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MASTER THESIS

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Quantum Aspects of Grand Unified Theories

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I would like to thank my supervisor Ing. Michal Malinský, Ph.D. for all his help and patience and my parents for all the support.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Kvantové aspekty teorií velkého sjednocení

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Abstrakt: V této práci se zaměřujeme na teorie velkého sjednocení s kalibrační grupou $SO(10)$, a to především na konzistenci tzv. minimálních nesupersymetrických $SO(10)$ modelů. Realistické chování takových modelů je podmíněno lokální stabilitou jejich vakua a s tím spojenou (ne)tachyonicitou jejich spektra. Jak bylo ukázáno, pro eliminaci tachyonického chování některých pseudo-Goldstoneových bosonů ve skalárním spektru daných teorií jsou důležité jednosmyčkové korekce k jejich hmotám. Hlavním předmětem předložené práce jsou dvě různé metody výpočtu těchto korekcí. Kromě zopakování výpočtu pomocí metod efektivního potenciálu pro případ Higgsova modelu $45 \oplus 16$ je zde proveden také výpočet pomocí standardní poruchové teorie. Dvojí přístup je motivován především realističtějším Higgsovým modelem $45 \oplus 126$, pro nějž je konstrukce efektivního potenciálu velmi komplikovaná, a proto je pro tento model vhodnější diagramatický postup. Tímto způsobem je v dané teorii také určen první člen vedoucí kvantové korekce ke hmotám problematických pseudo-Goldstoneových bosonů.

Klíčová slova: Teorie velkého sjednocení, efektivní potenciál, spontánní narušení kalibrační symetrie, rozšíření Standardního modelu

Title: Quantum Aspects of Grand Unified Theories

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Abstract: In this thesis we focus on Grand Unified Theories based on the $SO(10)$ gauge group and, in particular, on the viability of the minimal nonsupersymmetric $SO(10)$ models. Technically, this amounts to a detailed assessment of their vacuum stability and related (non-)tachyonicity of their scalar spectrum. It turns out that the one-loop scalar mass corrections are important for elimination of the tachyonic behaviour of certain pseudo-Goldstone bosons. In this work these issues are briefly reviewed and two distinct methods for the calculation of the critical radiative corrections are discussed. More specifically, besides the revision of the effective potential approach to the $45 \oplus 16$ Higgs model also the standard perturbative theory method is employed for this purpose. The latter approach is particularly suitable for the more realistic $45 \oplus 126$ Higgs model since it appears to be practically impossible to construct the corresponding effective potential in that case. Consequently, diagrammatic methods are used to calculate the $SO(10)$-invariant leading scalar quantum correction to the problematic pseudo-Goldstone masses in the $45 \oplus 126$ model.

Keywords: Grand unified theories, effective potential, spontaneous gauge symmetry breaking, Standard Model extensions

Contents

Introduction

> *"The effort to understand the Universe is one of the very few things which lifts human life a little above the level of farce and gives it some of the grace of tragedy."*
>
> – Steven Weinberg

The fundamental structure of matter has always been one of the most challenging questions of physics. The birth of the quantum field theory followed by the development of gauge theories can be viewed as one of the cornerstones of this endeavour. Consequently, the concept of continuous symmetry and the related theory of Lie algebras began to play a significant role in models of particle physics. These efforts resulted in the formulation of the current most complete theory of matter, the Standard Model of fundamental particles and interaction. Although all of these theories provided a number of outstanding results, there remained an intention to formulate a more general theory connecting together all the information contained in the particular gauge theories, thus resolving the issues of the Standard Model and revealing a more fundamental physics hidden beyond it. One of the paths of that quest was the attempt to formulate a universal unifying gauge theory, the so-called Grand Unified Theory.

Although the research on Grand Unified Theories (GUTs) flourished in the eighties and although this "golden era" has already passed, the GUTs still deserve significant attention because they can provide a number of testable predictions and they are related to a number of other fields of physics.

This thesis will focus specifically on the grand unification theory based on the $SO(10)$ gauge group. Namely, we will be interested in the minimal nonsupersymmetric versions of this theory. Despite the fact that these scenarios were among the earliest that were discussed and, thus, it may seem a hopeless task to find anything new to say in this context, the reality is surprisingly different. The point is that these minimal models were abandoned in early eighties because of their apparent "tachyonic instability" excluding the realistic symmetry breaking patterns. However, it was recently shown that this problematic behaviour is rather an artifact of the tree-level approximation than a serious problem of the theory. As a result, the minimal $SO(10)$ models were revived and they became again a subject of common interest.

The purpose of this thesis is to provide a detailed insight into the quantum structure of the scalar sector of the minimal nonsupersymmetric $SO(10)$ models and, in particular, to provide a method to compute the one-loop corrections to the scalar masses, which are central to the resurrection of these scenarios.

In the first chapter of the thesis a brief review is provided into the field of Grand Unified Theories, and a description of the main features in a number of models developed during several decades of research. In addition, an attempt is made to illustrate the interesting details of the minimal nonsupersymmetric $SO(10)$ scenarios based on the 45-dimensional representation.

The second chapter recapitulates the salient elements of the $SO(10)$ representation theory and the embedding of the Standard Model subgroup within it. It then describes the minimal nonsupersymmetric $45 \oplus 16$ model, its spontaneous symmetry breaking pattern and its tree level scalar spectrum. At that stage it also explains why this model has been ignored for such a long time. In the second part of this chapter a brief theoretical introduction is provided about the effective potential method.

The third chapter then deals with the calculation of the full leading radiative corrections to the masses of the problematic (i.e., potentially tachyonic at the tree level) pseudo-Goldstone bosons of the $45 \oplus 16$ model. First, the full leading corrections including the leading logarithmic terms are computed using the effective potential approach. In the second part of this chapter the diagrammatic calculation of the leading polynomial terms of these corrections is provided using the standard perturbative theory approach.

Finally, the fourth chapter outlines the structure and the main features of a more realistic $45 \oplus 126$ model. This framework focuses on the diagrammatic calculation of the $SO(10)$-invariant term of the leading one-loop scalar corrections to the masses of the potentially tachyonic pseudo-Goldstone bosons of the $45 \oplus 126$ model.

1. Theoretical and historical context

Most of our current knowledge of the structure of matter is encoded in the Standard Model of Particle Interactions. It includes the $SU(3)_C$ colour gauge theory of strong interactions and the Glashow-Salam-Weinberg electroweak theory based on a direct product of groups $SU(2)_L \otimes U(1)_Y$ [1, 2, 3, 4]. Despite the fact that it is the best theory of matter we have, experimental results show that it is not complete. A typical signal leading "beyond the Standard Model" is the non-zero neutrino mass representing one of the principal challenges for contemporary particle physics.

Moreover, there are many hints that the Standard Model is not the ultimate theory of matter. Above all, the Standard Model is quite complicated and arbitrary since the strong, weak and electromagnetic interactions are to a great extent independent of each other. This statement is supported by the well-known fact that the complete gauge group of the Standard Model is the direct product of three Lie groups $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ and consequently, one has also three different gauge couplings. In addition, there are other problems or 'unexplained features' of the Standard Model, for instance, the anomaly cancellation, which appears to be a consequence of a more general theory. Another difficulty is the pattern of fermions - the existence of fermion families cannot be explained, as well as the parity violation in case of the weak interactions, while the parity of the strong interactions remains intact. The Standard Model has many free parameters; i.e., even if one assigns the groups, representations and electric charges, there are many observable quantities (mixing angles, fermion masses, etc.) that remain undetermined.

In order to get a more general theory, which would solve at least some of the problems, it is necessary somehow to constrain the arbitrariness of the Standard Model. One possibility is to consider a model with greater symmetry. Therefore, it seems obvious to try to find a gauge theory based on a simple group G with single gauge coupling constant g_U containing the gauge group of the Standard Model as a subgroup. Such theories that intend to unify all three fundamental interactions are called Grand Unified Theories (GUTs).

In addition to the problems of the Standard Model there is a further very significant hint in favour of the grand unification: the running of the gauge couplings within the Standard Model show approximate convergence close to the energy of 10^{15}GeV (see Fig. 1.1). Therefore, it encourages a conclusion that all three corresponding interactions have a common origin.

In fact, the grand unification theories represent an improved narrative of already existing concepts. To be more specific, it is currently thought that our low-energy world is in a broken phase invariant under $SU(3)_C \otimes U(1)_{QED}$ and most of the low-energy phenomena can be described by the theory of strong interactions and effective 4-fermi contact interaction. At higher energies one can observe new degrees of freedom of a new dynamics, which can be described by a renormalizable $SU(2)_L \otimes U(1)_Y$ gauge theory, which is spontaneously broken to $U(1)_{QED}$. Similarly, as we moved from $U(1)_{QED}$ to $SU(2)_L \otimes U(1)_Y$ we can now

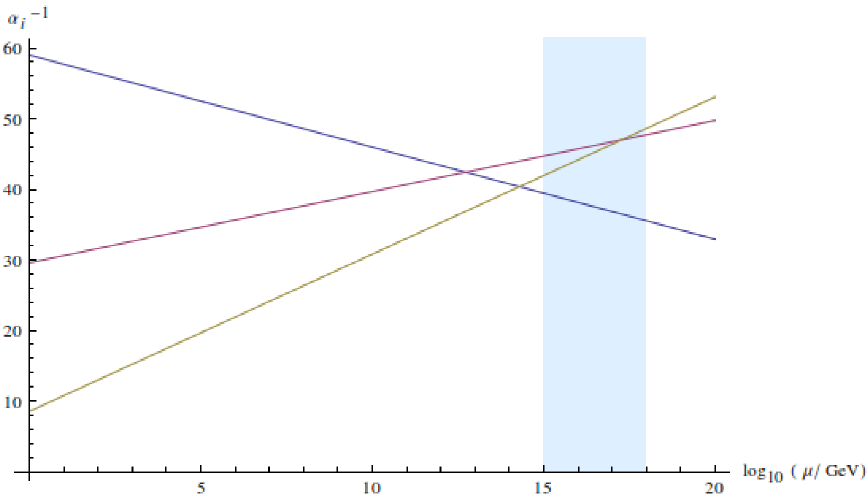


Figure 1.1: One-loop evolution of the Standard Model running gauge couplings. As usual we chose the $SU(5)$ compatible normalization of the $U(1)$ gauge coupling.

assume that the Standard Model gauge group is embedded into a simple group G. On the contrary, to get from the higher symmetry to the broken phase we implement a chain of spontaneous symmetry breaking. The concept of GUTs is similar to the Glashow-Salam-Weinberg electroweak theory. However, there is one significant difference, namely the fact that although we mostly speak about "electroweak unification", it is not a unification in the usual sense since instead of a single coupling constant there are still two distinct couplings e and g connected by the well-known "unification relation" [5]

$$\frac{e}{g} = \sin\theta_W,$$

where θ_W (electroweak mixing angle) is unfortunately a free parameter of the theory. Thus, within the electroweak theory we still deal with several different interactions.

The "Golden era of GUTs" was the period between years 1980 and 1986, when the foundations of these theories were given and the principal results were calculated. Above all, the first theoretical estimate of the proton lifetime was determined as $\tau_p \sim 10^{31}$ years [6], which was at that time very close to the experimental bound ($\sim 10^{30}$ years)[7].

Nevertheless, the pioneering idea of GUTs was first proposed by S. L. Glashow and H. Georgi even earlier, in 1974 [8]. Their unifying gauge theory was based on the gauge group $SU(5)$. The corresponding Lie algebra is the smallest simple algebra containing the algebra corresponding to the gauge group of the Standard Model. The rank of the $SU(5)$ group is equal to the rank of the Standard Model, but there are more degrees of freedom. In general, GUTs always bring new interactions into play; in other words, they predict the existence of additional gauge bosons residing in the coset of the corresponding gauge groups. Therefore, unified theories predict new phenomena, which can be correlated with the known ones. There are several model-independent predictions that are common to all unifications.

Probably the most striking prediction of GUTs is the instability of matter. The current experimental lower bound for proton lifetime is very large, about 10^{34} years [9]. As a result, the baryon number started to be considered as an exact symmetry of the Standard Model Lagrangian [10] and it is a global, anomalous symmetry within the Standard Model. However, a theory going beyond the Standard Model can naturally introduce the baryon number violation. Although there exist more hypotheses describing mechanisms of baryon number violation, the great thing about GUTs is that they provide a convenient framework and the predictions of proton lifetime are close to the experimental bounds. Explicitly, the proton lifetime τ_p can be roughly estimated as

$$\tau_p \sim \frac{1}{\alpha_u}\frac{M^4}{m_p^5},$$

where m_p is the mass of proton and the relevant baryon number violating process is mediated by gauge boson of mass M. For $\frac{1}{\alpha_u} \sim 40$ and $\tau_{proton} \sim 10^{34}$ we get $M \sim 10^{15}$ GeV, which is remarkably close to the intersection in Fig. 1.1.

The next general prediction of grand unification is magnetic monopoles [11, 12] because when the symmetry of a certain simple gauge group G is broken down to a subgroup containing a $U(1)$ factor, topologically nontrivial configurations of the Higgs and gauge fields can be found. Depending on the vacuum homotopy, stable monopole solutions of the gauge potential can exist. The monopoles appearing in GUTs have masses that are too big to be produced by accelerators. However, it is likely that they were produced during the symmetry breaking phase transition in the early universe thanks to the so-called Kibble mechanism [13].

Except for the matter instability and magnetic monopoles, the charge quantization is also a model independent prediction of grand unification. Nevertheless, if one focuses on a specific GUT model, other new physical topics can be inspected. For instance, the non-zero neutrino masses can be explained, which will be discussed later.

1.1 $SU(5)$ grand unified theory

As we have already mentioned, the Georgi-Glashow model is the first and the simplest phenomenologically viable GUT because its generators belong to the smallest simple algebra containing the algebra of the Standard Model. In the following paragraphs we outline the basic structure, principal results and highlight the problems of this theory [14].

The unification scale of the $SU(5)$ theory is quite low, as it lies approximately around 10^{15} GeV [15]. The breaking of the $SU(5)$ symmetry down to the symmetry of our low-energy world is achieved by introducing two Higgs multiplets. The first of them belongs to the 24-dimensional irreducible representation and the second to the 5-dimensional representations. Hence, the chain of symmetry breakdown reads

$$SU(5) \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \rightarrow SU(3)_C \times U(1)_{QED}.$$

Next we describe the ordering of elementary fields in the $SU(5)$ representations. The fermion families are within the framework of the $SU(5)$ theory accommodated in a reducible $\overline{5}+10$-dimensional representation. These representations can be decomposed with respect to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ group as follows

$$\begin{aligned} \overline{5} &= \left(\overline{3}, 1, \tfrac{1}{2}\right) \oplus \left(1, \overline{2}, -\tfrac{1}{2}\right), \\ 10 &= \left(\overline{3}, 1, -\tfrac{2}{3}\right) \oplus \left(3, 2, \tfrac{1}{6}\right) \oplus (1, 1, 1). \end{aligned}$$

Hence, the fields corresponding to quarks and leptons are placed together in these two representations. However, there is no place for neutrino mass because there is neither right-handed neutrino, nor a suitable Higgs field. In fact, this property of the $SU(5)$ theory was initially considered as an advantage. Nevertheless, observation of neutrino oscillations at the turn of the 21$^{\text{th}}$ century showed that neutrinos are massive and this fact should be accounted for in any viable beyond Standard Model theory. As a result, papers considering minimal extended models, which can fix the mentioned constrained, were published just a few years ago [16, 17].

Let us now comment on the quantization of the electric charge in the $SU(5)$ GUT. The operator associated with this physical quantity is one of the $SU(5)$ generators, which means its trace is equal to zero. In consequence, the charges of the particles in each multiplet have to cancel. For instance, applying this condition on the electric charge operator expressed in the $\overline{5}$ representation we find that $3Q_{d^c} + Q_e = 0$. Thus, $Q_d = \frac{1}{3}Q_e$ and the meaning of the factor 3 is, of course, the number of colours.

The 24 generators of the $SU(5)$ group correspond to 24 gauge bosons accommodated in the adjoint representation that can be decomposed with respect to $SU(3)_C \times SU(2)_L \times U(1)_Y$ as

$$24 = (8, 1, 0) \oplus (1, 3, 0) \oplus (1, 1, 0) \oplus \left(3, \overline{2}, -\tfrac{5}{6}\right) \oplus \left(\overline{3}, 2, \tfrac{5}{6}\right).$$

Obviously, the first multiplet represents the gluon octet G^{α}_{β}, the second multiplet is the weak triplet and the third term of the sum is nothing else than the Abelian field B. The last two terms are multiplets of new gauge bosons. These twelve new fields relate the $SU(2)$ and colour quantum numbers and they mediate baryon number violating processes.

An important example of a process violating the baryon number in case of the $SU(5)$ GUT is the decay of the proton into a positron and a neutral pion [18]

$$p \to e^+ + \pi^0.$$

Unfortunately, the $SU(5)$ theory has a very significant problem: it does *not* unify. It was shown that the running couplings of the $SU(5)$ theory in fact do not meet. Consequently, this theory does not unify the three interactions, which is the basic condition for a GUT that is required to be satisfied.

Another inconvenience of the $SU(5)$ theory is that the fermions of a single family cannot belong to a single fermion multiplet (irreducible representation) and the family structure of fermions is not described in a satisfactory way. Moreover, the $SU(5)$ theory does not provide any deeper explanation for the so called left-right asymmetry (the dominance of V-A currents over V+A currents) that is observed in Nature [19, 20, 21].

1.2 $SO(10)$ GUT

Although the $SU(5)$ theory is revealed to be quite problematic and eventually wrong, since it yields results which contradict experiments, it is by no means the end of the unification quest. The next candidate gauge group for grand unification is the special orthogonal group $SO(10)$. This gauge theory was proposed by H. Fritsch and P. Minkowski in 1975 [22, 23]. $SO(10)$ has rank 5; therefore, there are typically more intermediate stages along the physically interesting symmetry breaking chains.

Among the most significant virtues of the $SO(10)$ theory (when compared with the $SU(5)$) is the fact that all fermions of a single family including the desired right-handed neutrinos can be assigned to a single 16-dimensional spinorial representation [24]. Similarly, as in the case of $SU(5)$, it is possible to quantize the charges of fermions using the appropriate combination of Cartan operators of the $SO(10)$ group. However, it remains unexplained why the fermion families repeat. Anyway, the $SO(10)$ model satisfactorily describes a complete spectrum of the known elementary particles.

As we know, at the renormalizable level in the Standard Model neutrinos are strictly massless because of the absence of their right-handed components, which would allow the neutrino Dirac mass terms come into play. However, in the $SO(10)$ grand unified theory, it is possible to include naturally both Dirac and Majorana neutrino mass terms, which can consequently imply the existence of very massive right-handed neutrinos along with their already present and almost massless left-handed partners. Explicitly, the addition of the singlet Majorana mass term corresponding to a right-handed neutrino implies that all the neutrino mass terms can be expressed in a compact matrix form

$$\mathcal{L} \ni \left(\nu_L^T \;\; (\nu_L^c)^T\right) C^{-1} \begin{pmatrix} 0_{3\times3} & \left(m_\nu^D\right)_{3\times3} \\ \left(m_\nu^D\right)^T_{3\times3} & \left(m_\nu^M\right)^T_{3\times3} \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_L^c \end{pmatrix},$$

where m_ν^D is the 3×3 matrix corresponding to Dirac mass terms and m_ν^M is the 3×3 matrix corresponding to Majorana mass terms. After the diagonalization we get the famous relation for the neutrino mass

$$m_\nu \sim -m_\nu^{DT}(m_\nu^M)^{-1}m_\nu^D = -v^2 Y_\nu^T (m_\nu^M)^{-1} Y_\nu,$$

where Y_ν denotes the neutrino Yukawa coupling constant and v is the electroweak scale. It is anticipated that the neutrino Yukawa couplings have a similar size to other Standard Model Yukawas. Therefore, if one wants to get the right value of the light neutrino mass (~ 0.1eV), it is necessary to take $m_\nu^M \approx 10^{14}$GeV, which means that there is a new large physical scale. It lies suspiciously close to the scale at which the Standard Model gauge couplings converge.

This so-called see-saw mechanism [25], which introduces and describes the neutrino mass generation, is naturally included in the $SO(10)$ theory. The presence of this feature means a significant improvement in comparison with the $SU(5)$ theory. Since it is already known that left-handed neutrinos have tiny masses, the concept of the right-handed neutrino becomes very attractive. Its natural occurrence in the $SO(10)$ theory is a very pleasant feature, which started to be appreciated especially during the "neutrino revolution" in the late nineties, when oscillations of atmospheric [26] and solar [27] neutrinos were first observed and on the basis of those experiments the non-zero neutrino mass was confirmed [28, 29, 30, 31].

Moreover, besides the gauge bosons responsible for proton decay, in the $SO(10)$ theory there are also gauge bosons mediating the baryon number processes and bosons mediating the lepton number processes (independently). Within the frame of the $SO(10)$ theory we can also study the CP violation.

The $SO(10)$ group theory is discussed in Appendix A.

1.2.1 Flipped $SU(5)$

The research of the $SO(10)$ GUT led also to the formulation of another model based on the $SU(5)$ group - the so-called *flipped* $SU(5)$ *model* first contemplated by S. Barr in 1982 [32] and two years later by J. Derendinger, J. Kim and D. Nanopoulos [33]. The gauge group of this model is $SU(5) \otimes U(1)_X$, where the "flipped" embedding of the Standard Model hypercharge is considered instead of the "standard" one. Obviously, it is not a fully unified model because the gauge group is not simple.

The two-fold nature of the $SU(5)$ group springs from the fact that it is possible to embed the Standard Model hypercharge into the corresponding algebra in two different ways. While the "standard" embedding means we take $Y = T_{24}$ (T_{24} is the generator of $SU(5)$), in the "flipped" case the hypercharge reads $Y = \frac{1}{5}(X - T_{24})$. This assignment in fact interchanges $\nu^c \leftrightarrow e^c$, $u^c \leftrightarrow d^c$ etc. with respect to the "standard" field identification. As a result, the right-handed neutrino belongs to the 10-dimensional representation and the VEV of the scalar version of the multiplet $(10, 1)$ can spontaneously break the $SU(5) \otimes U(1)_X$ symmetry down to the Standard Model.

This model has the following virtues. First, there is no problem with unification of gauge couplings as it was in "standard" $SU(5)$, because now we require just the two non-Abelian couplings of the Standard Model to unify. Therefore, in contrast to the "standard" $SU(5)$ model the gauge coupling does not need the "help" of the TeV-scale supersymmetry. Second, relations between fermion masses predicted by the Yukawa sector do not contradict the observed flavour structure (which is the case of the "standard" $SU(5)$). Third, as the large-scale VEV fall into the 10-dimensional scalar representation, we cannot couple it to the pair of gauge field tensors $F_{\mu\nu}$. Hence, there is no problem with shifts of the effective gauge couplings caused by the Planck scale effects, which also implies a more accurate determination of the unification scale [34].

The interesting aspect of this model is the fact that $SU(5) \otimes U(1)$ is a subgroup of the $SO(10)$ group and the above representations can be obtained as a decomposition of the 16-dimensional spinorial representation of $SO(10)$

$$16 \to \overline{5} \oplus 10 \oplus 1.$$

The major shortcoming of the flipped $SU(5)$ model is probably the absence of a convenient mechanism of the Majorana neutrino mass term generation. One way is the inclusion of the 50-dimensional scalar field $(50, -2)$. This however leads to an unpleasant reduction of the effective Planck scale [35, 36]. In addition, it does not provide us with any insight into the neutrino mass generation.

1.3 Supersymmetry

Despite the promising predictions of GUTs published in eighties neither proton decay, nor magnetic monopoles were observed since then. As a result, the "Golden era of GUTs" was followed by a big depression of the research focused on grand unification.

Nevertheless, it did not mean the end of the GUTs. Another significant theoretical concept, which has much to do with the grand unification, is supersymmetry (SUSY). For instance, the convergence of running couplings of the Standard Model becomes perfect once supersymmetry is employed. To be more specific, in a supersymmetric framework new states occurring around the TeV scale, especially the so called higgsinos and gauginos, the supersymmetric partners of the Higgs scalars and gauge bosons, contribute to the beta-functions of the Standard Model so that all of the three gauge couplings meet at approximately 10^{16}GeV.

Consequently, at the beginning of nineties when the new data coming from the Large Electron-Positron Collider (LEP) indicated that the low-energy supersymmetry could be the right cure for the gauge coupling unification, it attracted attention of many theorists again. In the framework of supersymmetry the bosonic and fermionic degrees of freedom are treated as two sides of one fundamental object called superfield. This concept was supposed not only to unify matter and interactions, but it could be also very promising for solving other drawbacks of the Standard Model.

For all of the reasons mentioned in the last paragraphs, supersymmetry has always been closely related to unifications and from the early nineties on, almost all of the theorists' attention was paid to supersymmetric GUTs. Hence, in this section we will comment on particular SUSY GUTs and describe their principal features as well as the main differences with respect to the non-SUSY GUTs.

1.3.1 Minimal Supersymmetric Standard Model

Let us start with the minimal supersymmetric extension of the Standard Model, which represents an essential part of the historical development of supersymmetric models - so called Minimal Supersymmetric Standard Model (MSSM) [37, 38], which was first proposed in 1981. One of the nice features of this model is that it shows that supersymmetry can help resolve the hierarchy problem. This problem consists in the fact that the gauge symmetry does not protect the electroweak scale from receiving big radiative corrections. However, in the supersymmetric framework we have for each fermionic field its bosonic partner having similar mass and vice versa. Hence, the big contributions from the fermionic and bosonic loop factors vanish.

Another nice feature supporting the MSSM was the gauge coupling unification. If the superpartners of the Standard Model fields were around TeV scale, the unification of the gauge coupling at high energies would be very accurate - specifically, the accuracy is about 1%. This fact supported indirectly not only the MSSM but also the SUSY GUTs.

Yet another great theoretical motivation of the MSSM is that it introduces the R-parity to explain the proton stability. Naturally, in the MSSM the baryon and lepton numbers are no longer conserved by the renormalizable couplings.

However, the experiments show that these couplings have to be very small, as we still have not observed any processes violating the conservation of baryon or lepton number. Hence, the concept of R-parity is shown in the addition of the $\mathcal{Z}_2$ symmetry acting on the MSSM fields, which ensures that the couplings allowing the baryon or lepton number violation are forbidden.

Moreover, the lightest superparticle of this model is a stable and weakly interacting massive particle (it does not interact via electromagnetic or strong interactions), which makes it a suitable cold dark matter candidate.

Except for that, a class of supersymmetrized extensions of the Standard Model including the MSSM can explain the spontaneous breakdown of the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ symmetry at the electroweak level.

Although the MSSM is a nice SUSY extension of the Standard Model, its position is quite questionable. Apart from the fact that it suffers from several drawbacks, there is an absence of any evidence for supersymmetry at the Large Hadron Collider (LHC).

1.3.2 SUSY $SU(5)$

The simplest SUSY GUT is the minimal supersymmetric $SU(5)$ theory first proposed by S. Dimopoulos and H. Georgi in 1981 [39]. The Higgs sector of the minimal SUSY $SU(5)$ model comprises a 24-dimensional superfield and two sets of superfields belonging to both $\overline{5}$- and 5-dimensional supermultiplets in order to give both the down and the up quark masses. These two Higgs fields are also important for the cancellation of the gauge anomalies. All of the other fields are just supersymmetric generalizations of the appropriate fields present in the non-SUSY $SU(5)$ theory. Using all the relevant superfields the superpotenital of the theory can be constructed, which should give a realistic breaking of the full symmetry down to the Standard Model and, consequently, to $SU(3)_c \times U(1)_{QED}$ group.

As well as the non-supersymmetric GUTs, the SUSY $SU(5)$ also predicts the proton decay. However, an important outcome of the supersymmetry is that the dominant decay mode differs from that of the non-SUSY $SU(5)$ theory since the preferred reaction is

$$p \to K^+ + \overline{\nu}.$$

The reason is that usually new 5-dimensional baryon and lepton number violating operators can be constructed using the new coloured scalars required by supersymmetry, namely, the proton decay is in this case mediated by the Higgs colour triplets of the 5- and $\overline{5}$-dimensional representations. As a result, the proton decay amplitudes are significantly bigger than in the non-SUSY model, where operators of dimension $d = 6$ contribute.

Although some predictions about proton lifetimes have been made in the past, they contradict current experimental data. However, the uncertainties of the predicted numbers are always huge. Moreover, some later works on this topic show that if we employ the effective operators or we take into account the effects of GUT scale "Yukawa mismatch" [40], then it is still possible to find a model that does not violate the experimental limits. Consequently, although the minimal SUSY $SU(5)$ theory gives quite fast proton decay, it cannot be completely excluded on the basis of this prediction.

Anyway, the next large difficulty of the minimal SUSY $SU(5)$ theory is also the fact (similarly as it was in case of the non-supersymmetric $SU(5)$) that there is no natural mechanism for generating neutrino masses within this model. Although there have been several suggestions how to achieve this goal by adding the effective operators corresponding to the R-parity violating interactions, the required tiny couplings would still have to be put in by hand.

1.3.3 SUSY $SO(10)$

In the same way as the $SU(5)$ model was supersymmetrically extended the supersymmetrization can be applied also to the $SO(10)$ theory. The first papers on the minimal supersymmetric $SO(10)$ theory were published in the early eighties by Clark, Kuo and Nakagawa [41] and Aulakh and Mohapatra [42]. Within the framework of this model there are just 7 free parameters when no constraints are imposed; hence, the SUSY $SO(10)$ theory is very predictive. The supersymmetry is again involved to achieve the proper gauge unification pattern and to avoid the gauge hierarchy problem.

