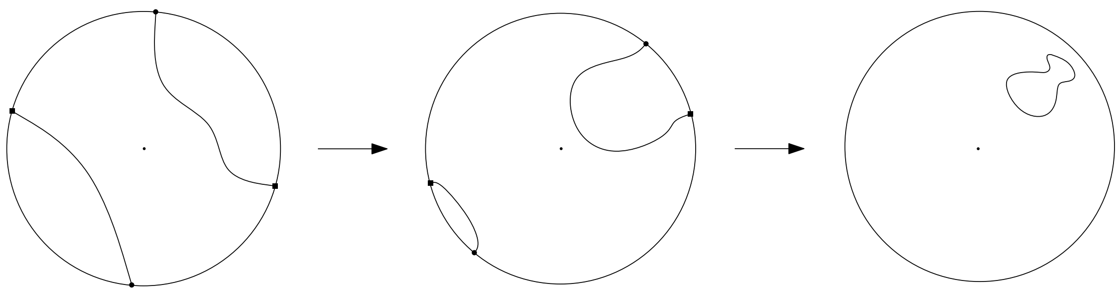

Figure 1.3: Consider the closed curve in the leftmost picture above. It intersects the boundary in two points. We can move the curve inside the ball by the means of continuous deformation, as is illustrated in the other two pictures. Then it is clearly possible to deform it further in order to make it arbitrarily close to the identity rotation. Notice that points of crossing disappear in pair, i.e. there are two such points in the first two pictures, and zero in the third picture.

[12]The terminology may appear little confusing. We use a word rotation to label a certain automorphism of Euclidean space, and a rotation $\mathbf{R}$, when applied to some geometric object, therefore determines only its *final* position. In common language term rotation may also refer to a continuous movement. To describe such a movement we need a map $t \mapsto \mathbf{R}(t)$. Euler's theorem states that for any such continuous rotation there is another continuous rotation about a fixed axis.

1.5 Operations on spinors

In this section we will define basic operations on spinors. Space of spin-vectors is a vector space over complex numbers, therefore among basic operations to consider there are scalar multiplication of a spin-vector, addition of two spin-vectors and some kind of an inner product. We impose one requirement on these operations: we want them to have geometric meaning, i.e. to be coordinate independent. Thus we require them to be invariant under passive spin transformations. In the case of the first two operations it is straightforward to both choose their definition and to check the invariance: given the spin-vectors $\boldsymbol{\kappa}$ and $\boldsymbol{\omega}$ with components κ^0, κ^1 and ω^0, ω^1,

$$\begin{aligned}\boldsymbol{\kappa} &= (\kappa^0, \kappa^1)\,,\\ \boldsymbol{\omega} &= (\omega^0, \omega^1)\,,\end{aligned}$$

and a complex number λ, we define scalar multiplication and addition simply by

$$\lambda\boldsymbol{\kappa} = \lambda(\kappa^0, \kappa^1) = (\lambda\kappa^0, \lambda\kappa^1)\,, \tag{1.34}$$

$$\boldsymbol{\kappa} + \boldsymbol{\omega} = (\kappa^0, \kappa^1) + (\omega^0, \omega^1) = (\kappa^0 + \omega^0, \kappa^1 + \omega^1)\,. \tag{1.35}$$

Covariance under spin transformations

$$\begin{pmatrix} \kappa^0 \\ \kappa^1 \end{pmatrix} \mapsto \begin{pmatrix} \kappa^{\hat{0}} \\ \kappa^{\hat{1}} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \kappa^0 \\ \kappa^1 \end{pmatrix}$$

follows from the linearity of matrix multiplication:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \lambda\kappa^0 \\ \lambda\kappa^1 \end{pmatrix} = \lambda \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \kappa^0 \\ \kappa^1 \end{pmatrix} \right],$$

and likewise for the operation of addition.

A search for a suitable inner product would be little more tricky. We will simply check that the choice

$$\{\boldsymbol{\kappa}, \boldsymbol{\omega}\} = \{(\kappa^0, \kappa^1), (\omega^0, \omega^1)\} = \kappa^0\omega^1 - \kappa^1\omega^0 \tag{1.36}$$

yields the desired invariance, as follows from the fact that determinant of a spin-matrix equals one:

$$\kappa^{\hat{0}}\omega^{\hat{1}} - \kappa^{\hat{1}}\omega^{\hat{0}} = \det \begin{pmatrix} \kappa^{\hat{0}} & \omega^{\hat{0}} \\ \kappa^{\hat{1}} & \omega^{\hat{1}} \end{pmatrix} = \left| \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \kappa^0 & \omega^0 \\ \kappa^1 & \omega^1 \end{pmatrix} \right| = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \begin{vmatrix} \kappa^0 & \omega^0 \\ \kappa^1 & \omega^1 \end{vmatrix} = \kappa^0\omega^1 - \kappa^1\omega^0\,. \tag{1.37}$$

We will expand on these results in the next sections dedicated to the spinor algebra, but before we do so, we need to discuss the notation that this work follows.

1.6 Abstract-index notation

Classical approach to the tensor formalism employs notation where tensors are represented by their components. For example an array V^α represents a (contravariant) vector. Index α takes values from 1 to n, where n stands for dimension of the considered space, and for each such

α, the number[13] V^α is α-th component of the vector. Disadvantage of such formalism is that tensors can only be accessed via their components. A more modern, coordinate-free approach deals with tensors directly, i.e. it denotes the aforementioned vector simply as $\mathbf{V}$. The problem with this formalism is that it lacks an ability to effectively deal with various index permutations and contractions, something at which the classical approach is very efficient.

The formalism used in this thesis is that of the abstract indices. It retains the functionality of the classical approach while also allows us to deal with tensors directly. Formally the notation is nearly identical to the classical one, but the symbol V^α represents the vector itself, not its components. Symbols with other indices are also needed. But while symbols V^α and V^β correspond to the same vector $\mathbf{V}$, we do not want them to represent the very same mathematical object. For then identities like $V^\alpha = V^\beta$ would hold, rendering the formalism unusable. Even if we were to introduce rules forbiding substitutions like $V^\alpha V_\alpha = V^\beta V_\alpha$, other problems would remain. For example antisymmetric product $V^\alpha W^\beta - V^\beta W^\alpha$ would identically be null. Therefore we create several distinct copies $\mathcal{V}^\alpha$, $\mathcal{V}^\beta$, $\mathcal{V}^\gamma$... for each relevant vector space $\mathcal{V}^\circ$. Vectors V^α, W^α... belong to the space $\mathcal{V}^\alpha$, while V^β, W^β... belong to $\mathcal{V}^\beta$ and so on. This way we achieve that although V^α and V^β correspond to the same vector $\mathbf{V}$, they are different objects. Similarly we create copies $\mathcal{V}_\alpha$, $\mathcal{V}_\beta$, $\mathcal{V}_\gamma$... for the space dual to $\mathcal{V}^\circ$. Then we construct copies of tensor spaces of higher valence.

While it is convenient to be able to work with tensors directly, we often wish to use components anyway. Let us denote vectors of some chosen basis by $\delta^\alpha_{\mathbf{i}}$, with the bold Latin index distinguishing between different vectors of the basis and the normal Greek index used to determine the copy of the relevant vector space[14]. Then, using usual summation convention for bold Latin indices, we can express any vector $\mathbf{V}$ from that vector space as

$$V^\alpha = V^{\mathbf{i}} \delta^\alpha_{\mathbf{i}} \,, \tag{1.38}$$

where $V^{\mathbf{i}}$ are components of the vector $\mathbf{V}$ with the respect to the basis $\delta^\alpha_{\mathbf{i}}$. Basis dual to $\delta^\alpha_{\mathbf{i}}$ consists of covariant vectors $\delta^{\mathbf{i}}_\alpha$ satisfying

$$\delta^{\mathbf{i}}_\alpha V^\alpha = V^{\mathbf{i}} \,, \tag{1.39}$$

for any $V^\alpha \in \mathcal{V}^\alpha$. Particularly, we have

$$\delta^{\mathbf{i}}_\alpha \delta^\alpha_{\mathbf{j}} = \delta^{\mathbf{i}}_{\mathbf{j}}. \tag{1.40}$$

Kronecker delta symbol $\delta^{\mathbf{i}}_{\mathbf{j}}$ equals one whenever $\mathbf{i} = \mathbf{j}$ and is zero otherwise.

Now consider some other basis $\delta^\alpha_{\hat{\mathbf{i}}}$ for the same vector space $\mathcal{V}^\alpha$. With the help of the dual base $\delta^{\hat{\mathbf{i}}}_\alpha$ we can obtain components $V^{\hat{\mathbf{i}}} = V^\alpha \delta^{\hat{\mathbf{i}}}_\alpha$. Similarly, we may express components of $\delta^\alpha_{\mathbf{i}}$ with respect to $\delta^\alpha_{\hat{\mathbf{i}}}$ and components of $\delta^\alpha_{\hat{\mathbf{i}}}$ with respect to the basis $\delta^\alpha_{\mathbf{i}}$ and arrive at

$$\delta^{\hat{\mathbf{i}}}_{\mathbf{i}} = \delta^\alpha_{\mathbf{i}} \delta^{\hat{\mathbf{i}}}_\alpha \,, \quad \delta^{\mathbf{i}}_{\hat{\mathbf{i}}} = \delta^\alpha_{\hat{\mathbf{i}}} \delta^{\mathbf{i}}_\alpha \,. \tag{1.41}$$

These two matrices are useful when we wish to compute components of a tensor with respect to one basis from the components with respect to the other basis. Consider tensor $A^\alpha_{\beta\gamma}$, we have

$$A^{\mathbf{i}}_{\mathbf{jk}} \delta^\alpha_{\mathbf{i}} \delta^{\mathbf{j}}_\beta \delta^{\mathbf{k}}_\gamma = A^\alpha_{\beta\gamma} = A^{\hat{\mathbf{i}}}_{\hat{\mathbf{j}}\hat{\mathbf{k}}} \delta^\alpha_{\hat{\mathbf{i}}} \delta^{\hat{\mathbf{j}}}_\beta \delta^{\hat{\mathbf{k}}}_\gamma \,. \tag{1.42}$$

Contracting with $\delta^{\hat{\mathbf{l}}}_\alpha \delta^\beta_{\hat{\mathbf{m}}} \delta^\gamma_{\hat{\mathbf{n}}}$ (and renaming indices) we get

$$A^{\hat{\mathbf{i}}}_{\hat{\mathbf{j}}\hat{\mathbf{k}}} = A^{\mathbf{i}}_{\mathbf{jk}} \delta^{\hat{\mathbf{i}}}_{\mathbf{i}} \delta^{\mathbf{j}}_{\hat{\mathbf{j}}} \delta^{\mathbf{k}}_{\hat{\mathbf{k}}} \,. \tag{1.43}$$

[13] Or the function of the coordinates, if we are concerned with vector *fields*.

[14] Thus the basis consists of vectors δ^α_1, δ^α_2,..., δ^α_n, where n stands for dimension of vector space $\mathcal{V}^\alpha$.

1.7 Spinor algebra

After short interlude on abstract indices we can resume our discussion of spinor algebra. We will denote the vector space of spin-vectors by $\mathcal{G}^\circ$ and use non-bold uppercase Latin letters to indicate copies of that space, i.e. we will use symbols $\mathcal{G}^A$, $\mathcal{G}^B$ etc. for copies of $\mathcal{G}^\circ$. To denote components, bold uppercase Latin letters will be used. Symbol κ^A will therefore denote a spin-vector from the space $\mathcal{G}^A$ and for its components we will write $\kappa^{\mathbf{A}}$, where $\mathbf{A}$ takes values from $\{0,1\}$. As usually, lowered indices will indicate covariant spinors.

In an earlier section we have found the Lorentz invariant inner product of two spin-vectors given by

$$\{\boldsymbol{\kappa},\boldsymbol{\omega}\} = \begin{vmatrix} \kappa^0 & \omega^0 \\ \kappa^1 & \omega^1 \end{vmatrix} . \tag{1.44}$$

It is an antisymmetric bilinear map from $\mathcal{G}^\circ\times\mathcal{G}^\circ$ into complex numbers $\mathbb{C}$. As such, it provides us with a natural correspondence between the space of spin-vectors $\mathcal{G}^\circ$ and its dual, since it allows us to assign a map $\{\boldsymbol{\kappa},\circ\} : \boldsymbol{\omega}\mapsto\{\boldsymbol{\kappa},\boldsymbol{\omega}\}$ to any spin-vector $\boldsymbol{\kappa}$. Because the map $\{\boldsymbol{\kappa},\circ\}$ is linear in its argument it belongs to the space $\mathcal{G}_\circ$. Using index notation we will denote both the map $\{\boldsymbol{\kappa},\circ\}$ and its preimage κ^A by the same kernel letter κ, i.e. $\{\boldsymbol{\kappa},\circ\} = \kappa_A$ and $\{\boldsymbol{\kappa},\boldsymbol{\omega}\} = \kappa_A\omega^A$.

Since the inner product itself is a multilinear map, it corresponds to a spinor of valence $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$. We will denote it by Greek letter ϵ:

$$\{\boldsymbol{\kappa},\boldsymbol{\omega}\} = \epsilon_{AB}\kappa^A\omega^B . \tag{1.45}$$

Because the inner product is antisymmetric, we have $\epsilon_{AB} = -\epsilon_{BA}$. The relation between spinors κ^A and κ_A defined in the last paragraph can be expressed in the form

$$\kappa_B = \epsilon_{AB}\kappa^A . \tag{1.46}$$

Due to the antisymmetry of ϵ_{AB} we need to be careful about what indices we contract through. When we contract through the second index we get $\epsilon_{AB}\kappa^B = -\epsilon_{BA}\kappa^B = -\kappa_A$, instead of κ_A.

To find a formula that relates the components of spin-vector κ^A to components of κ_A, we use (1.44):

$$\begin{aligned} \{\boldsymbol{\kappa},\boldsymbol{\omega}\} &= \kappa^0\omega^1 - \kappa^1\omega^0 \\ = \kappa_A\omega^A &= \kappa_{\mathbf{A}}\omega^{\mathbf{A}} = \kappa_0\omega^0 + \kappa_1\omega^1 . \end{aligned} \tag{1.47}$$

Thus we have $\kappa_0 = -\kappa^1$ and $\kappa_1 = \kappa^0$. This result shows that a map $\kappa^A \mapsto \epsilon_{AB}\kappa^A$ is one-to-one. Consequently, there must be the inverse map from $\mathcal{G}_B$ to $\mathcal{G}^A$. Since that map is clearly linear, there exists a spinor ϵ^{AB} that effects it:

$$\epsilon^{AB}\kappa_B = \kappa^A . \tag{1.48}$$

We expect ϵ^{AB} to be antisymmetric. To show that, we use the antisymmetry of ϵ_{AB}. We have

$$\epsilon^{AB}\kappa_A\omega_B = \kappa_A\omega^A = \epsilon_{BA}\kappa^B\omega^A = -\epsilon_{AB}\kappa^B\omega^A = -\kappa^B\omega_B = -\epsilon^{BA}\kappa_A\omega_B \tag{1.49}$$

for arbitrary spinors κ_A and ω_B, which proves the desired. Now, substituting from (1.46) for κ_B into (1.48), and similarly, substituting from (1.48) into (1.46), we get two equations:

$$\kappa^A = \epsilon^{AC}\kappa_C = \epsilon^{AC}\epsilon_{BC}\kappa^B = \delta^A_B\kappa^B , \tag{1.50}$$

$$\kappa_B = \epsilon_{CB}\kappa^C = \epsilon_{CB}\epsilon^{CA}\kappa_A = \delta^A_B\kappa_A , \tag{1.51}$$

where δ_B^A of the first equation is the canonical isomorphism from $\mathcal{G}^B$ into $\mathcal{G}^A$ and δ_B^A is the canonical isomorphism between $\mathcal{G}_A$ and $\mathcal{G}_B$. One can easily check that, as the notation suggests, both spinors are actually the same[15].

We have seen that spinors ϵ_{AB} and ϵ^{AB} can be used for lowering and raising the spinor indices. In accordance with that, we can write $\delta_A^B = \epsilon_{AC}\epsilon^{BC} = \epsilon_{CA}\epsilon^{CB} = \epsilon_A{}^B$. Thus we can consider $\epsilon_A{}^B$ to be either ϵ_{AB} with the second index raised or ϵ^{AB} with the first index lowered. Subsequently, we may consider ϵ^{AB} to be ϵ_{AB} with both indices raised. In the following we shall often use symbol $\epsilon_A{}^B$ instead of δ_A^B.

At the first glance, the rules (1.46) and (1.48) for lowering and raising of spinor indices may seem rather confusing. It may be therefore worthwhile to pause here and introduce some mnemonics to ease the remembering of the correct rules. First observe, that due to the antisymmetry of $\boldsymbol{\epsilon}$ a general spinor changes the sign whenever we switch positions of contracted indices. The mechanism behind this is the same as in (1.49), i.e. for a general spinor $\xi_A{}^{BC}$ we have $\xi_A{}^{AC} = \epsilon_{DA}\xi^{DAC} = -\epsilon_{AD}\xi^{DAC} = -\xi^D{}_D{}^C$. This means that it is sufficient to remember that $\epsilon_A{}^B = \delta_A^B$. Then it is easy to reconstruct the correct rules. For example, we may proceed as follows:

$$\epsilon^{AB}\xi_{CA}{}^D = -\epsilon_A{}^B\xi_C{}^{AD} = -\delta_A^B\xi_C{}^{AD} = -\xi_C{}^{BD}\,. \tag{1.52}$$

It may also be helpful to observe that indices of ϵ_{AB} function in a sense inversely than those of ϵ^{AB}. When we use ϵ_{AB} to lower the index of some spinor ξ^{AC}, we get $\epsilon_{AB}\xi^{AC} = \xi_B{}^C$. Notice that the index we lower is the same as the first index of ϵ_{AB} and we rewrite it as B, which is the second index of ϵ_{AB}. On the other hand, when raising indices, as in $\epsilon^{AB}\xi_B{}^C = \xi^{AC}$, we raise the index that is the same as the second index of ϵ^{AB} and then we 'change' it into A, the first index of ϵ^{AB}. Thus we only need to remember that in ϵ_{AB} "A (of the effected spinor is lowered) and goes into B ", while in ϵ^{AB} "index B goes into A ". If we imagine a circle around ϵ, both these movements are in the direction of its positive rotation.

Let us now turn our attention back to equation (1.44). The inner product given by that relation satisfies the Jacobi identity

$$\{\boldsymbol{\kappa},\boldsymbol{\omega}\}\boldsymbol{\tau} + \{\boldsymbol{\omega},\boldsymbol{\tau}\}\boldsymbol{\kappa} + \{\boldsymbol{\tau},\boldsymbol{\kappa}\}\boldsymbol{\omega} = 0\,, \tag{1.53}$$

as can be seen by Laplace expansion of

$$\begin{vmatrix} \tau^0 & \kappa^0 & \omega^0 \\ \tau^1 & \kappa^1 & \omega^1 \\ \tau^{\mathbf{A}} & \kappa^{\mathbf{A}} & \omega^{\mathbf{A}} \end{vmatrix} = 0\,, \quad \mathbf{A} = 0,1\,, \tag{1.54}$$

with respect to the last row. As a corrolary to (1.53), we see that any pair of spin-vectors whose inner product is nonzero form a basis. For example, if $\boldsymbol{\kappa}$ and $\boldsymbol{\omega}$ satisfy $\{\boldsymbol{\kappa},\boldsymbol{\omega}\} \neq 0$, then the Jacobi identity (1.53) shows how to express any other spin-vector $\boldsymbol{\tau}$ as their linear combination.

We shall refer to a pair of spin-vectors that form a basis as a dyad and usually denote those spinors by letters $\boldsymbol{o}$ and $\boldsymbol{\iota}$ (omikron and iota). A dyad normalized so that $o_A\iota^A = 1$ will be called a spin-frame. Let κ^0 and κ^1 be components of a spin-vector $\boldsymbol{\kappa}$ with respect to the spin-frame $(\boldsymbol{o},\,\boldsymbol{\iota})$:

$$\kappa^A = \kappa^0 o^A + \kappa^1 \iota^A\,. \tag{1.55}$$

Transvecting the last equation with ι_A and o_A we obtain relations

$$\kappa^0 = -\kappa^A\iota_A \quad \text{and} \quad \kappa^1 = \kappa^A o_A\,. \tag{1.56}$$

[15] δ_B^A (of the first equation) $= \epsilon^{AC}\epsilon_{BC} = (-\epsilon^{CA})(-\epsilon_{CB}) = \delta_B^A$ (of the second equation)

To continue the discussion we introduce a rather useful new symbol $\epsilon_{\mathbf{A}}{}^{A}$ which collectively denotes spin-vectors of the dyad:

$$\epsilon_0{}^A = o^A\,, \quad \epsilon_1{}^A = \iota^A\,. \tag{1.57}$$

Using this notation we can rewite (1.55) as[16]

$$\kappa^A = \kappa^{\mathbf{A}}\epsilon_{\mathbf{A}}{}^A\,. \tag{1.58}$$

We write $\epsilon_A{}^{\mathbf{A}}$ for the basis dual to $\epsilon_{\mathbf{A}}{}^A$. By definition, it satisfies

$$\epsilon_A{}^{\mathbf{B}}\epsilon_{\mathbf{A}}{}^A = \delta_{\mathbf{A}}^{\mathbf{B}} = \epsilon_{\mathbf{A}}{}^{\mathbf{B}}\,, \tag{1.59}$$

where we define $\epsilon_{\mathbf{A}}{}^{\mathbf{B}}$ to be an equivalent of the Kronecker delta. Solving the last equation we arrive at

$$\epsilon_A{}^0 = -\iota_A\,, \quad \epsilon_A{}^1 = o_A\,, \tag{1.60}$$

or, in a case of general dyad when $\{\boldsymbol{o}, \boldsymbol{\iota}\} = \chi \neq 0$,

$$\epsilon_A{}^0 = -\frac{1}{\chi}\iota_A\,, \quad \epsilon_A{}^1 = \frac{1}{\chi}o_A\,. \tag{1.61}$$

The notation we have just introduced allows us to write

$$\xi_{\mathbf{A}}{}^{\mathbf{BC}} = \xi_A{}^{BC}\epsilon_{\mathbf{A}}{}^A\epsilon_B{}^{\mathbf{B}}\epsilon_C{}^{\mathbf{C}} \tag{1.62}$$

for the components of a general spinor $\xi_A{}^{BC}$. Applying this to spinors ϵ_{AB} and ϵ^{AB} yields

$$\begin{aligned}
\epsilon_{AB}\epsilon_{\mathbf{A}}{}^A\epsilon_{\mathbf{B}}{}^B &= \epsilon_{\mathbf{A}B}\epsilon_{\mathbf{B}}{}^B = \epsilon_{\mathbf{AB}} = \begin{pmatrix} 0 & \chi \\ -\chi & 0 \end{pmatrix}, \\
\epsilon^{AB}\epsilon_A{}^{\mathbf{A}}\epsilon_B{}^{\mathbf{B}} &= \epsilon_A{}^{\mathbf{A}}\epsilon^{A\mathbf{B}} = \epsilon^{\mathbf{AB}} = \begin{pmatrix} 0 & \frac{1}{\chi} \\ -\frac{1}{\chi} & 0 \end{pmatrix},
\end{aligned} \tag{1.63}$$

where $\chi = 1$ in the case of spin-frame. Similarly, we may use $\epsilon_{\mathbf{A}}{}^{\mathbf{B}} = \epsilon_A{}^B\epsilon_{\mathbf{A}}{}^A\epsilon_B{}^{\mathbf{B}} = \epsilon_{\mathbf{A}}{}^A\epsilon_A{}^{\mathbf{B}}$, thus obtaining $\epsilon_{\mathbf{A}}{}^{\mathbf{B}} = \delta_{\mathbf{A}}^{\mathbf{B}}$ in accordance with our earlier definition.

For the relation (1.58) or its generalization

$$\xi_A{}^{BC} = \xi_{\mathbf{A}}{}^{\mathbf{BC}}\epsilon_A{}^{\mathbf{A}}\epsilon_{\mathbf{B}}{}^B\epsilon_{\mathbf{C}}{}^C \tag{1.64}$$

to be consistent with (1.62), the following relation must hold:

$$\epsilon_{\mathbf{A}}{}^B\epsilon_A{}^{\mathbf{A}} = \delta_A^B\,. \tag{1.65}$$

This is just a condition for completeness of the basis $\epsilon_{\mathbf{A}}{}^A$.

We can employ matrices $\epsilon_{\mathbf{AB}}$ and $\epsilon^{\mathbf{AB}}$ to lower or raise bold (component) indices. We may also freely contract over bold indices. Consistency is ensured again by relations (1.59) and (1.65). For example we have

$$\xi^A\eta_A = \left(\xi^{\mathbf{A}}\epsilon_{\mathbf{A}}{}^A\right)\eta_A = \xi^{\mathbf{A}}\left(\epsilon_{\mathbf{A}}{}^A\eta_A\right) = \xi^{\mathbf{A}}\eta_{\mathbf{A}}\,,$$

[16] Remember that we sum over the identical bold indices.

or

$$\left(\xi_A \epsilon^{AB}\right) \epsilon_B{}^{\mathbf{B}} = \xi_{\mathbf{A}} \epsilon_A{}^{\mathbf{A}} \epsilon^{AB} \epsilon_B{}^{\mathbf{B}} = \xi_{\mathbf{A}} \epsilon^{\mathbf{AB}} \,.$$

Let us now return back to the Jacobi identity (1.53). Using the abstract index notation the equation acquires the form

$$\epsilon_{AB} \epsilon_C{}^D \kappa^A \omega^B \tau^C + \epsilon_{BC} \epsilon_A{}^D \omega^B \tau^C \kappa^A + \epsilon_{CA} \epsilon_B{}^D \tau^C \kappa^A \omega^B = 0 \,. \tag{1.66}$$

Because the above relation holds true for arbitrary spin-vectors $\boldsymbol{\kappa}$, $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$, spinor $\boldsymbol{\epsilon}$ must satisfy the following equation:

$$\epsilon_{AB} \epsilon_C{}^D + \epsilon_{BC} \epsilon_A{}^D + \epsilon_{CA} \epsilon_B{}^D = 0 \,. \tag{1.67}$$

After we contract the previous equation with ϵ^{EB} and rename a few indices, we arrive at

$$\epsilon_A{}^C \epsilon_B{}^D - \epsilon_B{}^C \epsilon_A{}^D = \epsilon_{AB} \epsilon^{CD} \,. \tag{1.68}$$

Applying this result on arbitrary spinor χ_{CD} we find that antisymmetric part of any such spinor is proportional to ϵ_{CD}:[17]

$$\frac{1}{2} \left(\chi_{AB} - \chi_{BA}\right) = \frac{1}{2} \chi_X{}^X \epsilon_{AB} \,. \tag{1.69}$$

Symmetry operations play an important role in spinor as well as tensor calculus. We shall use a common notation where round brackets symbolize symmetrization over the enclosed indices, for example:

$$\gamma_{AB}{}^{(C_1 C_2 \ldots C_n)} = \frac{1}{n!} \sum_\sigma \gamma_{AB}{}^{C_{\sigma(1)} C_{\sigma(2)} \ldots C_{\sigma(n)}} \,, \tag{1.70}$$

where the summation is taken over all $n!$ permutations of the set $\{1, \ldots, n\}$. Antisymmetrization is denoted by square brackets:

$$\gamma_{AB}{}^{[C_1 C_2 \ldots C_n]} = \frac{1}{n!} \sum_\sigma \operatorname{sgn}(\sigma) \gamma_{AB}{}^{C_{\sigma(1)} C_{\sigma(2)} \ldots C_{\sigma(n)}} \,. \tag{1.71}$$

We can enclose a group of indices in vertical bars to exclude them from the operation of symmetry. For example, the symbol $\beta_{AB}{}^{[C|DE|F]}$ denotes

$$\beta_{AB}{}^{CDEF} - \beta_{AB}{}^{FDEC} \,. \tag{1.72}$$

Now we can rewrite (1.69) as $\chi_{[AB]} = \frac{1}{2} \chi_X{}^X \epsilon_{AB}$. Since any spinor χ_{AB} can be decomposed into a sum of its symmetric and its antisymmetric part, we have

$$\chi_{AB} = \chi_{(AB)} + \chi_{[AB]} = \chi_{(AB)} + \frac{1}{2} \chi_X{}^X \epsilon_{AB} \,. \tag{1.73}$$

Thus we see that all infromation on spinor χ_{AB} is contained in two spinors $\chi_{(AB)}$ and $\frac{1}{2}\chi_X{}^X$, both of them symmetric. (The latter, being a scalar, can be considered symmetric, since it has no indices.) This is a special case of general fact that any spinor can be decomposed into

[17]This is the consequence of a general fact, that in a space of dimension n, any two n-forms (totally antisymmetric covariant tensors of valence n) may differ only by a common factor.

the sum of outer products of symmetric spinors with ϵs. To prove this[18] we first show that spinors[19] $\phi_{\mathcal{I}AB...F}$ and $\phi_{\mathcal{I}(AB...F)}$ differ only by the sum of outer products of ϵs with spinors of lower valence. We shall use a symbol '$\sim$' to denote such a relation, i.e. we wish to show that $\phi_{\mathcal{I}AB...F} \sim \phi_{\mathcal{I}(AB...F)}$. Clearly, the relation '$\sim$' defines an equivalence class. Therefore, it suffices to show that $\phi_{\mathcal{I}(AB...F)} \sim \phi_{\mathcal{I}A(B...F)}$ for any spinor $\phi_{\mathcal{I}AB...F}$, because then we can reiterate the argument to obtain $\phi_{\mathcal{I}(AB...F)} \sim \phi_{\mathcal{I}A(B...F)} \sim \phi_{\mathcal{I}AB(C...F)} \sim ... \sim \phi_{\mathcal{I}AB...F}$. To prove that $\phi_{\mathcal{I}(AB...F)} \sim \phi_{\mathcal{I}A(B...F)}$ we expand the symmetrization of $\phi_{\mathcal{I}(AB...F)}$ as follows:

$$\phi_{\mathcal{I}(AB...F)} = \frac{1}{r}\left(\phi_{\mathcal{I}A(BC...F)} + \phi_{\mathcal{I}B(AC...F)} + \phi_{\mathcal{I}C(AB...F)} + \dots + \phi_{\mathcal{I}F(AB...E)}\right) , \tag{1.74}$$

where r is the number of indices A,B, ... , F. Next we rewrite the right hand side as

$$\phi_{\mathcal{I}A(BC...F)} + \frac{1}{r}\left[\left(\phi_{\mathcal{I}B(AC...F)} - \phi_{\mathcal{I}A(BC...F)}\right) + \dots + \left(\phi_{\mathcal{I}F(AB...E)} - \phi_{\mathcal{I}A(BC...F)}\right)\right] . \tag{1.75}$$

Consider the term $\phi_{\mathcal{I}B(AC...F)} - \phi_{\mathcal{I}A(BC...F)}$. By (1.69) we have

$$\phi_{\mathcal{I}B(AC...F)} - \phi_{\mathcal{I}A(BC...F)} = \epsilon_{AB}\phi_{\mathcal{I}}{}^{X}{}_{(XC...F)} , \tag{1.76}$$

and we get a similar result for each other of such terms. This establishes that $\phi_{\mathcal{I}(AB...F)} \sim \phi_{\mathcal{I}A(B...F)}$ and repeating the procedure we get $\phi_{\mathcal{I}AB...F} \sim \phi_{\mathcal{I}(AB...F)}$. The difference $\phi_{\mathcal{I}AB...F} - \phi_{\mathcal{I}(AB...F)}$ consists of terms that are outer products of ϵs with spinors of lower valence. These spinors of lower valence are not necessarily symmetric. But because aforementioned argument applies to them as well, they can too be expressed as their symmetrized versions plus outer products of ϵs with spinors of lower valence. We can repeat the process until only symmetric spinors and ϵs remain. Thus we have proved that any spinor $\phi_{\mathcal{I}AB...F}$ is a sum of the symmetric spinor $\phi_{\mathcal{I}(AB...F)}$ and of outer products of ϵs with symmetric spinors of lower valence.

To illustrate this result we perform the decomposition for a spinor χ_{ABC}. Proceeding along the lines of the given proof, we first get

$$\chi_{(ABC)} = \frac{1}{3}\left(\chi_{A(BC)} + \chi_{B(AC)} + \chi_{C(AB)}\right) = \chi_{A(BC)} + \frac{1}{3}\left(\epsilon_{AB}\chi^{X}{}_{(XC)} + \epsilon_{AC}\chi^{X}{}_{(XB)}\right) .$$

We repeat the computation for the spinor $\chi_{A(BC)}$, arriving at

$$\chi_{A(BC)} = \frac{1}{2}\left(\chi_{ABC} + \chi_{ACB}\right) = \chi_{ABC} + \frac{1}{2}\epsilon_{BC}\chi_{A}{}^{X}{}_{X} . \tag{1.77}$$

Thus we have

$$\chi_{ABC} = \chi_{(ABC)} - \frac{1}{3}\epsilon_{AB}\chi^{X}{}_{(XC)} - \frac{1}{3}\epsilon_{AC}\chi^{X}{}_{(XB)} - \frac{1}{2}\epsilon_{BC}\chi_{A}{}^{X}{}_{X} . \tag{1.78}$$

1.8 World-tensors in spinor formalism

As we have seen at the beginning of this chapter, a general Hermitian matrix

$$V^{\mathbf{AB}} = \begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} \tag{1.79}$$

[18]The proof is taken from [11].

[19]Symbol $\mathcal{I}$ stands for set of indices which do not directly enter the following argument and therefore need not to be stated individually. We will call such a symbol, which stands for several indices, a composite index.

defines a four-vector V^α whose components are

$$V^{\mathbf{a}} = \begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix} . \tag{1.80}$$

We have also found that if a vector V^α is null, it is possible to partition the matrix $M_{\mathbf{AB}}$ into an outer product of two spin-vectors:

$$V^{\mathbf{AB}} = \begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} . \tag{1.81}$$

While it is tempting to simply write

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} = \kappa^{\mathbf{A}} \bar{\kappa}^{\mathbf{B}} \tag{1.82}$$

for the rightmost part of the last equation, we ought to be more careful. Clearly, an operation of complex conjugation needs to be included into the spinor calculus to make it capable of handling world-vectors. But are we able to simply add it as a map from $\mathcal{G}^\circ$ onto $\mathcal{G}^\circ$, the way equation (1.82) suggests? It turns out that such an operation would spoil the Lorentz invariance of the formalism. The problem is that complex conjugation does not treat all complex numbers equally — imaginary numbers are multiplied by -1 under complex conjugation, while real numbers are not affected at all — and that Lorentz transformations do not in general respect that structure. For example a rotation about the axis given by spin-vector's flagpole results in a phase transformation $\boldsymbol{\kappa} \mapsto e^{\mathrm{i}\frac{\phi}{2}} \boldsymbol{\kappa}$ of that spin-vector. Hence it is possible to transform a purely imaginary spin-vector into a real one. Surely, complex conjugation does not commute with such a transformation, thus violating the Lorentz symmetry.

We see that the complex conjugate of spin-vector $\kappa^A \in \mathcal{G}^A$ cannot be a quantity of the same type. We define a new space $\mathcal{G}^{A'}$ that consists of complex conjugates of elements from $\mathcal{G}^A$. Complex conjugation thus does not map $\mathcal{G}^A$ onto $\mathcal{G}^A$, but $\mathcal{G}^A$ onto $\mathcal{G}^{A'}$:

$$\overline{\kappa^A} = \bar{\kappa}^{A'} . \tag{1.83}$$

Applying the complex conjugation twice should result in the identity, i.e. $\overline{\overline{\kappa^A}} = \overline{\bar{\kappa}^{A'}} = \bar{\bar{\kappa}}^A = \kappa^A$. Therefore, complex conjugation applied on an element of $\mathcal{G}^{A'}$ results in an appropriate element of $\mathcal{G}^A$.

Operations of addition and scalar multiplication in $\mathcal{G}^{A'}$ are defined so that

$$\lambda\, \kappa^A + \mu\, \omega^A = \tau^A \iff \bar{\lambda}\, \bar{\kappa}^{A'} + \bar{\mu}\, \bar{\omega}^{A'} = \bar{\tau}^{A'} . \tag{1.84}$$

That means that complex conjugation defines an anti-isomorphism between spaces $\mathcal{G}^A$ and $\mathcal{G}^{A'}$.

We define the dual to $\mathcal{G}^{A'}$ and spaces of higher valence analogously as we did for $\mathcal{G}^A$. We also create copies of those spaces so that we can employ the abstract index formalism. Here we need to make sure that correspondences between elements of various spaces are chosen correctly. For example there is an isomorphism between $\mathcal{G}^{A'}$ and $\mathcal{G}^{B'}$, but there also is an anti-isomorphism between $\mathcal{G}^B$ and $\mathcal{G}^{B'}$ and we want them to be consistent so that any index substitution will commute with complex conjugation, i.e. $\overline{\kappa^B} = \bar{\kappa}^{B'}$ for each B. Then there is a complication due to the fact that e.g. $\mathcal{G}_{B'}$ arises both as a dual to $\mathcal{G}^{B'}$ and as a complex conjugate of $\mathcal{G}_B$. We fix these relations by the definition

$$\bar{\tau}_{X'} \bar{\kappa}^{X'} = \overline{\tau_X \kappa^X} , \tag{1.85}$$

where the bar on the right hand side denotes ordinary complex conjugation of scalars. As the equation (1.82) demonstrates, we will need to deal with spinors that possess both primed and unprimed indices. While it is important to preserve relative positions of primed indices as well as relative positions of unprimed idices, we do not need to uphold relative order between primed and unprimed indices due to the fact that we cannot substitute an unprimed index for primed one and neither the other way around. Thus we have $\chi_{ABC'} \neq \chi_{BAC'}$, while $\chi_{ABC'} = \chi_{AC'B}$.

Spinor spaces $\mathcal{G}^A$ and $\mathcal{G}^{A'}$ are isomorphic. When we consider them separately, i.e. we ignore how they are related by complex conjugation, they have the same properties. Therefore all the results on 'unprimed' spinors we got in previous sections hold for 'primed' spinors as well. Particularly, any spinor of primed indices can be decomposed as a sum of outer products of symmetric (primed) spinors with (primed) ϵs. Consequently, any spinor can be written as a sum where each term is an outer product of spinor symmetric in both its primed and unprimed indices and ϵs. For example for a spinor $\xi_{ABA'B'}$ we have

$$\xi_{ABA'B'} = \xi_{(AB)(A'B')} - \frac{1}{2}\epsilon_{AB}\xi^X{}_{X(A'B')} - \frac{1}{2}\epsilon_{A'B'}\xi_{(AB)}{}^{X'}{}_{X'} + \frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}\xi^X{}_X{}^{X'}{}_{X'} \,. \tag{1.86}$$

Now that we have learned how to deal with complex conjugation, we can return to our original discussion and rewrite the equation (1.81) as

$$V^{\mathbf{AA}'} = \begin{pmatrix} T+Z & X+\mathrm{i}Y \\ X-\mathrm{i}Y & T-Z \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \overline{\xi} & \overline{\eta} \end{pmatrix} = \kappa^{\mathbf{A}}\overline{\kappa}^{\mathbf{A}'} \,. \tag{1.87}$$

This suggests that even in a case of general Hermitian matrix (representing a general four-vector) we should use $\mathbf{AA}'$ instead of $\mathbf{AB}$ for its indices. Now we may regard both a list of numbers $V^{\mathbf{a}}$ and a Hermitian matrix $V^{\mathbf{AA}'}$ as different coordinate representations of the same object, the four-vector V^α. The matrix $V^{\mathbf{AA}'}$ does also represent a spinor $V^{AA'}$, so we may equate that spinor with the vector V^α. If we think of the index α as a composite index that stands for AA'[20], we may regard a tensor algebra as embedded in the spinor one. In a similar fashion, β may be considered to be an composite index standing for BB', γ an composite index for CC' and so forth. Spinors that can be rewritten using just these composite indices will be called *complex world-tensors.* An example of such a spinor would be

$$\sigma^{AA'BB'CC'}{}_{DD'EE'} = \sigma^{\alpha\beta\gamma}{}_{\delta\epsilon} \,. \tag{1.88}$$

The complex conjugate of complex world-tensor is another complex world-tensor:

$$\overline{\sigma^{AA'BB'CC'}{}_{DD'EE'}} = \overline{\sigma}^{A'AB'BC'C}{}_{D'DE'E} = \overline{\sigma}^{\alpha\beta\gamma}{}_{\delta\epsilon} \,. \tag{1.89}$$

Certain complex world-tensors are invariant under complex conjugation and we shall refer to them as *real world-tensors* or simply *world-tensors.* For a real world-tensor we therefore have

$$\overline{\sigma}^{\alpha\beta\gamma}{}_{\delta\epsilon} = \sigma^{\alpha\beta\gamma}{}_{\delta\epsilon} \,. \tag{1.90}$$

Spinors of the type $\kappa^A\overline{\kappa}^{A'}$ corresponding to null four-vectors are examples of such real world-vectors.

[20] In this work we use bold lowercase latin letters to indicate component indices, i.e. we write $V^{\mathbf{a}}$ for components of a vector $\mathbf{V}$. As an abstract index corresponding to the index $\mathbf{a}$ we use the greek letter α instead of its latin variant. We do so in order to ease distinguishing between abstract and component indices, since lowercase bold indices are rather hard to tell apart from their non-bold variants. At this moment, however, it would be more convenient to use the more straightforward convention, since then we would have a as a composite index standing for AA', b standing for BB' and so on.

Now that we have found how to equate certain spinors to world-tensors, we may ask whether it is possible to express arbitrary spinorial expression in terms of tensor formalism. A natural place to start is the simplest of spinors, a spin-vector κ^A. It is clear that it is unachievable to transcribe κ^A directly into a tensor expression — κ^A simply lacks indices. But we may still try to find some spinorial expression that would have informational content identical to that of the κ^A[21] and which would be expressible in tensor formalism. We have already seen how to assign the four-vector $\kappa^A\overline{\kappa}^{A'}$ to spinor κ^A, but such correspondence is ambiguous. The four-vector $\kappa^A\kappa^{A'}$ is unchanged under the phase transformation $\kappa^A \mapsto \exp(\mathrm{i}\phi)\kappa^A\,, \phi \in \mathbb{R}$. Since we need to pair primed and unprimed indices to form a world-tensor, it clearly is not possible to construct suitable tensorial expression using only κ^A and $\overline{\kappa}^{A'}$ as its building blocks. Fortunately, there is one more ingredient we can use: the canonical spinor ϵ_{AB} (and its complex conjugate). The simplest world tensor we can create using κ_A, ϵ_{AB} and their complex conjugates is spinor

$$P^{ab} = \kappa^A\kappa^B\epsilon^{A'B'} + \overline{\kappa}^{A'}\overline{\kappa}^{B'}\epsilon^{AB}\,. \tag{1.91}$$

We can easily see that it is antisymmetric in indices a and b. To interpret this tensor, we will need the relation

$$\epsilon^{AB} = o^A\iota^B - \iota^A o^B\,, \tag{1.92}$$

which holds if $o_A\iota^A = 1$. We can prove it simply by expanding formula $\epsilon^{AB} = \epsilon_{\mathbf{A}}{}^A\epsilon^{\mathbf{AB}}\epsilon_{\mathbf{B}}{}^B$ for a spin-frame $\epsilon_{\mathbf{A}}{}^A$. Because for any nonzero κ^A there exists a spinor τ^A such that the pair κ^A, τ^A form a spin-frame[22], we can rewrite ϵ^{AB} as $\kappa^A\tau^B - \tau^A\kappa^B$. Thus we have

$$\begin{aligned} P^{ab} =& \kappa^A\kappa^B\left(\overline{\kappa}^{A'}\overline{\tau}^{B'} - \overline{\tau}^{A'}\overline{\kappa}^{B'}\right) + \overline{\kappa}^{A'}\overline{\kappa}^{B'}\left(\kappa^A\tau^B - \tau^A\kappa^B\right) \\ =& \kappa^A\overline{\kappa}^{A'}\left(\kappa^B\overline{\tau}^{B'} + \tau^B\overline{\kappa}^{B'}\right) - \kappa^B\overline{\kappa}^{B'}\left(\kappa^A\overline{\tau}^{A'} + \tau^A\overline{\kappa}^{A'}\right) = K^\alpha L^\beta - K^\beta L^\alpha\,, \end{aligned} \tag{1.93}$$

where the vector $L^\alpha = \kappa^A\overline{\tau}^{A'} + \tau^A\overline{\kappa}^{A'}$ is space-like and orthogonal to the vector $K^\alpha = \kappa^A\overline{\kappa}^{A'}$. As can be shown, these two vectors determine the flag plane of κ^A.

When dealing with general spinor, we proceed similarly. Basically, there are three distinct cases which may occur. The first case is when a spinor $\chi_{A\ldots FA'\ldots F'}$ has equal number of primed and unprimed indices. Then it is possible to transcribe it directly. For example we can write $\kappa^A\tau^{A'} = V^\alpha$, where V^α is a complex four-vector. If desired, we can consider its real and imaginary part separately, thus obtaining two real world-tensors. In the second case the spinor has unequal number of primed and unprimed indices, while the total number of indices is even. In such a case we multiply it with suitable ϵs to make the numbers of unprimed and primed indices same, and then proceed as in the first case. An example would be a spinor $\chi_{ABCDA'B'}$. We simply make an outer product of it with $\epsilon_{C'D'}$ to obtain the spinor $\chi_{ABCDA'B'}\epsilon_{C'D'}$ which can be readily interpreted as a tensor. The last case is of a spinor that possesses odd number of indices. This is the only case when true tensorial representation is actually impossible, since a spinor with odd number of indices is a true spinorial object, i.e. it changes sign when rotated through 2π. A tensor is able to represent it only up to the sign ambiguity. To progress, we multiply such spinor with itself to obtain an expression having an even number of indices. Such expression is of one of

[21] So that that expression would be both fully determined by and would fully determine the spin-vector κ^A. Strictly speaking, it is not always possible to achieve such an equivalence, since world-tensors are not spinorial objects in a sense that a full rotation returns them into their original state. But even if the spinor under consideration is a true spinorial object, we can still try to find a world-tensor representing it uniquely up to the sign ambiguity.

[22] Notice that the requirement $\kappa_A\tau^A = 1$ does not fix τ^A uniquely, since if τ^A satisfies $\kappa_A\tau^A = 1$, then same holds for any $\tilde{\tau}^A = \tau^A + c\kappa^A\,, c \in \mathbb{C}$ as well.

previous cases and it determines the original spinor up to the sign. Probably the best example is that of a spin-vector κ^A, which we have discussed above. Its square, the spinor $\kappa^A\kappa^B$ is of the second type. Taking an outer product with $\epsilon^{A'B'}$ we arrive at $\kappa^A\kappa^B\epsilon^{A'B'}$, which is a complex world-tensor. In this particular example it suffices to consider only its real part, since it contains all the information on $\kappa^A\kappa^B\epsilon^{A'B'}$[23].

There are two essential tensors that can be constructed using just ϵs. The first is the symmetric tensor

$$g_{\alpha\beta} = \epsilon_{AB}\epsilon_{A'B'} \,. \tag{1.94}$$

Applying the tensor $g_{\alpha\beta}$ on a four-vector V^α results in the lowering of its index:

$$g_{\alpha\beta}V^\alpha = \epsilon_{AB}\epsilon_{A'B'}V^{AA'} = V_{BB'} = V_\beta \,. \tag{1.95}$$

Thus we see that $g_{\alpha\beta}$ acts like the metric tensor and so we shall define the latter by the equation (1.94).

The other tensor is the alternating tensor $e_{\alpha\beta\gamma\delta}$. We shall simply state its form here, without proving that it possesses the desired attributes[24]:

$$e_{\alpha\beta\gamma\delta} = \mathrm{i}\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \mathrm{i}\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'} \,. \tag{1.96}$$

If we were to transcribe a general spinorial relation as a relation between tensors, knowing how to find tensorial analogue of any spinor would in most cases not suffice. After all, symbols that represent spinors or tensors contain very little valuable information on their own. Its through an explicit use of operations on and between spinors or tensors that we usualy capture the content. Therefore it is important to know how to transcribe operations on spinors: the addition of spinors, the multiplication of spinor by scalar, the outer product of spinors, operations of contraction and of index permutation/substitution. Hence we should put each one of those operations under close scrutiny, but that is exactly what we won't do, since it would require prolonged discussion. We shall only briefly mention the case of index permutation, since that one is actually quite important from the conceptual perspective, while the discussion of the other cases would revolve mostly about technical subtleties. To illustrate such technical difficulties, consider for example the outer product of spin-vectors μ^A and $\nu^{A'}$, i.e. the relation

$$\mu^A\nu^{A'} = \chi^{AA'} \,. \tag{1.97}$$

Tensorial analogues for these spinors (obtained by the aforementioned procedure) are: $\mu^A\mu^B\epsilon^{A'B'} = M^{\alpha\beta}$ for μ^A, $\nu^{C'}\nu^{D'}\epsilon^{CD} = N^{\gamma\delta}$ for $\nu^{A'}$, but simply $\chi^{AA'} = \chi^\alpha$ for $\chi^{AA'}$. Thereby we see that the outer multiplication of spinors does not necessarily results in the simple outer multiplication of relevant tensors, since here clearly $M^{\alpha\beta}N^{\gamma\delta} \neq \chi^\alpha$. The way out of this problem depends on circumstances. Expressions $M^{\alpha\beta}$ and $N^{\alpha\beta}$ tell us nothing about the overall sign of spin-vectors μ^A and $\nu^{A'}$, and unless we have access to some other tensor which provides us with information on their relative sign, the overall sign of χ^α is also unknown to us. Thus knowing just $M^{\alpha\beta}$ and $N^{\alpha\beta}$ we may determine only the square $\chi^\alpha\chi^\beta$, and so we should look for the way to relate $M^{\alpha\beta}N^{\gamma\delta}$ with $\chi^\alpha\chi^\beta$. This may be done by applying suitable (spinor) index permutation on expression $M^{\alpha\beta}N^{\gamma\delta}$, obtaining $\chi^\alpha\chi^\beta g^{\gamma\delta}$.

[23] As we will see later, the imaginary part of $\kappa^A\kappa^B\epsilon^{A'B'}$ can be obtained from the real part by dualization.

[24] We want $e_{\alpha\beta\gamma\delta}$ to be totally antisymmetric and to satisfy the normalization condition $e_{\alpha\beta\gamma\delta}t^\alpha x^\beta y^\gamma z^\delta = 1$. We actually are not prepared to show that the latter is satisfied by the definition given above, since we are yet to introduce the required tetrad.

Let us now turn our attention to the operation of index permutation. Consider some spinor $\chi_{AA'BB'} = \chi_{\alpha\beta}$. Switching the whole pair AA' with BB' clearly results in $\chi_{\beta\alpha}$, but what is an effect of permutation $A \leftrightarrow B$? To find an answer, we need to consider the case of symmetric tensor $T_{\alpha\beta}$ and the case of the antisymmetric one, the bivector $F_{\alpha\beta}$, separately.

Let us start with a symmetric tensor $T_{\alpha\beta} = T_{(\alpha\beta)}$. We are interested in the relation between $T_{BA'AB'}$ (which, due to the symmetry of $T_{\alpha\beta}$, equals $T_{AB'BA'}$) and the original tensor $T_{\alpha\beta}$. When we split $T_{AA'BB'}$ into parts symmetric and antisymmetric in indices A and B

$$T_{AA'BB'} = \frac{1}{2}\left(T_{AA'BB'} + T_{BA'AB'}\right) + \frac{1}{2}\left(T_{AA'BB'} - T_{BA'AB'}\right) , \tag{1.98}$$

we may observe that, because of the symmetry of $T_{\alpha\beta}$, the expression in the first brackets is actually symmetric in both AB and $A'B'$, and the expression in the second brackets is fully antisymmetric. Thus we can write

$$T_{AA'BB'} = S_{AA'BB'} + \frac{1}{4} T_{CC'}{}^{CC'} \epsilon_{AB}\epsilon_{A'B'} , \tag{1.99}$$

where $\frac{1}{4}T_{CC'}{}^{CC'} = \frac{1}{4}T_{\gamma}{}^{\gamma}$ is the trace of $T_{\alpha\beta}$, and $S_{AA'BB'}$, which is totally symmetric, is the trace-free part of $T_{\alpha\beta}$, i.e. $S_{\gamma}{}^{\gamma} = 0$. The permutation $A \leftrightarrow B$ (or $A' \leftrightarrow B'$) leaves $S_{AA'BB'}$ unchanged, while it reverses the sign of the second term of (1.99). Hence, interchanging indices A and B (or A' and B') in symmetric tensor $T_{\alpha\beta}$ amounts to the tensorial operation of trace reversal.

Next we investigate the case of bivector $F_{\alpha\beta} = F_{[\alpha\beta]}$. We proceed similarly as we did before, and decompose $F_{\alpha\beta}$ into parts symmetric and antisymmetric in AB:

$$F_{AA'BB'} = \frac{1}{2}\left(F_{AA'BB'} + F_{BA'AB'}\right) + \frac{1}{2}\left(F_{AA'BB'} - F_{BA'AB'}\right) . \tag{1.100}$$

Because of antisymmetry of $F_{\alpha\beta}$, the first parenthesis is symmetric in AB and antisymmetric in $A'B'$, while the second parenthesis is antisymmetric in AB and symmetric in $A'B'$. Therefore we have

$$F_{\alpha\beta} = \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\psi_{A'B'} . \tag{1.101}$$

Spinors ϕ_{AB} and $\psi_{A'B'}$ are both symmetric, and related to $F_{\alpha\beta}$ by

$$\phi_{AB} = \frac{1}{2} F_{ABC'}{}^{C'} , \quad \psi_{A'B'} = \frac{1}{2} F_{C}{}^{C}{}_{A'B'} . \tag{1.102}$$

When we interchange indices AB or $A'B'$, one of the terms in decomposition (1.101) switches the sign. This is related to the tensorial operation of dualization. The dual ${}^{*}F_{\alpha\beta}$ of bivector $F_{\alpha\beta}$ is defined as

$${}^{*}F_{\alpha\beta} = \frac{1}{2} e_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta} . \tag{1.103}$$

Substituting for $F_{\gamma\delta}$ from (1.101) and for the alternating tensor from (1.96), we arrive at

$${}^{*}F_{AA'BB'} = -\mathrm{i}\phi_{AB}\epsilon_{A'B'} + \mathrm{i}\epsilon_{AB}\psi_{A'B'} \tag{1.104}$$

and, finally,

$${}^{*}F_{AA'BB'} = \mathrm{i}F_{AB'BA'} = -\mathrm{i}F_{BA'AB'} . \tag{1.105}$$

While we are discussing dualization, it may be worthwhile to introduce some useful terminology. A bivector, which when multiplied by the imaginary unit equals its dual, i.e. $\mathrm{i}F_{\alpha\beta} = {}^*F_{\alpha\beta}$, is said to be *self-dual*. By (1.104), we have $F_{ABA'B'} = \epsilon_{AB}\psi_{A'B'}$ for self-dual bivector. On the other hand, we say that a bivector $F_{ABA'B'} = \phi_{AB}\epsilon_{A'B'}$ is *anti-self-dual*. Such a bivector satisfies the relation ${}^*F_{\alpha\beta} = -\mathrm{i}F_{\alpha\beta}$. For an arbitrary complex bivector $F_{\alpha\beta}$, the expression

$$ {}^-F_{\alpha\beta} = \frac{1}{2}\left(F_{\alpha\beta} + \mathrm{i}^*F_{\alpha\beta}\right) = \phi_{AB}\epsilon_{A'B'} \tag{1.106}$$

is anti-self-dual, and expression

$$ {}^+F_{\alpha\beta} = \frac{1}{2}\left(F_{\alpha\beta} - \mathrm{i}^*F_{\alpha\beta}\right) = \epsilon_{AB}\psi_{A'B'} \tag{1.107}$$

is self-dual.

Let us now turn back to the issue of index permutation. So far, we have seen which tensorial operations correspond to permutation of spinorial indices only in special cases of symmetric and antisymmetric tensors. But because any tensor $H_{\alpha\beta}$ can be decomposed into the sum of its symmetric and antisymmetric parts, we can easily deduce the rules for such a general case. Consider the permutation $A \leftrightarrow B$. The symmetric part of $H_{\alpha\beta}$ undergoes the change[25]

$$H_{(\alpha\beta)} \mapsto H_{(\alpha\beta)} - \frac{1}{2}H_\gamma{}^\gamma g_{\alpha\beta}\,, \tag{1.108}$$

while the antisymmetric part is transformed according to

$$H_{[\alpha\beta]} \mapsto \mathrm{i}^*H_{[\alpha\beta]}\,. \tag{1.109}$$

As a result, we may write

$$H_{BAA'B'} = H_{(\alpha\beta)} - \frac{1}{2}H_\gamma{}^\gamma g_{\alpha\beta} + \mathrm{i}^*H_{[\alpha\beta]} = \frac{1}{2}\left(H_{\alpha\beta} + H_{\beta\alpha}\right) - \frac{1}{2}H_\gamma{}^\gamma g_{\alpha\beta} + \frac{1}{2}\mathrm{i}e_{\alpha\beta}{}^{\gamma\delta}H_{\gamma\delta}\,, \tag{1.110}$$

or

$$H_{BAA'B'} = \frac{1}{2}\left(g_\alpha{}^\gamma g_\beta{}^\delta + g_\beta{}^\gamma g_\alpha{}^\delta - g_{\alpha\beta}g^{\gamma\delta} + \mathrm{i}e_{\alpha\beta}{}^{\gamma\delta}\right)H_{\gamma\delta}\,. \tag{1.111}$$

Thus we see that the tensor

$$U_{\alpha\beta}{}^{\gamma\delta} = \frac{1}{2}\left(g_\alpha{}^\gamma g_\beta{}^\delta + g_\beta{}^\gamma g_\alpha{}^\delta - g_{\alpha\beta}g^{\gamma\delta} + \mathrm{i}e_{\alpha\beta}{}^{\gamma\delta}\right) \tag{1.112}$$

effectuates the desired permutation. It is not hard to check that to perform the permutation $A' \leftrightarrow B'$, it suffices to apply the complex conjugate of the tensor $U_{\alpha\beta}{}^{\gamma\delta}$, the tensor $\overline{U}_{\alpha\beta}{}^{\gamma\delta} = \frac{1}{2}\left(g_\alpha{}^\gamma g_\beta{}^\delta + g_\beta{}^\gamma g_\alpha{}^\delta - g_{\alpha\beta}g^{\gamma\delta} - \mathrm{i}e_{\alpha\beta}{}^{\gamma\delta}\right)$. Thus, to summarize, we have

$$H_{BAA'B'} = U_{\alpha\beta}{}^{\gamma\delta}H_{\gamma\delta}\,, \quad H_{ABB'A'} = \overline{U}_{\alpha\beta}{}^{\gamma\delta}H_{\gamma\delta}\,. \tag{1.113}$$

Finally, since any index permutation can be achieved by successively applying suitable permutations on pairs of indices, any permutation of spinor indices can be captured by the virtue of tensors $U_{\alpha\beta}{}^{\gamma\delta}$ and $\overline{U}_{\alpha\beta}{}^{\gamma\delta}$.