In a realistic SUSY $SO(10)$ model there should be at least three different Higgs multiplets in order to get the proper spontaneous symmetry breaking chain down to the $SU(3)_c \times U(1)_{QED}$ symmetry of our low-energy world. The important building block of the Higgs sector which singles out a class of economical $SO(10)$ models is the $\overline{126}$-dimensional representation. It is capable of breaking the GUT-scale symmetries and together with 10-dimensional representation it also gives the renormalizable Yukawa couplings. However, to get the minimal renormalizable SUSY $SO(10)$ GUT the 126- and 210-dimensional representations have to be added. While the 126-dimensional representation is necessary to cancel the D-term driving the SUSY breakdown generated by GUT-scale VEV of the $\overline{126}$-dimensional representation, the 210-dimensional representation triggers the spontaneous $SO(10)$ symmetry breaking and allows (after electroweak symmetry breaking) a realistic fermion spectrum by mixing of the weak doublet components of the 10- and $\overline{126}$-dimensional representations. Hence, the minimal SUSY $SO(10)$ model possesses the Higgs sector of the form $10 \oplus \overline{126} \oplus 126 \oplus 210$.

In contrast to the non-SUSY $SO(10)$ scenarios, the requirement of zero F-terms do not allow us to use solely (on top of the $\overline{126}_S$ representation and its conjugate) the adjoint 45-dimensional representation to break $SO(10)$ through the left-right groups, thus avoiding the $SU(5)$ intermediate step [43]. In other words, the 45_S alone does not develop a SUSY-preserving VEV; hence, usually the pair $45_S \oplus 54_S$ is considered. The alternative for this choice is the abovementioned 210-dimensional representation.

The description of the proton decay in the minimal SUSY $SO(10)$ model is technically similar to that in the minimal SUSY $SU(5)$ theory. The biggest difference is the fact that now the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ coloured Higgs triplets, thanks to which the new set of $d = 5$ baryon and lepton number violating operators can be constructed, can mix, when the GUT-scale symmetry is broken. This can consequently lead to suppression of the proton decay rate. In other words, properly adjusted mixing can cause the rise of the predicted proton lifetime above the experimental bound.

As for the neutrino masses, the SUSY $SO(10)$ represents a significant improvement in comparison with the minimal SUSY $SU(5)$. Although the minimal SUSY $SU(5)$ model has 21 free parameters, when we add the neutrino Majorana terms, this number of parameters is approximately doubled [44]. In contrast, within the minimal SUSY $SO(10)$ model we deal only with 26 free parameters and the right-handed neutrino is naturally included. For this reason the minimal SUSY $SO(10)$ model got the label of the simplest viable grand unified theory. Moreover, the R-parity is automatically conserved along with all of the symmetry breaking chains in this model. Hence, for the model with 126-dimensional representation the lightest supersymmetric particle is stable and it is a good cold dark matter candidate.

Despite all the nice features of the minimal renormalizable SUSY $SO(10)$ model we have described, it is already known that it does not represent a viable realistic unification theory. In 2006 it was shown by S. Bertolini, M. Malinsky and T. Schwetz [45] and by C. Aulakh and S. Garg [46] that the minimal renormalizable SUSY $SO(10)$ model with the Higgs sector $10 \oplus \overline{126} \oplus 126 \oplus 210$ is not consistent with available data. If it is required that the proton decay experimental bounds are respected it is possible to identify a very limited area in the parameter space of the model, where all fermion data are consistently reproduced. As was found out, in all of these cases the gauge coupling unification is strongly affected by lighter than GUT states, which are present in the model. Hence, the minimal supersymmetric $SO(10)$ scenario does not work.

1.4 Motivation for the $45 \oplus 16$ model

Within the previous sections a rather sketchy outline of the pros and cons of the most discussed grand unified theories was provided and we saw there are possibly viable models as well as some no-go scenarios. The current situation is not very clear. The critical thing, which would certainly help to clarify the standing of particular theories, is the observation of the proton decay and other phenomena revealing the physics beyond the Standard Model. Unfortunately, there is no experimental evidence of such a desired phenomenon so far.

Despite the scarcity of the restrictive data, one can still assume certain theoretical requirements which should be met by a possibly realistic model and which could consequently show the right direction one should choose and focus on. Of course, there is no definite guarantee in the sense of a formal proof that these assumptions are completely necessary; however, it is possible to justify them by a number of reasonable arguments.

The main guiding principle in the field of unification models is the already mentioned "minimality", which is usually translated as simplicity, predictivity or tractability. On grounds of minimality we can compare the models or even eliminate some of them. Although there is no direct experimental motivation for this conception, we do not posses any better restrictions. Moreover, physical experience has taught us many times that the successful and realistic theories and models are simple and elegant rather than complicated and cumbersome.

As may be inferred from the previous paragraphs, the term of minimality is quite vague and although GUTs have been around for forty years, there is still no solid, widely accepted explication. One of the possible interpretations is the minimality in sense of the rank of the group and therefore (to some extent) also of the number of degrees of freedom. This was evidently one of the main clues that led to the pioneering model proposed by Georgi and Glashow, since it is well-known that the $SU(5)$ group is the minimal and the only simple group of rank 4 with complex representations, which contains the group of the Standard Model as a subgroup.

Another possible criterion to judge on the minimality of particular model, is its predictivity. The potentially realistic model should not provide too much freedom and it should offer an elegant and viable description of the observed reality. In this respect, the $SO(10)$ unification represents a suitable choice. The nice feature that all the fermions of one family (inclusive of the desired right-handed neutrino) appears in a single spinor representation constrains the Yukawa sector of this model much more than in the case of the $SU(5)$ gauge group.

Although all of the matter fields fit perfectly into three 16-dimensionl spinor representations of $SO(10)$, one still has to deal with the Higgs sector of the theory, which does not share such a uniqueness. The structure of the symmetry-breaking sector is for most of the GUTs, including the $SO(10)$ theory, the biggest source of arbitrariness. The reason is that large higher-dimensional Higgs representations usually have to be considered so that it is possible to break spontaneously the gauge symmetry down to the Standard Model. The exception from this rule is the SUSY flipped $SU(5)$, which we outlined a few pages back. However, it is not a typical GUT because the gauge group is not simple and it still suffers from several problems.

In what follows we will focus on the non-SUSY $SO(10)$ unification and corresponding models. First, let us discuss the Higgs sector and suggest its possible realizations on grounds of minimality.

To achieve the full breaking of $SO(10)$ to the Standard Model at least two Higgses are employed.

- Either 16_S, or 126_S representation is necessary for reduction of the $SO(10)$ rank.
- One of the representations 45_S, 54_S and 210_S must be considered to break the symmetry down to small groups containing Standard Model group, which are different from $SU(5) \times U(1)$.

The minimal possibility from the 45_S, 54_S and 210_S representations is the first one. The 16_S and 126_S representations relates to the Yukawa sector and we always have to choose the convenient one in dependence on a particular model. As a result, the 45_S representation, in combination with either 16_S, or 126_S, can form the minimal non-SUSY $SO(10)$ model.

At first, it was observed [47, 48, 49, 50] that the phenomenologically viable breaking chains

$$SO(10) \to SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \to \text{Standard Model},$$
$$SO(10) \to SU(4)_C \otimes SU(2)_L \otimes U(1)_R \to \text{Standard Model}$$

are allowed by the gauge coupling unification. The first breaking is controlled by the 45_S VEVs, while the following breakdown to the Standard Model is driven by the 16_S (or 126_S) VEV. Additionally, one of the 45_S VEVs can contribute also to the second symmetry breaking.

Nevertheless, in the early eighties a series of studies [51, 52, 53, 54] indicated that the scalar sector dynamics allows just the $SU(5) \times U(1)$ intermediate stage for dominant $\langle 45_S \rangle$ and $SU(5)$ intermediate stage for leading $\langle 16_S \rangle$. Since the unification constraints (even without proton decay) do not allow any $SU(5)$-symmetric intermediate scale, the simplest (minimal) Higgs sectors responsible for the breaking of the $SO(10)$ symmetry down to the Standard Model were excluded from any realistic considerations.

Surprisingly, two decades later these minimal models were revived. It was shown that ignoring these minimal scenarios was wrong since the minimal non-SUSY $SO(10)$ models can be cured if the quantum aspects of these scenarios are taken into account [55].

Within this thesis we will reproduce some of these calculations. Particularly, the one-loop quantum corrections of the certain pseudo-Goldstone bosons contained by the scalar spectrum of the $45 \oplus 16$ model will be computed using the effective potential approach. Later on, we will calculate these corrections using standard perturbative theory approach. The main motivation for this work relates to the more complicated minimal $SO(10)$ model with Higgs sector composed of the $45_S \oplus 126_S$ representations. The point is that the gauge coupling unification within the $45 \oplus 126$ model can be (in contrast to the $45 \oplus 16$ model) compatible with the neutrino data [56]. However, one can hardly apply the effective potential approach to the $45 \oplus 126$ model, as it is very complicated to construct the effective potential for such a big algebraic object. As a result, a more elegant and simple way of calculation of the scalar radiative corrections would be extremely useful. Consequently, the standard perturbative theory approach applied to the $45 \oplus 16$ model, which is the principal goal of this thesis, can represent a great way how to determine the one-loop corrections in case of the $45 \oplus 126$ model.

2. Model description and technical preliminaries

In this chapter we will focus on the technical preliminaries needed for the calculations presented in the rest of the thesis. First, we will comment on the model of our interest, i.e., the minimal model with scalar potential spanned on $45_S \oplus 16_S$ scalar representations. After that, we will get acquainted with the effective potential approach describing its relation to the standard diagrammatic methods of calculation of quantum corrections.

The $SO(10)$ group theory is provided in Appendix A.

2.1 Embedding of the Standard Model

The generators relevant to a certain gauge theory enter the corresponding Lagrangian most obviously in the covariant derivatives. As known, the covariant derivative term of any gauge theory Lagrangian determines the dynamics of the corresponding gauge fields describing their interaction with other objects. In case of the $SO(10)$ theory, the covariant derivative reads

$$D_\mu^{(\phi)} = \partial_\mu - igA_\mu^{ij}(x)T_\phi^{ij}.$$

To find in the dynamics encoded in the covariant derivatives the familiar Standard Model structures it is necessary to find such linear combination of the $SO(10)$ generators, whose form allows us to distinguish the blocks of irreducible representations corresponding to the Standard Model subgroup. In other words, one has to pass from the $SO(10)$ irreducible representation of generators to a different one, which is reducible under the Standard Model gauge group. This step is essential for all the following calculations since it is essential to label the physical fields present in the $SO(10)$ theory by quantum numbers of the Standard Model.

The best generators to start with are those belonging to the complete set of commuting operators of $SO(10)$, i.e., the Cartan operators. The rank of $SO(10)$, defined as dimension of the Cartan subalgebra, is 5. Taking into account the $SO(10)$ commutation relations (A.1) it is possible to show that the set of simultaneously commuting generators can be chosen for instance as

$$T_{12},\ T_{34},\ T_{56},\ T_{78},\ T_{90}. \tag{2.1}$$

Although this choice is very straightforward, for further development it would be suitable to construct the Cartan operators with eigenvalues corresponding to the quantum numbers of the Standard Model. We know that the Standard Model group is a subgroup of $SO(10)$ and thus it must be possible to find the generators with the familiar eigenvalues as a linear combination of the $SO(10)$ generators. However, since the rank of $SO(10)$ is 5 (while the Standard Model gauge group has rank 4) it is necessary to consider a subgroup with equal rank (instead of the Standard Model subgroup) to cover all of the coordinates. Hence, it is convenient to consider a subgroup with rank 5 similar to the Standard Model gauge group.

A suitable choice is the left-right group $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$, in which the identification of the Standard Model multiplets is trivial as can be seen from the following decompositions of the $SO(10)$ representations.

2.1.1 Decompositions of the $SO(10)$ representations

The weak hypercharge operator reads

$$Y = T_R^3 + \frac{1}{2}(B - L),$$

where T_R^3 is the Cartan operator corresponding to the $SU(2)_R$ group and B and L denote the baryon and lepton numbers, respectively. These numbers we define in standard way, i.e., $B = \frac{1}{3}$ for quarks and $L = 1$ for leptons.

The defining vector 10-dimensional representation of $SO(10)$ decomposes under the subgroup $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ as

$$10 = (3, 1, 1, -\tfrac{2}{3}) \oplus (\overline{3}, 1, 1, +\tfrac{2}{3}) \oplus (1, 2, 2, 0). \tag{2.2}$$

The submultiplets on the right hand side of the equation (2.2) can be decomposed under the Standard Model gauge group as

$$\begin{aligned}
(3, 1, 1, -\tfrac{2}{3}) &= (3, 1, -\tfrac{1}{3}),\\
(\overline{3}, 1, 1, +\tfrac{2}{3}) &= (3, 1, +\tfrac{1}{3}),\\
(1, 2, 2, 0) &= (1, 2, +\tfrac{1}{2}) \oplus (1, 2, -\tfrac{1}{2}).
\end{aligned}$$

The adjoint 45-dimensional representation can be decomposed under the subgroup $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ of $SO(10)$ as

$$\begin{aligned}
45 = {} & (1, 1, 3, 0) \oplus (1, 3, 1, 0) \oplus (3, 2, 2, -\tfrac{2}{3}) \oplus (\overline{3}, 2, 2, +\tfrac{2}{3})\\
& \oplus (1, 1, 1, 0) \oplus (3, 1, 1, +\tfrac{4}{3}) \oplus (\overline{3}, 1, 1, -\tfrac{4}{3}) \oplus (8, 1, 1, 0)
\end{aligned}$$

and these multiplets then decompose into the submultiplets of the Standard Model as

$$\begin{aligned}
(1, 1, 3, 0) &= (1, 1, +1) \oplus (1, 1, 0) \oplus (1, 1, -1),\\
(1, 3, 1, 0) &= (1, 3, 0),\\
(3, 2, 2, -\tfrac{2}{3}) &= (3, 2, +\tfrac{1}{6}) \oplus (3, 2, -\tfrac{5}{6}),\\
(\overline{3}, 2, 2, +\tfrac{2}{3}) &= (\overline{3}, 2, -\tfrac{1}{6}) \oplus (\overline{3}, 2, +\tfrac{5}{6}),\\
(1, 1, 1, 0) &= (1, 1, 0),\\
(3, 1, 1, +\tfrac{4}{3}) &= (3, 1, +\tfrac{2}{3}),\\
(\overline{3}, 1, 1, -\tfrac{4}{3}) &= (\overline{3}, 1, -\tfrac{2}{3}),\\
(8, 1, 1, 0) &= (8, 1, 0).
\end{aligned}$$

The decomposition of the 16-dimensional spinor representation under the subgroup $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ reads

$$16 = (3, 2, 1, +\tfrac{1}{3}) \oplus (1, 2, 1, -1) \oplus (\overline{3}, 1, 2, -\tfrac{1}{3}) \oplus (1, 1, 2, +1)$$

and these can be decomposed under the Standard Model group as

$$
\begin{aligned}
&(3,2,1,+\tfrac{1}{3}) = (3,2,+\tfrac{1}{6}),\\
&(1,2,1,-1) = (1,2,-\tfrac{1}{2}),\\
&(\overline{3},1,2,-\tfrac{1}{3}) = (\overline{3},1,+\tfrac{1}{3}) \oplus (\overline{3},1,-\tfrac{2}{3}),\\
&(1,1,2,+1) = (1,1,+1) \oplus (1,1,0).
\end{aligned}
$$

2.1.2 $SO(10)$ Cartan operators in the vector and adjoint representations

Let us now look for the set of Cartan operators of $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$. They can be denoted as

$$T^3_C,\ T^8_C,\ T^3_L,\ T^3_R,\ T_{B-L},$$

where the first two operators correspond to $SU(3)_C$, while the next three operators correspond to $SU(2)_L$, $SU(2)_R$ and $U(1)_{B-L}$, respectively.

To construct these Cartan operators, it is useful consider the $SO(6) \otimes SO(4)$ maximal subalgebra of $SO(10)$. Furthermore, the local isomorphisms

$$
\begin{aligned}
&SO(4) \sim SU(2) \otimes SU(2),\\
&SO(6) \sim SU(4)
\end{aligned}
$$

can be employed.

Therefore we can work with group $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$. Moreover, one can use the relation

$$SU(3)_C \otimes U(1)_{B-L} \subset SU(4)_C.$$

Using these facts a convenient 10-dimensional representation of the five $SO(10)$ Cartan operators can be constructed.

The generators relevant to the $SU(3) \otimes U(1)_{B-L}$ can be represented by 6×6 matrices as one can verify from the above decomposition of the 10-dimensional $SO(10)$ representation. Hence, these generators can be built as block matrices with two 3×3 blocks formed by the Gell-Mann matrices and their complex conjugates multiplied by factor (-1). Both these sets of 3×3 matrices have to be transformed to the basis in which they are purely imaginary. Taking the two commuting matrices from the constructed set and completing them by zeros to 10-dimensional matrices we get the first two $SO(10)$ Cartan operators.

On the other hand, the $SU(2)_L \otimes SU(2)_R$ part can be represented by 4×4 matrices obtained in a similar fashion as one usually constructs the corresponding irreducible representation of the Lorentz algebra. Again, chosing two mutually commuting and purely imaginary matrices (one corresponding to the $SU(2)_L$ group and one respective to the $SU(2)_R$ group) and completing them by zeros to 10-dimensional matrices we have two more $SO(10)$ Cartan operators, which obviously do commute with the first two.

The last $SO(10)$ Cartan operator corresponding to $U(1)_{B-L}$ can be built as a properly normalized diagonal 6×6 matrix commuting with the two Cartans constructed from the Gell-Mann matrices. After it is transformed to the new basis, so that it has a purely imaginary form, we complete it trivially to 10×10 matrix getting the fifth Cartan operator of the $SO(10)$ group in its 10-dimensional representation.

For decomposition of the $SO(10)$ representation to the irreducible representations of the Standard Model group it is helpful to construct also the Casimir operators. Summing squares of the generators in the 6×6 sector we get the C_C. Likewise, we can construct two Casimir operators, which are non-zero just in the 4×4 sector (C_L for the $SU(2)_L$ group and C_R for the $SU(2)_R$ group). The concrete realization of the Cartan and Casimir operators of the $SO(10)$ group in the 10-dimensional representation is printed in the Appendix B.

Using these definitions it is already easy to construct their $10 \otimes 10$ versions since

$$T = (T^{10} \otimes 1) + (1 \otimes T^{10}).$$

2.1.3 $SO(10)$ Cartan operators in the spinor representation

In the 16-dimensional spinor representation (and its conjugate) the $SO(10)$ Cartan operators can be written in the $3_C 2_L 2_R 1_{B-L}$ basis as follows

$$T_{C3} = \tilde{T}_{C3} = \frac{1}{4}\left(\sigma_{12} - \sigma_{34}\right), \tag{2.3}$$

$$T_{C8} = \tilde{T}_{C8} = \frac{1}{4\sqrt{3}}\left(\sigma_{12} + \sigma_{34} - 2\sigma_{56}\right), \tag{2.4}$$

$$T_{B-L} = \tilde{T}_{B-L} = -\frac{2}{6}\left(\sigma_{12} + \sigma_{34} + \sigma_{56}\right), \tag{2.5}$$

$$T_{R3} = \frac{1}{4}\left(\sigma_{78} + \sigma_{910}\right), \ \tilde{T}_{R3} = \frac{1}{4}\left(\sigma_{78} - \sigma_{910}\right) \tag{2.6}$$

$$T_{L3} = \frac{1}{4}\left(\sigma_{78} - \sigma_{910}\right), \ \tilde{T}_{R3} = \frac{1}{4}\left(\sigma_{78} + \sigma_{910}\right). \tag{2.7}$$

The T-operators act on spinor χ, while the $\tilde{T}$-operators act on its conjugate χ^C.

2.1.4 Compatibility of bases

The important thing to mention is that the basis, in which we express the Cartan operators in the 16-dimensional spinor representation must be "compatible" with the basis of the adjoint 45-dimensional representation (or equivalently the basis of the 10-dimensional representation we discussed above). To ensure this "compatibility" let us consider the contraction $\sum_{i=1}^{10} \chi^C \Gamma_i \chi \phi_i$, which represents the Standard Model singlet (for definition of the Γ-matrices see Appendix A).

By derivation of the contraction with respect to chosen pairs of fields we always get a combination of components ϕ_i. This combination then always has to represent the physical vector of the 10-dimensional physical basis labelled by quantum numbers, which cancel out, when summed together with the quantum numbers of the fields with respect to which we took the derivative of the contraction. Hence, if we express this 10-dimensional physical vector in terms of ϕ_i-components of the defining basis using the transformation matrix between the defining and the physical bases of the 10-dimensional representation, we get a set of equations relating components ϕ_i. Solving this system we found out that the solution is trivial - some of the components remained unchanged and some of them just interchanged. Therefore, to make the basis of the 16-dimensional representation compatible with the basis of the 10-dimensional representation (and thus also with that of the adjoint representation), it is enough to relabel mutually some of the Γ-matrices.

Applying this relabelling to the Cartan operators (2.3)-(2.7) one obtains a new ("tensor-compatible") definition

$$
\begin{aligned}
T'_{C3} &= \tilde{T}'_{C3} = \frac{1}{4}\left(\sigma_{21} - \sigma_{43}\right), \\
T'_{C8} &= \tilde{T}'_{C8} = \frac{1}{4\sqrt{3}}\left(\sigma_{21} + \sigma_{43} - 2\sigma_{56}\right), \\
T'_{B-L} &= \tilde{T}'_{B-L} = -\frac{2}{6}\left(\sigma_{21} + \sigma_{43} + \sigma_{56}\right), \\
T'_{R3} &= \frac{1}{4}\left(\sigma_{98} + \sigma_{107}\right), \ \tilde{T}'_{R3} = \frac{1}{4}\left(\sigma_{98} - \sigma_{107}\right), \\
T'_{L3} &= \frac{1}{4}\left(\sigma_{98} - \sigma_{107}\right), \ \tilde{T}'_{R3} = \frac{1}{4}\left(\sigma_{98} + \sigma_{107}\right).
\end{aligned}
$$

2.2 The Higgs sector

In the previous chapter we have already highlighted the most interesting features of the $SO(10)$ theory, among which we emphasized the elegant embedding of the Standard Model matter content - especially that all the elementary fields nicely fit into 16-dimensional spinor representation. Unfortunately, this great unification aspect must be compensated by more complicated spontaneous symmetry-breaking pattern.

As we have already mentioned, the Higgs sector brings quite a lot of arbitrariness to the model building and to choose the most convenient possibility usually represents a challenging problem. Although there are also other ways to attain the desired result [57], the Higgs mechanism is the simplest possibility that we shall stick to.

In the previous chapter we have explained the concept of minimality, which can serve as a convenient criterion for construction of the Higgs sector. Based on that we concluded that the best way is to break the symmetry down to the Standard Model using the 45-dimensional representation plus the 16-dimensional spinor representation. Although this model has been disregarded for a long time, a recent paper [55] revived its fortunes. An important fact is that the revivification applies not only to the $45 \oplus 16$ model but to every non-SUSY $SO(10)$ model with the adjoint 45_S representation responsible for the first symmetry-breaking step. The only extra thing we need is one additional scalar representation to reduce the rank. Hence, for instance, the 126-dimensional tensor representation can be considered instead of the 16-dimensional one. The selected spontaneous symmetry-breaking patterns including the possible intermediate symmetries between the $SO(10)$ and the Standard Model gauge symmetry $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ are depicted in Fig. 2.1.

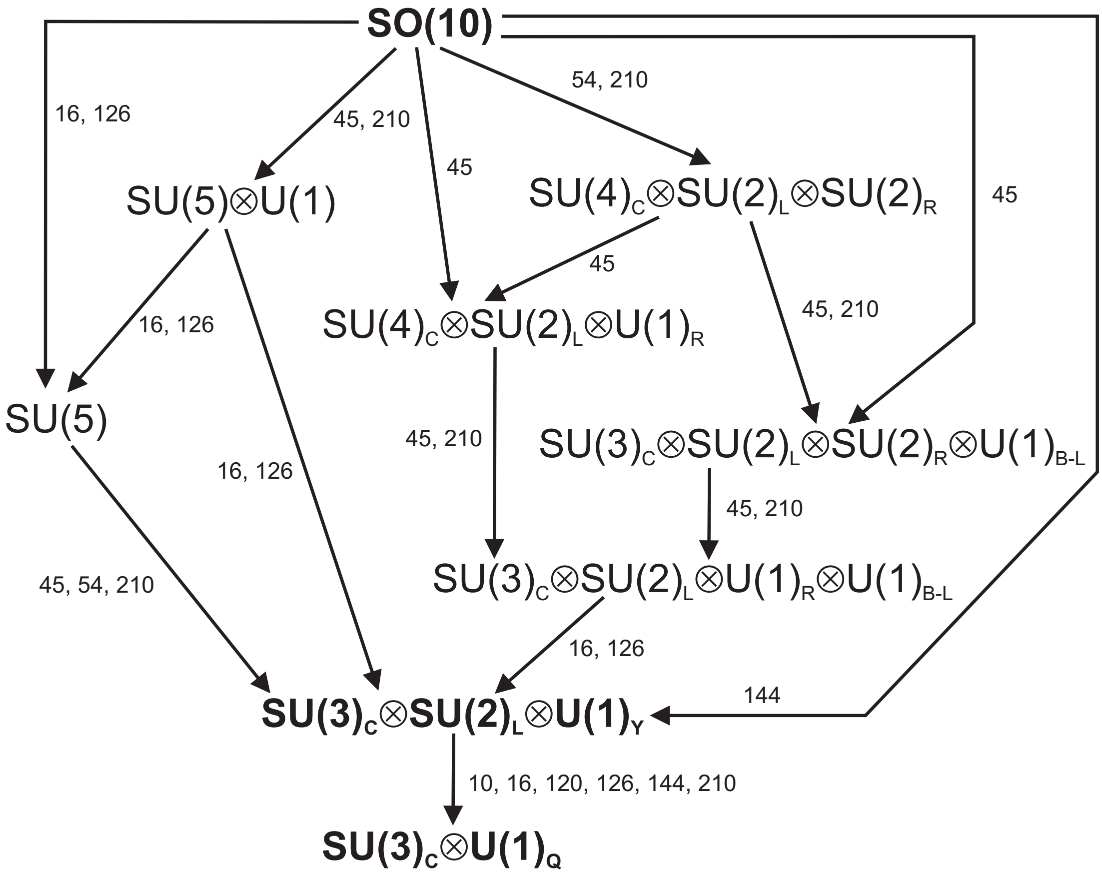


Figure 2.1: $SO(10)$ breaking patterns with representations up to 210 are depicted in this figure. The $SU(5) \otimes U(1)$ represents either the standard, or the flipped realization. In the former case either 16, or 126 representation breaks it down to $SU(5)$. In the latter case it is broken into the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

Another possibility how to break the $SO(10)$ down to the Standard Model in a single step is to employ the 144-dimensional Higgs representation. However, this option is not really minimal, as it requires an extension of the matter sector. Hence, to break the symmetry using the 144 representation and to guarantee the realistic masses and mixing of the Standard Model fermions, an extra 120-dimensional representation would have to be included except for the 45_F multiplet [58, 59], which is quite an unpleasant fact.

As a result, it seems that the convenient realistic non-SUSY $SO(10)$ models include either the $45_S \oplus 16_S$, or the $45_S \oplus 126_S$ representations. Therefore, the phenomenologically preferable breaking patterns allowed by the gauge unification are the two following scenarios

$$SO(10) \xrightarrow{M_U} 3_C 2_L 2_R 1_{B-L} \xrightarrow{M_I} 3_C 2_L 1_R 1_{B-L} \xrightarrow{M_{B-L}} 3_C 2_L 1_Y$$
$$SO(10) \xrightarrow{M_U} 4_C 2_L 1_R \xrightarrow{M_I} 3_C 2_L 1_R 1_{B-L} \xrightarrow{M_{B-L}} 3_C 2_L 1_Y,$$

where we have denoted $SU(3)_C$, $SU(2)_{L,R}$ and $U(1)_{B-L,Y}$ by a more compact notation, i.e., by 3_C, $2_{L,R}$ and 1_{B-L} respectively. The first two breaking stages M_U and M_I are controlled by VEVs of the 45_S and the last breaking from the intermediate scale M_{B-L} down to the Standard Model is driven by VEV of the

16_S. The gauge unification yields the following relation of the particular scales

$$M_U \gg M_I > M_{B-L}.$$

In this thesis we will focus especially on the study of the non-SUSY $SO(10)$ model with the $45 \oplus 16$ Higgs sector. The symbol ϕ will be used for the fields belonging to the 45-dimensional adjoint Higgs representation. Similarly, the fields belonging to the 16-dimensional spinor Higgs representation will be denoted by χ; consequently, the relevant multiplets transforming as positive and negative chirality components of the 32-dimensional $SO(10)$ spinor will be denoted by χ^+ and χ^-, respectively.

2.3 The tree level scalar potential

Following to [60, 61] the most general renormalizable $SO(10)$ invariant tree level scalar potential for one 45_S and one $16_S \oplus \overline{16}_S$ representation can be written as

$$V = V_\phi + V_{\phi\chi} + V_\chi, \tag{2.8}$$

where

$$V_\phi = -\frac{\mu^2}{2}\mathrm{Tr}\Phi^2 + \frac{a_1}{4}\left(\mathrm{Tr}\Phi^2\right)^2 + \frac{a_2}{4}\mathrm{Tr}\Phi^4, \tag{2.9}$$

$$V_{\phi\chi} = \alpha(\chi^\dagger\chi)\mathrm{Tr}\Phi^2 + \beta\left(\chi^\dagger\Phi^2\chi\right) + \tau\left(\chi^\dagger\Phi\chi\right), \tag{2.10}$$

$$V_\chi = -\frac{\nu^2}{2}\chi^\dagger\chi + \frac{\lambda_1}{4}\left(\chi^\dagger\chi\right)^2 + \frac{\lambda_2}{4}\left(\chi_+^\dagger\Gamma_j\chi_-\right)\left(\chi_-^\dagger\Gamma_j\chi_+\right), \tag{2.11}$$

where the definition $\Phi = \frac{\phi_{ij}\sigma_{ij}}{4}$ (see Appendix A) was used. All the constants and masses are real because of hermiticity. As can be seen, there are no linear and cubic terms in ϕ since the matrices of the $SO(10)$ adjoint representation are antisymmetric ($\phi_{ij} = -\phi_{ji}$).

2.4 The symmetry breaking

With the scalar potential at hand we can start to discuss the vacuum structure of the model.

2.4.1 Standard Model singlets

Generally, in the $45 \oplus 16$ model there are three Standard Model singlets in the scalar representations - two of them reside in the 45_S and one in 16_S. If we label the fields by the $3_C 2_L 2_R 1_{B-L}$ quantum numbers, the singlets of the 45-dimensional representation reside in the $(1,1,1,0)$ and $(1,1,3,0)$ submultiplets, while the singlet accommodated in the 16-dimensional representation resides in the $(1,1,2,+1)$ submultiplet. For the relevant VEVs of these fields we will use the following notation

$$\omega_R \equiv \langle (1,1,0,0)_{(1,1,1,0)} \rangle, \tag{2.12}$$

$$\omega_Y \equiv \langle (1,1,0,0)_{(1,1,3,0)} \rangle, \tag{2.13}$$

$$\chi_R \equiv \langle \left(1,1,-\tfrac{1}{2},+1\right)_{(1,1,2,+1)} \rangle, \tag{2.14}$$

where the subscripts indicate the relevant submultiplets. The VEVs ω_R and ω_Y are real and the VEV χ_R can be also taken as real if the phase of the 16_S representation is redefined.

The symmetry breaking patterns of the $SO(10)$ theory varies for different VEV configurations. For different settings of the vacuum expectation values the symmetry breaks to different subgroups. If we set $\chi_R = 0$, these possibilities are following

$$\omega_R = 0, \omega_Y \neq 0 \rightarrow 3_C 2_L 2_R 1_{B-L}, \tag{2.15}$$
$$\omega_R \neq 0, \omega_Y = 0 \rightarrow 4_C 2_L 1_R, \tag{2.16}$$
$$\omega_R \neq 0, \omega_Y \neq 0 \rightarrow 3_C 2_L 1_R 1_{B-L}, \tag{2.17}$$
$$\omega_R = -\omega_Y \neq 0 \rightarrow \text{flipped } 5'1_{Z'}, \tag{2.18}$$
$$\omega_R = \omega_Y \neq 0 \rightarrow \text{standard } 51_Z, \tag{2.19}$$

where $5'1_{Z'}$ and 51_Z represent the "flipped" and the "standard" embedding of the $SU(5)$ group into $SO(10)$, respectively. For $\chi_R \neq 0$ the first four options of the intermediate scale are broken down to the Standard Model gauge group, just the last one maintains the whole $SU(5)$ subgroup unbroken. The vacuum configurations depend on the phase conventions used for the parametrization of the subspace corresponding to the Standard Model singlets.

2.4.2 The breaking chains

In the section dedicated to the Higgs sector of $SO(10)$ we already presented the two phenomenologically viable breaking patterns. These can be related to the corresponding settings of the VEVs of the $45 \oplus 16$ model as follows

- $\omega_Y \gg \omega_R > \chi_R$
  $$\Rightarrow SO(10) \xrightarrow{M_U} 3_C 2_L 2_R 1_{B-L} \xrightarrow{M_I} 3_C 2_L 1_R 1_{B-L} \xrightarrow{M_{B-L}} 3_C 2_L 1_Y,$$
- $\omega_R \gg \omega_Y > \chi_R$
  $$\Rightarrow SO(10) \xrightarrow{M_U} 4_C 2_L 1_R \xrightarrow{M_I} 3_C 2_L 1_R 1_{B-L} \xrightarrow{M_{B-L}} 3_C 2_L 1_Y.$$

If we consider that one of the 45_S VEVs (either ω_R, or ω_Y) equals to zero, a really two-step $SO(10)$ symmetry breaking pattern can be obtained [62]. However, the settings when $\omega_R < \chi_R$ or $\omega_Y < \chi_R$ imply the effective two-step breaking, for which a different set of scalars survive at the intermediate scale.

2.5 The tree-level vacuum

Before we start with the calculation of the quantum corrections needed for the local vacuum stability, it is convenient to recapitulate also the tree level results. In particular, we will compute and discuss the tree level scalar spectrum of the $45 \oplus 16$ model, which will be needed for the construction of the effective potential later on.

2.5.1 The stationarity conditions

The physical masses corresponding to the particular fields can be obtained as the second derivatives of the scalar potential shifted into the asymmetric vacuum of the model. Hence, at first, we have to compute the stationarity condition determining the minimum on the vacuum manifold. Luckily, it is not needed to take all the 45 derivatives of the scalar potential with respect to all the fields residing in the adjoint representation and to solve this large system of equations because we are interested in the vacuum form of these conditions. In other words, the stationarity conditions are the first derivatives expressed in the vacuum; therefore, there are just three nontrivial equations corresponding to the three non-zero VEVs. All of the other fields have zero VEVs and thus they do not affect the calculations. Hence, there is much less work to do, as it is enough to calculate the three derivatives of the vacuum expectation value of the scalar potential with respect to the three non-zero vacuum expectation values.

Expressing the scalar potential (2.8) in terms of the physical fields and substituting the non-zero VEVs of the relevant fields by (2.12)-(2.14) (other vacuum expectation values are zero), we get the formula representing the vacuum manifold

$$\begin{aligned}\langle V_0\rangle = &-2\mu^2\left(2\omega_R^2+3\omega_Y^2\right)+4a_1\left(2\omega_R^2+3\omega_Y^2\right)^2\\&+\frac{1}{4}a_2\left(8\omega_R^4+36\omega_R^2\omega_Y^2+21\omega_Y^4\right)-\frac{1}{2}\nu^2\chi_R^2+\frac{1}{4}\lambda_1\chi_R^4\\&+4\alpha\chi_R^2\left(2\omega_R^2+3\omega_Y^2\right)+\frac{1}{4}\beta\chi_R^2\left(2\omega_R^2+3\omega_Y^2\right)^2-\frac{1}{2}\tau\chi_R^2\left(2\omega_R+3\omega_Y\right)\end{aligned}$$

Now we can take the first derivative of the potential (2.8) with respect to the three fields with non-zero VEVs (2.12)-(2.14). In fact, these derivatives are in this case equivalent to the derivatives of the vacuum manifold with respect to the VEVs themselves. Therefore, the three stationarity conditions read

$$\begin{aligned}0=\frac{\partial\langle V_0\rangle}{\partial\omega_R}&=2\omega_R\left(4\mu^2+32a_1\omega_R^2+48a_1\omega_Y^2+4a_2\omega_R^2+9a_2\omega_Y^2+8\alpha\chi_R^2\right)\\&\quad-\chi_R^2(2\beta\omega_R+3\beta\omega_Y+\tau),\end{aligned}\tag{2.20}$$

$$\begin{aligned}0=\frac{\partial\langle V_0\rangle}{\partial\omega_Y}&=2\omega_Y\left(4\mu^2+32a_1\omega_R^2+48a_1\omega_Y^2+6a_2\omega_R^2+6a_2\omega_R^2+7a_2\omega_Y^2\right.\\&\quad\left.+8\alpha\chi_R^2\right)-\chi_R^2\left(2\beta\omega_R+3\beta\omega_Y+\tau\right),\end{aligned}\tag{2.21}$$

$$\begin{aligned}0=\frac{\partial\langle V_0\rangle}{\partial\chi_R}&=\chi_R\left(-2\nu^2+2\lambda_1\chi_R^2-32\alpha\omega_R^2-48\alpha\omega_Y^2+4\beta\omega_R^2+9\beta\omega_Y^2\right.\\&\quad\left.+12\beta\omega_R\omega_Y+4\tau\omega_R+6\tau\omega_Y\right).\end{aligned}\tag{2.22}$$

The first two conditions can be re-expressed in a more suitable form, which factors out the vacuum configuration $\omega_R=\omega_Y$ leading to the intermediate scale containing the "standard" $SU(5)$ group

$$0 = \frac{1}{8}\left(\frac{\partial\langle V_0\rangle}{\partial\omega_R} - \frac{2}{3}\frac{\partial\langle V_0\rangle}{\partial\omega_Y}\right) = (\omega_R - \omega_Y)\left(- \mu^2 + 4a_1\left(2\omega_R^2 + 3\omega_Y^2\right)\right.$$
$$\left. + \frac{1}{4}a_2\left(4\omega_R^2 + 7\omega_Y^2 - 2\omega_R\omega_Y\right) + 2\alpha\chi_R^2\right) \tag{2.23}$$
$$0 = \omega_Y\left(\frac{\partial\langle V_0\rangle}{\partial\omega_R}\right) - \omega_R\frac{2}{3}\left(\frac{\partial\langle V_0\rangle}{\partial\omega_Y}\right) = (\omega_R - \omega_Y)\left(4a_2(\omega_R + \omega_Y)\,\omega_R\omega_Y\right.$$
$$\left. + \beta\chi_R^2\left(2\omega_R + 3\omega_Y - \tau\chi_R^2\right)\right) \tag{2.24}$$

It is not difficult to see that for $\chi_R = 0$ the conditions (2.23) and (2.24) allow for the vacuum settings (2.15)-(2.19). However, it will be revealed that the first two options of the vacuum configurations in fact are not the tree level minima since at the three level these scenarios suffer from tachyonicity.

2.5.2 The tree level scalar spectrum and constraints on the scalar potential parameters

The scalar spectrum at the tree level is given by the second derivatives of the scalar potential (2.8) with respect to the physical fields and into the resulting expressions the stationarity conditions calculated above must be substituted. The full scalar spectrum for the the Standard Model vacuum configuration (the case when either all of the three VEVs, or at least χ_R and one of the omegas are non-zero) is calculated in Appendix D.

For what follows the most important elements of the scalar spectrum are the 45_S submultiplets transforming as $(8,1,0)$ and $(1,3,0)$ under the Standard Model gauge group. The corresponding squared masses are given by

$$M^2(8,1,0) = 2a_2\left(\omega_R - \omega_Y\right)\left(\omega_R + 2\omega_Y\right), \tag{2.25}$$
$$M^2(1,3,0) = 2a_2\left(\omega_Y - \omega_R\right)\left(\omega_Y + 2\omega_R\right). \tag{2.26}$$

Obviously, these tightly correlated masses can be negative, i.e., tachyonic. This fact implies that the theory does not lie in the local minimum, or, in other words, the stationary points do not represent the physical minima. Hence, the corresponding vacuum is unstable and the VEVs occurring in the above expressions must be constrained in order to avoid the tachyonicity and to satisfy the requirement of boundedness. As a result, the following constraints on the potential parameters can be obtained

$$a_2 < 0 \text{ and } -2 < \frac{\omega_Y}{\omega_R} < -\frac{1}{2}. \tag{2.27}$$

Moreover, it is possible to show that for $\tau = 0$ the combination of restrictions yielded by equations (2.24), (2.25), (2.26) and the mass eigenvalues of the $(1,1,1)$ and $(3,2,\frac{1}{6})$ fields the constraint on ω_Y/ω_R is even more restrictive and reads [51, 52, 53, 54]

$$-1 < \frac{\omega_Y}{\omega_R} < -\frac{2}{3}. \tag{2.28}$$

If $\chi_R = 0$, the only vacuum configuration allowed by equation (2.24) and by the constraints (2.27), (2.28) is the one that leads to the breaking to the 51_Z subgroup. For $\chi \neq 0$, it is possible to fine-tune the parameters so that $\omega_Y/\omega_R \sim -1$ and χ_R is fixed at an intermediate scale. However, this setting does not reproduce the Standard Model couplings [62]. For $\omega_Y \sim \omega_R \sim \chi_R \sim M_G$ and for $\chi_R \gg \omega_{R,Y}$ analogical conclusions can be obtained.

As a result, it was concluded in 1980s that the minimal $SO(10)$ models containing the adjoint Higgs representation are not realistic. The major problem is the fact that the large hierarchy between the adjoint VEVs cannot be set to allow a phenomenologically viable breaking. Nevertheless, we will see that the situation is significantly improved at the quantum level.

2.5.3 "Tachyonic" masses revis(it)ed

The explicit form of the masses corresponding to the $(8,1,0)$ and $(1,3,0)$ submultiplets of the representation 45_S is quite suspicious, as it depends only on a single potential parameter a_2. It is clear that there are no terms proportional to λ_2 or τ since they simply cannot contribute to these masses at the tree level. However, the χ_R^2-dependent contribution proportional to the parameter β could be expected. The relevant term of the potential reads

$$\frac{\beta}{16}\chi_R^2 \sigma_{ij}\sigma_{kl}\phi_{ij}\phi_{kl}, \tag{2.29}$$

but after the defining fields are projected onto directions corresponding to the physical states labelled as $(8,1,0)$ or $(1,3,0)$, these potential contributions vanish. This can be seen from a more general point of view. The term (2.29) resembles the structure of the gauge boson mass term. Since the gauge bosons residing in the submultiplets $(8,1,0)_G$ and $(1,3,0)_G$ do not get their masses at the $SU(5)\otimes U(1)$ level, neither do their scalar counterparts. Hence, the tree level mass β-dependent contribution to all scalars belonging to the adjoint representation and labelled by quantum numbers of the standard $SU(5)$ is zero.

Let us explain also the fact that the masses of the fields accommodated in the submultiplets $(8,1,0)$ and $(1,3,0)$ are proportional to the potential parameter a_2. It can be observed that if $a_2 = 0$, the potential (2.8) exhibits an enhanced global symmetry $O(45)$. Hence, the a_2-dependent term of the scalar potential explicitly breaks this accidental global symmetry and the fields belonging to $(8,1,0)$ and $(1,3,0)$ are the corresponding pseudo-Goldstone bosons.

To sum up, although the β and τ couplings do not contribute to the masses (2.25)-(2.26) at the tree level, one can still anticipate their contribution of order $O(M_G/4\pi)$ at the quantum level. Except for these scalar couplings also the gauge interactions will contribute to the "problematic" masses.

2.6 Coleman-Weinberg effective potential

The effective potential technique was introduced by J. Goldstone, A. Salam, S. Weinberg [63] and by G. Jona-Lasinio [64] as a method for treating spontaneous symmetry breaking within the quantum field theory. It was further developed by S. Coleman and E. Weinberg [65], whose contribution will be mainly reported in this section. We will recapitulate the salient features of this method, describe its formalism, meaning, application and also correspondence to the classical diagrammatic framework.

2.6.1 The effective potential formalism

For the sake of simplicity let us assume that we have a theory of a single scalar field ϕ with dynamics given by the Lagrange density $\mathscr{L}\left(\phi, \partial_\mu \phi\right) = \partial_\mu \phi \partial^\mu \phi - V(\phi)$. Let us consider a c-number function of space and time representing an external source $J(x)$. This source is coupled linearly to ϕ and added to the Lagrange density, i.e.,

$$\mathscr{L}(\phi, \partial_\mu \phi) \rightarrow \mathscr{L} + J(x)\phi(x).$$

The transition amplitude from the vacuum asymptotic state $|0^-\rangle$ in the far past to the vacuum asymptotic state $\langle 0^+|$ in the far future when the source $J(x)$ is present defines the connected generating functional $W(J)$ as

$$\langle 0^+|0^-\rangle_J = \exp(iW(J)).$$

This functional can be expanded in a functional Taylor series

$$W = \sum_n \frac{1}{n!} \int \mathrm{d}^4 x_1 \ldots x_n G^{(n)}\left(x_1, \ldots, x_n\right) J(x_1) \ldots J(x_n), \tag{2.30}$$

where $G^{(n)}\left(x_1, \ldots, x_n\right)$ are the Green's functions of the theory corresponding to the sum of all connected Feynman diagrams with n external lines.

The classical field ϕ_c is defined as

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = \left(\frac{\langle 0^+|\phi(x)|0^-\rangle}{\langle 0^+|0^-\rangle}\right)_J. \tag{2.31}$$

The defining relation for the effective action $\Gamma(\phi_c)$ reads

$$\Gamma(\phi_c) = W(J) - \int \mathrm{d}^4 x \phi_c(x) J(x),$$

which is in fact just a Legendre transformation of the generating functional and it easily implies

$$J(x) = -\frac{\delta \Gamma}{\delta \phi_c(x)}. \tag{2.32}$$

We will see that this relation is very important for the study of spontaneous symmetry breaking. In the same way as we expanded the generating functional in equation (2.30) we can expand also the effective action, i.e.,

$$\Gamma = \sum_n \frac{1}{n!} \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n \Gamma^{(n)}\left(x_1, \ldots, x_n\right) \phi_c\left(x_1\right) \ldots \phi_c\left(x_n\right), \tag{2.33}$$

where the coefficients (one-point irreducible Green's functions called sometimes "proper vertices") can be shown to be the sum of all the one-particle-irreducible Feynman diagrams with n external lines.

While in equation (2.33) the effective action was expanded in powers of the classical field ϕ_c, it is also possible to write the expansion alternatively in terms of momentum around the point where are all the external momenta zero. Such an expansion in position space explicitly reads

$$\Gamma = \int \mathrm{d}^4 x \left(-V\left(\phi_c\right) + \frac{1}{2}\left(\partial_\mu \phi_c\right) Z\left(\phi_c\right) + \dots \right), \tag{2.34}$$

where the ordinary function $V\left(\phi_c\right)$, the so called effective potential, represents the quantum version of the classical scalar potential $V(\phi)$. The derivative of order n of the effective potential V is in fact the sum of all one-particle irreducible graphs with n vanishing external momenta. It is straightforward to see that at the tree level the effective potential equals to the ordinary scalar potential.

Using the functions in the expansion of the effective action the usual renormalization condition of the perturbation theory can be formulated; therefore, if m^2 denotes the scalar field squared running mass in the $\overline{MS}$ scheme, it holds that

$$m^2 = \left. \frac{\mathrm{d}^2 V}{\mathrm{d}\phi_c^2} \right|_0 \tag{2.35}$$

and similarly if we consider the well-known example of the scalar interaction $\lambda\phi^4$, then the running coupling λ can be obtained as

$$\lambda = \left. \frac{\mathrm{d}^4 V}{\mathrm{d}\phi_c^4} \right|_0 . \tag{2.36}$$

The normalization of the field is in this case ensured by the condition

$$Z(0) = 1. \tag{2.37}$$

After all these definitions we now apply the formalism of the effective potential to the spontaneous symmetry breaking. Let us assume a Lagrange density with a certain internal symmetry. If the quantum field ϕ has a non-zero vacuum expectation value, it triggers the spontaneous symmetry breakdown even in case that the external source $J(x)$ equals to 0. Taking into account the equations (2.31) and (2.32) this situation appears when

$$\frac{\delta \Gamma}{\delta \phi_c} = 0,$$

which, assuming translational invariance, can be simplified to

$$\frac{\delta V}{\delta \phi_c} = 0.$$

The minimum arises for certain value of ϕ_c, which is denoted by $\langle \phi \rangle$ - the expectation value of ϕ in the new, asymmetric minimum.

As usual, we shift the field ϕ by its vacuum expectation value ϕ in order to get new field ϕ' with zero VEV, i.e.,

$$\phi' = \phi - \langle \phi \rangle$$

and this redefinition implies also the change of the classical field

$$\phi_c' = \phi_c - \langle\phi\rangle.$$

Consequently, the new masses, coupling constants, etc. can be calculated from relations analogous to (2.35) and (2.36). The only difference is that now the derivatives have to be evaluated at point $\langle\phi\rangle$ rather than at zero.

Although we have seen that the theoretical derivation of the effective potential approach is quite straightforward, the explicit construction of the effective potential corresponding to specific model may be rather difficult. The calculation of the effective potential is in fact an infinite sum of Feynman diagrams.

However, an approximation method for V can be derived. To be specific, the loop expansion can be used. It means that we sum all the tree graphs, then all graphs with one closed loop, etc. This expansion corresponds to the expansion in a parameter multiplying the total Lagrange density. In other words, if we denote such a parameter as a and we define

$$\mathcal{L}\left(\phi, \partial_\mu\phi, a\right) = a^{-1}\mathcal{L}\left(\phi, \partial_\mu\phi\right),$$

it can be shown that the loop expansion is equivalent to a power-series expansion in a. Therefore, the Lagrange density remains unaffected by any shifts of fields as well as by different divisions of the Lagrangian into free and interacting parts. Naturally, the first term in the effective potential expansion is the classical scalar potential representing the sum of all nonderivative terms present in the scalar Lagrange density.

2.6.2 Sample application

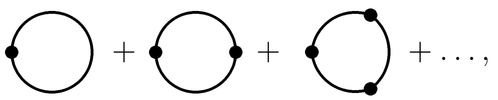
Let us now assume a simple model of a single scalar field with Lagrange density

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - U(\phi), \tag{2.38}$$

where U denotes the classical potential and we assume it to be polynomial. The zero loop approximation then gives

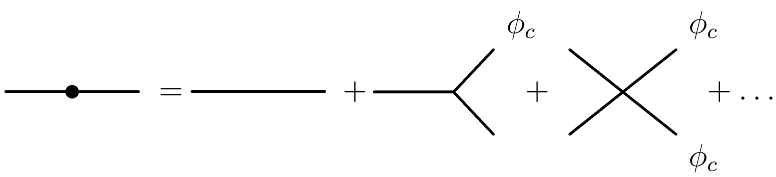
$$V = U\left(\phi_c\right).$$

Furthermore, we can assume that the free part of the Lagrange density (2.38) is just the first term $\frac{1}{2}\left(\partial_\mu\phi\right)^2$, while the interacting part is the whole rest given by the potential $U(\phi)$. All graphs contributing to the one-loop effective potential of this theory are



(2.39)

where each dot stands for a sum of terms with zero, one, two, etc. external lines, which arise from the terms in U of the second, third, fourth, etc. order in ϕ. All the external lines carry ϕ_c with a zero external momentum. Pictorially,



and this vertex can expressed as

$$i\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\bigg|_{\phi=\phi_c} = iU''(\phi_c)\,.$$

Each line between two vertices represents the massless propagator, i.e.,

$$\frac{i}{k^2+i\varepsilon}.$$

where k is the loop momentum.

Performing the summation in (2.39) one obtains

$$(2.39) = i\int \mathrm{d}^4k\frac{1}{(2\pi)^4}\sum_{n=1}^{+\infty}\frac{1}{2n}\left(\frac{U''(\phi_c)}{k^2+i\varepsilon}\right)^n, \tag{2.40}$$

where the i at the beginning is from the definition of the generating functional and the prefactor $\frac{1}{2n}$ in the sum comes from the factorial in Dyson's formula and the fact that a rotation and a reflection of the n-point graph does not lead to a new contraction in the Wick's expansion.

The sum above can be easily evaluated, while the result

$$V = U + \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left(k^2+U''(\phi_c)-i\varepsilon\right), \tag{2.41}$$

where the rotation of the integral into the Euclidean space was performed for convenience.

Obviously, the integral (2.40) diverges. Imposing an integration momentum cutoff Λ the last formula can be rewritten as

$$V = U + \frac{\Lambda^2}{32\pi^2}U'' + \frac{(U'')^2}{64\pi^2}\left(\ln\left(\frac{U''-i\varepsilon}{\Lambda^2}\right)-\frac{1}{2}\right). \tag{2.42}$$

Naturally, the behaviour of the resulting formula depends on the fact whether the interaction is renormalizable. If U is a polynomial of up to fourth order, then the cut-off dependence of the equation (2.42) can be absorbed into a counterterm. On the other hand, if the considered theory possesses a potential U of higher than fourth order, the counterterms have to be of yet higher order and so on, which results in the drawbacks characteristic for non-renormalizable theories.

The important implication of the above calculation, is that even when the spontaneous symmetry breakdown is included, no extra counterterms than those required by the theory without symmetry breaking are needed. Hence, the spontaneous symmetry breakdown cannot affect the structure of the divergences present in a renormalizable field theory.

2.6.2.1 Explicit computation in the simplest case

Let us present a more specific computation of the one-loop effective potential in the simplest possible case of the theory of a massless, self-interacting meson field given by the Lagrange density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}A(\partial_\mu\phi)^2 - \frac{1}{2}B\phi^2 - \frac{1}{4!}C\phi^4,$$

where A, B and C denote the wave-function, mass and coupling renormalization counterterms, respectively. Imposing the definitions of the scale of the renormalized field, the renormalized mass and the renormalized coupling these counterterms will be determined order by order in the expansion. Although we are dealing with the massless theory, the mass counterterm is present since there is no symmetry that would guarantee that the bare mass vanishes when the renormalized mass goes to zero.

In the tree approximation of the considered theory we get

$$V = \frac{\lambda}{4!}\phi_c^4.$$

At one-loop level we have to sum the infinite series of the polygon graphs plus the contributions from the mass and the coupling counterterms, i.e.,

$$V = \frac{\lambda}{4!}\phi_c^4 - \frac{1}{2}B\phi_c^2 - \frac{1}{4!}C\phi_c^4 + i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{1}{2}\frac{\lambda\phi_c^2}{k^2+i\varepsilon}\right)^n. \tag{2.43}$$

The prefactor of i in front of the integral is from the definition of the generating functional W, the factor of $\frac{1}{2}$ in the integrand represents the Bose statistics factor and the factor of $\frac{1}{2n}$ is the combination of the $\frac{1}{n!}$ factor in Dyson's formula and the combinatoric factor generated by the fact that the rotation (or reflection) of the n-sided polygon does not lead to a new graph.

The formula (2.43) is infrared divergent, but by summing the series we get an improved expression

$$V = \frac{\lambda}{4!}\phi_c^4 - \frac{1}{2}B\phi_c^2 - \frac{1}{4!}C\phi_c^4 + \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\log\left(1+\frac{\lambda\phi_c^2}{2k^2}\right). \tag{2.44}$$

The integral in the above equation was rotated into Euclidean space and the factor of $i\varepsilon$ in the denominator of the integrand was dropped. Obviously, the infrared divergence was transformed into a logarithmic singularity at $\phi_c = 0$. The equivalent behaviour we would get if tried to compute the radiative correction to the propagator in this theory.

However, the integral in (2.44) is still ultraviolet divergent. Considering the cut off at $k^2 = \Lambda^2$ we obtain

$$V = \frac{\lambda}{4!}\phi_c^4 + \frac{1}{2}B\phi_c^2 + \frac{1}{4!}C\phi_c^4 + \lambda\frac{\Lambda^2}{64\pi^2}\phi_c^2 + \lambda^2\frac{\Lambda^2}{256\pi^2}\left(\log\frac{\lambda^2\phi_c^4}{2\Lambda^2} - \frac{1}{2}\right),$$

where the terms going to zero for big Λ^2 were omitted. Now we can already express the renormalization counterterms. Applying the equation (2.35) we get

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}\phi_c^2}\right|_0 = 0,$$

which implies

$$B = -\lambda\frac{\Lambda^2}{32\pi^2}.$$

However, to determine the renormalized coupling the equation (2.36) cannot be used since the fourth derivative of V does not exist in the origin because of the infrared divergence. To avoid this trouble we can define the coupling λ at a point away from the singularity, i.e., we can write

$$\left.\frac{\mathrm{d}^4 V}{\mathrm{d}\phi_c^4}\right|_M = \lambda,$$

where M denotes an arbitrary scalar with dimension of a mass. Imposing this condition it is obtained

$$C = -\frac{3\lambda^2}{32\pi^2}\left(\log\frac{\lambda M^2}{2\Lambda^2} + \frac{11}{3}\right).$$

The infrared divergence affects also the standard condition for defining the scale of the field (2.37), but we can avoid it in the same way as we did above, i.e.,

$$Z(M) = 1.$$

As a result, substituting the derived expressions into the original potential we get the final formula for the one-loop effective potential

$$V = \frac{\lambda\phi_c^4}{4!} + \frac{\lambda^2\phi_c^2}{256\pi^2}\left(\log\frac{\phi_c^2}{M^2} - \frac{25}{6}\right).$$

2.6.3 The physical meaning of the effective potential

As we have seen, the effective potential in fact represents a quantum analogue of the classical potential; therefore, its physical meaning is analogical as well. The ordinary potential $U(\phi)$ appearing in the classical field theory is physically an energy density, i.e., the energy per unit volume for the state, in which the field has the value ϕ. Similarly, it is obvious in quantum field theory that the effective potential $V(\phi_c)$ is also an energy density. Namely, *it is the expectation value of the energy per unit volume in a state, in which the field has the expectation value* ϕ_c. Consequently, if the effective potential has more than just one local minimum, then the real ground state of the theory (that is the state of the lowest energy) has energy density of the the global (absolute) minimum of V.

Let us now prove the above statements using a beautiful quantum-mechanical analogy. Expanding $W(J)$ in a similar fashion as the effective action Γ in (2.34) it is obtained

$$W = \int \mathrm{d}^4x \left(-\mathcal{E}(J) + \frac{1}{2}(\partial_\mu J)\, X(J) + \dots\right), \tag{2.45}$$

where $J = J(x)$ is again the external source. Let us assume that this source is during a time T constant within a certain finite volume V. This constant will be denoted J and we will assume that the value of $J(x)$ goes smoothly to zero outside the boundary of the region defined by T and V. Consequently, the term $\mathcal{E}$ in intergral on the right hand side of the equation (2.45) becomes dominant and we can write

$$\exp(iW) = \langle 0^+|0^-\rangle \approx \exp(-iVT\mathcal{E}(J)).$$

This can be physically described as a departure from the original Hamiltonian density $\mathcal{H}$ to a new one which is smoothly changed by a small perturbation as

$$\mathcal{H} \to \mathcal{H} - J\phi.$$

Hence, the ground state of the theory within the volume defined by parameters V and T can be expected to go adiabatically to the ground state of the theory, whose Hamiltonian density contains the additional term. Naturally, the time development of the ground state is driven by the Schrödinger equation; hence, this state develops a certain phase. If we subsequently turn the perturbation down, the perturbed ground state returns back to the unperturbed one; however, it keeps the generated phase. Therefore, the quantity $\mathcal{E}(J)$ represents the energy per unit volume of the ground state corresponding to the perturbed theory.