We continue this section with introducing some tetrads which we can construct from spin-frame, and close it with some remarks on transforming between components with respect to

[25]The trace of a tensor and the trace of its symmetric part are of course the same.

spin-frame and components with respect to some tetrad. Let us consider a spin-frame o^A, ι^A with normalization $o_A\iota^A = 1$. A *null tetrad* is defined by

$$l^\alpha = o^A\bar{o}^{A'}\,, \quad n^\alpha = \iota^A\bar{\iota}^{A'}\,, \quad m^\alpha = o^A\bar{\iota}^{A'}\,, \quad \overline{m}^\alpha = \iota^A\bar{o}^{A'}\,. \tag{1.114}$$

These are all null four-vectors. Nearly all scalar products between them vanish, with exception of

$$l_\alpha n^\alpha = 1\,, \quad m_\alpha \overline{m}^\alpha = -1\,. \tag{1.115}$$

Vectors l^α, n^α, m^α, $\overline{m}^\alpha$ are linearly independent, hence constituting a basis. The dual basis is n_α, l_α, $-\overline{m}_\alpha$, $-m_\alpha$. Thus we may write

$$g_{\alpha\beta} = \delta^{\mathbf{a}}_\alpha \delta_{\beta\mathbf{a}} = 2l_{(\alpha}n_{\beta)} - 2m_{(\alpha}\overline{m}_{\beta)}\,. \tag{1.116}$$

Sometimes it is more convenient to use tetrad that contains only real four-vectors. Starting either from l^α, n^α, m^α, $\overline{m}^\alpha$, or from o^A, ι^A, we can easily construct a *Minkowski tetrad*:

$$\begin{aligned}
t^\alpha =& \frac{1}{\sqrt{2}}\left(l^\alpha + n^\alpha\right) = \frac{1}{\sqrt{2}}\left(o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}\right)\,,\\
x^\alpha =& \frac{1}{\sqrt{2}}\left(m^\alpha + \overline{m}^\alpha\right) = \frac{1}{\sqrt{2}}\left(o^A\bar{\iota}^{A'} + \iota^A\bar{o}^{A'}\right)\,,\\
y^\alpha =& \frac{\mathrm{i}}{\sqrt{2}}\left(m^\alpha - \overline{m}^\alpha\right) = \frac{\mathrm{i}}{\sqrt{2}}\left(o^A\bar{\iota}^{A'} - \iota^A\bar{o}^{A'}\right)\,,\\
z^\alpha =& \frac{1}{\sqrt{2}}\left(l^\alpha - n^\alpha\right) = \frac{1}{\sqrt{2}}\left(o^A\bar{o}^{A'} - \iota^A\bar{\iota}^{A'}\right)\,.
\end{aligned} \tag{1.117}$$

Four-vectors t^α, x^α, y^α and z^α, defined as above, satisfy the desired conditions (for constituting Minkowski tetrad)

$$t^\alpha t_\alpha = 1\,, \quad x^\alpha x_\alpha = y^\alpha y_\alpha = z^\alpha z_\alpha = -1\,, \quad t^\alpha x_\alpha = t^\alpha y_\alpha = t^\alpha z_\alpha = x^\alpha y_\alpha = x^\alpha z_\alpha = y^\alpha z_\alpha = 0\,. \tag{1.118}$$

In previous sections, we have introduced symbols $\epsilon_{\mathbf{A}}{}^A$ and $\epsilon_A{}^{\mathbf{A}}$ for spin-frame. Such a notation allows us to lower and raise component indices in a similar way that we do with the abstract ones. Analogously, we may introduce symbols $g_{\mathbf{a}}{}^\alpha$ and $g_\alpha{}^{\mathbf{a}}$ for tetrad and its dual:[26]

$$\begin{aligned}
g_0{}^\alpha =& t^\alpha\,, \quad g_1{}^\alpha = x^\alpha\,, \quad g_2{}^\alpha = y^\alpha\,, \quad g_3{}^\alpha = z^\alpha\,,\\
g_\alpha{}^0 =& t_\alpha\,, \quad g_\alpha{}^1 = -x_\alpha\,, \quad g_\alpha{}^2 = -y_\alpha\,, \quad g_\alpha{}^3 = -z_\alpha\,.
\end{aligned} \tag{1.119}$$

Now suppose that we are given coordinates $V^{\mathbf{a}}$ of some four-vector, taken with respect to the tetrad $g_{\mathbf{a}}{}^\alpha$, and we want to find coordinates with respect to the spin-frame $\epsilon_{\mathbf{A}}{}^A$. The abstract index formalism provides us with a very plain solution. We may proceed in the virtually same manner as we did when we transformed bewteen coordinates taken with respect to different vector bases. Given components $V^{\mathbf{a}}$, we can recover the vector $V^\alpha = V^{\mathbf{a}} g_{\mathbf{a}}{}^\alpha$, and since in our formalism $V^\alpha = V^{AA'}$, we may simply use the basis dual to $\epsilon_{\mathbf{A}}{}^A$ to obtain components $V^{\mathbf{AA'}} = V^\alpha \epsilon_A{}^{\mathbf{A}} \epsilon_{A'}{}^{\mathbf{A'}}$. So the transformation is achieved by means of the object[27] $g_{\mathbf{a}}{}^\alpha \epsilon_A{}^{\mathbf{A}} \epsilon_{A'}{}^{\mathbf{A'}}$.

[26]Although here the notation $\delta^\alpha_{\mathbf{a}}$, which does not contain information on relative order of indices $\mathbf{a}\alpha$, would suffice, since the metric is symmetric.

[27]Remember that component index $\mathbf{a}$ is not a composite index, i.e. it does not equal $\mathbf{AA'}$. Therefore there is no summation carried out between indices $\mathbf{a}$ and $\mathbf{AA'}$.

The same object would be used, if we started from $V_{\mathbf{AA'}}$ and wanted to obtain $V_{\mathbf{a}}$. In the remaining two cases the object $g_\alpha{}^{\mathbf{a}}\epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{A'}}{}^{A'}$ would be used. These two objects,

$$
\begin{aligned}
g_{\mathbf{a}}{}^{\mathbf{AA'}} &= g_{\mathbf{a}}{}^{\alpha}\epsilon_{A}{}^{\mathbf{A}}\epsilon_{A'}{}^{\mathbf{A'}} , \\
g_{\mathbf{AA'}}{}^{\mathbf{a}} &= g_{\alpha}{}^{\mathbf{a}}\epsilon_{\mathbf{A}}{}^{A}\epsilon_{\mathbf{A'}}{}^{A'} ,
\end{aligned}
\tag{1.120}
$$

are called *Infeld - van der Waerden symbols.* Components of a general world-tensor $\chi_{\alpha\ldots\beta}{}^{\mu\ldots\nu}$ are related by

$$
\chi_{\mathbf{a}\ldots\mathbf{c}}{}^{\mathbf{m}\ldots\mathbf{n}} = g_{\mathbf{a}}{}^{\mathbf{AA'}} \cdots g_{\mathbf{c}}{}^{\mathbf{CC'}} g_{\mathbf{MM'}}{}^{\mathbf{m}} \cdots g_{\mathbf{NN'}}{}^{\mathbf{n}} \chi_{\mathbf{AA'}\ldots\mathbf{BB'}}{}^{\mathbf{MM'}\ldots\mathbf{NN'}}
\tag{1.121}
$$

and

$$
\chi_{\mathbf{AA'}\ldots\mathbf{BB'}}{}^{\mathbf{MM'}\ldots\mathbf{NN'}} = g_{\mathbf{AA'}}{}^{\mathbf{a}} \cdots g_{\mathbf{BB'}}{}^{\mathbf{b}} g_{\mathbf{m}}{}^{\mathbf{MM'}} \cdots g_{\mathbf{n}}{}^{\mathbf{NN'}} \chi_{\mathbf{a}\ldots\mathbf{c}}{}^{\mathbf{m}\ldots\mathbf{n}} .
\tag{1.122}
$$

If $\epsilon_{\mathbf{A}}{}^{A}$ and $g_{\mathbf{a}}{}^{\alpha}$ are related by (1.117), we have

$$
g_0{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad g_1{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,
\tag{1.123}
$$

$$
g_2{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} , \quad g_3{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\tag{1.124}
$$

These are proportional to the well known Pauli matrices and the unit matrix.

2. Spinor analysis

So far we have considered only spinors at a point. In this chapter we will introduce covariant derivative on spinors, which will provide us with means to compare spinors at different points. Then we will briefly describe spinor formulation of general relativity. We assume that reader is already familiar with differential geometry and tensor formulation of general theory of relativity.

2.1 Spinor covariant derivative

Tensors on manifold are given their geometric meaning through isomorphism between the space of vector fields and the space of derivatives of scalar fields on the manifold. In this chapter we adopt an axiomatic approach where we postulate the existence of spinor structure on manifold and demand that the space of real world-vectors constructed from spinors is isomorphic with derivatives of (complex) scalar field on manifold. (This for example fixes the dimension of manifold.) We require a covariant derivative $\nabla_{AA'}$ to have the following properties:

1. The basic properties of any derivative operator, i.e.

   - Linearity: Let $\alpha^{\mathcal{A}}$, $\beta^{\mathcal{A}}$ be two spinor fields of the same valence[1]. Then

   $$\nabla_{AA'}\left(\alpha^{\mathcal{A}}+\beta^{\mathcal{A}}\right)=\nabla_{AA'}\alpha^{\mathcal{A}}+\nabla_{AA'}\beta^{\mathcal{A}}\,. \tag{2.1}$$

   - The Leibniz rule: For any two spinor fields $\alpha^{\mathcal{A}}$, $\beta^{\mathcal{B}}$

   $$\nabla_{AA'}\left(\alpha^{\mathcal{A}}\beta^{\mathcal{B}}\right)=\left(\nabla_{AA'}\alpha^{\mathcal{A}}\right)\beta^{\mathcal{B}}+\alpha^{\mathcal{A}}\nabla_{AA}\beta^{\mathcal{A}}\,. \tag{2.2}$$

   - The derivative annihilates constant scalar fields.

   We also require that an operation of covariant derivative commutes with the operation of index substitution (not involving indices AA' of its operator $\nabla_{AA'}$) and with the operation of contraction.

2. We demand that $V^{\alpha}\nabla_{\alpha}f=\mathbf{V}\left(f\right)$ for any world-vector $\mathbf{V}$ and scalar f.

3. We want the operation of covariant derivative to commute with complex conjugation, i.e. if $\beta^{\mathcal{B}}{}_{AA'}=\nabla_{AA'}\alpha^{\mathcal{B}}$ then $\overline{\beta^{\mathcal{B}}{}_{AA'}}=\nabla_{AA'}\overline{\alpha^{\mathcal{B}}}$. Formally, this means that the operator ∇_{α} is real.

4. We want the covariant derivative to be torsion free[2], i.e. $\nabla_{[\alpha}\nabla_{\beta]}f=\frac{1}{2}\Delta_{\alpha\beta}f=0$ for any scalar field f, and to annihilate the spinor ϵ_{AB}, i.e. $\nabla_{AA'}\epsilon_{BC}=\nabla_{AA'}\epsilon_{B'C'}=0$.

Proof of the existence of such a covariant derivative may be found for example in chapter 4 of [11]. Here we only prove the uniqueness of such a construction, following the notation used in aforementioned book. Suppose we have two covariant derivatives $\nabla_{AA'}$ and $\widetilde{\nabla}_{AA'}$, both of which satisfy the above requirements. Since the relation $\nabla_{AA'}f=\widetilde{\nabla}_{AA'}f$ holds[3] for any scalar f, the

[1] Symbols $\mathcal{A}$, $\mathcal{B}$ stand for composite indices.
[2] Torsion tensor $T_{\alpha\beta}{}^{\gamma}$ is defined by the relation $2\nabla_{[\alpha}\nabla_{\beta]}f=T_{\alpha\beta}{}^{\gamma}\nabla_{\gamma}f$, for any scalar field f.
[3] As follows directly from our requirement that $V^{AA'}\nabla_{AA'}f=\mathbf{V}\left(f\right)$ for any scalar f and world-vector $\mathbf{V}$.

difference $\widetilde{\nabla}_{AA'} - \nabla_{AA'}$ annihilates scalar fields. Thus we have

$$\begin{aligned}
&\left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\left(f\alpha^B + \beta^B\right) = \\
&\alpha^B \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right) f + f \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\alpha^B + \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\beta^B = \\
&f \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\alpha^B + \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\beta^B
\end{aligned}$$

for any scalar field f and spinor fields α^B and β^B. Consequently, $\widetilde{\nabla}_{AA'} - \nabla_{AA'}$ is equivalent to some spinor field $\Theta_{AA'B}{}^C$:

$$\left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\kappa^C = \Theta_{AA'B}{}^C \kappa^B \,, \tag{2.3}$$

and[4]

$$\left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\lambda^{C'} = \overline{\Theta}_{AA'B'}{}^{C'} \lambda^{B'} \,. \tag{2.4}$$

By a familiar procedure we extend these rules on covariant spin-vectors and spinors of higher valence. Using the Leibniz rule and the property that the derivative commutes with contraction, we have $0 = \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\left(\alpha^B\beta_B\right) = \alpha^B \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\beta_B + \beta_B \left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\alpha^B$, or

$$\left(\widetilde{\nabla}_{AA'} - \nabla_{AA'}\right)\kappa_C = -\Theta_{AA'C}{}^B \kappa_B \,, \tag{2.5}$$

and similarly for $\lambda_{B'}$. In a case of a spinor of higher valence, e.g. $\chi_{AB'}{}^C$, we find

$$\left(\widetilde{\nabla}_{XX'} - \nabla_{XX'}\right)\chi_{AB'}{}^C = -\Theta_{XX'A}{}^D \chi_{DB'}{}^C - \overline{\Theta}_{XX'B'}{}^{D'}\chi_{AD'}{}^C + \Theta_{XX'D}{}^C \chi_{AB'}{}^D \,, \tag{2.6}$$

again by the Leibniz rule. Applying these results on the spinor ϵ_{AB}, we get

$$\left(\widetilde{\nabla}_{XX'} - \nabla_{XX'}\right)\epsilon_{AB} = -\Theta_{XX'A}{}^D \epsilon_{DB} - \Theta_{XX'B}{}^D \epsilon AD = -\Theta_{XX'AB} + \Theta_{XX'AB} \,. \tag{2.7}$$

If, as we required, ϵ_{AB} is covariantly constant, i.e. $\nabla_{XX'}\epsilon_{AB} = \widetilde{\nabla}_{XX'}\epsilon_{AB} = 0$, we find that $\Theta_{XX'AB}$ is symmetric in AB. To prove the uniqueness we will have to use the requirement on the derivative to be torsion free. To do so, we first need to find how the operator $\widetilde{\nabla}_{AA'} - \nabla_{AA'}$ acts on world-vectors. Employing (2.6) we arrive at

$$\begin{aligned}
\left(\widetilde{\nabla}_{CC'} - \nabla_{CC'}\right) V^{AA'} &= \Theta_{CC'B}{}^A V^{BA'} + \overline{\Theta}_{CC'B'}{}^{A'} V^{AB'} \\
&= \left(\Theta_{CC'B}{}^A \epsilon_{B'}{}^{A'} + \overline{\Theta}_{CC'B'}{}^{A'} \epsilon_B{}^A\right) V^{BB'} \\
&= Q_{\gamma\beta}{}^\alpha V^\beta \,.
\end{aligned} \tag{2.8}$$

Now we are able to work out what restrictions on $Q_{\alpha\beta}{}^\gamma$ does the torsion free property of covariant derivative imply. Effects of operators $\Delta_{\alpha\beta}$ and $\widetilde{\Delta}_{\alpha\beta}$ on scalar field differ by

$$\begin{aligned}
\widetilde{\Delta}_{\alpha\beta} f &= 2\widetilde{\nabla}_{[\alpha}\widetilde{\nabla}_{\beta]} f = 2\widetilde{\nabla}_{[\alpha}\nabla_{\beta]} f \\
&= 2\nabla_{[\alpha}\nabla_{\beta]} f - 2Q_{[\alpha\beta]}{}^\gamma \nabla_\gamma f \\
&= \Delta_{\alpha\beta} f - 2Q_{[\alpha\beta]}{}^\gamma \nabla_\gamma f \,.
\end{aligned} \tag{2.9}$$

[4]Because $\nabla_{AA'}\lambda^{B'}$ is actually defined as $\overline{\nabla_{AA'}\overline{\lambda}^B}$.

Therefore, if $\Delta_{\alpha\beta} = \widetilde{\Delta}_{\alpha\beta} = 0$ holds, we obtain symmetry of $Q_{\alpha\beta}{}^{\gamma}$ in the first two indices. From the requirement that the covariant derivative annihilates ϵ_{AB} we got the symmetry of $\Theta_{AA'BC}$ in the last two indices. Therefore, the tensor $Q_{\alpha\beta\gamma} = \Theta_{AA'BC}\epsilon_{B'C'} + \overline{\Theta}_{AA'B'C'}\epsilon_{BC}$ is antisymmetric in the pair $\beta\gamma$. To conclude, we have found that $Q_{\alpha\beta\gamma}$ is symmetric in indices $\alpha\beta$ and antisymmetric in indices $\beta\gamma$. This means that $Q_{\alpha\beta\gamma}$ is identically zero[5]. Thus we have proved that torsion free spinor covariant derivative that annihilates ϵ_{AB} is unique.

2.2 Spin coefficients and the Newman-Penrose formalism

Once we have chosen spin-frame or tetrad field (at least locally), we may project all the other geometric objects onto it, obtaining a number of scalar quantities. This way we can rewrite relations between spinorial objects in terms of relations between scalar fields. Such a description may offer certain advantages over the tensorial one, especially if the choice of spin-frame is of some physical significance. Here we shall outline one special case of such description, the Newman-Penrose formalism.

Suppose there are two spin-vector fields $\epsilon_0{}^A$, $\epsilon_1{}^A$ on some region of a manifold, and that these fields form a spin-frame at each point of that region. Not unexpectedly, we shall also use symbol o^A and ι^A for those fields. From these spinors we can construct a null tetrad l^α, n^α, m^α, $\overline{m}^\alpha$:

$$l^\alpha = o^A\overline{o}^{A'}\,, \quad n^\alpha = \iota^A\overline{\iota}^{A'}\,, \quad m^\alpha = o^A\overline{\iota}^{A'}\,, \quad \overline{m}^\alpha = \iota^A\overline{o}^{A'}\,. \tag{2.10}$$

Let us now consider some spinor field, say $\chi_{ABB'}$. At points where the spin-frame is known, $\chi_{ABB'}$ is fully described by the set of scalar fields $\chi_{\mathbf{ABB'}} = \chi_{ABB'}\epsilon_{\mathbf{A}}{}^A\epsilon_{\mathbf{B}}{}^B\epsilon_{\mathbf{B'}}{}^{B'}$. They could be used in the formalism, but it is often more convenient to decompose the spinor into (the sum of direct product of ϵs with) symmetric spinors, project them onto the spin-frame, and work with these scalar fields instead of the components of the original spinor. When we contract a symmetric spinor with spin-vectors of spin-frame, the results depends only on the overall number of os and ιs entering the contraction, and not on their relative order, e.g. for a symmetric spinor Φ_{ABC}, $\Phi_{ABC}o^Ao^B\iota^C = \Phi_{ABC}o^A\iota^bo^c$ and so on. Therefore it is common to use a notation where independent components are designated by the number of ιs entering the contraction, or eventually by two numbers, if there are both primed and unprimed indices. For example, we would write

$$\begin{aligned}
\Phi_0 =&\Phi_{ABC}o^Ao^Bo^C\,,\\
\Phi_1 =&\Phi_{ABC}o^Ao^B\iota^C\,,\\
\Phi_2 =&\Phi_{ABC}o^A\iota^B\iota^C\,,\\
\Phi_3 =&\Phi_{ABC}\iota^A\iota^B\iota^C,
\end{aligned}$$

for independent components of the symmetric spinor Φ_{ABC}.

Perhaps the most important element of the formalism is the scalar description of the covariant derivative. Once we know what is the action of the derivative on spin-vectors of spin-frame, we are able to evaluate the covariant derivative of any spinor. Consider for example the derivative

[5] If tensor (or spinor) is symmetric in group of indices that overlaps with a group of indices in which that tensor is antisymmetric, the tensor equals zero. This statement is nontrivial only in cases when those two groups overlap in exactly one index. In such a case, e.g. $Q_{(\alpha\beta)\gamma} = Q_{\alpha[\beta\gamma]} = Q_{\alpha\beta\gamma}$, we proceed as follows:

$$Q_{\alpha\beta\gamma} = Q_{\beta\alpha\gamma} = -Q_{\beta\gamma\alpha} = -Q_{\gamma\beta\alpha} = Q_{\gamma\alpha\beta} = Q_{\alpha\gamma\beta} = -Q_{\alpha\beta\gamma}\,.$$

of a simple spin-vector:

$$\nabla_{AA'}\kappa^B = \nabla_{AA'}\left(\kappa^{\mathbf{B}}\epsilon_{\mathbf{B}}{}^B\right) = \epsilon_{\mathbf{B}}{}^B\nabla_{AA'}\kappa^{\mathbf{B}} + \kappa^{\mathbf{B}}\nabla_{AA'}\epsilon_{\mathbf{B}}{}^B\,. \tag{2.11}$$

This illustrates that with spinors $\nabla_{AA'}\epsilon_{\mathbf{B}}{}^B$ (and their complex conjugates) at our disposal, we may completely reconstruct the derivative. But the quantities that we are really interested in are the so-called spin coefficients

$$\gamma_{\mathbf{AA'C}}{}^{\mathbf{B}} = \epsilon_A{}^{\mathbf{B}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{C}}{}^A\,, \tag{2.12}$$

rather than spinors $\nabla_{AA'}\epsilon_{\mathbf{B}}{}^B$. (This is of course due to our intent to work with scalar quantities.) Employing these spin coefficients, we may write

$$\begin{aligned}\epsilon_B{}^{\mathbf{B}}\epsilon_{\mathbf{A}}{}^A\epsilon_{\mathbf{A'}}{}^{A'}\nabla_{AA'}\kappa^B =&\epsilon_B{}^{\mathbf{B}}\nabla_{\mathbf{AA'}}\left(\kappa^{\mathbf{C}}\epsilon_{\mathbf{C}}{}^B\right) = \epsilon_B{}^{\mathbf{B}}\epsilon_{\mathbf{C}}{}^B\nabla_{\mathbf{AA'}}\kappa^{\mathbf{C}} + \kappa^{\mathbf{C}}\epsilon_B{}^{\mathbf{B}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{C}}{}^B\\ =&\nabla_{\mathbf{AA'}}\kappa^{\mathbf{B}} + \gamma_{\mathbf{AA'C}}{}^{\mathbf{B}}\kappa^{\mathbf{C}}\end{aligned} \tag{2.13}$$

for the components of $\nabla_{AA'}\kappa^B$. We would obtain a similar equation for the covariant derivative of spin-vector from the complex conjugated space, only this time the required scalars would be $\epsilon_{A'}{}^{\mathbf{B'}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{C'}}{}^{A'}$, instead of $\epsilon_A{}^{\mathbf{B}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{C}}{}^A$. These two sets of scalars are related by complex conjugation:

$$\overline{\gamma}_{\mathbf{AA'C'}}{}^{\mathbf{B'}} = \epsilon_{A'}{}^{\mathbf{B'}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{C'}}{}^{A'}\,. \tag{2.14}$$

Because we work with spin-frame, i.e. the condition $o_A\iota^A = 1$ holds and hence $\epsilon_{\mathbf{AB}}$ are constant scalars, the sixteen quantities $\gamma_{\mathbf{AA'C}}{}^{\mathbf{B}}$ are not fully independent, for we have[6]

$$\begin{aligned}0 =&\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{BC}} = \nabla_{\mathbf{AA'}}\left(\epsilon_{\mathbf{B}}{}^A\epsilon_{A\mathbf{C}}\right) = \epsilon_{A\mathbf{C}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{B}}{}^A + \epsilon_{\mathbf{B}}{}^A\nabla_{\mathbf{AA'}}\epsilon_{A\mathbf{C}}\\ =&\epsilon_{A\mathbf{C}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{B}}{}^A - \epsilon_{\mathbf{B}A}\nabla_{\mathbf{AA'}}\left(\epsilon^{AB}\epsilon_{B\mathbf{C}}\right) = \epsilon_{A\mathbf{C}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{B}}{}^A - \epsilon_{A\mathbf{B}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{C}}{}^A\\ =&\gamma_{\mathbf{AA'BC}} - \gamma_{\mathbf{AA'CB}}\,.\end{aligned} \tag{2.15}$$

Quantities $\gamma_{\mathbf{AA'BC}}$ are therefore symmetric in the last two indices, which means that there are only 12 independent (complex) scalars characterizing the covariant derivative.

Similarly as we derived the symmetry of $\gamma_{\mathbf{AA'BC}}$, we can find the components of the derivative of covariant spin-vector. This time, however, we need just the constancy of $\epsilon_{\mathbf{A}}{}^{\mathbf{B}}$, which holds quite generally. We obtain:

$$\begin{aligned}\epsilon_{\mathbf{B}}{}^A\nabla_{\mathbf{AA'}}\kappa_A =&\epsilon_{\mathbf{B}}{}^A\nabla_{\mathbf{AA'}}\left(\kappa_{\mathbf{C}}\epsilon_A{}^{\mathbf{C}}\right) = \nabla_{\mathbf{AA'}}\kappa_{\mathbf{B}} + \kappa_{\mathbf{C}}\epsilon_{\mathbf{B}}{}^A\nabla_{\mathbf{AA'}}\epsilon_A{}^{\mathbf{C}}\\ =&\nabla_{\mathbf{AA'}}\kappa_{\mathbf{B}} + \kappa_{\mathbf{C}}\left(-\epsilon_A{}^{\mathbf{C}}\nabla_{\mathbf{AA'}}\epsilon_{\mathbf{B}}{}^A\right)\\ =&\nabla_{\mathbf{AA'}}\kappa_{\mathbf{B}} - \gamma_{\mathbf{AA'B}}{}^{\mathbf{C}}\kappa_{\mathbf{C}}\,.\end{aligned} \tag{2.16}$$

Generalising the previous results straightforwardly to the case of the derivative of a general spinor, we arrive at

$$\begin{aligned}\epsilon_{\mathbf{A}}{}^A\ldots\epsilon_{\mathbf{B'}}{}^{B'}\ldots\epsilon_C{}^{\mathbf{C}}\ldots\epsilon_{D'}{}^{\mathbf{D'}}\nabla_{\mathbf{XX'}}\Phi_{A\ldots B'\ldots}{}^{C\ldots D'\ldots} =& \nabla_{\mathbf{XX'}}\Phi_{\mathbf{A}\ldots\mathbf{B'}\ldots}{}^{\mathbf{C}\ldots\mathbf{D'}\ldots}\\ &-\gamma_{\mathbf{XX'A}}{}^{\mathbf{Y}}\Phi_{\mathbf{Y}\ldots\mathbf{B'}\ldots}{}^{\mathbf{C}\ldots\mathbf{D'}\ldots} - \ldots - \overline{\gamma}_{\mathbf{XX'B'}}{}^{\mathbf{Y'}}\Phi_{\mathbf{A}\ldots\mathbf{Y'}\ldots}{}^{\mathbf{C}\ldots\mathbf{D'}\ldots} - \ldots\\ &+\gamma_{\mathbf{XX'Y}}{}^{\mathbf{C}}\Phi_{\mathbf{A}\ldots\mathbf{B'}\ldots}{}^{\mathbf{Y}\ldots\mathbf{D'}\ldots} + \ldots + \overline{\gamma}_{\mathbf{XX'Y'}}{}^{\mathbf{D'}}\Phi_{\mathbf{A}\ldots\mathbf{B'}\ldots}{}^{\mathbf{C}\ldots\mathbf{Y'}\ldots} + \ldots\,.\end{aligned} \tag{2.17}$$

It is often convenient to employ a special symbol for each of the twelve independent rotation coefficients:

[6] Here we use that $\epsilon_{A\mathbf{B}} = \epsilon_A{}^{\mathbf{C}}\epsilon_{\mathbf{CB}} = \epsilon_A{}^{\mathbf{C}}\left(-\epsilon_{\mathbf{BC}}\right) = -\epsilon_{\mathbf{B}A}$.

$$
\begin{array}{llll}
\epsilon = \gamma_{00'0}{}^{0}\,, & \gamma = \gamma_{11'0}{}^{0}\,, & \beta = \gamma_{01'0}{}^{0}\,, & \alpha = \gamma_{10'0}{}^{0}\,, \\
\kappa = -\gamma_{00'0}{}^{1}\,, & \tau = -\gamma_{11'0}{}^{1}\,, & \sigma = -\gamma_{01'0}{}^{1}\,, & \rho = -\gamma_{10'0}{}^{1}\,, \\
\pi = \gamma_{00'1}{}^{1}\,, & \nu = \gamma_{11'1}{}^{1}\,, & \mu = \gamma_{01'1}{}^{1}\,, & \lambda = \gamma_{10'1}{}^{1}\,.
\end{array}
$$

To conclude the presentation of the formalism we introduce special notation for covariant derivative in each of the directions of the null tetrad:

$$
\begin{array}{ll}
D = o^{A}\bar{o}^{A'}\nabla_{AA'} = l^{\alpha}\nabla_{\alpha}\,, & \Delta = \iota^{A}\bar{\iota}^{A'}\nabla_{AA'} = n^{\alpha}\nabla_{\alpha}\,, \\
\delta = o^{A}\bar{\iota}^{A'}\nabla_{AA'} = m^{\alpha}\nabla_{\alpha}\,, & \bar{\delta} = \iota^{A}\bar{o}^{A'}\nabla_{AA'} = \overline{m}^{\alpha}\nabla_{\alpha}\,.
\end{array}
$$

Putting all this new notation to work, we may express a derivative of a basis spinor, e.g. $o^{A}\bar{o}^{A'}\nabla_{AA'}o^{B}$, as follows:

$$
o^{A}\bar{o}^{A'}\nabla_{AA'}o^{B} = Do^{B} = \epsilon_{\mathbf{B}}{}^{B}\left(\epsilon^{\mathbf{B}}_{A}Do^{A}\right) = \epsilon o^{B} - \kappa\iota^{B}\,, \tag{2.18}
$$

and by similar calculations for other such derivatives we arrive at the relations

$$
\begin{aligned}
Do^{A} &= \epsilon\, o^{A} - \kappa\, \iota^{A}, && (2.19a)\\
D\iota^{A} &= \pi\, o^{A} - \epsilon\, \iota^{A}, && (2.19b)\\
\Delta o^{A} &= \gamma\, o^{A} - \tau\, \iota^{A}, && (2.19c)\\
\Delta\iota^{A} &= \nu\, o^{A} - \gamma\, \iota^{A}, && (2.19d)\\
\delta o^{A} &= \beta\, o^{A} - \sigma\, \iota^{A}, && (2.19e)\\
\delta\iota^{A} &= \mu\, o^{A} - \beta\, \iota^{A}, && (2.19f)\\
\bar{\delta} o^{A} &= \alpha\, o^{A} - \rho\, \iota^{A}, && (2.19g)\\
\bar{\delta}\iota^{A} &= \lambda\, o^{A} - \alpha\, \iota^{A}. && (2.19h)
\end{aligned}
$$

2.3 Spinorial equations of gravitational field

Einstein equation, which describes the dynamics of gravitational field, involves contractions of the Riemann tensor, metric tensor and the energy-momentum tensor of the source fields. To take the first step towards acquiring the spinorial equivalent of the Einstein equation we decompose the Riemann tensor into the sum of symmetric spinors. We define (the sign of) the Riemann tensor[7] by the equation

$$
\left(\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha} - T_{\alpha\beta}{}^{\gamma}\nabla_{\gamma}\right)V^{\delta} = R_{\alpha\beta\gamma}{}^{\delta}V^{\gamma}\,, \tag{2.20}
$$

which holds for any vector V^{α} and where $T_{\alpha\beta}{}^{\gamma}$ stands for the torsion tensor. The covariant derivative we use is torsion free, therefore $T_{\alpha\beta}{}^{\gamma} = 0$ throughout this chapter.

The Riemann tensor $R_{\alpha\beta\gamma\delta}$ (with the fourth index lowered) possesses several symmetries. It is antisymmetric in the first two and in the last two indices, and it is symmetric under the permutation $\alpha\beta \leftrightarrow \gamma\delta$:

$$
R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]}\,, \tag{2.21}
$$

$$
R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}\,. \tag{2.22}
$$

[7] We also refer to the Riemann tensor $R_{\alpha\beta\gamma}{}^{\delta}$ as the *curvature tensor*.

Furthermore it satisfies two Bianchi identities[8]:

$$R_{[\alpha\beta\gamma]\delta} = 0\,, \tag{2.23}$$
$$\nabla_{[\epsilon} R_{\alpha\beta]\gamma\delta} = 0\,. \tag{2.24}$$

These relations fully describe all symmetries of the Riemann tensor. Note that the Einstein equation on its own does not capture these properties. Nevertheless, we need to reflect them in the spinorial description of gravitation and they will result in additional equations for spinor fields associated with gravitation. But let us first briefly discuss some aspects of these properties of the curvature tensor. The above symmetries are not independent. The interchange symmetry (2.22) is a consequence of symmetries (2.21) and of the Bianchi symmetry (2.23). Bianchi identities are consequence of the fact that the Riemann tensor is a commutator of covariant derivatives. A general procedure to derive such a relation is to express the operator $\nabla_{[\alpha}\nabla_{\beta}\nabla_{\gamma]}$ (acting on some object) once as $\nabla_{[[\alpha}\nabla_{\beta]}\nabla_{\gamma]}$ and then as $\nabla_{[\alpha}\nabla_{[\beta}\nabla_{\gamma]]}$ and compare the resulting expressions.

To prove the relation (2.23), we apply the operator $\nabla_{[\alpha}\nabla_{\beta}\nabla_{\gamma]}$ on an arbitrary function f and compare the expression[9] $\nabla_{[\alpha}\nabla_{[\beta}\nabla_{\gamma]]}f = \nabla_{[\alpha}\left(\frac{1}{2}T_{\beta\gamma]}{}^{\delta}\nabla_{\delta}f\right)$ with the expression $\nabla_{[[\alpha}\nabla_{\beta]}\nabla_{\gamma]}f = -\frac{1}{2}R_{[\alpha\beta\gamma]}{}^{\delta}\nabla_{\delta}f$. We then easily get (2.23) by the assumption of vanishing torsion.

Searching for a proof of (2.24) one should apply $\nabla_{[\alpha}\nabla_{\beta}\nabla_{\gamma]}$ on a vector. The outlined procedure then leads to equations

$$\nabla_{[\alpha}\nabla_{\beta}\nabla_{\gamma]}V^{\delta} = \nabla_{[\alpha}\nabla_{[\beta}\nabla_{\gamma]]}V^{\delta} = \nabla_{[\alpha}\left(\frac{1}{2}R_{\beta\gamma]\rho}{}^{\delta}V^{\rho}\right) = \frac{1}{2}\left(\nabla_{[\alpha}R_{\beta\gamma]\rho}{}^{\delta}\right)V^{\rho} + \frac{1}{2}\left(\nabla_{[\alpha}V^{\rho}\right)R_{\beta\gamma]\rho}{}^{\delta} \tag{2.25}$$

and[10]

$$\nabla_{[\alpha}\nabla_{\beta}\nabla_{\gamma]}V^{\delta} = \nabla_{[[\alpha}\nabla_{\beta]}\nabla_{\gamma]}V^{\delta} = \frac{1}{2}R_{[\alpha\beta|\rho|}{}^{\delta}\nabla_{\gamma]}V^{\rho} - \frac{1}{2}R_{[\alpha\beta\gamma]}{}^{\rho}\nabla_{\rho}V^{\delta}\,. \tag{2.26}$$

The term $\frac{1}{2}\left(\nabla_{[\alpha}V^{\rho}\right)R_{\beta\gamma]\rho}{}^{\delta}$ of the first equation, (2.25), equals the term $\frac{1}{2}R_{[\alpha\beta|\rho|}{}^{\delta}\nabla_{\gamma]}V^{\rho}$ of the second equation (2.26). Furthermore, the second term of (2.26) is zero by Bianchi symmetry (2.23). Therefore, equating (2.25) with (2.26), we obtain the Bianchi identity (2.24).

The procedure described above is applicable beyond the realms of general relativity as well. For example, the electromagnetic (Maxwell) tensor $F_{\alpha\beta}$ may be regarded as the commutator of gauge-covariant derivatives, and progressing as above, we obtain one half of Maxwell equations, the equation $\nabla_{[\gamma}F_{\alpha\beta]} = 0$. Therefore, we may consider the equation $\nabla_{[\gamma}F_{\alpha\beta]} = 0$ to be an analogue of the Bianchi identity (2.24) for the case of electromagnetic field.

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[8] If the torsion was non-zero, Bianchi symmetries would acquire a little more complicated form:

$$R_{[\alpha\beta\gamma]\delta} + \nabla_{[\alpha}T_{\beta\gamma]\delta} + T_{[\alpha\beta}{}^{\epsilon}T_{\gamma]\epsilon\delta} = 0\,,$$
$$\nabla_{[\epsilon}R_{\alpha\beta]\gamma\delta} + T_{[\epsilon\alpha}{}^{\zeta}R_{\beta]\zeta\gamma\delta} = 0\,.$$

[9] We first use $\nabla_{[\beta}\nabla_{\gamma]}f = \frac{1}{2}T_{\beta\gamma}{}^{\delta}\nabla_{\delta}f$, and then apply the anti-symmetrization in all three indices. Similarly, in the second case, we first express the commutator of derivatives acting on covariant vector $\nabla_{\gamma}f$ using the Riemann tensor, and only then apply anti-symmetrization in all indices.

[10] Here we apply the commutator $\nabla_{[\alpha}\nabla_{\beta]}$ on a tensor of higher valence, namely $\nabla_{\gamma}V^{\delta}$. We do not examine such an operation (and many other important aspects of the theory) in this work, since we assume that the reader is already familiar with the tensor calculus and general relativity. Nevertheless, using the Leibniz rule and the fact that the covariant derivative does commute with the operation of contraction, we may find out what an action of $\nabla_{[\alpha}\nabla_{\beta]}$ on a general tensor looks like in similar fashion as we generalized the action of covariant derivative from the action on contravariant spin-vectors to an action on general spinors. Specifically, one can easily check that $\nabla_{[\alpha}\nabla_{\beta]}\left(V^{\gamma}W^{\delta}\right) = W^{\delta}\nabla_{[\alpha}\nabla_{\beta]}V^{\gamma} + V^{\gamma}\nabla_{[\alpha}\nabla_{\beta]}W^{\delta}$ and that $\nabla_{[\alpha}\nabla_{\beta]}V_{\gamma} = -\frac{1}{2}R_{\alpha\beta\gamma}{}^{\delta}V_{\delta}$.

Before we finally rewrite the curvature tensor in spinor terms and find what properties of curvature spinors do the above relations imply, we need to introduce one rather helpful identity. We shall make use of it while searching for spinorial analogues of Bianchi identities. Let us consider a tensor $G_{\alpha\beta\mathcal{B}}$ that is skew in indices $\alpha\beta$. We define its dual ${}^*G_{\alpha\beta\mathcal{B}}$ by

$$ {}^*G_{\alpha\beta\mathcal{B}} = \frac{1}{2} e_{\alpha\beta}{}^{\gamma\delta} G_{\gamma\delta\mathcal{B}} \,. $$

Now write $\gamma\mathcal{A}$ for the composite index $\mathcal{B}$. We wish to prove the following equivalence:

$$ G_{[\alpha\beta\gamma]\mathcal{A}} = 0 \Leftrightarrow {}^*G^{\alpha\beta}{}_{\alpha\mathcal{A}} = 0 \,. \tag{2.27} $$

To prove the implication from right to left, rewrite $G_{[\alpha\beta\gamma]\mathcal{A}}$ as

$$ G_{[\alpha\beta\gamma]\mathcal{A}} = \delta^{\kappa}_{[\alpha}\delta^{\lambda}_{\beta}\delta^{\mu}_{\gamma]} G_{\kappa\lambda\mu\mathcal{A}} = -\frac{1}{6} e_{\alpha\beta\gamma\delta} e^{\kappa\lambda\mu\delta} G_{\kappa\lambda\mu\mathcal{A}} = -\frac{1}{3} e_{\alpha\beta\gamma\delta} {}^*G^{\mu\delta}{}_{\mu\mathcal{A}} \,. \tag{2.28} $$

The implication in the other direction is given by

$$ {}^*G^{\alpha\beta}{}_{\alpha\mathcal{A}} = \frac{1}{2} e^{\alpha\beta\gamma\delta} G_{\gamma\delta\alpha\mathcal{A}} = \frac{1}{2} e^{\alpha\beta\gamma\delta} G_{[\gamma\delta\alpha]\mathcal{A}} \,. \tag{2.29} $$

Let us now rewrite the Riemann tensor in spinor terms. We want to decompose it into the sum of direct products of ϵs with symmetric spinors, but we shall not directly apply the procedure that we outlined when we proved that such decomposition is always possible. Rather, we will follow the symmetries of the tensor. The Riemann tensor $R_{\alpha\beta\gamma\delta}$ is skew in $\alpha\beta$ and in $\gamma\delta$. By the antisymmetry in the first two indices, we have

$$ R_{\alpha\beta\gamma\delta} = \frac{1}{2} R_{AX'B}{}^{X'}{}_{\gamma\delta}\epsilon_{A'B'} + \frac{1}{2} R_{XA'}{}^{X}{}_{B'\gamma\delta}\epsilon_{AB} \,. \tag{2.30} $$

The antisymmetry in the last two indices then implies

$$ \begin{aligned} R_{\alpha\beta\gamma\delta} = & \frac{1}{4} R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'}\epsilon_{A'B'}\epsilon_{C'D'} + \frac{1}{4} R_{AX'B}{}^{X'}{}_{YC'}{}^{Y}{}_{D'}\epsilon_{A'B'}\epsilon_{CD} \\ & + \frac{1}{4} R_{XA'}{}^{X}{}_{B'CY'D}{}^{Y'}\epsilon_{AB}\epsilon_{C'D'} + \frac{1}{4} R_{XA'}{}^{X}{}_{B'YC'}{}^{Y}{}_{D'}\epsilon_{AB}\epsilon_{CD} \,. \end{aligned} \tag{2.31} $$

Using the symbol X_{ABCD} for $\frac{1}{4} R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'}$ and the symbol $\Phi_{ABC'D'}$ for $\frac{1}{4} R_{AX'B}{}^{X'}{}_{YC'}{}^{Y}{}_{D'}$, the previous equation acquires the form[11] [12]

$$ R_{\alpha\beta\gamma\delta} = X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \overline{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \overline{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \,. \tag{2.32} $$

[11] Since the Riemann tensor is real, we have

$$ \begin{aligned} \overline{X}_{A'B'C'D'} = & \overline{X_{ABCD}} = \overline{\frac{1}{4} R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'}} = \frac{1}{4} \overline{R_{AA'BB'CC'DD'}\epsilon^{A'B'}\epsilon^{C'D'}} \\ = & \frac{1}{4} \overline{R}_{AA'BB'CC'DD'}\epsilon^{AB}\epsilon^{CD} = \frac{1}{4} R_{XA'}{}^{X}{}_{B'YC'}{}^{Y}{}_{D'} \,, \end{aligned} $$

and similarly for $\overline{\Phi}_{A'B'CD}$.

[12] What follows is a rather long and technical discussion on properties of spinors X_{ABCD} and $\Phi_{ABC'D'}$. Unless the reader is already familiar with the spinorial description of the general relativity, it may beneficial to reveal some results in advance. We will decompose the spinor X_{ABCD} in the terms of totally symmetric spinor Ψ_{ABCD} (the Weyl spinor) and the real scalar Λ. The Weyl spinor does not enter the Einstein equation and describes the "free gravitational field". On the other hand, the Ricci spinor $\Phi_{ABC'D'}$ and the scalar Λ are fully determined by the matter field through the Einstein equation, which — in the spinorial formalism — is a purely algebraical relation. Differential equations for the gravitational spinor fields will be provided by the Bianchi identity (2.24). Therefore, the results of the following exposition, while they may appear to be overly technical, they are not devoid of physical significance.

Because of the symmetry (2.21) of the curvature tensor, we have

$$X_{ABCD} = X_{(AB)(CD)}\,, \tag{2.33}$$

$$\Phi_{ABC'D'} = \Phi_{(AB)(C'D')}\,, \tag{2.34}$$

for spinors X_{ABCD} and $\Phi_{ABC'D'}$. Spinor $\Phi_{ABC'D'}$ is therefore symmetric, which means that as far as only terms involving spinors $\Phi_{ABC'D'}$ and $\overline{\Phi}_{A'B'CD}$ in equation (2.32) are concerned, we have already achieved the form we desire. On the other hand, spinor X_{ABCD} does not possess such a full symmetry and we wish to decompose it further. But before we attempt to do so, let us examine what are the consequences of the interchange symmetry (2.22) on these two spinors. Equating (2.32) and the same expression with the indices $\alpha\beta$ and $\gamma\delta$ interchanged, and then contracting the resulting equation with suitable products of ϵs, we obtain:

$$X_{ABCD} = X_{CDAB}\,, \tag{2.35}$$

$$\Phi_{ABC'D'} = \overline{\Phi}_{C'D'AB}\,. \tag{2.36}$$

Hence, the spinor $\Phi_{ABA'B'}$ is a real world-tensor, and, due to its symmetry (2.34), it is trace-free, i.e. $\Phi_\alpha{}^\alpha = 0$. Another outcome of symmetries (2.21), (2.22) — a one that we shall immediately put in use — is that $X_{A(BCD)} = X_{(ABCD)}$. To see this, simply observe that — employing symmetries (2.33) and (2.35) — each term in symmetrization $X_{(ABCD)}$ is equal to some term $X_{A\circ\circ\circ}$ with the index A in the first position. The second index of such a $X_{A\circ\circ\circ}$ is determined unambiguously by the term of $X_{(ABCD)}$ under consideration. The order of the last two indices is not fixed, but also — due to symmetry (2.33) — does not matter. Lastly, both $X_{(ABCD)}$ and $X_{A(BCD)}$ are totally symmetric in indices BCD. Therefore it is clear that shifting indices A in all terms of the $X_{(ABCD)}$ to the first position results in the tensor $X_{A(BCD)}$.

The last result suggests that in order to decompose the spinor X_{ABCD}, we should try to isolate the tensor $X_{A(BCD)}$. Since X_{ABCD} is already symmetric in CD, we have

$$\begin{aligned} X_{A(BCD)} =& \frac{1}{3}\left(X_{ABCD} + X_{ACBD} + X_{ADCB}\right) \\ =& X_{ABCD} + \frac{1}{3}\left(X_{ACDB} - X_{ABDC}\right) + \left(X_{ADCB} - X_{ABCD}\right) \\ =& X_{ABCD} + \frac{1}{3}X_{AYD}{}^{Y}\epsilon_{CB} + \frac{1}{3}X_{AYC}{}^{Y}\epsilon_{DB}\,. \end{aligned} \tag{2.37}$$

The interchange symmetry (2.35) implies that the spinor $X_{AYB}{}^{Y}$ is antisymmetric. Hence it equals $\frac{1}{2}X_{ZY}{}^{ZY}\epsilon_{AB}$. Denoting $\frac{1}{6}X_{ZY}{}^{ZY}$ by Λ, the previous equation acquires the form

$$X_{ABCD} = \Psi_{ABCD} + \Lambda\left(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD}\right)\,, \tag{2.38}$$

where Ψ_{ABCD} stands for $X_{(ABCD)}$. Substituting this result into (2.32), we obtain the Riemann tensor in the terms of totally symmetric spinors Λ, $\Phi_{ABC'D'}$, Ψ_{ABCD} multiplied by ϵs, which concludes the first part of our program.

We still haven't found what properties must spinors Λ, $\Phi_{ABC'D'}$ and Ψ_{ABCD} possess in order to satisfy the Bianchi identities (2.23) and (2.24). Consider first the symmetry (2.23):

$$R_{[\alpha\beta\gamma]\delta} = 0\,.$$

The identity (2.27) shows us that the above equation is equivalent to the equation

$$^{*}R^{\alpha\beta}{}_{\alpha\delta} = 0\,, \tag{2.39}$$

where dualization is applied to the first two indices of the curvature tensor. It is much easier to spinorially interpret the Bianchi symmetry in this form than in the original one (2.23). That is because the dualization of the tensor $G_{\alpha\beta\mathcal{A}}$ antisymmetric in the indices $\alpha\beta$ amounts to a simple interchange of spinor indices $A'B'$ followed by the multiplication by the imaginary unit i (cf. (1.105)). Applying this on (2.32), the Bianchi symmetry may be expressed as

$$\begin{aligned}
&{}^*R^{\alpha\beta}{}_{\alpha\delta} = \\
&= \mathrm{i}\left(X^{AB}{}_{AD}\epsilon^{B'A'}\epsilon_{A'D'} + \Phi^{AB}{}_{A'D'}\epsilon^{B'A'}\epsilon_{AD} + \overline{\Phi}^{B'A'}{}_{AD}\epsilon^{AB}\epsilon_{A'D'} + \overline{X}^{B'A'}{}_{A'D'}\epsilon^{AB}\epsilon_{AD}\right) \\
&= \mathrm{i}\left(-X^{AB}{}_{AD}\epsilon_{D'}{}^{B'} + \Phi_D{}^{BB'}{}_{D'} - \overline{\Phi}_{D'}{}^{B'B}{}_D + \overline{X}^{A'B'}{}_{A'D'}\epsilon_D{}^B\right) \\
&= \mathrm{i}\left(-X^{AB}{}_{AD}\epsilon_{D'}{}^{B'} + \overline{X}^{A'B'}{}_{A'D'}\epsilon_D{}^B\right) = 0\,,
\end{aligned} \tag{2.40}$$

where we have used the reality of the world-tensor $\Phi_{ABA'B'}$[13]. Using the decomposition (2.38), one can easily see that the last line of the previous equation may be transcribed as $\Lambda - \overline{\Lambda} = 0$. The Bianchi symmetry (2.23) is therefore equivalent to the reality condition on Λ.

We approach the Bianchi identity (2.24), $\nabla_{[\epsilon}R_{\alpha\beta]\gamma\delta} = 0$, in a similar fashion. Tensor $\nabla_\epsilon R_{\alpha\beta\gamma\delta}$ is antisymmetric in indices $\alpha\beta$ and $\gamma\delta$, but not in any pair involving the index ϵ. Therefore, to apply the identity (2.27), we must dualize it with respect to indices $\alpha\beta$. The Bianchi identity is thus equivalent to

$$\nabla_\alpha {}^*R^{\alpha\beta}{}_{\gamma\delta} = 0\,. \tag{2.41}$$

Substituting for $R_{\alpha\beta\gamma\delta}$ from (2.32), we obtain

$$\begin{aligned}
&\nabla_\alpha {}^*R^{\alpha\beta}{}_{\gamma\delta} = \\
&= \mathrm{i}\nabla_{AA'}\left(X^{AB}{}_{CD}\epsilon^{B'A'}\epsilon_{C'D'} + \Phi^{AB}{}_{C'D'}\epsilon^{B'A'}\epsilon_{CD} + \Phi^{B'A'}{}_{CD}\epsilon^{AB}\epsilon_{C'D'} + \overline{X}^{B'A'}{}_{C'D'}\epsilon^{AB}\epsilon_{CD}\right) \\
&= \mathrm{i}\left(\epsilon_{C'D'}\nabla^{B'}_A X^{AB}{}_{CD} + \epsilon_{CD}\nabla^{B'}_A\Phi^{AB}{}_{C'D'} - \epsilon_{C'D'}\nabla^B_{A'}\Phi^{A'B'}{}_{CD} - \epsilon_{CD}\nabla^B_{A'}X^{A'B'}{}_{C'D'}\right) = 0\,.
\end{aligned} \tag{2.42}$$

Contracting the last line with ϵ^{CD} (or $\epsilon^{C'D'}$), we find that the spinor equivalent of the Bianchi identity (2.24) is the equation

$$\nabla^A_{B'}X_{ABCD} = \nabla^{A'}_B\Phi_{A'B'CD}\,, \tag{2.43}$$

or, substituting for X_{ABCD} from (2.38), the equation

$$\nabla^A_{B'}\Psi_{ABCD} + \epsilon_{BD}\nabla_{CB'}\Lambda + \epsilon_{BC}\nabla_{DB'}\Lambda = \nabla^{A'}_B\Phi_{A'B'CD}\,. \tag{2.44}$$

We may isolate the part containing the spinor Ψ_{ABCD} from the part containing the scalar Λ by symmetrizing the above equation in indices BCD, thus obtaining

$$\nabla^A_{B'}\Psi_{ABCD} = \nabla^{A'}_{(B}\Phi_{CD)A'B'}\,. \tag{2.45}$$

The rest of the Bianchi identity (2.44) is then equivalent to the condition[14]

$$\nabla^{AA'}\Phi_{ADA'B'} + 3\nabla_{DB'}\Lambda = 0\,. \tag{2.46}$$

[13] In the following text we shall no longer differentiate between spinors $\overline{\Phi}_{A'B'CD}$ and $\Phi_{CDA'B'}$.

[14] To prove it, simply decompose the spinor $\nabla^{A'}_B\Phi_{A'B'CD}$ as follows:

$$\nabla^{A'}_B\Phi_{A'B'CD} = \nabla^{A'}_{(B}\Phi_{CD)A'B'} + \frac{1}{3}\epsilon_{BC}\nabla^{A'}_A\Phi^A{}_{DA'B'} + \frac{1}{3}\epsilon_{BD}\nabla^{A'}_A\Phi^A{}_{CA'B'}\,.$$

Now that we have concluded the discussion of the curvature tensor and its spinorial description, we are ready to investigate what is the spinorial equivalent of the Einstein equation

$$\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) + \lambda g_{\alpha\beta} = -8\pi G T_{\alpha\beta}\,, \tag{2.47}$$

where λ stands for the cosmological constant and G for the gravitational constant. Tensor $T_{\alpha\beta}$ is the energy-momentum tensor of the matter fields. Here we shall be not concerned with its origin or any properties other than that it is symmetric. The symbol $R_{\alpha\beta}$ represents the Ricci tensor $R_{\alpha\beta} = R_{\alpha\delta\beta}{}^{\delta}$ and R is the scalar curvature R^{α}_{α}. Since both these quantities are derived from the curvature tensor, we are readily able to express them in spinor terms. By (2.32) and (2.38), we get

$$\begin{aligned}R_{\alpha\beta} =&R_{\alpha\delta\beta}{}^{\delta}\\ =&X_{ADB}{}^{D}\epsilon_{A'B'} - 2\Phi_{ABA'B'} + \overline{X}_{A'D'B'}{}^{D'}\epsilon_{AB}\\ =&6\Lambda\epsilon_{AB}\epsilon_{A'B'} - 2\Phi_{ABA'B'}\end{aligned} \tag{2.48}$$

for the Ricci tensor[15], and

$$R = R^{\alpha}_{\alpha} = 24\Lambda \tag{2.49}$$

for the scalar curvature. Substituting these results into (2.47), we obtain[16]

$$\Phi_{ABA'B'} + 3\left(\Lambda - \frac{1}{6}\lambda\right)\epsilon_{AB}\epsilon_{A'B'} = 4\pi G T_{ABA'B'}\,. \tag{2.50}$$

We may split the equation (2.50) into its trace-free part and its trace, thus obtaining

$$\Phi_{\alpha\beta} = 4\pi G\left(T_{\alpha\beta} - \frac{1}{4}T^{\gamma}_{\gamma}g_{\alpha\beta}\right) = 4\pi T_{(AB)(A'B')}\,, \tag{2.51}$$

and

$$\Lambda = \frac{1}{3}\left(\pi G T^{\gamma}_{\gamma} + \frac{1}{2}\lambda\right)\,. \tag{2.52}$$

To summarize, using the symmetries (2.21), (2.22) and (2.23), we have managed to rewrite the curvature tensor $R_{\alpha\beta\gamma\delta}$ in terms of a real scalar Λ, a real symmetric spinor $\Phi_{ABC'D'}$ and a symmetric spinor Ψ_{ABCD}:

$$\begin{aligned}R_{\alpha\beta\gamma\delta} = &\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}\\ &+ \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \Phi_{CDA'B'}\epsilon_{AB}\epsilon_{C'D'}\\ &+ \Lambda\left[\left(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}\right)\epsilon_{A'B'}\epsilon_{C'D'} + \left(\epsilon_{A'C'}\epsilon_{B'D'} + \epsilon_{A'D'}\epsilon_{B'C'}\right)\epsilon_{AB}\epsilon_{CD}\right].\end{aligned} \tag{2.53}$$

Substituting this into (2.44) and using (2.45), we arrive at the equation

$$3\epsilon_{B(C}\nabla_{D)B'}\Lambda = -\epsilon_{B(C}\nabla^{AA'}\Phi_{D)AA'B'}\,,$$

which, after contraction with ϵ^{BC}, gives (and is equivalent to the) (2.46).

[15]Equation (2.48) reveals that the spinor $\Phi_{ABA'B'}$ is proportional to the trace-free part of the Ricci tensor.

[16]In principle, we do not need to substitute for the scalar curvature, since its only function in the Einstein equation is to reverse the trace of the Ricci tensor. Trace reversal of the symmetric tensor $T_{\alpha\beta}$ amounts to simple interchange of spinorial indices A and B (or A' and B'), c.f. (1.99).

If the tensor $R_{\alpha\beta\gamma\delta}$ is to describe the curvature of the physical space, then, according to the theory of general relativity, gravitational spinors Λ, $\Phi_{ABC'D'}$ and Ψ_{ABCD} must satisfy the Einstein equations

$$\Phi_{\alpha\beta} = 4\pi G\left(T_{\alpha\beta} - \frac{1}{4}T^{\gamma}_{\gamma}g_{\alpha\beta}\right), \tag{2.54}$$

$$\Lambda = \frac{1}{3}\pi G T^{\gamma}_{\gamma} + \frac{1}{6}\lambda\,. \tag{2.55}$$

We see that in the absence of matter ($T_{\alpha\beta} = 0$), and if the gravitational constant λ is zero, the only non-vanishing part of the curvature tensor may be the spinor Ψ_{ABCD}, which is therefore called *the Weyl spinor*. The first part of the decomposition (2.53) is the well-known *Weyl tensor*

$$C_{\alpha\beta\gamma\delta} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}\,. \tag{2.56}$$

Its anti-self-dual part will be denoted by

$$\psi_{\alpha\beta\gamma\delta} = \Psi_{ABCD}\epsilon_{A'B}\epsilon_{C'D'}\,. \tag{2.57}$$

The spinor $\Phi_{ABA'B'}$, which is — up to the factor of proportionality — equal to the trace-free part of the Ricci tensor, is often referred to as *the Ricci spinor*. We introduce symbol ϕ_{abcd} for anti-self-dual form

$$\phi_{\alpha\beta\gamma\delta} = \Phi_{ABC'D'}\,\epsilon_{A'B'}\,\epsilon_{CD}\,. \tag{2.58}$$

Anti-self-dual form related to the scalar curvature will be denoted by

$$\lambda_{\alpha\beta\gamma\delta} = \Lambda\,(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})\,\epsilon_{A'B'}\epsilon_{C'D'}. \tag{2.59}$$

Finally, there are two more equations that the above gravitational spinors need to satisfy:

$$\nabla^{A}_{B'}\Psi_{ABCD} = \nabla^{A'}_{(B}\Phi_{CD)A'B'}\,, \tag{2.60}$$

$$\nabla^{AA'}\Phi_{ADA'B'} + 3\nabla_{DB'}\Lambda = 0\,. \tag{2.61}$$

These equations constitute the spinorial analogue of the Bianchi identity (2.24). In the spinorial formulation of Einstein's theory of gravitation, these are the differential (field) equations for the gravitational field[17].

To close this section, we will discuss how to reformulate the above results into the language of the Newman-Penrose formalism. As we have already mentioned, we use the following notation for the projections of a symmetric spinor $\xi_{A\ldots DA'\ldots D'}$ onto the spin-frame[18]:

$$\xi_{n,m} = \overbrace{o^A\ldots o^B}^{N-n}\,\overbrace{\iota^C\ldots\iota^D}^{n}\,\overbrace{\bar{o}^{A'}\ldots\bar{o}^{B'}}^{M-m}\,\overbrace{\bar{\iota}^{C'}\ldots\bar{\iota}^{D'}}^{m}\,\xi_{A\ldots BC\ldots DA'\ldots B'C'\ldots D'}\,. \tag{2.62}$$

For example, we write Φ_{01} for $\Phi_{ABC'D'}o^A o^B \bar{o}^{C'}\bar{\iota}^{D'}$, or Ψ_3 for $\Psi_{ABCD}o^A\iota^B\iota^C\iota^D$[19].

[17] Consider, for example, the Einstein equation in vacuum. By (2.51) and (2.52) we have $\Phi_{ABA'B'} = 0$, $\Lambda = \frac{1}{6}\lambda$. The Bianchi identity is then equivalent to the differential equation

$$\nabla^{AA'}\Psi_{ABCD}\,.$$

Formally, the above equation is the wave equation for a zero rest-mass spin 2 particle (in this case the graviton).

[18] Where the spinor $\xi_{A\ldots DA'\ldots D'}$ has N unprimed and M primed indices.

[19] If the spinor does not possess both primed and unprimed indices, we use only one suffix to denote its projections. As an example, for spinor Ψ_{ABCD} — which does not have any primed indices — we write Ψ_3, not Ψ_{30}, for $\Psi_{ABCD}o^A\iota^B\iota^C\iota^D$.

We wish to find projections of the Einstein equation (2.50) and the Bianchi identity (2.44). The Einstein equation is purely algebraic and thus it is straightforward to project it. The Bianchi identity is little more tricky, since it involves differentiation. Consider a spinor $\nabla_{XX'}\xi_{A\dots DA'\dots D'}$. To find its N-P components, let us first decompose the spinor $\xi_{A\dots DA'\dots D'}$ as follows:

$$\xi_{A\dots BC\dots DA'\dots B'C'\dots D'} = \sum_{n,m} N!\,M!\,\xi_{n,m}\,\overbrace{(-\iota)_{(A}\dots(-\iota)_{B}}^{N-n}\,\overbrace{o_{C}\dots o_{D)}}^{n}\,\overbrace{(-\bar{\iota})_{(A'}\dots(-\bar{\iota})_{B'}}^{M-m}\,\overbrace{\bar{o}_{C'}\dots\bar{o}_{D')}}^{m}\,. \tag{2.63}$$

Applying the operator $\nabla_{XX'}$ and projecting onto the spin-frame, we obtain[20]

$$\begin{aligned}\overbrace{o^{A}\dots o^{B}}^{N-n}\overbrace{\iota^{C}\dots\iota^{D}}^{n}\overbrace{\bar{o}^{A'}\dots\bar{o}^{B'}}^{M-m}\overbrace{\bar{\iota}^{C'}\dots\bar{\iota}^{D'}}^{m}\nabla_{XX'}\xi_{A\dots DA'\dots D'} &= \nabla_{XX'}\xi_{n,m}\\ +n\xi_{n,m}\iota^{A}\nabla_{XX'}o_{A} &+ (N-n)\,\xi_{n,m}o^{A}\nabla_{XX'}(-\iota_{A})\\ +(N-n)\,\xi_{(n+1),m}o^{A}\nabla_{XX'}o_{A} &+ n\xi_{(n-1),m}\iota^{A}\nabla_{XX'}(-\iota_{A})\\ +m\xi_{n,m}\bar{\iota}^{A'}\nabla_{XX'}\bar{o}_{A'} &+ (M-m)\,\xi_{n,m}\bar{o}^{A'}\nabla_{XX'}(-\bar{\iota}_{A'})\\ +(M-m)\,\xi_{n,(m+1)}\bar{o}^{A'}\nabla_{XX'}\bar{o}_{A'} &+ m\xi_{n,(m-1)}\bar{\iota}^{A'}\nabla_{XX'}(-\bar{\iota}_{A'})\,.\end{aligned} \tag{2.64}$$

Projecting the operator $\nabla_{XX'}$ of the last equation onto the directions of the null tetrad and using equations (2.19a) - (2.19h) (or directly the definitions of the rotation coefficients), we get the equations

$$\begin{aligned}&\overbrace{o^{A}\dots o^{B}}^{N-n}\overbrace{\iota^{C}\dots\iota^{D}}^{n}\overbrace{\bar{o}^{A'}\dots\bar{o}^{B'}}^{M-m}\overbrace{\bar{\iota}^{C'}\dots\bar{\iota}^{D'}}^{m}D\xi_{A\dots DA'\dots D'} = D\xi_{n,m} + (2n-N)\,\epsilon\xi_{n,m}\\ &+(2m-M)\,\bar{\epsilon}\xi_{n,m} + (N-n)\,\kappa\xi_{(n+1),m} - n\pi\xi_{(n-1),m} + (M-m)\,\bar{\kappa}\xi_{n,(m+1)} - m\bar{\pi}\xi_{n,(m-1)}\,,\end{aligned} \tag{2.65}$$

$$\begin{aligned}&\overbrace{o^{A}\dots o^{B}}^{N-n}\overbrace{\iota^{C}\dots\iota^{D}}^{n}\overbrace{\bar{o}^{A'}\dots\bar{o}^{B'}}^{M-m}\overbrace{\bar{\iota}^{C'}\dots\bar{\iota}^{D'}}^{m}\Delta\xi_{A\dots DA'\dots D'} = \Delta\xi_{n,m} + (2n-N)\,\gamma\xi_{n,m}\\ &+(2m-M)\,\bar{\gamma}\xi_{n,m} + (N-n)\,\tau\xi_{(n+1),m} - n\nu\xi_{(n-1),m} + (M-m)\,\bar{\tau}\xi_{n,(m+1)} - m\bar{\nu}\xi_{n,(m-1)}\,,\end{aligned} \tag{2.66}$$

$$\begin{aligned}&\overbrace{o^{A}\dots o^{B}}^{N-n}\overbrace{\iota^{C}\dots\iota^{D}}^{n}\overbrace{\bar{o}^{A'}\dots\bar{o}^{B'}}^{M-m}\overbrace{\bar{\iota}^{C'}\dots\bar{\iota}^{D'}}^{m}\delta\xi_{A\dots DA'\dots D'} = \delta\xi_{n,m} + (2n-N)\,\beta\xi_{n,m}\\ &+(2m-M)\,\bar{\alpha}\xi_{n,m} + (N-n)\,\sigma\xi_{(n+1),m} - n\mu\xi_{(n-1),m} + (M-m)\,\bar{\rho}\xi_{n,(m+1)} - m\bar{\lambda}\xi_{n,(m-1)}\,,\end{aligned} \tag{2.67}$$

$$\begin{aligned}&\overbrace{o^{A}\dots o^{B}}^{N-n}\overbrace{\iota^{C}\dots\iota^{D}}^{n}\overbrace{\bar{o}^{A'}\dots\bar{o}^{B'}}^{M-m}\overbrace{\bar{\iota}^{C'}\dots\bar{\iota}^{D'}}^{m}\bar{\delta}\xi_{A\dots DA'\dots D'} = \bar{\delta}\xi_{n,m} + (2n-N)\,\alpha\xi_{n,m}\\ &+(2m-M)\,\bar{\beta}\xi_{n,m} + (N-n)\,\rho\xi_{(n+1),m} - n\lambda\xi_{(n-1),m} + (M-m)\,\bar{\sigma}\xi_{n,(m+1)} - m\bar{\mu}\xi_{n,(m-1)}\,.\end{aligned} \tag{2.68}$$

[20] The second and the fourth line of the equation (2.64) may be further simplified. We have $\iota^{A}\nabla_{XX'}o_{A} = o^{A}\nabla_{XX'}\iota_{A}$ due to the normalization of the spin-frame. The second line therefore equals $(2n-N)\xi_{n,m}\iota^{A}\nabla_{XX'}o_{A}$, while the fourth is $(2m-M)\xi_{n,m}\bar{\iota}^{A'}\nabla_{XX'}\bar{o}_{A'}$.

With the above results at our disposal, we can easily (though tediously) find all projections of the Bianchi identity. Let us maybe illustrate this by finding the projection of the equation (2.44) onto spinors $o^B\bar{o}^{B'}o^C o^D$. To project the left hand side, rewrite it[21] as follows:

$$\begin{aligned}\nabla^A_{B'}\Psi_{ABCD} =&\epsilon^{AX}\nabla_{XB'}\Psi_{ABCD} = \epsilon^{A\mathbf{A}}\epsilon_{\mathbf{A}}{}^X\nabla_{XB'}\Psi_{ABCD}\\ =&-\iota^A\nabla_{0B'}\Psi_{ABCD} + o^A\nabla_{1B'}\Psi_{ABCD}\,.\end{aligned} \tag{2.69}$$

Employing (2.65) and (2.68) we straightforwardly obtain

$$\begin{aligned}o^B\bar{o}^{B'}o^C o^D\nabla^A_{B'}\Psi_{ABCD} =&-\iota^A o^B\bar{o}^{B'}o^C o^D\nabla_{0B'}\Psi_{ABCD} + o^A o^B\bar{o}^{B'}o^C o^D\nabla_{1B'}\Psi_{ABCD}\\ =&-D\Psi_1 + 2\epsilon\Psi_1 - 3\kappa\Psi_2 + \pi\Psi_0 + \bar{\delta}\Psi_0 - 4\alpha\Psi_0 + 4\rho\Psi_1\,.\end{aligned} \tag{2.70}$$

Analogously, with the help of equations (2.65) and (2.67), we find that

$$\begin{aligned}o^B\bar{o}^{B'}o^C o^D\nabla^{A'}_B\Phi_{CDA'B'} =&-\bar{\iota}^{A'}o^B\bar{o}^{B'}o^C o^D\nabla_{B0'}\Phi_{A'B'CD} + \bar{o}^{A'}o^B\bar{o}^{B'}o^C o^D\nabla_{B1'}\Phi_{A'B'CD}\\ =&-D\Phi_{01} + 2\epsilon\Phi_{01} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} + \bar{\pi}\Phi_{00} + \delta\Phi_{00} - 2\beta\Phi_{00} - 2\bar{\alpha}\Phi_{00} + 2\sigma\Phi_{10} + 2\bar{\rho}\Phi_{01}\,.\end{aligned} \tag{2.71}$$

Equating the last two results gives us one of the projections of the Bianchi identity. Proceeding along similar lines for the other projections of equation (2.44) we obtain the following equations:

$$\begin{aligned}D\Psi_1 - \bar{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} =& (\pi - 4\alpha)\Psi_0 + 2(2\rho + \varepsilon)\Psi_1 - 3\kappa\Psi_2 + 2\kappa\Phi_{11}\\ &- (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} - 2\sigma\Phi_{10} - 2(\bar{\rho} + \varepsilon)\Phi_{01} + \bar{\kappa}\Phi_{02},\end{aligned} \tag{2.72a}$$

$$\begin{aligned}D\Psi_2 - \bar{\delta}\Psi_1 + \Delta\Phi_{00} - \bar{\delta}\Phi_{01} + 2D\Lambda =& -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3\\ &+2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} + (2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} - 2\tau\Phi_{10},\end{aligned} \tag{2.72b}$$

$$\begin{aligned}D\Psi_3 - \bar{\delta}\Psi_2 - D\Phi_{21} + \delta\Phi_{20} - 2\bar{\delta}\Lambda =& -2\lambda\Psi_1 + 3\pi\Psi_2 + 2(\rho - \varepsilon)\Psi_3 - \kappa\Psi_4\\ &+2\mu\Phi_{10} - 2\pi\Phi_{11} - (2\beta + \bar{\pi} - 2\bar{\alpha})\Phi_{20} - 2(\bar{\rho} - \varepsilon)\Phi_{21} + \bar{\kappa}\Phi_{22},\end{aligned} \tag{2.72c}$$

$$\begin{aligned}D\Psi_4 - \bar{\delta}\Psi_3 + \Delta\Phi_{20} - \bar{\delta}\Phi_{21} =& -3\lambda\Psi_2 + 2(\alpha + 2\pi)\Psi_3 + (\rho - 4\varepsilon)\Psi_4 + 2\nu\Phi_{10}\\ &-2\lambda\Phi_{11} - (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} - 2(\bar{\tau} - \alpha)\Phi_{21} + \bar{\sigma}\Phi_{22},\end{aligned} \tag{2.72d}$$

$$\begin{aligned}\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} =& (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2\\ &+(\bar{\rho} + 2\varepsilon - 2\bar{\varepsilon})\Phi_{02} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01},\end{aligned} \tag{2.72e}$$

$$\begin{aligned}\Delta\Psi_1 - \delta\Psi_2 - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} - 2\delta\Lambda =& \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3\\ &-\bar{\nu}\Phi_{00} + 2(\bar{\mu} - \gamma)\Phi_{01} + (2\alpha + \bar{\tau} - 2\bar{\beta})\Phi_{02} + 2\tau\Phi_{11} - 2\rho\Phi_{12},\end{aligned} \tag{2.72f}$$

$$\begin{aligned}\Delta\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\Delta\Lambda =& 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4\\ &-2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + 2(\beta + \bar{\pi})\Phi_{21} + (\bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\Phi_{22},\end{aligned} \tag{2.72g}$$

$$\begin{aligned}\Delta\Psi_3 - \delta\Psi_4 - \Delta\Phi_{21} + \bar{\delta}\Phi_{22} =& 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 - 2\nu\Phi_{11}\\ &-\bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22},\end{aligned} \tag{2.72h}$$

2.4 Ricci identities

The action of the commutator $\nabla_{[\alpha}\nabla_{\beta]}$ on a tensor can be expressed by the means of the Riemann tensor, i.e.

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\,\omega_\gamma = -R_{\alpha\beta\gamma\delta}\,\omega^\delta\,. \tag{2.73}$$

[21] Here we have already omitted terms involving ϵs since they are annihilated by the projection under consideration. (Spinor $o^B o^C o^D$ is of course symmetric in BCD.)

The above relation (and a similar one for a contravariant vector) is sometimes called *the Ricci identity*. In this section we shall show that when the commutator $\nabla_{[\alpha}\nabla_{\beta]}$ is applied on a spinor, the result is determined by the curvature spinors through equations which are formally very similar to the Ricci identity we just mentioned. For that reason, we shall call those equations the (spinorial) Ricci identities as well.

Let us first take a closer look at the commutator $\nabla_{[\alpha}\nabla_{\beta]}$. It is antisymmetric in indices $\alpha\beta$ and therefore may be decomposed as follows:

$$(\nabla_\alpha \nabla_\beta \,-\, \nabla_\beta \,\nabla_\alpha) = 2\nabla_{[\alpha}\nabla_{\beta]} = \epsilon_{AB}\Box_{A'B'} + \epsilon_{A'B'}\Box_{AB}\,, \tag{2.74}$$

where operators $\Box_{AB}$ and $\Box_{A'B'}$ are defined by

$$\Box_{AB} \;=\; \nabla_{X'(A}\nabla^{X'}_{B)}, \quad \Box_{A'B'} \;=\; \nabla_{X(A'}\nabla^{X}_{B')}. \tag{2.75}$$

These operators are merely contractions of the commutator $\nabla_{[\alpha}\nabla_{\beta]}$ with ϵ^{AB} and $\epsilon^{A'B'}$, respectively, e.g.

$$\begin{aligned}\epsilon^{AB}[\nabla_\alpha,\nabla_\beta] \;&=\; \epsilon^{AB}\,(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'}) \;=\; \nabla_{AA'}\nabla^{A}_{B'} - \nabla^{A}_{B'}\nabla_{AA'}\\ &=\; \nabla_{AA'}\nabla^{A}_{B'} \,+\, \nabla_{AB'}\nabla^{A}_{A'} \;=\; 2\,\nabla_{A(A'}\nabla^{A}_{B')} \;=\; 2\,\Box_{A'B'}.\end{aligned} \tag{2.76}$$

Consequently, $\Box_{AB}$ and $\Box_{A'B'}$ annihilate scalar quantities,

$$\Box_{AB}\phi \;=\; \Box_{A'B'}\phi \;=\; 0. \tag{2.77}$$

If we employ the notation we just introduced, we may write the Ricci identities in the form

$$\begin{aligned}&\Box_{AB}\xi_C = \Psi_{ABCD}\,\xi^D \,-\, 2\,\Lambda\,\xi_{(A}\epsilon_{B)C}, \quad \Box_{AB}\xi^B \;=\; -\,3\,\Lambda\,\xi_A,\\ &\Box_{(AB}\xi_{C)} = \Psi_{ABCD}\,\xi^D, \quad \Box_{A'B'}\xi_A \;=\; \Phi_{ABA'B'}\,\xi^B, \quad \Box_{AB}\xi_{A'} \;=\; \Phi_{ABA'B'}\,\xi^{B'}.\end{aligned} \tag{2.78}$$

The above identities are not all independent, but we list them for convenience.

Let us now prove the Ricci identities. It is a well known fact that while the operator $\nabla_a\nabla_b$ does not satisfy the Leibniz rule, the commutator $[\nabla_a,\nabla_b]$ *does*:

$$[\nabla_\alpha,\nabla_\beta](k_\gamma\omega_\delta) \;=\; k_\gamma[\nabla_\alpha,\nabla_\beta]\omega_\delta + \omega_\delta[\nabla_\alpha,\nabla_\beta]k_\gamma\,. \tag{2.79}$$

Since the operators $\Box_{AB}$ and $\Box_{A'B'}$ are just contractions of the commutator $[\nabla_\alpha,\nabla_\beta]$, they also satisfy the Leibniz rule. Although we do not know how the commutator acts on spinors, we know its action on *vectors*. Hence, we apply the commutator (2.74) to the null vector $\xi_C\bar{\xi}_{C'}$:

$$\bar{\xi}_{C'}\,(\epsilon_{AB}\Box_{A'B'} + \epsilon_{A'B'}\Box_{AB})\,\xi_C \,+\, \xi_C\,(\epsilon_{AB}\Box_{A'B'} + \epsilon_{A'B'}\Box_{AB})\,\bar{\xi}_{C'} \;=\; -\,R_{\alpha\beta\gamma\delta}\,\xi^D\,\bar{\xi}^{D'}. \tag{2.80}$$

We transvect this equation with ξ^C. It serves us in two ways. First, it annihilates the term $\xi_C\,(\epsilon_{AB}\Box_{A'B'} + \epsilon_{A'B'}\Box_{AB})\,\bar{\xi}_{C'}$, and secondly, it annihilates all terms in the decomposition (2.53) of the Riemann tensor that are antisymmetric in indices CD. Consequently, we arrive at

$$\begin{aligned}\xi^C\epsilon_{AB}\Box_{A'B'}\xi_C \,+\, \xi^C\epsilon_{A'B'}\Box_{AB}\xi_C \;&=\; \Psi_{ABCD}\,\epsilon_{A'B'}\,\xi^C\,\xi^D\\ &\quad+\, \Phi_{CDA'B'}\,\epsilon_{AB}\,\xi^C\,\xi^D \,+\, 2\Lambda\xi_A\xi_B\epsilon_{A'B'}.\end{aligned} \tag{2.81}$$

The last term can be written in the form of contraction with ξ^C preserving manifest symmetry in AB:

$$2\,\Lambda\,\xi_A\,\xi_B \;=\; 2\,\Lambda\,\epsilon_{C(A}\xi_{B)}\,\xi^C.$$

Thus we obtain

$$\xi^C\left(\epsilon_{AB}\Box_{A'B'}\xi_C+\epsilon_{A'B'}\Box_{AB}\xi_C\right)=$$
$$\xi^C\left(\Psi_{ABCD}\epsilon_{A'B'}\xi^D+\Phi_{CDA'B'}\epsilon_{AB}\xi^D+2\Lambda\epsilon_{C(A}\xi_{B)}\epsilon_{A'B'}\right).\tag{2.82}$$

Now we only need to show that the expression in the parentheses on the left hand side of the equation (2.82) equals the bracketed expression on the right hand side of the same equation. For then we could obtain the Ricci identities by simply separating the part symmetric in AB and the part symmetric in $A'B'$ of the resulting equation. Unfortunately, spinor ξ^C, though being arbitrary, is not independent of the bracketed expressions and therefore it is not obvious from (2.82) that those expressions are equal. To overcome this problem, substitute the expression $\kappa^C+k\lambda^C$ for spinor ξ^C, with k an arbitrary real (or complex) number. Both sides of the equation (2.82) then obtain a form of a second order polynomial in k. Obviously, the coefficients of each k^0, k^1 and k^2 on the left hand side must be equal to the corresponding coefficients on the right hand side. For coefficients of k^1 we get the equation

$$\kappa^C\epsilon_{AB}\Box_{A'B'}\lambda_C+\kappa^C\epsilon_{A'B'}\Box_{AB}\lambda_C+\lambda^C\epsilon_{AB}\Box_{A'B'}\kappa_C+\lambda^C\epsilon_{A'B'}\Box_{AB}\kappa_C = 2(\Psi_{ABCD}\,\epsilon_{A'B'}\,\kappa^C\,\lambda^D + \Phi_{CDA'B'}\,\epsilon_{AB}\,\kappa^C\,\lambda^D + 2\Lambda\,\epsilon_{C(A}\lambda_{B)}\kappa^C\epsilon_{A'B'}).\tag{2.83}$$

Due to the Leibniz rule (2.79) and the property (2.77) we have

$$\lambda^C\epsilon_{AB}\Box_{A'B'}\kappa_C=\epsilon_{AB}\Box_{A'B'}\left(\lambda^C\kappa_C\right)-\kappa_C\epsilon_{AB}\Box_{A'B'}\lambda^C=\kappa^C\epsilon_{AB}\Box_{A'B'}\lambda_C\,,\tag{2.84}$$

and similarly for $\lambda^C\epsilon_{A'B'}\Box_{AB}\kappa_C$. Applying these results, the equation (2.83) obtains the form

$$\kappa^C\left(\epsilon_{AB}\Box_{A'B'}\lambda_C + \epsilon_{A'B'}\Box_{AB}\lambda_C\right) = \kappa^C(\Psi_{ABCD}\,\epsilon_{A'B'}\,\lambda^D + \Phi_{CDA'B'}\,\epsilon_{AB}\,\lambda^D + 2\Lambda\,\epsilon_{C(A}\lambda_{B)}\epsilon_{A'B'}).\tag{2.85}$$

This time the (arbitrary) spinor κ^C is independent of the expressions in brackets (since it is independent of λ^C). Thus we have

$$\epsilon_{AB}\Box_{A'B'}\xi_C + \epsilon_{A'B'}\Box_{AB}\xi_C = \epsilon_{A'B'}\Psi_{ABCD}\xi^D + \Phi_{CDA'B'}\epsilon_{AB}\,\xi^D + 2\Lambda\epsilon_{C(A}\xi_{B)}\epsilon_{A'B'}.\tag{2.86}$$

Symmetrizing this identity first in AB, then in $A'B'$ gives us the Ricci identities

$$\Box_{A'B'}\xi_C = \Phi_{CDA'B'}\xi^D,\qquad \Box_{AB}\xi_C = \Psi_{ABCD}\xi^D + 2\Lambda\epsilon_{C(A}\xi_{B)}.$$

Remaining equations in (2.78) are merely variations on these results.

We may be interested to find the action of the whole commutator $2\nabla_{[\alpha}\nabla_{\beta]}$ on a spinor. By (2.74) and (2.78) we get

$$\begin{aligned}[\nabla_\alpha,\nabla_\beta]\xi_C &= (\epsilon_{AB}\Box_{A'B'} + \epsilon_{A'B'}\Box_{AB})\,\xi_C\\ &= \epsilon_{A'B'}\,\Psi_{ABCD}\,\xi^D + \epsilon_{AB}\,\Phi_{CDA'B'}\,\xi^D + 2\,\Lambda\,\epsilon_{A'B'}\,\epsilon_{C(A}\xi_{B)}.\end{aligned}\tag{2.87}$$

The expression on the right hand side is similar to the anti-self-dual-part of the Riemann tensor. Indeed, let us define spinor

$$\begin{aligned}R_{ABCDA'B'} = R_{\alpha\beta CD} &= \frac{1}{2}\,\epsilon^{C'D'}\,R_{\alpha\beta\gamma\delta}\\ &= \Psi_{ABCD}\,\epsilon_{A'B'} + \Phi_{CDA'B'}\,\epsilon_{AB} - 2\,\Lambda\,\epsilon_{A(C}\,\epsilon_{D)B}\,\epsilon_{A'B'}.\end{aligned}\tag{2.88}$$

Then we can write

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\,\xi_C = -\,R_{\alpha\beta C}{}^D\,\xi_D = R_{\alpha\beta CD}\,\xi^D.\tag{2.89}$$

2.5 Differential equations for the spin coefficients

The spin coefficients, which were introduced in the second section of this chapter, are defined by the equation (2.12) which reads

$$\gamma_{\mathbf{AA'C}}{}^{\mathbf{B}} = \epsilon_A{}^{\mathbf{B}} \nabla_{\mathbf{AA'}} \epsilon_{\mathbf{C}}{}^A \,.$$

The action of the covariant derivative in the Newman-Penrose formalism is fully specified through these coefficients. Consequently, they need to suitably reflect the properties of the derivative.

There are two sets of differential equations which the scalar fields $\gamma_{\mathbf{AA'C}}{}^{\mathbf{B}}$ satisfy due to their relation to the derivative. The first set are the commutation relations. The covariant derivative we use is torsion-free, which means that the commutator $2\nabla_{[\alpha}\nabla_{\beta]}$ annihilates scalar fields:

$$2\nabla_{[\alpha}\nabla_{\beta]}\phi = 0\,, \tag{2.90}$$

for any scalar ϕ. Projecting the last equation onto the null tetrad and employing (2.65) – (2.68) gives us the following relations:

$$\begin{aligned}
\Delta D - D\Delta &= (\gamma + \overline{\gamma})D + (\varepsilon + \overline{\varepsilon})\Delta - (\pi + \overline{\tau})\delta - (\tau + \overline{\pi})\overline{\delta}, && \text{(2.91a)}\\
D\delta - \delta D &= (\overline{\pi} - \beta - \overline{\alpha})D - \kappa\Delta + (\overline{\rho} + \varepsilon - \overline{\varepsilon})\delta + \sigma\overline{\delta}, && \text{(2.91b)}\\
\Delta\delta - \delta\Delta &= \overline{\nu}D + (\beta - \tau + \overline{\alpha})\Delta + (\gamma - \overline{\gamma} - \mu)\delta - \overline{\lambda\delta}, && \text{(2.91c)}\\
\overline{\delta}\delta - \delta\overline{\delta} &= (\overline{\mu} - \mu)D + (\overline{\rho} - \rho)\Delta + (\alpha - \overline{\beta})\delta + (\beta - \overline{\alpha})\overline{\delta}. && \text{(2.91d)}
\end{aligned}$$

Two remarks are appropriate here. First, notice that the operators of the above equations are meant to act on scalars. We do not write that explicitly since in the Newman-Penrose formalism we do not deal with any quantities other than scalars. Second, there are six different projections of the equation $2\nabla_{[\alpha}\nabla_{\beta]}\phi = 0$ onto the spin-frame. We list only four since the remaining two are just complex conjugates of equations (2.91b) and (2.91c).

To obtain the second set of differential equantions for spin coefficients, consider the action of the commutator $2\nabla_{[\alpha}\nabla_{\beta]}$ on a general spinor. It must be in accordance with the Ricci identity (2.87). Projecting that equation onto the spin-frame and using (2.65) – (2.68) and definitions of the spin coefficients, we obtain equations

$$D\rho - \overline{\delta}\kappa = \rho^2 + (\epsilon + \overline{\epsilon})\,\rho - \kappa\left(3\alpha + \overline{\beta} - \pi\right) - \tau\overline{\kappa} + \sigma\overline{\sigma} + \Phi_{00}, \tag{2.92a}$$
$$D\sigma - \delta\kappa = (\rho + \overline{\rho} + 3\varepsilon - \overline{\varepsilon})\sigma - (\tau - \overline{\pi} + \overline{\alpha} + 3\beta)\kappa + \Psi_0, \tag{2.92b}$$
$$D\tau - \Delta\kappa = \rho(\tau + \overline{\pi}) + \sigma(\overline{\tau} + \pi) + (\varepsilon - \overline{\varepsilon})\tau - (3\gamma + \overline{\gamma})\kappa + \Psi_1 + \Phi_{01}, \tag{2.92c}$$
$$D\alpha - \overline{\delta}\varepsilon = (\rho + \overline{\varepsilon} - 2\varepsilon)\alpha + \beta\overline{\sigma} - \overline{\beta}\varepsilon - \kappa\lambda - \overline{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10}, \tag{2.92d}$$
$$D\beta - \delta\varepsilon = (\alpha + \pi)\sigma + (\overline{\rho} - \overline{\varepsilon})\beta - (\mu + \gamma)\kappa - (\overline{\alpha} - \overline{\pi})\varepsilon + \Psi_1, \tag{2.92e}$$
$$D\gamma - \Delta\varepsilon = (\tau + \overline{\pi})\alpha + (\overline{\tau} + \pi)\beta - (\varepsilon + \overline{\varepsilon})\gamma - (\gamma + \overline{\gamma})\varepsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11}, \tag{2.92f}$$
$$D\lambda - \overline{\delta}\pi = (\rho - 3\varepsilon + \overline{\varepsilon})\lambda + \overline{\sigma}\mu + (\pi + \alpha - \overline{\beta})\pi - \nu\overline{\kappa} + \Phi_{20}, \tag{2.92g}$$
$$D\mu - \delta\pi = (\overline{\rho} - \varepsilon - \overline{\varepsilon})\mu + \sigma\lambda + (\overline{\pi} - \overline{\alpha} + \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda, \tag{2.92h}$$
$$D\nu - \Delta\pi = (\pi + \overline{\tau})\mu + (\overline{\pi} + \tau)\lambda + (\gamma - \overline{\gamma})\pi - (3\varepsilon + \overline{\varepsilon})\nu + \Psi_3 + \Phi_{21}, \tag{2.92i}$$
$$\Delta\lambda - \overline{\delta}\nu = -(\mu + \overline{\mu} + 3\gamma - \overline{\gamma})\lambda + (3\alpha + \overline{\beta} + \pi - \overline{\tau})\nu - \Psi_4, \tag{2.92j}$$
$$\Delta\mu - \delta\nu = -(\mu + \gamma + \overline{\gamma})\mu - \lambda\overline{\lambda} + \overline{\nu}\pi + (\overline{\alpha} + 3\beta - \tau)\nu - \Phi_{22}, \tag{2.92k}$$
$$\Delta\beta - \delta\gamma = (\overline{\alpha} + \beta - \tau)\gamma - \mu\tau + \sigma\nu + \varepsilon\overline{\nu} + (\gamma - \overline{\gamma} - \mu)\beta - \alpha\overline{\lambda} - \Phi_{12}, \tag{2.92l}$$
$$\Delta\sigma - \delta\tau = -(\mu - 3\gamma + \overline{\gamma})\sigma - \overline{\lambda}\rho - (\tau + \beta - \overline{\alpha})\tau + \kappa\overline{\nu} - \Phi_{02}, \tag{2.92m}$$
$$\Delta\rho - \overline{\delta}\tau = (\gamma + \overline{\gamma} - \overline{\mu})\rho - \sigma\lambda + (\overline{\beta} - \alpha - \overline{\tau})\tau + \nu\kappa - \Psi_2 - 2\Lambda, \tag{2.92n}$$
$$\Delta\alpha - \overline{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\overline{\gamma} - \overline{\mu})\alpha + (\overline{\beta} - \overline{\tau})\gamma - \Psi_3, \tag{2.92o}$$
$$\delta\rho - \overline{\delta}\sigma = (\overline{\alpha} + \beta)\rho - (3\alpha - \overline{\beta})\sigma + (\rho - \overline{\rho})\tau + (\mu - \overline{\mu})\kappa - \Psi_1 + \Phi_{01}, \tag{2.92p}$$
$$\delta\alpha - \overline{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\alpha\beta + (\rho - \overline{\rho})\gamma + (\mu - \overline{\mu})\varepsilon - \Psi_2 + \Lambda + \Phi_{11}, \tag{2.92q}$$
$$\delta\lambda - \overline{\delta}\mu = (\rho - \overline{\rho})\nu + (\mu - \overline{\mu})\pi + (\alpha + \overline{\beta})\mu + (\overline{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}. \tag{2.92r}$$

3. GHP formalism

In previous chapter we introduced the Newman-Penrose formalism and used it to reformulate equations of Einstein's theory of gravitation. Despite the fact that the formalism is very economic — there are only 12 independent (complex) spin coefficient as opposed to 40 independent (real) Christoffel symbols of the standard formalism — it produced very complicated expressions. There are two reasons for such a complexity. The first one is of course the complexity of the equations themselves. Equations of Einstein's theory are fairly complicated, they deal with tensors with many degrees of freedom, and thus it may be expected that any description of such a theory in terms of scalar quantities will yield a large number of complicated equations. But there is also another cause for such a complexity: scalars of the Newman-Penrose formalism depend on the choice of the null tetrad (or the spin-frame). In addition to the trivial dependence of all Newman-Penrose scalars due to them being obtained through projections onto the spin-frame, there is a deeper dependency which holds for spin coefficients[1]. It stems from the fact that the spin coefficients are projections of the covariant derivatives of basis spinors, and those derivatives of course depend not only on the curvature of the space-time, but also on the spin-frame field itself. Consequently, a significant portion of the information contained in the spin coefficients tells us nothing about physical curvature, but rather about properties of the basis fields. This — in a sense unphysical — content of the formalism reflects the gauge freedom we have in the choice of the spin-frame.

It should be pointed out that specific physical problems often offer us some natural way to fix a portion of that gauge freedom. (For example there may be unique time-like Killing field, in which case it is quite natural to choose the null tetrad so as to make $l^\alpha + n^\alpha$ parallel to the Killing field.) If such a possibility is exploited, then the spin coefficients acquire additional physical or geometric significance. Unfortunately, it is usually not possible to fix the spin-frame fully just on a basis of some physical reasoning. This suggests that it may advantageous to develop some kind of a partially covariant formalism which would not require a fully specified choice of the spin-frame, but which would still be relatively similar to the NP-formalism. GHP(Geroch-Held-Penrose) formalism which we present in this chapter possesses such a properties. Formally, it strongly resembles the Newman-Penrose formalism. However, it does not deal with true scalars, but rather with indeterminate quantities subject to a gauge freedom[2].

In GHP formalism one chooses two null directions at each point. These two directions play a role similar to the role of the null tetrad in the Newman-Penrose formalism. Now consider a null tetrad consisting of null vectors l^α, n^α and of complex space-like vectors m^α, $\overline{m}^\alpha$. We require that vectors l^α, n^α lie in those two null directions which we have chosen at each point. This requirement significantly reduces the freedom in the choice of the basis. But there is of course still some gauge freedom left. We can rescale the vector l^α (the vector n^α must then be rescaled suitably to preserve the normalization) and we can rotate space-like vectors m^α, $\overline{m}^\alpha$ in the space-like 2-plane which they define. GHP formalism is then basically the Newman-Penrose formalism adjusted so as to be covariant with respect to the remaining 2-parameter gauge freedom.

[1]The last statement is maybe little misleading. It is of course precisely the dependence of Newman-Penrose projections on the choice of the tetrad which is reflected in this "deeper" dependency of the spin coefficients.

[2]Strictly speaking, we will work with scalar quantities, but that is only because our approach will be similar to the classical approach to the tensor formalism where tensors are represented by their components. In that formalism, when we write T^{ab} for some tensor, we of course presuppose that some basis has been chosen with respect to which the components T^{ab} are taken. Nevertheless, we do not really care what specific basis has been chosen. We are allowed to ignore those details because the formalism is covariant with respect to all such choices. Similarly, in GHP formalism we work with projections onto some fixed null tetrad, but we are not interested in what exact gauge has been chosen, for the formalism is gauge covariant.

3.1 Null tetrad and gauge freedom

As in the NP formalism, consider the null (NP) tetrad consisting of two real and two complex null vectors

$$l^\alpha = o^A \bar{o}^{A'}, \quad n^\alpha = \iota^A \bar{\iota}^{A'}, \quad m^\alpha = o^A \bar{\iota}^{A'}, \quad \overline{m}^\alpha = \iota^A \bar{o}^{A'} \tag{3.1}$$

and introduce operators of covariant derivative associated with these vectors by

$$D = l^\alpha \nabla_\alpha, \quad \Delta = n^\alpha \nabla_\alpha, \quad \delta = m^\alpha \nabla_\alpha, \quad \overline{\delta} = \overline{m}^\alpha \nabla_\alpha. \tag{3.2}$$

Vectors of the null tetrad satisfy relations

$$l^\alpha n_\alpha = -m^\alpha \overline{m}_\alpha = 1, \quad l^\alpha m_\alpha = n^\alpha m_\alpha = l^\alpha l_\alpha = n^\alpha n_\alpha = m^\alpha m_\alpha = 0. \tag{3.3}$$

In GHP formalism we do not require the null tetrad to be fixed completely. We only demand that vectors l^α and n^α lie in the fixed null directions. What then is the remaining freedom in the choice of the NP tetrad? The most general transformation that preserves the direction of both l^α and n^α is of the form

$$o^A \mapsto \lambda o^A, \quad \iota^A \mapsto \mu \iota^A, \tag{3.4}$$

where λ and μ are arbitrary (nonzero) complex numbers. The normalization (3.3) then requires $|\lambda\mu| = 1$, but since we also wish to preserve the normalization of the spin-frame, we simply put $\mu = \lambda^{-1}$. The gauge transformation therefore reads

$$o^A \mapsto \lambda\, o^A, \quad \iota^A \mapsto \lambda^{-1}\, \iota^A. \tag{3.5}$$

It will prove useful to write $R\exp(i\theta)$ for λ^2, with R and θ real. Consider a gauge transformation with $\theta = 0$, i.e. a transformation where λ is real. This transformation of the spin basis results in the following transformation of the null tetrad:

$$l^\alpha \mapsto R\, l^\alpha, \quad n^\alpha \mapsto R^{-1} n^\alpha, \quad m^\alpha \mapsto m^\alpha, \quad \overline{m}^\alpha \mapsto \overline{m}^\alpha. \tag{3.6}$$

We see the transformation rescales the null vectors l^α,n^α and leaves the space-like vectors m^α, $\overline{m}^\alpha$ unchanged. Such a transformation will therefore be called the *boost*[3].

On the other hand, if λ of the gauge transformation (3.5) has the unit length, i.e. if $R = 1$, then the null tetrad is transformed as follows:

$$l^\alpha \mapsto l^\alpha, \quad n^\alpha \mapsto n^\alpha, \quad m^\alpha \mapsto e^{i\theta} m^\alpha, \quad \overline{m}^\alpha \mapsto e^{-i\theta}\overline{m}^\alpha. \tag{3.7}$$

A transformation of this kind is called the *spin*[4].

In GHP formalism, all physical quantities and equations are projected onto the NP tetrad. Thus, instead of tensorial (spinorial) equations, in the GHP formalism we work with scalar quantities and scalar equations. This is similar to NP formalism, but in GHP formalism the NP tetrad is determined only up to the gauge transformation (3.5). Hence we need to study how do scalar quantities transform under those transformation.

Suppose that some quantity transforms according to

$$\eta \mapsto \lambda^p \overline{\lambda}^q \eta.$$

[3] If we rewrote the transformation (3.6) in terms of the Minkowski tetrad, we would see that it results in a boost in the z-direction.

[4] Now again, if we rewrote the transformation (3.7) in terms of the Minkowski tetrad, we would find that it results in a rotation of the 2-plane determined by the vectors m^α and $\overline{m}^\alpha$.

Then we call it a weighted quantity of type (p,q). We say it has a *spin weight* $\frac{1}{2}(p-q)$ and a *boost weight* $\frac{1}{2}(p+q)$. A quantity η of spin weight s and boost weight b transforms as $\eta \mapsto R^b \exp(is\theta)\eta$ under the gauge transformation.

We would expect most scalars that result from projecting a tensor onto the NP tetrad to be weighted. But as we shall see, it is not always the case. Non-weighted quantities, i.e. quantities that transform inhomogenenously under (3.5), may emerge if the projected tensor itself does depend on the gauge, as is the case with the spin coefficients. But before we discuss this critical issue more closely, let us turn our attention to the "prime-transformation" which helps us to simplify the notation.

3.2 Spin coefficients

In the Newman-Penrose formalism, the connection is described by twelve complex quantities called spin coefficients. We introduced them in section 2.2. They are essentially the Ricci rotation coefficients with respect to the null tetrad, see the following table (adopted from [17]).

∇	$o^A\nabla o_A$	$\iota^A\nabla o_A$	$\iota^A\nabla\iota_A$
D	κ	ε	π
Δ	τ	γ	ν
δ	σ	β	μ
$\overline{\delta}$	ρ	α	λ

Thus, for example, $\rho = o^A\overline{\delta}o_A$, $\gamma = \iota^A\Delta o_A$, etc.

In this section, we wish to discuss the following transformation

$$o^A \mapsto i\iota^A, \quad \iota^A \mapsto io^A, \quad \overline{o}^{A'} \mapsto -i\overline{\iota}^{A'}, \quad \overline{\iota}^{A'} \mapsto -i\overline{o}^{A'}. \tag{3.8}$$

It will be denoted by prime $'$ and we shall refer to it as the *prime-transformation*.

Let us now apply the prime-transformation on the differential operators D, Δ, δ, $\overline{\delta}$ and on the spin coefficients. For the operator D we get

$$D' = \left(o^A\overline{o}^{A'}\nabla_{AA'}\right)' = \iota^A\overline{\iota}^{A'}\nabla_{AA'} = \Delta\,.$$

Proceeding similarly for the remaining operators, we obtain

$$D' = \Delta, \;\; \Delta' = D, \;\; \delta' = \overline{\delta}, \;\; \overline{\delta}' = \delta. \tag{3.9}$$

The spin coefficients transform between themselves under the prime operation. For one half of them, we have

$$\begin{aligned}
\kappa' &= -\,\iota^A\Delta\iota_A = -\nu,\\
\tau' &= -\iota^A D\iota_A = -\,\pi,\\
\sigma' &= -\iota^A\overline{\delta}\iota_A = -\,\lambda,\\
\rho' &= -\iota^A\delta\iota_A = -\,\mu,\\
\beta' &= -\iota^A\overline{\delta}o_A = -\alpha,\\
\varepsilon' &= -\iota^A\Delta o_A = -\,\gamma.
\end{aligned} \tag{3.10}$$

The other half would yield similar relations[5]. We therefore can see that it is not necessary to introduce all twelve spin coefficients, since the half of them can be obtained by the prime operation. Thus, in the GHP formalism we define only spin coefficients

$$\kappa, \tau, \sigma, \rho, \varepsilon, \beta. \tag{3.11}$$

Because the prime operation commutes with complex conjugation, we obtain similar results for conjugated coefficients.

Let us now turn back to the study of the gauge transformation (3.5). What are the boost weights of the spin coefficients? For the coefficient κ, for example, we get

$$\begin{aligned}\kappa = o^A o^B \bar{o}^{B'} \nabla_{BB'} o_A \mapsto (\lambda o^A)(\lambda o^B)(\bar{\lambda}\bar{o}^{B'})\nabla_{BB'}(\lambda o_A) \ &= \ \lambda^2 \bar{\lambda} o^A \left(\lambda D o_A \ + \ o_A D\lambda\right) \\ &= \lambda^3 \,\bar{\lambda}\kappa,\end{aligned}$$

where term containing $D\lambda$ vanishes by $o^A o_A = 0$. Therefore, coefficient κ has boost weight $(3,1)$. By similar calculation we find

$$\begin{aligned} \kappa :\ (3,1), \quad & \tau :\ (1,-1), \\ \sigma :\ (3,-1), \quad & \rho :\ (1,1). \end{aligned} \tag{3.12}$$

On the other hand, coefficients ε and β transform as

$$\begin{aligned} \varepsilon &\mapsto \ \lambda\bar{\lambda}\varepsilon \ + \ \bar{\lambda} D\lambda, \\ \beta &\mapsto \ \lambda\bar{\lambda}^{-1}\beta \ + \ \bar{\lambda}^{-1}\delta\lambda, \end{aligned} \tag{3.13}$$

which means that these quantities do not have a boost weight.

3.3 Boost weighted operators

Operators D, Δ, δ and $\bar{\delta}$ under the gauge transformation (3.5) transform according to

$$D \mapsto \lambda\bar{\lambda} D, \quad \Delta \mapsto \lambda^{-1}\bar{\lambda}^{-1}\Delta, \quad \delta \mapsto \lambda\bar{\lambda}^{-1}\delta, \quad \bar{\delta} \mapsto \lambda^{-1}\overline{\lambda\delta}. \tag{3.14}$$

Nevertheless, when acting on the scalar of weight (p,q), these operator transform inhomogeneously, e.g.

$$D\eta \mapsto \lambda\bar{\lambda} D(\lambda^p \bar{\lambda}^q \eta) \ = \ \lambda^{p+1}\bar{\lambda}^{q+1} D\eta \ + \ p\,\lambda^p \bar{\lambda}^{q+1}\,\eta\, D\lambda \ + \ q\,\lambda^{p+1}\bar{\lambda}^q\,\eta\, D\bar{\lambda}.$$

That is, if η is the scalar of weight (p,q), its derivative $D\eta$ is not a weighted scalar anymore. However, we have seen that two of the spin coefficients, namely ε and β, also transform inhomogeneously. We can guess that by appropriate combination of operators D, Δ, δ and $\bar{\delta}$ and non-weighted spin coefficients it would be possible to obtain weighted scalars even after differentiation. From (3.13) we find

$$\begin{aligned} \varepsilon\,\eta &\mapsto \ \lambda^{p+1}\,\bar{\lambda}^{q+1}\,\varepsilon\,\eta \ + \ \lambda^p\,\bar{\lambda}^{q+1}\,\eta\, D\lambda, \\ \bar{\varepsilon}\,\eta &\mapsto \ \lambda^{p+1}\,\bar{\lambda}^{q+1}\,\bar{\varepsilon}\,\eta \ + \ \lambda^{p+1}\,\bar{\lambda}^q\,\eta\, D\bar{\lambda}. \end{aligned} \tag{3.15}$$

Comparing the behavior of operator D to the behavior of the product $\varepsilon\,\eta$ we can see that object

$$D\eta \ - \ p\,\varepsilon\,\eta \ - \ q\,\bar{\varepsilon}\,\eta \mapsto \lambda^{p+1}\,\bar{\lambda}^{q+1}\,\left(D\eta \ - \ p\,\varepsilon\,\eta \ - \ q\,\bar{\varepsilon}\,\eta\right)$$

[5]As far as the spin coefficients are concerned, the prime-transformation is involutory.

already *is* a weighted scalar with the weight $(p+1, q+1)$. Thus, we define the operator Þ (pronounced as "thorn") by

$$Þ\eta \;=\; (D \,-\, p\,\varepsilon \,-\, q\,\overline{\varepsilon})\,\eta$$

for $\eta : (p, q)$. Since $Þ\eta$ has weight $(p+1, q+1)$, the operator Þ itself has weight $(1, 1)$.

Next, consider transformation of $\delta-$operator:

$$\delta\eta \;\mapsto\; \lambda\overline{\lambda}^{-1}\delta\lambda^p\overline{\lambda}^q\eta \;=\; \lambda^{p+1}\overline{\lambda}^{q-1}\delta\eta + p\lambda^p\overline{\lambda}^{q-1}\eta\delta\lambda + q\lambda^{p+1}\overline{\lambda}^{q-2}\delta\overline{\lambda}.$$

It is not a big surprise that in order to obtain weighted operator constructed from δ, we have to add an appropriate expression containing the spin coefficient β. Again, we find

$$\begin{aligned}
\beta\,\eta &= \lambda^{p+1}\,\overline{\lambda}^{q-1}\,\beta\,\eta \;+\; \lambda^p\overline{\lambda}^{q-1}\,\eta\,\delta\lambda,\\
\overline{\beta}\,\eta &= \lambda^{p-1}\,\overline{\lambda}^{q+1}\,\overline{\beta}\,\eta \;+\; \lambda^{p-1}\overline{\lambda}^{q}\,\eta\,\overline{\delta\lambda}.
\end{aligned} \tag{3.16}$$

It is clear that inhomogeneous term in $p\,\beta\eta$ nicely cancels corresponding term in $\delta\eta$, but inhomogeneous term in $q\overline{\beta}\eta$ has wrong powers of λ and wrong operator ($\overline{\delta}$ instead od δ). The problem is that complex conjugation turns $\delta\lambda$ into $\overline{\delta\lambda}$ while the desired term is $\delta\overline{\lambda}$.

The "prime" operation, however, does exactly what we need. Recall that the prime turns o^A into $i\iota^A$ and vice versa. On the other hand, o^A transforms as λo^A under boost while ι^A transforms as $\lambda^{-1}\iota^A$. That means that the prime effectively turns λ into λ^{-1}:

$$\left(o^A\right)' \;=\; i\,\iota^A \;\mapsto\; i\,\lambda^{-1}\,\iota^A \;=\; \lambda^{-1}\,\left(o^A\right)'.$$

In other words, we can immediately write down transformation rule for β' and its complex conjugate:

$$\begin{aligned}
\beta' &\mapsto\; \lambda^{-1}\,\overline{\lambda}\,\beta' \;+\; \overline{\lambda}\,\overline{\delta}\lambda^{-1} \;=\; \lambda^{-1}\,\overline{\lambda}\,\beta' \;-\; \lambda^{-2}\,\overline{\lambda}\,\overline{\delta}\lambda,\\
\overline{\beta}' &\mapsto\; \lambda\,\overline{\lambda}^{-1}\,\overline{\beta}' \;-\; \lambda\,\overline{\lambda}^{-2}\,\delta\overline{\lambda}.
\end{aligned} \tag{3.17}$$

Thus, correct transformation can be achieved by defining operator ð (pronounced as "eth") as

$$ð\eta = \left(\delta \,-\, p\,\beta \,+\, q\,\overline{\beta}'\right)\eta. \tag{3.18}$$

It is clear from the equations above, that ð has weight $(1, -1)$, i.e. it transforms η of weight (p, q) into $ð\eta$ of weight $(p+1, q-1)$.

Weighted operators Þ and ð are associated with NP operators D and δ. Operators associated to NP operators Δ and $\overline{\delta}$ can be obtained easily by taking the prime of Þ and ð. The only thing to realize is that some signs must be adjusted because the prime effectively turns λ into λ^{-1}.

Now we may take the equations written in the Newman-Penrose formalism and substitute for derivative operators D, Δ, δ, $\overline{\delta}$ from the expressions that we found for weighted operators Þ, Þ′, ð, ð′. Doing so, we obtain equations which are manifestly covariant with respect to the gauge transformation (3.5). As a consequence, they do not contain non-weighted spin coefficients and are therefore considerably simpler than their NP counterparts. Let us illustrate this on the projection (2.72a) of the Bianchi identity, which — when we write $-\tau'$ and $-\beta'$ for coefficients π and α — has the form

$$\begin{aligned}
D\Psi_1 - \overline{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} = {} & (-\tau' + 4\beta')\Psi_0 + 2(2\rho + \varepsilon)\Psi_1 - 3\kappa\Psi_2 + 2\kappa\Phi_{11}\\
& (\overline{\tau}' - 2\overline{\beta}' + 2\beta)\Phi_{00} - 2\sigma\Phi_{10} - 2(\overline{\rho} + \varepsilon)\Phi_{01} + \overline{\kappa}\Phi_{02}\,.
\end{aligned}$$

The equation contains several non-weighted terms, namely derivative terms and terms involving coefficients β, $\overline{\beta}$, $\overline{\beta}'$ and ε. The rest are terms with the weight $(3,1)$. If we arrange all non-weighted terms on the left hand side of the equation, the right hand side becomes weighted. Consequently, the left hand side — now containing all non-weighted terms — is a weighted quantity as well. Since there is no non-zero weighted combination of non-weighted coefficient, we see that those non-weighted terms must combine into the weighted operators Þ, Þ′, ð, ð′. Specifically, for equation (2.72a) we obtain

$$
\begin{aligned}
&\text{Þ}\Psi_1 - \eth'\Psi_0 - \text{Þ}\Phi_{01} + \eth\Phi_{00} \\
&\quad = -\tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2 + 2\kappa\Phi_{11} + \overline{\tau}'\Phi_{00} - 2\sigma\Phi_{10} - 2\overline{\rho}\Phi_{01} + \overline{\kappa}\Phi_{02}\,.
\end{aligned}
$$

3.4 Résumé

In this chapter we briefly introduced the essentials of the GHP formalism. Since this chapter contains some auxiliary equations and some motivation, here we merely summarize basic relations.

In GHP formalism we define four boost-weighted spin coefficients

$$
\begin{aligned}
\kappa &= o^A D o_A : (3,1), & \tau &= o^A \Delta o_A : (1,-1), \\
\sigma &= o^A \delta o_A : (3,-1), & \rho &= o^A \overline{\delta} o_A : (1,1),
\end{aligned}
\tag{3.19}
$$

and two non-weighted spin coefficients

$$
\begin{aligned}
\varepsilon &= \iota^A D o_A, \\
\beta &= \iota^A \overline{\delta} o_A.
\end{aligned}
\tag{3.20}
$$

Weighted derivative operators read:

$$
\begin{aligned}
\text{Þ} &= D - p\,\varepsilon - q\,\overline{\varepsilon}, & \text{Þ}' &= \Delta + p\,\varepsilon' + q\overline{\varepsilon}', \\
\eth &= \delta - p\,\beta + q\,\overline{\beta}', & \eth' &= \overline{\delta} + p\,\beta' - q\overline{\beta}.
\end{aligned}
\tag{3.21}
$$

4. Twistor equation and twistors

Twistor theory, originally proposed by Roger Penrose, was intended as an alternative attempt to unify general theory of relativity with the quantum theory. After more than 40 years of research in the area of twistor theory it seems that formidable mathematical difficulties connected with the formulation of the twistor theory in curved spacetimes are convincing enough to abandon the theory as an alternative to quantum gravity. Nevertheless, the twistor theory as a mathematical tool inspired research both in pure mathematics (e.g. integrable systems) and physics. The problem of quasi-local mass is one of the most remarkable successes of the twistor theory in the area of physics.

In this thesis we do not intend to provide the description of the twistor theory and the twistor geometry, although it is an engaging part of differential geometry on complex manifolds. Our aim is just to present technical tools necessary for our calculation of the Bondi mass of the spacetime with electro-scalar sources. Hence, we skip the motivation for the twistor equation which arises naturally when one tries to solve the so-called zero-rest-mass equations using the contour integral in the complex plane. In this chapter we simply present the twistor equation and solve it. In Chapter 6, however, the twistor equation arises naturally in a different context to be explained in detail.

4.1 1-valence twistor equation

The twistor equation of valence 1 or *univalent twistor equation* is equation

$$\nabla_{A'}^{(A}\omega^{B)} = 0. \tag{4.1}$$

An interesting and important feature of this equation is its *conformal invariance*. Indeed, consider the conformal transformation

$$\hat{\epsilon}_{AB} \;=\; \Omega\,\epsilon_{AB}, \qquad \hat{\epsilon}^{AB} \;=\; \Omega^{-1}\,\epsilon^{AB}$$

where the hat denotes conformally rescaled quantities. If we set

$$\hat{\omega}_A \;=\; \Omega\,\omega_A, \qquad \hat{\omega}^A \;=\; \omega^A,$$

the rule for conformal transformation of the covariant derivative

$$\nabla_{AA'}\omega_B \;=\; \hat{\nabla}_{AA'}\omega_B \;+\; \Omega^{-1}\,\omega_A\,\nabla_{BA'}\Omega$$

immediately gives

$$\nabla_{A'(A}\omega_{B)} \;=\; 0 \quad\rightarrow\quad \Omega^{-1}\,\hat{\nabla}_{A'(A}\hat{\omega}_{B)} \;=\; 0.$$

Hence, the twistor equation is conformally invariant with the conformal weight -1.
By standard spinorial decomposition we find

$$\nabla_{A'}^{A}\omega^{B} \;=\; \nabla_{A'}^{(A}\omega^{B)} \;+\; \frac{1}{2}\,\epsilon^{AB}\,\nabla_{XA'}\omega^{X}.$$

Denoting

$$\pi_{A'} \;=\; \frac{1}{2}\,i\,\nabla_{XA'}\omega^{X}$$

and using (4.1) we can rewrite the twistor equation in the form

$$\nabla^A_{A'}\omega^B = -\,i\,\epsilon^{AB}\,\pi_{A'}. \tag{4.2}$$

Unfortunately, (4.2) does not have a solution in a general curved spacetime. It has a solution only in the spacetimes of type N, flat spacetime and conformally flat spacetimes. Therefore, in this chapter we assume the flat spacetime in which the covariant derivatives commute when acting on any spinorial quantity:

$$\nabla_{A(A'}\nabla^A_{B')}\omega_B \;=\; 0, \qquad \nabla_{A'(A}\nabla^{A'}_{B)}\omega_C \;=\; 0. \tag{4.3}$$

Substituting (4.3) into (4.2) we find two conditions

$$\nabla_{B(A'}\pi_{B')} \;=\; 0 \quad \text{and} \quad \nabla^{A'}_B\,\pi_{A'} \;=\; 0.$$

Since

$$\nabla_{AA'}\pi_{B'} \;=\; \nabla_{A(A'}\pi_{B')} \;+\; \frac{1}{2}\,\epsilon_{A'B'}\,\nabla^{X'}_A\,\pi_{X'},$$

these conditions imply

$$\nabla_{AA'}\pi_{B'} = 0. \tag{4.4}$$

That is, $\pi_{B'}$ is a constant spinor, a notion which exists only in type N spacetimes. Now that we know the spinor $\pi_{A'}$ is constant, we can easily integrate (4.2):

$$\omega^B = \tilde{\omega}^B \;-\; i\,x^{BA'}\,\pi_{A'} \tag{4.5}$$

where $\tilde{\omega}^A$ is the "integration constant", i.e. it is a constant spinor.