Now we turn to the quantum mechanics providing a derivation, which will be possible to generalize easily to the field case and thus complete the proof. Let us consider a stationary state $|o\rangle$ of the quadratic form

$$\langle o|H|o\rangle,$$

assuming

$$\langle o|o\rangle = 1.$$

To find such a state the standard method of Lagrange multipliers can be used. If the Lagrange multiplier is denoted as E and the following form is varied

$$\langle o|\left(H - E\right)|o\rangle,$$

we get the equation

$$\left(H - E\right)|o\rangle = 0.$$

Now we will assume an extra constraint

$$\langle o|O|o\rangle = o,$$

where we defined a Hermitian operator O with eigenvalue o. As a result, we will now have to use two Lagrange multipliers denoted as E' and J. Again, varying the form

$$\langle o|\left(H - E' - Jo\right)|o\rangle$$

we obtain the equation

$$\left(H - E' - Jo\right)|o\rangle = 0.$$

Obviously, on grounds of the last equation we can define the perturbed Hamiltonian as $H - Jo$, whose eigenstate $|o\rangle$ corresponds to the energy E'. Hence, the eigenvalue o can be expressed as

$$o = \langle o|O|o\rangle = -\frac{\mathrm{d}E'}{\mathrm{d}J},$$

which allows us to express the form $\langle o|H|o\rangle$ as

$$\langle o|H|o\rangle = E' + Jo = E' - J\frac{\mathrm{d}E'}{\mathrm{d}J}.$$

If we now substitute the quantities according to the following "dictionary" relating the quantum mechanics with the effective potential

$$\begin{aligned}
\textbf{Quantum Mechanics} &\leftrightarrow \textbf{Effective Potential}\\
\text{Hamiltonian } H &\leftrightarrow \text{Hamiltonian density } \mathcal{H}\\
\text{energy } E &\leftrightarrow \text{energy density } \mathcal{E}\\
\text{source } J &\leftrightarrow \text{source density } \mathcal{J}\\
\text{operator } O &\leftrightarrow \text{field } \phi\\
\text{stationary state } |o\rangle &\leftrightarrow \text{vacuum } |0\rangle\\
\text{eigenvalue } o &\leftrightarrow \text{classical field (VEV) } \phi_c
\end{aligned}$$

we find that the line of arguments above matches exactly the formulas we went through when defining the effective potential. Hence, the physical meaning of the effective potential can be easily inferred from the resulting relation

$$V(\phi_c) = \langle o|\mathcal{H}|o\rangle,$$

where $|o\rangle$ is the state, for which

$$\delta\langle o|\mathcal{H}|o\rangle = 0$$

and the constraints

$$\begin{aligned}
\langle o|o\rangle &= 1,\\
\langle o|\phi|o\rangle &= \phi_c
\end{aligned}$$

are satisfied.

The physical interpretation given above can be explained also by another argument. If we consider just one-dimensional space, the Lagrange density can be substituted by the Lagrangian corresponding to a particle with a unit mass. Similarly, ϕ becomes x and consequently $U(\phi)$ is substituted by the potential $U(x)$, which influences the motion of the particle. Consequently, the equation (2.41) can be rewritten as

$$V = U + \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} \log\left(\omega^2 + U'' - i\varepsilon\right) = U + \frac{1}{2}\left(U'' - i\varepsilon\right)^{\frac{1}{2}}, \tag{2.46}$$

which can be interpreted as follows: While in the classical case the particle is situated in the potential minimum having the energy equal to U in this minimum, in the quantum case the potential has to be approximated around the minimum by the potential of the harmonic oscillator and the particle energy must be increased by the value of the ground state energy of the harmonic oscillator (the second term in equation (2.46)).

3. The $45 \oplus 16$ model at the quantum level

As we have described in detail in the first part of the previous chapter, the minimal $SO(10)$ models with Higgs sector formed either by $45_S \oplus 16_S$, or by $45_S \oplus 126_S$ representation were for a long time considered to be excluded by the phenomenology [51, 52, 53, 54] since it was acknowledged that just the inconsistent symmetry breaking patterns in these scenarios were allowed. However, recently it was shown that these models can be revived [55] if the radiative corrections to the scalar mass spectrum are calculated.

In this chapter we will first reproduce the calculations of the full one-loop scalar mass corrections to the potentially tachyonic pseudo-Goldstone bosons using the effective potential approach from section 2.6. Except for the computation itself, we will present also the behaviour of the results in interesting limits.

Later on, we will use the standard diagrammatic methods for the calculation of the leading polynomial one-loop corrections to the masses of the problematic pseudo-Goldstone bosons. This approach will turn out to be very smart and particularly much easier than the effective potential machinery.

3.1 Effective potential and its applications

The main purpose of this part is to understand the quantum structure of the $45 \oplus 16$ Higgs model. We will therefore mostly reproduce the calculations of the recently published results [55].

3.1.1 One-loop scalar spectrum of the $45 \oplus 16$ model

Let us first focus on the calculation of the full scalar corrections of the masses corresponding to the potentially tachyonic states labelled as $(8,1,0)$ and $(1,3,0)$ in the basis $3_C 2_L 1_Y$.

Since the vacuum expectation value χ_R is not important for the production of significant mass corrections, it does not influence the viability of the phenomenologically viable breaking chains at the quantum level. Thanks to this fact it is possible to set $\chi_R = 0$ and perform the one-loop level calculations in this limit. The zero value of the χ_R decouples the mass matrices of the 45_S and the 16_S sectors, which are used in the effective potential approach. Consequently, in the following text we will focus only on vacuum configurations

$$\begin{aligned}
\chi_R = 0, \omega_R = 0, \omega_Y \neq 0 &\to 3_C 2_L 2_R 1_{B-L}, \\
\chi_R = 0, \omega_R \neq 0, \omega_Y = 0 &\to 4_C 2_L 1_R, \\
\chi_R = 0, \omega_R = -\omega_Y \neq 0 &\to \text{flipped } 5' 1_{Z'}.
\end{aligned}$$

Vanishing χ_R means that the condition (2.22) is trivially satisfied and the tree-level scalar spectrum also changes in this limit. However, to obtain the scalar spectrum for $\chi_R = 0$ it is not enough to send this VEV to zero in the expressions corresponding to the fully general spectrum. The calculation of the spectrum has to be performed anew. Nevertheless, the potentially tachyonic states $(8,1,0)$ and $(1,3,0)$, in which we are interested, do not change.

3.1.2 The effective potential at the one-loop level

Now, we are ready to start the calculation of the scalar quantum corrections to the potentially tachyonic masses $M^2(8,1,0)$ and $M^2(1,3,0)$ employing the effective potential approach. The one-loop effective potential of the minimal $45\oplus 16$ model can be written as

$$V_{eff} = V + \Delta V_{scalar} + \Delta V_{gauge} + \Delta V_{fermion}, \tag{3.1}$$

where V is the scalar potential at the tree level, while the terms ΔV_{scalar}, ΔV_{gauge} and $\Delta V_{fermion}$ denote the one-loop quantum corrections corresponding to scalars, gauge bosons and fermions in the loop, respectively. Using the Landau gauge and the dimensional regularization with the modified minimal subtraction $\overline{MS}$, the one-loop corrections to the tree-level potential read

$$\begin{aligned}
\Delta V_{scalar}\left(\phi,\chi,\mu\right) &= \frac{\zeta}{64\pi^2}\mathrm{Tr}\left[M^4\left(\phi,\chi\right)\left(\log\left(\frac{M^2\left(\phi,\chi\right)}{\mu^2}\right)-\frac{3}{2}\right)\right], \\
\Delta V_{gauge}\left(\phi,\chi,\mu\right) &= \frac{3}{64\pi^2}\mathrm{Tr}\left[\mathcal{M}^4\left(\phi,\chi\right)\left(\log\left(\frac{\mathcal{M}^2\left(\phi,\chi\right)}{\mu^2}\right)-\frac{3}{2}\right)\right], \\
\Delta V_{fermion}\left(\phi,\chi,\mu\right) &= \frac{\eta}{64\pi^2}\mathrm{Tr}\left[\mathscr{M}^4\left(\phi,\chi\right)\left(\log\left(\frac{\mathscr{M}^2\left(\phi,\chi\right)}{\mu^2}\right)-\frac{3}{2}\right)\right],
\end{aligned} \tag{3.2}$$

where the coefficient ζ equals to 1 or 2 for real or complex scalars, respectively, while the coefficient η is equal to 2 or 4 for Weyl or Dirac spinors, respectively. The M, $\mathcal{M}$, $\mathscr{M}$ represent the functional mass matrices of scalars, gauge bosons and fermions. In this thesis we are interested especially in the scalar spectrum of the $45\oplus 16$ model; hence, let us now build the functional mass matrix $M^2(\phi,\chi)$ corresponding to the scalars. Given the basis of $45+(2\times 16)$ fields, the dimension of the matrix $M^2(\phi,\chi)$ is 77. This matrix is hermitian and the relevant term in the Lagrangian reads

$$\mathcal{L} \ni \frac{1}{2}\psi^\dagger M^2(\phi,\chi)\psi,$$

where we accommodated all the fields into a single vector $\psi = (\phi,\chi,\chi^*)$. Therefore, $M^2(\phi,\chi)$ is a block matrix of the form

$$M^2(\phi,\chi) = \begin{pmatrix} V_{\phi\phi} & V_{\phi\chi} & V_{\phi\chi^*} \\ V_{\chi^*\phi} & V_{\chi^*\chi} & V_{\chi^*\chi^*} \\ V_{\chi\phi} & V_{\chi\chi} & V_{\chi\chi^*} \end{pmatrix}, \tag{3.3}$$

where the particular blocks contain the derivatives of the scalar potential V with respect to the fields denoted by the corresponding subscripts; for instance, $V_{\phi\chi^*}$ stands for $\frac{\partial^2 V}{\partial\phi\partial\chi^*}$ and so on.

3.1.3 The stationarity conditions at one loop

As in the tree level case we first have to calculate the stationarity conditions given the new, quantum-level potential. In what follows we shall focus on the calculation of the scalar corrections eqrefscalcor; consequently, just the relevant part of the effective potential (3.1) will be used, namely with $V + \Delta V_{scalar}$. Let us therefore take the derivative of ΔV_{scalar} with respect to the field component ψ_i. The following expression is obtained

$$\frac{\partial \Delta V_{scalar}}{\partial \psi_i} = \frac{1}{64\pi^2} \mathrm{Tr} \left[M^2 M^2_{\psi_i} + \{M^2_{\psi_i}, M^2\} \left(\log \left(\frac{M^2}{\mu^2} \right) - \frac{3}{2} \right) \right], \tag{3.4}$$

where we used the shorthand notation $M^2_{\psi_i} = \frac{\partial M^2(\phi,\chi)}{\partial \psi_i}$. In case of the gauge bosons or fermions we could proceed in the same way.

Using the scalar spectrum computed in the previous chapter the matrix $M^2(\phi, \chi)$ and its derivatives can be calculated quite easily. They are not presented here explicitly because they are too large. The most important thing they imply is that the corrected condition allow for the same vacuum settings as the tree-level conditions did.

3.1.4 The one-loop scalar mass corrections

Having expressed the stationarity condition, we can now turn to the mass correction itself. Taking the second derivative of the ΔV_{scalar} a much more complicated formula than was the case of the first derivative (3.4) emerges

$$\begin{aligned} \frac{\partial^2 \Delta V_{scalar}}{\partial \psi_i \partial \psi_j} = \frac{1}{64\pi^2} \mathrm{Tr} \Big[& M^2_{\psi_i} M^2_{\psi_j} + M^2 M^2_{\psi_j} \\ & + \left(\{M^2, M^2_{\psi_i \psi_j}\} + \{M^2_{\psi_i}, M^2_{\psi_j}\} \right) \log \left(\frac{M^2}{\mu^2} - \frac{3}{2} \right) \\ & + \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \sum_{k=1}^{n} \binom{n}{k} \{M^2, M^2_{\psi_i}\} \\ & \times \left[M^2, \ldots \left[M^2, M^2_{\psi_j} \right] \ldots \right] \left(M^2 - 1 \right)^{n-k} \Big], \end{aligned} \tag{3.5}$$

where the series in the last line involves $(k-1)$ nested commutators descending from the commutation properties of the M^2 matrix with its derivatives. Obviously, the situation is now more complicated than it was in case of the first derivative (3.4). There were no nested commutators since the trace properties allowed to avoid them. Hence, the formula was obtained in quite a simple form independently on commutation properties of M^2 with its derivative. It is worth mentioning that the right hand side of the above equation is really symmetric in subscripts ψ_i and ψ_j.

3.1.5 From the running to the pole mass

Evaluating the equation (3.5) in the vacuum of the potential one finds that the numerators of the arguments of certain logarithms equal exactly to the tree stationarity condition (i.e., equation obtained as the first derivative of the tree-level scalar potential). Hence, the infrared singularities occur once the stationarity condition is substituted into (3.5).

This instabilities has the origin in the fact that the effective potential is represented by the first term in the momentum expansion of the effective action around zero momentum [66]. Hence, the expression

$$\overline{m}_{ij}^2 = \left\langle \frac{\partial^2 V_{eff}}{\partial\psi_i \partial\psi_j} \right\rangle \tag{3.6}$$

does not represent the physical (i.e., pole) mass corresponding to a massive scalar. In general, the physical mass is in any renormalization scheme S given by the root of the renormalized inverse propagator

$$\Gamma_S^{(2)}(p^2) \equiv p^2 - \mu_S^2 - \Sigma_S(p^2) = 0,$$

where μ_S^2 represents the renormalized mass, while $\Sigma_S(p^2)$ is the self-energy corresponding to the momentum p.

The masses (3.6) given by the second derivatives of the (dimensionally regularized $\overline{MS}$) effective potential correspond to $\left(\Gamma^{(2)}_{\overline{MS}}(0)\right)_{ij}$, i.e.,

$$\overline{m}_{ij}^2 = -\left(\mu^2_{\overline{MS}}\right)_{ij} - \left(\Sigma_{\overline{MS}}(0)\right)_{ij}. \tag{3.7}$$

Consequently, the physical mass can be obtained as solution of the secular equation of the relevant eigenvalue problem

$$\det\left[\delta_{ij}p^2 - \overline{m}_{ij}^2 - \left(\Sigma_{\overline{MS}}(p^2)\right)_{ij} + \left(\Sigma_{\overline{MS}}(0)\right)_{ij}\right] = 0, \tag{3.8}$$

where $\overline{m}_{ij}^2$ is the zero-momentum mass matrix given by (3.6) and $\Sigma_{\overline{MS}}$ stands for the matrix of the scalar self-energies.

From the above expression it is obvious why the instabilities cannot occur in the Goldstone sector. For zero p^2-eigenvalues the self-energies cancel out from the equation (3.8) and, thus, there are no extra contributions.

In case that the physical mass m_i^2 is small when compared to the GUT-scale fields contributing to $\Sigma(0)$, then the corresponding correction $\Delta\Sigma \equiv \Sigma(p^2) - \Sigma(0)$ is of the order of $O\left(\frac{m_i^4}{M_G^2}\right)$ and the running mass (3.6) contains the leading gauge independent corrections.

3.1.6 The scalar mass corrections to the tree-level tachyonic masses

The analysis of the tree level scalar spectrum showed that the masses corresponding to the fields $(8, 1, 0)$ and $(1, 3, 0)$ depend just on a single potential parameter a_2 and they turn out to be tachyonic for the viable symmetry breaking patterns. Nevertheless, also the terms proportional to τ and β could be expected. Although they do not contribute at the tree level, they could give a significant contribution at the quantum level. As we have also argued, the corrections dependent on χ_R are negligible. Therefore, the one-loop mass correction can be calculated in the limit $\chi_R = 0$ in which the matrix (3.3) has a block diagonal form as the 45_S and 16_S sectors effectively decouple. Moreover, the relevant leading one-loop corrections are determined by the block $V_{\chi^*\chi}$, which significantly simplifies the calculation.

Another fact which makes the calculation of the corrections to the masses of $(8,1,0)$ and $(1,3,0)$ easier is that for $\chi_R = 0$ the matrix $M^2(\phi,\chi)$ and its first derivative with respect to these pseudo-Goldostone-boson states do commute. As a result, the complicated series containing the nested commutators vanish (except the first term of the series corresponding to $k = 1$) and the relevant formula is thus much simpler.

3.1.6.1 Full leading corrections and the $SU(5)$ limit

Now we can finally perform the explicit calculation of the leading scalar one-loop corrections $\Delta M^2(8,1,0)$ and $\Delta M^2(1,3,0)$. The full stationarity equations we get by merging the tree (2.20)-(2.22) and one-loop (3.4) stationarity conditions. Similarly, the corrected mass then can be obtained as a sum of the tree (2.25)-(2.26) and the one-loop (3.5) parts. Moreover, the cumbersome part containing the series with the nested operators can be omitted since for the interesting potentially tachyonic submultiplets $(8,1,0)$ and $(1,3,0)$, the commutation $[M^2, M^2_{\psi_i}] = 0$ is satisfied. Hence, the relevant formula reads

$$
\begin{aligned}
\frac{\partial^2 \left(V + \Delta V_{scalar}\right)}{\partial\psi_i\partial\psi_j} = \frac{\partial^2 V}{\partial\psi_i\partial\psi_j} + \frac{1}{64\pi^2}\mathrm{Tr}\Bigg[& M^2_{\psi_i}M^2_{\psi_j} + M^2 M^2_{\psi_j} \\
& + \left(\{M^2, M^2_{\psi_i\psi_j}\} + \{M^2_{\psi_i}, M^2_{\psi_j}\}\right)\log\left(\frac{M^2}{\mu^2} - \frac{3}{2}\right) \\
& + \{M^2, M^2_{\psi_i}\}\Bigg],
\end{aligned}
\tag{3.9}
$$

where the derivatives are taken with respect to the submultiplets $(8,1,0)$ and $(1,3,0)$ of the 45_S.

Remarkably, to determine the leading one-loop mass corrections it is not necessary to take the full second derivative (3.9) and substitute into it the full stationarity condition. In fact, after explicit computation of the equations (3.9) and (3.4) we find that they both consist of a tree-level part, a leading polynomial one-loop part, a leading logarithmic one-loop part and a subleading logarithmic part. Inspecting the explicit forms of the stationarity conditions and the one-loop masses it turns out that to get the leading loop correction it is enough to substitute the tree-level parts of the stationarity conditions (2.20) and (2.21) into the leading one-loop parts of the mass formula (3.5) and the leading one-loop part of the stationarity condition (3.4) into the tree-level term plus into the leading one-loop parts of the mass formula (3.5).

Let us now present the resulting quantum-level leading order expressions (including the leading logarithmic terms) for the masses of $(8,1,0)$ and $(1,3,0)$. Although they are quite complicated, we will write them here and use them to show a beautiful cross-check consisting in the calculation of the $SU(5)$ limit.

The leading part of the one-loop mass of the submultiplet $(8,1,0)$ reads

$$\begin{aligned}
M^2_{full}(8,1,0) = \frac{1}{16\pi(\omega_R-\omega_Y)} \big[\, & 4(\omega_R-\omega_Y)(\tau^2+\beta^2(\omega_R^2-\omega_R\omega_Y+3\omega_Y^2) \\
& + 8a_2\pi(\omega_R-\omega_Y)(\omega_R+2\omega_Y)) \\
& + (\tau+\beta\omega_R)(\omega_R+3\omega_Y)(\tau-3\beta\omega_Y)\log_\mu[2(\tau+\beta\omega_R)(\omega_R+3\omega_Y)] \\
& + (-\tau^2(\omega_R+3\omega_Y)+2\beta\tau(-2\omega_R^2+3\omega_R\omega_Y+3\omega_Y^2) \\
& \quad + \beta^2(4\omega_R^3+3\omega_R\omega_Y^2+5\omega_Y^3))\log_\mu[4(\omega_R+\omega_Y)(\tau+\beta\omega_Y)] \\
& - 2\omega_R(\tau+3\beta\omega_Y)(\tau-2\beta\omega_R+3\beta\omega_Y)\log_\mu[4\omega_R(\tau+3\beta\omega_Y)] \\
& + (\tau^2\omega_R+4\beta\tau\omega_R^2+4\beta^2\omega_R^3+3\tau^2\omega_Y+6\beta\tau\omega_R\omega_Y \\
& \quad + 6\beta\tau\omega_Y^2+9\beta^2\omega_R\omega_Y^2-\beta^2\omega_Y^3)\log_\mu[4\omega_Y(\tau+\beta(2\omega_R+\omega_Y))] \\
& + (\tau^2(\omega_R-3\omega_Y)+\beta^2(-4\omega_R^3-9\omega_R^2\omega_Y+3\omega_R\omega_Y^2+4\omega_Y^3) \\
& \quad - \beta\tau(5\omega_R^2+3\omega_Y^2))\log_\mu[2(\omega_R+\omega_Y)(\tau+\beta(\omega_R+2\omega_Y))] \, \big] \, .
\end{aligned} \tag{3.10}$$

Similarly, the leading order quantum-level mass of $(1,3,0)$ reads

$$\begin{aligned}
M^2_{full}(1,3,0) = \frac{1}{16\pi(\omega_R-\omega_Y)} \big[\, & 4(\omega_R-\omega_Y)(\tau^2+\beta^2(2\omega_R^2-\omega_R\omega_Y \\
& + 2\omega_Y^2)+8a_2\pi(-2\omega_R^2+\omega_R\omega_Y+\omega_Y^2)) \\
& + (2\tau^2(\omega_R+\omega_Y)-\beta\tau(\omega_R^2+10\omega_R\omega_Y-3\omega_Y^2)-\beta^2(2\omega_R^3 \\
& \quad + 7\omega_R^2\omega_Y-6\omega_R\omega_Y^2+9\omega_Y^3))\log_\mu[2(\tau+\beta\omega_R)(\omega_R+3\omega_Y)] \\
& + 2(\omega_R+\omega_Y)(\tau+\beta\omega_Y)(-\tau+\beta(2\omega_R+\omega_Y)) \\
& \quad \times \log_\mu[4(\omega_R+\omega_Y)(\tau+\beta\omega_Y)] \\
& - 2\omega_R(\tau+3\beta\omega_Y)(\tau-2\beta\omega_R+3\beta\omega_Y)\log_\mu[4\omega_R(\tau+3\beta\omega_Y)] \\
& + 4\omega_Y((\tau+2\beta\omega_R)^2-\beta^2\omega_Y^2)\log_\mu[4\omega_Y(\tau+\beta(2\omega_R+\omega_Y))] \\
& + (2\tau^2(\omega_R-2\omega_Y)+\beta\tau(-7\omega_R^2+2\omega_R\omega_Y-3\omega_Y^2) \\
& \quad + \beta^2(-6\omega_R^3-13\omega_R^2\omega_Y+6\omega_R\omega_Y^2+7\omega_Y^3)) \\
& \qquad \times \log_\mu[2(\omega_R+\omega_Y)(\tau+\beta(\omega_R+2\omega_Y))] \, \big] \, .
\end{aligned} \tag{3.11}$$

In both results (3.10) and (3.11) the redefined logarithm

$$\log_\mu(x) \equiv \log\left(\frac{x}{\mu^2}\right)$$

was used.

As expected, these expressions (3.10) and (3.11) depend on potential parameters τ and β and they contribute significantly to the potentially tachyonic masses of the $(8,1,0)$ and $(1,3,0)$ pseudo-Goldstone bosons. Therefore, on grounds of the above results, the symmetry breaking patterns which seemed to be forbidden at the tree level can be revived at the quantum level.

Let us note that both these multiplets should remain massless in the $SU(5)$ limit $\omega_R \to \omega_Y$ because at the $SU(5)$ level they are massless Goldstone bosons belonging to the $(24,0)$ representation. Indeed, the complicated expressions presented above really reduce to zero

$$\lim_{\omega_R\to\omega_Y} M^2_{full}(8,1,0) = 0, \tag{3.12}$$

$$\lim_{\omega_R\to\omega_Y} M^2_{full}(1,3,0) = 0, \tag{3.13}$$

which can be viewed as a nice consistency check of the obtained results.

In the $SU(5)$ limit we could also simply replace the redefined logarithm $\log_\mu$ by the standard log because the prefactors of logarithms $\log(\mu^2)$ cancel for $\omega_R = \omega_Y$ as can be easily verified. Vanishing of remaining logarithms is more complicated since they have different arguments. Hence, for their complete vanishing in the $SU(5)$ limit the leading polynomial part of the corrections is essential.

3.1.6.2 The leading polynomial terms

Although the full leading one-loop masses of the submultiplets $(8,1,0)$ and $(1,3,0)$ were presented above, for our further proceeding the leading polynomial one-loop corrections to these masses are the most interesting parts. As usually, the leading nonlogarithmic corrections are of higher order than the logarithmic ones.

The leading polynomial one-loop corrections to the masses of the fields $(8,1,0)$ and $(1,3,0)$ can be written as

$$\Delta M^2(8,1,0) = \frac{\tau^2 + \beta^2\left(\omega_R^2 - \omega_R\omega_Y + 3\omega_Y^2\right)}{4\pi^2} + \ldots, \tag{3.14}$$

$$\Delta M^2(1,3,0) = \frac{\tau^2 + \beta^2\left(2\omega_R^2 - \omega_R\omega_Y + 2\omega_Y^2\right)}{4\pi^2} + \ldots. \tag{3.15}$$

Therefore, the masses of the potentially tachyonic pseudo-Goldstone bosons including the nonlogarithmic parts of the quantum corrections read

$$\begin{aligned} M^2(8,1,0) &= 2a_2\left(\omega_R - \omega_Y\right)\left(\omega_R + 2\omega_Y\right) \\ &\quad + \frac{\tau^2 + \beta^2\left(\omega_R^2 - \omega_R\omega_Y + 3\omega_Y^2\right)}{4\pi^2} + \ldots, \end{aligned} \tag{3.16}$$

$$\begin{aligned} M^2(1,3,0) &= 2a_2\left(\omega_Y - \omega_R\right)\left(\omega_Y + 2\omega_R\right) \\ &\quad + \frac{\tau^2 + \beta^2\left(2\omega_R^2 - \omega_R\omega_Y + 2\omega_Y^2\right)}{4\pi^2} + \ldots. \end{aligned} \tag{3.17}$$

3.1.6.3 Potentially tachyonic masses in various limits

Just for a later convenience we now rewrite here the above results (3.16) and (3.17) in the three limits corresponding to the three interesting vacuum configurations. These expressions will be important particularly for comparison with the results yielded by the diagrammatic method of calculation of the leading polynomial quantum corrections.

First, for the setting with $\chi_R = 0, \omega_R = 0, \omega_Y \neq 0$ the intermediate breaking scale is $3_C 2_L 2_R 1_{B-L}$ and the mass corrections in this limit are

$$M^2(8,1,1,0) = -4a_2\omega_Y^2 + \frac{\tau^2 + 3\beta^2\omega_Y^2}{4\pi^2} + \ldots,$$

$$M^2(1,3,1,0) = 2a_2\omega_Y^2 + \frac{\tau^2 + 2\beta^2\omega_Y^2}{4\pi^2} + \ldots.$$

In the case of the $\chi_R = 0, \omega_R \neq 0, \omega_Y = 0$ configuration the intermediate symmetry is $4_C 2_L 1_R$ and the mass corrections have the form

$$M^2(15,1,0) = 2a_2\omega_R^2 + \frac{\tau^2 + \beta^2\omega_R^2}{4\pi^2} + \ldots,$$

$$M^2(1,3,0) = -4a_2\omega_R^2 + \frac{\tau^2 + 2\beta^2\omega_R^2}{4\pi^2} + \ldots.$$

Finally, let us consider the vacuum setting with $\chi_R = 0, \omega \equiv \omega_R = -\omega_Y \neq 0$ corresponding to the flipped $SU(5)$ symmetry. In this case one has

$$M^2(24,0) = -4a_2\omega^2 + \frac{\tau^2 + 5\beta^2\omega^2}{4\pi^2} + \ldots .$$

There is just a single expression valid for both submultiplets $(8,1,0)$ and $(1,3,0)$ because they are both contained in the $(24,0)$ multiplet of the flipped $SU(5)$.

3.1.7 Gauge corrections

For the sake of completeness, let us make a brief mention of the gauge corrections. Although we have described just the calculation of the scalar corrections to the masses of the pseudo-Goldstone bosons labelled by $(8,1,0)$ and $(1,3,0)$, the relevant leading gauge corrections can be also calculated, as we already have all the prerequisities at hand. Hence, according to [55] the total one-loop mass corrections of the potentially tachyonic fields read

$$\begin{aligned}\Delta M^2(8,1,0) &= \frac{1}{4\pi^2}\left[\tau^2 + \beta^2\left(\omega_R^2 - \omega_R\omega_Y + 3\omega_Y^2\right)\right. \\ &\quad \left. + g^4\left(16\omega_R^2 + \omega_R\omega_Y + 19\omega_Y^2\right)\right] + \ldots , \end{aligned} \tag{3.18}$$

$$\begin{aligned}\Delta M^2(1,3,0) &= \frac{1}{4\pi^2}\left[\tau^2 + \beta^2\left(2\omega_R^2 - \omega_R\omega_Y + 2\omega_Y^2\right)\right. \\ &\quad \left. + g^4\left(13\omega_R^2 + \omega_R\omega_Y + 22\omega_Y^2\right)\right] + \ldots . \end{aligned} \tag{3.19}$$

It is easy to determine how these corrections change for various vacuum settings as we did it just for the scalar corrections.

3.1.8 Necessary conditions for the local vacuum stability

Let us now proceed similarly as in case of the scalar potential at the tree level in the previous chapter and look at the conditions of the local vacuum stability. Considering the equations (3.18) and (3.19) in various limits we can get relations among the potential parameters a_2, β, τ and g at the scale $\mu = M_G$.