4.2 Twistor space

In the previous section we have shown that the solution of the twistor equation in the flat spacetime is determined by two constant spinors $\tilde{\omega}^A$ and $\pi_{A'}$, each of them having two independent complex components, and reads

$$\omega^A \;=\; \tilde{\omega}^A \;-\; i\,x^{AA'}\,\pi_{A'}.$$

Thus, the space of the solutions of twistor equation is complex four-dimensional space called *twistor space* and denoted by $\mathbb{T}^\alpha$. Element $\mathsf{Z}^\alpha \in \mathbb{T}^\alpha$ is called the *twistor*. Twistor determined by spinors $\tilde{\omega}^A$ and $\tilde{\pi}_{A'}$ is denoted by

$$\mathsf{Z}^\alpha \;=\; (\tilde{\omega}^A,\;\pi_{A'}), \qquad \mathsf{Z}^A \;=\; \tilde{\omega}^A, \qquad \mathsf{Z}_{A'} \;=\; \pi_{A'}.$$

Twistor space has a natural structure of the vector space given by

$$(\tilde{\omega}^A, \pi_{A'}) \;+\; \lambda\,(\tilde{\xi}^A, \eta_{A'}) \;=\; (\tilde{\omega}^A + \lambda\tilde{\xi}^A, \pi_{A'} + \lambda\tilde{\eta}_{A'}).$$

Notice that ω^A is a *spinor field* rather than a single spinor, unlike the constant spinor $\tilde{\omega}^A$ which is, in a sense, a coordinate of the twistor Z^α. Obviously, $\tilde{\omega}^A$ is the value of field ω^A at the origin of the coordinate system $x^{AA'} = 0$. If we choose a different origin at $a^{AA'}$, the value of ω^A at this point will be

$$\omega^A(a) \;=\; \tilde{\omega}^A \;-\; i\, a^{AA'}\, \pi_{A'} .$$

Thus, instead of writing $\mathsf{Z}^\alpha = (\tilde{\omega}^A, \pi_{A'})$ we can write $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ and regard ω^A and $\pi_{A'}$ as spinors which transform according to

$$\begin{aligned} &\omega^A \mapsto \omega^A \;-\; i\, x^{AA'}\, \pi_{A'}, \\ &\pi_{A'} \mapsto \pi_{A'} \end{aligned} \tag{4.6}$$

under the shift of the origin by vector $x^{AA'}$.

As in the spinor algebra, we can introduce *dual twistor space* $\mathbb{T}_\alpha$ and complex conjugated spaces $\overline{\mathbb{T}}^{\alpha'}$ and $\overline{\mathbb{T}}_{\alpha'}$. Let us denote the components of a twistor as

$$\mathsf{Z}^\alpha = (\omega^A, \pi_{A'}) \;=\; (\omega^0, \omega^1, \pi_{0'}, \pi_{1'}) \;=\; (Z^0, Z^1, Z^2, Z^3). \tag{4.7}$$

A natural norm on the twistor space is defined by

$$\begin{aligned} H(\mathsf{Z},\mathsf{Z}) &= \omega^A\, \overline{\pi}_A \,+\, \overline{\omega}^{A'}\, \pi_{A'} \;=\; \omega^0\, \overline{\pi}_0 \,+\, \omega^1\, \overline{\pi}_1 \,+\, \overline{\omega}^{0'}\, \pi_{0'} \,+\, \overline{\omega}^{1'}\, \pi_{1'} \\ &= Z^0\, \overline{Z^2} \,+\, Z^1\, \overline{Z^3} \,+\, Z^2\, \overline{Z^0} \,+\, Z^3\, \overline{Z^1} \end{aligned} \tag{4.8}$$

where $\overline{Z^\alpha} = \overline{Z}^{\alpha'}$. Equivalently we can write

$$H(\mathsf{Z},\mathsf{Z}) = H_{\alpha\beta'}\, \mathsf{Z}^\alpha\, \overline{\mathsf{Z}}^{\beta'} \tag{4.9}$$

where

$$H_{\alpha\beta'} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \det H_{\alpha\beta'} \;=\; 1. \tag{4.10}$$

Since $H_{\alpha\beta'}$ is non-degenerate, it introduces a metric on the twistor space by

$$H(\mathsf{Z}_1,\mathsf{Z}_2) \;=\; H_{\alpha\beta'}\, \mathsf{Z}_1^\alpha\, \overline{\mathsf{Z}}_2^{\beta'} .$$

Notice that the diagonal form of $H_{\alpha\beta'}$ is $\mathrm{diag}(-1,-1,1,1)$, so the metric $H_{\alpha\beta'}$ has signature $(-2,2)$.

Non-degenerate $H_{\alpha\beta'}$ allows us to introduce an isomorphism $\overline{\mathbb{T}}^{\beta'} \mapsto \mathbb{T}_\alpha$ by

$$\overline{\mathsf{Z}}_\alpha \;= H_{\alpha\beta'}\, \overline{\mathsf{Z}}^{\beta'} .$$

For $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ we have

$$\overline{\mathsf{Z}}_\alpha = H_{\alpha\beta'}\, \overline{\mathsf{Z}}^{\beta'} \;=\; (\overline{Z^2}, \overline{Z^3}, \overline{Z^0}, \overline{Z^1}) \;=\; (\overline{\pi}_A, \overline{\omega}^{A'}). \tag{4.11}$$

Since the spaces $\overline{\mathbb{T}}^{\alpha'}$ and $\mathbb{T}_\alpha$ are isomorphic, complex conjugated twistor with primed upper indices can be always mapped to a dual twistor with unprimed lower indices. That is, we never need primed twistors; a difference to the spinor case.

Clearly, the norm $H(\mathsf{Z},\mathsf{Z}) = H_{\alpha\beta'}\mathsf{Z}^\alpha \overline{\mathsf{Z}}^{\beta'} = \overline{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha$ is invariant under the shift of the origin, for we have

$$\begin{aligned}\overline{\mathsf{Z}}_\alpha\,\mathsf{Z}^\alpha \;&=\; \omega^A\,\overline{\pi}_A \,+\, \overline{\omega}^{A'}\,\pi_{A'} \mapsto \left(\omega^A \,-\, i\,x^{AB'}\pi_{B'}\right)\overline{\pi}_A \,+\, \left(\overline{\omega}^{A'}\,\pi_{A'} \,+\, i\,x^{BA'}\,\overline{\pi}_B\right)\pi_{A'}\\ &=\; \omega^A\,\overline{\pi}_A \,+\, \overline{\omega}^{A'}\,\pi_{A'}.\end{aligned} \tag{4.12}$$

4.3 2-valence twistor equation

Twistor equation of valence 2 is the spinorial equation

$$\nabla^{(A}_{A'}\omega^{BC)} = 0 \tag{4.13}$$

where ω^{BC} is a *symmetric* 2-valent spinor. Before actually solving the equation we derive an auxiliary identity. Solution of the twistor equation then follows.

We start with the relation for the total symmetrization of the object $\nabla^A_{A'}\omega^{BC}$ and use the symmetry of ω^{BC}:

$$\begin{aligned}\nabla^{(A}_{A'}\omega^{BC)} &= \frac{1}{3}\left[\nabla^{(A}_{A'}\omega^{B)C} \,+\, \nabla^{(B}_{A'}\omega^{C)A} \,+\, \nabla^{(A}_{A'}\omega^{C)B}\right]\\ &= \nabla^{(A}_{A'}\omega^{B)C} \,+\, \frac{1}{6}\epsilon^{CA}\nabla_{A'X}\omega^{XB} \,+\, \frac{1}{6}\epsilon^{CB}\nabla_{A'X}\omega^{XA}\,,\end{aligned}$$

So far we have eliminated symmetrization in three indices in favour of symmetrization in two indices which can be, however, eliminated further using the spinorial identity

$$\nabla^A_{A'}\omega^{BC} = \nabla^{(A}_{A'}\omega^{B)C} \,+\, \frac{1}{2}\,\epsilon^{AB}\,\nabla_{XA'}\omega^{XC}\,.$$

For brevity we introduce spinor

$$\lambda^A_{A'} \;=\; \nabla_{XA'}\omega^{XA}.$$

In this notation, total symmetrization of $\nabla^A_{A'}\omega^{BC}$ can be written in the form

$$\nabla^{(A}_{A'}\omega^{BC)} = \nabla^A_{A'}\omega^{BC} - \frac{1}{6}\left[3\,\epsilon^{AB}\,\lambda^C_{A'} \,+\, \epsilon^{BC}\,\lambda^A_{A'} \,+\, \epsilon^{AC}\,\lambda^B_{A'}\right].$$

After some arrangements and using the identity

$$\epsilon^{B[A}\lambda^{C]}_{A'} \;=\; -\frac{1}{2}\,\epsilon^{AC}\,\lambda^B_{A'}$$

we finally arrive at desired relation

$$\nabla^{(A}_{A'}\omega^{BC)} = \nabla^A_{A'}\omega^{BC} \,-\, \frac{2}{3}\,\epsilon^{A(B}\lambda^{C)}_{A'}. \tag{4.14}$$

For the sake of Chapter 6 we introduce vector field

$$K^A_{A'} = \frac{2}{3}\,i\,\lambda^A_{A'} \;=\; \frac{2}{3}\,i\,\nabla_{XA'}\omega^{XA} \tag{4.15}$$

in terms of which the identity (4.14) acquires the form

$$\nabla^{(A}_{A'}\omega^{BC)} = \nabla^{A}_{A'}\omega^{BC} + i\,\epsilon^{A(B}\,K^{C)}_{A'}. \tag{4.16}$$

Twistor equation (4.13) then implies

$$\nabla^{A}_{A'}\omega^{BC} = -\,i\,\epsilon^{A(B}\,K^{C)}_{A'}. \tag{4.17}$$

Let us investigate some properties of the vector field K_a. As in the case of univalent case, twistor equation has non-trivial solutions only in the *flat spacetime* where covariant derivatives commute,

$$\nabla^{X}_{X'}\nabla^{A}_{A'}\omega^{BC} \;=\; \nabla^{A}_{A'}\nabla^{X}_{X'}\omega^{BC}.$$

Substituting from (4.17) and writing the symmetrization explicitly we find

$$\epsilon^{AB}\nabla^{X}_{X'}K^{C}_{A'} + \epsilon^{AC}\nabla^{X}_{X'}K^{B}_{A'} = \epsilon^{XB}\nabla^{A}_{A'}K^{C}_{X'} + \epsilon^{XC}\nabla^{A}_{A'}K^{B}_{X'}. \tag{4.18}$$

In order to simplify this expression we contract it with ϵ_{AB},

$$3\nabla^{X}_{X'}K^{C}_{A'} = \nabla^{X}_{A'}K^{C}_{X'} + \epsilon^{XC}\,\nabla_{CA'}K^{C}_{X'}. \tag{4.19}$$

Symmetrization in indices $X'A'$ and XC immediately yields

$$\nabla^{(X}_{(X'}K^{C)}_{A')} = 0. \tag{4.20}$$

Next, contraction of (4.19) with ϵ_{XC} gives

$$\nabla_{CX'}K^{C}_{A'} \;=\; \nabla_{CA'}K^{C}_{X'}$$

which means that antisymmetric part of $\nabla_{CX'}K^{C}_{A'}$ vanishes. However, antisymmetric part must be always proportional to symplectic form,

$$\nabla_{C[X'}K^{C}_{A']} \;=\; \frac{1}{2}\,\epsilon_{X'A'}\,\nabla_{CC'}K^{CC'},$$

so that the divergence of vector field K^c is identically zero:

$$\nabla_c K^c \;=\; 0. \tag{4.21}$$

We have found that symmetric part of tensor $\nabla_a K_b$ is zero. Indeed, using standard decomposition we can write

$$\nabla_a K_b = \nabla_{(A(A'}K_{B')B)} + \frac{1}{2}\,\epsilon_{AB}\nabla_{X(A'}K^{X}_{B')} + \frac{1}{2}\,\epsilon_{A'B'}\nabla_{X'(A}K^{X'}_{B)} + \frac{1}{4}\,\epsilon_{AB}\,\epsilon_{A'B'}\,\nabla_c K^c. \tag{4.22}$$

The first and the last term on the right hand side is symmetric in ab while the middle two terms are antisymmetric. However, symmetric terms vanish by (4.20) and (4.21) and only antisymmetric part remains:

$$\nabla_a K_b \;=\; \frac{1}{2}\,\epsilon_{AB}\nabla_{X(A'}K^{X}_{B')} + \frac{1}{2}\,\epsilon_{A'B'}\nabla_{X'(A}K^{X'}_{B)}.$$

In other words, since the symmetric part vanishes, vector K_a satisfies the Killing equations

$$\nabla_a K_b \;+\; \nabla_b K_a \;=\; 0$$

and hence it is a Killing vector!

But the equation (4.18) does actually restrict the quantity $\nabla_a K_b$ even further. If we contract it by $\epsilon^{A'X'}$ and ϵ_{XC}, we find

$$-\epsilon^{AB}\nabla_{A'C}K^{A'C} = 4\,\nabla^A_{A'}K^{BA'}\,, \tag{4.23}$$

which, after the symmetrization in AB, yields

$$0 = \nabla^{(A}_{X'}K^{B)X'}\,. \tag{4.24}$$

Thus we see that only one term in (4.22) survives:

$$\nabla_a K_b = \frac{1}{2}\,\epsilon_{AB}\,\nabla_{X(A'}K^X_{B')}. \tag{4.25}$$

Therefore, $\nabla_a K_b$ is *self-dual* two form.

Let us recapitulate what we have found in this section. We started with the twistor equation in the flat spacetime

$$\nabla^{(A}_{A'}\omega^{BC)} \;=\; 0.$$

We have shown that any spinor ω^{BC} satisfying the twistor equation must satisfy also equation

$$\nabla^A_{A'}\omega^{BC} \;=\; -\,i\,\epsilon^{A(B}K^{C)}_{A'}$$

where $K^{AA'}$ is *self-dual* Killing vector in the sense, that its derivative $\nabla_a K_b$ is self-dual. Now it is clear that the twistor equation cannot have non-trivial solutions in arbitrary spacetime, because general spacetime possesses no Killing vectors.

At the end of this section we finally solve the twistor equation in a way similar to solution of univalent twistor equation. Notice that, by (4.25), $\nabla_{AA'}K_{BB'}$ is antisymmetric in AB and symmetric in $A'B'$. Consider object

$$\nabla^A_{A'}\nabla^B_{B'}K^C_{C'}.$$

Since in the flat spacetime covariant derivatives commute, this object is automatically antisymmetric in AC and BC by (4.25). Antisymmetry in AB follows from simple calculation:

$$\nabla^A_{A'}\nabla^B_{B'}K^C_{C'} \;=\; -\,\nabla^A_{A'}\nabla^C_{B'}K^B_{C'} \;=\; \nabla^B_{A'}\nabla^C_{B'}K^A_{C'} \;=\; -\,\nabla^B_{A'}\nabla^A_{B'}K^C_{C'}.$$

Thus, $\nabla^A_{A'}\nabla^B_{B'}K^C_{C'}$ is antisymmetric in each pair of unprimed indices. Since the maximal possible rank of antisymmetric form built on two-dimensional space of spinors is 2, this object must vanish identically:

$$\nabla^A_{A'}\nabla^B_{B'}K^C_{C'} = 0. \tag{4.26}$$

The rest of derivation is straightforward. Equation (4.26) integrates to

$$\nabla^A_{A'}K^B_{B'} = \lambda^{AB}_{A'B'} \tag{4.27}$$

where $\lambda^{AB}_{A'B'}$ is a constant spinor. However, using the symmetries of $\nabla^A_{A'} K^B_{B'}$ we can decompose this spinor as

$$\lambda^{AB}_{A'B'} \;=\; 2\,\epsilon^{AB}\, M_{A'B'}$$

where $M_{A'B'}$ is symmetric constant spinor. After first integration we therefore have

$$\nabla^A_{A'} K^B_{B'} \;=\; 2\,\epsilon^{AB}\, M_{A'B'}.$$

Next integration gives

$$K^{AA'} = T^{AA'} \,+\, 2\,x^{AB'}\, M^{A'}_{B'} \tag{4.28}$$

where $T^{AA'}$ is an arbitrary constant spinor. This result can be inserted into (4.17) and the equation obtained can be integrated for the last time. Solution of the twistor equation can be finally written in the form

$$\omega^{BC} = -\,i\,x^{BA'}\,x^{CC'}\,M_{A'C'} \,+\, i\,T^{(B}_{A'}x^{C)A'} \,+\, \Omega^{BC} \tag{4.29}$$

where Ω^{BC} is an arbitrary symmetric constant spinor.

Notice that there are exactly ten complex constants entering the solution (3 constants for symmetric $M_{A'B'}$, 3 constants for symmetric Ω^{AB} and 4 constants for $T_{AA'}$). Thus, there are ten independent solutions to the twistor equations which coincides with the number of the Killing vectors in the flat spacetime. We have already mentioned that the twistor equation indeed has non-trivial solutions only in the Minkowski spacetime. In the derivation of the solution we have seen that the solution of the twistor equation always implies the existence of the Killing vector. Next we have seen that the solution of the twistor equation was based on the fact that $\nabla_a K_b$ is a constant tensor; such tensors, however, exist only in the flat spacetime. And finally, we have found that the number of independent solutions to the 2-valent twistor equation is equal to the number of the Killing vectors of the Minkowski spacetime. Hence, the Killing equation and the twistor equation are intimately related.

4.4 Twistor equation in GHP formalism

In this section we project the twistor equation

$$\nabla^{(A}_{A'}\omega^{B)} \;=\; 0$$

onto the null tetrad and obtain the set of scalar differential equations between the components of the spinor ω^A and the spin coefficients. Next we use the GHP formalism to reduce the number of equation and to see the character of the twistor equation more explicitly.

We know that whenever ω^A is a solution to the twistor equation, it must satisfy

$$\nabla^A_{A'}\omega^B \;=\; -\,i\,\epsilon^{AB}\,\pi_{A'}$$

for some spinor $\pi_{A'}$ (which must be constant in the flat spacetime). Let us introduce the components ω^0 and ω^1 by

$$\omega^A = \omega^0\,o^A \,+\, \omega^1\,\iota^A \tag{4.30}$$

so that

$$\omega^0 \;=\; -\,\iota_A\,\omega^A \;:\; (-1,0), \qquad \omega^1 \;=\; o_A\,\omega^A \;:\; (1,0),$$

where the pair (p, q) after the colon denotes the weight of corresponding quantity (recall the definition of weight in 3.1). Similarly we decompose the spinor $\pi_{A'}$ as

$$\pi_{A'} = \pi_{1'}\,\bar{o}_{A'} \;-\; \pi_{0'}\,\bar{\iota}_{A'} \tag{4.31}$$

so that

$$\pi_{0'} \;=\; \pi_{A'}\,\bar{o}^{A'}, \qquad \pi_{1'} \;=\; \pi_{A'}\,\bar{\iota}^{A'}.$$

Projections of the twistor equation onto the null tetrad in the Newman-Penrose formalism follow.

$$\begin{aligned}
D\omega^0 &= -\,\varepsilon\,\omega^0 \;-\; \pi\,\omega^1 \;-\; i\,\pi_{0'} & \qquad \Delta\omega^0 &= -\,\gamma\,\omega^0 \;-\; \nu\,\omega^1 \\
D\omega^1 &= \kappa\,\omega^0 \;+\; \varepsilon\,\omega^1 & \qquad \Delta\omega^1 &= \tau\,\omega^0 \;+\; \gamma\,\omega^1 \;-\; i\,\pi_{1'} \\
\\
\delta\omega^0 &= -\,\beta\,\omega^0 \;-\; \mu\,\omega^1 \;-\; i\,\pi_{1'} & \qquad \bar{\delta}\omega^0 &= -\,\alpha\,\omega^0 \;-\; \lambda\,\omega^1 \\
\delta\omega^1 &= \sigma\,\omega^0 \;+\; \beta\,\omega^1 & \qquad \bar{\delta}\omega^1 &= \rho\,\omega^0 \;+\; \alpha\,\omega^1 \;+\; i\,\pi_{0'}
\end{aligned} \tag{4.32}$$

Now we rewrite equations (4.32) in the Geroch-Held-Penrose formalism. Since the weight of ω^0 is $(p = -1, q = 0)$ and the weight of ω^1 is $(p = 1, q = 0)$, we can use definitions (3.21) to find the action of þ and ð on these quantities:

$$\begin{aligned}
\text{þ}\omega^0 &= (D + \varepsilon)\omega^0, & \qquad \text{þ}'\omega^0 &= (\Delta + \gamma)\omega^0, \\
\text{þ}\omega^1 &= (D - \varepsilon)\omega^1, & \qquad \text{þ}'\omega^1 &= (\Delta - \gamma)\omega^1, \\
\\
\eth\omega^0 &= (\delta + \beta)\omega^0, & \qquad \eth'\omega^0 &= (\bar{\delta} + \alpha)\omega^0, \\
\eth\omega^1 &= (\delta - \beta)\omega^1, & \qquad \eth'\omega^1 &= (\bar{\delta} - \alpha)\omega^1.
\end{aligned} \tag{4.33}$$

In this notation, projections (4.32) simplify to (4.34).

$$\begin{aligned}
\text{þ}\omega^0 &= -\,\pi\,\omega^1 \;-\; i\,\pi_{0'} & \qquad \text{þ}'\omega^0 &= -\,\nu\,\omega^1 \\
\text{þ}\omega^1 &= \kappa\,\omega^0 & \qquad \text{þ}'\omega^1 &= \tau\,\omega^0 \;-\; i\,\pi_{1'} \\
\\
\eth\omega^0 &= -\,\mu\,\omega^1 \;-\; i\,\pi_{1'} & \qquad \eth'\omega^0 &= -\,\lambda\,\omega^1 \\
\eth\omega^1 &= \sigma\,\omega^0 & \qquad \eth'\omega^1 &= \rho\,\omega^0 \;+\; i\,\pi_{0'}
\end{aligned} \tag{4.34}$$

Although the set of equations (4.34) is not too clumsy, compared to usual Newman-Penrose equations, it is still redundant. What is the action of "prime" operation introduced in (3.2)? Behaviour of the spin coefficients is given by (3.10), behaviour of the components of spinors ω^A and $\pi_{A'}$ is found easily. For example, the prime of ω^0 reads

$$\big(\omega^0\big)' \;=\; \big(-\,\iota_A\,\omega^A\big)' \;=\; -\,i\,o_A\,\omega^A \;=\; -\,i\,\omega^1.$$

In a similar way we find

$$\big(\omega^1\big)' \;=\; -\,i\,\omega^0, \qquad (\pi_{0'})' \;=\; -\,i\,\pi_{1'}, \qquad (\pi_{1'})' \;=\; -\,i\,\pi_{0'}.$$

Now, take, e.g. the first equation in (4.34) and perform the "prime"-operation:

$$\text{Þ}\omega^1 = \kappa\omega^0 \quad \mapsto \quad -i\,\text{Þ}'\omega^0 = -i\,\kappa'\,\omega^1.$$

However, according to (3.10) we have $\kappa' = -\nu$. Thus, under the prime, our equation transforms to

$$\text{Þ}'\omega^0 = -\nu\,\omega^1$$

which is just the first equation in the second column in (4.34)! Hence it is not necessary to explicitly write down all of equations (4.34) since the half of them can be obtained simply by taking the prime. In fact, the only independent equations are (4.35) and (4.36).

$$\text{Þ}\omega^0 = \omega^1\,\tau' - i\,\pi_{0'} \qquad \text{Þ}\omega^1 = \kappa\,\omega^0 \tag{4.35}$$

$$\eth\omega^0 = \rho'\,\omega^1 - i\,\pi_{1'} \qquad \eth\omega^1 = \sigma\,\omega^0 \tag{4.36}$$

As will be explained in Chapter 6, the Penrose mass is associated with the 3-volume enclosed in a topological sphere $\mathcal{S}$. In this context, vectors m^a and $\overline{m}^a$ are chosen to be tangent to the sphere $\mathcal{S}$ and vectors l^a and n^a are orthogonal to the sphere. Equations (4.35) are then *normal projections* of the twistor equation while equations (4.36) are tangential projections. Solutions of both (4.35) and (4.36) do not exist in general but, by the Atiyah-Singer index theorem, equations (4.36) can be solved if $\mathcal{S}$ has certain properties to be discussed in Chapter 6. The solutions to (4.36) are called *two-surface twistors*.

5. The geometry of spacelike 2-surfaces

After introducing some basic constructions of the twistor theory in the previous chapter, now we present some mathematical tools to analyze the geometry of spacelike two-dimensional surfaces. These tools will be necessary in our construction of the Penrose and the Bondi mass. The reason is obvious: quasi-local mass or angular momentum should be associated with such 2-surfaces.

The analysis of spacelike 2-surfaces is analogous to the 3+1 formulation of general relativity, see e.g. [21]. Three-dimensional spacelike hypersurface Σ has a unique normal vector n^a which defines the projection operator from full four-dimensional spacetime to Σ. Three-dimensional spatial metric h_{ab} is then projection of metric tensor g_{ab} onto Σ. Intrinsic geometry of the hypersurface can be characterized in terms of covariant derivative D_a associated with spatial metric h_{ab}, extrinsic geometry is encoded in the extrinsic curvature $K_{ac} = h^b_a \nabla_b n_c$. Both these objects, derivative D_a and extrinsic curvature K_{ac} can be used to calculate the Riemann tensor of hypersurface Σ.

On the other hand, in the case of spacelike 2-surface, we have *two* independent normal directions which will be denoted by t^a and v_a. Again, we define the projector Π^a_b to the subspace orthogonal to t^a and v^a which induces metric q_{ab} on the 2-surface. However, now there are two natural connections derived from the Levi-Civita connection and they will be denoted by δ_a and Δ_a, the latter being two dimensional version of the so-called Sen connection.

In this chapter we briefly review some aspect of the geometry of spacelike 2-surfaces. These tools have been developed gradually by Szabados in papers [19, 18] which we follow very closely.

5.1 2-surface tensor fields

Let (M, g_{ab}) be a manifold and $T^a M$ its tangent bundle, let $\mathcal{S}$ be a closed orientable spacelike 2-surface. Then the restriction of $T^a M$ to $\mathcal{S}$ will be denoted by $V^a\mathcal{S}$ and it can be decomposed uniquely into tangent bundle $T^a\mathcal{S}$ and the bundle of vectors normal to $\mathcal{S}$:

$$V^a\mathcal{S} \;=\; T^a\mathcal{S} \oplus N^a\mathcal{S}.$$

Elements of $T^a\mathcal{S}$ will be called *2-surface vector fields* and elements of general space $T^{a..b}_{c..d}\mathcal{S}$ will be called *2-surface tensor fields*. Let $t^a \in N^a\mathcal{S}$ be a timelike unit normal, let $v^a \in N^a\mathcal{S}$ be a spacelike unit normal, so that

$$t^a t_a \;=\; -\,v^a v_a \;=\; 1, \qquad t^a n_a \;=\; 0.$$

Clearly, there is a boost gauge freedom in the choice of t^a and v^a:

$$\begin{aligned} t^a &\mapsto \cosh u\, t^a + \sinh u\, v^a, \\ v^a &\mapsto \sinh u\, t^a + \cosh u\, v^a. \end{aligned}$$

Any vector $X^a \in V^a\mathcal{S}$ can be written in the form

$$X^a \;=\; \Pi^a_b\, X^b \;+\; O^a_b\, X^b$$

where Π^a_b and O^a_b are projection operators

$$\begin{aligned} &\Pi^a_b : V^a\mathcal{S} \mapsto T\mathcal{S}, \quad \Pi^a_b \;=\; \delta^a_b - t^a\, t_b + v^a\, v_b, \\ &O^a_b : V^a\mathcal{S} \mapsto N\mathcal{S}, \quad O^a_b \;=\; t^a\, t_b - v^a\, v_b. \end{aligned}$$

Trivial calculation shows that projection operators are gauge invariant. Condition that tensor $T^{a..b}_{c..d}$ be a 2-surface tensor is

$$\Pi^a_{a'} \dots \Pi^b_{b'} \, \Pi^{c'}_c \dots \Pi^{d'}_d \, T^{a'..b'}_{c'..d'} \;=\; T^{a..b}_{c..d},$$

i.e. its projection to $\mathcal{S}$ must be equal to the original tensor.

Two-dimensional metric q_{ab} induced on $\mathcal{S}$ is given by the projection of four-dimensional metric g_{ab} to $\mathcal{S}$:

$$q_{ab} \;=\; \Pi^c_a \, \Pi^d_b \, g_{cd} \;=\; g_{ab} \;-\; t_a \, t_b \;+\; v_a \, v_b.$$

Volume 2-form induced on surface $\mathcal{S}$ is naturally given by

$$\epsilon_{ab} \;=\; \epsilon_{abcd} \, t^c \, v^d$$

where ϵ_{abcd} is the volume 4-form on M. Again, it is easy to show that q_{ab} and ϵ_{ab} are gauge invariant quantities.

5.2 The two dimensional Sen operator

There are two natural operators of the covariant derivative characterizing the geometry of 2-surfaces. Let X^a be a surface vector field. The first operator is usual covariant derivative δ_a induced on $\mathcal{S}$ by familiar relation

$$\delta_a X^b = \Pi^c_a \, \Pi^b_d \, \nabla_c X^d. \tag{5.1}$$

This operator is compatible both with spacetime metric g_{ab} and with induced metric q_{ab} in the sense that

$$\delta_a q_{bc} \;=\; \delta_a g_{bc} \;=\; 0.$$

The connection associated with operator δ_a can be represented by the following quantities:

$$\tau_{ab} \;=\; \Pi^e_a \Pi^f_b \nabla_e t_f, \qquad \nu_{ab} \;=\; \Pi^e_a \Pi^f_b \nabla_e v_f. \tag{5.2}$$

Since the projector Π^a_b is boost invariant and since $\Pi^a_b t^b = \Pi^a_b v^b = 0$, these quantities under the boost transform as

$$\begin{aligned} \tau_{ab} &\mapsto \cosh u \, \tau_{ab} \;+\; \sinh u \, \nu_{ab}, \\ \nu_{ab} &\mapsto \sinh u \, \tau_{ab} \;+\; \cosh u \, \nu_{ab}. \end{aligned} \tag{5.3}$$

The Riemann tensor ${}^{\mathcal{S}}R_{abcd}$ characterizing the curvature of 2-surface is derived from connection δ_a by

$$(\delta_a \delta_b \;-\; \delta_b \delta_a) \, X_c \;=\; -\,{}^{\mathcal{S}}R_{abcd} \, X^d.$$

Quantities τ_{ab} and ν_{ab} are 2-surface tensors by definition. Hence, action of these tensors on fields X^a and Y^a which are *not* 2-surface vectors is equivalent to action of these tensors on their tangential projections $\Pi^a_c X^c$ and $\Pi^a_c Y^c$. Suppose, now, that X^a and Y^a *are* 2-surface tensors. Then we have

$$\tau_{ab} X^a Y^b \;=\; X^a Y^b \nabla_a t_b$$

and similarly for ν_{ab}. Since X^a and Y^a are orthogonal to both t^a and v^a, we have

$$Y^b \nabla_a t_b \;=\; -\,t_b \nabla_a Y^b \quad \text{and} \quad X^b \nabla_a t_b \;=\; -\,t_b \nabla_a X^b.$$

Thus, the action of antisymmetric part $t_{[ab]}$ on $X^a Y^b$ is

$$2\,\tau_{[ab]}\,X^a\,Y^b \;=\; (X^a\,Y^b - X^b\,Y^a)\nabla_a t_b \;=\; -X^a\,t_b\,\nabla_a Y^b + Y^a\,t_b\,\nabla_a X^b \;=\; -\,t_b\,[X,Y]^b \;=\; 0$$

by the orthogonality of t_b and commutator $[X,Y]^b$. Hence, antisymmetric part of τ_{ab} vanishes and so tensor τ_{ab} is symmetric. The same consideration applies to tensor ν_{ab}.

Let us now turn to another operator of covariant derivative which can be defined on $\mathcal{S}$. The *2-dimensional Sen operator* Δ_a is defined by

$$\Delta_a = \Pi^b_a \nabla_b. \tag{5.4}$$

Connection defined by Δ_a is again compatible with spacetime metric g_{ab} but not with the induced two-dimensional metric q_{ab}. In terms of the Sen operator, relations (5.2) can be written as

$$\tau_{ab} \;=\; \Pi^d_b\,\Delta_a t_d, \qquad \nu_{ab} \;=\; \Pi^d_b\,\Delta_a n_d. \tag{5.5}$$

Suppose that X^a is a surface vector, i.e. $\Pi^a_b X^b = X^a$; we want to find the difference between δ_a and Δ_a on X^b. We find

$$\delta_a X_b \;=\; \Pi^c_a \Pi^d_b \nabla_c X_d \;=\; \Pi^d_b \Delta_a X_d \;=\; \Delta_a(\Pi^d_b X_d) \;-\; X_d \Delta_a \Pi^d_b \;=\; \Delta_a X_b - X_e \Pi^e_d \Delta_a \Pi^d_b.$$

Hence, we define

$$Q^e{}_{ab} = -\Pi^e_d \Delta_a \Pi^d_b \tag{5.6}$$

and write

$$\delta_a X_b = \Delta_a X_b \;+\; Q^e{}_{ab}\,X_e \quad \text{for 2-surface field } X_a. \tag{5.7}$$

Simple calculation shows that $Q^e{}_{ab}$ can be expressed in terms of τ_{ab} and ν_{ab} as

$$Q^e{}_{ab} = t_b\,\tau_a{}^e \;-\; v_b\,\nu_a{}^e. \tag{5.8}$$

Notice that $Q^e{}_{ab}$ is, unlike τ_{ab} and ν_{ab}, independent of the boost gauge.

In order to derive the relation between the Riemann tensor and the Sen operator, let us write (for arbitrary X_c)

$$\Delta_a \Delta_b X_c \;=\; (\Delta_a \Pi^d_b)\nabla_d X_c \;+\; \Pi^e_a \Pi^f_b \nabla_e \nabla_f X_c.$$

The first term can be rearranged as follows:

$$(\Delta_a \Pi^d_b)\nabla_d X_c \;=\; -Q^e{}_{ab}\Delta_e X_c \;-\; \tau_{ab} t^e \nabla_e X_c \;+\; \nu_{ab} v^e \nabla_e X_c.$$

Thus, by the symmetry of τ_{ab} and ν_{ab}, the commutator of the Sen operators reads

$$(\Delta_a \Delta_b \;-\; \Delta_b \Delta_a)\,X_c = \Pi^e_a \Pi^f_b\,(\nabla_e \nabla_f - \nabla_f \nabla_e)\,X_c \;-\; (Q^e{}_{ab} - Q^e{}_{ba})\,\Delta_e X_c. \tag{5.9}$$

In the conventions used in the thesis, the Riemann curvature tensor is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\,X_c \;=\; -\,R_{abc}{}^d\,X_d.$$

Similarly we define the curvature tensor F_{abcd} and torsion $T^e{}_{ab}$ tensor associated with the Sen operator Δ_a. Relation (5.9) can be then written in the form

$$(\Delta_a \Delta_b \;-\; \Delta_b \Delta_a)\,X_c = -F_{abc}{}^d\,X_d \;-\; T^e{}_{ab}\,\Delta_e X_c, \tag{5.10}$$

where

$$\begin{aligned} F_{abc}{}^d &= \Pi^e_a\,\Pi^f_b\,R_{efc}{}^d, \\ T^e{}_{ab} &= 2\,Q^e{}_{[ab]}. \end{aligned} \tag{5.11}$$

5.3 2-surface spinors

Let us now investigate the geometry of 2-surfaces in terms of spinors. Recall that the Newman-Penrose (or Geroch-Held-Penrose) null tetrad consists of vectors l^a, n^a, m^a and $\overline{m}^a$ satisfying relations (3.3). Null vectors l^a and n^a can be constructed from our vectors t^a and v^a normal to 2-surface by

$$l^a = \frac{1}{\sqrt{2}} (t^a + v^a), \quad n^a = \frac{1}{\sqrt{2}} (t^a - v^a). \tag{5.12}$$

Now we can employ the spin basis (o^A, ι^A) associated with l^a and n^a via relations

$$l^a = o^A \bar{o}^{A'}, \quad n^a = \iota^A \bar{\iota}^{A'}. \tag{5.13}$$

Conversely, we can express normals t^a and v^a in terms of basis spinors as

$$t^a = \frac{1}{\sqrt{2}} \left(o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'} \right), \quad v^a = \frac{1}{\sqrt{2}} \left(o^A \bar{o}^{A'} - \iota^A \bar{\iota}^{A'} \right). \tag{5.14}$$

The basis of $T^a\mathcal{S}$ is then formed by null vectors

$$m^a = o^A \bar{\iota}^{A'}, \quad \overline{m}^a = \iota^A \bar{o}^{A'}.$$

The symplectic form $\epsilon_{AB} = o_A \iota_B - o_B \iota_A$ is related to the spacetime metric by usual formula

$$\epsilon_{AB}\, \epsilon_{A'B'} = g_{ab}.$$

Notice that ϵ_{AB} is in fact independent of the choice of the spin basis, because *all* two-forms on two-dimensional spinor space must be proportional to each other. Without an additional structure, ϵ_{AB} is the only canonical object on the space of spinors.

However, when we fix the normal vectors t^a and v^a, and thus fix the spin basis (up to gauge freedom), there is another canonical object which can be constructed: we will denote it as $\gamma^A{}_B$ and define it as

$$\gamma^A{}_B = 2\, t^{AA'} v_{BA'} = o^A \iota_B + o_B \iota^A. \tag{5.15}$$

Obvious properties of $\gamma^A{}_B$ are

$$\gamma^A{}_A = 0, \quad \gamma^A{}_B \gamma^B{}_C = \epsilon_C{}^A, \quad \det \gamma^A{}_B = -1. \tag{5.16}$$

Let ξ^A be an eigenspinor of $\gamma^A{}_B$, i.e. $\gamma^A{}_B \xi^B = \lambda \xi^A$ for some eigenvalue λ. Multiplying this equation with $\gamma^C{}_D$ and using (5.16) we find that eigenvalues of $\gamma^A{}_B$ are only $\lambda = \pm 1$. Eigenspinors with $\lambda = 1$ will be called *right-handed* while the eigenspinors with $\lambda = -1$ will be called *left-handed.*

Each spinor $\xi^A \in S^A$ can be then decomposed into right-handed and left-handed part using the projection operators

$$\pi_+^A{}_B = \frac{1}{2} \left(\epsilon^A{}_B + \gamma^A{}_B \right), \quad \pi_-^A{}_B = \frac{1}{2} \left(\epsilon^A{}_B - \gamma^A{}_B \right). \tag{5.17}$$

It is straightforward to verify that for arbitrary spinor ξ^A, its projections $\pi_\pm^A{}_B \xi^B$ are right handed and left handed eigenspinors of $\gamma^A{}_B$, respectively.

The projection operators Π^a_b and O^a_b in terms of the null tetrad can be found from well-known expression for the metric tensor or, equivalently, the Kronecker delta which reads

$$\delta^a_b \;=\; l^a n_b + l_b n^a - m^a \overline{m}_b - m_b \overline{m}^a.$$

Using (5.14) and (5.12) we obtain

$$O^a_b \;=\; n^a l_b \,+\, n_b l^a, \quad \Pi^a_b \;=\; -m^a \overline{m}_b \,-\, m_b \overline{m}^a. \tag{5.18}$$

By the definition (5.15) we have

$$\gamma^A{}_B \overline{\gamma}^{A'}{}_{B'} \;=\; l^a n_b + l_b n^a + m^a \overline{m}_b + \overline{m}^a m_b.$$

Hence, combining these relations we find the projection operators Π^a_b and O^a_b to be

$$\begin{aligned} \Pi^a_b &= \frac{1}{2}\left(\epsilon_B{}^A\,\epsilon_{B'}^{A'} \,-\, \gamma^A{}_B\,\overline{\gamma}^{A'}_{B'}\right), \\ O^a_b &= \frac{1}{2}\left(\epsilon_B{}^A\,\epsilon_{B'}^{A'} \,+\, \gamma^A{}_B\,\overline{\gamma}^{A'}_{B'}\right). \end{aligned} \tag{5.19}$$

5.4 Spinor decomposition of the Sen curvature

We have seen that the curvature tensor F_{abcd} associated with the Sen operator Δ_a is given by (5.11). In previous section we have derived the spinor from of the projection operators (5.19), while the spacetime Riemann tensor can be decomposed as in (2.53). Recall that when we derived the Ricci identities in the spinor form, section 2.4, we have found that the commutator of derivatives $[\nabla_a, \nabla_b]$ acting on the *spinor* ξ_C is given by anti-self-dual part of the Riemann tensor $R_{abCD} = 1/2\,\epsilon^{C'D'} R_{abcd}$. Similarly we define

$$F_{abCD} = \frac{1}{2}\,\epsilon^{C'D'}\,F_{abcd} \;=\; \frac{1}{2}\,\epsilon^{C'D'}\,\Pi^e_a\,\Pi^f_b\,R_{efcd}. \tag{5.20}$$

Substituting (2.88) for anti-self-dual part of the Riemann tensor and (5.19) for the projection operator we find

$$\begin{aligned} 4\,F_{abCD} = \;& \Big[\Psi_{ABCD} \,-\, \gamma^E{}_A\,\gamma^F{}_B\,\Psi_{EFCD} \,+\, \gamma_{AB}\,\overline{\gamma}^{E'F'}\,\Phi_{CDE'F'}\Big]\,\epsilon_{A'B'} \\ & -\,2\,\Lambda\left[\gamma_{A(C}\gamma_{D)B} \,+\, \epsilon_{A(C}\,\epsilon_{D)B}\right]\,\epsilon_{A'B'} \\ & \Big[\overline{\gamma}_{A'B'}\,\gamma^{EF}\,\Psi_{EFCD}\,\epsilon_{AB} \,+\, \Phi_{CDA'B'} \,-\, \Phi_{CDE'F'}\,\overline{\gamma}^{E'}_{A'}\,\overline{\gamma}^{F'}{}_{B'}\Big]\,\epsilon_{AB} \\ & +\,2\,\Lambda\,\gamma_{CD}\,\overline{\gamma}_{A'B'}\,\epsilon_{AB}. \end{aligned} \tag{5.21}$$

This expression can be simplified significantly. Spinor F_{abCD} is antisymmetric in ab which are indices tangent to surface $\mathcal{S}$ (because they arose from the projection $\Pi^e_a \Pi^f_b R_{efCD}$). Hence, F_{abCD} must be proportional to the volume 2-form ϵ_{ab} induced on $\mathcal{S}$ by pull-back of ϵ_{abcd} from four-dimensional spacetime. Volume 2-form ϵ_{ab} can be written as

$$\epsilon_{ab} \;=\; \epsilon_{abcd}\,t^c\,v^d \;=\; \epsilon_{abcd}\,n^c\,l^d \;=\; i\,\left(\epsilon_{AB}\overline{o}_{A'}\overline{\iota}_{B'} \,-\, \epsilon_{A'B'}o_A\iota_B\right).$$

Since now we have the symmetric spinor $\gamma_{AB} = 2o_{(A}\iota_{B)}$, the product $o_A\iota_B$ can be decomposed into the sum

$$o_A\,\iota_B \;=\; \frac{1}{2}\gamma_{AB} \,+\, \frac{1}{2}\,\epsilon_{AB}$$

so that the induced volume 2-form reads

$$\epsilon_{ab} = \frac{i}{2}\left(\epsilon_{AB}\,\overline{\gamma}_{A'B'} - \epsilon_{A'B'}\,\gamma_{AB}\right). \tag{5.22}$$

Now we can write

$$F_{abCD} = \lambda_{CD}\,\epsilon_{ab}$$

from which, by contraction with ϵ^{ab} and using $\epsilon_{ab}\epsilon^{ab} = 2$, we get

$$F_{abCD} = \left[-\frac{i}{2}\,\gamma^{AB}\,\Psi_{ABCD} + \frac{i}{2}\,\overline{\gamma}^{A'B'}\,\Phi_{CDA'B'} - i\,\Lambda\,\gamma_{CD}\right]\epsilon_{ab}. \tag{5.23}$$

Let us now return to commutator (5.10). In order to find the action of commutator $[\Delta_a, \Delta_b]$ on the spinor field, let us put $X_c = \xi_C\overline{\xi}_{C'}$ and contract the equation with ξ^C:

$$\xi^C\,\overline{\xi}_{C'}\,[\Delta_a, \Delta_b]\xi_C = -\,F_{abcd}\,\xi^C\,\xi^D\,\overline{\xi}^{D'} - 2\,Q^e{}_{[ab]}\,\xi^C\,\overline{\xi}_{C'}\,\Delta_e\xi_C. \tag{5.24}$$

Now we eliminate the self-dual part of F_{abcd} by decomposition

$$F_{abcd}\,\xi^C\,\xi^D = F_{abCD(C'D')}\,\xi^C\,\xi^D + \frac{1}{2}\,\epsilon_{C'D'}\,\epsilon^{X'Y'}\,F_{abCDX'Y'}\,\xi^C\,\xi^D. \tag{5.25}$$

The second term on the right hand side is obviously anti-self-dual part $\epsilon_{C'D'}F_{abCD}\xi^C\xi^D$ while the first term reads

$$\begin{aligned} F_{abCD(C'D')}\,\xi^C\,\xi^D &= \Pi^e_a\,\Pi^f_b\,R_{abCD(C'D')}\xi^C\xi^D \\ &= \Pi^e_a\,\Pi^f_b\,\left(\Psi_{E'F'C'D'}\epsilon_{CD}\epsilon_{E'F'} + \Phi_{EFC'D'}\epsilon_{E'F'}\epsilon_{CD} - 2\Lambda\epsilon_{E'(C'}\epsilon_{D')F'}\epsilon_{EF}\epsilon_{CD}\right)\xi^C\,\xi^D \end{aligned} \tag{5.26}$$

which vanishes by the symmetry of $\xi^C\xi^D$. Hence, we can write

$$\xi^C\,[\Delta_a, \Delta_b]\xi_C = F_{abCD}\,\xi^C\,\xi^D - 2Q^e{}_{[ab]}\,\xi^C\,\Delta_e\xi_C \tag{5.27}$$

and therefore[1]

$$[\Delta_a, \Delta_b]\xi_C = F_{abCD}\,\xi^D - 2Q^e{}_{[ab]}\,\Delta_e\xi_C. \tag{5.28}$$

Next we find the spinor form of the torsion represented by tensor $Q^e{}_{ab}$. Recalling the definition (5.6) and employing the spinor form of projector (5.19) we find

$$4\,Q^e{}_{ab} = \gamma^E{}_B\,\Delta_a\,\overline{\gamma}^{E'}{}_{B'} + \overline{\gamma}^{E'}{}_{B'}\,\Delta\,\gamma^E{}_B - \delta^E_B\,\overline{\gamma}^{E'}{}_{D'}\,\Delta_a\,\overline{\gamma}^{D'}{}_{B'} - \delta^{E'}_{B'}\,\gamma^E{}_D\,\Delta_a\,\gamma^D{}_B. \tag{5.29}$$

[1] Similarly as in section 2.4 we put $\xi^C = \alpha^C + k\beta^C$, where α^C, β^C are arbitrary independent spinors and k is an arbitrary real or complex number. The equation (5.27) will then contain terms quadratic in k, terms linear in k, and terms not dependent on k. The equation must hold for each group separately. Collecting terms linear in k, we obtain the equation

$$k\beta^C[\Delta_a, \Delta_b]\alpha_C + \alpha^C[\Delta_a, \Delta_b]\,(k\beta_C) = 2kF_{abCD}\alpha^C\beta^D - 2Q^e{}_{[ab]}k\beta^C\Delta_e\alpha_C - 2Q^e{}_{[ab]}\alpha^C\Delta_e\,(k\beta_C)\,.$$

Now, using $[\Delta_a, \Delta_b]l = -2Q^e{}_{[ab]}\Delta_e l$, where $l = k\beta_C\alpha^C$, we arrive at

$$\beta^C[\Delta_a, \Delta_b]\alpha_C = \beta^C\left(F_{abCD}\alpha^D - 2Q^e{}_{[ab]}\Delta_e\alpha_C\right)\,,$$

which yields the desired result.

If we define

$$Q^E{}_{aF} = \frac{1}{2}\,\gamma^X{}_F\,\Delta_a\,\gamma^E{}_X, \tag{5.30}$$

the Sen-derivative of $\gamma^A{}_B$ can be written in the form

$$\Delta_a\gamma^E{}_B = 2\,Q^E{}_{aF}\,\gamma^F{}_B. \tag{5.31}$$

Inserting this back to (5.29) we find

$$2\,Q^e{}_{ab} = Q^E{}_{aX}\gamma^X{}_B\,\overline{\gamma}^{E'}_{B'} + \overline{Q}^{E'}_{aX'}\overline{\gamma}^{X'}_{B'}\,\gamma^E{}_B + \delta^E_B\,\overline{Q}^{E'}_{aB'} + \delta^{E'}_{B'}\,Q^E{}_{aB}. \tag{5.32}$$

5.5 Decomposition of spinors

In the spinor formalism we frequently use the fact that any spinor ϕ_{AB} can be decomposed into totally symmetric and totally antisymmetric part according to

$$\phi_{AB} \;=\; \phi_{(AB)} \;+\; \phi_{[AB]} \;=\; \phi_{(AB)} \;+\; \frac{1}{2}\,\epsilon_{AB}\,\phi_X{}^X.$$

In general, both these parts are irreducible, unless an additional structure is present. In our case, this structure is provided by the symmetric spinor γ_{AB}. Symmetric part $\phi_{(AB)}$ can be then decomposed further into part representing the trace of $\phi_{(AB)}$ with respect to γ_{AB} and the trace-free part. By the trace we mean the contraction of ϕ_{AB} with γ_{AB}, i.e. the complex number

$$\gamma^{AB}\,\phi_{(AB)} \;=\; \gamma^{AB}\,\phi_{AB}.$$

The trace-free part of $\phi_{(AB)}$ is then spinor of the form

$$\mathcal{T}_{AB}{}^{CD}\phi_{CD} \;=\; \phi_{(AB)} - k\,\gamma_{AB}\,\gamma^{CD}\,\phi_{CD}$$

with k chosen so that

$$\gamma^{AB}\,\mathcal{T}_{AB}{}^{CD}\phi_{CD} \;=\; 0.$$

Relation $\gamma^{AB}\gamma_{AB} = -2$ then implies $k = -1/2$ and hence we define operator

$$\mathcal{T}_{AB}{}^{CD} = \delta^C_{(A}\delta^D_{B)} \;+\; \frac{1}{2}\,\gamma_{AB}\,\gamma^{CD} \tag{5.33}$$

which projects spinor ϕ_{AB} into its γ-trace-free part

$$\begin{aligned} \mathcal{T}_{AB}{}^{CD}\Phi_{CD} &= \phi_{(AB)} \;+\; \frac{1}{2}\,\gamma_{AB}\,\gamma^{CD}\phi_{CD}, \\ \gamma^{AB}\,\mathcal{T}_{AB}{}^{CD}\phi_{CD} &= 0. \end{aligned} \tag{5.34}$$

Full decomposition of the spinor ϕ_{AB} into irreducible parts then reads

$$\phi_{AB} = \frac{1}{2}\,\epsilon_{AB}\,\phi_X{}^X \;-\; \frac{1}{2}\,\gamma_{AB}\,\gamma^{CD}\,\phi_{CD} \;+\; \mathcal{T}_{AB}{}^{CD}\,\phi_{CD}. \tag{5.35}$$

Let us apply this decomposition to the Sen derivative $\Delta_{AA'}$ of arbitrary spinor λ_B:

$$\Delta_{A'A}\lambda_B = \frac{1}{2}\,\epsilon_{AB}\,\Delta_{A'X}\lambda^X \;-\; \frac{1}{2}\,\gamma_{AB}\,\gamma^{CD}\,\Delta_{A'C}\lambda_D \;+\; \mathcal{T}_{AB}{}^{CD}\Delta_{A'C}\lambda_D. \tag{5.36}$$

By definition of $\mathcal{T}_{AB}{}^{CD}$, the trace-free part of the Sen derivative $\Delta_{AA'}\lambda_B$ is

$$\mathcal{T}_{AB}{}^{CD}\Delta_{A'C}\lambda_D = \Delta_{A'(A}\lambda_{B)} + \frac{1}{2}\gamma_{AB}\,\gamma^{CD}\,\Delta_{A'C}\lambda_D. \tag{5.37}$$

This can be written in the form of action of new operator $\mathcal{T}_{A'AB}{}^C$ on the spinor λ_C, where

$$\begin{aligned} \mathcal{T}_{A'AB}{}^C &= \delta^C_{(A}\Delta_{B)A'} + \frac{1}{2}\gamma_{AB}\gamma^{CD}\Delta_{DA'}, \\ \mathcal{T}_{A'AB}{}^C\lambda_C &= \Delta_{A'(A}\lambda_{B)} + \frac{1}{2}\gamma_{AB}\gamma^{CD}\Delta_{A'C}\lambda_D. \end{aligned} \tag{5.38}$$

In terms of this operator, decomposition of $\Delta_{AA'}\lambda_B$ reads

$$\Delta_{AA'}\lambda_B = \frac{1}{2}\epsilon_{AB}\,\Delta_{A'C}\lambda^C - \frac{1}{2}\gamma_{AB}\,\gamma^{CD}\,\Delta_{A'C}\lambda_D + \mathcal{T}_{A'AB}{}^C\lambda_C. \tag{5.39}$$

We will show that in the case of spinor $\Delta_{AA'}\lambda_B$, part proportional to γ_{AB} in expression (5.39) is not an independent irreducible part. Using the definition of the Sen derivative, $\Delta_a = \Pi^c_a\nabla_c$ and the spinorial form of the projection operator (5.19) we find

$$\Delta_{AA'}\lambda_B = \frac{1}{4}\left[\nabla_{AA'}\lambda_B - \gamma^C{}_A\,\overline{\gamma}^{C'}{}_{A'}\,\nabla_{CC'}\lambda_B\right]. \tag{5.40}$$

Contracting this equation with $\epsilon^{AB}\overline{\gamma}^{A'}{}_{B'}$ we find (after relabeling some dummy indices)

$$\overline{\gamma}^{B'}{}_{A'}\,\Delta_{AB'}\lambda^A = \frac{1}{4}\left[\overline{\gamma}^{B'}{}_{A'}\nabla_{AB'}\lambda^A + \gamma^{AB}\,\nabla_{AA'}\lambda_B\right]. \tag{5.41}$$

On the other hand, contracting equation (5.40) with γ^{AB} yields

$$\gamma^{AB}\Delta_{AA'}\lambda_B = \frac{1}{4}\left[\gamma^{AB}\nabla_{AA'}\lambda_B + \overline{\gamma}^{C'}{}_{A'}\nabla_{CC'}\lambda^C\right]. \tag{5.42}$$

Since the right hand sides of equations (5.41) and (5.42) coincide, we arrive at identity

$$\gamma^{AB}\Delta_{AA'}\lambda_B = \overline{\gamma}^{B'}{}_{A'}\Delta_{AB'}\lambda^A. \tag{5.43}$$

Now, let us rewrite the trace parts (both ϵ and γ) in decomposition (5.39) using the identity (5.43):

$$\frac{1}{2}\epsilon_{AB}\Delta_{A'C}\lambda^C - \frac{1}{2}\gamma_{AB}\,\gamma^{CD}\Delta_{CA'}\lambda_D = \frac{1}{2}\left(\epsilon_{AB}\,\epsilon_{A'}{}^{B'} - \gamma_{AB}\,\overline{\gamma}^{B'}{}_{A'}\right)\Delta_{CB'}\lambda^C.$$

Obviously, the term in the bracket is the projection operator and so the spinor decomposition (5.39) can be brought into the form of the sum of two irreducible parts,

$$\Delta_{A'A}\lambda_B = \Pi^{CB'}_{AA'}\,\epsilon_{CB}\,\Delta_{DB'}\lambda^D + \mathcal{T}_{A'AB}{}^C\lambda_C. \tag{5.44}$$

Now we can see why the $\gamma-$trace is not an independent irreducible part: it appears as one term in the projection of the $\epsilon-$trace $\Delta_{DB'}\lambda^D$.

In fact, the operator $\mathcal{T}_{A'AB}{}^C$ (or its chiral projections $\overline{\pi}^{D'}_{\pm}{}_{A'}\mathcal{T}_{D'AB}{}^C$) constitutes tangential part of the twistor equation (4.1). In the twistor theory, it was introduced as a tool to construct solutions of the zero-rest-mass equations. Here we have shown that the twistor operator appears naturally as a part of the geometry intrinsic to $\mathcal{S}$.

6. Penrose's mass

In this chapter we briefly review Penrose's construction of quasilocal mass, following [7, 20].

6.1 Motivation

Let us start our discussion with rather trivial case of Newton's gravity. Newtonian gravitational potential ϕ is subject to the Poisson equation

$$\Delta\phi \;=\; 4\,\pi\,G\,\rho\,,$$

where ρ is the mass density of gravitating source. Strength of gravitational field $\boldsymbol{K}$ is related to gravitational potential by familiar relation

$$\boldsymbol{K} \;=\; -\,\nabla\phi.$$

These formulae allow us to pass from well-defined notion of local mass-density to quasilocal mass. Let us choose any finite three-dimensional volume V with boundary $\mathcal{S} = \partial V$. The mass contained in volume V is, by the definition of ρ, equal to

$$m[V] = \int\limits_V \rho\,\mathrm{d}V\,, \tag{6.1}$$

where we use square brackets to emphasize that $m[V]$ is mass associated to a volume V. Now, using the Poisson equation and the Gauss divergence theorem we arrive at the expression

$$m[V] = \frac{1}{4\pi G}\int\limits_V \Delta\phi\,\mathrm{d}V \;=\; \frac{1}{4\pi G}\oint\limits_{\mathcal{S}} n^a\nabla_a\phi\,\mathrm{d}\mathcal{S} \;\equiv\; m[\mathcal{S}]\,, \tag{6.2}$$

where n^a is the normal to 2-surface $\mathcal{S}$.