If the 45_S VEVs have the values $\omega_R = 0$, $\omega_Y \neq 0$ (χ_R is still zero), which implies the spontaneous breaking of the full symmetry down to $3_C 2_L 2_R 1_{B-L}$, the condition on the coefficient a_2 is obtained

$$a_2 > -\frac{1}{8\pi^2}\left(\tau^2\frac{1}{\omega_Y^2} + 2\beta^2 + 19g^4\right).$$

In case that the vacuum is set as $\omega_R = 0$, $\omega_Y \neq 0$, the resulting condition is quite similar, explicitly

$$a_2 > -\frac{1}{8\pi^2}\left(\tau^2\frac{1}{\omega_R^2} + \beta^2 + 13g^4\right).$$

For the vacuum configuration $\omega_R = -\omega_Y$ corresponding to the intermediate scale with the flipped $5'1_{Z'}$ symmetry the same condition reads

$$a_2 < 0.$$

Numerically, if we assume $\tau \sim \omega_{R,Y}$ as indicated by naturalness, we get

$$|a_2| < 10^{-2}.$$

3.2 Diagrammatic calculation of the leading one-loop mass corrections

In the previous chapter we performed a calculation of the quantum corrections to the scalar masses using the technique of the effective potential. In this chapter we will show how to recalculate most of these results by means of the standard perturbative theory, thus demonstrating their connection to the Coleman-Weinberg approach. First, we will focus on the Abelian Higgs model. Later we will apply the same methods to the $45 \oplus 16$ model to get the leading polynomial terms of the radiative corrections to the scalar masses of our interest.

3.2.1 Abelian Higgs Model

Let us consider the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V$$

describing a single complex scalar field charged under $U(1)$ gauge group with a selfinteraction given at the classical level by the scalar potential

$$V = m^2 |\phi|^2 + \lambda |\phi|^4. \tag{3.20}$$

As usual, we denote $\langle \phi \rangle = v$; with that at hand one can substitute ϕ with φ, for which $\langle \varphi \rangle = 0$, explicitly

$$\phi = \frac{1}{\sqrt{2}} (\varphi + v) \, e^{i\rho}.$$

The potential (3.20) can then be rewritten as

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2}(\varphi + v)\partial_\mu \rho \partial^\mu \rho + \frac{1}{2} m^2 v^2 + \frac{1}{4}\lambda v^4 + \left(m^2 v + \lambda v^3\right) \varphi \\ &+ \left(\frac{1}{2} m^2 + \frac{3}{2}\lambda v^2\right) \varphi^2 + \lambda v \varphi^3 + \frac{1}{4}\lambda \varphi^4, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} V = &\frac{1}{2} m^2 v^2 + \frac{1}{4}\lambda v^4 + \left(m^2 v + \lambda v^3\right) \varphi \\ &+ \left(\frac{1}{2} m^2 + \frac{3}{2}\lambda v^2\right) \varphi^2 + \lambda v \varphi^3 + \frac{1}{4}\lambda \varphi^4, \end{aligned} \tag{3.22}$$

where the standard $\lambda\varphi^4$-like terms

$$\frac{1}{2} m^2 \varphi^2 + \frac{1}{4}\lambda \varphi^4$$

can be found.

3.2.1.1 Calculation of the tree-level mass of the field φ

Let us start with the mass given by

$$m'^2 = \left\langle \frac{\partial^2 V}{\partial \varphi^2} \right\rangle,$$

where V is the tree-level potential of the theory. However, to get the right mass corresponding to the field φ we have to shift to the new asymmetric minimum. The mass of the field φ is therefore given by

$$m_\varphi^2 = \left\langle \frac{\partial^2 V}{\partial \varphi^2} \right\rangle \bigg|_{\langle \frac{\partial V}{\partial \varphi} \rangle = 0}.$$

In other words, we are looking for the value of $\langle \frac{\partial^2 V}{\partial \varphi^2} \rangle$ in the minimum of the potential.

Given $\langle \varphi \rangle$ the tree-level stationarity condition (SC) $\langle \frac{\partial V}{\partial \varphi} \rangle = 0$ in the case of potential (3.22) reads

$$m^2 v + \left(\lambda v^3\right) = 0, \tag{3.23}$$

which can be rewritten as

$$m^2 = -\frac{1}{v}\left(\lambda v^3\right). \tag{3.24}$$

The second derivative of the potential (3.22) expressed in vacuum explicitly reads

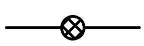
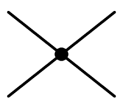
$$\left\langle \frac{\partial^2 V}{\partial \varphi^2} \right\rangle = m^2 + 3\lambda v^2. \tag{3.25}$$

Therefore, by substitution of the stationarity condition (3.24) into the equation (3.25) the squared tree-level mass m_φ^2 is obtained

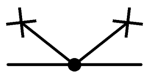
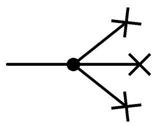
$$m_\varphi^2 = 3\lambda v^2 - \frac{1}{v}\left(\lambda v^3\right).$$

3.2.1.2 Diagrammatic form of the calculation

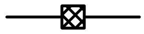
Let us now rewrite the presented calculation in terms of diagrams. The basic $\lambda\varphi^4$-terms can be defined as

 $\equiv \frac{1}{2} m^2,$  $\equiv \frac{1}{4}\lambda.$ (3.26)

The terms with two or three contracted legs can be depicted as

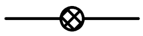
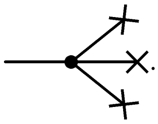
 $\equiv \frac{3}{2}\lambda v^2,$  $\equiv \lambda v^3.$ (3.27)

The mass term correspoding to the field φ will be diagrammatically represented as

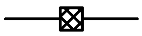
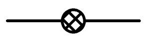
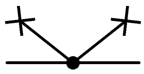
 $\equiv \frac{1}{2} m_\varphi^2.$ (3.28)

To distinguish the relations valid at the one-loop level from the tree-level expressions the label $\big|_{\text{tree/1-loop}}$ will be used.

Therefore, the expression (3.24) implied by the stationarity condition can be expressed diagramatically as

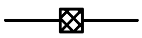
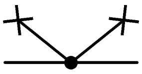
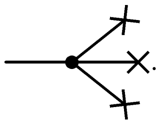
 $\Big|_{\text{tree}} = -\frac{1}{2v}$  .

The mass of the field φ at the tree level obtained as the second derivative of the potential (3.22) has the following diagrammatic form

$m_\varphi^2\big|_{\text{tree}} = 2$  $\Big|_{\text{tree}} = \Big(2$  $\Big|_{\text{tree}} + 2$  $\Big)\Big|_{\text{tree SC}}$,

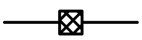
where the extra factor of 2 in front of the diagrams have to be added here in order to keep the calculation consistent with the above definitions (3.26)-(3.28).

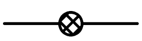
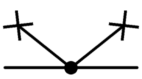
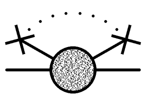
After substitution of the tree-level stationarity condition we get the explicit diagrammatic equation for the tree-level mass of the field φ

$m_\varphi^2\big|_{\text{tree}} = 2$  $\Big|_{\text{tree}} = 2$  $- \frac{1}{v}$  .

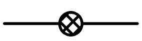
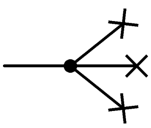
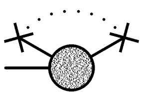
3.2.1.3 One-loop level mass of the field φ

To calculate the mass of φ at the one-loop level, the relevant radiative corrections have to be taken into account. For that sake, our diagramatic formula can be extended as follows

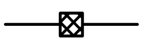
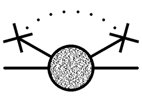
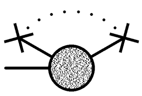
$m_\varphi^2\big|_{\text{1-loop}} = 2$  $\Big|_{\text{1-loop}}$

$= \Big(2$  $\Big|_{\text{1-loop}} + 2$  $+ 2$  $\Big)\Big|_{\text{1-loop SC}}$.

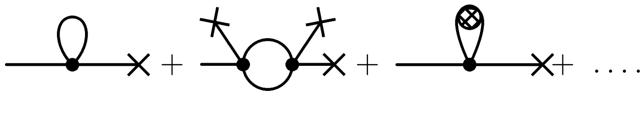
The stationarity condition at the one-loop level has the form

 $\Big|_{\text{1-loop}} = -\frac{1}{2v}$  $- \frac{1}{2v}$ 

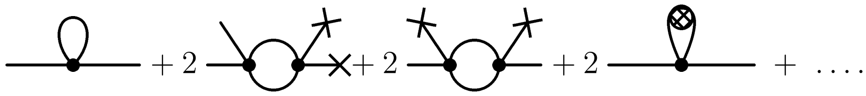
Consequently, the one-loop level mass term can be schematically written as

$m_\varphi^2\big|_{\text{1-loop}} = 2$  $\Big|_{\text{tree}} - \frac{1}{v}$  $+ \Big(2$  $\Big)\Big|_{\text{1-loop SC}}$. (3.29)

The one point irreducible diagrams we are interested in look like

$\Gamma^{(1)} =$  (3.30)

As for the two point irreducible graphs we have to sum contributions represented by the diagrams of form

$\Gamma^{(2)} = 2$  (3.31)

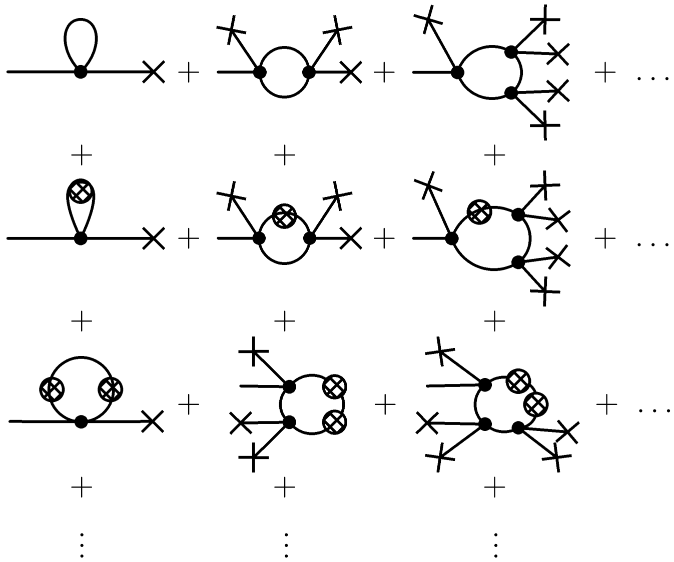
Obviously, we do not have to take into account the one point reducible graphs since we are calcaulating the loop correction to mass, i.e., we are interested in the p^2-independent part of one particle irreducible two point Green's function $\Gamma^{(2)}(p^2)$.

Let us now outline how to calculate the contribution from $\Gamma^{(1)}$. The propagators in tadpole diagrams can be always constructed out of r massive propagators corresponding to $\frac{1}{2}m^2$ (first diagram in (3.26)) and s second order interaction vertices of the form $\frac{3}{2}\lambda v^2$ (first diagram in (3.27)). Therefore, all the contributing diagrams can be sorted out into sets labelled as $(r, s)_1$ (the subscript 1 labels that we are interested in 1-point irreducible graphs) and to get the total contribution one has to sum the "series of series"

$(0, s)_1 \ni$ (3.32)

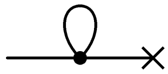
$(1, s)_1 \ni$ (3.33)

$(2, s)_1 \ni$ (3.34)



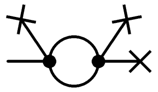
However, to calculate the contribution from the diagrams containing both types of the terms ($\frac{1}{2}m^2$ and $\frac{3}{2}\lambda v^2$) all the possible permutations of these terms have to be taken into account. The above diagrammatic "series of series" is just schematical since there are not displayed all these permutations. There is always just one diagram of each type $(r, s)_1$ representing all the possible graphs containing various permutations of the included terms.

As a result, to calculate the complete contribution from $\Gamma^{(1)}$ would be very difficult. However, let us for a later convenience calculate just the leading polynomial contribution given by the first series $(0, s)_1$ without any mass term in the propagator. The important thing to notice is that the series $(0, s)_1$ is a geometric series. Therefore, to calculate its sum one has to determine the first term and the quotient. The first diagram of the series $(0, s)_1$, i.e., diagram $(0, 0)_1$, equals to the expression



$$= 3\lambda v \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 + i\varepsilon}.$$

The second diagram of the (o, s) series, i.e. diagram $(0, 1)$, will contribute as



$$= 9\lambda^2 v^3 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(\frac{1}{k^2 + i\varepsilon}\right)^2$$

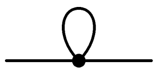
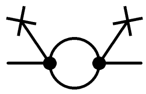
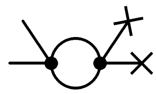
and similarly one could evaluate the following diagrams from the $(0, s)$ series. Consequently, the contribution to the mass of the field φ given by the $(0, s)$ series reads

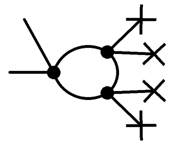
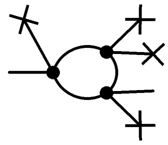
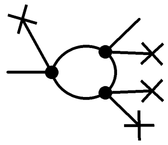
$$\begin{aligned}
i\Delta m_\varphi^2 \left((o, s)_1\right) &= \int \frac{\mathrm{d}k^4}{(2\pi)^4} \left(3\lambda v \frac{1}{k^2 + i\varepsilon} + 9\lambda^2 v^3 \left(\frac{1}{k^2 + i\varepsilon}\right)^2 + \dots\right) \\
&= \int \frac{\mathrm{d}k^4}{(2\pi)^4} \frac{3\lambda v \frac{1}{k^2+i\varepsilon}}{1 - 3\lambda v^2 \frac{1}{k^2+i\varepsilon}} \\
&= -\frac{i}{16\pi^2} 9\lambda^2 v^3 \left[C_{UV} + 1 - \log\left(\frac{9\lambda^2 v^3}{\mu^2}\right)\right],
\end{aligned}$$

where the well-known formula for the sum of the infinite geometric series was used. Taking into account the factor of 1 in the square bracket of the resulting expression the factor standing in front of the square bracket corresponds to the polynomial contribution of the series (3.32).

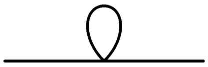
The contribution from $\Gamma^{(2)}$ could be depicted similarly as the above "series of series" corresponding to $\Gamma^{(1)}$. Each diagram would have two "free legs" (i.e., two legs without VEVs) instead of one, which was the case of $\Gamma^{(1)}$. Consequently, there would be more different diagrams than in case of $\Gamma^{(1)}$ because one has to take into account graphs with various positions of the two free legs (see the second and the third diagram in the equation (3.31)). Additionally, the overall factor of 2 would appear for the permutation of these two free legs (similarly as in (3.31)). Anyway, it would be very complicated to determine the complete contribution from $\Gamma^{(2)}$.

However, for a later convenience we can again calculate the leading polynomial contribution given by the corresponding $(0, s)_2$ series (the subscript 2 denotes that the set contains the 2-point irreducible graphs), i.e., by the following diagrams

$(0, s)_2 \ni 2$  $+ 2$  $+ 2$  (3.35)

$+ 2$  $+ 2$  $+ 2$  $+ \ \dots .$ (3.36)

Since we are interested just in the leading polynomial mass corrections, just the diagrams of the seagull type will contribute. Hence, the two ”free legs” must be attached to the same vertex. This statement can be simply clarified if we calculate the following type of graph



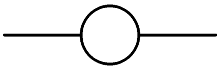
$$\propto \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)} = -\frac{i}{16\pi^2} m^2 \left[C_{UV} + 1 - \log\left(\frac{m^2}{\mu^2}\right) \right], \tag{3.37}$$

where

$$C_{UV} = \frac{1}{\varepsilon} - \gamma + \log(4\pi).$$

Hence, only the above diagrams will contribute to the desired leading polynomial (nonlogarithmic) terms, since they contain the constant factor 1 in the bracket (3.37).

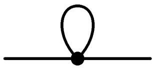
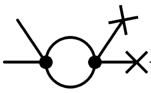
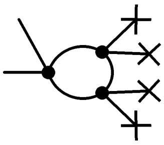
On the other hand, diagrams of type



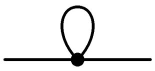
$$\propto \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)\left[(p-k)^2 - m^2\right]} = \frac{i}{16\pi^2} \left[C_{UV} - \int_0^1 \mathrm{d}x \log\left(\frac{m^2 - q^2 x(1-x)}{\mu^2}\right) \right]$$

cannot contribute to the terms of our interest, as they consist only of divergent and logarithmic parts.

As a result, to calculate the leading polynomial contribution from $\Gamma^{(2)}$ one does not have to consider all the diagrams in (3.36). The relevant contribution will be determined just by the following diagrams

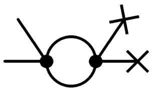
$(0, s)_2 \ni 2$  $+\,2$  $+\,2$  $+ \ \ldots .$ (3.38)

The first diagram in (3.38), i.e., the diagram $(0, 0)$, gives the contribution

2 

$$= 3\lambda \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 + i\varepsilon}.$$

The second diagram of the series (3.38), i.e. diagram $(0, 1)_2$, will give the contribution

2 

$$= 9\lambda^2 v^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left(\frac{1}{k^2 + i\varepsilon}\right)^2$$

and similarly one could evaluate the following diagrams in the $(0, s)_2$ series. Consequently, the leading polynomial correction to the mass of the field φ given by the $(0,s)_2$ series reads

$$
\begin{aligned}
i\Delta m_\varphi^2\left((o,s)_2\right) &= \int \frac{\mathrm{d}k^4}{(2\pi)^4}\left(3\lambda\frac{1}{k^2+i\varepsilon}+9\lambda^2v^2\left(\frac{1}{k^2+i\varepsilon}\right)^2+\ldots\right)\\
&= \int \frac{\mathrm{d}k^4}{(2\pi)^4}\frac{3\lambda\frac{1}{k^2+i\varepsilon}}{1-3\lambda v^2\frac{1}{k^2+i\varepsilon}}\\
&= -\frac{i}{16\pi^2}9\lambda^2v^2\left[C_{UV}+1-\log\left(\frac{9\lambda^2v^2}{\mu^2}\right)\right].
\end{aligned}
$$

As a result, the leading polynomial correction to the mass of the field φ given by the $(0,s)_1$ and $(0,s)_2$ series can be calculated. Taking into account the general formula (3.29), the leading polynomial correction to the mass of the field φ given by the $(0,s)_{1,2}$ series of diagrams reads

$$
\Delta m_\varphi^2\left((o,s)_{1,2}\right)_{\text{polynomial}} = -\frac{1}{16\pi^2}\left(-\frac{1}{v}9\lambda^2v^3+9\lambda^2v^2\right)=0,
$$

which agrees with the results in [66].

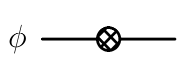
Of course, we have not calculated the complete leading polynomial correction to the mass of the field φ because we have omitted a large number of diagrams containing further polynomial contributions. Hence, if we calculated also the geometric series of diagrams containing the massive propagator (i.e., diagrams of type (3.33), (3.34), etc.), we would get also the term $4m^2$ of the leading polynomial correction, which appears in formula for one-point Green's function in [66]. However, also this contribution would be cancelled by equivalent term occuring in $\Gamma^{(2)}$ (see [66]).

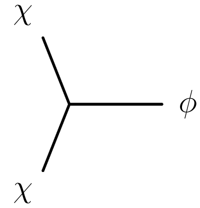
3.2.2 The $45 \oplus 16$ Higgs Model

Let us now focus on the main point of our interest, i.e., the $45 \oplus 16$ Higgs model. In the following, we will try to proceed similarly as in the simple Abelian Higgs model case and to determine diagramatically the leading polynomial (nonlogarithmic) parts of the radiative corrections to the scalar mass spectrum of the $45 \oplus 16$ model, namely, the masses $M^2(8,1,0)$ and $M^2(1,3,0)$ of the interesting multiplets $(8,1,0)$ and $(1,3,0)$. Since we are interested just in the leading polynomial mass corrections, we need to inspect only the loop diagrams giving such terms. Therefore, similarly as in case of the Abelian Higgs model, just the seagull type of diagrams (3.37) will contribute.

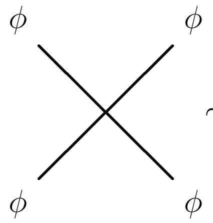
As we have mentioned in previous chapter, in case of the $45 \oplus 16$ Higgs model the tree level potential can be written as $V = V_\phi + V_{\phi\chi} + V_\chi$, where the individual components are give by (2.9), (2.10) and (2.11), respectively.

At the moment just the first two parts V_ϕ and $V_{\phi\chi}$ of the potential are of our interest since they include the vertices that can be used for construction of the seagulls, i.e.,



$\sim \mu^2$,


$\sim \tau\left(\Gamma\Gamma\right)$, (3.39)



$$\sim \beta \left(\Gamma\Gamma\Gamma\Gamma\right) . \tag{3.40}$$

Hence, we will take into account just tadpole diagrams built up from vertices corresponding to feynman rules with τ and β.

The important aspect that makes things more complicated in comparison with the Abelian model is the fact that now we have two different VEVs. However, to determine just the leading corrections we can do the following trick: We can calculate the contributions in three defferent limits

- $\omega_R = 0$, $\omega_Y \neq 0$, $\chi_R = 0$,
- $\omega_R \neq 0$, $\omega_Y = 0$, $\chi_R = 0$,
- $\omega_R = -\omega_Y$, $\chi_R = 0$ ($SU(5)$ limit).

Taking into account the Feynman rules (3.39) and (3.40), the non-logarithmic radiative corrections will include polynomials with terms proportional to ω_R^2, $\omega_R\omega_Y$ or ω_Y^2. Consequently, matching the three results calculated in the three mentioned limits will allow us to reconstruct the corrections in a complete form.

3.2.2.1 The τ^2 term

First, we try to reproduce the "simple" correction proportional to τ^2, which does not depend on any VEV and, as such, contributes uniformly to the masses of both $(8, 1, 0)$ and $(1, 3, 0)$. It is easy to see that it emerges from the trilinear interaction depicted above (3.39).

Let us rewrite the potential similarly as in the Abelian case using substitution

$$\phi_{ij} \to \left(\phi'_{ij} + \langle\phi_{ij}\rangle\right) ,$$

where $\langle\phi_{ij}\rangle$ are the VEVs of the fields ϕ_{ij} and ϕ'_{ij} are fields with zero VEVs. Hence, the relevant part of the potential reads

$$V_\tau = \mu^2 \left(\phi'_{ij} + \langle\phi_{ij}\rangle\right)\left(\phi'_{ji} + \langle\phi_{ji}\rangle\right) + \frac{\tau}{4}\left(\chi^\dagger\sigma_{ij}\chi\right)\left(\phi'_{ij} + \langle\phi_{ij}\rangle\right) . \tag{3.41}$$

In consequence, to obtain the relevant part of the stationarity condition one should take the first derivative of V_τ with respect to any of the Standard Model singlets and express it in the vacuum of the theory. However, from the calculation of the tree-level scalar spectrum we know that the Standard Model singlets belonging to the physical basis project onto the states ϕ_{12}, ϕ_{34}, ϕ_{56}, ϕ_{78} and ϕ_{910} from the defining basis. Hence, the first derivative of (3.41) can be taken with respect to one of these fields because it will be equal to the first derivative of (3.41) with respect to any of the Standard Model singlet up to an overall factor, which does not influence the resulting stationarity condition.

Let us therefore choose the field ϕ_{12} and perform the first derivative of (3.41) with respect to it. The resulting relation reads

$$\left\langle \frac{\partial V_\tau}{\partial \phi_{12}} \right\rangle = -2\mu^2 \langle \phi_{12} \rangle + \frac{\tau}{4} \langle \left(\chi^\dagger \sigma_{12} \chi \right) \rangle. \tag{3.42}$$

Consequently, the following stationarity condition is obtained

$$\mu^2 = \frac{\tau}{4} \frac{1}{2\langle \phi_{12} \rangle} \langle \left(\chi^\dagger \sigma_{12} \chi \right) \rangle. \tag{3.43}$$

As can be checked, its form is very similar to that of the stationarity condition (3.24) calculated in case of the Abelian Higgs model.

To get the τ^2-dependent mass correction corresponding to pseudo-Goldstone bosons $(8, 1, 0)$ and $(1, 3, 0)$ it is necessary to calculate the vacuum expectation value of the second derivative of (3.41) with respect to the appropriate field. Hence, it is necessary to substitute in (3.41) for the fields ϕ_{ij} the physical fields 45^a_{phys} (where $a = 1, \ldots, 45$) defined in Appendix C. The defining basis ϕ_{ij} can be obtained from the physical one using the relation

$$\phi_{ij} = C^a_{ij} 45^a_{phys},$$

where C^a_{ij} is the matrix of the corresponding unitary transformation. As a result, the equation (3.41) can be rewritten as

$$\begin{aligned} V_\tau = {} & \mu^2 C^a_{ij} C^{b*}_{ji} \left(45^a_{phys} + \langle 45^a_{phys} \rangle \right) \left(45^{b*}_{phys} + \langle 45^{b*}_{phys} \rangle \right) \\ & + \frac{\tau}{4} C^a_{ij} \left(\chi^\dagger \sigma_{ij} \chi \right) \left(45^a_{phys} + \langle 45^a_{phys} \rangle \right). \end{aligned}$$

The second derivative of the above expression with respect to the physical field of our interest reads

$$\left\langle \frac{\partial^2 V_\tau}{\partial 45^a_{phys} \partial 45^{a*}_{phys}} \right\rangle = C^a_{ij} C^{a*}_{ji} \mu^2 = -2\mu^2,$$

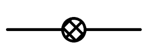
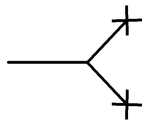
where the unitarity of the matrix C_{ij} was used ($\sum_{i,j=1}^{10} C^a_{ij} C^{*b}_{ij} = 2\delta^{ab}$).

By substitution of the stationarity condition (3.43) into this simple result the formula for the tree-level τ-dependent mass term is obtained

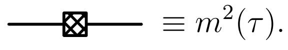
$$m^2(\tau)\big|_{\text{tree}} = -\frac{\tau}{4\langle \phi_{12} \rangle} \langle \left(\chi^\dagger \sigma_{12} \chi \right) \rangle. \tag{3.44}$$

Diagrammatic form

Similarly as we proceeded in the case of the Abelian Higgs model, we can now express the performed calculation using corresponding Feynman diagrams. Let us therefore define

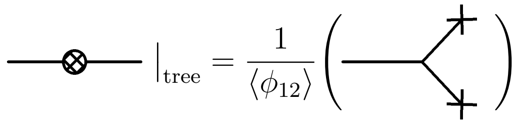
 $\equiv \mu^2,$  $\equiv \frac{\tau}{4} \langle \left(\chi^\dagger \sigma_{12} \chi \right) \rangle.$

The mass term m_τ^2 proportional to τ will be diagrammatically represented as

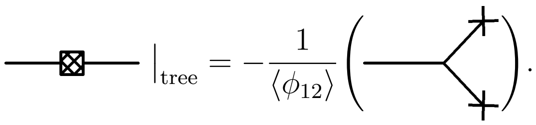


Again, to distinguish the relations valid at the one-loop level from the tree-level expressions the label $\big|_{\text{tree/1-loop}}$ will be used.

Employing these definitions, the stationarity condition can be expressed pictorially as



and the numerical result (3.44) in the diagrammatic form reads

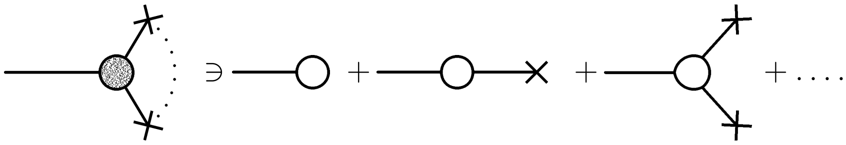


(3.45)

One-loop extension

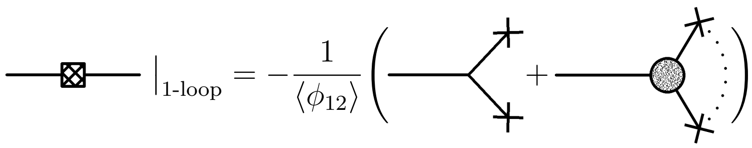
In analogy with the calculation performed for the Abelian Higgs model, the above diagrammatic equation can be extended to get their one-loop-level form.

Since τ appears just in the trilinear interaction, the only kind of diagram that will contribute to the τ^2 term is



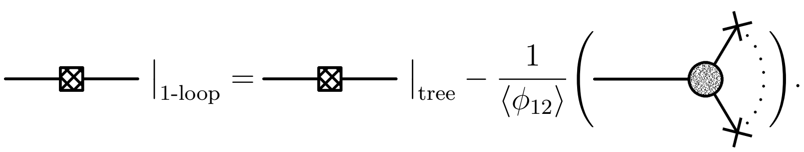
(3.46)

As a result, the equation (3.45) can be rewritten simply in the one-loop-level form as



(3.47)

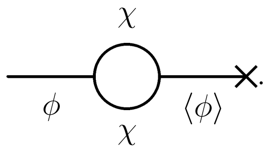
and equivalently



(3.48)

The resulting relation for the τ-dependent scalar mass correction is obviously very similar to the one obtained for the Abelian Higgs model (3.29). The difference between current calculations and those performed for the case of the Abelian Higgs model is that the φ field in the loop was massive, while now the loop field χ is massless. One could object to the fact that since we take $\chi = 0$, the ν^2 parameter plays the role of the mass corresponding to the field χ. Although this consideration is correct, it is not exactly our case, as we assume that χ is turned on; however, it is very small, so one can neglect it.

The series of diagrams on the right hand side of the equation (3.46) is in fact a geometric series that can be summed up into a single one-point diagram with a massive propagator with the mass $\propto \tau^2\omega$ similarly as we demonstrated in case of the Abelian Higgs Model. Technically, to determine this resulting mass one just has to calculate the quotient of this series; in other words, we have to calculate just the prefactor of the contribution given by the following diagram

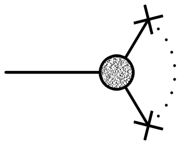


(3.49)

This quotient then plays the role of the mass of the massive propagator in the resulting (obtained by summation of the geometric series) one-point diagram (see the calculation of the leading polynomial mass corrections in case of the Abelian model). The mass of this massive propagator then corresponds exactly to the polynomial terms in the resulting mass correction obtained by integration (see the general formula for the integration of the seagull diagram (3.37)).

Therefore, the mass correction proportional to τ^2 reads

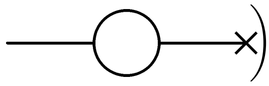
$$\Delta m^2(\tau) = -\frac{1}{\langle\phi_{12}\rangle}\Bigg(\qquad\Bigg). \tag{3.50}$$



Obviously, this correction will be the same for both the submultiplets $(8,1,0)$ and $(1,3,0)$ since the performed calculation do not depend on these fields. Hence, this mass contribution is $SO(10)$-invariant.

As we have already mentioned, instead of the diagram on the right hand side of (3.50) it is enough to focus on the quotient of these series, i.e., our aim is to calculate the expression

$$-\frac{1}{\langle\phi_{12}\rangle}Q \equiv -\frac{1}{\langle\phi_{12}\rangle}\Bigg(\qquad\Bigg). \tag{3.51}$$



Hence, the relevant term of the second order of the perturbation theory reads

$$\frac{1}{2!}\mathcal{L}^2 \ni \frac{1}{2!}\left(\frac{\tau}{4}\left(\chi_r^\dagger(\sigma_{ij})_{rs}\chi_s\right)\phi_{ij}\right)\left(\frac{\tau}{4}\left(\chi_u^\dagger(\sigma_{kl})_{uv}\chi_v\right)\phi_{kl}\right), \tag{3.52}$$

where the well-known overall factor $\frac{1}{2!}$ comes from the Dyson series and the indicated contractions give the propagator of the massless scalar field

$$\langle 0|\mathcal{T}\left(\chi(x)\chi^\dagger(y)\right)|0\rangle = \int \frac{\mathrm{d}k^4}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}e^{-ik(x-y)}.$$

However, to get the relevant mass contribution it is enough to determine the prefactor of the momentum-dependent integral as one could see in case of the Abelian Higgs model. At the leading order the τ^2 mass term (3.51) equals to the expression

$$-\frac{1}{\langle\phi_{12}\rangle}Q = -\frac{1}{\langle\phi_{12}\rangle}2\left[\frac{1}{2!}\frac{\tau}{4}\frac{\tau}{2}\mathrm{Tr}\left[\sigma_{12}\sigma_{kl}\right]\langle\phi_{kl}\rangle\right]\int\frac{\mathrm{d}k^4}{(2\pi)^4}\left(\frac{i}{k^2+i\varepsilon}\right)^2,$$

where we again took the only "free leg" of the diagram (3.49) to be the component ϕ_{12} corresponding to the direction in the defining basis onto which the Standard Model singlet is projected. One additional factor of $\frac{1}{2}$ was dropped since now we sum just over the indices $k, l = 1, \ldots, 10$. The extra overall factor of 2 in front of the square bracket reflects the fact that the VEV $\langle\phi_{12}\rangle$ can be taken also in place of the field ϕ_{kl} instead of ϕ_{ij}. Hence, there are in result two equal contributions.

To get the mass correction (3.50) we could repeat the calculation we performed in case of the Abelian Higgs model (3.35). Hence, the resulting mass correction will equal to the prefactor of the quotient of the corresponding geometric series multiplied by factor of $\left(-\frac{i}{16\pi^2}\right)$ coming from the integration. Thus, explicitly, the mass correction $\Delta m^2(\tau)$ reads

$$\begin{aligned} i\Delta m^2(\tau) &= \frac{i}{16\pi^2}\frac{1}{\langle\phi_{12}\rangle}\left(\frac{\tau^2}{8}\langle\phi_{kl}\rangle\right)\mathrm{Tr}[\sigma_{12}\sigma_{kl}] \\ &= \frac{i}{8\pi^2}\frac{1}{\langle\phi_{12}\rangle}\left(\tau^2\delta_{1[k}\delta_{2l]}\right)\langle\phi_{kl}\rangle = i\frac{\tau^2}{4\pi^2}, \end{aligned}$$

where the relation (A.8) was used. The above τ^2 term of the mass correction is in agreement with our previous results (3.14), (3.15).

3.2.2.2 The β^2 terms

While the tau term did not include any VEV and, thus, it was the same for both multiplets, the β^2 terms depend on ω_R and ω_Y and they differ for different fields. However, the derivation of the general formula for the β^2-dependent leading polynomial correction is formally same as in case of the τ^2 term in the previous section. However, now both one-point and two-point irreducible Green functions will be involved. The resulting relation will be analogous to the one we found in case of the Abelian model.

The relevant part of the scalar potential reads

$$\begin{aligned} V_\beta =&\mu^2\left(\phi'_{ij}+\langle\phi_{ij}\rangle\right)\left(\phi'_{ji}+\langle\phi_{ji}\rangle\right) \\ &+\frac{\beta}{16}\left(\chi^\dagger\sigma_{ij}\sigma_{kl}\chi\right)\left(\phi'_{ij}+\langle\phi_{ij}\rangle\right)\left(\phi'_{kl}+\langle\phi_{kl}\rangle\right) \end{aligned} \tag{3.53}$$

On grounds of the same arguments we used in the calculation of the τ^2 term let us take the first derivative of this part of the potential with respect to ϕ'_{12}

$$\left\langle\frac{\partial V_\beta}{\partial\phi_{12}}\right\rangle = -2\mu^2\langle\phi_{12}\rangle + \frac{\beta}{8}\langle(\chi^\dagger\sigma_{12}\sigma_{kl}\chi)\rangle\langle\phi_{kl}\rangle \tag{3.54}$$

and this stationarity condition implies the following formula

$$\mu^2 = \frac{1}{2\langle\phi_{12}\rangle}\left(\frac{\beta}{8}\left(\chi^\dagger\sigma_{12}\sigma_{kl}\chi\right)\langle\phi_{kl}\rangle\right). \tag{3.55}$$

To calculate the second derivative of the equation (3.53) with respect to the fields $(8,1,0)$ and $(1,3,0)$ it is again necessary to substitute for the fields ϕ_{ij} the physical fields 45^a_{phys} (where $a = 1, \ldots, 45$) defined in Appendix C, i.e.,

$$\begin{aligned} V_\beta =&\mu^2 C^a_{ij}C^{b*}_{ji}\left(45^a_{phys}+\langle 45^a_{phys}\rangle\right)\left(45^{b*}_{phys}+\langle 45^{b*}_{phys}\rangle\right) \\ &+\frac{\beta}{16}C^a_{ij}C^{b*}_{kl}\left(\chi^\dagger\sigma_{ij}\sigma_{kl}\chi\right)\left(45^a_{phys}+\langle 45^a_{phys}\rangle\right)\left(45^{b*}_{phys}+\langle 45^{b*}_{phys}\rangle\right) \end{aligned} \tag{3.56}$$

Therefore, the second derivative of the above expression with respect to a physical field 45^a_{phys} yields

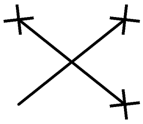
$$\left\langle \frac{\partial^2 V_\beta}{\partial 45^a_{phys} \partial 45^{a*}_{phys}} \right\rangle = -\mu^2 C^a_{ij} C^{a*}_{ij} + \frac{\beta}{16} C^a_{ij} C^{a*}_{kl} \langle (\chi^\dagger \sigma_{ij} \sigma_{kl} \chi) \rangle$$
$$= -2\mu^2 + \frac{\beta}{16} C^a_{ij} C^{a*}_{kl} \langle (\chi^\dagger \sigma_{ij} \sigma_{kl} \chi) \rangle .$$

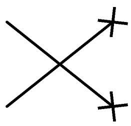
By the substitution of the stationarity condition (3.55) into the last equation one gets the tree-level β-dependent mass term

$$\left\langle \frac{\partial^2 V_\beta}{\partial 45^a_{phys} \partial 45^{a*}_{phys}} \right\rangle = \frac{\beta}{16} C^a_{ij} C^{a*}_{kl} \langle (\chi^\dagger \sigma_{ij} \sigma_{kl} \chi) \rangle - \frac{1}{\langle \phi_{12} \rangle} \frac{\beta}{8} \langle (\chi^\dagger \sigma_{12} \sigma_{kl} \chi) \rangle \langle \phi_{kl} \rangle . \quad (3.57)$$

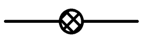
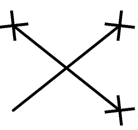
Diagrammatic form

Let us now again express the calculation in terms of Feynman diagrams. For mass terms we will use the same diagrams as we did in case of the τ^2 term calculation. However, instead of the trilinear interaction we now deal with the quadrilinear one. Hence, we define

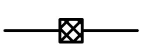
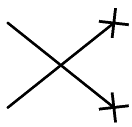
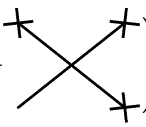
 $$\equiv \frac{\beta}{8} \langle (\chi^\dagger \sigma_{12} \sigma_{kl} \chi) \rangle \langle \phi_{kl} \rangle ,$$

 $$\equiv \frac{\beta}{16} C^a_{ij} C^{a*}_{kl} \langle (\chi^\dagger \sigma_{ij} \sigma_{kl} \chi) \rangle .$$

Employing these definitions, the stationarity condition (3.55) can be expressed as

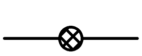
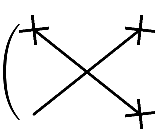
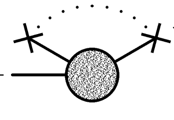
 $\Big|_{\text{tree}} = \frac{1}{2\langle \phi_{12} \rangle} \Bigg($  $\Bigg)$ (3.58)

and the resulting formula (3.57) in the diagrammatic form reads

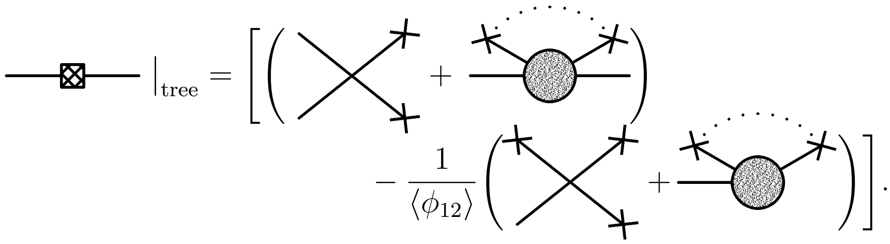
 $\Big|_{\text{tree}} = \Bigg($  $- \frac{1}{\langle \phi_{12} \rangle}$  $\Bigg).$ (3.59)

One-loop level

As usually, the next step is the extension of the diagrammatic equations (3.58) and (3.59) to the one-loop level. The stationarity condition then reads

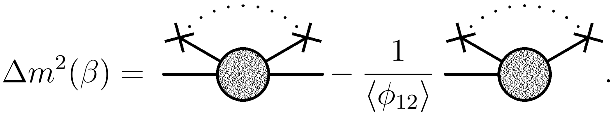
 $\Big|_{\text{tree}} = \frac{1}{2\langle \phi_{12} \rangle} \Bigg($  $+$  $\Bigg)$ (3.60)

and the formula (3.59) can be extended as



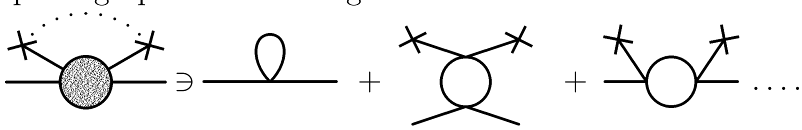
(3.61)

Hence, the β-dependent terms of the mass correction are given by the diagrammatic equation



(3.62)

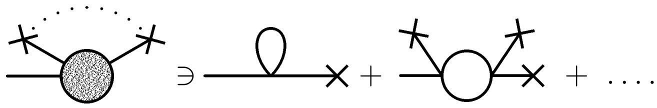
Two point graphs contributing to the first term in the above expression are



(3.63)

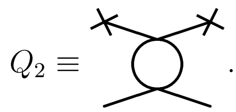
Since we are interested just in seagulls, the two free legs (those without VEVs) of the graph have to be attached to the same vertex. Hence, the last diagram in the above equation may be dropped from the series.

Analogically, one-point graphs that contribute to the second diagram in figure (3.62) can be depicted as

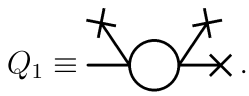


(3.64)

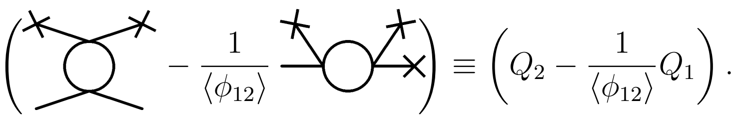
Obviously, the diagrams on the right hand side of equations (3.63) and (3.64) represent similarly as in case of tau term a geometric series. Therefore, we are again interested just in the relevant quotients. For two point graphs we define



For one point graphs we have

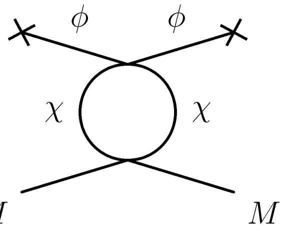


As a result, the relevant formula necessary for the calculation of the β-dependent leading polynomial scalar mass correction has the following form



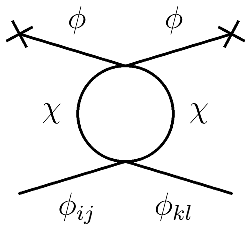
(3.65)

Now we apply the derived diagrammatic expressions to the fields of our interest, i.e., to a component M from one of the multiplets $(8,1,0)$ and $(1,3,0)$. In other words, we will calculate two point diagrams of type



(3.66)

rather than generic



,

because we want to calculate the contributions to the masses of the physical fields transforming as $(8,1,0)$ and $(1,3,0)$ and we can use the knowledge of the transfer matrix C connecting the physical basis with the defining one.

The relevant term of the second order of the perturbative theory is

$$\frac{1}{2!}\mathcal{L}^2 \ni \frac{1}{2!}\left(\frac{\beta}{16}\chi^\dagger\sigma_{ij}\sigma_{kl}\chi\phi_{ij}\phi_{kl}\right)\left(\frac{\beta}{16}\chi^\dagger\sigma_{mn}\sigma_{op}\chi C^M_{mn}45^M_{phys}C^{M*}_{op}45^{M*}_{phys}\right), \quad (3.67)$$

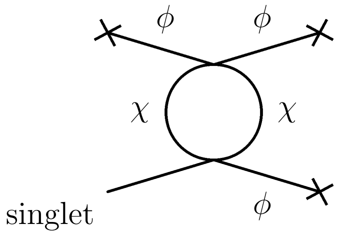
where the upper index M stands for the number corresponding to the field either from $(8,1,0)$, or from $(1,3,0)$ submultiplet. Therefore, the two free legs $C^M_{mn}45^M_{phys}$ and $C^{M*}_{op}45^{M*}_{phys}$ represent the field transforming either as $(8,1,0)$, or as $(1,3,0)$ projected onto vectors of the defining basis $\phi_{ij} = C^a_{ij}45^a_{phys}$. The indicated contractions again produce the propagators of the massless scalar field.

Similarly as in case of the Abelian Higgs model or in case of the calculation of the τ^2 term, the leading polynomial correction is given by the geometric series (3.63) equals to the prefactor of the quotient of the series times the factor generated by integration. The contribution corresponding to the series (3.63) therefore reads

$$-\frac{i}{16\pi^2}Q_2 = -\frac{i}{16\pi^2}2\frac{1}{2!}\left(\frac{\beta}{16}\right)^2 \mathrm{Tr}\left[\sigma_{ij}\sigma_{kl}\sigma_{mn}\sigma_{op}\right]\langle\phi_{ij}\rangle\langle\phi_{kl}\rangle C^M_{mn}45^M_{phys}C^{M*}_{op}45^{M*}_{phys},$$

where the factor of $\left(-\frac{i}{16\pi^2}\right)$ comes (as before) from integration. The extra overall factor of 2 reflects the fact that the fields $C^M_{mn}45^M_{phys}$ and $C^{M*}_{op}45^{M*}_{phys}$ can be attached to the second vertex (in place of the fields ϕ_{ij} and ϕ_{kl}). In other words, instead of the two VEVs $\langle\phi_{ij}\rangle$ and $\langle\phi_{kl}\rangle$ there would be VEVs of the fields ϕ_{mn} and ϕ_{op}. Hence, there are two equivalent contributions.

Analogically, in case of one point graphs we will evaluate



, (3.68)

where the only free leg represents a Standard Model singlet field.

Similarly as before, the relevant term of the second order of the perturbative theory reads

$$\frac{1}{2!}\mathcal{L}^2 \ni \frac{1}{2!}\left(\frac{\beta}{16}\chi^\dagger\sigma_{ij}\sigma_{kl}\chi\phi_{ij}\phi_{kl}\right)\left(\frac{\beta}{16}\chi^\dagger\sigma_{mn}\sigma_{op}\chi\phi_{mn}C^S_{op}45^S_{phys}\right). \tag{3.69}$$

As well as in the previous case we projected the singlet field 45^S_{phys} onto the defining basis ϕ_{ij} obtaining the contribution of our interest.

Hence, the contribution corresponding to the series (3.64) equals to

$$-\frac{i}{16\pi^2}Q_1 = -\frac{i}{16\pi^2}4\frac{1}{2!}\left(\frac{\beta}{16}\right)^2 \mathrm{Tr}\left[\sigma_{ij}\sigma_{kl}\sigma_{mn}\sigma_{op}\right]\langle\phi_{ij}\rangle\langle\phi_{kl}\rangle\langle\phi_{mn}\rangle C^S_{op}45^S_{phys},$$

where the factor of $\left(-\frac{i}{16\pi^2}\right)$ comes again from integration. The extra overall factor of 4 in the front reflects the fact that the Standard Model singlet $C^S_{op}45^S_{phys}$ can be inserted also in place of the fields ϕ_{ij}, ϕ_{kl}, or ϕ_{mn}. Consequently, there are four equivalent contributions.

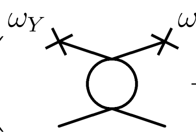
Let us finally apply the derived relations and formulae to the particular fields getting concrete results. At first, we focus on radiative corrections of $M^2(8,1,0)$, then we repeat the same proceeding for $M^2(1,3,0)$. All the presented results were calculated using *Wolfram Mathematica*.

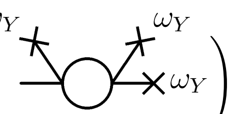
3.2.2.3 Leading one-loop β^2-dependent correction to $M^2(8,1,0)$

To reproduce the full radiative correction proportional to β^2 it is convenient to calculate the contributions in the three limits mentioned at the beginning of this section (3.2.2) and consequently fit a polynom of second order in ω_R and ω_Y using the obtained results.

Limit $\omega_R = 0$

First, let us consider the limit $\omega_R = 0$. In such a case the diagramatic formula (3.65) has the form

$i\Delta m^2_{\beta^2}(\omega_R = 0) = -\frac{i}{16\pi^2}\Bigg($ 

$- \frac{1}{\omega_Y}$ 

$\Bigg)$

$$\equiv -\frac{i}{16\pi^2}\left(Q_2(\omega_Y) - \frac{1}{\omega_Y}Q_1(\omega_Y)\right) \tag{3.70}$$

and performing the calculations in *Mathematica* it is obtained

$$
\begin{aligned}
Q_1(\omega_Y) &= 14\beta^2\omega_Y^3 \text{ for } (1,1,1,0),\\
Q_2(\omega_Y) &= 2\beta^2\omega_Y^2 \text{ for } (8,1,1,0).
\end{aligned}
$$

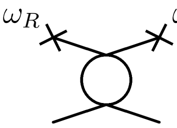
In consequence, the β^2-correction in this limit equals

$$
i\Delta m^2_{\beta^2}(\omega_R=0) = -\frac{i}{4\pi^2}\left(\frac{1}{2}\beta^2\omega_Y^2 - \frac{1}{\omega_Y}\frac{7}{2}\beta^2\omega_Y^3\right) = i\frac{3\beta^2\omega_Y^2}{4\pi^2}.
$$

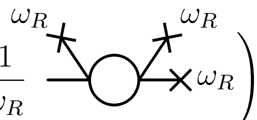
Limit $\omega_Y = 0$

Second, let us determine the correction in the limit $\omega_Y = 0$, i.e., we have to evaluate

$$
i\Delta m^2_{\beta^2}(\omega_Y=0) = -\frac{i}{16\pi^2}\Bigg(
$$



$$
-\frac{1}{\omega_R}
$$



$$
\Bigg) \equiv -\frac{i}{16\pi^2}\left(Q_2(\omega_R) - \frac{1}{\omega_R}Q_1(\omega_R)\right). \tag{3.71}
$$

In this limit we get

$$
\begin{aligned}
Q_1(\omega_R) &= 4\beta^2\omega_R^3 \text{ for } (1,1,0),\\
Q_2(\omega_R) &= 8\beta^2\omega_R^2 \text{ for } (15,1,0).
\end{aligned}
$$

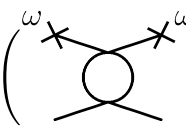
Therefore, the β^2-correction for $\omega_Y = 0$ reads

$$
i\Delta m^2_{\beta^2}(\omega_Y=0) = -\frac{i}{4\pi^2}\left(\beta^2\omega_R^2 - \frac{1}{\omega_Y}2\beta^2\omega_R^3\right) = i\frac{\beta^2\omega_R^2}{4\pi^2}.
$$

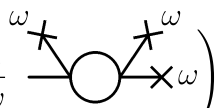
Limit $\omega_R = -\omega_Y$

We have found the two corrections quadratic in ω_R and ω_Y; however, there can be also a mixed term proportional to $\omega_R\omega_Y$. Thus we need to get some more information about the β^2 correction and a convenient way how to achieve this is the calculation of the correction in the $SU(5)$ limit, i.e., for $\omega_Y = -\omega_R \equiv \omega$. Consequently, we compute the following contribution

$$
i\Delta m^2_{\beta^2}(\omega_Y=-\omega_R) = -\frac{i}{16\pi^2}\Bigg(
$$



$$
-\frac{1}{\omega}
$$



$$
\Bigg) \equiv -\frac{i}{16\pi^2}\left(Q_2(\omega) - \frac{1}{\omega}Q_1(\omega)\right). \tag{3.72}
$$

In this limit we get

$$
\begin{aligned}
Q_1(\omega) &= 26\pi^2\beta^2\omega_R^3 \text{ for } (1,0),\\
Q_2(\omega) &= 6\pi^2\beta^2\omega_R^2 \text{ for } (24,0).
\end{aligned}
$$

Hence, the β^2-correction in the $SU(5)$ limit is

$$i\Delta m^2_{\beta^2}(\omega_Y = -\omega_R) = -\frac{i}{4\pi^2}\left(\frac{3}{2}\beta^2\omega^2 - \frac{1}{\omega}\frac{13}{2}\beta^2\omega^3\right) = i\frac{5\beta^2\omega^2}{4\pi^2}.$$

Finally, using the results obtained for the three limits we can reconstruct the complete β^2 correction appropriate to $M^2(8,1,0)$. The mixed term can be easily determined thanks to the knowledge of the $SU(5)$ limit. The resulting formula for the β-dependent part of the leading correction therefore reads

$$\Delta m^2_{\beta^2}(8,1,0) = \frac{\beta^2\left(\omega_R^2 - \omega_R\omega_Y + 3\omega_Y^2\right)}{4\pi^2}.$$

3.2.2.4 Leading one-loop β^2-dependent correction to $M^2(1,3,0)$

Let us now proceed in the same way for $M^2(1,3,0)$.

Limit $\omega_R = 0$

Again, first the limit $\omega_R = 0$ is considered. The diagramatic formula (3.65) has the same form as in equation (3.70). Performing the calculations of relevant quotients using our *Mathematica* code we obtain

$$Q_1(\omega_Y) = 14\beta^2\omega_Y^3 \text{ for } (1,1,1,0),$$
$$Q_2(\omega_Y) = 6\beta^2\omega_Y^2 \text{ for } (1,3,1,0).$$

Therefore, the β^2-correction in the $\omega_R = 0$ limit is

$$i\Delta m^2_{\beta^2}(\omega_R = 0) = -\frac{i}{4\pi^2}\left(\frac{3}{2}\beta^2\omega_Y^2 - \frac{1}{\omega_Y}\frac{7}{2}\beta^2\omega_Y^3\right) = i\frac{2\beta^2\omega_Y^2}{4\pi^2}.$$

Limit $\omega_Y = 0$

To calculate the correction in the limit $\omega_Y = 0$ we reuse the equation (3.71). The quotients equal to

$$Q_1(\omega_R) = 8\beta^2\omega_R^3 \text{ for } (1,1,0),$$
$$Q_2(\omega_R) = 0 \text{ for } (1,3,0).$$

Consequently, the β^2 correction reads

$$i\Delta m^2_{\beta^2}(\omega_Y = 0) = -\frac{i}{4\pi^2}\left(0 - \frac{1}{\omega_Y}2\beta^2\omega_R^3\right) = i\frac{\beta^2\omega_R^2}{4\pi^2}.$$

Limit $\omega_R = -\omega_Y$

Lastly, let us determine the β-dependent contribution to the leading correction in the $SU(5)$ limit using the formula (3.72). For $\omega_R = -\omega_Y$ limit our code yields the following values of quotients

$$Q_1(\omega) = 26\beta^2\omega_R^3 \text{ for } (1,0),$$
$$Q_2(\omega) = 6\beta^2\omega_R^2 \text{ for } (24,0).$$

Therefore, the leading β^2-correction in the $SU(5)$ limit equals to

$$i\Delta m^2_{\beta^2}(\omega_Y = -\omega_R) = -\frac{i}{4\pi^2}\left(\frac{3}{2}\beta^2\omega^2 - \frac{1}{\omega}\frac{13}{2}\beta^2\omega^3\right) = i\frac{5\beta^2\omega^2}{4\pi^2}.$$

Eventually, using the results presented in the above paragraphs for the three limits we can reproduce the complete β^2 correction appropriate to $M^2(1,3,0)$. The resulting formula for the β dependent part of the leading correction is

$$\Delta m^2_{\beta^2}(1,3,0) = \frac{\beta^2\left(2\omega_R^2 - \omega_R\omega_Y + 2\omega_Y^2\right)}{4\pi^2}.$$

3.2.2.5 Complete leading polynomial scalar corrections

To conclude, with the results derived in previous subsections we can rewrite the leading polynomial one loop mass scalar corrections to the masses of $(8,1,0)$ and $(1,3,0)$ in the form

$$\Delta M^2(8,1,0) = \frac{\tau^2 + \beta^2\left(\omega_R^2 - \omega_R\omega_Y + 3\omega_Y^2\right)}{4\pi^2}, \tag{3.73}$$

$$\Delta M^2(1,3,0) = \frac{\tau^2 + \beta^2\left(2\omega_R^2 - \omega_R\omega_Y + 2\omega_Y^2\right)}{4\pi^2}. \tag{3.74}$$

As one can easily check, these formulae are the same as those found using the effective potential approach in previous chapter ((3.14) and (3.15)). Consequently, the described method of calculation of the leading radiative corrections can represent a convenient alternative for the effective potential approach in models for which the construction of the effective potential would be technically too complicated.

The leading polynomial gauge corrections, which were mentioned in the first section of this chapter could be also calculated using the standard perturbative theory approach. However, as we outline in Appendix E, such a calculation would be very complicated.

4. The quantum level of the $45 \oplus 126$ Higgs model

So far we have been studying the minimal $45 \oplus 16$ Higgs model. However, as we have already mentioned, it was shown that this model is not realistic [56] since it is disfavoured by neutrino oscillation and cosmology data. The simplest potentially viable alternative is the minimal $45 \oplus 126$ model, which can be set to be compatible with the available experimental data.

We have already shown that the $45 \oplus 16$ can be revived computing the quantum corrections to the masses of the problematic pseudo-Goldstone bosons first using the effective potential approach and after that using the standard perturbative theory approach. Although the mass corrections to the masses of the problematic pseudo-Goldstone bosons in the $45 \oplus 126$ model could be principally calculated also using the effective potential, in practice one finds out that the computation becomes too difficult.

4.1 Why the $45 \oplus 126$ model?

As we have already claimed, both variants of the $SO(10)$ minimal model with scalar sectors formed by either $45 \oplus 16$, or $45 \oplus 126$ representations can be revived because the spurious tachyonic instabilities which had been believed to exist for more than thirty years, can be shown to be just an artifact of the tree-level calculation.

However, for a realistic model building there are other criteria which have to be fulfilled. According to the renormalization group studies [62, 48, 49, 50] the $B - L$ breaking scale has to be smaller than 10^{12} GeV in case of the $45 \oplus 16$ model and below 10^{10} GeV for the $45 \oplus 126$ model. The problem with these values, however, is that they are at variance with the experimental data. Namely, they are in disagreement with neutrino masses data from the β-decay and cosmology.

Let us first focus on the $45 \oplus 16$ model and explain the problem in more detail. The non-zero VEV of the 16_S representation causes the breaking of $B - L$ symmetry by 1 unit; hence, there should be two VEV insertions to give rise to the $\Delta(B - L) = 2$ seesaw operator. To achieve it at the renormalizable level one can attempt to apply the Witten's radiative mechanism [67, 68, 69]. Alternatively, if one does not demand renormalizability (at the expense of the model predictivity), the $d = 5$ operator can be employed instead. Anyway, both these possibilities lead to the situation when the effective $\Delta(B - L) = 2$ seesaw scale is further suppressed with respect to the scale of the $B - L$ breaking. As a result, the masses of the light neutrinos are too big (by many orders of magnitude) and thus unrealistic.

On the other hand, in case of the $45 \oplus 126$ model the $B - L$ symmetry is broken by two units. Consequently, the right-handed neutrinos get their masses via the renormalizable Yukawa interaction $16_F 16_F 126^*_S$ yet at the tree level [70, 71]. Nevertheless, the limit on the 126_S VEV mentioned above still implies too high masses of light neutrinos.

The situation described in the last paragraphs is not insoluable. The minimal

$SO(10)$ realistic unification can be saved if an extensive fine-tuning is used in the seesaw formula.