Thus, we have two equivalent expressions (6.1) and (6.2) for the mass contained in volume V or, equivalently, the mass enclosed in surface $\mathcal{S}$. Notice that while the volume integral (6.1) is expressed in terms of the source (mass density), surface integral (6.2) is expressed in terms of field quantity, potential ϕ. In general relativity, the mass density ρ must be replaced by the energy-momentum tensor T_{ab} and we can expect that some notion of quasi-local mass(energy) can be obtained by integrating T_{ab} over some 3-volume V. On the other hand, T_{ab} decribes only non-gravitational fields and vanishes in the regions where only gravitational field propagates. Hence, using the energy-momentum tensor it is impossible to define quasilocal energy associated to pure gravitational field and the volume integral like (6.1) is expected to vanish.

However, one can still hope to find reasonable notion of quasilocal mass in terms of surface integral like (6.2) which contains the field quantities instead of sources. Energy-momentum tensor is related to the curvature tensor R_{abcd} via Einstein's equations. The difference is that R_{abcd} does not vanish even in the vacuum if the gravitational field is present. So, in order to find an appropriate notion of quasilocal mass(energy), we are looking for a suitable surface integral of the Riemann tensor.

6.2 Quasilocal quantities in Minkowski spacetime

In this section we briefly discuss quasilocal quantities associated to matter fields(rather than to pure gravitational field) in the flat spacetime possessing 10 Killing symmetries. Let K^a be a Killing vector of the spacetime subject to the Killing equation

$$\nabla_a K_b + \nabla_b K_a = 0 \quad \text{or} \quad 2\nabla_{(a} K_{b)} = 0 \tag{6.3}$$

and let T_{ab} be (symmetric) energy-momentum tensor of the field. Recall that T_{ab} must satisfy equation

$$\nabla^a T_{ab} = 0 \tag{6.4}$$

by the requirement that the theory be invariant under diffeomorphisms.

Then we can associate a conserved quantity to each of the Killing vectors of the spacetime. Define the four-current j_a by

$$j_a = T_{ab} K^b. \tag{6.5}$$

This current automatically satisfies the continuity equation:

$$\nabla^a j_a = (\nabla^a T_{ab}) K^b + T_{ab} \nabla^a K^b = 0 ,$$

where we used (6.3), (6.4) and the symmetry of T_{ab} (so that $T_{ab}\nabla^a K^b = T_{ab}\nabla^{(a} K^{b)}$).

Now we can choose a spacelike three-dimensional hypersurface Σ with the boundary $\mathcal{S} = \partial\Sigma$ and integrate j_a over Σ to obtain charge $Q[K]$ associated with Killing vector K^a which is conserved, constant in time. Of course, j_a cannot be integrated over Σ directly because it is a three-dimensional hypersurface while j_a is a one-form. For this reason we have to introduce the Hodge dual ω_{abc} of j_a given by standard relation

$$\omega_{abc} = \epsilon_{abcd}\, j^d. \tag{6.6}$$

The Hodge dual ω_{abc} is a three-form and thus can be integrated over three-surface Σ.

Form ω_{abc} is closed, i.e. $\nabla_{[e}\omega_{abc]} = 0$. This can be shown as follows. Since the exterior derivative of ω_{abc} is the four-form, it must be proportional to ϵ_{abcd}:

$$\nabla_{[e}\omega_{abc]} = \lambda\, \epsilon_{abce}.$$

Coefficient λ can be obtained by contraction of the last equation with ϵ^{abce}:

$$4!\,\lambda = \epsilon^{abce}\nabla_e \epsilon_{abcd} j^d = 3!\,\delta^e_d \nabla_e j^d = 3!\,\nabla_d j^d = 0$$

where we have used relation $\epsilon^{abce}\epsilon_{abcd} = 3!\delta^e_d$ and the continuity equation. Hence, $\lambda = 0$ and form ω_{abc} is closed.

By the Poincaré lemma, any closed form in flat spacetime is also exact and therefore there exists a two-form K_{ab} such that

$$\omega_{abc} = 3\,\nabla_{[a} K_{bc]}. \tag{6.7}$$

Two-form K_{ab} is called the *superpotential*[1] for the current j^a. Now we can define the charge by

$$Q_{\mathcal{S}}[K] = \int_\Sigma \omega_{abc} = \int_\Sigma \epsilon_{abcd} T^{de} K_e\,, \tag{6.8}$$

[1] Sometimes, term superpotential is reserved for the Hodge dual of K_{ab}, i.e. for quantity $(1/2)\epsilon^{abcd}K_{cd}$.

which, by the Stokes theorem, is equivalent to

$$Q_{\mathcal{S}}[K] = \oint_{\mathcal{S}} K_{ab}. \tag{6.9}$$

Suppose that Σ is chosen to be a Cauchy hypersurface with boundary $\mathcal{S}$. Since all Cauchy hypersurfaces have the same boundary $\mathcal{S}$ at spacelike infinity, the value of $Q_{\mathcal{S}}[K]$ does not depend on the choice of Cauchy hypersurface Σ. This can be interpreted as the conservation of the charge in time.

6.3 Penrose's construction and twistors

Let us now turn to full general relativity. In the previous section we obtained expression (6.8) for the charge associated with the Killing vector K^a. This expression is analogous to Newtonian expression (6.1) in the sense that it is a volume integral of object representing the source of gravity, mass. By source we mean the mass-density in the Newtonian case and energy-momentum tensor in the relativistic case. However, as noted above, charge $Q_{\mathcal{S}}[K]$ necessarily vanishes if only gravitational field is present.

Hence, we would like to re-express (6.8) in terms of curvature tensor rather than in terms of T_{ab}. The first possibility is to replace the energy-momentum tensor by the left hand side of Einstein's equations, i.e. by the Einstein tensor. An obvious drawback of this suggestion is that the Einstein tensor consists of contractions of the Riemann tensor and therefore does not contain information about "pure" gravitational field which is represented by the (trace-free) Weyl part of the Riemann tensor. For this reason, charge $Q_{\mathcal{S}}[K]$ must be expressed by the integral of Riemann tensor rather than by the integral of the Ricci tensor. On the other hand, tensor R_{abcd} cannot be integrated over 2-surface $\mathcal{S}$ and so we introduce new tensor field f^{ab} to construct a 2-form $R_{abcd}f^{cd}$ which already can be integrated.

Thus, Penrose's suggestion is that the volume integral (6.8) can be expressed as a surface integral of the Riemann tensor as follows:

$$\int_{\Sigma} \epsilon_{abcd} T^{de} K_e = \varkappa \oint_{\mathcal{S}} R_{abcd} f^{cd}\,, \tag{6.10}$$

where $\varkappa$ is a constant to be determined later. Let us find conditions under which the above relation holds.

Without the loss of generality we may assume that f^{cd} is antisymmetric and thus can be written in the form

$$f^{cd} \;=\; \omega^{CD}\epsilon^{C'D'} + \overline{\omega}^{C'D'}\epsilon^{CD}$$

with ω^{CD} symmetric. In addition, we assume that f^{cd} is anti-self-dual[2](and hence necessarily complex) in order to simplify the calculations:

$$f^{cd} \;=\; \omega^{CD}\,\epsilon^{C'D'}.$$

Integrands in (6.10) cannot be compared directly, because the integral on the left hand side is taken over the volume Σ while the right hand side integral is taken over the boundary $\mathcal{S} = \partial\Sigma$.

[2]A two-form is self-dual, if it is multiplied by i under Hodge dualization. It is self-dual, if it is multiplied by $-i$ under dualization. General real 2-form is a sum of self-dual part $\overline{\omega}^{C'D'}\epsilon^{CD}$ and anti-self-dual part $\omega^{CD}\epsilon^{C'D'}$.

Applying the Stokes theorem, the right hand side transforms to the volume integral

$$\varkappa \oint_{\mathcal{S}} R_{abcd} f^{cd} \;=\; 3\,\varkappa \int_{\Sigma} \nabla_{[e}\left(R_{ab]cd} f^{cd}\right)$$

where the covariant derivative acts on both quantities R_{abcd} and f^{cd}. However, by the Bianchi identity we have $\nabla_{[e}R_{ab]cd} = 0$ and so the covariant derivative acts only on f^{cd}. Now that we have written the right hand side integral in (6.10) as a volume integral, we can compare both integrands:

$$\epsilon_{eabc} T^{cd} K_d \;=\; 3\,\varkappa\, \nabla_{[e} f^{cd} R_{ab]cd}. \tag{6.11}$$

At this stage we could insert spinorial equivalents into (6.11) and find desired conditions. However, it is more convenient to multiply (6.11) by $\epsilon_f{}^{eab}$ and remove the antisymmetrization in the operation of exterior derivative. Let us evaluate more complicated right hand side first:

$$\text{r.h.s.} = 3\varkappa\epsilon_f{}^{eab}\nabla_{[e} f^{cd} R_{ab]cd} \;=\; 3\varkappa\epsilon_f{}^{eab} R_{abcd}\,\nabla_e f^{cd} \;=\; 6\varkappa\,{}^*R_{fecd}\,\nabla^e f^{cd} \tag{6.12}$$

where ${}^*R_{abcd}$ is the Hodge dual of the Riemann tensor defined by

$${}^*R_{abcd} \;=\; \frac{1}{2}\,\epsilon_{ab}{}^{ef} R_{efcd}.$$

Using the notation of section 2.3, namely definitions (2.57), (2.58) and (2.59), decomposition of dual Riemann tensor reads

$${}^*R_{abcd} \;=\; -i\psi_{abcd} + i\overline{\psi}_{abcd} - i\phi_{abcd} + i\,\overline{\phi}_{abcd} - i\lambda_{abcd} + i\overline{\lambda}_{abcd}.$$

Recall that $f^{cd} = \omega^{CD}\epsilon^{C'D'}$. Thus, contraction of ${}^*R_{efcd}$ with $\nabla^e f^{cd}$ will annihilate all terms antisymmetric in CD and terms symmetric in $C'D'$. Direct calculation then reveals

$$\text{r.h.s} = 6\varkappa\left[2\,i\,\Psi_{FECD}\,\nabla^{(E}_{F'}\omega^{CD)} \;+\; 2\,i\,\Phi_{CDF'E'}\nabla^{E'}_{F}\,\omega^{CD} \;+\; 4\,i\,\Lambda\,\nabla^{E}_{F'}\omega_{EF}\right]. \tag{6.13}$$

Evaluation of the left hand side of (6.11) multiplied by $\epsilon_f{}^{eab}$ is much easier:

$$\text{l.h.s.} \;=\; \epsilon_f{}^{eab}\,\epsilon_{eabc}\,T^{cd}\,K_d \;=\; 3!\,g_{cf}\,T^{cd}\,K_d \;=\; -\,\frac{3!}{8\pi G}G_{fd}\,K^d.$$

Using the spinor equivalent of the Einstein tensor, or substituting for T^{cd} directly from (2.50), we arrive at

$$\text{l.h.s.} \;=\; \frac{6}{4\pi G}\left[\Phi_{FDF'D'}\,K^{DD'} \;+\; 3\,\Lambda\,K_{FF'}\right].$$

Finally we compare the left hand side to the right hand side and obtain equality

$$\begin{aligned}\Phi_{FDF'D'}\,K^{DD'} \;+\; 3\,\Lambda\,K_{FF'} \;&=\\ &=\; 8\,\pi\,G\,\varkappa\left[i\,\Psi_{FECD}\,\nabla^{(E}_{F'}\omega^{CD)} \;+\; i\,\Phi_{CDF'E'}\nabla^{E'}_{F}\,\omega^{CD} \;+\; 2\,i\,\Lambda\,\nabla^{E}_{F'}\omega_{EF}\right]\end{aligned} \tag{6.14}$$

We have brought both integrands in (6.11) to the form such that they can be compared and conditions under which they equal to each other can be discussed.

First, in order to eliminate the constants we set

$$\varkappa \;=\; (8\,\pi\, G)^{-1}.$$

Let us equate terms containing the Ricci spinor:

$$\Phi_{FDF'D'}\, K^{DD'} \;=\; i\,\Phi_{CDF'E'}\, \nabla^{E'}_{F}\, \omega^{CD}. \tag{6.15}$$

On the left hand side, the Ricci spinor is contracted with the two-index object while on the right hand side there is a contraction over three indices. Hence, we rewrite the left hand side as

$$\Phi_{FDF'D'}\, K^{DD'} \;=\; \Phi_{CDF'E'}\, \epsilon_{F}{}^{C}\, K^{DE'} \;=\; \Phi_{CDF'E'}\, \epsilon_{F}{}^{(C}\, K^{D)E'}$$

where we have used the symmetry of the Ricci spinor in the last step. Equation (6.15) then yields

$$\nabla^{CC'}\omega^{AB} = -\,i\,\epsilon^{C(A}K^{B)C'}. \tag{6.16}$$

So, we arrived at first condition which ω^{AB} must satisfy. However, symmetrization of this equation in all (unprimed) indices leads to

$$\nabla^{(A}_{C'}\omega^{BC)} = 0. \tag{6.17}$$

This is an example of the *twistor equation*. We can see that ω^{BC} satisfying (6.16) automatically satisfies also the twistor equation (6.17) and so the only term in (6.14) containing the Weyl spinor vanishes. Then, single remaining condition follows from the comparison of terms containing Λ:

$$\nabla^{B}_{A'}\omega_{BC} \;=\; -\,\frac{3}{2}\,i\,K_{CA'}.$$

It is straightforward to show that even this condition is a consequence of (6.16), for we have

$$\nabla^{C}_{C'}\omega_{CB} = \;-\,\frac{i}{2}\,\left(\epsilon^{C}{}_{C}\,K_{BC'} \,+\, \epsilon^{C}{}_{B}\,K_{CC'}\right) \;=\; -\,\frac{3}{2}\,i\,K_{BC'}. \tag{6.18}$$

Let us recapitulate the results. Following Penrose, we suggested to define the quasilocal charge associated with the Killing vector K^a by (cf. 6.10)

$$Q_{\mathcal{S}}[K] = \frac{1}{8\,\pi\, G}\oint R_{abcd}\, f^{cd} \tag{6.19}$$

where

$$f^{cd} \;=\; \omega^{CD}\,\epsilon^{C'D'}$$

is anti-self-dual form. Then we required that, applying the Stokes theorem, integral (6.19) should reduce to the volume integral

$$Q_{\mathcal{S}}[K] \;=\; \int\limits_{\Sigma} \epsilon_{abcd}\, T^{de}\, K_e.$$

This requirement implied that symmetric spinor ω^{CD} must satisfy the twistor equation (6.16) or, equivalently, twistor equation in the form (6.17).

6.4 Explicit expression for charge

Let us write the expression for quasi-local charge (6.19) explicitly in terms of the Newman-Penrose quantities. First, the integrand reads

$$R_{abcd}\, f^{cd} = \left(\psi_{abcd} + \overline{\psi}_{abcd} + \phi_{abcd} + \overline{\phi}_{abcd} + \lambda_{abcd} + \overline{\lambda}_{abcd}\right) \omega^{CD}\, \epsilon^{C'D'}. \tag{6.20}$$

Contraction of the expression in the brackets with $\omega^{CD}\epsilon^{C'D'}$ annihilates all terms symmetric in $C'D'$ and antisymmetric in CD, i.e. terms containing $\overline{\psi}_{abcd}$, ϕ_{abcd} and $\overline{\lambda}_{abcd}$. Remaining non-zero terms are

$$\mathcal{R}_{ab} := R_{abcd}\, f^{cd} = 2\,\epsilon_{A'B'}\,\omega^{CD}\,\Psi_{ABCD} + 2\,\epsilon_{AB}\,\omega^{CD}\,\Phi_{CDA'B'} + 4\,\Lambda\,\epsilon_{A'B'}\,\omega_{AB}. \tag{6.21}$$

Quantity $\mathcal{R}_{ab}$ is a two-form. However, in the integral

$$Q_{\mathcal{S}}[\omega^{AB}] \;=\; \oint_{\mathcal{S}} \mathcal{R}_{ab},$$

the integrand must be proportional to a two-dimensional volume-form ${}^{(2)}\epsilon_{ab}$ induced on the surface $\mathcal{S}$:

$$Q_{\mathcal{S}}[\omega^{AB}] \;=\; \oint_{\mathcal{S}} \mathcal{R}\; {}^{(2)}\epsilon_{ab} \;\equiv\; \oint_{\mathcal{S}} \mathcal{R}\, \mathrm{d}\mathcal{S}.$$

Thus, we have to find an expression for $\mathcal{R}$.

We now introduce the Newman-Penrose null tetrad consistently with previous chapters in a following way. Let l^a and n^a be null vectors orthogonal to surface $\mathcal{S}$ and let m^a and $\overline{m}^a$ be null vectors tangent to $\mathcal{S}$. A spinor dyad is chosen in a standard way so that

$$l^a \;=\; o^A\,\overline{o}^{A'}, \quad n^a \;=\; \iota^A\,\overline{\iota}^{A'}, \quad m^a \;=\; o^A\,\overline{\iota}^{A'}, \quad \overline{m}^a \;=\; \iota^A\,\overline{o}^{A'}.$$

Two-dimensional volume-form on the surface $\mathcal{S}$ is therefore a contraction of four-dimensional ϵ_{abcd} with vectors normal to $\mathcal{S}$:

$${}^{(2)}\epsilon_{ab} = \epsilon_{abcd}\, l^c\, n^d. \tag{6.22}$$

Using the spinor equivalent of Levi-Civita symbol (1.96) we find

$${}^{(2)}\epsilon_{ab} = i\,\left(\epsilon_{AB}\,\overline{o}_{A'}\,\overline{\iota}_{B'} - \epsilon_{A'B'}\,o_A\,\iota_B\right). \tag{6.23}$$

Normalization is chosen so that

$${}^{(2)}\epsilon^{ab}\;{}^{(2)}\epsilon_{ab} \;=\; 2.$$

Finally, form $\mathcal{R}_{ab}$ must be proportional to ${}^{(2)}\epsilon_{ab}$,

$$\mathcal{R}_{ab} \;=\; \mathcal{R}\,{}^{(2)}\epsilon_{ab}.$$

Contracting this equation with ${}^{(2)}\epsilon^{ab}$ and using (6.21) we arrive at

$$\mathcal{R} = 2\,i\,\omega^{CD}\,\overline{o}^{A'}\,\overline{\iota}^{B'}\,\Phi_{CDA'B'} - 2\,i\,o^A\,\iota^B\,\omega^{CD}\,\Psi_{ABCD} - 4\,i\,\Lambda\,o^A\,\iota^B\,\omega_{AB}. \tag{6.24}$$

In order to write this expression down in terms of NP scalars, we use the fact that arbitrary symmetric spinor can be factorized to a symmetrized direct product of uni-valent spinors. Namely, we decompose ω^{CD} as

$$\omega^{CD} \;=\; \alpha^{(C}\,\beta^{D)}.$$

Spinors α^C and β^C in terms of basis spinors o^C and ι^C read

$$\alpha^C = \alpha^0\,o^C \,+\, \alpha^1\,\iota^C, \quad \beta^C \;=\; \beta^0\,o^C \,+\, \beta^1\,\iota^C.$$

Now, performing contractions in (6.24) we find

$$\mathcal{R} = 2\,i\,\alpha^0\,\beta^0\,(\Phi_{01} \,-\, \Psi_1) \,+\, 4\,i\,\alpha^{(0}\,\beta^{1)}\,(\Phi_{11} \,-\, \Psi_2 \,+\, \Lambda) \,+\, 2\,i\,\alpha^1\,\beta^1\,(\Phi_{21} \,-\, \Psi_3).$$

We can conclude that the quasi-local charge is given by relation

$$\begin{aligned} Q_{\mathcal{S}}[\alpha^A,\beta^B] \;&=\; \frac{1}{8\pi G}\oint\limits_{\mathcal{S}} R_{abcd}f^{cd} \\ &=\; \frac{i}{4\pi G}\oint\limits_{\mathcal{S}} \big[\alpha^0\beta^0(\Phi_{01}-\Psi_1) + (\alpha^0\beta^1+\alpha^1\beta^0)(\Phi_{11}-\Psi_2+\Lambda) + \alpha^1\beta^1(\Phi_{21}-\Psi_3)\big]\,\mathrm{d}\mathcal{S}. \end{aligned} \tag{6.25}$$

We conclude this chapter with few general remarks on applicability of Penrose mass as introduced herein. We have seen that if the surface integral $\frac{1}{8\pi G}\oint_{\mathcal{S}} R_{abcd}f^{cd}$ is to correspond to the charge integral $\int_\Sigma \epsilon_{abcd}T^{de}K_e$, the anti-self-dual 2-form $f^{cd} = \omega^{CD}\epsilon^{C'D'}$ must be a solution of the twistor equation $\nabla_{A'}^{(A}\omega^{BC)} = 0$. But the twistor equation has only the trivial solution in general space-time. In order to make the construction viable in general space-times, we thus need to weaken the condition on ω^{CD}. Penrose's suggestion is to consider a solution of a projection of the twistor equation onto the 2-surface $\mathcal{S}$. Specifically, to consider $\omega^{CD} = \alpha^{(C}\beta^{D)}$, where α^A, β^A are solutions of the so-called 2-surface twistor equation $\mathcal{T}_{A'AB}{}^{C}\omega_C = 0$. (The solution of "tangential valence 2 twistor equation" is under-determined.) Penrose mass given by this modified construction will not yield the integral $\int_\Sigma \epsilon_{abcd}T^{de}K_e$ in general space-time, since that integral can not even be generally defined. But this is actually a good news. As we will discuss in the next chapter, a charge integral of the matter energy-momentum T^{ab} is not very useful quantity in general space-time, because we also need to take account of the gravitational energy. The "modified" Penrose mass is one of several promising candidates for a quasi-local (total) energy in the general relativity.

7. Mass in General Relativity

In this chapter we will briefly review the issue of mass in general relativity, mostly following the article [20] by László B. Szabados. It turns out that a notion of energy is very elusive in general relativity. Let us first consider the energy of non-gravitational fields. It should be described by the energy-momentum tensor T_{ab} which satisfies the condition $\nabla^a T_{ab} = 0$ and, as a consequence, the energy defined by it is *locally* conserved. Considering however the energy in a finite domain we generally find that it is *not* conserved, because it is not possible to write the covariant derivative as a coordinate derivative over a finite region in a curved space. An apparent explanation for this gain or loss of the energy is the interaction between the gravitational and non-gravitational fields, which is actually *expected* to result in a transfer of the energy between the two. The line of reasoning we just followed thus brings us to the concept of the energy of the gravitational field.

How can we describe this gravitational energy? Perhaps the most natural attempt at this point is to search for some tensor ${}_{\mathrm{grav}}T_{ab}$ that would represent the gravitational energy-momentum so that the sum ${}_{\mathrm{grav}}T_{ab} + T_{ab}$ — encompassing the total energy and momentum of a system — would be conserved. It turns out, however, that such a tensor cannot exist. The reason for this interesting trait of the gravitation lies in the very heart of the general relativity, in the equivalence principle. The equivalence principle requires that for any given point (i.e. a space-time event) there exist coordinates in which the gravitational field vanishes locally at that point. Components of tensor ${}_{\mathrm{grav}}T_{ab}$ with respect to those coordinates are therefore also zero there[1]. But because a tensor that vanishes in one coordinates is also zero in any other coordinates, we have ${}_{\mathrm{grav}}T_{ab} = 0$ at the point we just considered. Since the same analysis can be carried out at any other point, the tensor ${}_{\mathrm{grav}}T_{ab}$ is identically zero. Nevertheless, it is possible to define quantities t_{ab} which satisfy[2] the requirement $\partial^a (T_{ab} + t_{ab}) = 0$, but such quantities must be intrinsically coordinate-dependent. Quantities of this type are called *pseudotensors*. While pseudotensors are an interesting subject, they may as well be regarded as a symptom of a problem rather than a solution. Their basic disadvantage — when interpreted as strictly local quantities — is a lack of geometric meaning unless some background structure is specified[3]. We shall not focus on pseudotensors in this chapter and will turn our attention to different approaches to the notion of gravitational energy.

As the preceding discussion suggests, the equivalence principle sets up significant obstacles to any local geometric description of the gravitational energy. Fortunately, even non-local objects may possess a clear geometric meaning. Consider a covariant derivative. Christoffel symbols, which are used in coordinate description of covariant derivative, are functions of the first coordinate derivatives of the metric and we can make them locally vanish at any single point due to the equivalence principle. Christoffel symbols therefore do not have a local geometric interpretation. Nevertheless, if we consider them over a finite *region* of space-time, they obtain geometric meaning as a covariant derivative. This suggests that a non-local approach to the gravitational energy may be a viable alternative. There are basically two kinds of non-local quantities, namely global quantities which are taken over the whole of a manifold and quasi-local quantities which are taken over a finite region, possibly limiting such a region to a single point. But before we

[1] By vanishing of a gravitational field we mean the vanishing of the gravitational force, i.e. a quantity which depends on the first derivatives of the metric. We presume ${}_{\mathrm{grav}}T_{ab}$ to be a function of this gravitational force.

[2] In fact, complex $T_{ab} + t_{ab}$ usually needs to be multiplied by a suitable power of the metric determinant to be (coordinate-)divergence-free.

[3] Lack of a background structure is a recurrent theme in gravitational physics. Most of the physics we are used to is formulated on some fixed background, usually on a background of the Minkowski space-time. General relativity is different, since it describes dynamics of the background geometry itself. This lack of primordial geometric structure is again a consequence of the equivalence principle and the resulting universality of the gravitation.

inspect these two possibilities more closely, let us take a detour and give a few remarks on a concept of energy.

What we have said so far might suggest that the object we are searching for is some kind of a gravitational counterpart to the energy-momentum tensor T_{ab}. This seems however not to be the case. The canonical way to obtain the symmetric energy-momentum tensor T_{ab} in general relativity is through variational derivative of the action I_{m} for the matter fields with respect to the background metric g^{ab}. But the gravitational action I_{g} is a functional of the metric alone, and taking its variational derivative with respect to g^{ab} yields equations for gravitational field. There is no apparent background structure with respect to which we could vary the action I_{g} and obtain a gravitational analogue to the energy-momentum tensor of the matter fields. Moreover, the energy-momentum tensor should be — because of its definition $T_{ab} \sim \delta I_{\mathrm{m}}/\delta g^{ab}$ — regarded foremost as the source field for the gravitation and not as the energy-momentum.

By contrast, canonical energy-momentum originates as *Noether current* in a flat space-time, ensuing from a symmetry of the physical system with respect to space-time translations. Thus it seems appropriate to look for a suitable analogue to this Noether current in a curved space. A translation of a flat space-time is a special case of continuous symmetry. When searching for the energy of a gravitating system, it is therefore natural to consider continuous symmetries of the space-time. Such symmetries are generated by Killing vector fields and we should perhaps expect Killing fields associated with the energy to be time-like. And indeed, for stationary[4] asymptotically flat space-times a fully satisfactory construction for the total mass is known and it is closely connected with time-like Killing fields which those space-times possess. The construction was first given by Arthur Komar and is quite analoguous to the Newtonian case, where the mass enclosed in a 2-surface is proportional to the flux of the gravitational force through the 2-surface[5]. In Komar's construction we consider stationary observers who fly along the time-like Killing field. They do not follow geodesics, since they need to accelerate against the gravitational force to stay at "the same place". If K^a is the time-like Killing field, then the acceleration is basically $K^b \nabla_b K^a$, apart from some factor arising from the normalization of K^a. Now similarly to the Newtonian case, we may integrate this acceleration over a closed space-like 2-surface (lying in a hypersurface orthogonal to K^a) and obtain a quantity which may be interpreted as a mass. Following the procedure we have just sketched, one would arrive at the expression

$$M = -\frac{1}{8\pi} \oint_S \left(\nabla^a K^b\right) \varepsilon_{abcd} \tag{7.1}$$

for the mass enclosed by the 2-surface S. An important feature of the above expression is that as long as all the matter is inside the 2-surface S, the quantity M does not depend on the exact choice of S. This is the reason why the construction yields a well defined mass for asymptotically flat space-times[6].

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[4] By definition, the existence of a time-like Killing field is equivalent to stationarity.

[5] In Newtonian gravity the Laplacian of the gravitational potential ϕ is proportional to the matter density ρ:

$$\nabla^2 \phi = 4\pi\rho\,.$$

Consider a compact region Σ of a 3-space with a boundary $S = \partial\Sigma$. By the Gauss law the following holds true for the mass M contained in the region Σ:

$$M = \frac{1}{4\pi} \int_\Sigma \left(\nabla^2 \phi\right) \varepsilon_{abc} = \frac{1}{4\pi} \oint_S \left(\nabla^a \phi\right) \varepsilon_{abc}\,.$$

The mass is therefore proportional to the flux of the gradient $\vec{\nabla}\phi$ over the surface S.

[6] Even if space-time is not vacuous near the infinity, a limit of the quantity (7.1) when the 2-surface S is

While Komar's construction is very appealing, it has a great shortcoming — namely the very limited applicability. A general space-time simply does not possess any Killing fields and we therefore need to look for some more general construction for the gravitational energy.

7.1 Global energy and momentum

In previous paragraphs we touched upon serious difficulties that stand in a way of obtaining a reasonable energy-momentum for gravitating systems. There is, however, an important class of space-times for which we are able to define a well-behaved *global* energy and momentum. If a space-time is asymptotically flat, we know how to obtain a reasonable energy-momentum of the whole space-time, even if it does not have any Killing field. But before we discuss these global energy-momenta, we ought to specify what we actually mean by an asymptotically flat space-time.

7.1.1 Asymptotic simplicity

One expects that for an asymptotically flat space-time there exists some notion of infinity such that when we approach that infinity the geometry is gradually becoming more and more flat-like. This unfortunately does not make for a very usable definition. To obtain the results advertised above a stricter definition must be adopted. Here we will introduce the concept of asymptotic simplicity which was first proposed by Penrose. It relies on a conformal transformation that allows us to compactify the original space-time and thus directly access the infinity and its neighborhood. First, we elucidate this procedure in Minkowski space-time.

Minkowski metric in standard coordinates reads

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \,. \tag{7.2}$$

We will introduce an appropriate conformal transformation

$$ds^2 \mapsto d\hat{s}^2 = \Omega^2 ds^2 \tag{7.3}$$

of the physical metric, so that the "infinity" will be finitely close with respect to the transformed metric $\hat{g}_{ab}$. Let us first suitably rewrite the Minkowski metric. Our construction will respect the spherical symmetry, so we start by rewriting the metric using spherical coordinates:

$$ds^2 = dt^2 - dr^2 - r^2 d\Sigma^2 \,, \tag{7.4}$$

where

$$d\Sigma^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \,.$$

Next we employ the null coordinates u, v,

$$u = t - r \,, \quad v = t + r \,, \tag{7.5}$$

in a place of the pair t, r. The metric thus acquires the form

$$ds^2 = du\,dv - \frac{1}{4}\,(v-u)^2\, d\Sigma \,. \tag{7.6}$$

continuously stretched to the infinity is well defined for asymptotically flat space-times.

Now we may apply the conformal factor

$$\Omega^2 = \frac{4}{(1+u^2)(1+v^2)}\,, \tag{7.7}$$

obtaining

$$d\hat{s}^2 = \frac{4}{(1+u^2)(1+v^2)}\,du\,dv - \frac{(v-u)^2}{(1+u^2)(1+v^2)}\,d\Sigma^2 \tag{7.8}$$

for the rescaled metric $\hat{g}_{ab}$. To make the significance of this result more apparent, we employ new coordinates

$$p = \arctan u\,, \quad q = \arctan v\,, \quad -\frac{\pi}{2} < p \le q < \frac{\pi}{2}\,, \tag{7.9}$$

in which $\hat{g}_{ab}$ has the form

$$d\hat{s}^2 = 4\,dp\,dq - \sin^2(p-q)\;d\Sigma^2\,. \tag{7.10}$$

We see that $\hat{g}_{ab}$ in these coordinates looks nearly same as the physical metric in the form (7.6), the only significant difference being that there is the factor $\sin^2(p-q)$ in front of the spherical element in metric (7.10), instead of the factor $(v-u)^2$ in front of the element $d\Sigma^2$ in the physical metric (7.6). Switching to the coordinates

$$\begin{gathered} T = q + p\,, \quad R = q - p\,, \\ -\pi < T < \pi\,, \quad 0 < R < \pi\,, \quad |T| < \pi - R\,, \end{gathered} \tag{7.11}$$

we arrive at

$$d\hat{s}^2 = dT^2 - dR^2 - \sin^2 R\,d\Sigma^2\,. \tag{7.12}$$

Now we can see clearly that the correspondence $\sin^2(p-q) \leftrightarrow (v-u)^2$ simply means that while a space-like hypersurface $t = \text{const}$ is flat with respect to the physical metric g_{ab}, space-like hypersurface $T = \text{const}$ with respect to the rescaled metric $\hat{g}_{ab}$ is a 3-space of positive (elliptic) constant curvature. Actually, the space-time we obtained is locally identical to the Einstein static universe, which is topologically a product of the real line with the 3-sphere and can be described by the metric (7.12) with $-\infty < T < \infty$, $0 < R < \pi$. Thus we see that the Minkowski space-time can be conformally transformed to a region of the Einstein static universe (see figure 7.1). This provides us with a neat way of representing the infinity of the Minkowski spacetime. Minkowski space-time is conformal to an open region of the Einstein static universe and the boundary of that region may be interpreted as a representation of the infinity.

Let us now describe the structure of this infinity. Conformally transformed Minkowski space-time is sketched in a picture 7.2. It should be interpreted, especially when considering the infinity, as a region of the Einstein static universe (fig. 7.1). Now consider a future-pointing time-like radial geodesics. Both of the coordinates t and r increase to infinity along them, but since they are time-like, we also have $t - r \to \infty$. As a consequence, both u and v grow to infinity, and therefore $p \to \pi/2$, $q \to \pi/2$, or, if coordinates T, R are used, $T \to \pi$, $R \to 0$. This means that all future-pointing time-like geodesics approach the point i^+ of the compactified Minkowski space-time (7.2). For this reason we call it the *future time-like infinity* and, considering its location in the Einstein static universe, it is topologically a point. Similarly, past-pointing radial time-like geodesics approach the point i^-, which is called *past time-like infinity*. It is easy to

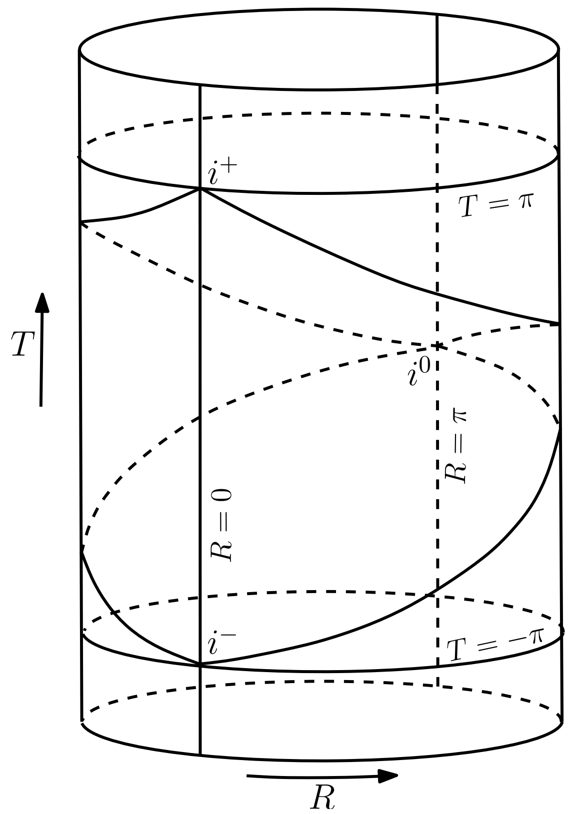

Figure 7.1: Conformally transformed Minkowski space-time as a region of the Einstein static universe. (The angular coordinates have been suppressed.)

see that non-radial time-like geodesics also approach either i^+ or i^-. In the case of space-like radial geodesics, we have $t+r \to \infty$ and $t-r \to -\infty$, which means that $u \to -\infty$ while $v \to \infty$. Thus we have $p \to -\pi/2$, $q \to \pi/2$, or in other words $T \to 0$ and $R \to \pi$. Non-radial space-like geodesics behave asymptotically same and thus we see that all space-like geodesics tend to i^0, or the *space-like infinity*, which is again topologically a point. For null geodesics we have $t+r \to \infty$ and $t-r \to \text{const}$ as $t \to \infty$, and $t+r \to \text{const}$ and $t-r \to -\infty$ as $t \to -\infty$. In other words, they approach points $q = \pi/2$, $p = c$ with $-\pi/2 < c < \pi/2$, in the future, and come "from" points $p = -\pi/2$, $q = c'$, where $-\pi/2 < c' < \pi/2$, of the past. All those points lie either in the null hypersurface $\mathcal{I}^+$ ($\mathcal{I}$ is pronounced as "scri") called the *future null infinity*, or in the null hypersurface $\mathcal{I}^-$ called the *past null infinity*. Both $\mathcal{I}^+$ and $\mathcal{I}^-$ can be shown to be topologically a product of the line and the 2-sphere.

Now that we have introduced a tangible representation of infinity of the flat space-time, we are prepared to introduce the definition of asymptotic simplicity[7]. Definitions given here are taken from [17] (where they were adopted from [6]).

A space-time (M, g) is said to be *asymptotically simple* if there exists another manifold $(\hat{M}, \hat{g})$ such that:

1. M is an open submanifold of $\hat{M}$ with a smooth boundary ∂M,

[7]We do not define an asymptotical flatness, because it has been defined in many incompatible ways in the literature.

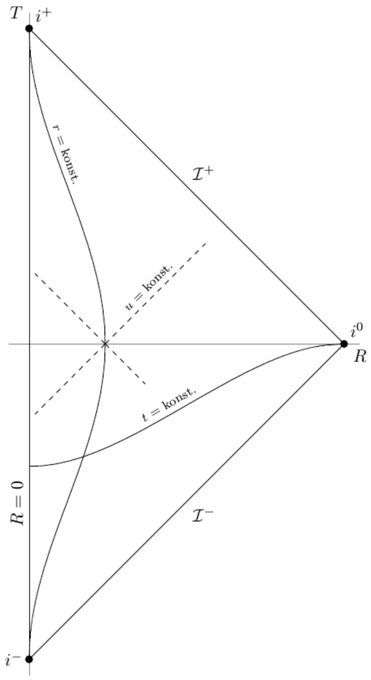

Figure 7.2: Conformal diagram of Minkowski space-time. (Adopted from [15].)

2. there exists a real function Ω on $\hat{M}$ such that $\hat{g}_{ab} = \Omega^2 g_{ab}$ on M and $\Omega = 0$, $\nabla_a \Omega \neq 0$ on ∂M,

3. every null geodesic has two endpoints on ∂M.

The manifold $(\hat{M}, \hat{g})$ is called the *unphysical space-time.*

The restrictions given above are rather strong. For example the Schwarzschild space-time is not asymptotically simple, for there are trapped null geodesics that will never reach the infinity. This indicates that we should probably weaken our conditions. We say that a space-time (M, g) is *weakly asymtotically simple* if there is another space-time (M', g') which *is* asymptotically simple and which is isometric to (M, g) near the infinity. Specifically, there must exist a neighborhood U' of $\partial M'$ in M' such that $M \cap U'$ is isometric to an open subspace of M'.

7.1.2 ADM and Bondi energy-momenta

We are able to associate satisfactory energy-momenta with (weakly) asymptotically simple space-times, even if they do not possess any exact Killing fields. Their asymptotic simplicity does however allow us to define *asymptotic* translations (vector fields that are translations at the infinity), so that we are able to talk about energy and momentum in a manner analogous to the Noether currents. In ADM (Arnowitt, Deser, Misner) construction we measure the energy (and momentum) over a space-like hypersurface. Original construction relies on Hamiltonian formalism and energy-momentum arises as a value of the Hamiltonian associated with a suitable asymptotic translation. The integral over the region of the space-like hypersurface is expressed by an integral over its two-dimensional boundary, which is then stretched to the infinity. In present, we know several different approaches to the computation of the ADM energy-momentum. Let us mention approaches based on background tetrad fields. If the tetrad field is constructed from a spin-frame $\epsilon_A{}^{\mathbf{A}} = \lambda_A^{\mathbf{A}}$, we obtain the energy-momentum in the form of the 2-surface integral of the *Nester-Witten 2-form*:

$$P^{\mathbf{AB}'} = \frac{1}{4\pi} \oint_S \frac{i}{2} \left(\overline{\lambda}^{\mathbf{B}'}_{A'} \nabla_{BB'} \lambda^{\mathbf{A}}_A - \overline{\lambda}^{\mathbf{B}'}_{B'} \nabla_{AA'} \lambda^{\mathbf{A}}_B \right) . \tag{7.13}$$

The Nester-Witten 2-form

$$u(\alpha, \overline{\beta})_{ab} = \frac{i}{2} \left(\overline{\beta}_{A'} \nabla_{BB'} \alpha_A - \overline{\beta}_{B'} \nabla_{AA'} \alpha_B \right) \tag{7.14}$$

will be encountered repeatedly in the remainder of this chapter. Many other results for the global or quasilocal energy-momentum can be written in its terms and it has some pleasant properties. For example, it is "essentially Hermitian", i.e. it differs from Hermitian only by an exact form:

$$u(\alpha, \overline{\beta})_{ab} - \overline{u(\beta, \overline{\alpha})}_{ab} = -i \nabla_{[a} X_{b]} , \tag{7.15}$$

where

$$X_a = \alpha_A \overline{\beta}_{A'} . \tag{7.16}$$

This property is useful (e.g.) when proving the positivity of the energy.

The basic difference between the ADM and the Bondi energy-momentum is that the Bondi energy-momentum is measured over a hypersurface that is asymptotically null. This distinction has notable consequences. Consider an isolated system which at some time t_0 sent a wave of radiation towards the infinity (see fig. 7.3). We would expect the energy of the isolated system to decrease at the time t_0, since the radiation is carrying away a portion of its energy. But the radiation intersects every space-like hypersurface that is in the future of the radiative event t_0. This means that ADM mass does not decrease in time for radiating systems. On the other hand, if we measure the energy over a family of asymptotically null hypersurfaces, there are hypersurfaces not intersected by the radiation. The Bondi energy of a radiative system is therefore decreasing.

Now again, there are several methods of calculating the Bondi energy-momentum and we shall not discuss them more closely in this chapter. However, we should point out that such calculations generally result in an integration over a space-like 2-surface lying in the null infinity. (As opposed to an integration over a space-like 2-surface that is being stretched towards the space-like infinity in the case of ADM energy-momentum.) The structure of null infinity gives rises to the so-called BMS (Bondi, Metzner, Sachs) group, which preserves the asymptotic form of the metric. It is a semidirect product of the Lorentz group and the infinite-dimensional commutative group of *supertranslations*. Supertranslation are connected with particular reparametrizations of curves along the null generators of the null infinity and it has a group of translations as its four-parameter subgroup.

7.2 Quasi-Local Energy and Momentum

While it is nice that we are able to define an energy-momenta for a whole space-time (if it is weakly asymptotically simple), there is no apparent reason why it should be not possible to speak of energy-momentum on a quasi-local level, moreover, the energy-momentum as a quasi-local quantity seems to be both more in accordance with our intuition and more theoretically interesting than a global energy-momentum.

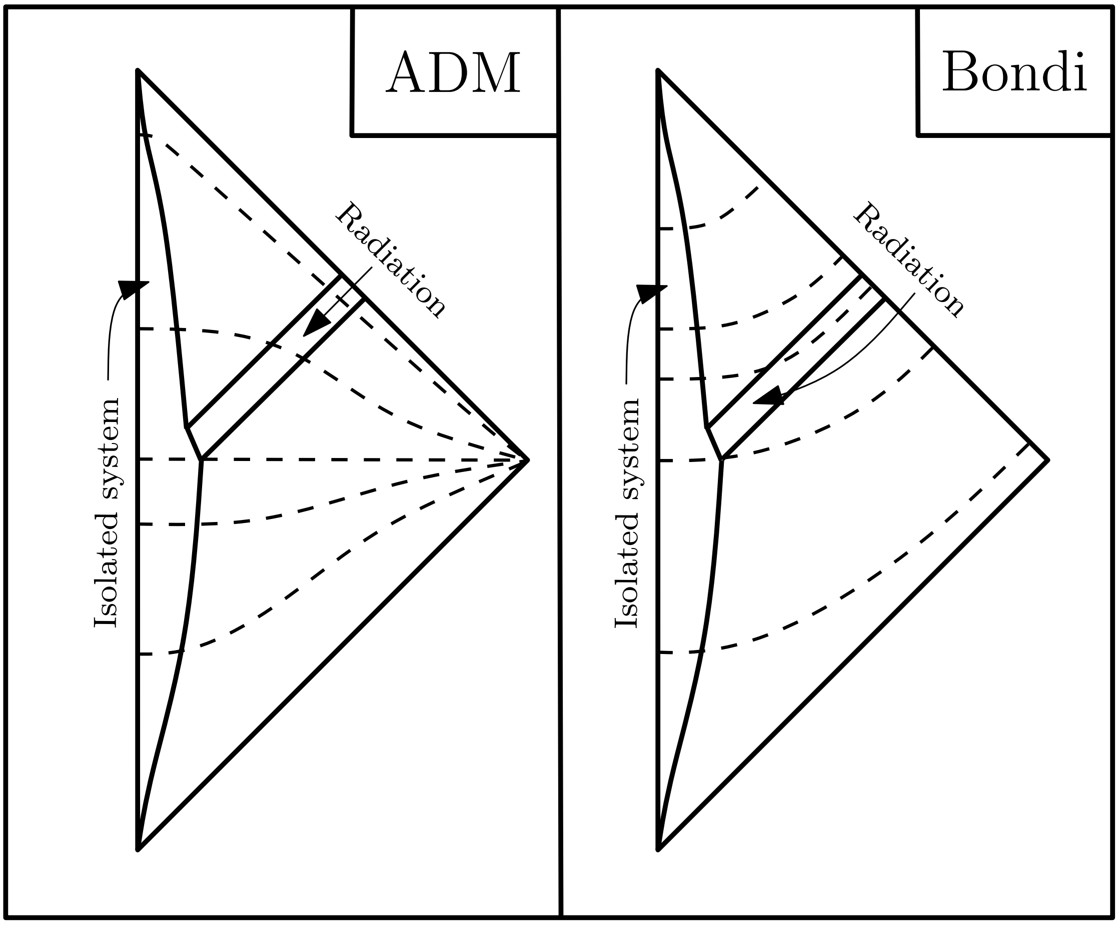

Figure 7.3: Conformal diagram of an idealised radiative system. An isolated system is located at the "center" of the space-time. It is inactive except for a finite interval when it radiates energy towards the null infinity. A family of space-like hypersurfaces is illustrated by dashed lines in the left picture, while the dashed lines in the picture on the right represent hypersurfaces that are asymptotically null. Energy measured over space-like hypersurfaces is not expected to change in time, since the radiation intersects each such hypersurface. On the other hand, if we measure the energy over asymptotically null hypersurfaces, the energy should decrease with time, because "the later" hypersurfaces are not intersected by the outgoing radiation.

All the energies-momenta that we have mentioned so far could be rewritten as an integral over a closed (orientable) space-like 2-surface. This seems natural for charge quantities and we will look for quasi-local energy-momentum in such a form.

When constructing a quasi-local quantity we would typically start with searching for some suitable Lagrangian or Hamiltonian formulation. Both approaches do however require us to choose some gauge reduction. Langrangian formulation with Noether-like analysis yields only the *divergence* of conserved current, while the Hamiltonian action may be altered by various boundary terms. Lastly, we need to specify what the transformations which generate the conserved current are exactly. For example, in the case of energy-momentum, we need to define what is meant by space-time translations — something not obvious in a curved space.

Here we shall not discuss the procedure outlined above any more deeply. We shall rather adopt a more heuristical description of the problem. Our focus here will be on two specific constructions based on the Nester-Witten 2-form.

Consider the following integral over a closed orientable space-like 2-surface S:

$$H_S[\lambda,\overline{\mu}] := \frac{1}{4\pi}\oint_S u(\lambda,\overline{\mu})_{ab} = \frac{1}{4\pi}\oint_S \overline{\gamma}^{A'B'}\overline{\mu}_{A'}\Delta_{B'B}\lambda^B dS\,, \tag{7.17}$$

with λ^A, $\overline{\mu}^{A'}$ an arbitrary spinor fields. Operator $\Delta_{AA'}$ stands for the Sen derivative and $\gamma^A{}_B$ is a spinor related to the geometry of the 2-surface analogous to the Dirac spinor γ_5. Both quantities are defined in Chapter 5. The quantity $H_S[\lambda,\overline{\mu}]$ has certain favorable features. First, observe from the rightmost expression of equation (7.17) that $H_S[\lambda,\overline{\mu}]$ depends only on the values of the fields λ^A, $\overline{\mu}^{A'}$ *on* the 2-surface S, because the Sen derivative is tangential to the surface. Next, we have $H_S[\lambda,\overline{\mu}] = \overline{H_S[\mu,\overline{\lambda}]}$ as a consequence of the equation (7.15) and the integration being taken over a *closed* 2-surface. Thus we see that equation (7.17) defines a Hermitian scalar product on the vector space of smooth spinor fields on the 2-surface S.

Consider now a pair $\lambda^{\mathbf{A}}_A = (\lambda^0_A, \lambda^1_A)$ of smooth spinor fields on S. If $\Lambda_{\mathbf{A}}{}^{\mathbf{B}}$ is a constant $SL(2,\mathbb{C})$ matrix, then $H_S[\Lambda_{\mathbf{A}}{}^{\mathbf{C}}\lambda^{\mathbf{A}}, \overline{\Lambda}_{\mathbf{B}'}{}^{\mathbf{D}'}\overline{\lambda}^{\mathbf{B}'}] = \Lambda_{\mathbf{A}}{}^{\mathbf{C}}\overline{\Lambda}_{\mathbf{B}'}{}^{\mathbf{D}'}H_S[\lambda^{\mathbf{A}},\overline{\lambda}^{\mathbf{B}'}]$, i.e. numbers $H_S[\lambda^{\mathbf{A}},\overline{\lambda}^{\mathbf{B}'}]$ transform as spinorial components of a Lorentz four-vector under a spin-transformation $\Lambda_{\mathbf{A}}{}^{\mathbf{B}}$. Moreover, because of the hermicity of the scalar product $H_S[\circ,\circ]$, the four-vector represented by numbers $H_S[\lambda^{\mathbf{A}},\overline{\lambda}^{\mathbf{B}'}]$ is *real*.

Thus we see that after choosing a two-dimensional subspace of the infinite-dimensional vector space of smooth spinor fields on S, i.e. specifying two fields $\lambda^{\mathbf{A}}_A$, and choosing a symplectic structure $\epsilon_{\mathbf{AB}}$ thereon, we obtain a promising candidate for an energy-momentum in terms of the integral (7.17):

$$\begin{aligned} P_S^{\mathbf{AB}'} &= H_S[\lambda^{\mathbf{A}},\overline{\lambda}^{\mathbf{B}'}]\,, \\ m_S^2 &= \epsilon_{\mathbf{AB}}\,\epsilon_{\mathbf{A}'\mathbf{B}'}\,P_S^{\mathbf{AA}'}P_S^{\mathbf{BB}'}\,. \end{aligned} \tag{7.18}$$

Various proposals for quasi-local energy-momentum based on Nester-Witten 2-form differ by the choice of metric $\epsilon_{\mathbf{AB}}$ and spinor fields $\lambda^{\mathbf{A}}_A$. In the remainder of this chapter we shall present two such constructions, the Ludvigsen-Vickers construction based on propagating the fields $\lambda^{\mathbf{A}}_A$ from null infinity, and the Dougan-Mason construction where $\lambda^{\mathbf{A}}_A$ are independent fields satisfying a particular (anti-)holomorphicity condition.

7.2.1 Ludvigsen-Vickers construction

Ludvigsen-Vickers construction [8] is related to the notion of Bondi mass. One of the accomplishments of the original paper was actually the proof of positivity of the Bondi mass under certain circumstances and the unmodified construction strongly relies on the structure of the null infinity (and thus is applicable only in weakly asymptotically simple space-times).

Consider a closed space-like 2-surface in a (weakly) asymptotically simple space-time and suppose that the 2-surface can be connected to the future null infinity by a smooth null hypersurface $\mathcal{N}$. The intersection of $\mathcal{N}$ with $\mathcal{I}^+$ defines another 2-surface S_∞ and $\mathcal{N}$ provides us with a smooth bijection between the cut S_∞ of the $\mathcal{I}^+$ and the 2-surface S. We will utilize this bijection to transport a particular structure of the null infinity onto S.

Let us introduce a coordinate r as an affine parameter along the null generators of $\mathcal{N}$ and a null four-vector l^a tangential to those generators. Lastly we define n^a as a null vector field on $\mathcal{N}$ orthogonal to space-like cuts of $\mathcal{N}$ of constant r and normalized so that $l^a n_a = 1$. Thus we may write $l^a = o^A \bar{o}^{A'}$, $n^a = \iota^A \bar{\iota}^{A'}$ for some spin-frame o^A, ι^A on $\mathcal{N}$.

A general radiative space-time does not admit asymptotically constant spinors, but there is a weaker condition — the asymptotic twistor equation — that can be solved in such general (weakly) asymptotically simple space-times, and which implies asymptotic constancy if the space-time is non-radiative. Moreover, the space of solutions to the asymptotic twistor equation is 2-complex-dimensional[8], i.e. it has a structure of a spin space — exactly the structure that is needed for the definition of energy-momentum based on Nester-Witten 2-form. We shall denote it by $S_\infty^{\mathbf{A}}$.

Thus we just need to associate a spinor field on S with a solution of asymptotic twistor equation. An asymptotic form of a spinor does however not determine that spinor uniquely in the whole space-time. Therefore, an additional condition needs to be imposed that will allow us to associate a *unique* spinor field on S with a solution of the asymptotic twistor equation on S_∞. In other words, we need to choose an equation of propagation, so that we can transport a spinor along the null generators of $\mathcal{N}$. Unfortunately, it is not obvious what propagation equation should one choose and results depend strongly on this, in a sense arbitrary, choice.

The equation of propagation chosen by Ludvigsen and Vickers reads

$$o^A o^{A'} \nabla_{BA'} \lambda_A = 0 \,. \tag{7.19}$$

They considered it to be a natural choice, since it possesses the following desirable properties:

- The resulting energy-momentum enclosed in a 2-surface S is non-decreasing when S is stretched over a larger region. More specifically, the mass-gain formula $P_a(S_{r_1})k^a \le P_a(S_{r_2})k^a$ holds for $r_1 \le r_2$ and a future-pointing time-like k^a, if dominant energy condition is satisfied on $\mathcal{N}$.
- Equation (7.19) reduces to parallel transport if space-time is flat.
- In the case of linearized gravity, the resulting $P_a(S)$ reduces to the "correct" energy-momentum enclosed in the 2-surface S.

In GHP formalism, the equation (7.19) acquires the form

$$\text{þ}\lambda_0 = 0 \tag{7.20}$$

$$\eth' \lambda_0 + \rho \lambda_1 = 0 \,. \tag{7.21}$$

We use the propagation equation specified above to transport two independent solutions $\lambda_A^{\mathbf{A}}$ of asymptotic twistor equation from the cut S_∞ onto the 2-surface S. This establishes a 2-complex-dimensional vector space[9] (i.e. a spin space) of spinor fields on S which we shall denote

[8] Which means that if spinors of a spin-frame $\epsilon_A{}^{\mathbf{A}}$ are solutions of the asymptotic twistor equation, then components of any other such solution taken with respect to that spin-frame are constant on the 2-surface S_∞.

[9] Recall that a space of solutions of asymptotic twistor equation is 2-complex-dimensional.

$S^{\mathbf{A}}$. We define a metric $\epsilon_{\mathbf{AB}}$ on space $S^{\mathbf{A}}$ to be a 2-form with respect to which do the fields $\lambda_A^{\mathbf{A}}$ form a spin-frame if they are normalized at S_∞. Thus we have all the components needed to define the energy-momentum as in equations (7.18).

The quasi-local energy-momentum of Ludvigsen and Vickers has two important drawbacks. First, it is not genuinely quasi-local. It depends on global properties of space-time, because the null infinity is needed in construction. The other problem is existence of caustics in general curved space-times, which may prevent us from joining the 2-surface S to $\mathcal{I}^+$ by a smooth null hypersurface $\mathcal{N}$. Thus for the construction to be viable, either the 2-surface S must be "close enough" to the null infinity, or the space-time cannot be "too curved". For small spheres, where the original construction is usually not possible because of the caustics, a modified procedure may be employed. A small sphere can be described as a space-like cut of a null cone emanating from some point p. We then may propagate the spin structure that exists at the vertex p along the null cone onto the 2-surface S.

7.2.2 Dougan-Mason construction

In construction of Dougan and Mason the desired spin space of spinor fields over S is obtained as a space of solutions for specific anti-holomorphicity or holomorphicity condition. Consider a null tetrad adapted to the 2-surface S, i.e. vectors m^a, $\overline{m}^a$ are tangential to the surface S. Then we will say that a spinor field λ_A on S is holomorphic if $\overline{m}^b \nabla_b \lambda_A = 0$ and anti-holomorphic if $m^b \nabla_b \lambda_A = 0$. Now suppose there are two holomorphic fields λ_A, μ_A on S and consider their inner product $\epsilon^{AB}\lambda_A\mu_B$. By holomorphicity of λ_A and μ_A we have $\overline{m}^c \nabla_c \left(\epsilon^{AB}\lambda_A\mu_B\right) = 0$ and thus, by Liouville's theorem, the quantity $\epsilon^{AB}\lambda_A\mu_B$ is constant on S. A closed space-like 2-surface which allows for two holomorphic fields λ_A and μ_A such that their inner product $\epsilon^{AB}\lambda_A\mu_B$ is non-zero will be called *generic*. Space of holomorphic spinor fields on a generic 2-surface S is a 2-complex-dimensional vector space. This follows from constancy of inner product of two holomorphic spinors on S, for components of holomorphic spinor field $\alpha_A$with respect to holomorphic dyad λ_A, μ_A are $\epsilon^{AB}\alpha_A\mu_B/\lambda_C\mu^C$ and $\epsilon^{AB}\alpha_A\lambda_B/\lambda^C\mu_C$ and therefore constants on S. We shall say that a surface which is not generic is *exceptional*. Similar remarks hold for anti-holomorphic spinors, i.e. their inner product is also constant, we can define generic and exceptional surfaces, and anti-holomorphic spinors on such generic surfaces form a spin space.

We have seen that if S is generic, Dougan-Mason construction defines a 2-complex-dimensional subspace of the space of spinor fields on S, and thus provides a background to employ the Nester-Witten 2-form and obtain a quasi-local energy-momentum. But are generic 2-surfaces common? Unfortunately, we do not know the answer for a general closed 2-surface. Nevertheless, a 2-surface with topology of 2-sphere is either generic or can be made generic by a small perturbation.

The Dougan-Mason construction has nice positivity properties. It can also be shown, that under certain circumstances (under the condition of the positivity proof), vanishing of Dougan-Mason energy-momentum is equivalent to the flatness of the space Σ enclosed by S, while vanishing of the Dougan-Mason mass means that $D(\Sigma)$ has a *pp*-wave geometry and the matter (inside) may be only radiation.

Part II

Calculations and results

8. Einstein-electro-scalar equations

Our ultimate goal is to calculate the Bondi mass of the spacetime where interacting charged scalar fields and electromagnetic fields are present. For this purpose we employ twistorial methods reviewed in the first part of the thesis.

We have seen that the Penrose mass given by (6.25) in terms of the Newman-Penrose quantities, namely by the components of the Ricci spinor, Weyl spinor and scalar curvature. Thus, in order to evaluate the mass we have to find expressions for these NP quantities first. The Bondi mass is the limit of the Penrose mass and corresponds to the case when the surface $\mathcal{S}$ in (6.25) is taken to be the two-sphere of $\mathcal{I}$, i.e. sphere at infinity. For the evaluation of the Bondi mass we need asymptotic solution of Einstein's equations with the energy-momentum tensor representing interacting scalar and electromagnetic fields.

Coupled scalar and electromagnetic fields will be referred to as the *electro-scalar fields* and corresponding set of Einstein's equations together with equations for electro-scalar fields will be called *Einstein-electro-scalar equations*. In this chapter we first derive the Lagrangian of electro-scalar fields in the flat spacetime, derive the equations of motion and generalize them to curved spacetime. This is a standard topic [13, 6] and we present it only briefly just for completeness and in order to fix the conventions.

Next we translate all equations to the spinor formalism and rewrite them as a first-order system of equations. Finally we project these equations onto the Newman-Penrose null tetrad and rewrite them using the Newman-Penrose formalism. This treatment is believed to be new. In the next chapter we present asymptotic solution of Einstein-electro-scalar equations.

8.1 Lagrangian of electro-scalar field

In this section we derive the Lagrangian of electro-scalar fields in the flat spacetime by demanding the local gauge invariance of non-interacting complex scalar field. We follow the conventions of [13]. In the signature $(1, -1, -1, -1)$ used in the thesis, the Lagrangian of complex scalar field reads

$$\mathcal{L}_0 = (\partial^\mu \phi)(\partial_\mu \overline{\phi}) \,-\, m^2\, \phi\, \overline{\phi}. \tag{8.1}$$

This Lagrangian is invariant under global gauge transformation

$$\phi \mapsto e^{-i\theta}\phi, \qquad \overline{\phi} \mapsto e^{i\theta}\phi, \qquad \theta \in \mathbb{R},$$

but is not invariant under *local* gauge transformation when θ is an arbitrary real function (rather than constant). In order to preserve local gauge invariance of $\mathcal{L}_0$ we have to add several terms to this Lagrangian.

Let us investigate how the Lagrangian $\mathcal{L}_0$ transforms under local gauge transformation. We restrict ourselves to infinitesimal gauge transformation so that

$$\begin{aligned} \delta\phi &= -i\theta\phi, & \delta\partial_\mu\phi &= -i\phi\partial_\mu\theta - i\,\theta\partial_\mu\phi, \\ \delta\overline{\phi} &= i\theta\overline{\phi}, & \delta\partial_\mu\overline{\phi} &= i\overline{\phi}\partial_\mu\theta + i\,\theta\partial_\mu\overline{\phi}. \end{aligned} \tag{8.2}$$

Then the Lagrangian transforms according to

$$\delta\mathcal{L}_0 \;=\; j^\mu\,\partial_\mu\theta$$

where

$$j^\mu = i\left(\overline{\phi}\partial^\mu\phi - \phi\partial^\mu\overline{\phi}\right) \tag{8.3}$$

can be interpreted as the four-current density of free scalar field. This interpretation (on classical level) comes from the fact that j^μ is conserved in the sense $\partial_\mu j^\mu = 0$ (which follows from the Klein-Gordon equation for free scalar field) and hence

$$\rho = i\,(\overline{\phi}\dot{\phi} - \phi\dot{\overline{\phi}})$$

can be interpreted as the charge density. Obviously, for any real scalar field j^μ vanishes and ϕ is uncharged. Under the gauge transformation, the four-current j^μ transforms according to

$$\delta j^\mu = 2\phi\overline{\phi}\,\partial^\mu\theta. \tag{8.4}$$

Now we wish to add terms to the Lagrangian $\mathcal{L}_0$ which will cancel terms in $\delta\mathcal{L}_0$. Let us introduce vector field A_μ and construct function

$$\mathcal{L}_1 = -\,e\,A^\mu j_\mu \tag{8.5}$$

so that

$$\delta\,(\mathcal{L}_0 + \mathcal{L}_1) = j^\mu\partial_\mu\theta - e\,j^\mu\,\delta A_\mu - 2\,e\,\phi\,\overline{\phi}\,A^\mu\partial_\mu\theta. \tag{8.6}$$

Clearly, first two terms will cancel each other if we postulate

$$\delta A_\mu = \frac{1}{e}\,\partial_\mu\theta$$

which is a usual gauge transformation of the four-potential of electromagnetic field. With this transformation law we have

$$\delta\,(\mathcal{L}_0 + \mathcal{L}_1) = -\,2\,e\,\phi\,\overline{\phi}\,A^\mu\,\partial_\mu\theta.$$

This term can be canceled easily by adding the third term

$$\mathcal{L}_2 = e^2\,\phi\,\overline{\phi}\,A^\mu\,A_\mu$$

to the Lagrangian. Hence, entire Lagrangian reads

$$\mathcal{L}_\mathrm{S} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 = (\partial^\mu\phi)(\partial_\mu\overline{\phi}) - m^2\,\phi\,\overline{\phi} - e\,A^\mu\,j_\mu + e^2\,\phi\,\overline{\phi}\,A^\mu A_\mu \tag{8.7}$$

and its variation under the gauge transformation is zero:

$$\delta\mathcal{L} = 0.$$

To summarize, we required local gauge invariance of original Lagrangian $\mathcal{L}_0$. In order to ensure it, we had to introduce new field A_μ and postulate its transformation properties. By adding appropriate terms to Lagrangian we arrived at invariant Lagrangian which therefore represents scalar field coupled to the field A_μ. However, we have seen that postulated transformation law is identical with the transformation law for the potential of electromagnetic field. Lagrangian (8.7) still does not describe the dynamics of the field A_μ, as it does not contain its derivatives. Clearly, $\partial_\mu A_\nu$ is not a gauge invariant object, for we have

$$\delta\partial_\mu A_\nu = \frac{1}{e}\,\partial_\mu\partial_\nu\theta.$$

In order to form a gauge invariant tensor we antisymmetrize the last expression and define *electromagnetic tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu \, - \, \partial_\nu A_\mu . \tag{8.8}$$

Kinetic term representing the dynamics of electromagnetic field must be quadratic in $F_{\mu\nu}$ and the only choice is

$$\mathcal{L}_{\mathrm{EM}} \; = \; -\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu}.$$

To conclude, we have arrived at the Lagrangian describing the scalar field, electromagnetic field and their interaction in the form

$$\mathcal{L}_{\mathrm{S+EM}} = \left[(\partial_\mu + ieA_\mu)\,\phi\right]\left[(\partial^\mu - ieA^\mu)\,\overline{\phi}\right] \, - \, m^2\,\phi\,\overline{\phi} \, - \, \frac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu}. \tag{8.9}$$

From the field-theoretical point of view it is useful to introduce the notion of *gauge covariant-derivative* D_μ by

$$\begin{aligned} D_\mu\phi &= \partial_\mu\phi \, + \, i\,e\,A_\mu\,\phi, \\ D_\mu\overline{\phi} &= \partial_\mu\overline{\phi} \, - \, i\,e\,A_\mu\,\overline{\phi}. \end{aligned} \tag{8.10}$$

In terms of covariant derivatives, the Lagrangian reads

$$\mathcal{L}_{\mathrm{S+EM}} = (D_\mu\phi)(D^\mu\overline{\phi}) \, - \, m^2\,\phi\,\overline{\phi} \, - \, \frac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu} \tag{8.11}$$

Equations of motion of fields ϕ, $\overline{\phi}$ and $F_{\mu\nu}$ can be derived in a straightforward way from the Euler-Lagrange equations

$$\begin{aligned} \partial_\mu \frac{\partial\mathcal{L}_{\mathrm{S+EM}}}{\partial\partial_\mu\phi} - \frac{\partial\mathcal{L}_{\mathrm{S+EM}}}{\partial\phi} &= 0, \\ \partial_\mu \frac{\partial\mathcal{L}_{\mathrm{S+EM}}}{\partial\partial_\mu\overline{\phi}} - \frac{\partial\mathcal{L}_{\mathrm{S+EM}}}{\partial\overline{\phi}} &= 0, \\ \partial_\mu \frac{\partial\mathcal{L}_{\mathrm{S+EM}}}{\partial\partial_\mu A_\nu} - \frac{\partial\mathcal{L}_{\mathrm{S+EM}}}{\partial A_\nu} &= 0. \end{aligned} \tag{8.12}$$

We find following field equations:

$$\begin{aligned} \left(D_\mu D^\mu \, + \, m^2\right)\phi &= 0, \\ \left(D_\mu D^\mu \, + \, m^2\right)\overline{\phi} &= 0, \\ \partial^\mu F_{\mu\nu} &= e\,J_\nu, \end{aligned} \tag{8.13}$$

where the four-current J_ν is now given by

$$J_\mu = i\left(\overline{\phi}D_\mu\phi \, - \, \phi D_\mu\overline{\phi}\right). \tag{8.14}$$

In curved spacetime we take the Lagrangian in the form

$$\mathcal{L} = (D_a\phi)\left(D^a\overline{\phi}\right) \, - \, m^2\,\phi\,\overline{\phi} \, - \, \frac{1}{4}\, F_{ab}\, F^{ab}, \tag{8.15}$$

where D_μ is a gauge covariant derivative defined by

$$\begin{aligned} D_a\phi &= \nabla_a\phi \,+\, i\,e\,A_a\,\phi, \\ D_a\overline{\phi} &= \nabla_a\overline{\phi} \,-\, i\,e\,A_a\,\overline{\phi}. \end{aligned} \tag{8.16}$$

and the electromagnetic field tensor is

$$F_{ab} = \nabla_a A_b \,-\, \nabla_b A_a. \tag{8.17}$$

Unknown variables are the scalar fields $\phi, \overline{\phi}$ and the vector potential of electromagnetic field A_a. Euler-Lagrange equations derived from Lagrangian (8.15) are

$$\begin{aligned} \left(D_a D^a \,+\, m^2\right)\phi &= 0, \\ \left(D_a D^a \,+\, m^2\right)\overline{\phi} &= 0, \\ \nabla_a F^{ab} &= j^b, \end{aligned} \tag{8.18}$$

where the four-current j^b is given by

$$j^b = i\,e\,\left(\overline{\phi} D^b \phi \,-\, \phi D^b \overline{\phi}\right). \tag{8.19}$$

8.2 Electromagnetic field

In the flat spacetime, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which implies $\partial_{[\alpha} F_{\mu\nu]} = 0$. In the language of differential forms, F is an exact form, $F = \mathrm{d}A$, and thus it is automatically closed, $\mathrm{d}F = 0$. This property holds also in curved spacetime where F_{ab} is defined by

$$F_{ab} = \nabla_a A_b \,-\, \nabla_b\, A_a. \tag{8.20}$$

Then, using the definition of the Riemann tensor, its exterior derivative is

$$\begin{aligned} \nabla_{[c} F_{ab]} &= \frac{2}{3}\,\left[\nabla_{[c}\nabla_{a]} A_b \,+\, \nabla_{[b}\nabla_{c]} A_a \,+\, \nabla_{[a}\nabla_{b]} A_c\right] \\ &= -\,\frac{1}{3}\,\left[R_{cab}{}^d \,+\, R_{bca}{}^d \,+\, R_{abc}{}^d\right]\,A_d \\ &= A^d\, R_{[abc]d} \;=\; 0 \end{aligned} \tag{8.21}$$

by the symmetries of the Riemann tensor. Hence, by the definition of F_{ab}, it satisfies half of Maxwell's equations

$$\nabla_{[c} F_{ab]} = 0. \tag{8.22}$$

Antisymmetric electromagnetic tensor F_{ab} can be decomposed in a usual way as

$$F_{ab} = \phi_{AB}\,\epsilon_{A'B'} \,+\, \overline{\phi}_{A'B'}\,\epsilon_{AB}. \tag{8.23}$$

On the other hand, F_{ab} is a curl of A_a, i.e.

$$F_{ab} = \nabla_a A_b \,-\, \nabla_b\, A_a. \tag{8.24}$$

Comparing both expressions and contracting it with ϵ^{AB} and $\epsilon^{A'B'}$, we arrive at relations between the potential A_a and symmetric spinor ϕ_{AB} (see, e.g. [14]):

$$\phi_{AB} = \nabla_{X'(A} A^{X'}_{B)}, \quad \overline{\phi}_{A'B'} \;=\; \nabla_{X(A'} A^{X}_{B')}. \tag{8.25}$$

By (2.27), the equation $\nabla_{[c}F_{ab]} = 0$ is equivalent to

$$\nabla^{a\,*}F_{ab} = 0, \tag{8.26}$$

where ${}^*F_{ab}$ is the Hodge dual of F_{ab},

$${}^*F_{ab} = \frac{1}{2}\,\epsilon_{abcd}\,F^{cd} \;=\; -\,i\,\phi_{AB}\,\epsilon_{A'B'} \;+\; i\,\epsilon_{AB}\,\overline{\phi}_{A'B'}. \tag{8.27}$$

Substituting spinor equivalent of ${}^*F_{ab}$ into (8.26) yields

$$\nabla^A_{B'}\phi_{AB} = \nabla^{A'}_B\overline{\phi}_{A'B'}. \tag{8.28}$$

On the other hand, equation (8.26) guarantees that F_{ab} is an exact form and thus can be written in the form (8.24) so that the spinors ϕ_{AB} and $\overline{\phi}_{A'B'}$ are given by (8.25). Since (8.28) is just a spinor equivalent of (8.26), equation (8.25) should imply (8.28). We will prove this statement as a separate lemma.

Lemma 8.2.1. *Let spinors ϕ_{AB} and $\overline{\phi}_{A'B'}$ be given by (8.25), i.e.*

$$\phi_{AB} \;=\; \nabla_{A'(A}A^{A'}_{B)}, \quad \overline{\phi}_{A'B'} \;=\; \nabla_{A(A'}A^{A}_{B')}.$$

Then equation (8.28) is automatically satisfied, i.e.

$$\nabla^A_{B'}\phi_{AB} \;=\; \nabla^{A'}_B\overline{\phi}_{A'B'}.$$

Proof. We have to show that

$$\nabla^A_{B'}\phi_{AB} \;-\; \nabla^{A'}_B\overline{\phi}_{A'B'} \;=\; \nabla^A_{B'}\nabla_{X'(A}A^{X'}_{B)} \;-\; \nabla^{A'}_B\nabla_{X(A'}A^{X}_{B')} \;=\; 0.$$

Let us write

$$\nabla^A_{B'}\nabla_{X'(A}A^{X'}_{B)} = \frac{1}{2}\left[\nabla^A_{B'}\nabla_{X'A}A^{X'}_{B} \;+\; \nabla^A_{B'}\nabla_{X'B}A^{X'}_{A}.\right] \tag{8.29}$$

By standard decomposition into symmetric and antisymmetric parts we obtain

$$\begin{aligned}
\nabla^A_{B'}\nabla_{X'A}A^{X'}_{B} &= -\,\square_{A'B'}A^{A'}_{B} \;+\; \frac{1}{2}\square A_b,\\
\nabla^A_{B'}\nabla_{X'B}A^{X'}_{A} &= -\,\nabla_{(A(A'}\nabla_{B')B)}A^a \;+\; \frac{1}{2}\,\square_{AB}A^A_{B'} \;-\; \frac{1}{2}\,\square_{A'B'}A^{A'}_B \;+\; \frac{1}{4}\,\square A_b.
\end{aligned} \tag{8.30}$$

Recall the definition of operators $\square_{AB}$ and $\square_{A'B'}$ given by (2.75). Thus, we have

$$\nabla^A_{B'}\phi_{AB} = \frac{3}{8}\square A_b \;-\; \frac{1}{2}\nabla_{(A(A'}\nabla_{B')B)}A^a \;+\; \frac{1}{4}\square_{AB}A^A_{B'} \;-\; \frac{3}{4}\square_{A'B'}A^{A'}_B. \tag{8.31}$$

Expression for $\nabla^{A'}_B\overline{\phi}_{A'B'}$ can be obtained by simple complex conjugation of the previous relation, so that

$$i\nabla^{a\,*}F_{ab} = \nabla^A_{B'}\phi_{AB} \;-\; \nabla^{A'}_B\overline{\phi}_{A'B'} \;=\; \square_{AB}A^A_{B'} \;-\; \square_{A'B'}A^{A'}_B. \tag{8.32}$$

By the spinor form of the Bianchi identities (2.78) we have

$$\square_{AB}A^A_{B'} \;=\; -\,3\,\Lambda A_b \;+\; \Phi_{ab}\,A^b$$

which is manifestly real expression and immediately implies

$$\nabla^A_{B'}\phi_{AB} \;-\; \nabla^{A'}_B\overline{\phi}_{A'B'} \;=\; \square_{AB}A^A_{B'} \;-\; \square_{A'B'}A^{A'}_B \;=\; 0.$$

We can conclude that (8.25) implies (8.26) or, equivalently, (8.28). $\square$

8.3 Einstein-electro-scalar equations

In this section we rewrite field equations (8.18) and gravitational field equations as the system of first-order equations in order to simplify the procedure of asymptotical solution of these equations. Unknown variables are:

- Φ_{ABCD}, the Ricci spinor;
- Ψ_{ABCD}, the Weyl spinor;
- $\phi_{AB}, \overline{\phi}_{A'B'}$, electromagnetic spinors;
- A_a, the vector potential, subject to the Lorenz condition $\nabla_a A^a = 0$;
- $\phi, \overline{\phi}$, scalar fields.

We start with the case of equations for the scalar fields ϕ and $\overline{\phi}$. According to (8.18), these fields satisfy the second-order equations

$$\begin{aligned} &(\nabla_a + ieA_a)(\nabla^a + ieA^a)\,\phi + m^2\,\phi = 0, \\ &(\nabla_a - ieA_a)(\nabla^a - ieA^a)\,\overline{\phi} + m^2\,\overline{\phi} = 0. \end{aligned} \tag{8.33}$$

Expanding the operator on the left hand side of the first equation we arrive at equation

$$\Box\phi + 2eiA^a\nabla_a\phi + ie\phi\nabla_a A^a - e^2 A^a A_a\phi + m^2\phi = 0.$$

Imposing the Lorenz gauge condition, $\nabla_a A^a = 0$, and introducing the notation

$$\varphi_a = \nabla_a\phi, \quad \varphi_{AA'} = \nabla_{AA'}\phi, \tag{8.34}$$

the last equation acquires the form

$$\Box\phi = -\,2\,i\,e\,A^a\,\varphi_a + \left(e^2\,A^a\,A_a - m^2\right)\phi. \tag{8.35}$$

Now we can treat the vector field φ_a, the gradient of ϕ, as a new variable, equations for which can be derived in a following way (cf. [14]). Object $\nabla^A_{A'}\varphi_{AB'}$ can be decomposed in a usual way as

$$\nabla^A_{A'}\varphi_{AB'} = \nabla^A_{(A'}\varphi_{B')A} + \frac{1}{2}\,\epsilon_{A'B'}\,\nabla^A_{X'}\varphi^{X'}_A = -\,\Box_{A'B'}\phi - \frac{1}{2}\,\epsilon_{A'B'}\,\Box\phi.$$

The first term vanishes by (2.77) while the second one is given by (8.35). Thus, we obtain first-order equation for $\varphi_{AA'}$:

$$\nabla^A_{A'}\varphi_{AB'} = i\,e\,A^c\varphi_c\,\epsilon_{A'B'} + \frac{1}{2}\left(m^2 - e^2\,A^cA_c\right)\phi\,\epsilon_{A'B'}. \tag{8.36}$$

Now we turn to equations for the four-potential A_a. Again, we decompose object $\nabla^{A'}_A A_{BA'}$ as

$$\nabla^{A'}_A A_{BA'} = \nabla^{A'}_{(A}A_{B)A'} + \frac{1}{2}\,\epsilon_{AB}\,\nabla^{A'}_X A^X_{A'}.$$

The first term is (up to sign) equal to ϕ_{AB}, cf. (8.25), while the second term vanishes by the Lorenz condition. Hence, equation for the potential reads

$$\nabla^{A'}_A A_{BA'} = -\,\phi_{AB}. \tag{8.37}$$

Finally we take the Maxwell equations $\nabla^a F_{ab} = j_b$ and express F_{ab} in terms of ϕ_{AB} according to equation (8.23):

$$\nabla^a F_{ab} \;=\; \nabla^A_{B'}\phi_{AB} \,+\, \nabla^{A'}_{B}\overline{\phi}_{A'B'} \;=\; j_b.$$

However, by lemma 8.2.1 we have

$$\nabla^A_{B'}\phi_{AB} \;=\; \nabla^{A'}_{B}\overline{\phi}_{A'B'}$$

and thus the Maxwell equation acquires the form

$$\nabla^A_{B'}\phi_{AB} \;=\; \frac{1}{2}\,j_b \;=\; \frac{ie}{2}\left(\overline{\phi}D_b\phi \,-\, \phi D_b\overline{\phi}\right).$$

Using the definition of gauge covariant derivative (8.16) we arrive at the final form of equation for spinor ϕ_{AB}:

$$\nabla^A_{B'}\phi_{AB} = \frac{ie}{2}\left(\overline{\phi}\,\varphi_b \,-\, \phi\,\overline{\varphi}_b\right) \,-\, e^2\,\phi\,\overline{\phi}\,A_b. \tag{8.38}$$

The last ingredient is the energy-momentum tensor entering the right hand side of Einstein's equations. In general relativity, the energy-momentum tensor is derived from the action of non-gravitational fields

$$I[g^{ab}] \;=\; \int \mathcal{L}_{\mathrm{S+EM}}(g^{ab})\,\sqrt{-\,g}\,\mathrm{d}^4x$$

where g is the determinant of metric. Then we define the energy-momentum tensor by[21]

$$T_{ab} = \frac{\alpha}{4\pi}\,\frac{1}{\sqrt{-g}}\,\frac{\delta I}{\delta g^{ab}}. \tag{8.39}$$

Conservation law $\nabla^a T_{ab} = 0$ follows from the requirement that the theory be invariant under diffeomorphisms, regardless of particular form of gravitational part of the action. In terms of the Lagrangian, the energy-momentum tensor reads[3]

$$T_{ab} = \frac{\alpha}{4\pi}\left[\frac{\partial\mathcal{L}_{\mathrm{S+EM}}}{\partial g^{ab}} \,-\, \frac{1}{2}\,g_{ab}\,\mathcal{L}_{\mathrm{S+EM}}\right]. \tag{8.40}$$

For Lagrangian (8.15) we find (cf. [6])

$$\begin{aligned} T_{ab} &= \frac{\alpha}{4\pi}\left[\left(D_{(a}\phi\right)\left(D_{b)}\overline{\phi}\right) \,-\, \frac{1}{2}\,F_{ac}\,F_b{}^{c} \,-\, \frac{1}{2}\,g_{ab}\,\mathcal{L}_{\mathrm{S+EM}}\right] \\ &= \frac{\alpha}{4\pi}\left[\frac{1}{2}\left(\varphi_a\,\overline{\varphi}_b \,+\, \varphi_b\,\overline{\varphi}_a\right) \,+\, \frac{ie}{2}\left(\phi A_a\overline{\varphi}_b \,+\, \phi A_b\overline{\varphi}_a \,-\, \overline{\phi}A_a\varphi_b \,-\, \overline{\phi}A_b\varphi_a\right)\right. \\ &\qquad \left. +\,e^2\phi\overline{\phi}A_aA_b \,-\, \frac{1}{2}\,F_{ac}\,F_b{}^{c} \,-\, \frac{1}{2}\,g_{ab}\,\mathcal{L}_{\mathrm{S+EM}}\right]. \end{aligned} \tag{8.41}$$

Components of the energy-momentum tensor are related to the Ricci spinor via Einstein's equations in the spinor form (2.50).

Let us summarize unknown variables and equations they satisfy.

- scalar field ϕ and its gradient $\varphi_{AA'}$;

$$\nabla_{AA'}\phi \;=\; \varphi_{AA'}, \quad \nabla^{A}_{A'}\varphi_{AB'} = i\,e\,A^{c}\varphi_{c}\,\epsilon_{A'B'} + \frac{1}{2}\left(m^{2} - e^{2}\,A^{c}A_{c}\right)\phi\,\epsilon_{A'B'}, \tag{8.42}$$

- potential A_a of electromagnetic field constrained by the gauge condition $\nabla_a A^a = 0$;

$$\nabla^{A'}_{A} A_{BA'} = -\,\phi_{AB}. \tag{8.43}$$

- electromagnetic spinor ϕ_{AB};

$$\nabla^{A}_{B'}\phi_{AB} = \frac{ie}{2}\left(\overline{\phi}\,\varphi_{b} - \phi\,\overline{\varphi}_{b}\right) - e^{2}\,\phi\,\overline{\phi}\,A_{b}. \tag{8.44}$$

- spin coefficients $\gamma_{\boldsymbol{A}}{}^{\boldsymbol{B}}{}_{CC'}$ satisfying the Ricci identities;
- Weyl spinor Ψ_{ABCD} and Ricci spinor $\Phi_{ABA'B'}$ satisfying the Bianchi identities;

$$\begin{aligned} \nabla^{D}_{A'}\Psi_{ABCD} &= \nabla^{D'}_{(A}\Phi_{BC)A'D'}; \\ \nabla^{BB'}\Phi_{ABA'B'} &= -\,3\,\nabla_{AA'}\Lambda. \end{aligned} \tag{8.45}$$

9. Electro-scalar fields in NP formalism

In this chapter we project Einstein-electro-scalar equations obtained in chapter 8 onto the Newman-Penrose null tetrad.

Four-potential $A_a = A_{AA'}$ is a real vector field and its components with respect to the spin basis will be denoted by

$$A_0 \;=\; A_{XX'} o^X \bar{o}^{X'}, \quad A_1 \;=\; A_{XX'} o^X \bar{\iota}^{X'}, \quad A_{\bar{1}} \;=\; A_{XX'} \iota^X \bar{o}^{X'}, \quad A_2 \;=\; A_{XX'} \iota^X \bar{\iota}^{X'}. \tag{9.1}$$

Since A_a is a real field, its components transform under complex conjugation according to

$$\overline{A_0} \;=\; A_0, \quad \overline{A_1} \;=\; A_{\bar{1}}, \quad \overline{A_{\bar{1}}} \;=\; A_1, \quad \overline{A_2} \;=\; A_2. \tag{9.2}$$

The potential can be then written in the form

$$A_{AA'} \;=\; A_0\, \iota_A \bar{\iota}_{A'} \;-\; A_1\, \iota_A\, \bar{o}_{A'} \;-\; A_{\bar{1}}\, o_A\, \bar{\iota}_{A'} \;+\; A_2\, o_A\, \bar{o}_{A'}.$$

Similarly we introduce NP-components of electromagnetic spinor ϕ_{AB} by

$$\phi_0 \;=\; \phi_{AB}\, o^A\, o^B, \quad \phi_1 \;=\; \phi_{AB}\, o^A\, \iota^B, \quad \phi_2 \;=\; \phi_{AB}\, \iota^A\, \iota^B. \tag{9.3}$$

Spinor ϕ_{AB} is then[17]

$$\phi_{AB} \;=\; \phi_0\, \iota_A\, \iota_B \;-\; 2\,\phi_1\, o_{(A}\, \iota_{B)} \;+\; \phi_2\, o_A\, o_B.$$

Potential A_a is governed by equation (8.43),

$$\nabla^{A'}_A A_{BA'} \;=\; -\,\phi_{AB}.$$

NP-projections of this equation are

$$DA_1 - \delta A_0 = (\bar{\pi} - \bar{\alpha} - \beta)A_0 + (\varepsilon - \bar{\varepsilon} + \bar{\rho})A_1 + \sigma A_{\bar{1}} - \kappa A_2 + \phi_0, \tag{9.4a}$$

$$DA_2 - \delta A_{\bar{1}} = -\mu A_0 + \pi A_1 + (\bar{\pi} - \bar{\alpha} + \beta)A_{\bar{1}} + (\bar{\rho} - \varepsilon - \bar{\varepsilon})A_2 + \phi_1, \tag{9.4b}$$

$$\Delta A_0 - \bar{\delta} A_1 = (\gamma + \bar{\gamma} - \bar{\mu})A_0 + (\bar{\beta} - \alpha - \bar{\tau})A_1 - \tau A_{\bar{1}} + \rho A_2 - \phi_1, \tag{9.4c}$$

$$\Delta A_{\bar{1}} - \bar{\delta} A_2 = \nu A_0 - \lambda A_1 + (\bar{\gamma} - \gamma - \bar{\mu})A_{\bar{1}} + (\alpha + \bar{\beta} - \bar{\tau})A_2 - \phi_2. \tag{9.4d}$$

In chapter 8 we introduced notation ϕ for the scalar field itself, $\bar{\phi}$ for its complex conjugate and $\varphi_{AA'} = \nabla_{AA'}\phi$ for the gradient of the scalar field. Projections of the gradient will be denoted in agreement with [2] as

$$\begin{aligned} \varphi_0 \;&=\; D\phi, \quad \varphi_2 \;=\; \Delta\phi, \quad \varphi_1 \;=\; \delta\phi, \quad \varphi_{\bar{1}} \;=\; \bar{\delta}\phi, \\ \bar{\varphi}_0 \;&=\; D\bar{\phi}, \quad \bar{\varphi}_2 \;=\; \Delta\bar{\phi}, \quad \bar{\varphi}_1 \;=\; \delta\bar{\phi}, \quad \bar{\varphi}_{\bar{1}} \;=\; \bar{\delta}\,\bar{\phi} \end{aligned} \tag{9.5}$$

So, under complex conjugation, index 1 consistently transforms to $\bar{1}$ and vice versa, while indices 0 and 2 are "real" in the sense $\bar{0} = 0, \bar{2} = 2$. Equations(9.5) can be regarded as dynamical equations for scalar field ϕ. Fields $\varphi_{AA'}$ and $\bar{\varphi}_{AA'}$ can be written in terms of the spin basis as

$$\begin{aligned} \varphi_{AA'} &= \varphi_0\, \iota_A\, \bar{\iota}_{A'} \;-\; \varphi_1\, \iota_A\, \bar{o}_{A'} \;-\; \varphi_{\bar{1}}\, o_A\, \bar{\iota}_{A'} \;+\; \varphi_2\, o_A\, \bar{o}_{A'}, \\ \bar{\varphi}_{AA'} &= \bar{\varphi}_0\, \iota_A\, \bar{\iota}_{A'} \;-\; \bar{\varphi}_{\bar{1}}\, o_A\, \bar{\iota}_{A'} \;-\; \bar{\varphi}_1\, \iota_A\, \bar{o}_{A'} \;+\; \bar{\varphi}_2\, o_A\, \bar{o}_{A'}. \end{aligned} \tag{9.6}$$

Now we can complete equations for electromagnetic field. Equation (8.44),

$$\nabla^A_{B'}\phi_{AB} = \frac{ie}{2}\left(\overline{\phi}\,\varphi_b \;-\; \phi\,\overline{\varphi}_b\right) \;-\; e^2\,\phi\,\overline{\phi}\,A_b,$$

is the spinor version of Maxwell's equations with four-current j^a on the right hand side. NP-projections of this equation follow:

$$D\phi_1 - \overline{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 + \frac{ie}{2}\left(\phi\overline{\varphi}_0 - \overline{\phi}\varphi_0\right) + e^2\phi\overline{\phi}A_0, \tag{9.7a}$$

$$D\phi_2 - \overline{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2 + \frac{ie}{2}\left(\phi\overline{\varphi}_{\overline{1}} - \overline{\phi}\varphi_{\overline{1}}\right) + e^2\phi\overline{\phi}A_{\overline{1}}, \tag{9.7b}$$

$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2 + \frac{ie}{2}\left(\overline{\phi}\varphi_1 - \phi\overline{\varphi}_1\right) - e^2\phi\overline{\phi}A_1, \tag{9.7c}$$

$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2 + \frac{ie}{2}\left(\overline{\phi}\varphi_2 - \phi\overline{\varphi}_2\right) - e^2\phi\overline{\phi}A_2. \tag{9.7d}$$

Dynamical equation for the gradient $\varphi_{AA'}$ is provided by equation (8.42)

$$\nabla^A_{A'}\varphi_{AB'} = i\,e\,A^c\varphi_c\,\epsilon_{A'B'} \;+\; \frac{1}{2}\left(m^2 \;-\; e^2\,A^cA_c\right)\phi\,\epsilon_{A'B'}. \tag{9.8}$$

Projected on the spin basis, this equation is equivalent to following four scalar equations:

$$D\varphi_{\overline{1}} - \overline{\delta}\varphi_0 = (\pi - \alpha - \overline{\beta})\varphi_0 + \overline{\sigma}\varphi_1 + (\rho + \overline{\varepsilon} - \varepsilon)\varphi_{\overline{1}} - \overline{\kappa}\varphi_2, \tag{9.9a}$$

$$\begin{aligned} D\varphi_2 - \overline{\delta}\varphi_1 &= -\overline{\mu}\varphi_0 + (\pi - \alpha + \overline{\beta})\varphi_1 + \overline{\pi}\varphi_{\overline{1}} + (\rho - \varepsilon - \overline{\varepsilon})\varphi_2 \\ &\quad + e^2\phi\,(A_0A_2 - A_1A_{\overline{1}}) + ie\,(A_1\varphi_{\overline{1}} + A_{\overline{1}}\varphi_1 - A_0\varphi_2 - A_2\varphi_0) - \phi m^2/2, \end{aligned} \tag{9.9b}$$

$$\begin{aligned} \Delta\varphi_0 - \delta\varphi_{\overline{1}} &= (\gamma + \overline{\gamma} - \mu)\varphi_0 - \overline{\tau}\varphi_1 + (\beta - \overline{\alpha} - \tau)\varphi_{\overline{1}} + \overline{\rho}\varphi_2 \\ &\quad + e^2\phi\,(A_0A_2 - A_1A_{\overline{1}}) + ie\,(A_1\varphi_{\overline{1}} + A_{\overline{1}}\varphi_1 - A_0\varphi_2 - A_2\varphi_0) - \phi m^2/2, \end{aligned} \tag{9.9c}$$

$$\Delta\varphi_1 - \delta\varphi_2 = \overline{\nu}\varphi_0 + (\gamma - \overline{\gamma} - \mu)\varphi_1 - \overline{\lambda}\varphi_{\overline{1}} + (\overline{\alpha} + \beta - \tau)\varphi_2. \tag{9.9d}$$

9.1 Einstein's equations

Einstein's equations in the spinor form (2.51), (2.52) are

$$\begin{aligned} \Phi_{ABA'B'} &= 4\pi\,T_{(AB)(A'B')}, \\ 3\Lambda &= \pi T_{XY}{}^{XY}. \end{aligned}$$

In order to avoid the symmetrization, we can write them in the form

$$\Phi_{ABA'B'} = 4\pi\,T_{ABA'B'} \;-\; 3\,\Lambda\,\epsilon_{AB}\,\epsilon_{A'B'}. \tag{9.10}$$

In chapter 8 we have derived the energy-momentum tensor for electro-scalar fields (8.41). From this expression it follows that the trace of energy-momentum tensor is

$$\begin{aligned} 12\pi\alpha^{-1}\Lambda &= -\varphi_a\overline{\varphi^a} + ie\overline{\phi}A^a\varphi_a - ie\phi A^a\overline{\varphi}_a - e^2\phi\overline{\phi}A_aA^a + 2m^2\phi\overline{\phi} \\ &= 2\left(\varphi_{(1}\overline{\varphi}_{\overline{1})} - \varphi_{(0}\overline{\varphi}_{2)}\right) \;+\; 2ie\overline{\phi}\left(A_{(0}\varphi_{2)} - A_{(1}\varphi_{\overline{1})}\right) \\ &\quad -\; 2ie\phi\left(A_{(0}\overline{\varphi}_{2)} - A_{(1}\overline{\varphi}_{\overline{1})}\right) - 2e^2\phi\overline{\phi}\,(A_0A_2 - A_{\overline{1}}A_1) + 2m^2\phi\overline{\phi}. \end{aligned} \tag{9.11}$$

Components of the Ricci spinor in the Newman-Penrose formalism are (we set $\alpha = 1$)

$$
\begin{aligned}
\Phi_{00} &= \phi_0\overline{\phi}_0 + \varphi_0\overline{\varphi}_0 + e^2 A_0^2 \phi\overline{\phi} + ieA_0\left(\phi\overline{\varphi}_0 - \overline{\phi}\varphi_0\right), \\
\Phi_{01} &= \phi_0\overline{\phi}_1 + \varphi_{(0}\overline{\varphi}_{1)} + e^2\phi\overline{\phi}A_0A_1 + ie\phi A_{(0}\overline{\varphi}_{1)} - ie\overline{\phi}A_{(0}\varphi_{1)}, \\
\Phi_{11} + 3\Lambda &= \phi_1\overline{\phi}_1 + \varphi_{(1}\overline{\varphi}_{\overline{1})} + e^2\phi\overline{\phi}A_1A_{\overline{1}} + ie\phi A_{(1}\overline{\varphi}_{\overline{1})} - ie\overline{\phi}A_{(1}\varphi_{\overline{1})} + m^2\phi\overline{\phi}/2, \\
\Phi_{02} &= \phi_0\overline{\phi}_2 + \varphi_1\overline{\varphi}_1 + e^2\phi\overline{\phi}A_1^2 + ie\phi A_1\overline{\varphi}_1 - ie\overline{\phi}A_1\varphi_1, \\
\Phi_{12} &= \phi_1\overline{\phi}_2 + \varphi_{(1}\overline{\varphi}_{2)} + e^2\phi\overline{\phi}A_1A_2 + ie\phi A_{(2}\overline{\varphi}_{1)} - ie\overline{\phi}A_{(2}\varphi_{1)}, \\
\Phi_{22} &= \phi_2\overline{\phi}_2 + e^2\phi\overline{\phi}A_2^2 + ieA_2\left(\phi\overline{\varphi}_2 - \overline{\phi}\varphi_2\right).
\end{aligned}
\tag{9.12}
$$

10. Asymptotic solution of Einstein-electro-scalar equations

After introducing all necessary equations, in this chapter we solve them asymptotically, i.e. in the neighbourhood of the null infinity $\mathcal{I}$. Following the procedure developed in [9] and explained in [17], we work in the physical spacetime rather than in the unphysical spacetime. However, we use construction [1] (adapted from $\mathcal{I}^-$ to $\mathcal{I}^+$) which differs slightly from [17].

We use conventions established in [1] and [2], where conformal field equations for free electromagnetic field and for free scalar fields have been analyzed. More general discussion of conformal Einstein-Maxwell-Yang-Mills equations can be found in fundamental paper by Friedrich[4].

10.1 Asymptotic flatness

Our aim is to find asymptotic solution of Einstein-electro-scalar equations near null infinity under assumption that the spacetime (M, g_{ab}) with electro-scalar fields is asymptotically flat. Hence, we assume that there exists a conformal factor Ω and unphysical spacetime $(\hat{M}, \hat{g}_{ab})$ such that

- $\hat{g}_{ab} \;=\; \Omega^2\, g_{ab}$ where $\Omega > 0$ in M;
- there exists embedding $\psi : M \mapsto \hat{M}$;
- hypersurface $I : \Omega = 0$ is a boundary of M embedded in $\hat{M}$; this hypersurface is called *infinity* and can be divided into spacelike infinity i^0, future and past timelike infinity $i^\pm$ and future and past null infinity $\mathcal{I}^+$;
- $\nabla_a \Omega \neq 0$ on I.

We will be interested in asymptotic behaviour of several geometrical quantities near $\mathcal{I}^+$. We use the notation $X = \mathcal{O}\left(\Omega^m\right)$ whenever $\Omega^{-m} X$ is finite on $\mathcal{I}^+$, i.e.

$$X \;=\; \mathcal{O}\left(\Omega^m\right) \quad \Leftrightarrow \quad \lim_{\Omega \to 0} |\Omega^{-m} X| < \;\infty.$$

We assume that all geometrical quantities in the unphysical spacetime are analytic and regular on $\mathcal{I}^+$ in the sense that all quantities are of order $\mathcal{O}\left(1\right)$. Thus, in the unphysical spacetime, all quantities can be expanded into the series in the neighbourhood of the form

$$X = \sum_{m=0}^{\infty} X^{(m)}\, \Omega^m \tag{10.1}$$

where $X^{(m)}$ is independent of Ω.

Under conformal rescaling, the covariant derivative transforms according to relations

$$\nabla_{AA'} \xi_B \;=\; \hat{\nabla}_{AA'} \xi_B \;+\; \Omega^{-1}\, \xi_A\, \hat{\nabla}_{BA'} \Omega, \quad \nabla_{AA'} \xi_{B'} \;=\; \hat{\nabla}_{AA'} \xi_{B'} \;+\; \Omega^{-1}\, \xi_{A'}\, \hat{\nabla}_{AB'} \Omega. \tag{10.2}$$

Moreover, for any scalar ϕ we have $\nabla_{AA'} \phi = \hat{\nabla}_{AA'} \phi$.

10.2 Construction of the coordinate system

We will work in the neighbourhood of future null infinity $\mathcal{I}^+$, see figure 10.2. Null infinity has topology[5] $\mathcal{S}^2 \times \mathbb{R}$. Let us choose arbitrary cut $\mathcal{S}_0$ with the topology $\mathcal{S}^2$ and introduce coordinates x^2 and x^3 on this cut. Since $\mathcal{S}_0$ is a two-sphere, a natural choice of the coordinates are standard spherical coordinates

$$x^2 = \theta, \quad x^3 = \phi.$$

Coordinates on spheres will be labelled by indices $I, J, \cdots = 2, 3$. So, we have coordinatized the initial cut $\mathcal{S}_0$.

Next we wish to introduce coordinates on entire $\mathcal{I}^+$. Since $\mathcal{I}^+$ is a null surface, it is generated by a congruence of null geodesics. Let $\gamma_x = \gamma_x(u)$ be a null geodesic crossing $\mathcal{S}_0$ at point with coordinates x^I and parametrized by the affine parameter u. We propagate coordinates x^I from $\mathcal{S}_0$ to $\mathcal{I}^+$ by condition

$$\nabla_{\dot{\gamma}_x} x^I = 0,$$

i.e. we require coordinates x^I to be constant along null generators of $\mathcal{I}^+$. Hence, a triple

$$(u, x^2, x^3)$$

constitutes a coordinate system on $\mathcal{I}^+$. Notice that since u is an affine parameter, we have a gauge freedom expressed by transformation

$$u \mapsto a(x^I)\, u + b(x^I)$$

where a and b are arbitrary functions. This freedom can be reduced by demanding $u = 0$ on $\mathcal{S}_0$, so only freedom in rescaling $u \mapsto a(x^I)u$ remains.

Now we construct a congruence of null geodesics coming from interior of the spacetime and crossing $\mathcal{I}^+$ and orthogonal to spheres $u =$ constant. Having done this, for each point $P(u, x^2, x^3)$ there is a null geodesic $\gamma'_{u,x} = \gamma'_{u,x}(r)$ coming from the interior of the spacetime and crossing $\mathcal{I}^+$ at point P; let r be an affine parameter of this geodesic. We propagate coordinates u and x^I into the spacetime by conditions

$$\nabla_{\dot{\gamma}'_{u,x}} x^I = \nabla_{\dot{\gamma}'_{u,x}} u = 0$$

i.e. we require that these coordinates be constant along geodesics $\gamma'_{u,x}$. Again, we have a freedom in the choice of parameter r

$$r \mapsto c(u, x^I)\, r + d(u, x^I).$$

Thus, we have constructed a coordinate chart

$$x^\mu = (u, r, x^2, x^3)$$

on $\mathcal{I}^+$ and its neighbourhood. Remaining gauge freedom is in coordinate transformation on $\mathcal{S}_0$,

$$x^I \mapsto \hat{x}^I = \hat{x}^I(x^2, x^3),$$

in the rescaling of coordinate u,

$$u \mapsto \hat{u} = a(x^2, x^3)\, u$$

and in the rescaling affine parameter r mentioned above.

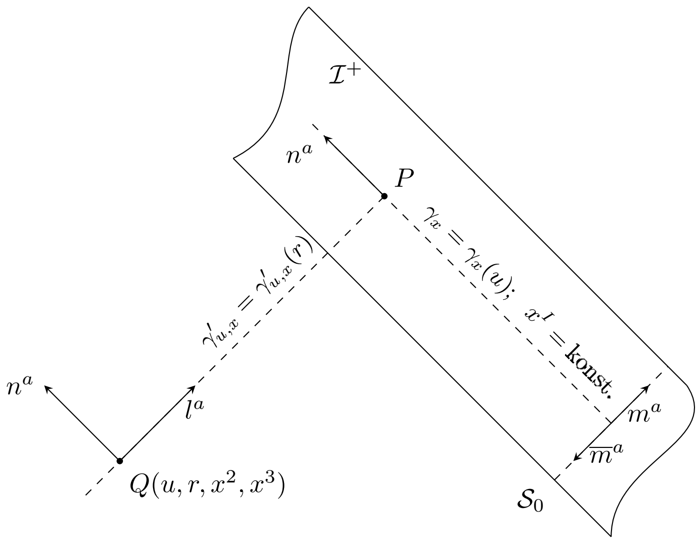

Figure 10.1: Construction of the coordinate system adapted to $\mathcal{I}^+$. Figure taken from [15].

10.3 Null tetrad

On the tangent space $T\mathcal{S}_0$ we can define a pair of mutually complex conjugated null vectors m^a and $\overline{m}^a$ satisfying

$$m^a\, m_a \;=\; 0, \quad m^a\, \overline{m}_a \;=\; -1$$

and introduce usual NP operators δ and $\overline{\delta}$ associated with them. Since these operators act on the sphere where coordinates u and r are constant, they can be expressed in the form

$$\delta \;=\; P^I\, \nabla_I, \quad \overline{\delta} \;=\; \overline{P}^I\, \nabla_I. \tag{10.3}$$

Vectors m^a and $\overline{m}^a$ can be propagated onto $\mathcal{I}^+$ and next to the interior of the spacetime by conditions

$$\nabla_{\dot\gamma_x}\, m^a \;=\; 0, \qquad \nabla_{\dot\gamma'_{u,x}} m^a \;=\; 0.$$

Let us define the gradient

$$l_a \;=\; \nabla_a u, \quad l_\mu \;=\; (1,0,0,0).$$

Surface $u =$ constant consists of null geodesics intersecting $\mathcal{I}^+$ at cut $\mathcal{S}_u$. Hence, l^a is a null vector, $l_a l^a = 0$. Then it is a tangent vector to a null geodesic, for we have

$$l^a \nabla_a l_b \;=\; l^a \nabla_a \nabla_b u \;=\; l^a \nabla_b \nabla_a u \;=\; l^a \nabla_b l_a \;=\; \frac{1}{2}\, \nabla_b (l^a l_a) \;=\; 0$$

by nullity of l^a. The last equation shows that l^a is in fact affinely parametrized. Since it is tangent to null surface generated by affinely parametrized geodesics $\gamma'_{u,x}$ along which only r is varying, we have

$$l^a = \frac{\partial}{\partial r}, \quad l^\mu \;=\; (0,1,0,0). \tag{10.4}$$

By the choice of r, vector l^a is pointing towards $\mathcal{I}^+$.

Now, at each point of the spacetime we have real vector l^a and pointing towards $\mathcal{I}^+$ and two complex vectors m^a and $\overline{m}^a$ spanning the tangent space of spacelike spheres $\mathcal{S}$, both orthogonal to l^a. Then, there is unique real n^a orthogonal to l^a and m^a satisfying

$$l^a n_a = 1.$$

In general, let components of n^a with respect to coordinates (u, r, x^2, x^3) be

$$n^\mu = (Q, H, C^2, C^3).$$

Normalization condition $l_\mu n^\mu = 1$, however, implies

$$Q = 1.$$

Notice that, by construction, vector n^a is the null vector tangent to generators of null hypersurfaces. In particular, n^a is tangent to generators of $\mathcal{I}^+$ along which only coordinate u varies, and therefore

$$n^a = \frac{\partial}{\partial u} \quad \text{on } \mathcal{I}^+.$$

Consequently, functions C^I must vanish on $\mathcal{I}^+$. Term $H\partial_r$ must vanish there as well but this will be a consequence of the fact that ∂_r is zero on $\mathcal{I}^+$, see below. This does not hold necessarily in the neighbourhood of $\mathcal{I}^+$ anymore.

The components of metric tensor can be recovered from the vectors of null tetrad via relation

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \overline{m}^\nu - \overline{m}^\mu m^\nu \tag{10.5}$$

and they read (cf. equation (31) in [1])

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2H & C^2 & C^3 \\ 0 & C^2 & -2P^2\overline{P}^2 & -P^3\overline{P}^2 - P^2\overline{P}^3 \\ 0 & C^3 & -P^3\overline{P}^2 - P^2\overline{P}^3 & -2P^3\overline{P}^3 \end{pmatrix}. \tag{10.6}$$

Hence, functions H, C^I and P^I are not only components of tetrad vectors, but they also form components of the metric tensor and they will be referred to as *metric functions*.

Thus, we have established the Newman-Penrose null tetrad $(l^a, n^a, m^a, \overline{m}^a)$. Their components with respect to basis induced by coordinates x^μ are

$$\begin{aligned} l^\mu &= (0,1,0,0), & m^\mu &= (0,0,P^2,P^3), \\ n^\mu &= (1,H,C^2,C^3), & \overline{m}^\mu &= (0,0,\overline{P}^2,\overline{P}^3). \end{aligned} \tag{10.7}$$

Since l^a and n^a are real null vectors, they can be written in the form

$$l^a = o^A \bar{o}^{A'}, \quad n^a = \iota^A \bar{\iota}^{A'} \tag{10.8}$$

where $|o_A \iota^A| = 1$ by $l^a n_a = 1$. Nevertheless, transformation of the phases of spinors o^A and ι^A leaves vectors l^a and n^a unchanged, so we can always choose this phase so as to achieve $o_A \iota^A = 1$. Remaining null vectors orthogonal to both l^a and n^a and normalized to -1 must be of the form

$o^A\bar{\iota}^{A'}$ and $\iota^A\bar{o}^{A'}$, respectively. Without the loss of generality, we set (relabelling original vectors m^a and $\overline{m}^a$ if necessary)

$$m^a \;=\; o^A\,\bar{\iota}^{A'}, \quad \overline{m}^a \;=\; \iota^A\,\bar{o}^{A'}. \tag{10.9}$$

Hence, we have established the spin basis adapted to the null tetrad. Available gauge freedom in the choice of the tetrad is the rotation of vectors m^a and $\overline{m}^a$, i.e. the transformation

$$o^A \mapsto e^{i\theta}o^A,\; \iota^A \mapsto e^{-i\theta}\iota^A, \quad \text{or} \quad m^a \mapsto e^{2i\theta}m^a,\; \overline{m}^a \mapsto e^{-2i\theta}\overline{m}^a. \tag{10.10}$$

Clearly, this transformation leaves l^a and n^a invariant.

10.4 Spin coefficients and frame equations

With the tetrad introduced above, several spin coefficients simplify or even vanish. We have shown that l^a satisfies the geodesic equation $Dl^a = 0$. However, from (2.19a) it follows

$$Dl^a \;=\; D(o^A\bar{o}^{A'}) \;=\; (\varepsilon+\bar{\varepsilon})l^a \;-\; \bar{\kappa}\,m^a \;-\; \kappa\,\overline{m}^a,$$

which immediately implies

$$\kappa \;=\; 0, \quad \varepsilon+\bar{\varepsilon} \;=\; 0.$$

By rotation (10.10), ε transforms as

$$\varepsilon \mapsto \varepsilon + iD\theta,$$

so that the sum $\varepsilon+\bar{\varepsilon}$ is invariant under this rotation. However, quantity $\varepsilon-\bar{\varepsilon}$ transforms as

$$\varepsilon-\bar{\varepsilon} \mapsto \varepsilon-\bar{\varepsilon}+2iD\theta$$

and hence solving equation

$$D\theta \;=\; \frac{\partial\theta}{\partial r} \;=\; \frac{i}{2}\,(\varepsilon-\bar{\varepsilon})$$

we can achieve $\varepsilon=\bar{\varepsilon}$. Equation for $D\theta$ is a first-order initial value problem for which arbitrary initial conditions can be imposed on the surface $r = \text{constant}$. Thus, we have set

$$\varepsilon \;=\; 0$$

but θ is still not fixed completely.

Further restrictions on the spin coefficients follow from the commutation relations. Applying commutator (2.91d) on coordinates u and r yields

$$\bar{\rho} \;=\; \rho, \quad \bar{\mu} \;=\; \mu. \tag{10.11}$$

Similarly, commutator (2.91b) acting on r and commutator (2.91c) acting on u give

$$\bar{\pi} \;=\; \tau \;=\; \bar{\alpha} \,+\, \beta. \tag{10.12}$$

Remaining commutators (acting on x^μ) are either trivial or represent equations for the metric functions. Commutator (2.91a) applied to coordinates r and x^I, commutator (2.91b) applied to r and x^I and commutator (2.91c) to x^I gives

$$DH = -\gamma - \overline{\gamma}, \tag{10.13a}$$

$$DC^I = 2\pi P^I + 2\overline{\pi}\overline{P}^I, \tag{10.13b}$$

$$DP^I = \rho P^I + \sigma\overline{P}^I, \tag{10.13c}$$

$$\Delta P^I - \delta C^I = (\gamma - \overline{\gamma} - \mu)P^I - \overline{\lambda}\overline{P}^I, \tag{10.13d}$$

$$\delta H = -\overline{\nu}, \tag{10.13e}$$

$$\overline{\delta}P^I - \delta\overline{P}^I = (\alpha - \overline{\beta})P^I + (\beta - \overline{\alpha})\overline{P}^I,. \tag{10.13f}$$

These equations will be referred to as the *frame equations*.

10.5 Asymptotic behaviour

The last ingredient necessary to find asymptotic solution of Einstein-electro-scalar equations is to establish asymptotic behaviour of the null tetrad and other geometrical quantities. Let us make a remark on the construction introduced above. Strictly speaking, null infinity $\mathcal{I}^+$ is not a part of physical spacetime M since it is well-defined only in the unphysical, conformally rescaled spacetime as the boundary ∂M. Under conformal rescaling, geodesics of the physical spacetime are mapped into curves in the unphysical spacetime which are not geodesics anymore, although their causal type is preserved. For example, timelike geodesics are mapped into timelike curves which are not geodesics. The only exception are *null geodesics*, i.e. null geodesics of physical spacetime are mapped to null geodesics of the unphysical spacetime. Since our coordinates near $\mathcal{I}^+$ are based on the families of null geodesics generating $\mathcal{I}^+$ and null geodesics generating hypersurfaces intersecting $\mathcal{I}^+$, this construction can be performed in the unphysical spacetime and then translated to coordinates in the physical spacetime (with $\mathcal{I}^+$ itself removed).

In section 10.1 we explained that all geometrical quantities can be expanded in the series (10.1) in conformal factor Ω. However, we do not know a priori, what is the asymptotic behaviour of geometrical quantities and thus we do not know what is the leading term in series (10.1) for particular X. On the other hand, by assumption made in section 10.1, we assume all *unphysical* geometrical quantities to be regular on $\mathcal{I}^+$, i.e. we assume that any unphysical quantity $\hat{X}$ is of order $\mathcal{O}(1)$. Using this assumption and behaviour of quantity X under conformal rescaling we can deduce the asymptotic behaviour of physical quantity X in the physical spacetime. Hence, in this section we perform the analysis sketched in previous lines.

Recall that unphysical metric $\hat{g}_{ab}$ and physical metric g_{ab} are related by conformal rescaling

$$\hat{g}_{ab} = \Omega^2\, g_{ab}. \tag{10.14}$$

In terms of symplectic form ϵ_{AB}, the spinor equivalent of metric tensor, the conformal rescaling reads

$$\hat{\epsilon}_{AB} = \Omega\,\epsilon_{AB}. \tag{10.15}$$

Symplectic form can be constructed from the basis spinors as

$$\epsilon_{AB} = o_A\,\iota_B - o_B\,\iota_A.$$

Hence, we can prescribe arbitrary conformal transformation of basis spinors which is consistent with (10.15). A natural choice (see, e.g. discussion in [15]) is

$$\begin{aligned} \hat{o}^A &= \Omega^{-1}\, o^A, \quad \hat{\iota}^A = \iota^A, \\ \hat{o}_A &= o_A, \quad \hat{\iota}_A = \Omega\, \iota_A. \end{aligned} \tag{10.16}$$

Vectors of the null tetrad then transform according to

$$\hat{l}^a = \Omega^{-2}\, l^a, \quad \hat{n}^a = n^a, \quad \hat{m}^a = \Omega^{-1}\, m^a. \tag{10.17}$$

We emphasize again that *unphysical* spinors $\hat{o}^A$ and $\hat{\iota}^A$ are assumed to be regular on $\mathcal{I}^+$ which implies that physical spinor $o^A = \Omega \hat{o}^A$ vanishes on $\mathcal{I}^+$ while the spinor ι^A remains non-vanishing there.

Now, recall the construction of coordinate r in section 10.2. Let $\hat{r}$ be an affine parameter of null geodesics intersecting $\mathcal{I}^+$ in the unphysical spacetime and let $\hat{l}^a$ be a tangent to these geodesics. Then, by definition,

$$\hat{D}\hat{l}_a = 0$$

where $\hat{D} = \hat{l}^a \hat{\nabla}_a$. Acting on scalars, we have

$$\hat{D} = \frac{\partial}{\partial \hat{r}}.$$

Since $\hat{r}$ is an affine parameter, we can rescale it in such a way that

$$\hat{r} = 0, \quad \text{and} \quad \hat{D}\Omega = \frac{\partial \Omega}{\partial \hat{r}} = -1 \quad \text{on } \mathcal{I}^+. \tag{10.18}$$

Thus, in the neighbourhood of $\mathcal{I}^+$, $\hat{r} = -\Omega + \mathcal{O}\left(\Omega^2\right)$. However, following [17], we identify coordinate $\hat{r}$ with conformal factor Ω as we are interested in the limit $\Omega \to 0$ which corresponds to limit $\hat{r} \to 0$. The Bondi mass is not affected by this difference.