Obviously, the minimal non-SUSY $SO(10)$ models are rather different with respect to the SUSY $SO(10)$ scenarios since in the minimal supersymmetric models the neutrino masses are typically predicted to be lower than one would need. The reason is as follows: because of the rigidity of the Higgs potential in the minimal SUSY $SO(10)$ models there happen to be extra pseudo-Goldstone bosons much below the GUT scale (in case with more than one-stage spontaneous symmetry breaking [45, 46]) and, as a result, the unification in the Minimal Supersymmetric Standard Model is destroyed.

It is important to say that the bounds on the scale of the $B-L$ breaking obtained for the non-SUSY $SO(10)$ scenarios are based on the so called minimal survival hypothesis. Specifically, these bounds were derived under the assumption that just the minimal number of necessary intermediate fields is dusted around the scale of the corresponding symmetry breaking. However, this assumption does not have to be fulfilled in general; hence, the $B-L$ scale can be in principle much higher than the bounds quoted above. This could, consequently, make the unification compatible with the experimental data. In addition, a "big" Higgs sector like $45_S \oplus 126_S$ possesses more room for the violation of the minimal survival hypothesis. Moreover, another great feature favouring the $45 \oplus 126$ model is that the renormalizable seesaw mechanism constrains the Yukawa sector of the theory, which makes it potentially testable in the future.

As was recently shown [56], there are, indeed, several domains in the parametric space of this minimal non-SUSY model allowing consistent unification with $B-L$ scale as high as 10^{14} GeV without occurrence of any tachyonic instabilities or problems related to the proton lifetime. This is made possible thanks to an accidentally light multiplet appearing in the unification desert that influences conveniently the unification picture.

To be more specific, there turn out to be two classes of the viable solutions. The first one includes an intermediate-scale multiplet transforming as $(6, 3, \frac{1}{3})$ with respect to the Standard Model gauge group and it supports the breaking of the $SO(10)$ symmetry down to the Standard Model via the $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ intermediate symmetry. The second solution involves a relatively light $(8, 2, \frac{1}{2})$ multiplet and supports the $SO(10)$ breakdown via the $SU(4)_C \otimes SU(2)_L \otimes U(1)_R$ intermediate stage. It is welcome that in all the interesting cases the predicted unification scale lies quite close to the current proton lifetime lower bound set by Super-Kamiokande. A detailed study of the minimal non-SUSY $SO(10)$ unification including the light colour octet $(8, 2, \frac{1}{2})$ was performed quite recently [72]. This scenario is particularly interesting since the unification constraints allow for the existence of a coloured scalar octet near the electroweak scale, which can be appealing for the collider physics. In addition, the mass of the scalar octet is anticorrelated with the masses of the GUT-scale vector bosons mediating the $d = 6$ proton decay. Consequently, from the lower bound of the proton lifetime an upper bound for the mass of the coloured scalar octet can be derived.

To sum up, the revived minimal realistic $SO(10)$ models open a large space for further model building providing a number of predictions which could be testable in the near-future experiments. In particular, the up-coming large volume facilities such as Hyper-Kamiokande could seek through the physically interesting region of the parameter space of the considered class of models.

4.2 The $45 \oplus 126$ model in a nutshell

Before we start any calculations, let us first introduce the minimal non-SUSY $45 \oplus 126$ model.

4.2.1 The tree-level scalar potential

The scalar potential of the minimal $45 \oplus 126$ Higgs model is analogical to the scalar potential of the minimal $45 \oplus 16$ Higgs model (2.8). Again, the most general renormalizable scalar potential can be written as a sum of three terms

$$V = V_\phi + V_{\phi\Sigma} + V_\Sigma,$$

which

$$\begin{aligned}
V_\phi &= -\frac{\mu^2}{2}\left(\phi_{ij}\phi_{ij}\right) + \frac{a_0}{4}\left(\phi_{ij}\phi_{ij}\right)\left(\phi_{ij}\phi_{ij}\right) + \frac{a_2}{4}\left(\phi_{ij}\phi_{kl}\right)\left(\phi_{ij}\phi_{kl}\right), \\
V_{\phi\Sigma} &= \frac{i\tau}{4!}\phi_{ij}\left(\Sigma\Sigma^*\right)_2 + \frac{\alpha}{2(5!)}\left(\phi_{ij}\phi_{ij}\right)\left(\Sigma\Sigma^*\right)_0 + \frac{\beta_4}{4(3!)}\left(\phi_{ij}\phi_{kl}\right)\left(\Sigma\Sigma^*\right)_4 \\
&+ \frac{\beta_4'}{3!}\left(\phi_{ij}\phi_{kl}\right)\left(\Sigma\Sigma^*\right)_{4'} + \frac{\gamma_2}{4!}\left(\phi_{ij}\phi_{ik}\right)\left(\Sigma\Sigma\right)_2 + \frac{\gamma_2^*}{4!}\left(\phi_{ij}\phi_{ik}\right)\left(\Sigma^*\Sigma^*\right)_2, \\
V_\Sigma &= -\frac{\nu^2}{5!}\left(\Sigma\Sigma^*\right)_0 + \frac{\lambda_0}{(5!)^2}\left(\Sigma\Sigma^*\right)_0\left(\Sigma\Sigma^*\right)_0 + \frac{\lambda_2}{(4!)^2}\left(\Sigma\Sigma^*\right)_2\left(\Sigma\Sigma^*\right)_2 \\
&+ \frac{\lambda_4}{(3!)^2(2!)^2}\left(\Sigma\Sigma^*\right)_4\left(\Sigma\Sigma^*\right)_4 + \frac{\lambda_4'}{(3!)^2}\left(\Sigma\Sigma^*\right)_{4'}\left(\Sigma\Sigma^*\right)_{4'} \\
&+ \frac{\eta_2}{(4!)^2}\left(\Sigma\Sigma\right)_2\left(\Sigma\Sigma\right)_2 + \frac{\eta_2^*}{(4!)^2}\left(\Sigma^*\Sigma^*\right)_2\left(\Sigma^*\Sigma^*\right)_2,
\end{aligned}$$

where ϕ was used for components of the adjoint 45-dimensional representation while Σ denotes the components of the 126-dimensional representation.

The contractions of the Σ components are defined as follows

$$\Sigma\Sigma^* \equiv \Sigma_{ijklm}\Sigma^*_{ijklm},$$

$$\begin{aligned}
\phi_{ij}(\Sigma\Sigma^*)_2 &\equiv \phi_{ij}\Sigma_{klmni}\Sigma^*_{klmnj},\\
\phi_{ij}\phi_{ij}(\Sigma\Sigma^*)_0 &\equiv \phi_{ij}\phi_{ij}\Sigma_{klmno}\Sigma^*_{klmno},\\
\phi_{ij}\phi_{kl}(\Sigma\Sigma^*)_4 &\equiv \phi_{ij}\phi_{kl}\Sigma_{mnoij}\Sigma^*_{mnokl},\\
\phi_{ij}\phi_{kl}(\Sigma\Sigma^*)_{4'} &\equiv \phi_{ij}\phi_{kl}\Sigma_{mnoik}\Sigma^*_{mnojl},\\
\phi_{ij}\phi_{ik}(\Sigma\Sigma)_2 &\equiv \phi_{ij}\phi_{ik}\Sigma_{lmnoj}\Sigma^*_{lmnok},\\
\phi_{ij}\phi_{ik}(\Sigma^*\Sigma^*)_2 &\equiv \phi_{ij}\phi_{ik}\Sigma^*_{lmnoj}\Sigma^*_{lmnok},
\end{aligned}$$

$$\begin{aligned}
(\Sigma\Sigma^*)_0(\Sigma\Sigma^*)_0 &\equiv \Sigma_{ijklm}\Sigma^*_{ijklm}\Sigma_{nopqr}\Sigma^*_{nopqr},\\
(\Sigma\Sigma^*)_2(\Sigma\Sigma^*)_2 &\equiv \Sigma_{ijklm}\Sigma^*_{ijkln}\Sigma_{opqrm}\Sigma^*_{opqrn},\\
(\Sigma\Sigma^*)_4(\Sigma\Sigma^*)_4 &\equiv \Sigma_{ijklm}\Sigma^*_{ijkno}\Sigma_{pqrlm}\Sigma^*_{pqrno},\\
(\Sigma\Sigma^*)_{4'}(\Sigma\Sigma^*)_{4'} &\equiv \Sigma_{ijklm}\Sigma^*_{ijkno}\Sigma_{pqrln}\Sigma^*_{pqrmo},\\
(\Sigma\Sigma)_2(\Sigma\Sigma)_2 &\equiv \Sigma_{ijklm}\Sigma_{ijkln}\Sigma_{opqrm}\Sigma_{opqrn},\\
(\Sigma^*\Sigma^*)_2(\Sigma^*\Sigma^*)_2 &\equiv \Sigma^*_{ijklm}\Sigma^*_{ijkln}\Sigma^*_{opqrm}\Sigma^*_{opqrn}.
\end{aligned}$$

Let us note that the couplings η_2 and γ_2 present in the scalar potential are complex, all the other coefficients are real.

4.2.2 The Standard Model singlets

As in case of the minimal $45 \oplus 16$ Higgs model there are in general three Standard Model singlets in the reducible $45_S \oplus 126_S$ representation of the $SO(10)$ group. If we again use the standard labelling with respect to the $3_C 2_L 2_R 1_{B-L}$ subgroup of $SO(10)$, the two singlets belonging to the 45_S are the submultiplets $(1,1,1,0)$ and $(1,3,1,0)$ while the third singlet belongs to the $(1,1,3,+2)$ submultiplet of 126_S. As before, we shall denote the vacuum expectation values of these singlets as

$$\begin{aligned}
\omega_Y &\equiv \langle(1,1,0,0)_{(1,1,1,0)}\rangle,\\
\omega_Y &\equiv \langle(1,1,0,0)_{(1,1,3,0)}\rangle,\\
\sigma &\equiv \langle(1,1,-1,+2)_{(1,1,3,+2)}\rangle.
\end{aligned}$$

The first two VEVs ω_Y and ω_R are real. The σ VEV can be made real using a phase redefinition of 126_S.

4.2.3 The symmetry breaking patterns

Different vacuum configurations lead to different symmetry breaking patterns, which reduce the original full $SO(10)$ symmetry into its various subgroups. For $\sigma = 0$ we distinguish the same vacuum settings and the corresponding breakings as we did in case of the minimal $45 \oplus 16$ model, namely, (2.15)-(2.19).

For $\sigma \neq 0$ the intermediate symmetries (2.15)-(2.18) are further broken down to the Standard Model gauge group. Only the vacuum configuration (2.19) corresponding to intermediate scale containing the $SU(5)$ subgroup is not affected by the non-zero value of σ; therefore, this vacuum configuration does not trigger a full breaking of the $SO(10)$ down to the Standard Model. As a result, again, only the four vacuum settings and intermediate symmetries (2.15)-(2.18) are physically interesting.

4.2.4 The tree level scalar spectrum

Defining the 297-dimensional basis formed by the fields belonging to the Higgs representations as $\psi = (\phi, \Sigma^*, \Sigma)$, the Lagrangian mass term can be written as

$$\mathcal{L} \ni \frac{1}{2}\psi^T M^2(\phi, \Sigma, \Sigma^*)\psi,$$

where the matrix $M^2(\phi, \Sigma, \Sigma^*)$ represents the functional scalar mass matrix evaluated on the Standard Model vacuum. This matrix can be obtained by taking the second derivatives of the scalar potential with respect to the 297 fields assigned to the vector ψ. Therefore, it can be schematically expressed in the following block form

$$M^2(\phi, \Sigma, \Sigma^*) = \begin{pmatrix} V_{\phi\phi} & V_{\phi\Sigma} & V_{\phi\Sigma^*} \\ V_{\Sigma^*\phi} & V_{\Sigma^*\Sigma} & V_{\Sigma^*\Sigma^*} \\ V_{\Sigma\phi} & V_{\Sigma\Sigma} & V_{\Sigma\Sigma^*} \end{pmatrix}, \tag{4.1}$$

where the subscripts of V denote the fields, with respect to which the second derivative is taken in the relevant block.

Applying a unitary transformation on the vacuum expectation value of this matrix we can express it in a block-diagonal form in the Standard Model basis.

The important aspect of the tree-level scalar spectrum is that there is again (as in case of the minimal $45 \oplus 16$ model) the pair of the pseudo-Goldstone bosons whose tree-level masses are proportional solely to the a_2 parameter, namely,

$$M^2(8,1,0) = 2a_2\,(\omega_R - \omega_Y)\,(\omega_R + 2\omega_Y)\,, \tag{4.2}$$
$$M^2(1,3,0) = 2a_2\,(\omega_Y - \omega_R)\,(\omega_Y + 2\omega_R)\,. \tag{4.3}$$

As a result, at the tree level the tachyonic masses are again generated if the fraction $\frac{\omega_Y}{\omega_R}$ lies outside the interval $[-2, -\frac{1}{2}]$. This fact then again excludes the physically interesting patterns of spontaneous symmetry breakdown corresponding to either $\omega_Y \gg \omega_R$, or $\omega_Y \ll \omega_R$.

An important aspect of the formulae (4.2) and (4.3) is that they do not contain any contributions proportional to the β parameter. This may seem analogous to the $45 \oplus 16$ model; however, the number of possible contractions $\phi^2\Sigma\Sigma^*$ is now much larger than it was in case of the $45 \oplus 16$ model. Hence, the analogous behaviour of the models does not have to be so obvious. Anyway, the absence of the σ-dependence in relations (4.3) and (4.2) can be explained as follows. The tensors of type $(\Sigma\Sigma^*)$ always posses the $SU(5)$ symmetry (it does not matter how many indices of the Σ's are contracted). The index structure of both ϕ's to Σ's have the same form. In consequence, the pair of ϕ's look like quadratic covariant-derivative term for the fields with the Standard Model gluon and A-field quantum numbers. However, at the $SU(5)$ level these fields are massless.

As before, the tachyonic behaviour of the submultiplets $(8,1,0)$ and $(1,3,0)$ can be fixed at the quantum level, which again revives the forbidden realistic symmetry breaking patterns as in case of the $45 \oplus 16$ model.

4.2.5 (In)effective potential

While in case of the minimal $45 \oplus 16$ model we used the effective potential approach to calculate the quantum corrections to the $M^2(8,1,0)$ and $M^2(1,3,0)$ masses, the minimal $45 \oplus 126$ model is more difficult to be treated in the same way. This fact can be quite easily understood if we look at the structure of the scalar potential of the $45 \oplus 126$ model. It contains contractions of tensors with five indices, which makes the tree-level calculations themselves more complicated, let alone the calculation of the one-loop effective potential.

While in case of the $45 \oplus 16$ model we worked with a 77-dimensional matrix $M^2(\phi,\chi,\chi^*)$ now, to compute the quantum correction to the tree-level scalar potential in the $45 \oplus 126$ model, we have to deal with a matrix $M^2(\phi,\Sigma,\Sigma^*)$ which is 297-dimensional. As a result, the explicit evaluation of the radiative corrections to the scalar masses becomes very cumbersome.

Consequently, it would be very desirable to have an easier method for the calculation of the scalar mass corrections in order to confirm the viability of the realistic spontaneous symmetry breaking patterns in this scenario. In the following section we will therefore apply the standard diagrammatic methods on the scalar sector of the minimal $45 \oplus 16$ model and we will try to figure out the desired one-loop mass corrections to set the stage for performance of analogous calculations in more complicated cases.

4.3 Diagrammatic calculation of the leading polynomial scalar mass corrections

Finally, we can try to calculate diagrammatically part of the leading polynomial correction of the potentially tachyonic masses appearing in the scalar spectrum of the realistic minimal non-SUSY $45 \oplus 126$ model. It is quite predictable that the mass corrections will contain a uniform term proportional to τ^2 as well as it was in case of the $45 \oplus 16$ model. The great thing about this term is that in fact we do not need to know the explicit form of the matrix of transformation from the defining basis to the physical one and the whole derivation of this term is not much different from the way we proceeded in case of the simpler $45 \oplus 16$ model. On the other hand, even this single uniform term is enough to prove the viability of the minimal non-SUSY $45 \oplus 126$ model. Hence, within this section we will determine this term to serve our needs. Unfortunately, the calculation of the further terms of the leading polynomial corrections turned out to be beyond the time possibilities of this thesis. Nevertheless, they will be very likely the subject of our further work.

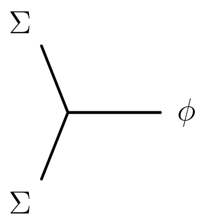
4.3.1 The τ^2 term

As we will see, the calculation of this term in case of the $45 \oplus 126$ model is very analogical to the computation we performed for the $45 \oplus 16$ model. The relevant part of the Lagrangian reads

$$V_\tau \ni -\frac{\mu^2}{2}\left(\phi'_{ij} + \langle\phi_{ij}\rangle\right)\left(\phi'_{ij} + \langle\phi_{ij}\rangle\right) + \frac{i\tau}{4!}\Sigma_{mnopi}\Sigma^*_{mnopj}\left(\phi'_{ij} + \langle\phi_{ij}\rangle\right), \tag{4.4}$$

where we substituted the fields ϕ with non-zero VEVs for the fields ϕ' with zero vacuum expectation values.

The corresponding Feynman rule thus can be depicted as



$$\sim \frac{i\tau}{4!}. \tag{4.5}$$

Given the arguments used in the calculation of the τ-dependent term in the previous chapter let us take the first derivative of (4.4) with respect to ϕ'_{12}

$$\left\langle \frac{\partial V_\tau}{\partial \phi_{12}} \right\rangle = -\mu^2\langle\phi_{12}\rangle + \frac{i\tau}{4!}\Sigma_{mnop1}\Sigma^*_{mnop2} \tag{4.6}$$

and this stationarity condition implies the formula for the parameter μ^2

$$\mu^2 = \frac{1}{\langle\phi_{12}\rangle}\left(\frac{i\tau}{4!}\Sigma_{mnop1}\Sigma^*_{mnop2}\right). \tag{4.7}$$

To determine the second derivative of the equation (4.4) with respect to the pseudo-Goldstone bosons from submultiplets $(8, 1, 0)$ and $(1, 3, 0)$ it is again necessary to transform the defining fields ϕ_{ij} to the physical fields 45^a_{phys} (where $a = 1, \ldots, 45$), i.e.,

$$\begin{aligned} V_\tau = & -\frac{\mu^2}{2}C^a_{ij}C^{b*}_{ij}\left(45^a_{phys} + \langle 45^a_{phys}\rangle\right)\left(45^{b*}_{phys} + \langle 45^{b*}_{phys}\rangle\right) \\ & + \frac{i\tau}{4!}C^a_{ij}\Sigma_{mnopi}\Sigma^*_{mnopj}\left(45^a_{phys} + \langle 45^a_{phys}\rangle\right). \end{aligned} \tag{4.8}$$

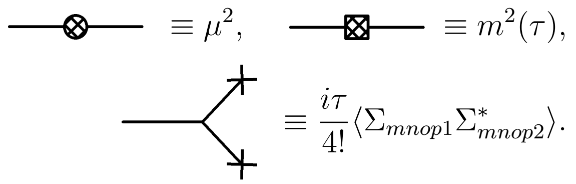
The second derivative of this expression with respect to a physical field 45^a_{phys} reads

$$\left\langle \frac{\partial^2 V_\tau}{\partial 45^a_{phys}\partial 45^{a*}_{phys}} \right\rangle = -\frac{\mu^2}{2}C^a_{ij}C^{a*}_{ij} = -\mu^2.$$

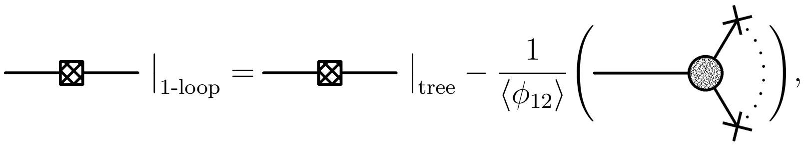
By the substitution of the stationarity condition (4.7) into the above equation one gets the tree-level τ-dependent mass term

$$m^2(\tau) = -\frac{1}{\langle\phi_{12}\rangle}\frac{i\tau}{4!}\langle\Sigma_{mnop1}\Sigma^*_{mnop2}\rangle \tag{4.9}$$

To express the above calculation in terms of Feynman diagrams the same steps as in case of the $45 \oplus 16$ model can be followed since the structures of these two models are analogous. Let us define

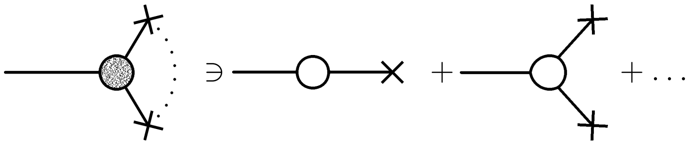


To avoid the repetition of the formulae already presented in the previous chapter, let us now skip to the resulting formula for the one-loop-level mass term proportional to τ^2 of the $45_S \oplus 126_S$ scenario, which reads

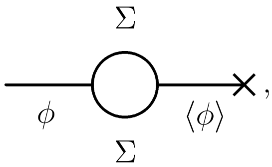


(4.10)

The only diagram contributing to the leading polynomial mass correction proportional to τ^2 is again

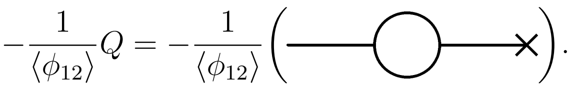


To sum the series on the right-hand side of the above equation it is enough to compute the following diagram



(4.11)

which represents the quotient of the series. Hence, our aim is to calculate the expression



Consequently, the relevant term in the second order of the perturbation theory reads

$$\frac{1}{2!}\mathcal{L}^2 \ni \frac{1}{2!}\left(\frac{i\tau}{4!}\Sigma_{mnopi}\Sigma^*_{mnopj}\phi_{ij}\right)\left(\frac{i\tau}{4!}\Sigma_{efghk}\Sigma^*_{efghl}\phi_{kl}\right). \tag{4.12}$$

The only free outer leg in diagram (4.11) is a Standard Model singlet. As we have already argued when computing the leading τ-dependent mass term in case of the $45 \oplus 16$ model, this singlet field can be substituted for instance by the field ϕ_{12}. Therefore, to determine the desired leading τ-dependent mass correction it is necessary to calculate the formula

$$-\frac{1}{\langle \phi_{12}\rangle} Q = -\frac{1}{\langle \phi_{12}\rangle}\left[2\frac{1}{2!}2\left(\frac{i\tau}{4!}\Sigma_{mnop1}\Sigma^*_{mnop2}\right)\left(\frac{i\tau}{4!}\Sigma_{efghk}\Sigma^*_{efghl}\langle \phi_{kl}\rangle\right)\right]. \tag{4.13}$$

There are two extra factors of 2 inside the square bracket. The first is present because before choosing $i = 1$ and $j = 2$ the sum ran over all indices $i, j = 1, \ldots, 10$. Hence, in the sum there must have been two terms of type (4.13). The second extra overall factor of 2 reflects the fact that the VEV $\langle\phi_{12}\rangle$ can be taken also in place of the field ϕ_{kl} instead of ϕ_{ij}. Hence, again one gets two equal contributions.

To evaluate the above expression one has to sum all the non-zero contractions of the Σ tensors. The only non-zero contractions we can get if we set $k = 2$ and $l = 1$. This statement can be justified in the following way: in case that $k = 1$ and $l = 2$, the indices e, f, g and h must have values from $\{3, 4, \ldots, 10\}$. However, then the 126^*_{efgh2} can hardly contract with 126_{mnop1}. Similarly, all the other values of k and l can be excluded except for the already mentioned case $(kl) = (21)$, when one gets

$$-\frac{1}{\langle\phi_{12}\rangle}Q = -\frac{1}{\langle\phi_{12}\rangle}\left[2\left(\frac{i\tau}{4!}\Sigma_{mnop1}\Sigma^*_{mnop2}\right)\left(\frac{i\tau}{4!}\Sigma_{efgh2}\Sigma^*_{efgh1}\langle\phi_{21}\rangle\right)\right],$$

where the indices a, b, c, d and e, f, g, h can take on values from the set $\{3, 4, \ldots, 10\}$. Moreover, it is quite easy to see that all the possible contractions are not only non-zero but also positive. Since the signs of permutations $(mnop1)$ and $(mnop2)$ are the same, the resulting sign of contraction $\Sigma_{mnop1}\Sigma^*_{mnop2}$ will be always positive. Naturally, for the permutations $(efgh2)$ and $(efgh1)$ the same arguments can be used. As a result, the only thing we have to do is to determine the number of all the possible contractions. At first, there are 4! possible permutations of indices $mnop$ and $efgh$. Furthermore, although the permutations $(mnop)$ and $(efgh)$ have to contain the same numbers, we can still choose them from the set $\{3, 4, \ldots, 10\}$; consequently, we get the factor $\binom{8}{4}$. Taking into account all these facts and the factor $\left(-\frac{i}{16\pi^2}\right)$ coming from integration (see e.g. the calculation of τ^2 term in case of the $45 \oplus 16$ model), the resulting expression for the τ^2 term of the mass correction reads

$$\begin{aligned} i\Delta m^2(\tau) &= -\frac{i}{16\pi^2}\left(-\frac{1}{\langle\phi_{12}\rangle}Q\right) = -\frac{i}{16\pi^2}\left[4!4!\binom{8}{4}\frac{1}{\langle\phi_{12}\rangle}2\left(\frac{\tau}{4!}\right)^2\langle\phi_{21}\rangle\right] \\ &= -\frac{i}{8\pi^2}\left[4!4!\binom{8}{4}\left(\frac{\tau}{4!}\right)^2\frac{1}{\langle\phi_{12}\rangle}\left(-\langle\phi_{12}\rangle\right)\right] = i\frac{35}{4\pi^2}\tau^2. \end{aligned} \tag{4.14}$$

Although the computation of this term of the leading mass corrections appears to be straightforward, the calculation of other terms depending on β^2 would be much less trivial. Similarly as we saw in the case of the $45 \oplus 16$ model, we would have to determine the explicit form of the matrix transforming from the defining basis to the physical one, etc.

However, the τ^2 term that we have just calculated proves that the realistic spontaneous symmetry breaking patterns of the $45 \oplus 126$ can be safely revived.

Conclusions and Outlook

In this thesis we have studied the $SO(10)$ grand unified theories with emphasis on their quantum aspects. The grand unifications based on the $SO(10)$ gauge group are attractive for a number of reasons. The first of them is the minimality, which in general translates as simplicity and predictivity of the particular model. There are $SO(10)$ scenarios having (at least to certain extent) all of these desired features. For instance, the seesaw mechanism essential for the neutrino mass generation can be successfully implemented into most $SO(10)$ scenarios. Next, the proton lifetime calculated within by these models is typically close to the current experimental bounds, which represents a promising beyond-Standard-Model prediction. Apart from that, there is enough room for cosmological aspects in $SO(10)$ theories. For example, they typically contain a suitable cold dark matter candidate.

Despite the fact that the grand unified theories based on $SO(10)$ gauge group have been studied for decades and various scenarios have been proposed, no "hot" candidate on "the" theory emerged. As we described in more detail in previous chapters the models preferred by the criteria of minimality were quite soon ruled out because of the supposed tachyonic behaviour.

The development of supersymmetry gave rise to a number of SUSY $SO(10)$ models. Nevertheless, the concept of supersymmetry was significantly impaired by the lack of any SUSY evidence at the LHC and the reasonable minimal supersymmetric $SO(10)$ scenario was definitely buried on grounds of phenomenological requirements [45, 46].

Given this, it seemed that the minimal $SO(10)$ unifications were excluded from any realistic considerations; however, quite surprisingly, the situation has recently changed. After years of oblivion the minimal nonsupersymmetric $SO(10)$ scenarios were put back into play.

Therefore, motivated by the recent progress in the field of the minimal non-SUSY $SO(10)$ models we focussed in particular on the study of the one-loop scalar spectrum and its connection with the local vacuum stability of the $45_S \oplus 16_S$ model. The main goal of this work was to calculate the scalar radiative corrections to the masses of problematic pseudo-Goldstone bosons transforming as $(8,1,0)$ and $(1,3,0)$ under the Standard Model gauge group (i.e., those with tachyonic tree-level behaviour) using the standard perturbative theory methods. This calculation was motivated especially by a possible application of analogical methods to the more realistic (but also more complicated) model with a Higgs sector formed by the $45_S \oplus 126_S$ representation. Hence, we first calculated the tree-level scalar spectrum of the $45_S \oplus 16_S$ model (see Appendix D) and the full leading one-loop corrections to the masses of $(8,1,0)$ and $(1,3,0)$ using the effective potential approach (equations (3.10) and (3.11)). Besides that, we demonstrated the trivial behaviour of the corrected one-loop masses corresponding to these potentially tachyonic pseudo-Goldstone bosons in the $SU(5)$ limit (see (3.12) and (3.13)) that constitutes a highly nontrivial consistency check of our results.

After that, we performed the calculation of the leading polynomial corrections to the masses of $(8,1,0)$ and $(1,3,0)$ using the standard perturbative theory approach, which was first applied (for a better comprehension) to the Abelian

Higgs model. A smart trick based on the behaviour of the mass corrections in the flipped $SU(5)$ limit was employed to determine the leading polynomial corrections to the masses of $(8, 1, 0)$ and $(1, 3, 0)$ in their complete form (equations (3.73) and (3.74)). The results obtained by the diagrammatic calculation were found to be in agreement with the results computed by the effective potential approach.

Finally, the minimal $45_S \oplus 126_S$ model was introduced. The experience acquainted in the $45_S \oplus 16_S$ model was consequently used for the calculation of the $SO(10)$-invariant τ^2-proportional term of the leading polynomial corrections to the masses of $(8, 1, 0)$ and $(1, 3, 0)$ in the $45_S \oplus 126_S$ model (equation (4.14)). This is a novel result that has never been calculated in detail before. Therefore, the stage was set for the calculation of the β^2-dependent terms of the leading scalar corrections in this model, which is to be the subject of our future work.

This knowledge may be useful for several reasons. In particular, it would allow us to recalculate the phenomenological results presented in [72] and thus get more accurate results. In this sense, the calculation of the β^2-dependent terms in the $45_S \oplus 126_S$ model is important especially because the minimal $SO(10)$ scenario allows one to make relatively accurate predictions of the proton lifetime, one of the quantities of main interest for the upcoming generation of the megaton-scale experimental facilities.