Let us now see what happens in the physical spacetime. First, it is important that relations (10.16) and (10.17) imply that also l_a is a geodesic. Indeed, using (10.2) we find

$$Dl_b = \Omega^2\, \hat{o}^A\, \hat{\bar{o}}^{A'} \left(\hat{\nabla}_{AA'} \hat{l}_b + \Omega^{-1} o_A \bar{o}_{B'} \nabla_{BA'} \Omega + \Omega^{-1} o_B \bar{o}_{A'} \nabla_{AB'} \Omega \right) = \Omega^2 \hat{D} \hat{l}_b = 0.$$

Next we find relation between $D\Omega$ and $\hat{D}\Omega$, which follows immediately from (10.17):

$$D\Omega = \frac{\partial \Omega}{\partial r} = \Omega^2\, \hat{D}\Omega = \Omega^2\, \frac{\partial \Omega}{\partial \hat{r}}.$$

If we treat $\hat{r}$ as a function of r, we can write

$$\frac{\partial}{\partial \hat{r}} = \frac{\mathrm{d}r}{\mathrm{d}\hat{r}} \frac{\partial}{\partial r}$$

and applying this on Ω we find

$$\mathrm{d}\hat{r} = \Omega^2\, \mathrm{d}r\,. \tag{10.19}$$

Comparing this result with (10.18) we see that near the $\mathcal{I}^+$ we may set

$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} = -\Omega^2\,. \tag{10.20}$$

Consider derivative of quantity X along the null geodesic l^a. Since only r varies along this geodesic, we have

$$l^a \nabla_a X = \frac{\partial X}{\partial r} = \Omega^2 \frac{\partial X}{\partial \hat{r}} = -\Omega^2 \frac{\partial X}{\partial \Omega}\,.$$

Thus, in the neighbourhood of $\mathcal{I}^+$ we have

$$D \;=\; \frac{\partial}{\partial r} = -\,\Omega^2 \frac{\partial}{\partial \Omega}. \tag{10.21}$$

In other words, we can use Ω as a coordinate instead of r, so that the NP operators (acting on scalars) read

$$D \;=\; -\,\Omega^2 \partial_\Omega, \quad \Delta \;=\; \partial_u \;-\; \Omega^2\, H\, \partial_\Omega \;+\; C^I \partial_I, \quad \delta \;=\; P^I \partial_I, \quad \bar{\delta} \;=\; \overline{P}^I \partial_I. \tag{10.22}$$

In particular, we have

$$D\Omega \;=\; -\,\Omega^2, \quad \Delta\Omega \;=\; -\,\Omega^2\, H, \quad \delta\Omega \;=\; \bar{\delta}\Omega \;=\; 0. \tag{10.23}$$

In addition, by (10.16) we have

$$C^I \;=\; \mathcal{O}\,(\Omega)\,, \quad P^I \;=\; \mathcal{O}\,(\Omega)\,. \tag{10.24}$$

Now we establish asymptotic behaviour of the spin coefficients under assumption that unphysical spin coefficients are regular on $\mathcal{I}^+$. Under conformal rescaling, the spin coefficients transform as

$$\begin{aligned}
&\kappa = \Omega^3\,\hat{\kappa}, \quad \tau \;=\; \Omega\,\hat{\tau} \;+\; \hat{\delta}\Omega, \quad \sigma \;=\; \Omega^2\,\hat{\sigma}, \quad \rho \;=\; \Omega^2\,\hat{\rho} \;+\; \Omega\,\hat{D}\Omega,\\
&\varepsilon = \Omega^2\,\hat{\varepsilon}, \quad \gamma \;=\; \hat{\gamma} \;+\; \Omega^{-1}\hat{\Delta}\Omega, \quad \beta \;=\; \Omega\,\hat{\beta}, \quad \alpha \;=\; \Omega\,\hat{\alpha} \;+\; \hat{\bar{\delta}}\Omega,\\
&\pi = \Omega\,\hat{\pi} \;-\; \hat{\bar{\delta}}\Omega, \quad \nu \;=\; \Omega^{-1}\,\hat{\nu}, \quad \mu \;=\; \hat{\mu} \;-\; \Omega^{-1}\,\hat{\Delta}\Omega, \quad \lambda \;=\; \hat{\lambda}.
\end{aligned} \tag{10.25}$$

These relations have been derived using the definitions of spin coefficients, see table on page 55, the rule for transformation of covariant derivative (10.2) and the behaviour of the spin basis (10.16). Derivatives with the hat are operators associated with the unphysical spin basis $\hat{o}^A$ and $\hat{\iota}^A$. We assume order $\mathcal{O}\,(1)$ for all unphysical quantities. Nevertheless, *tangential* derivatives of Ω vanish on $\mathcal{I}^+$ where $\Omega = 0$, and thus we can use estimates

$$\hat{D}\Omega \;=\; \mathcal{O}\,(1)\,, \quad \hat{\Delta}\Omega \;=\; \mathcal{O}\,(\Omega)\,, \quad \hat{\delta}\Omega \;=\; \mathcal{O}\,(\Omega)\,, \quad \hat{\bar{\delta}}\Omega \;=\; \mathcal{O}\,(\Omega)\,.$$

In the tetrad introduced above, coefficients ε and κ vanish and thus, by (10.25), their unphysical counterparts $\hat{\varepsilon}$ and $\hat{\kappa}$ vanish as well. Moreover, by (10.23) we have

$$\tau \;=\; \Omega\hat{\tau}, \quad \pi \;=\; \Omega\hat{\pi}$$

and thus

$$\tau \;=\; \mathcal{O}\,(\Omega)\,, \quad \pi \;=\; \mathcal{O}\,(\Omega)\,. \tag{10.26}$$

Similarly, $\alpha \;=\; \Omega\hat{\alpha}$ and

$$\alpha \;=\; \mathcal{O}\,(\Omega)\,, \quad \beta \;=\; \mathcal{O}\,(\Omega)\,. \tag{10.27}$$

For coefficients γ, μ and λ we find

$$\begin{aligned}
\gamma &= \hat{\gamma} + \mathcal{O}(1) = \mathcal{O}(1), \\
\mu &= \hat{\mu} - \mathcal{O}(1) = \mathcal{O}(1), \\
\lambda &= \hat{\lambda} = \mathcal{O}(1).
\end{aligned} \tag{10.28}$$

Coefficient ν is manifestly of order

$$\nu = \Omega^{-1}\,\hat{\nu} = \mathcal{O}\left(\Omega^{-1}\right) \tag{10.29}$$

and coefficient σ is

$$\sigma = \Omega^2\,\hat{\sigma} = \mathcal{O}\left(\Omega^2\right). \tag{10.30}$$

Finally, for coefficient ρ we have

$$\rho = \Omega^2\,\hat{\rho} - \Omega = -\Omega + \mathcal{O}\left(\Omega^2\right).$$

Using the remaining freedom in the choice of origin of r we can set[17]

$$\rho = -\Omega + \mathcal{O}\left(\Omega^3\right). \tag{10.31}$$

Let us now turn to asymptotic behaviour of physical fields. First we find appropriate law for conformal transformation of the potential A_a. Let us put

$$A_a = \Omega^w\,\hat{A}_a \tag{10.32}$$

where w is the conformal weight to be determined. By the rule (10.2) we have

$$\nabla_{AA'}A_{BB'} = \hat{\nabla}_{AA'}A_{BB'} + \Omega^{-1}A_{AB'}\hat{\nabla}_{BA'}\Omega + \Omega^{-1}A_{BA'}\hat{\nabla}_{AB'}\Omega.$$

Contracting with $\epsilon^{A'B'} = \Omega\hat{\varepsilon}^{A'B'}$ and rearranging terms we find

$$\nabla_A^{A'}A_{BA'} = \Omega^{w+1}\hat{\nabla}_A^{A'}\hat{A}_{BA'} + w\Omega^w\hat{A}_{BA'}\hat{\nabla}_A^{A'}\Omega + \epsilon_{AB}\,\Omega^w\hat{A}_c\hat{\nabla}^c\Omega. \tag{10.33}$$

Potential is related to electromagnetic spinor ϕ_{AB} by relation (8.25):

$$\phi_{AB} = \nabla_{A'(A}A_{B)}^{A'} = \Omega^{w+1}\hat{\nabla}_{A'(A}\hat{A}_{B)}^{A'} + w\Omega^w\hat{A}_B^{A'}\hat{\nabla}_{AA'}\Omega.$$

Thus, if we set $w = 0$, equation (8.25) will be conformally invariant with weight 1 in the sense

$$\phi_{AB} = \Omega\,\hat{\nabla}_{A'(A}\hat{A}_{B)}^{A'}.$$

Moreover we can define

$$\hat{\phi}_{AB} = \hat{\nabla}_{A'(A}\hat{A}_{B)}^{A'}.$$

To summarize, we postulate following behaviour of electromagnetic field and the potential:

$$\begin{aligned}
A_a &= \hat{A}_a, \quad & \phi_{AB} &= \nabla_{A'(A}A_{B)}^{A'}, \\
\phi_{AB} &= \Omega\,\hat{\phi}_{AB}, \quad & \hat{\phi}_{AB} &= \hat{\nabla}_{A'(A}\hat{A}_{B)}^{A'}.
\end{aligned} \tag{10.34}$$

Again, we assume that unphysical quantities are of order $\mathcal{O}(1)$ near $\mathcal{I}^+$. Then, for physical components of the potential we obtain

$$
\begin{aligned}
A_0 &= A_{AA'}\, o^A \bar{o}^{A'} \;=\; \Omega^2\, \hat{A}_{AA'} \hat{o}^A \hat{\bar{o}}^{A'} \;=\; \mathcal{O}\left(\Omega^2\right), \\
A_1 &= A_{AA'}\, o^A \bar{\iota}^{A'} \;=\; \Omega\, \hat{A}_{AA'} \hat{o}^A \hat{\bar{\iota}}^{A'} \;=\; \mathcal{O}(\Omega), \\
A_{\bar{1}} &= A_{AA'}\, \iota^A \bar{o}^{A'} \;=\; \Omega\, \hat{A}_{AA'} \hat{\iota}^A \hat{\bar{o}}^{A'} \;=\; \mathcal{O}(\Omega), \\
A_2 &= A_{AA'}\, \iota^A \bar{\iota}^{A'} \;=\; \Omega\, \hat{A}_{AA'} \hat{\iota}^A \hat{\bar{\iota}}^{A'} \;=\; \mathcal{O}(1).
\end{aligned} \tag{10.35}
$$

Similarly, for electromagnetic spinor we find

$$
\begin{aligned}
\phi_0 &= \phi_{AB}\, o^A\, o^B \;=\; \Omega^3\, \hat{\phi}_{AB}\, \hat{o}^A\, \hat{o}^B \;=\; \mathcal{O}\left(\Omega^3\right), \\
\phi_1 &= \phi_{AB}\, o^A\, \iota^B \;=\; \Omega^2\, \hat{\phi}_{AB}\, \hat{o}^A\, \hat{\iota}^B \;=\; \mathcal{O}\left(\Omega^2\right), \\
\phi_2 &= \phi_{AB}\, \iota^A\, \iota^B \;=\; \Omega\, \hat{\phi}_{AB}\, \hat{\iota}^A\, \hat{\iota}^B \;=\; \mathcal{O}(\Omega).
\end{aligned} \tag{10.36}
$$

Equation (8.42) is genuinely non-conformally-invariant and so we have to prescribe conformal behaviour of the scalar field on physical grounds. Natural requirement is that the scalar field vanishes at infinity, so we postulate

$$
\phi \;=\; \Omega\, \hat{\phi} \tag{10.37}
$$

and therefore

$$
\phi \;=\; \mathcal{O}(\Omega)\,, \tag{10.38}
$$

provided that $\hat{\phi}$ is regular on $\mathcal{I}^+$. Components of the gradient $\varphi_a = \nabla_a \phi$ then behave according to formulae (recall (10.22) and (10.24))

$$
\varphi_0 \;=\; \mathcal{O}\left(\Omega^2\right), \quad \varphi_1 \;=\; \mathcal{O}\left(\Omega^2\right), \quad \varphi_{\bar{1}} \;=\; \mathcal{O}\left(\Omega^2\right), \quad \varphi_2 \;=\; \mathcal{O}(\Omega). \tag{10.39}
$$

We expand these quantities as follows:

$$
\phi = \phi^0\, \Omega \,+\, \phi^1\, \Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.40a}
$$

$$
\varphi_0 = \varphi_0^0\, \Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right) \;=\; -\,\phi^0\, \Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.40b}
$$

$$
\varphi_1 = \varphi_1^0\, \Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.40c}
$$

$$
\varphi_{\bar{1}} = \varphi_{\bar{1}}^0\, \Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.40d}
$$

$$
\varphi_2 = \varphi_2^0\, \Omega \,+\, \varphi_2^1\, \Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right). \tag{10.40e}
$$

The Weyl spinor is conformally invariant with zero weight[1]:

$$
\Psi_{ABCD} \;=\; \hat{\Psi}_{ABCD}.
$$

[1] This depends on the conventions used. What is convention-independent is the behaviour of the Weyl tensor $C_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'}$. In the non-abstract index formalism, components of tensors are related to components of spinors via van der Waerden symbols $\sigma_a^{AA'}$ which can have a conformal weight and thus they affect the conformal weight of Ψ_{ABCD}, as in, e.g. [10].

Under certain weak assumptions it is possible to show[17] that $\hat{\Psi}_{ABCD}$ vanishes on $\mathcal{I}^+$ and therefore is of order $\mathcal{O}(\Omega)$. Then the components of the Weyl tensor behave as

$$\begin{aligned}
\Psi_0 = \Psi_{ABCD} o^A o^B o^C o^D \;&=\; \Omega^4 \hat{\Psi}_0 \;=\; \mathcal{O}\left(\Omega^5\right), \\
\Psi_1 = \Psi_{ABCD} o^A o^B o^C \iota^D \;&=\; \Omega^3 \hat{\Psi}_1 \;=\; \mathcal{O}\left(\Omega^4\right), \\
\Psi_2 = \Psi_{ABCD} o^A o^B \iota^C \iota^D \;&=\; \Omega^2 \hat{\Psi}_2 \;=\; \mathcal{O}\left(\Omega^3\right), \\
\Psi_3 = \Psi_{ABCD} o^A \iota^B \iota^C \iota^D \;&=\; \Omega^1 \hat{\Psi}_3 \;=\; \mathcal{O}\left(\Omega^2\right), \\
\Psi_4 = \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D \;&=\; \hat{\Psi}_4 \;=\; \mathcal{O}(\Omega).
\end{aligned} \tag{10.41}$$

Asymptotic behaviour of the components of the Ricci spinor can be found from Einstein's equations (9.12):

$$\begin{aligned}
\Phi_{00} &= \mathcal{O}\left(\Omega^4\right), \\
\Phi_{01} &= \mathcal{O}\left(\Omega^4\right), \\
\Phi_{11} + 3\Lambda &= \mathcal{O}\left(\Omega^2\right), \\
\Phi_{02} &= \mathcal{O}\left(\Omega^4\right), \\
\Phi_{12} &= \mathcal{O}\left(\Omega^3\right), \\
\Phi_{22} &= \mathcal{O}\left(\Omega^2\right).
\end{aligned} \tag{10.42}$$

Behaviour of scalar curvature Λ is found from (9.11) to be

$$\Lambda = \mathcal{O}\left(\Omega^2\right). \tag{10.43}$$

This completes our discussion of conformal behaviour of physical and geometrical quantities used in the calculation. In the following section we expand all quantities in the series in Ω, substitute them into the field equations in the NP formalism introduced in the previous sections and find the coefficients of those expansions.

10.6 Asymptotic solution

Ricci identities (2.92a) and (2.92b) in our tetrad reduce to

$$D\rho = \rho^2 + \sigma\,\overline{\sigma} + \Phi_{00}, \tag{10.44a}$$

$$D\sigma = 2\,\rho\,\sigma + \Psi_0. \tag{10.44b}$$

We expand the scalar field ϕ into the series

$$\phi = \phi^0\,\Omega + \mathcal{O}\left(\Omega^2\right) \tag{10.45}$$

so that

$$\varphi_0 \;=\; D\phi = -\,\phi^0\,\Omega^2 + \mathcal{O}\left(\Omega^3\right) \tag{10.46}$$

and

$$\Phi_{00} = \phi^0\,\overline{\phi}^0\,\Omega^4 + \mathcal{O}\left(\Omega^5\right). \tag{10.47}$$

Similarly, we expand ρ and σ into

$$\rho = -\,\Omega \,+\, \rho^0\,\Omega^3 \,+\, \mathcal{O}\left(\Omega^4\right), \tag{10.48a}$$
$$\sigma = s\,\Omega^2 \,+\, s^1\,\Omega^3 \,+\, \mathcal{O}\left(\Omega^4\right). \tag{10.48b}$$

Substituting these expression into (10.44a) and (10.44b) we find

$$\rho^0 \;=\; -\,s\,\bar{s} - \phi^0\,\overline{\phi}^0, \qquad s^1 \;=\; 0.$$

Recall that when we constructed the coordinate system, we introduced arbitrary coordinates x^I on the cut $\mathcal{S}_0$ of $\mathcal{I}^+$. Since the cut $\mathcal{S}_0$ is a two-sphere, it is convenient to identify x^I with standard spherical coordinates[2] θ and ϕ. Metric tensor on the two-sphere has components

$$^{(2)}g_{IJ} \;=\; \mathrm{diag}\left(-1, -\sin^2\theta\right).$$

Let us define functions p^I by

$$p^2 \;=\; \frac{1}{\sqrt{2}}, \quad p^3 \;=\; -\,\frac{i}{\sqrt{2}}\,\frac{1}{\sin\theta} \tag{10.49}$$

so that vectors $\hat{m} = p^I\partial_I$ and $\hat{\overline{m}} = \overline{p}^I\partial_I$ satisfy

$$\hat{m}^I\,\hat{m}_I \;=\; 0, \quad \hat{\overline{m}}^I\,\hat{m}_I \;=\; -1.$$

We know by (10.24) that vector $m = P^I\partial_I$ is of order $\mathcal{O}\left(\Omega\right)$, so it can be expanded into

$$P^I \;=\; \Omega\,p^I \,+\, \Omega^2\,q^I \,+\, \mathcal{O}\left(\Omega^3\right) \tag{10.50}$$

where p^I can be specified freely and higher order terms are determined by the frame equation (10.13c). Hence, we choose p^I to be (10.49) which implies

$$\delta \;=\; \Omega\,\hat{\delta} \,+\, \Omega^2\,q^I\,\partial_I \,+\, \mathcal{O}\left(\Omega^3\right), \quad \text{where} \quad \hat{\delta} \;=\; \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta} \,-\, \frac{i}{\sin\theta}\,\frac{\partial}{\partial\phi}\right). \tag{10.51}$$

Finally, using the frame equation (10.13c) we find

$$\delta \;=\; \Omega\,\hat{\delta} \,-\, s\,\Omega^2\,\hat{\overline{\delta}} + \mathcal{O}\left(\Omega^3\right), \quad \overline{\delta} \;=\; \Omega\,\hat{\overline{\delta}} \,-\, \overline{s}\,\Omega^2\,\hat{\delta} \,+\, \mathcal{O}\left(\Omega^3\right). \tag{10.52}$$

Let us expand coefficients α and β into series as usually. Taking (10.12) into account we arrive at following expansions:

$$\alpha = a\,\Omega \,+\, a^1\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.53a}$$
$$\beta = b\,\Omega \,+\, b^1\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.53b}$$
$$\pi = (a+\overline{b})\Omega \,+\, (a^1+\overline{b}^1)\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.53c}$$
$$\tau = (\overline{a}+b)\Omega \,+\, (\overline{a}^1+b^1)\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right). \tag{10.53d}$$

Ricci identites (2.92d) and (2.92e) simplify to

$$D\alpha = \rho\alpha + \beta\overline{\sigma} + \rho\pi + \Phi_{10}, \tag{10.54a}$$
$$D\beta = (\alpha+\pi)\sigma + \rho\beta + \Psi_1. \tag{10.54b}$$

[2]Unfortunately, symbol ϕ represents both angle ϕ and the scalar field ϕ. However, the meaning of symbol ϕ should be clear from the context where it appears.

These identities now imply

$$b \;=\; -\,\overline{a}, \quad b^1 \;=\; -\,a\,s. \tag{10.55}$$

Coefficient a can be determined from the frame equation (10.13f) for $I = 2, 3$:

$$a \;=\; \overline{a}, \quad a \;=\; -\frac{\cot\theta}{2\sqrt{2}}. \tag{10.56}$$

In Chapter 3 we introduced operators Þ and ð acting on scalar quantities depending on their boost and spin weights. Consistently with this notation, we introduce operator ð associated with the leading term of operator δ, i.e. with operator $\hat{\delta}$. Acting on scalar η of spin weight w, operators ð and $\overline{ð}$ are defined by

$$ð\eta \;=\; \hat{\delta} + 2wa\eta, \quad \overline{ð}\eta \;=\; \hat{\overline{\delta}} - 2wa\eta. \tag{10.57}$$

It is straightforward to show that coefficient σ has the spin weight 2 and its complex conjugate has weight -2.

Higher term of α can be found from the Ricci identity (2.92p) which in our coordinates reads

$$\delta\rho - \overline{\delta}\sigma = (\overline{\alpha} + \beta)\rho - (3\alpha - \overline{\beta})\sigma - \Psi_1 + \Phi_{01}. \tag{10.58}$$

This equation gives

$$a^1 = ð\overline{s} \,+\, a\,\overline{s}. \tag{10.59}$$

Thus, coefficients α, β, π and τ have expansions

$$\alpha = a\,\Omega \,+\, (ð\overline{s} + a\overline{s})\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.60a}$$
$$\beta = -\,a\,\Omega \,-\, a\,s\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.60b}$$
$$\pi = ð\overline{s}\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.60c}$$
$$\tau = \overline{ð}s\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.60d}$$

where

$$a = -\frac{\cot\theta}{2\sqrt{2}}. \tag{10.61}$$

These expansion are, in fact, identical with corresponding expansions of solution to vacuum Einstein's equations, see [17].

Now we can use the Bianchi identity (2.72a) which simplifies to

$$D\Psi_1 - \overline{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} = (\pi - 4\alpha)\Psi_0 + 4\rho\Psi_1 + \overline{\pi}\Phi_{00} - 2\sigma\Phi_{10} - 2\rho\Phi_{01} \tag{10.62}$$

This equation is satisfied up to order $\mathcal{O}\left(\Omega^5\right)$ and in order $\mathcal{O}\left(\Omega^6\right)$ we reveal the equation

$$\begin{aligned}\Psi_1^1 = &-\overline{ð}\Psi_0^0 + 3\phi_0^0\overline{\phi}_1^0 + \phi^0\overline{\phi}^0\left(3e^2A_0^0A_1^0 - \overline{ð}s\right) - \phi^1\overline{ð}\overline{\phi}^0 - \overline{\phi}^1ð\phi^0\\ &+\frac{1}{2}\left(\phi^0ð\overline{\phi}^1 + \overline{\phi}^0ð\phi^1\right) - \frac{1}{2}s\overline{ð}(\phi^0\overline{\phi}^0)\\ &+\frac{3}{2}ie\left(A_0^0\phi^0ð\overline{\phi}^0 - A_0^0\overline{\phi}^0ð\phi^0 + A_1^0\phi^1\overline{\phi}^0 - A_1^0\phi^0\overline{\phi}^1\right).\end{aligned} \tag{10.63}$$

This is already a difference from the vacuum case, when only term $\overline{\eth}\Psi_0^0$ is present. For free electromagnetic field our result simplifies to $\Psi_1^1 = 3\phi_0^0\overline{\phi}_1^0 - \overline{\eth}\Psi_0^0$ which is consistent with expansions presented in [12], page 394.

Next, we expand functions C^I into the series

$$C^I \;=\; c_0^I\Omega \,+\, c_1^I\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right)$$

and the frame equation (10.13b) shows

$$c_0^2 \;=\; 0, \quad c_1^2 \;=\; -\frac{1}{\sqrt{2}}\left(\eth\overline{s} + \overline{\eth}s\right), \tag{10.64a}$$

$$c_0^3 \;=\; 0, \quad c_1^3 \;=\; -\frac{i}{\sqrt{2}\sin\theta}\;\left(\eth\overline{s} + \overline{\eth}s\right) \tag{10.64b}$$

so that

$$C^I\partial_I = -\,\Omega^2\left[(\overline{\eth}s)\hat{\overline{\delta}} + (\eth\overline{s})\hat{\delta}\right] \,+\, \mathcal{O}\left(\Omega^3\right). \tag{10.65}$$

Let us expand γ as

$$\gamma = \gamma^0 \,+\, \gamma^1\Omega \,+\, \gamma^2\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right) \tag{10.66}$$

and use the Ricci identity (2.92f) which now reduces to

$$D\gamma = 2\overline{\pi}\alpha + 2\pi\beta + \pi\tau + \Psi_2 - \Lambda + \Phi_{11}. \tag{10.67}$$

From this identity we find

$$\gamma^1 = 0, \tag{10.68a}$$

$$\gamma^2 = a\eth\overline{s} - a\overline{\eth}s - \frac{1}{2}\Psi_2^0 + \frac{1}{6}\partial_u\left(\phi^0\overline{\phi}^0\right). \tag{10.68b}$$

Similarly we expand μ and λ as

$$\mu = \mu^0 \,+\, \mu^1\,\Omega \,+\, \mu^2\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.69a}$$

$$\lambda = \lambda^0 \,+\, \lambda^1\,\Omega \,+\, \lambda^2\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right) \tag{10.69b}$$

and use the Ricci identities (2.92h) and (2.92g) which simplify to

$$D\mu - \delta\pi = \rho\mu + \sigma\lambda + 2\beta\pi + \Psi_2 + 2\Lambda, \tag{10.70a}$$

$$D\lambda - \overline{\delta}\pi = \rho\lambda + \mu\overline{\sigma} + 2\alpha\pi + \Phi_{20}. \tag{10.70b}$$

These equations give

$$\mu^0 \;=\; 0, \quad \lambda^0 \;=\; 0. \tag{10.71}$$

Ricci identity (2.92q),

$$\delta\alpha - \overline{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\alpha\beta - \Psi_2 + \Lambda + \Phi_{11}, \tag{10.72}$$

implies

$$\mu^1 = \frac{1}{2}. \tag{10.73}$$

Coefficient γ^0 now vanishes by the Ricci identity (2.92n):

$$\gamma^0 = 0. \tag{10.74}$$

From the Ricci identity (2.92m) we obtain

$$\lambda^1 = \dot{\overline{s}} \tag{10.75}$$

where the dot means differentiation with respect to coordinate u which represents the (retarded) time. The Ricci identity (2.92r) reduces to

$$\delta\lambda - \overline{\delta}\mu = \mu\pi + (\overline{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21} \tag{10.76}$$

and we find

$$\Psi_3^0 = -\hat{\delta}\dot{\overline{s}} + 4a\dot{\overline{s}}. \tag{10.77}$$

Recall that $\overline{s}$ has the spin weight -2. Since the time derivative does not involve contraction with m^a or $\overline{m}^a$, it does not change the spin weight and we can write

$$\Psi_3^0 = -\eth\dot{\overline{s}}. \tag{10.78}$$

Returning to equation (10.70b), we find

$$\lambda^2 = -\hat{\overline{\delta}}\eth\overline{s} + 2a\eth\overline{s} + \frac{1}{2}\overline{s}. \tag{10.79}$$

Since $\overline{s}$ has the spin weight -2, quantity $\eth\overline{s}$ has the spin weight -1 because operator $\eth$ acts as the spin-raising operator; we can write

$$\lambda^2 = \overline{\eth}\eth\overline{s} + \frac{1}{2}\overline{s}. \tag{10.80}$$

Finally, coefficient μ^2 is determined by the Ricci identity (10.72):

$$\mu^2 = -\Psi_2^0 - \eth^2\overline{s} - s\dot{\overline{s}} - \frac{1}{6}\partial_u(\phi^0\overline{\phi}^0). \tag{10.81}$$

Now we expand the metric function H which is of order $\mathcal{O}(1)$ into the series

$$H = h^0 + h^1\,\Omega + h^2\,\mathcal{O}\left(\Omega^3\right). \tag{10.82}$$

The leading term h^0 is found from the Ricci identity (2.92n) to be

$$h^0 = \frac{1}{2}. \tag{10.83}$$

Next term is naturally found from the frame equation (10.13a):

$$h^1 = -\frac{1}{2}\Psi_2^0 - \frac{1}{2}\overline{\Psi}_2^0 + \frac{1}{3}\partial_u(\phi^0\overline{\phi}^0). \tag{10.84}$$

The Bianchi identity (2.72g) implies

$$\begin{aligned}\dot{\Psi}_2^0 = \phi_2^0\overline{\phi}_2^0 - \frac{1}{3}\,\dot{\phi}^0\dot{\overline{\phi}}^0 - \eth^2\dot{\overline{s}} + s\,\Psi_4^0 - \frac{1}{6}\left(\overline{\phi}^0\ddot{\phi}^0 + \phi^0\ddot{\overline{\phi}}^0\right)\\ + ieA_2^0\left(\phi^0\dot{\overline{\phi}}^0 - \overline{\phi}^0\dot{\phi}^0\right) + e^2\,(A_2^0)^2\phi^0\,\overline{\phi}^0.\end{aligned} \tag{10.85}$$

10.7 Summary

In this final section of the chapter we summarize all expansions. Spin coefficients:

$$\rho = -\Omega \,-\, \left(s\overline{s} + \phi^0\overline{\phi}^0\right)\Omega^3 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.86a}$$

$$\sigma = s\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.86b}$$

$$\alpha = a\,\Omega \,+\, (\eth\overline{s} + a\overline{s})\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.86c}$$

$$\beta = -\,a\,\Omega \,-\, a\,s\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.86d}$$

$$\pi \;=\; \overline{\tau} = (\eth\overline{s})\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.86e}$$

$$\lambda = \dot{\overline{s}}\,\Omega \,+\, \left(\frac{\overline{s}}{2} + \overline{\eth}\eth\overline{s}\right)\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.86f}$$

$$\mu = \frac{1}{2}\,\Omega \,-\, \left(\eth^2\overline{s} + s\dot{\overline{s}} + \Psi_2^0 \,+\, \frac{1}{6}\partial_u(\phi^0\overline{\phi}^0)\right)\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right), \tag{10.86g}$$

$$\gamma = \left(a\eth\overline{s} - a\overline{\eth}s - \frac{1}{2}\Psi_2^0 + \frac{1}{6}\partial_u(\phi^0\overline{\phi}^0)\right)\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right) \tag{10.86h}$$

Ricci scalars:

$$\Phi_{11} = -\,\frac{1}{4}\partial_u\left(\phi^0\overline{\phi}^0\right)\Omega^3 \,+\, \mathcal{O}\left(\Omega^4\right), \tag{10.87a}$$

$$\Lambda = \frac{1}{12}\partial_u\left(\phi^0\overline{\phi}^0\right)\Omega^3 \,+\, \mathcal{O}\left(\Omega^4\right), \tag{10.87b}$$

Weyl scalars:

$$\Psi_3 = -\eth\dot{\overline{s}}\,\Omega^2 \,+\, \mathcal{O}\left(\Omega^3\right). \tag{10.88a}$$

10.8 The Bondi mass

Using the expansions found in the previous chapter, we can now turn to evaluation of the Penrose charge integral (6.25)

$$\begin{aligned} Q_{\mathcal{S}}[\alpha^A,\beta^B] \;&=\; \frac{1}{8\pi G}\oint\limits_{\mathcal{S}} R_{abcd}f^{cd} \\ &=\; \frac{i}{4\pi G}\oint\limits_{\mathcal{S}} \left[\alpha^0\beta^0(\Phi_{01}-\Psi_1) + (\alpha^0\beta^1+\alpha^1\beta^0)(\Phi_{11}-\Psi_2+\Lambda) + \alpha^1\beta^1(\Phi_{21}-\Psi_3)\right]\mathrm{d}\mathcal{S}. \end{aligned} \tag{10.89}$$

The Bondi mass arises as the limit of this integral for $\Omega \to 0$, i.e. when the surface $\mathcal{S}$ is chosen to be the cut of $\mathcal{I}^+$ at arbitrary time u. In order to find this limit we have to investigate asymptotic properties of the integrand. The main results were find in the previous chapter where we have found asymptotic expansions of all relevant components of the Riemann tensor. The only remaining thing is to determine asymptotic behaviour of the surface element $\mathrm{d}\mathcal{S}$ and the behaviour of components α^A and β^A.

By (6.23), the surface element $\mathrm{d}\mathcal{S}$ is given by

$$\mathrm{d}\mathcal{S} \;=\; {}^{(2)}\epsilon_{ab} \;=\; \Omega^{-2}\, i(\hat{\epsilon}_{AB}\hat{\bar{o}}_{A'}\hat{\bar{\iota}}_{B'} - \hat{\epsilon}_{A'B'}\hat{o}_A\hat{\iota}_B) \;=\; \Omega^{-2}\,{}^{(2)}\hat{\epsilon}_{ab} \;=\; \mathcal{O}\left(\Omega^{-2}\right).$$

Thus, the surface element diverges as Ω^{-2} at null infinity $\mathcal{I}^+$. In order to obtain any meaningful notion of the mass-energy, the integrand must be of order $\mathcal{O}\left(\Omega^2\right)$: for lower order integrand we would obtain diverging quantity, for higher order integrand we would obtain zero.

In chapter 6 we explained how to choose the spinors α^A and β^A; they are two independent solutions of tangential projections of the univalent twistor equation, i.e. they are 2-surface twistors satisfying tangential parts of equations

$$\nabla^{(A}_{A'}\alpha^{B)} \;=\; 0, \qquad \nabla^{(A}_{A'}\beta^{B)} \;=\; 0.$$

In section 4.1 we have established the conformal invariance of the twistor equation provided that spinors α^A and β^A transform as

$$\alpha_A \;=\; \Omega^{-1}\,\hat{\alpha}_A, \quad \beta_A \;=\; \Omega^{-1}\,\hat{\beta}_A, \quad \text{or, equivalently,} \quad \alpha^A \;=\; \hat{\alpha}^A, \quad \beta^A \;=\; \hat{\beta}^A. \tag{10.90}$$

As usually, we assume that unphysical spinors are regular on $\mathcal{I}^+$. We decompose spinor α^A as

$$\alpha^A \;=\; \alpha^0\, o^A \;+\; \alpha^1\, \iota^A$$

and simultaneously

$$\hat{\alpha}^A \;=\; \hat{\alpha}^0\, \hat{o}^A \;+\; \hat{\alpha}^1\, \hat{\iota}^A.$$

Contracting equation $\hat{\alpha}^A = \alpha^A$ with o^A and ι^A and having relations (10.16) on mind (and applying the same consideration to spinor β^A)

$$\begin{aligned} \alpha^0 &= \Omega^{-1}\hat{\alpha}^0, \quad & \alpha^1 &= \hat{\alpha}^1, \\ \beta^0 &= \Omega^{-1}\hat{\beta}^0, \quad & \beta^1 &= \hat{\beta}^1. \end{aligned} \tag{10.91}$$

Now, expanding the integrand in (10.89) we find

$$\alpha^0\beta^0\,(\Phi_{01} - \Psi_1)\,\mathrm{d}S = -\,\hat{\alpha}^0\,\hat{\beta}^0\,\left(\frac{1}{2}\eth(\phi^0\overline{\phi}^0) + \Psi_1^0\right)\mathrm{d}\hat{\mathcal{S}} \;+\; \mathcal{O}\left(\Omega\right), \tag{10.92a}$$

$$2\alpha^{(0}\beta^{1)}\,(\Phi_{11} - \Psi_2 + \Lambda)\,\mathrm{d}S = -\,2\hat{\alpha}^{(0}\hat{\beta}^{1)}\left(\frac{1}{6}\partial_u(\phi^0\overline{\phi}^0) \;+\; \Psi_2^0\right)\mathrm{d}\hat{\mathcal{S}} \;+\; \mathcal{O}\left(\Omega\right), \tag{10.92b}$$

$$\alpha^1\beta^1\,(\Phi_{21} - \Psi_3)\,\mathrm{d}S = \hat{\alpha}^1\hat{\beta}^1\,\left(\eth\dot{\bar{s}}\right)\mathrm{d}\hat{\mathcal{S}} \;+\; \mathcal{O}\left(\Omega\right). \tag{10.92c}$$

Thus, on $\mathcal{I}$ we obtain a finite and non-zero limit of the Penrose charge integral. The energy-momentum obtained in this way is called *the Bondi energy-momentum* and its zeroth component is usually refered to as the *Bondi mass.* The details of this construction are described in the paper [16] attached to this diploma thesis and will not be repeated here.

11. On the Bondi mass of Maxwell-Klein-Gordon spacetimes

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On the Bondi mass of Maxwell-Klein-Gordon spacetimes

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Abstract In this paper we calculate the Bondi mass of asymptotically flat spacetimes with interacting electromagnetic and scalar fields. The system of coupled Einstein-Maxwel-Klein-Gordon equations is investigated and corresponding field equations are written in the spinor form and in the Newman-Penrose formalism. Asymptotically flat solution of the resulting system is found near null infinity. Finally we use the asymptotic twistor equation to find the Bondi mass of the spacetime and derive the Bondi mass-loss formula. We compare the results with our previous work [4] and show that, unlike the conformal scalar field, the (Maxwell-)Klein-Gordon field has negatively semi-definite mass-loss formula.

Keywords Asymptotic flatness · Einstein-Maxwell-Klein-Gordon equations · Bondi mass

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1 Introduction

It is a well known fact that the energy-momentum of gravitational field cannot be introduced at the *local* level which is, after all, the consequence of the equivalence principle. Since it is highly desirable to have a meaningful notion of the energy and the momentum, many suggestions have been made in order to define the *quasi-local* energy-momentum which is associated with, e.g., a compact spacelike hypersurface Σ with boundary S, rather than with a spacetime point. The quasi-local quantities are usually expressed as the surface integrals over the 2-surface S. The most influential suggestions are, for example, those of Penrose [19], Hawking [11], Dougan and Mason [8] and Brown and York[6]. For extensive reviews on the subject, see [25, 16].

On the other hand, in the case of asymptotically flat spacetimes there is a well-defined notion of *global* energy-momentum associated with the entire spacetime (ADM mass [2] defined at spatial infinity) or energy-momentum associated with an isolated gravitating source (Bondi mass [5] defined at null infinity). Hence, one of the natural criteria of the plausibility of particular quasi-local energy-momentum is whether it coincides with the ADM mass or the Bondi mass in the limit of the large spheres near spatial or null infinity [25].

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Standard expression for the Bondi energy-momentum of electro-vacuum spacetimes in the Newman-Penrose formalism has the form

$$P^{\mathbf{AA}'} = -\oint_S \left(\Psi_2^{(0)} + \sigma^{(0)}\,\dot{\bar{\sigma}}^{(0)}\right)\,\omega_0^{\mathbf{A}}\,\overline{\omega}_0^{\mathbf{A}'}\,\mathrm{d}S,$$

where $\Psi_2^{(0)}$ is the leading $\mathscr{O}\left(r^{-3}\right)$ term in the asymptotic expansion of the Ψ_2-component of the Weyl spinor, $\sigma^{(0)}$ is the asymptotic shear of Newman and Penrose, $\omega_A^{\mathbf{0}}$ and $\omega_A^{\mathbf{1}}$ are asymptotic spinors [25] and $\omega_0^{\mathbf{A}} = \omega_A^{\mathbf{A}} o^A$, where o^A is the element of GHP spinor dyad [10]. The dot means the derivative with respect to (retarded) time u. In [4] we have shown that this result remains true for the spacetimes with conformally invariant scalar field sources. In the presence of the massless Klein-Gordon scalar field, however, the scalar field contributes to the Bondi energy and the correct expression for the Bondi mass (energy) is (in the conventions used in this paper)

$$M_B = -\frac{1}{2\sqrt{\pi}}\oint_S \left(\Psi_2^{(0)} + \sigma^{(0)}\,\dot{\bar{\sigma}}^{(0)} + \frac{1}{6}\partial_u(\phi^{(0)}\,\bar{\phi}^{(0)})\right)\mathrm{d}S, \tag{1}$$

where $\phi^{(0)}$ is now the leading $\mathscr{O}\left(r^{-1}\right)$ term in the asymptotic expansion of the scalar field.

A crucial property of the Bondi energy is that it should decrease whenever the system emits gravitational (or another) radiation. As we have shown in [4], in the case of massless Klein-Gordon field the mass-loss formula acquires the form

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}}\oint \left(\dot{\sigma}^{(0)}\,\dot{\bar{\sigma}}^{(0)} + \dot{\phi}^{(0)}\,\dot{\bar{\phi}}^{(0)}\right)\mathrm{d}S, \tag{2}$$

so that the Bondi mass is a non-increasing function of time u. For the conformally invariant scalar field, resulting "mass-loss" formula is indefinite and reads

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}}\oint_S \left(\dot{\sigma}^{(0)}\,\dot{\bar{\sigma}}^{(0)} + 2(\dot{\phi}^{(0)})^2 - \phi^{(0)}\,\ddot{\phi}^{(0)}\right)\mathrm{d}S. \tag{3}$$

Hence, in this case the Bondi mass is not a monotonic function of time, which can be traced back to the fact that the energy-momentum tensor for the conformally invariant scalar field does not obey the energy condition $T_{ab}l^a n^b \geq 0$ for any future null vectors l^a and n^a.

In this paper we investigate the natural generalization of these calculations and we calculate the Bondi mass of the spacetimes with interacting electromagnetic and scalar fields. The purpose is twofold. It seems that the analysis of the Bondi mass of Maxwell-Klein-Gordon spacetimes in the Newman-Penrose formalism is missing (see, however, [7,14] for some results on the scalar field in the Hamiltonian formalism). Hence, our first goal is to fill this gap.

The Penrose mass has been calculated for a wide class of spacetimes in [28,26,27], but the spacetimes with scalar field sources are not included. In fact, only a very few exact solutions of coupled Einstein-Maxwell-Klein-Gordon equations are known, e.g. [9]. On the other hand, there is a chance that at least some properties of the Penrose mass can be understood without having an exact solution. The idea is to apply standard 3+1 decomposition of the spacetime with electromagnetic and scalar field sources and analyse the constraints which must be satisfied on the initial Cauchy hypersurface. The 2-surface S can be chosen to lie in this initial hypersurface and one can hope that the constraints will be easier to solve than the full set of equations. In this context, the present paper is a preliminary work: the Penrose mass calculated by the analysis sketched in this paragraph can be examined to have the correct large sphere limit.

The paper is organized as follows. In the section 2 we introduce standard equations governing the system of coupled gravitational, electromagnetic and scalar fields and translate them into the spinor formalism. In the appendix A we present the Newman-Penrose projections of these equations. Next we consider an asymptotically flat spacetime with the electromagnetic and scalar field sources which is analytic at the future null infinity $\mathscr{I}^+$. The asymptotic behaviour of the Newman-Penrose quantities describing the gravitational, scalar and electromagnetic fields is investigated in the section 3. In the next section 4 we present the asymptotic solution of Einstein-Maxwell-Klein-Gordon equations and finally in the section 5 we calculate the Bondi mass of the spacetime and find corresponding mass-loss formula which is presented both in terms of the four-potential and in the gauge invariant form.

2 Field equations

In this section we introduce field equations of interacting electromagnetic, scalar and gravitational fields in the spinor form. Resulting system of equations will be referred to as the *Einstein-Maxwell-Klein-Gordon equations* and corresponding spacetime will be called *electro-scalar spacetime* for the sake of brevity.

The gauge invariant Lagrangian of the coupled scalar and electromagnetic fields can be written in the form [12]

$$\mathscr{L} = (\mathscr{D}_a\phi)(\mathscr{D}^a\overline{\phi}) - m^2\,\phi\,\overline{\phi} - \frac{1}{4}F^{ab}F_{ab}\,, \tag{4}$$

where ϕ is a scalar field with charge e, $\overline{\phi}$ its complex conjugate with charge $-e$, m is the mass of the scalar field and F_{ab} is standard Faraday 2-form. When acting on the uncharged fields, the gauge covariant derivative $\mathscr{D}_a$ coincides with the usual covariant derivative ∇_a, otherwise its action on an arbitrary tensor field $T^{a..b}_{c..d}$ with the charge e is defined by [20]

$$\mathscr{D}_f T^{a..b}_{c..d} = \nabla_f T^{a..b}_{c..d} + i\,e\,A_f\,T^{a..b}_{c..d}\,, \tag{5}$$

with A_a being the four-potential. The Lagrangian (4) yields, through the standard Euler-Lagrange equations, familiar field equations

$$\left(\mathscr{D}_a\mathscr{D}^a + m^2\right)\phi = 0, \qquad \left(\mathscr{D}_a\mathscr{D}^a + m^2\right)\overline{\phi} = 0, \qquad \nabla^a F_{ab} = i\,e\left(\overline{\phi}\,\mathscr{D}_b\phi - \phi\,\mathscr{D}_b\overline{\phi}\right). \tag{6}$$

Next we wish to rewrite these equations as a system of first-order spinorial equations. Electromagnetic spinor ϕ_{AB} is related to the potential A_a by

$$\phi_{AB} = \nabla_{X'(A}A^{X'}_{B)}. \tag{7}$$

We reduce the gauge freedom imposing standard Lorenz condition $\nabla_a A^a = 0$, so that the equation (7) simplifies to

$$\nabla^{A'}_{A}A_{BA'} = -\,\phi_{AB}\,. \tag{8}$$

Because we prefer our equations to be of the first order, we retain both ϕ_{AB} and $A_{AA'}$ in future formulae and equation (8) will be regarded as a dynamical equation for the potential A_a. Spinor form of (6) then implies the equation for ϕ_{AB}:

$$\nabla^{A}_{B'}\phi_{AB} = \frac{ie}{2}\left(\overline{\phi}\,\mathscr{D}_b\phi - \phi\,\mathscr{D}_b\overline{\phi}\right) = \frac{ie}{2}\left(\overline{\phi}\,\varphi_b - \phi\,\overline{\varphi}_b\right) - e^2\,\phi\,\overline{\phi}\,A_b. \tag{9}$$

In order to derive first-order equations for the scalar field, we introduce notation (cf. [4])

$$\varphi_a = \nabla_a \phi \quad \text{and} \quad \varphi_{AA'} = \nabla_{AA'} \phi \tag{10}$$

which eliminates formally the second derivatives of the scalar field ϕ that are present in the equations (6). These equations are equivalent to the wave equation

$$\Box\phi = -2\, i\, e A^a\, \varphi_a + \left(e^2 A^a A_a - m^2\right) \phi \tag{11}$$

and its complex conjugate. At this point we could employ the Newman-Penrose formalism and express $\Box\phi$ with the help of only the first derivatives of φ_a and the spin coefficients. However, it is more convenient to decompose spinor $\nabla^A_{A'} \varphi_{AB'}$ into its symmetric and antisymmetric parts,

$$\nabla^A_{A'} \varphi_{AB'} = \nabla^A_{(A'} \varphi_{B')A} + \frac{1}{2}\, \varepsilon_{A'B'}\, \nabla^A_{X'} \varphi^{X'}_A = -\Box_{A'B'}\phi - \frac{1}{2}\, \varepsilon_{A'B'}\, \Box\phi,$$

and use $\Box_{A'B'}\phi = 0$. (Commutator $\Box_{AB} = \nabla_{X'(A} \nabla^{X'}_{B)}$ annihilates scalar quantities.) Now the scalar equation (11) is equivalent to the spinor equation

$$\nabla^A_{A'} \varphi_{AB'} = i\, e A^c \varphi_c\, \varepsilon_{A'B'} + \frac{1}{2}\left(m^2 - e^2 A^c A_c\right) \phi\, \varepsilon_{A'B'}. \tag{12}$$

If, on the other hand, we apply the procedure of spinor decomposition to covariant derivatives $\mathscr{D}_a = \mathscr{D}_{AA'}$ in the Klein-Gordon equation (6), we arrive at somewhat more elegant formula

$$\mathscr{D}^{X'}_A \mathscr{D}_{BX'} \phi = \frac{1}{2}\, m^2\, \phi\, \varepsilon_{AB} - i\, e\, \phi\, \phi_{AB}. \tag{13}$$

Here, the Lorenz condition *has not* been imposed and equation (13) is manifestly gauge-invariant.

Now we turn our attention to equations of gravitational field which is described by the Newman-Penrose spin coefficients, the Weyl spinor Ψ_{ABCD}, the Ricci spinor $\Phi_{ABA'B'}$ and the scalar curvature $\Lambda = R/24$. Equations for the spin coefficients follow from the spinorial form of the Ricci identities [22]

$$\Box_{CD} \xi_A = \Psi_{ABCD}\, \xi^B - 2\Lambda\, \varepsilon_{A(C}\, \xi_{D)}, \qquad \Box_{C'D'} \xi_A = \Phi_{ABC'D'}\, \xi^B, \tag{14}$$

where ξ_A is chosen to be one of the basis spinors o_A and ι_A. The Weyl spinor and the Ricci spinor satisfy the Bianchi identities

$$\nabla^D_{A'} \Psi_{ABCD} = \nabla^{B'}_{(A} \Phi_{BC)A'B'}, \qquad \nabla^{BB'} \Phi_{ABA'B'} = -3\, \nabla_{AA'} \Lambda. \tag{15}$$

Moreover, the Ricci spinor and the scalar curvature are related to the energy-momentum tensor by the Einstein equations [3]

$$\Phi_{ABA'B'} = 4\pi\, T_{(AB)(A'B')}, \qquad 3\Lambda = \pi\, T_{XY'}{}^{XY'}. \tag{16}$$

In order to obtain the energy-momentum tensor T_{ab} we vary the action of the electro-scalar field with the Lagrangian (4) with respect to the metric g^{ab}. This yields (cf. [12])

$$\begin{aligned} T_{ab} &= \frac{1}{4\pi}\left[\left(\mathscr{D}_{(a}\phi\right)\left(\mathscr{D}_{b)}\overline{\phi}\right) - \frac{1}{2} F_{ac}\, F_b{}^c - \frac{1}{2}\, g_{ab}\, \mathscr{L}\right] \\ &= \frac{1}{4\pi}\left[\left(\mathscr{D}_{(a}\phi\right)\left(\mathscr{D}_{b)}\overline{\phi}\right) + \phi_{AB}\, \overline{\phi}_{A'B'} - \frac{1}{2}\, g_{ab}\left(\mathscr{D}_c\phi\right)\left(\mathscr{D}^c\overline{\phi}\right) + \frac{1}{2}\, m^2\, g_{ab}\, \phi\overline{\phi}\right]. \end{aligned} \tag{17}$$

where the factor $(4\pi)^{-1}$ has been included for convenience. Using the Einstein equations (16) we find that the Ricci spinor and the scalar curvature are given by relations

$$\Phi_{ABA'B'} = \left(\mathscr{D}_{(A(A'}\phi\right)\left(\mathscr{D}_{B')B)}\overline{\phi}\right) + \phi_{AB}\,\overline{\phi}_{A'B'}, \qquad \Lambda = \frac{1}{12}\left[-\left(\mathscr{D}_a\phi\right)\left(\mathscr{D}^a\overline{\phi}\right) + 2m^2\,\phi\,\overline{\phi}\right]. \tag{18}$$

To summarize, the unknown variables representing the matter fields are the potential A_a governed by equation (8), the electromagnetic spinor ϕ_{AB} governed by (9) and the scalar field satisfying (13). Corresponding Newman-Penrose projections are summarized in the appendix A, equations (73), (76) and (78). The components of the Ricci spinor and the scalar curvature are given by (18) and their are listed explicitly in the Newman-Penrose form in the appendix A, equations (79) and (80). The Weyl spinor and the Ricci spinor satisfy the Bianchi identities (15). Corresponding Newman-Penrose equations [20,22] are listed in the appendix for the reference purposes.

3 Asymptotic behaviour of the fields

We are interested in a weakly asymptotically simple solution of the Einstein-Maxwell-Klein-Gordon equations which is analytic[1] in the neighbourhood of the future null infinity $\mathscr{I}^+$. We employ the notation $(\widehat{M},\widehat{g}_{ab})$ for the unphysical spacetime and (M,g_{ab}) for the physical one, where, by assumption of weak asymptotic simplicity, the two metrics are related by conformal rescaling

$$\widehat{g}_{ab} = \Omega^2 g_{ab}. \tag{19}$$

To proceed further we need to establish a coordinate system and the Newman-Penrose null tetrad in a neighbourhood of $\mathscr{I}^+$. In accordance with [22] we introduce coordinates $x^\mu = (u,r,x^2,x^3)$, where $x^I, I = 2, 3$, are arbitrary coordinates on the 2-sphere, u is an affine parameter along null generators of $\mathscr{I}^+$ and r is an affine parameter along null hypersurfaces intersecting $\mathscr{I}^+$ in cuts $u = $ constant. Vector l^a is chosen to be tangent to these null hypersurfaces and orthogonal to the cuts of constant (both) u and r. Null vectors m^a and $\overline{m}^a$ are chosen so as to span the tangent space of these cuts. Resulting null tetrad has the following properties.

- l^a and n^a are real and null vectors normalized by $l^a n_a = 1$. Vector m^a and its complex conjugate $\overline{m}^a$ are null and complex, satisfying the condition $m^a\overline{m}_a = -1$. Remaining scalar products between these four vectors are all zero. Their components with respect to the basis induced by the coordinates (u,r,x^2,x^3) read

$$l^\mu = (0,1,0,0), \quad n^\mu = (1,H,C^2,C^3), \quad m^\mu = (0,0,P^2,P^3), \quad \overline{m}^\mu = (0,0,\overline{P}^2,\overline{P}^3). \tag{20}$$

- There exists a spin basis (o^A, ι^A) such that

$$l^a = o^A\overline{o}^{A'}, \quad n^a = \iota^A\overline{\iota}^{A'}, \quad m^a = o^A\overline{\iota}^{A'}, \quad \overline{m}^a = \iota^A\overline{o}^{A'}. \tag{21}$$

- Functions H, C^I and P^I are subject to the frame equations :

$$DH = -\gamma - \overline{\gamma}, \tag{22a}$$

$$DC^I = 2\pi P^I + 2\overline{\pi}\overline{P}^I, \tag{22b}$$

$$DP^I = \rho P^I + \sigma\overline{P}^I, \tag{22c}$$

$$\Delta P^I - \delta C^I = (\gamma - \overline{\gamma} - \mu)P^I - \overline{\lambda}\overline{P}^I, \tag{22d}$$

$$\delta H = -\overline{\nu}, \tag{22e}$$

$$\overline{\delta}P^I - \delta\overline{P}^I = (\alpha - \overline{\beta})P^I + (\beta - \overline{\alpha})\overline{P}^I, \tag{22f}$$

[1] In order to calculate the Bondi mass, the analyticity is not necessary and weaker assumptions on the differentiability of the solution could be imposed. In what follows we use the analyticity to argue that the mass of the Klein-Gordon field must be zero.

where we have used the standard Newman-Penrose notation

$$l^a\nabla_a = D, \quad n^a\nabla_a = \Delta, \quad m^a\nabla_a = \delta, \quad \overline{m}^a\nabla_a = \overline{\delta}. \tag{23}$$

– Some of the spin coefficients get simplified:

$$\varepsilon = 0, \quad \kappa = 0, \quad \mu = \overline{\mu}, \quad \rho = \overline{\rho}, \quad \overline{\pi} = \tau = \overline{\alpha} + \beta. \tag{24}$$

– In accordance with (19) we choose a spin basis in the unphysical spacetime

$$\widehat{o}^A = \Omega^{-1}\, o^A, \quad \widehat{\iota}^A = \iota^A, \quad \widehat{o}_A = o_A, \quad \widehat{\iota}_A = \Omega\, \iota_A. \tag{25}$$

Associated unphysical null tetrad then reads

$$\widehat{l}^a = \Omega^{-2}\, l^a, \quad \widehat{n}^a = n^a, \quad \widehat{m}^a = \Omega^{-1}\, m^a. \tag{26}$$

We assume that unphysical spinors $\widehat{o}^A$ and $\widehat{\iota}^A$ are regular on $\mathscr{I}^+$ which implies that physical spinor $o^A = \Omega\widehat{o}^A$ vanishes on $\mathscr{I}^+$ while the spinor ι^A remains non-vanishing there.

– In the neighbourhood of $\mathscr{I}^+$ we can use the conformal factor Ω as a coordinate instead of r by setting $\mathrm{d}\Omega/\mathrm{d}r = -\Omega^2$. The Newman-Penrose operators (acting on scalars) then read

$$D = -\Omega^2\partial_\Omega, \quad \Delta = \partial_u - \Omega^2 H\, \partial_\Omega + C^I\partial_I, \quad \delta = P^I\partial_I. \tag{27}$$

In particular, we have

$$D\Omega = -\,\Omega^2, \quad \Delta\Omega = -\,\Omega^2 H, \quad \delta\Omega = \overline{\delta}\Omega = 0. \tag{28}$$

In addition, by (25) we have

$$C^I = \mathscr{O}(\Omega), \quad P^I = \mathscr{O}(\Omega). \tag{29}$$

Next we establish the asymptotic behaviour of the spin coefficients under the assumption that unphysical spin coefficients are regular on $\mathscr{I}^+$, i.e. they are of order $\mathscr{O}(1)$. Under the conformal rescaling, the spin coefficients transform as

$$\begin{aligned}
&\kappa = \Omega^3\,\widehat{\kappa}, && \tau = \Omega\,\widehat{\tau} + \widehat{\delta}\Omega, && \sigma = \Omega^2\,\widehat{\sigma}, && \rho = \Omega^2\,\widehat{\rho} + \Omega\,\widehat{D}\Omega,\\
&\varepsilon = \Omega^2\,\widehat{\varepsilon}, && \gamma = \widehat{\gamma} + \Omega^{-1}\widehat{\Delta}\Omega, && \beta = \Omega\,\widehat{\beta}, && \alpha = \Omega\,\widehat{\alpha} + \widehat{\overline{\delta}}\Omega,\\
&\pi = \Omega\,\widehat{\pi} - \widehat{\overline{\delta}}\Omega, && \nu = \Omega^{-1}\,\widehat{\nu}, && \mu = \widehat{\mu} - \Omega^{-1}\,\widehat{\Delta}\Omega, && \lambda = \widehat{\lambda}.
\end{aligned} \tag{30}$$

These relations have been derived using the definitions of spin coefficients, the rule for the transformation of the covariant derivative [21,22] and the behaviour of the spin basis (25). Derivatives with the hats are operators associated with the unphysical spin basis $\widehat{o}^A$ and $\widehat{\iota}^A$. We assume the order $\mathscr{O}(1)$ for all unphysical quantities.

In the tetrad introduced above, coefficients ε and κ vanish and thus, by (30), their unphysical counterparts $\widehat{\varepsilon}$ and $\widehat{\kappa}$ vanish as well. Moreover, by (28) we have

$$\tau = \Omega\widehat{\tau} = \mathscr{O}(\Omega), \quad \pi = \Omega\widehat{\pi} = \mathscr{O}(\Omega), \quad \alpha = \Omega\widehat{\alpha} = \mathscr{O}(\Omega), \quad \beta = \Omega\widehat{\beta} = \mathscr{O}(\Omega).$$

For the coefficients γ, μ and λ we find

$$\gamma = \widehat{\gamma} - \Omega H = \mathscr{O}(1), \quad \mu = \widehat{\mu} + \Omega H = \mathscr{O}(1), \quad \lambda = \widehat{\lambda} = \mathscr{O}(1). \tag{31}$$

The coefficient ν is apparently divergent on $\mathscr{I}^-$,

$$\nu = \Omega^{-1}\,\widehat{\nu} = \mathscr{O}\left(\Omega^{-1}\right), \tag{32}$$

because of (25), but we will show that in fact $\nu = \mathscr{O}\left(\Omega^2\right)$. The coefficient σ is of the order

$$\sigma = \Omega^2\,\widehat{\sigma} = \mathscr{O}\left(\Omega^2\right). \tag{33}$$

Finally, for the coefficient ρ we have (see [22])

$$\rho = \Omega^2\,\widehat{\rho} - \Omega = -\Omega + \mathscr{O}\left(\Omega^3\right).$$

Let us now turn to the asymptotic behaviour of the matter fields. Appropriate conformal transformation of the four-potential A_a is $\widehat{A}_a = A_a$, so that the unphysical electromagnetic spinor $\widehat{\phi}_{AB}$ is

$$\widehat{\phi}_{AB} = \widehat{\nabla}_{X'(A}\widehat{A}^{X'}_{B)} = \Omega\,\phi_{AB}.$$

Assuming that the unphysical quantities are of the order $\mathscr{O}(1)$ near $\mathscr{I}^+$, for the Newman-Penrose components of the potential we obtain

$$A_0 = A_a l^a = \mathscr{O}\left(\Omega^2\right), \quad A_1 = A_a m^a = \mathscr{O}(\Omega), \quad A_{\bar{1}} = A_a \overline{m}^a = \mathscr{O}(\Omega), \quad A_2 = A_a n^a = \mathscr{O}(1). \tag{34}$$

Similarly, for the electromagnetic spinor we find standard asymptotic behaviour in the form

$$\phi_0 = \phi_{AB} o^A o^B = \mathscr{O}\left(\Omega^3\right), \quad \phi_1 = \phi_{AB} o^A \iota^B = \mathscr{O}\left(\Omega^2\right), \quad \phi_2 = \phi_{AB} \iota^A \iota^B = \mathscr{O}(\Omega). \tag{35}$$

The spinor form of the Klein-Gordon equation (12) is genuinely not conformally-invariant and so we have to prescribe the conformal behaviour of the scalar field on the physical grounds. Natural requirement [4] is that the physical scalar field vanishes at infinity, so we postulate

$$\phi = \Omega\,\widehat{\phi} = \mathscr{O}(\Omega), \tag{36}$$

assuming that $\widehat{\phi}$ is regular on $\mathscr{I}^+$. Components of the gradient $\varphi_a = \nabla_a \phi$ then behave according to the formulae (recall (27) and (29))

$$\varphi_0 = \mathscr{O}\left(\Omega^2\right), \quad \varphi_1 = \mathscr{O}\left(\Omega^2\right), \quad \varphi_{\bar{1}} = \mathscr{O}\left(\Omega^2\right), \quad \varphi_2 = \mathscr{O}(\Omega), \tag{37}$$

where the Newman-Penrose components of the field φ_a are defined by (75).

The Weyl spinor is conformally invariant with zero weight[2]:

$$\Psi_{ABCD} = \widehat{\Psi}_{ABCD}.$$

Under certain weak assumptions it is possible to show [22] that $\widehat{\Psi}_{ABCD}$ vanishes on $\mathscr{I}^+$ so that smoothness shows it is of order $\mathscr{O}(\Omega)$. Hence, for the Weyl tensor we obtain usual asymptotic behaviour

$$\Psi_0 = \mathscr{O}\left(\Omega^5\right), \quad \Psi_1 = \mathscr{O}\left(\Omega^4\right), \quad \Psi_2 = \mathscr{O}\left(\Omega^3\right), \quad \Psi_3 = \mathscr{O}\left(\Omega^2\right), \quad \Psi_4 = \mathscr{O}(\Omega). \tag{38}$$

Asymptotic behaviour of the components of the Ricci spinor can be found from Einstein's equations (79):

$$\begin{aligned} &\Phi_{00} = \mathscr{O}\left(\Omega^4\right), && \Phi_{01} = \mathscr{O}\left(\Omega^4\right), && \Phi_{11} = \mathscr{O}\left(\Omega^2\right), \\ &\Phi_{02} = \mathscr{O}\left(\Omega^4\right), && \Phi_{12} = \mathscr{O}\left(\Omega^3\right), && \Phi_{22} = \mathscr{O}\left(\Omega^2\right). \end{aligned}$$

Behaviour of the scalar curvature Λ is found from (18) to be

$$\Lambda = \mathscr{O}\left(\Omega^2\right). \tag{39}$$

This completes the discussion of the conformal behaviour of physical and geometrical quantities used in the calculation.

[2] This depends on the conventions used. What is convention-independent is the behaviour of the Weyl tensor $C_{abcd} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'}$. In the non-abstract index formalism, components of tensors are related to components of spinors via van der Waerden symbols $\sigma_a^{AA'}$ which can have a conformal weight and thus they affect the conformal weight of Ψ_{ABCD}, as in, e.g. [18].

4 Asymptotic solution

In this section we present the asymptotic solution of Einstein-Maxwell-Klein-Gordon equations introduced in the section 2. Let X be any Newman-Penrose scalar quantity which is of the order $\mathscr{O}(\Omega^n)$. Then, assuming analyticity of the solution, we expand this quantity into the series in coordinate Ω in the neighbourhood of $\mathscr{I}$:

$$X = \sum_{k=0}^{\infty} X^{(k)}\,\Omega^{n+k}. \tag{40}$$

Expanding all Newman-Penrose quantities[3] in this way and using the field equations we find the coefficients $X^{(0)}, X^{(1)}, \ldots$ in the leading terms of expansions (40).

At the first stage we employ the Ricci identities (81a), (81b), (81c), (81d) and (81r) and the frame equation (22f) which yield the following expansions of the spin coefficients ρ, σ, α and β:

$$\rho = -\Omega - \left(\sigma^{(0)}\overline{\sigma}^{(0)} + \phi^{(0)}\overline{\phi}^{(0)}\right)\Omega^3 - \left(\phi^{(0)}\,\overline{\phi}^{(1)} + \phi^{(1)}\,\overline{\phi}^{(0)}\right)\Omega^4 + \mathscr{O}\left(\Omega^5\right), \tag{41a}$$

$$\sigma = \quad \sigma^{(0)}\,\Omega^2 + \left(\sigma^{(0)2}\,\overline{\sigma}^{(0)} - \frac{1}{2}\Psi_0^{(0)} + \sigma^{(0)}\,\phi^{(0)}\,\overline{\phi}^{(0)}\right)\Omega^4 + \mathscr{O}\left(\Omega^5\right), \tag{41b}$$

$$\alpha = \quad a\,\Omega + \left(\bar{\eth}\overline{\sigma}^{(0)} + a\overline{\sigma}^{(0)}\right)\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{41c}$$

$$\beta = -a\,\Omega - a\,\sigma^{(0)}\,\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{41d}$$

$$\pi = \overline{\tau} = \quad (\bar{\eth}\overline{\sigma}^{(0)})\,\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{41e}$$

where $\sigma^{(0)}$ is the asymptotic shear of Newman and Penrose [1,17] and

$$a = -\frac{\cot\theta}{2\sqrt{2}}.$$

Operators $\eth$ and $\bar{\eth}$ are defined by relations [22]

$$\eth\eta = \widehat{\delta}\eta + 2\,w\,a\,\eta, \qquad \bar{\eth}\eta = \widehat{\overline{\delta}}\eta - 2\,w\,a\,\eta, \tag{42}$$

when acting on the scalar η of the spin weight w.

Now, the $\mathscr{O}\left(\Omega^2\right)$ terms in the Ricci identity (81g) give

$$m^2\,\phi^{(0)}\,\overline{\phi}^{(0)} = 0,$$

where m is the mass of the scalar field. The coefficient $\phi^{(0)}$ is the leading term in the asymptotic expansion of the scalar field and in fact represents the radiative component of the field. If we do not want to exclude the presence of the scalar radiation which is expected to contribute to the Bondi mass-loss formula, we are forced to set $m = 0$. This is in agreement with the fact that massive fields do not extend to $\mathscr{I}^+$, see [29,13,4]. Hence, in what follows we will consider only the massless scalar field.

Assuming now $m = 0$ and $\phi^{(0)} \neq 0$ and using all Ricci identities (81a)–(81r) and the frame equations (22a)–(22f) we find the asymptotic expansion of remaining spin coefficients:

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[3] By the Newman-Penrose quantities we mean five components Ψ_m, $m = 0, \ldots 4$, six independent components Φ_{mn}, $m, n = 0, 1, 2$, twelve spin coefficients, three electromagnetic components ϕ_m, $m = 0, 1, 2$, four components of the potential A_m, $m = 0, 1, \overline{1}, 2$, and the scalar field ϕ.

$$\lambda = \dot{\overline{\sigma}}^{(0)}\,\Omega + \left(\frac{1}{2}\overline{\sigma}^{(0)} - \overline{\eth}\eth\overline{\sigma}^{(0)}\right)\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{43a}$$

$$\mu = -\frac{1}{2}\,\Omega - \left(\eth^2\overline{\sigma}^{(0)} + \sigma^{(0)}\dot{\overline{\sigma}}^{(0)} + \Psi_2^{(0)} + \frac{1}{6}\partial_u(\phi^{(0)}\overline{\phi}^{(0)})\right)\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{43b}$$

$$\gamma = \left(a\eth\overline{\sigma}^{(0)} - a\overline{\eth}\sigma^{(0)} - \frac{1}{2}\Psi_2^{(0)} + \frac{1}{6}\partial_u(\phi^{(0)}\overline{\phi}^{0})\right)\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{43c}$$

$$\nu = \mathscr{O}\left(\Omega^2\right). \tag{43d}$$

Components of the metric tensor with respect to the coordinates (u, r, θ, ϕ) are given in terms of the metric functions H, C^I and P^I satisfying the frame equations (22). Their asymptotic expansions read

$$H = -\frac{1}{2} + \left(\frac{1}{3}\partial_u(\phi^{(0)}\overline{\phi}^{(0)}) - \frac{1}{2}\Psi_2^{(0)} - \frac{1}{2}\overline{\Psi}_2^{(0)}\right)\Omega + \mathscr{O}\left(\Omega^2\right), \tag{44a}$$

$$C^2 = -\frac{1}{\sqrt{2}}\left(\eth\overline{\sigma}^{(0)} + \overline{\eth}\sigma^{(0)}\right)\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{44b}$$

$$C^3 = \frac{i}{\sqrt{2}\sin\theta}\left(\eth\overline{\sigma}^{(0)} - \overline{\eth}\sigma^{(0)}\right)\Omega^2 + \mathscr{O}\left(\Omega^3\right). \tag{44c}$$

Similar expansions can be obtained for the components of the Ricci tensor and the Ricci scalar,

$$\Phi_{00} = \phi^{(0)}\,\overline{\phi}^{(0)}\,\Omega^4 + 2\left(\phi^{(1)}\,\overline{\phi}^{(0)} + \phi^{(0)}\,\overline{\phi}^{(1)}\right)\Omega^5 + \mathscr{O}\left(\Omega^6\right), \tag{45a}$$

$$\Phi_{01} = -\frac{1}{2}\eth(\phi^{(0)}\,\overline{\phi}^{(0)})\Omega^4 + \mathscr{O}\left(\Omega^5\right), \tag{45b}$$

$$\Phi_{02} = \left(-\phi_0^{(0)}\,\dot{A}_1^{(0)} + (\eth\phi^{(0)} + ieA_1^{(0)}\phi^{(0)})(\eth\overline{\phi}^{(0)} - ieA_1^{(0)}\overline{\phi}^{(0)})\right)\Omega^4 + \mathscr{O}\left(\Omega^5\right), \tag{45c}$$

$$\Phi_{11} = -\frac{1}{4}\partial_u\left(\phi^0\overline{\phi}^0\right)\Omega^3 + \mathscr{O}\left(\Omega^4\right), \tag{45d}$$

$$\Phi_{12} = \left(-\phi_1^{(0)}\,\dot{A}_1^{(0)} + \frac{1}{2}\dot{\overline{\phi}}^{(0)}(\eth\phi^{(0)} + ieA_1^{(0)}\phi^{(0)}) + \frac{1}{2}\dot{\phi}^{(0)}(\eth\overline{\phi}^{(0)} - ieA_1^{(0)}\overline{\phi}^{(0)})\right)\Omega^3 + \mathscr{O}\left(\Omega^4\right), \tag{45e}$$

$$\Phi_{22} = \left(\dot{A}_1^{(0)}\,\dot{A}_{\overline{1}}^{(0)} + \dot{\phi}^{(0)}\,\dot{\overline{\phi}}^{(0)}\right)\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{45f}$$

$$\Lambda = \frac{1}{12}\partial_u\left(\phi^0\overline{\phi}^0\right)\Omega^3 + \mathscr{O}\left(\Omega^4\right), \tag{45g}$$

and for the components of the Weyl spinor,

$$\Psi_0 = \Psi_0^{(0)}\,\Omega^5 + \Psi_0^{(1)}\,\Omega^6 + \mathscr{O}\left(\Omega^7\right), \tag{46a}$$

$$\Psi_1 = \Psi_1^{(0)}\,\Omega^4 + \Psi_1^{(1)}\,\Omega^5 + \mathscr{O}\left(\Omega^6\right), \tag{46b}$$

$$\Psi_2 = \Psi_2^{(0)}\,\Omega^3 + \mathscr{O}\left(\Omega^4\right), \tag{46c}$$

$$\Psi_3 = \Psi_3^{(0)}\,\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{46d}$$

$$\Psi_4 = \Psi_4^{(0)}\,\Omega + \Psi_4^{(1)}\,\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{46e}$$

where

$$\Psi_3^{(0)} = -\eth\dot{\overline{\sigma}}^{(0)}, \qquad \Psi_4^{(0)} = -\ddot{\overline{\sigma}}^0, \qquad \Psi_4^{(1)} = \overline{\eth}\eth\dot{\overline{\sigma}}^0 \tag{47a}$$

$$\Psi_1^{(0)} = -2\,\sigma^{(0)}\,\eth\overline{\sigma}^{(0)} + 2\,a\,\sigma^{(0)}\,\overline{\sigma}^{(0)} + (\eth + a)(\phi^{(0)}\,\overline{\phi}^{(0)}), \tag{47b}$$

$$\begin{aligned}\Psi_1^{(1)} = {} & 3\,\phi_0^{(0)}\,\overline{\phi}_1^{(0)} - \overline{\eth}\Psi_0^{(0)} - \sigma^{(0)}\eth(\phi^{(0)}\,\overline{\phi}^{(0)}) + \frac{1}{2}\left(\phi^{(0)}\,\eth\overline{\phi}^{(1)} + \overline{\phi}^{(0)}\,\eth\phi^{(1)}\right) + \frac{1}{2}\,\sigma^{(0)}\overline{\eth}(\phi^{(0)}\,\overline{\phi}^{(0)}) \\ & - \left(\phi^{(1)}\,\eth\overline{\phi}^{(0)} + \overline{\phi}^{(1)}\,\eth\phi^{(0)}\right) + \phi^{(0)}\,\overline{\phi}^{(0)}\left(3\,e^2 A_0^{(0)}\,A_1^{(0)} - \overline{\eth}\sigma^{(0)}\right) \\ & + \frac{3}{2}\,i\,e\left[A_0^{(0)}\,\phi^{(0)}\left(\eth\overline{\phi}^{(0)} - \overline{\phi}^{(1)}\right) - A_1^{(0)}\,\overline{\phi}^{(0)}\left(\eth\phi^{(0)} - \phi^{(1)}\right)\right],\end{aligned} \tag{47c}$$

$$\begin{aligned}\dot{\Psi}_2^{(0)} = {} & \frac{2}{3}\,\dot{\phi}^{(0)}\,\dot{\overline{\phi}}^{(0)} + \phi_2^{(0)}\,\overline{\phi}_2^{(0)} + \eth\Psi_3^{(0)} - \frac{1}{6}\left(\ddot{\overline{\phi}}^{(0)}\,\phi^{(0)} + \ddot{\phi}^{(0)}\,\overline{\phi}^{(0)}\right) + \sigma^{(0)}\,\Psi_4^{(0)} + e^2\,(A_2^{(0)})^2\,\phi^{(0)}\,\overline{\phi}^{(0)} \\ & + i\,e A_2{}^{(0)}\left(\phi^{(0)}\,\dot{\overline{\phi}}^{(0)} - \dot{\phi}^{(0)}\,\overline{\phi}^{(0)}\right).\end{aligned} \tag{47d}$$

For the components of electromagnetic spinor we find the following expansions:

$$\phi_0 = \phi_0^{(0)}\,\Omega^3 + \phi_1^{(1)}\,\Omega^4 + \mathscr{O}\left(\Omega^5\right), \tag{48a}$$

$$\phi_1 = \phi_1^{(0)}\,\Omega^2 + \phi_1^{(1)}\,\Omega^3 + \mathscr{O}\left(\Omega^4\right), \tag{48b}$$

$$\phi_2 = \phi_2^{(0)}\,\Omega + \phi_2^{(1)}\,\Omega^2 + \mathscr{O}\left(\Omega^3\right), \tag{48c}$$

where

$$\phi_0^{(0)} = -\sigma^{(0)}A_{\overline{1}}^{(0)} - \eth A_0^{(0)}, \tag{49a}$$

$$\phi_1^{(0)} = -\eth A_{\overline{1}}^{(0)}, \tag{49b}$$

$$\phi_2^{(0)} = \eth A_2^{(0)} - \dot{A}_{\overline{1}}^{(0)}. \tag{49c}$$

5 Bondi mass

In this section we finally construct the expression for the Bondi mass. We adopt the approach based on the asymptotic twistor equation as described in [22, 15]. The twistor equation reads

$$\nabla_{A'}{}^{(A}\omega^{B)} = 0. \tag{50}$$

Spinor ω^A can be written as a linear combination of the basis spinors,

$$\omega^A = \omega^0\,o^A + \omega^1\,\iota^A.$$

In the following we assume that the components

$$\omega^0 = -\iota_A\,\omega^A \quad \text{and} \quad \omega^1 = o_A\,\omega^A$$

are regular on $\mathscr{I}^+$. Null vector m^a has the spin weight 1 which, assuming ε_{AB} has the spin weight zero, implies that the spin weights of o^A and ι^A are $1/2$ and $-1/2$, respectively. Consequently, the components ω^0 and ω^1 have spin weights $-1/2$ and $1/2$.

Twistor equation is conformally invariant if the spinor ω^A has conformal weight zero, i.e.

$$\omega^A = \widehat{\omega}^A.$$

In order to obtain explicit form of the twistor equation (50), we project it onto the spin basis and arrive at

$$D\omega^1 = \kappa\,\omega^0 + \varepsilon\,\omega^1, \qquad \Delta\omega^0 = -\gamma\,\omega^0 - \nu\,\omega^1, \tag{51a}$$

$$\overline{\delta}\omega^0 = -\alpha\,\omega^0 - \lambda\,\omega^1, \qquad \delta\omega^1 = \sigma\,\omega^0 + \beta\,\omega^1, \tag{51b}$$

$$D\omega^0 - \overline{\delta}\omega^1 = -(\varepsilon+\rho)\omega^0 - (\alpha+\pi)\omega^1, \qquad \Delta\omega^1 - \delta\omega^0 = (\beta+\tau)\omega^0 + (\gamma+\mu)\omega^1. \tag{51c}$$

In general spacetimes, these equations do not possess a non-trivial solution. Thus, since we are interested in the Bondi mass which is defined at null infinity, we restrict the twistor equation to $\mathscr{I}$ in what follows.

Quantities ω^0 and ω^1 are regular by assumption and hence can be expanded in the neighbourhood of $\mathscr{I}$ into the series of the form

$$\omega^0 = \omega^0_0 + \omega^0_1\,\Omega + \mathscr{O}\left(\Omega^2\right), \qquad \omega^1 = \omega^1_0 + \omega^1_1\,\Omega + \mathscr{O}\left(\Omega^2\right). \tag{52}$$

Using expansions of the spin coefficients and the Newman-Penrose operators, we find that leading terms ω^0_0 and ω^1_0 satisfy relations

$$\eth\omega^1_0 = 0, \qquad \overline{\eth}\omega^1_0 = -\omega^0_0, \qquad \dot{\omega}^1_0 = 0, \qquad \omega^1_1 = 0, \tag{53}$$

where the dot denotes differentiation with respect to the variable u.

Next we define the symmetric spinor [24,23]

$$u_{AB} = \frac{1}{2}\left(\omega_{(A}\nabla^{C'}_{B)}\overline{\omega}_{C'} - \overline{\omega}_{C'}\nabla^{C'}_{(A}\omega_{B)}\right), \tag{54}$$

and the associated two form

$$\mathscr{F}_{ab} = u_{AB}\,\varepsilon_{A'B'} + \overline{u}_{A'B'}\,\varepsilon_{AB}. \tag{55}$$

Now, following [22], we choose a null hypersurface Σ which extends to $\mathscr{I}^+$ and define $S(\Omega)$ to be the two surface $\Omega =$ constant in Σ. Hence, the hypersurface Σ intersects $\mathscr{I}^+$ at the two sphere $S(0)$. In addition, we define

$$I(\Omega) = \oint_{S(\Omega)} \mathscr{F}_{ab}\, l^a n^b\, \mathrm{d}S$$

and

$$I_0 = \lim_{\Omega\to 0} I(\Omega) \tag{56}$$

if the limit exists. The Bondi four-momentum P^a is then defined by the equation

$$I_0 = P^a k_a,$$

where $k^a = \omega^A\overline{\omega}^{A'}$.

The induced volume form on the two surface $S(\Omega)$ is

$${}^{(2)}\varepsilon_{cd} = n^a l^b \varepsilon_{abcd} = i\left(\varepsilon_{C'D'}o_C\iota_D - \varepsilon_{CD}\overline{o}_{C'}\overline{\iota}_{D'}\right) = \mathscr{O}\left(\Omega^{-2}\right).$$

Thus, in order to show that the limit I_0 exists we have to show that the integrand behaves as

$$\mathscr{F}_{ab}l^a n^b = \mathscr{O}\left(\Omega^2\right).$$

Direct calculation shows

$$\mathscr{F}_{ab}l^a n^b = \rho\,\omega^0\,\overline{\omega}^0 + \mu\,\omega^1\,\overline{\omega}^1 + \Re\left(\pi\,\omega^1\,\overline{\omega}^0 + \overline{\omega}^1\,\delta\omega^0 - \overline{\omega}^0\overline{\delta}\omega^1\right).$$

Using expansions (41) and (52) we find

$$\mathscr{F}_{ab} l^a n^b = \Omega\, \Re \left[-\omega_0^0\, \overline{\varpi}_0^0 - \frac{1}{2}\, \omega_0^1\, \overline{\varpi}_0^1 + \overline{\varpi}_0^1 \eth \omega_0^0 - \overline{\varpi}_0^0 \overline{\eth} \omega_0^1 \right] + \mathscr{O}\left(\Omega^2\right). \tag{57}$$

By (53) we have

$$-\overline{\varpi}_0^0\, \overline{\eth} \omega_0^1 = \omega_0^0\, \overline{\varpi}_0^0$$

and so

$$\mathscr{F}_{ab} l^a n^b = \Omega\, \Re \left[-\frac{1}{2}\, \omega_0^1\, \overline{\varpi}_0^1 + \overline{\varpi}_0^1 \eth \omega_0^0 \right] + \mathscr{O}\left(\Omega^2\right). \tag{58}$$

Using the commutator

$$[\eth, \overline{\eth}]\, \omega_0^1 = -\frac{1}{2}\, \omega_0^1$$

and asymptotic twistor equation (53) we find

$$\eth \omega_0^0 = -\eth \overline{\eth} \omega_0^1 = -\overline{\eth} \eth \omega_0^1 + \frac{1}{2}\, \omega_0^1$$

which implies

$$\mathscr{F}_{ab} l^a n^b = \mathscr{O}\left(\Omega^2\right)$$

and hence the limit I_0 in (56) exists.

Expanding the quantity $\mathscr{F}_{ab} l^a n^b$ further we arrive at

$$\begin{aligned} \mathscr{F}_{ab} l^a n^b = \Omega^2\, \Re \Big[& -2\, \omega_0^0\, \overline{\varpi}_1^0 - \omega_0^1\, \overline{\varpi}_1^1 + \mu^1\, \omega_0^1\, \overline{\varpi}_0^1 + (\eth \overline{\sigma}^{(0)})\, \omega_0^1\, \overline{\varpi}_0^0 + \overline{\sigma}^{(0)}\, \overline{\varpi}_0^0\, \eth \omega_0^1 \\ & -\sigma^{(0)}\, \overline{\varpi}_0^1\, \overline{\eth} \omega_0^0 + \overline{\varpi}_1^1\, \eth \omega_0^0 + \overline{\varpi}_0^1\, \eth \omega_1^0 - \overline{\varpi}_1^0\, \overline{\eth} \omega_0^1 - \overline{\varpi}_0^0\, \overline{\eth} \omega_1^1 \Big] + \mathscr{O}\left(\Omega^3\right). \end{aligned} \tag{59}$$

Imposing (53) this simplifies to

$$\mathscr{F}_{ab} l^a n^b = \Omega^2\, \Re \left[-\omega_0^0\, \overline{\varpi}_1^0 + \mu^1\, \omega_0^1\, \overline{\varpi}_0^1 + (\eth \overline{\sigma}^{(0)})\, \omega_0^1\, \overline{\varpi}_0^0 - \sigma^{(0)}\, \overline{\varpi}_0^1\, \overline{\eth} \omega_0^0 + \overline{\varpi}_0^1\, \eth \omega_1^0 \right] + \mathscr{O}\left(\Omega^3\right). \tag{60}$$

Next we have

$$\Re \left[-\omega_0^0\, \overline{\varpi}_1^0 + \overline{\varpi}_0^1 \eth \omega_1^0 \right] = \Re \left[(\overline{\eth} \omega_0^1) \overline{\varpi}_1^0 + \overline{\varpi}_0^1 \eth \omega_1^0 \right] = \Re \left[\omega_0^1 \eth \overline{\varpi}_0^1 + \overline{\varpi}_0^1 \eth \omega_1^0 \right] = \Re \left[\eth (\omega_1^0 \overline{\varpi}_0^1) \right]$$

which vanishes on integration,

$$\oint \eth (\omega_1^0 \overline{\varpi}_0^1)\, \mathrm{d}\widehat{S} = 0,$$

because quantity $\omega_1^0 \overline{\varpi}_0^1$ has the spin weight -1. Thus,

$$I_0 = \oint \Re \left[\mu^1\, \omega_0^1 \overline{\varpi}_0^1 - \sigma^{(0)}\, \overline{\varpi}_0^1 \overline{\eth} \omega_0^0 + \omega_0^1\, \overline{\varpi}_0^0 \eth \overline{\sigma}^{(0)} \right] \mathrm{d}\widehat{S}. \tag{61}$$

Let us use equation (53) again to rearrange the third term of the integrand (61),

$$\oint \omega^1\, \overline{\varpi}_0^0\, \eth \overline{\sigma}^{(0)}\, \mathrm{d}\widehat{S} = -\oint \eth \left(\omega_0^1\, \overline{\sigma}^{(0)}\, \eth \overline{\varpi}_0^1 \right) \mathrm{d}\widehat{S} + \oint \overline{\sigma}^{(0)}\, \omega_0^1\, \eth \left(\eth \overline{\varpi}_0^1 \right) \mathrm{d}\widehat{S},$$

where the first integral on the right hand side vanishes because of the spin weight of the argument of the $\eth$ operator. Next we expand quantity ω_0^1 of the spin weight $1/2$ into the series in spin-weighted spherical harmonics,

$$\omega_0^1 = \sum_{l=0}^{\infty} \sum_{m=-l-1}^{l} a_{lm} \, {}_{\frac{1}{2}}Y_{l+\frac{1}{2},\, m+\frac{1}{2}},$$

where the coefficients $a_{lm}$are time-independent by (53). Since the operator $\eth$ ($\overline{\eth}$) acts as the spin raising (lowering) operator, we can write

$$\eth_s Y_{lm} = c_{slm\, s+1}Y_{lm}, \qquad \overline{\eth}_s Y_{lm} = d_{slm\, s-1}Y_{lm},$$

where particular form of coefficients c_{slm} and d_{slm} is not important. Applying $\eth$ on ω_0^1 and imposing (53) yields

$$\eth\omega_0^1 = \sum_{l=0}^{\infty} \sum_{m=-l-1}^{l} a_{lm} \, c_{\frac{1}{2}lm} \, {}_{\frac{3}{2}}Y_{l+\frac{1}{2},\, m+\frac{1}{2}} = 0.$$

Functions ${}_{\frac{3}{2}}Y_{\frac{1}{2}m}$ vanish by definition while the orthogonality of spin-weighted spherical harmonics implies

$$a_{lm} = 0 \quad \text{for} \quad l > 0.$$

The quantity ω_0^1 then acquires the form

$$\omega_0^1 = a \, {}_{\frac{1}{2}}Y_{\frac{1}{2},\, -\frac{1}{2}} + b \, {}_{\frac{1}{2}}Y_{\frac{1}{2},\, \frac{1}{2}}. \tag{62}$$

Application of $\overline{\eth}^2$ to this expansion immediately yields

$$\overline{\eth}^2 \omega_0^1 = \tilde{a} \, {}_{-\frac{3}{2}}Y_{\frac{1}{2},\, -\frac{1}{2}} + \tilde{b} \, {}_{-\frac{3}{2}}Y_{\frac{1}{2},\, \frac{1}{2}} = 0. \tag{63}$$

Hence,

$$\oint \omega_0^1 \, \overline{\omega}_0^0 \, \eth\overline{s} \, \mathrm{d}\widehat{S} = 0.$$

Finally, the last vanishing term in (61) is

$$\oint \sigma^{(0)} \, \overline{\omega}_0^1 \, \overline{\eth}\omega_0^0 \mathrm{d}\widehat{S} = -\oint \sigma^{(0)} \, \overline{\omega}_0^1 \overline{\eth}^2 \, \omega_0^1 \mathrm{d}\widehat{S} = 0$$

by (53) and (63).

Thus, we have found that the integral I_0 exists and reduces to

$$I_0 = \Re \oint \mu^1 \, \omega_0^1 \, \overline{\omega}_0^1 \mathrm{d}\widehat{S}, \tag{64}$$

where μ^1 is $\mathscr{O}\left(\Omega^2\right)$ term in (43b) so that the integral reads

$$I_0 = \oint \left(\Psi_2^0 + \sigma^{(0)} \, \dot{\overline{\sigma}}^{(0)} + \frac{1}{6} \frac{\partial}{\partial u} (\phi^0 \, \overline{\phi}^0) \right) \omega_0^1 \, \overline{\omega}_0^1 \, \mathrm{d}\widehat{S}, \tag{65}$$

where we have used

$$\oint \eth^2 \overline{s} \, \mathrm{d}\widehat{S} = 0.$$

Now, (62) implies that the spin weight zero quantity $\omega_0^1\overline{\omega}_0^1$ can be expanded as

$$\omega_0^1\,\overline{\omega}_0^1 = \alpha\,Y_{00} + \sum_{m=1}^{1} \beta_m\,Y_{1m} \tag{66}$$

for some coefficients α and β_m. Since the Bondi mass is a zeroth component of the four-momentum, we set $\beta_m = 0$ and $\alpha = 1$ which corresponds to the timelike direction. With this choice, we arrive at the expression for the Bondi mass of electro-scalar spacetimes in the form

$$M_B = \frac{1}{2\sqrt{\pi}} \oint \left(\Psi_2^0 + \sigma^{(0)}\dot{\overline{\sigma}}^{(0)} + \frac{1}{6}\frac{\partial}{\partial u}(\phi^0\,\overline{\phi}^0) \right) \mathrm{d}\widehat{S}. \tag{67}$$

Corresponding mass-loss formula is found by taking the derivative of (67) with respect to variable u and using equations (47d) and (47a):

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \oint \left[\dot{\sigma}^{(0)}\dot{\overline{\sigma}}^{(0)} + \phi_2^0\,\overline{\phi}_2^0 + \dot{\phi}^0\,\dot{\overline{\phi}}^0 + i\,e\,A_2^0 \left(\phi^0\,\dot{\overline{\phi}}^0 - \dot{\phi}^0\,\overline{\phi}^0 \right) + e^2 A_2^0 A_2^0\,\phi^0\,\overline{\phi}^0 \right] \mathrm{d}\widehat{S}. \tag{68}$$

Clearly, the first three terms represent the mass-loss by gravitational, electromagnetic and scalar radiation, while remaining terms represent the mass-loss by interactions between electromagnetic and scalar fields.

The Bondi mass-loss formula can be brought into simpler form when we define

$$\mathscr{D}_u\phi^{(0)} = \partial_u\phi^{(0)} + i\,e\,A_2^0\,\phi^{(0)}, \qquad \mathscr{D}_u\overline{\phi}^{(0)} = \partial_u\overline{\phi}^{(0)} - i\,e\,A_2^0\,\overline{\phi}^{(0)},$$

so that $\mathscr{D}_u$ is the projection of the gauge covariant derivative $n^a\mathscr{D}_a$ restricted to $\mathscr{I}$. In terms of the operator $\mathscr{D}_u$, the Bondi mass-loss formula reads

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \oint \left[\dot{\sigma}^{(0)}\dot{\overline{\sigma}}^{(0)} + \phi_2^0\,\overline{\phi}_2^0 + \left(\mathscr{D}_u\phi^{(0)}\right)\left(\mathscr{D}_u\overline{\phi}^{(0)}\right) \right] \mathrm{d}\widehat{S}. \tag{69}$$

This expression is manifestly *gauge invariant* and *negative semi-definite*. Hence, unlike the conformal scalar field with indefinite "mass-loss" formula (3), in the case of interacting electromagnetic and massless Klein-Gordon fields, the Bondi mass is either constant or decreasing function of time. Alternatively, expression (69) can be rewritten in terms of the four-potential A_a using the relation (49c):

$$\dot{M}_B = -\frac{1}{2\sqrt{\pi}} \oint \left[\dot{\sigma}^{(0)}\dot{\overline{\sigma}}^{(0)} + \dot{A}_1^{(0)}\dot{A}_{\overline{1}}^{(0)} + \dot{\phi}^{(0)}\dot{\overline{\phi}}^{(0)} \right] \mathrm{d}\widehat{S}. \tag{70}$$

In the absence of electromagnetic field, formulae (67) and (69) reduce to expressions (1) and (2) found in [4] for the massless Klein-Gordon field.

6 Conclusion

In this paper we have derived the spinor equations for the system of coupled gravitational, electromagnetic and scalar fields and found the asymptotic solution of this system in the neighbourhood of the future null infinity. The asymptotic solution reduces to the well-known expansions for electrovacuum spacetimes [21, 17] and our previous results on spacetimes with the scalar field sources [4]. Using this solution and the solution of asymptotic twistor equation, we have arrived at the expression for the Bondi mass of resulting electro-scalar spacetime, equation (67). This expression coincides with (1).

The Bondi mass-loss formula has been derived and expressed in terms of the four-potential (70) and in the gauge invariant form (69) which is manifestly negative semi-definite. This last result shows that in the case of electro-scalar spacetimes, the Bondi mass is a non-increasing function of time.

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A Field equations in the Newman-Penrose formalism

Four-potential $A_a = A_{AA'}$ is a real vector field and its components with respect to the spin basis will be denoted by

$$A_0 = A_{XX'} o^X \bar{o}^{X'}, \qquad A_1 = A_{XX'} o^X \bar{\iota}^{X'}, \tag{71a}$$
$$A_{\bar{1}} = A_{XX'} \iota^X \bar{o}^{X'}, \qquad A_2 = A_{XX'} \iota^X \bar{\iota}^{X'}. \tag{71b}$$

Similarly we introduce the Newman-Penrose components of electromagnetic spinor ϕ_{AB} by

$$\phi_0 = \phi_{AB}\, o^A\, o^B, \qquad \phi_1 = \phi_{AB}\, o^A\, \iota^B, \qquad \phi_2 = \phi_{AB}\, \iota^A\, \iota^B. \tag{72}$$

Potential A_a is governed by equation (8),

$$\nabla_A^{A'} A_{BA'} = -\,\phi_{AB}.$$

Projections of this equation onto the spin basis are

$$DA_1 - \delta A_0 = (\overline{\pi} - \overline{\alpha} - \beta)A_0 + (\varepsilon - \overline{\varepsilon} + \overline{\rho})A_1 + \sigma A_{\bar{1}} - \kappa A_2 + \phi_0, \tag{73a}$$
$$DA_2 - \delta A_{\bar{1}} = -\mu A_0 + \pi A_1 + (\overline{\pi} - \overline{\alpha} + \beta)A_{\bar{1}} + (\overline{\rho} - \varepsilon - \overline{\varepsilon})A_2 + \phi_1, \tag{73b}$$
$$\Delta A_0 - \overline{\delta} A_1 = (\gamma + \overline{\gamma} - \overline{\mu})A_0 + (\overline{\beta} - \alpha - \overline{\tau})A_1 - \tau A_{\bar{1}} + \rho A_2 - \phi_1, \tag{73c}$$
$$\Delta A_{\bar{1}} - \overline{\delta} A_2 = \nu A_0 - \lambda A_1 + (\overline{\gamma} - \gamma - \overline{\mu})A_{\bar{1}} + (\alpha + \overline{\beta} - \overline{\tau})A_2 - \phi_2. \tag{73d}$$

The Lorenz condition $\nabla^a A_a = 0$ in the Newman-Penrose formalism acquires the form

$$DA_2 - \Delta A_0 - \delta A_{\bar{1}} - \overline{\delta} A_1 = (\gamma + \overline{\gamma} - \mu - \overline{\mu})A_0 + (\pi - \alpha + \overline{\beta} - \overline{\tau})A_1 + (\overline{\pi} - \overline{\alpha} + \beta - \tau)A_{\bar{1}} + (\rho + \overline{\rho} - \varepsilon - \overline{\varepsilon})A_2 = 0. \tag{74}$$

Projections of the gradient $\varphi_{AA'} = \nabla_{AA'}\phi$ will be denoted by

$$\begin{aligned} &\varphi_0 = D\phi, \qquad \varphi_2 = \Delta\phi, \qquad \varphi_1 = \delta\phi, \qquad \varphi_{\bar{1}} = \overline{\delta}\phi, \\ &\overline{\varphi}_0 = D\overline{\phi}, \qquad \overline{\varphi}_2 = \Delta\overline{\phi}, \qquad \overline{\varphi}_1 = \delta\overline{\phi}, \qquad \overline{\varphi}_{\bar{1}} = \overline{\delta}\,\overline{\phi} \end{aligned} \tag{75}$$

Now we can complete equations for electromagnetic field. Equation (9),

$$\nabla^A_{B'}\phi_{AB} = \frac{ie}{2}\left(\overline{\phi}\,\varphi_b - \phi\,\overline{\varphi}_b\right) - e^2\,\phi\,\overline{\phi}\,A_b,$$

is the spinor version of Maxwell's equations with four-current j^a on the right hand side. Projections of this equation onto the spin basis follow:

$$D\phi_1 - \overline{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 + \frac{ie}{2}\left(\phi\overline{\varphi}_0 - \overline{\phi}\varphi_0\right) + e^2\phi\overline{\phi}A_0, \tag{76a}$$
$$D\phi_2 - \overline{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2 + \frac{ie}{2}\left(\phi\overline{\varphi}_{\bar{1}} - \overline{\phi}\varphi_{\bar{1}}\right) + e^2\phi\overline{\phi}A_{\bar{1}}, \tag{76b}$$
$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2 + \frac{ie}{2}\left(\overline{\phi}\varphi_1 - \phi\overline{\varphi}_1\right) - e^2\phi\overline{\phi}A_1, \tag{76c}$$
$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2 + \frac{ie}{2}\left(\overline{\phi}\varphi_2 - \phi\overline{\varphi}_2\right) - e^2\phi\overline{\phi}A_2. \tag{76d}$$

Dynamical equation for the gradient $\varphi_{AA'}$ is provided by equation (12)

$$\nabla^A_{A'}\varphi_{AB'} = i\,e A^c\varphi_c\,\varepsilon_{A'B'} + \frac{1}{2}\left(m^2 - e^2A^cA_c\right)\phi\,\varepsilon_{A'B'}. \tag{77}$$

Projected on the spin basis, this equation is equivalent to any of the following four scalar equations:

$$D\varphi_{\bar{1}} - \overline{\delta}\varphi_0 = (\pi - \alpha - \overline{\beta})\varphi_0 + \overline{\sigma}\varphi_1 + (\rho + \overline{\varepsilon} - \varepsilon)\varphi_{\bar{1}} - \overline{\kappa}\varphi_2, \tag{78a}$$
$$D\varphi_2 - \overline{\delta}\varphi_1 = -\overline{\mu}\varphi_0 + (\pi - \alpha + \overline{\beta})\varphi_1 + \overline{\pi}\varphi_{\bar{1}} + (\rho - \varepsilon - \overline{\varepsilon})\varphi_2 - \phi\, m^2/2 + e^2\phi\left(A_0A_2 - A_1A_{\bar{1}}\right) + ie\left(A_1\varphi_{\bar{1}} + A_{\bar{1}}\varphi_1 - A_0\varphi_2 - A_2\varphi_0\right), \tag{78b}$$
$$\Delta\varphi_0 - \delta\varphi_{\bar{1}} = (\gamma + \overline{\gamma} - \mu)\varphi_0 - \overline{\tau}\varphi_1 + (\beta - \overline{\alpha} - \tau)\varphi_{\bar{1}} + \overline{\rho}\varphi_2 - \phi\, m^2/2 + e^2\phi\left(A_0A_2 - A_1A_{\bar{1}}\right) + ie\left(A_1\varphi_{\bar{1}} + A_{\bar{1}}\varphi_1 - A_0\varphi_2 - A_2\varphi_0\right), \tag{78c}$$
$$\Delta\varphi_1 - \delta\varphi_2 = \overline{\nu}\varphi_0 + (\gamma - \overline{\gamma} - \mu)\varphi_1 - \overline{\lambda}\varphi_{\bar{1}} + (\overline{\alpha} + \beta - \tau)\varphi_2. \tag{78d}$$

The Ricci spinor is related to the electro-scalar fields by Einstein's equations (16) and is given by formula (18). The Newman-Penrose components of the Ricci spinor read:

$$\Phi_{00} = \phi_0\,\overline{\phi}_0 + (\mathscr{D}_0\phi)\left(\mathscr{D}_0\overline{\phi}\right) = \phi_0\overline{\phi}_0 + \varphi_0\overline{\varphi}_0 + e^2A_0^2\phi\overline{\phi} + ieA_0\left(\phi\overline{\varphi}_0 - \overline{\phi}\varphi_0\right), \tag{79a}$$

$$\Phi_{01} = \phi_0\,\overline{\phi}_1 + \left(\mathscr{D}_{(0}\phi\right)\left(\mathscr{D}_{1)}\overline{\phi}\right) = \phi_0\overline{\phi}_1 + \varphi_{(0}\overline{\varphi}_{1)} + e^2\phi\overline{\phi}A_0A_1 + ie\phi A_{(0}\overline{\varphi}_{1)} - ie\overline{\phi}A_{(0}\varphi_{1)}, \tag{79b}$$

$$\Phi_{11} = \phi_1\,\overline{\phi}_1 + \frac{1}{2}\left[\left(\mathscr{D}_{(0}\phi\right)\left(\mathscr{D}_{2)}\overline{\phi}\right) + \left(\mathscr{D}_{(1}\phi\right)\left(\mathscr{D}_{\overline{1})}\overline{\phi}\right)\right] \tag{79c}$$

$$= \phi_1\,\overline{\phi}_1 + \frac{1}{2}\left[\varphi_{(0}\overline{\varphi}_{2)} + \varphi_{(1}\overline{\varphi}_{\overline{1})} + ie\phi\left(A_{(0}\overline{\varphi}_{2)} + A_{(1}\overline{\varphi}_{\overline{1})}\right) - ie\overline{\phi}\left(A_{(0}\varphi_{2)} + A_{(1}\varphi_{\overline{1})}\right) + e^2\phi\overline{\phi}\left(A_0A_2 - A_1A_{\overline{1}}\right)\right], \tag{79d}$$

$$\Phi_{02} = \phi_0\overline{\phi}_2 + (\mathscr{D}_1\phi)\left(\mathscr{D}_1\overline{\phi}\right) = \phi_0\overline{\phi}_2 + \varphi_1\overline{\varphi}_1 + e^2\phi\overline{\phi}A_1^2 + ie\left(\phi A_1\overline{\varphi}_1 - \overline{\phi}A_1\varphi_1\right), \tag{79e}$$

$$\Phi_{12} = \phi_1\overline{\phi}_2 + \left(\mathscr{D}_{(1}\phi\right)\left(\mathscr{D}_{2)}\overline{\phi}\right) = \phi_1\overline{\phi}_2 + \varphi_{(1}\overline{\varphi}_{2)} + e^2\phi\overline{\phi}A_1A_2 + ie\left(\phi A_{(2}\overline{\varphi}_{1)} - \overline{\phi}A_{(2}\varphi_{1)}\right), \tag{79f}$$

$$\Phi_{22} = \phi_2\overline{\phi}_2 + (\mathscr{D}_2\phi)\left(\mathscr{D}_2\overline{\phi}\right) = \phi_2\overline{\phi}_2 + \varphi_2\,\overline{\varphi}_2 + e^2\phi\overline{\phi}A_2^2 + ieA_2\left(\phi\overline{\varphi}_2 - \overline{\phi}\varphi_2\right). \tag{79g}$$

$$6\Lambda = \varphi_{(1}\overline{\varphi}_{\overline{1})} - \varphi_{(0}\overline{\varphi}_{2)} + ie\overline{\phi}\left(A_{(0}\varphi_{2)} - A_{(1}\varphi_{\overline{1})}\right) + ie\phi\left(A_{(1}\overline{\varphi}_{\overline{1})} - A_{(0}\overline{\varphi}_{2)}\right) + e^2\phi\overline{\phi}\left(A_{\overline{1}}A_1 - A_0A_2\right) + m^2\phi\overline{\phi}. \tag{80}$$

The Ricci identities in the tetrad introduced in section 3 simplify to the following set of equations.

$$D\rho = \rho^2 + \sigma\,\overline{\sigma} + \Phi_{00}, \tag{81a}$$

$$D\sigma = 2\rho\,\sigma + \Psi_0, \tag{81b}$$

$$D\alpha = \rho\,\alpha + \beta\,\overline{\sigma} + \rho\,\pi + \Phi_{10},, \tag{81c}$$

$$D\beta = (\alpha + \pi)\,\sigma + \rho\,\beta + \Psi_1, \tag{81d}$$

$$D\gamma = 2\overline{\pi}\,\alpha + 2\pi\,\beta + \pi\,\overline{\pi} + \Psi_2 - \Lambda + \Phi_{11}, \tag{81e}$$

$$D\lambda - \overline{\delta}\pi = \rho\,\lambda + \mu\,\overline{\sigma} + 2\,\alpha\,\pi + \Phi_{20}, \tag{81f}$$

$$D\mu - \delta\pi = \rho\,\mu + \sigma\,\lambda + 2\,\beta\,\pi + \Psi_2 + 2\Lambda, \tag{81g}$$

$$D\nu - \Delta\pi = 2\pi\,\mu + 2\overline{\pi}\,\lambda + (\gamma - \overline{\gamma})\pi + \Psi_3 + \Phi_{21}, \tag{81h}$$

$$D\tau = 2\overline{\pi}\,\rho + 2\pi\,\sigma + \Psi_1 + \Phi_{01}, \tag{81i}$$

$$\Delta\rho - \overline{\delta}\tau = (\gamma + \overline{\gamma} - \mu)\rho - \sigma\,\lambda - 2\,\alpha\,\tau - \Psi_2 - 2\Lambda, \tag{81j}$$

$$\Delta\sigma - \delta\tau = -(\mu - 3\,\gamma + \overline{\gamma})\,\sigma - \overline{\lambda}\,\rho - 2\,\beta\,\tau - \Phi_{02}, \tag{81k}$$

$$\Delta\lambda - \overline{\delta}\nu = -(2\,\mu + 3\,\gamma - \overline{\gamma})\,\lambda - (3\,\alpha + \beta)\,\nu, \tag{81l}$$

$$\Delta\alpha - \overline{\delta}\gamma = \rho\,\nu - (\beta + \tau)\,\lambda + (\overline{\gamma} - \mu)\,\alpha + (\overline{\beta} - \overline{\tau})\,\gamma - \Psi_3, \tag{81m}$$

$$\Delta\beta - \delta\gamma = -\mu\,\tau + \sigma\,\nu + (\gamma - \overline{\gamma} - \mu)\,\beta - \alpha\,\overline{\lambda} - \Phi_{12}, \tag{81n}$$

$$\Delta\mu - \delta\nu = -(\mu + \gamma + \overline{\gamma})\,\mu - \lambda\,\overline{\lambda} + \overline{\nu}\,\pi + 2\,\beta\,\nu - \Phi_{22}, \tag{81o}$$

$$\delta\alpha - \overline{\delta}\beta = \mu\,\rho - \lambda\,\sigma + \alpha\,\overline{\alpha} + \beta\,\overline{\beta} - 2\,\alpha\,\beta - \Psi_2 + \Lambda + \Phi_{11}, \tag{81p}$$

$$\delta\lambda - \overline{\delta}\mu = \pi\,\mu + (\overline{\alpha} - 3\,\beta)\,\lambda - \Psi_3 + \Phi_{21},, \tag{81q}$$

$$\delta\rho - \overline{\delta}\sigma = (\overline{\alpha} + \beta)\,\rho - (3\,\alpha - \overline{\beta})\,\sigma - \Psi_1 + \Phi_{01}. \tag{81r}$$

References

1. Adamo, T.M., Kozameh, C., Newman, E.T.: Null Geodesic Congruences, Asymptotically-Flat Spacetimes and Their Physical Interpretation. Living Reviews in Relativity **12**, 6 (2009)
2. Arnowitt, R., Deser, S., Misner, C.W.: Republication of: The dynamics of general relativity. General Relativity and Gravitation **40**, 1997–2027 (2008)
3. Bičák, J., Scholtz, M., Tod, P.: On asymptotically flat solutions of Einstein's equations periodic in time: I. Vacuum and electrovacuum solutions. Classical and Quantum Gravity **27**(5), 055007 (2010). DOI 10.1088/0264-9381/27/5/055007

4. Bičák, J., Scholtz, M., Tod, P.: On asymptotically flat solutions of Einstein's equations periodic in time: II. Spacetimes with scalar-field sources. Classical and Quantum Gravity **27**(17), 175011 (2010). DOI 10.1088/0264-9381/27/17/175011
5. Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems. Royal Society of London Proceedings Series A **269**, 21–52 (1962)
6. Brown, J.D., York Jr., J.W.: Quasilocal energy and conserved charges derived from the gravitational action. Phys. Rev. D **47**, 1407–1419 (1993)
7. Chrusciel, P.T., Jezierski, J., Kijowski, J.: A Hamiltonian field theory in the radiating regime. NewYork, Springer (2002)
8. Dougan, A.J., Mason, L.J.: Quasilocal mass constructions with positive energy. Physical Review Letters **67**, 2119–2122 (1991)
9. Gegenberg, J.D., Das, A.: An exact stationary solution of the combined Einstein-Maxwell-Klein-Gordon equations. Journal of Mathematical Physics **22**, 1736–1739 (1981)
10. Geroch, R., Held, A., Penrose, R.: A space-time calculus based on pairs of null directions. Journal of Mathematical Physics **14**, 874–881 (1973)
11. Hawking, S.W.: Gravitational radiation in an expanding universe. Journal of Mathematical Physics **9**(4), 598–604 (1968)
12. Hawking, S.W., Ellis, G.F.R.: The Large Scale Structure of Space-Time. Cambridge University Press (1975)
13. Helfer, A.D.: Null infinity does not carry massive fields. Journal of Mathematical Physics **34**, 3478–3480 (1993)
14. Huang, C.: Bondi mass in scalar fields. In: The 28th International Cosmic Ray Conference in Tsukuba, pp. 3145–3148. Universal Academy Press, Inc., Tokyo, Japan (2013)
15. Huggett, S.A., Tod, K.P.: An Introduction to Twistor Theory. Cambridge University Press (1994)
16. Jaramillo, J.L., Gourgoulhon, E.: Mass and motion in general relativity. In: L. Blanchet, A. Spallicci, B. Whiting (eds.) Mass and Motion in General Relativity, chap. Mass and Angular Momentum in General Relativity. Springer (2011)
17. Newman, E.T., Penrose, R.: An approach to gravitational radiation by a method of spin coefficients. J. Math. Phys. **3**(3), 566–578 (1962)
18. Penrose, R.: Zero Rest-Mass Fields Including Gravitation: Asymptotic Behaviour. Royal Society of London Proceedings Series A **284**, 159–203 (1965). DOI 10.1098/rspa.1965.0058
19. Penrose, R.: Quasi-Local Mass and Angular Momentum in General Relativity. Royal Society of London Proceedings Series A **381**, 53–63 (1982). DOI 10.1098/rspa.1982.0058
20. Penrose, R., Rindler, W.: Spinors and space-time. Vol. 1: Two-spinor calculus and relativistic fields. Cambridge University Press (1984)
21. Penrose, R., Rindler, W.: Spinors and space-time. Volume 2: Spinor and twistor methods in space-time geometry. Cambridge University Press (1986)
22. Stewart, J.: Advanced General Relativity. Cambridge University Press (1993)
23. Szabados, L.B.: Two-dimensional Sen connections and quasi-local energy–momentum. Classical and Quantum Gravity **11**, 1847–1866 (1994). DOI 10.1088/0264-9381/11/7/020
24. Szabados, L.B.: Two-dimensional Sen connections in general relativity. Classical and Quantum Gravity **11**, 1833–1846 (1994). DOI 10.1088/0264-9381/11/7/019
25. Szabados, L.B.: Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article. Living Reviews in Relativity **7** (2004)
26. Tod, K.P.: Some Examples of Penrose's Quasi-Local Mass Construction. Royal Society of London Proceedings Series A **388**, 457–477 (1983)
27. Tod, K.P.: More on Penrose's quasi-local mass. Classical and Quantum Gravity **3**, 1169–1189 (1986). DOI 10.1088/0264-9381/3/6/016
28. Tod, K.P.: Twistors in mathematics and physics. In: T.N. Bailey, R.J. Baston (eds.) Twistors in mathematics and physics, chap. Penrose's quasi-local mass. Cambridge University Press (1990)
29. Winicour, J.: Massive fields at null infinity. Journal of Mathematical Physics **29**, 2117–2121 (1988)

Bibliography

[1] J. Bičák, M. Scholtz, and P. Tod. On asymptotically flat solutions of Einstein's equations periodic in time: I. Vacuum and electrovacuum solutions. *Classical and Quantum Gravity*, 27(5):055007, March 2010.

[2] J. Bičák, M. Scholtz, and P. Tod. On asymptotically flat solutions of Einstein's equations periodic in time: II. Spacetimes with scalar-field sources. *Classical and Quantum Gravity*, 27(17):175011, September 2010.

[3] M. Fecko. *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press, October 2006.

[4] H. Friedrich. On the global existence and the asymptotic behaviour of solutions to the einstein-maxwell-yang-mills equations. *Journal of Differential Geometry*, 34(2), 1991.

[5] R. Geroch. Asymptotic structure of space-times. In F. P. Esposito and Witten L., editors, *Asymptotic structure of space-times*. Plenum Press, 1977.

[6] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, 1975.

[7] S. A. Huggett and K. P. Tod. *An Introduction to Twistor Theory*. Cambridge University Press, 1994.

[8] M. Ludvigsen and J. A. G. Vickers. Momentum, angular momentum and their quasi-local null surface extensions. *Journal of Physics A*, 16:1155–1168, 1983.

[9] E. T. Newman and T. W. J. Unti. Behavior of Asymptotically Flat Empty Spaces. *Journal of Mathematical Physics*, 3:891–901, September 1962.

[10] R. Penrose. Zero Rest-Mass Fields Including Gravitation: Asymptotic Behaviour. *Royal Society of London Proceedings Series A*, 284:159–203, February 1965.

[11] R. Penrose and W. Rindler. *Spinors and space-time. Vol. 1: Two-spinor calculus and relativistic fields*. Cambridge University Press, 1984.

[12] R. Penrose and W. Rindler. *Spinors and space-time. Volume 2: Spinor and twistor methods in space-time geometry*. Cambridge University Press, 1986.

[13] L. H. Ryder. *Quantum Field Theory*. Cambridge University Press, June 1996.

[14] M. Scholtz. *Helical symmetry and the non-existence of asymptotically flat periodic solutions in general relativity*. PhD thesis, Faculty of Mathematics and Physics, Charles University in Prague, 2011.

[15] M. Scholtz. Periodická řešení Einsteinových rovnic. *Československý časopis pro fyziku*, 3, June 2012.

[16] M. Scholtz and L. Holka. On the Bondi mass of Maxwell-Klein-Gordon spacetimes. *General Relativity and Gravitation*, 46, 2014.

[17] J. Stewart. *Advanced General Relativity*. Cambridge University Press, 1993.

[18] L. B. Szabados. Two-dimensional Sen connections and quasi-local energy–momentum. *Classical and Quantum Gravity*, 11:1847–1866, July 1994.

[19] L. B. Szabados. Two-dimensional Sen connections in general relativity. *Classical and Quantum Gravity*, 11:1833–1846, July 1994.

[20] L. B. Szabados. Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article. *Living Reviews in Relativity*, 7, March 2004.

[21] R. M. Wald. *General relativity.* Chicago University Press, 1984.