A. Group theory of $SO(10)$

In general, the $SO(10)$ represents the special orthogonal group of rotations in a 10-dimensional vector space. The defining representation of this Lie group consists of 10×10 matrices M, which preserve the norm of arbitrary vector v from the 10-dimensional vector space V, on which they act. This explicitly means

$$\|Mv\| = (Mv)^T(Mv) = v^T v = \|v\| \text{ for } \forall v \in V \implies M^T M = I$$

and, hence, the matrix M is orthogonal. In addition, just matrices with $\det M = 1$ are considered, which is why the group is called "special". In fact, this condition ensures that each element of the group is continuously connected with the identity element (i.e., the group is connected). As every orthogonal group also the $SO(10)$ is simply connected, which means that it can be covered by exponential map. Therefore, the matrices belonging to this group can be written in the following form

$$M = \exp\left(\frac{1}{2}\alpha_{ij}T_{ij}\right),$$

where T_{ij} stands for the set of 45 generators ($i = 0, 1, \dots, 9$ and $j = i, \dots, 9$) of the corresponding Lie algebra $so(10)$ and α_{ij} are the parameters characterizing the transformation.

The standard defining basis of the generators reads

$$T_{ij} = -i(\delta_{i[j}\delta_{kl]})_{ab},$$

where all the indices run from 0 to 9 and the square brackets denote the antisymmetrization. These generators satisfy the following commutation relation

$$[T_{ij}, T_{kl}] = i\left(\delta_{ik}T_{jl} + \delta_{jl}T_{ik} - \delta_{il}T_{jk} - \delta_{jk}T_{il}\right). \tag{A.1}$$

The conditions mentioned above, $M^T M = 1$ and $\det M = 1$, imply the existence of two invariant operators. The first of them is the Kronecker tensor δ_{ij}, which is transformed as

$$M_{ii'}M_{jj'}\delta_{i'j'} = M_{ij'}M_{jj'} = \delta_{ij},$$

where, obviously, the condition $M^T M = 1$ was used. The second invariant operator is the Levi-Civita tensor with ten indices $\varepsilon_{ijklmnopqr}$ obeying

$$\det M \varepsilon_{ijklmnopqr} = M_{ii'}M_{jj'}M_{kk'}M_{ll'}M_{mm'}M_{nn'}M_{oo'}M_{pp'}M_{qq'}M_{rr'}\varepsilon_{i'j'k'l'm'n'o'p'q'r'}$$

and, taking into account the condition $\det M = 1$, we see that the transformation is trivial, indeed.

Let us now discuss the irreducible representations of the $SO(10)$ group. To begin with, we distinguish two different kinds of the irreducible $SO(10)$ representations: single-valued and double-valued representations. The first type denominates the representations, which transform simply in the same way as standard vectors in the 10-dimensional real vector space and in the symmetrized and antisymmetrized direct products of such spaces. On the other hand, the double-values representations are spinor representations transforming like spinors in the 10-dimensional space.

A.1 Tensor representations of the $SO(10)$ group

We start with the tensor representations of the $SO(10)$ group, which can be constructed as a tensor product of n fundamental vectors ϕ_i transforming according to the rule

$$\phi_j \xrightarrow{M} \phi_i = M_{ij}\phi_j.$$

The tensor product of two of them can be decomposed into symmetric, antisymmetric and scalar part as

$$\begin{aligned}\phi_i \otimes \phi_j =& \frac{1}{2}\left(\phi_i \otimes \phi_j - \phi_j \otimes \phi_i\right) + \frac{1}{2}\left(\phi_i \otimes \phi_j + \phi_j \otimes \phi_i\right) - \frac{1}{10}\delta_{ij}\left(\phi_k \otimes \phi_k\right) \\ &+ \frac{1}{10}\delta_{ij}\left(\phi_k \otimes \phi_k\right),\end{aligned}$$

where the antisymmetric, symmetric and scalar parts can be distinguished

$$\begin{aligned}\phi_{ij}^{antisym} &\equiv \frac{1}{2}\left(\phi_i \otimes \phi_j - \phi_j \otimes \phi_i\right), \\ \phi_{ij}^{sym} &\equiv \frac{1}{2}\left(\phi_i \otimes \phi_j + \phi_j \otimes \phi_i\right) - \frac{1}{10}\delta_{ij}\left(\phi_k \otimes \phi_k\right), \\ \phi_{ij}^{scalar} &\equiv \frac{1}{10}\delta_{ij}\left(\phi_k \otimes \phi_k\right).\end{aligned}$$

These three types of tensors obviously cannot transform into each other since the symmetry properties of the tensors under permutation of the indices are not touched by the group transformations. Hence, antisymmetric and symmetric tensors, together with the scalar, form three separate subspaces. As a result, these three kinds of tensors form the irreducible representations of the $SO(10)$ group. Their dimensionalities are

- dimension of the subspace of the antisymmetric tensors is $d^A = \frac{n(n-1)}{2} = 45$
- dimension of the subspace of the symmetric tensors is $d^S = \frac{n(n+1)}{2} - 1 = 54$
- the remaining one dimension corresponds to a singlet

In the same way, taking tensor products of more vectors the representations with higher dimensions could be built.

The 126-dimensional representation needed for the construction of the Higgs sector of the most promising minimal non-SUSY $SO(10)$ model can be obtained using 5-index tensors. The invariant Levi-Civita tensor with 10 indices $\varepsilon_{ijklmnopqr}$ allows to write the following duality map

$$\phi_{ijklm} \xrightarrow{\text{dual map}} \tilde{\phi}_{ijklm} \equiv -\frac{i}{5}\varepsilon_{ijklmnopqr}\phi_{nopqr}.$$

Thanks to this map the tensor ϕ_{ijklm} can be separated into the self-dual and antiself-dual parts defined as

$$\begin{aligned}\Lambda_{ijklm} = \tilde{\Lambda}_{ijklm} &\equiv \frac{1}{\sqrt{2}}\left(v_{ijklm} + \tilde{v}_{ijklm}\right), \\ \bar{\Lambda}_{ijklm} = -\tilde{\bar{\Lambda}}_{ijklm} &\equiv \frac{1}{\sqrt{2}}\left(v_{ijklm} - \tilde{v}_{ijklm}\right).\end{aligned}$$

The property of duality, again, cannot be changed by the application of the group transformation; consequently, the tensors Λ_{ijklm} and $\bar{\Lambda}_{ijklm}$ form irreducible representations of $SO(10)$. Their dimension is naturally given by $\frac{1}{2}\binom{n}{k}$ for $n = 10$ and $k = 5$, i.e., 126.

A.2 Spinor representations of the $SO(10)$ group

The $SO(10)$ group is defined as a group of transformations acting on the 10-dimensional space V described by coordinates $x_1, x_2, \ldots, x_{10}$, which leave the norm of every vector $\phi \in V$ intact. In other words, the quadratic form $x_1^2 + x_2^2 + \cdots + x_{10}^2$ is preserved.

Following the standard construction, the quadratic form can be expressed as a square of a certain linear form

$$x_1^2 + x_2^2 + \cdots + x_{10}^2 = (\Gamma_1 x_1 + \Gamma_2 x_2 + \cdots + \Gamma_{10} x_{10})^2 .$$

Hence, we conclude that the coefficients Γ_i of the coordinates have to satisfy the anticommutation relation

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij},$$

which is called the Clifford algebra of dimension $d = 10$ and the coefficients Γ_i, apparently, have to be matrices. Moreover, it can be easily proved that this relation implies that the dimension of these matrices has to be even.

The Γ_i matrices generate the basis S_{ij} of the spinor representation. If we consider a rotation of the coordinates $x_i' = R_{ij} x_j$, it implies the transformation of the Γ_i matrices

$$\Gamma_i' = R_{ij}\Gamma_j,$$

i.e., they transform as vectors. Obviously, this transformation does not affect the Clifford algebra

$$\{\Gamma_i', \Gamma_j'\} = R_{ik}R_{jl}\{\Gamma_k, \Gamma_l\} = 2\delta_{ij}$$

and there must exist a similarity transformation relating the new set of Γ' matrices to the old one as

$$\Gamma_i' = R_{ij}\Gamma_j = S(R)\Gamma_i S^{-1}(R). \tag{A.2}$$

The matrix of the similarity transformation corresponding to the rotation group is its 2^n-dimensional spinor representation and any quantity transforming as

$$\Xi_i' = S_{ij}(R)\Xi_j$$

is a spinor.

To derive the relation between Γ_i and S_{ij} an infinitesimal rotation

$$R_{ij} = \delta_{ij} + \alpha_{ij},$$

with $\alpha_{ij} = -\alpha_{ji}$ can be considered. It induces an infinitesimal similarity transformation

$$S(R) = 1 + \frac{1}{2} i S_{ij} \alpha_{ij}.$$

By substitution of last two relations into the formula (A.2) we get

$$i\left[S_{ij}, \Gamma_k\right] = \left(\Gamma_j \delta_{ki} - \Gamma_k \delta_{kj}\right).$$

The S_{ij} satisfying the equation above can be written as

$$S_{ij} = \frac{1}{4i}\left[\Gamma_i, \Gamma_j\right] \tag{A.3}$$

and it can be easily shown that $S(R(4\pi)) = 1$, which means that $S(R)$ is a double-valued representation.

Furthermore, it can be checked that the S_{ij} matrices satisfy the commutation relation

$$\left[S_{ij}, S_{kl}\right] = i\left(\delta_{ik} S_{jl} + \delta_{jl} S_{ik} - \delta_{il} S_{jk} - \delta_{jk} S_{il}\right),$$

from which it is obvious that they form the $SO(10)$ representation.

Generally, all the $SO(n)$ groups have spinorial representations. For $n = 1$ the matrices $\Gamma_i^{n=1}$ are simply the Pauli matrices σ_i, which naturally satisfy the anticommutation formula

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}.$$

If we take

$$\Gamma_1^{n=1} \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Gamma_2^{n=1} \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

then we can iteratively build also the "gammas" of higher dimensions using the tensor product. The rank of the resulting representation is increased by each added tensor product by one and the size of the matrix representation is doubled. The ranks of $SO(2n)$ and $SO(2n+1)$ are equal and they share also the spinorial basis produced by n iterations. The difference is that the basis of $SO(2n+1)$ includes one more Γ matrix. Moreover, while the spinorial representation of $SO(2n+1)$ is irreducible, the one of $SO(2n)$ is reducible into two 2^{n-1}-dimensional parts. The irreducible spinor is real, self-conjugated and it transforms under a matrix representation with size 2^n. The two parts of the irreducible spinor are real and self-conjugate for n even. On the other hand, if n is odd, both parts of these reducible spinors are complex and conjugate to each other - they are called chiral or Weyl spinors.

So called chiral projection operators can be defined as

$$\Gamma_* = (-i)^{\frac{n}{2}}\, \Gamma_1 \Gamma_2 \cdots \Gamma_n.$$

This operator has dimension 2^n and it can be block-reducible; hence, it can be represented as

$$\Gamma_* = \begin{pmatrix} -I_{2^{n-1}} & 0 \\ 0 & I_{2^{n-1}} \end{pmatrix}, \tag{A.4}$$

where $I_{2^{n-1}}$ is the identity matrix of dimension 2^{n-1}.

If a spinor Ξ transforms as $\Xi'_i = S(R)_{ij}\Xi_j$, the irreducible chiral spinors can be defined as

$$\chi_+ \equiv \Pi^+\Xi = \frac{1}{2}\left(I_{2n} + \Gamma_*\right)\Xi,$$
$$\chi_- \equiv \Pi^-\Xi = \frac{1}{2}\left(I_{2n} - \Gamma_*\right)\Xi,$$

where we defined projectors $\Pi^{\pm} = \frac{1}{2}\left(I_{32} \mp \Gamma_*\right)$. The corresponding representations, under which the chiral spinors transform, read

$$S_+ \equiv \frac{1}{2}\left(I_{2n} + \Gamma_*\right)S \quad \text{and} \quad S_- \equiv \frac{1}{2}\left(I_{2n} - \Gamma_*\right)S.$$

If we derive the Γ_* from a particular basis and it has the form (A.4), then Ξ and S attain the following form

$$\Xi = \begin{pmatrix} \chi \\ \chi^C \end{pmatrix}, \quad S = \begin{pmatrix} S^+ & 0 \\ 0 & S^- \end{pmatrix},$$

where $\chi^C = C(\chi^*)$, i.e., the chiral components are related by charge conjugation (see below).

The explicit representation of the Γ_i matrices we work with is

$$\Gamma_0 = \begin{pmatrix} 0 & I_{16} \\ I_{16} & 0 \end{pmatrix}, \quad \Gamma_r = \begin{pmatrix} 0 & is_r \\ -is_r & 0 \end{pmatrix},$$

where I_{16} denotes the 16-dimensional identity matrix and the submatrices s_r for $r = 1, \ldots, 9$ are defined as

$$s_k = \mu_k\tau_3, \quad s_{k+3} = \nu_k\tau_1, \quad s_{k+6} = \rho_k\tau_2,$$

where $k = 1, 2, 3$ and the matrices μ_k, ν_k, ρ_k and τ_k equal to

$$\begin{aligned} \mu_k &= I_2 \otimes I_2 \otimes I_2 \otimes \sigma_k, \\ \nu_k &= I_2 \otimes I_2 \otimes \sigma_k \otimes I_2, \\ \rho_k &= I_2 \otimes \sigma_k \otimes I_2 \otimes I_2, \\ \tau_k &= \sigma_k \otimes I_2 \otimes I_2 \otimes I_2, \end{aligned} \tag{A.5}$$

and σ_k are standard Pauli matrices.

Let us define

$$s_{pq} = \frac{1}{2i}\left[s_p, s_q\right]$$

for $p, q = 1, \ldots, 9$. Hence, the albegra (A.3) can be expressed as

$$S_{p0} = \frac{1}{2}\begin{pmatrix} s_p & 0 \\ 0 & -s_p \end{pmatrix}, \quad S_{pq} = \frac{1}{2}\begin{pmatrix} s_{pq} & 0 \\ 0 & s_{pq} \end{pmatrix}. \tag{A.6}$$

In the current notation the Cartan subalgebra can be spanned over the matrices S_{03}, S_{12}, S_{45}, S_{78} and S_{69}; hence, the Γ_* matrix can be expressed as

$$\Gamma_* = 2^{-5}S_{03}S_{12}S_{45}S_{78}S_{69} = \begin{pmatrix} -I_{16} & 0 \\ 0 & I_{16} \end{pmatrix}.$$

Obviously, it obeys $\Gamma_*^2 = I_{32}$ and $\{\Gamma_i, \Gamma_*\} = 0$ for any i. Consequently, the 16-dimensional chiral spinors $\chi_\pm = \Pi^\pm \Xi$ mentioned above can be defined as

$$\chi_+ = \left(\begin{array}{c} \chi \\ 0 \end{array} \right) \quad \text{and} \quad \chi_- = \left(\begin{array}{c} 0 \\ \chi^C \end{array} \right).$$

If we use the relations $[S_{ij}, \Pi^\pm] = 0$, $\left(\Pi^\pm\right)^2 = \Pi^\pm$ and $\Pi^+ + \Pi^- = I_{32}$, the S_{ij} representation can be expressed in the form

$$S_{ij} = \Pi^+ S_{ij} \Pi^+ + \Pi^- S_{ij} \Pi^- \equiv \frac{1}{2} \begin{pmatrix} \sigma_{ij} & 0 \\ 0 & \tilde{\sigma}_{ij} \end{pmatrix}, \tag{A.7}$$

where the σ_{ij} and $\tilde{\sigma}_{ij}$ are 16×16 matrices with normalization

$$\frac{1}{4} \mathrm{Tr} \sigma_{ij} \sigma_{kl} = \frac{1}{4} \mathrm{Tr} \tilde{\sigma}_{ij} \tilde{\sigma}_{kl} = 4 \delta_{i[k} \delta_{jl]}. \tag{A.8}$$

By comparison of matrices (A.6) with the equation (A.7) one gets

$$\sigma_{p0} = s_p, \; \sigma_{pq} = s_{pq}, \; \tilde{\sigma}_{p0} = -s_p, \; \tilde{\sigma}_{pq} = s_{pq}.$$

In the invariants built off the adjoint representation it is convenient to trace out the σ-matrices using the definition

$$\Phi \equiv \frac{\sigma_{ij} \phi_{ij}}{4}.$$

The traces of two and four σ-matrices then read

$$\begin{aligned} \mathrm{Tr} \Phi^2 &= -2 \mathrm{Tr} \phi^2, \\ \mathrm{Tr} \Phi^4 &= -\frac{3}{4} \left(\mathrm{Tr} \phi^2 \right)^2 - \mathrm{Tr} \phi^4. \end{aligned}$$

A.3 Charge conjugation

Using the above notation, the spinor and its complex conjugate obey the following transformations

$$\begin{aligned} \chi &\to \chi - \frac{i}{4} \lambda_{ij} \sigma_{ij} \chi, \\ \chi^* &\to \chi^* + \frac{i}{4} \lambda_{ij} \sigma_{ij}^T \chi^*. \end{aligned}$$

Hence, the charge conjugated spinor χ^C transforms as

$$\chi^C \to \chi^C - \frac{i}{4} \lambda_{ij} \tilde{\sigma}_{ij} \chi^C,$$

where the charge conjugation matrix C must satisfy

$$C^{-1} \tilde{\sigma}_{ij} C = -\sigma_{ij}^T.$$

Employing the formula (A.5) it can be found that

$$C = \mu_2 \nu_2 \rho_2 \tau_2,$$

which implies the explicit form

$$C = \mathrm{antidiag} \left(1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1 \right).$$

It is not difficult to see that

$$C = C^* = C^{-1} = C^T = C^\dagger.$$

B. Mutually commuting operators

B.1 Cartan operators in the 10_S representation

The concrete realization, which we used for the five Cartan operators of the $SO(10)$ group expressed in the 10-dimensional vector representation reads

$$T_{C3} = \begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_{C8} = \begin{pmatrix} 0 & \frac{i}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{i}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_{BL} = \begin{pmatrix} 0 & -\frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2i}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2i}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2i}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_{R3} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-\frac{i}{2}&0&0\\ 0&0&0&0&0&0&\frac{i}{2}&0&0&0\\ 0&0&0&0&0&0&0&0&0&-\frac{i}{2}\\ 0&0&0&0&0&0&0&0&\frac{i}{2}&0 \end{pmatrix},$$

$$T_{L3} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\frac{i}{2}&0&0\\ 0&0&0&0&0&0&-\frac{i}{2}&0&0&0\\ 0&0&0&0&0&0&0&0&0&-\frac{i}{2}\\ 0&0&0&0&0&0&0&0&\frac{i}{2}&0 \end{pmatrix}.$$

B.2 Casimir operators in the 10_S representation

The concrete realization of the three Casimir operators of the $SO(10)$ group expressed in the 10-dimensional vector representation can be written as

$$C_C = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{4}{3}&0&0&0&0&0\\ 0&0&0&0&0&\frac{4}{3}&0&0&0&0\\ 0&0&0&0&0&0&\frac{4}{3}&0&0&0\\ 0&0&0&0&0&0&0&\frac{4}{3}&0&0\\ 0&0&0&0&0&0&0&0&\frac{4}{3}&0\\ 0&0&0&0&0&0&0&0&0&\frac{4}{3} \end{pmatrix},$$

$$C_R = \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_L = \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

C. Physical basis of the adjoint representation

The vectors 45^a_{phys} for $a = 1, \ldots, 45$ are defined as follows

$$
\begin{aligned}
45^1_{phys} &\equiv (3, \tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}, 0, 0, 0, 0, 0) = \tfrac{1}{2}\phi_{35} + \tfrac{1}{2}i\phi_{36} - \tfrac{1}{2}i\phi_{45} + \tfrac{1}{2}\phi_{46}, \\
45^2_{phys} &\equiv (3, -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0, 0, 0, 0, 0) = \tfrac{1}{2}\phi_{35} - \tfrac{1}{2}i\phi_{36} + \tfrac{1}{2}i\phi_{45} + \tfrac{1}{2}\phi_{46}, \\
45^3_{phys} &\equiv (3, 0, 0, 0, 0, 0, 0, 0) = \sqrt{\tfrac{2}{3}}\phi_{12} - \tfrac{1}{\sqrt{6}}\phi_{34} - \tfrac{1}{\sqrt{6}}\phi_{56}, \\
45^4_{phys} &\equiv (3, 0, 0, 0, 0, 0, 0, 0) = \tfrac{1}{\sqrt{2}}\phi_{34} - \tfrac{1}{\sqrt{2}}\phi_{56}, \\
45^5_{phys} &\equiv (3, 1, 0, 0, 0, 0, 0, 0) = -\tfrac{1}{2}i\phi_{13} - \tfrac{1}{2}\phi_{14} + \tfrac{1}{2}\phi_{23} - \tfrac{1}{2}i\phi_{24}, \\
45^6_{phys} &\equiv (3, -1, 0, 0, 0, 0, 0, 0) = \tfrac{1}{2}i\phi_{13} - \tfrac{1}{2}\phi_{14} + \tfrac{1}{2}\phi_{23} + \tfrac{1}{2}i\phi_{24}, \\
45^7_{phys} &\equiv (3, \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0, 0, 0, 0, 0) = \tfrac{1}{2}\phi_{15} - \tfrac{1}{2}i\phi_{16} + \tfrac{1}{2}i\phi_{25} + \tfrac{1}{2}\phi_{26}, \\
45^8_{phys} &\equiv (3, -\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}, 0, 0, 0, 0, 0) = \tfrac{1}{2}\phi_{15} + \tfrac{1}{2}i\phi_{16} - \tfrac{1}{2}i\phi_{25} + \tfrac{1}{2}\phi_{26}, \\
45^9_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \tfrac{2}{3}, 0, 0, 0, 0) = -\tfrac{1}{2}\phi_{35} + \tfrac{1}{2}i\phi_{36} + \tfrac{1}{2}i\phi_{45} + \tfrac{1}{2}\phi_{46}, \\
45^{10}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = -\tfrac{1}{2}\phi_{39} - \tfrac{1}{2}i\phi_{310} - \tfrac{1}{2}i\phi_{49} + \tfrac{1}{2}\phi_{410}, \\
45^{11}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = -\tfrac{1}{2}\phi_{37} - \tfrac{1}{2}i\phi_{38} - \tfrac{1}{2}i\phi_{47} + \tfrac{1}{2}\phi_{48}, \\
45^{12}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = \tfrac{1}{2}\phi_{39} - \tfrac{1}{2}i\phi_{310} + \tfrac{1}{2}i\phi_{49} + \tfrac{1}{2}\phi_{410}, \\
45^{13}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = \tfrac{1}{2}\phi_{37} - \tfrac{1}{2}i\phi_{38} + \tfrac{1}{2}i\phi_{47} + \tfrac{1}{2}\phi_{48}, \\
45^{14}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, -\tfrac{2}{3}, 0, 0, 0, 0) = -\tfrac{1}{2}\phi_{35} - \tfrac{1}{2}i\phi_{36} - \tfrac{1}{2}i\phi_{45} + \tfrac{1}{2}\phi_{46}, \\
45^{15}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = \tfrac{1}{2}\phi_{39} + \tfrac{1}{2}i\phi_{310} - \tfrac{1}{2}i\phi_{49} + \tfrac{1}{2}\phi_{410}, \\
45^{16}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = \tfrac{1}{2}\phi_{37} + \tfrac{1}{2}i\phi_{38} - \tfrac{1}{2}i\phi_{47} + \tfrac{1}{2}\phi_{48}, \\
45^{17}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = -\tfrac{1}{2}\phi_{39} + \tfrac{1}{2}i\phi_{310} + \tfrac{1}{2}i\phi_{49} + \tfrac{1}{2}\phi_{410}, \\
45^{18}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = -\tfrac{1}{2}\phi_{37} + \tfrac{1}{2}i\phi_{38} + \tfrac{1}{2}i\phi_{47} + \tfrac{1}{2}\phi_{48}, \\
45^{19}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \tfrac{2}{3}, 0, 0, 0, 0) = -\tfrac{1}{2}\phi_{15} + \tfrac{1}{2}i\phi_{16} + \tfrac{1}{2}i\phi_{25} + \tfrac{1}{2}\phi_{26}, \\
45^{20}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = -\tfrac{1}{2}\phi_{19} - \tfrac{1}{2}i\phi_{110} - \tfrac{1}{2}i\phi_{29} + \tfrac{1}{2}\phi_{210}, \\
45^{21}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = -\tfrac{1}{2}\phi_{17} - \tfrac{1}{2}i\phi_{18} - \tfrac{1}{2}i\phi_{27} + \tfrac{1}{2}\phi_{28}, \\
45^{22}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = \tfrac{1}{2}\phi_{19} - \tfrac{1}{2}i\phi_{110} + \tfrac{1}{2}i\phi_{29} + \tfrac{1}{2}\phi_{210}, \\
45^{23}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = \tfrac{1}{2}\phi_{17} - \tfrac{1}{2}i\phi_{18} + \tfrac{1}{2}i\phi_{27} + \tfrac{1}{2}\phi_{28}, \\
45^{24}_{phys} &\equiv (\tfrac{4}{3}, \tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, -\tfrac{2}{3}, 0, 0, 0, 0) = -\tfrac{1}{2}\phi_{15} - \tfrac{1}{2}i\phi_{16} - \tfrac{1}{2}i\phi_{25} + \tfrac{1}{2}\phi_{26}, \\
45^{25}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = \tfrac{1}{2}\phi_{19} + \tfrac{1}{2}i\phi_{110} - \tfrac{1}{2}i\phi_{29} + \tfrac{1}{2}\phi_{210}, \\
45^{26}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = \tfrac{1}{2}\phi_{17} + \tfrac{1}{2}i\phi_{18} - \tfrac{1}{2}i\phi_{27} + \tfrac{1}{2}\phi_{28}, \\
45^{27}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = -\tfrac{1}{2}\phi_{19} + \tfrac{1}{2}i\phi_{110} + \tfrac{1}{2}i\phi_{29} + \tfrac{1}{2}\phi_{210}, \\
45^{28}_{phys} &\equiv (\tfrac{4}{3}, -\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = -\tfrac{1}{2}\phi_{17} + \tfrac{1}{2}i\phi_{18} + \tfrac{1}{2}i\phi_{27} + \tfrac{1}{2}\phi_{28}, \\
45^{29}_{phys} &\equiv (\tfrac{4}{3}, 0, \tfrac{1}{\sqrt{3}}, -\tfrac{2}{3}, 0, 0, 0, 0) = -\tfrac{1}{2}\phi_{13} - \tfrac{1}{2}i\phi_{14} - \tfrac{1}{2}i\phi_{23} + \tfrac{1}{2}\phi_{24},
\end{aligned}
$$

$$
\begin{aligned}
45^{30}_{phys} &\equiv (\tfrac{4}{3}, 0, -\tfrac{1}{\sqrt{3}}, \tfrac{2}{3}, 0, 0, 0, 0) = -\tfrac{1}{2}\phi_{13} + \tfrac{1}{2}i\phi_{14} + \tfrac{1}{2}i\phi_{23} + \tfrac{1}{2}\phi_{24},\\
45^{31}_{phys} &\equiv (0, 0, 0, 0, 0, 0, 0, 0) = \tfrac{1}{\sqrt{3}}\phi_{12} + \tfrac{1}{\sqrt{3}}\phi_{34} + \tfrac{1}{\sqrt{3}}\phi_{56},\\
45^{32}_{phys} &\equiv (\tfrac{4}{3}, 0, \tfrac{1}{\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = -\tfrac{1}{2}\phi_{57} + \tfrac{1}{2}i\phi_{58} + \tfrac{1}{2}i\phi_{67} + \tfrac{1}{2}\phi_{68},\\
45^{33}_{phys} &\equiv (\tfrac{4}{3}, 0, \tfrac{1}{\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = -\tfrac{1}{2}\phi_{59} + \tfrac{1}{2}i\phi_{510} + \tfrac{1}{2}i\phi_{69} + \tfrac{1}{2}\phi_{610},\\
45^{34}_{phys} &\equiv (\tfrac{4}{3}, 0, \tfrac{1}{\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = \tfrac{1}{2}\phi_{57} + \tfrac{1}{2}i\phi_{58} - \tfrac{1}{2}i\phi_{67} + \tfrac{1}{2}\phi_{68},\\
45^{35}_{phys} &\equiv (\tfrac{4}{3}, 0, \tfrac{1}{\sqrt{3}}, \tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = \tfrac{1}{2}\phi_{59} + \tfrac{1}{2}i\phi_{510} - \tfrac{1}{2}i\phi_{69} + \tfrac{1}{2}\phi_{610},\\
45^{36}_{phys} &\equiv (\tfrac{4}{3}, 0, -\tfrac{1}{\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = \tfrac{1}{2}\phi_{57} - \tfrac{1}{2}i\phi_{58} + \tfrac{1}{2}i\phi_{67} + \tfrac{1}{2}\phi_{68},\\
45^{37}_{phys} &\equiv (\tfrac{4}{3}, 0, -\tfrac{1}{\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{1}{2}) = \tfrac{1}{2}\phi_{59} - \tfrac{1}{2}i\phi_{510} + \tfrac{1}{2}i\phi_{69} + \tfrac{1}{2}\phi_{610},\\
45^{38}_{phys} &\equiv (\tfrac{4}{3}, 0, -\tfrac{1}{\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = -\tfrac{1}{2}\phi_{57} - \tfrac{1}{2}i\phi_{58} - \tfrac{1}{2}i\phi_{67} + \tfrac{1}{2}\phi_{68},\\
45^{39}_{phys} &\equiv (\tfrac{4}{3}, 0, -\tfrac{1}{\sqrt{3}}, -\tfrac{1}{3}, \tfrac{3}{4}, -\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{1}{2}) = -\tfrac{1}{2}\phi_{59} - \tfrac{1}{2}i\phi_{510} - \tfrac{1}{2}i\phi_{69} + \tfrac{1}{2}\phi_{610},\\
45^{40}_{phys} &\equiv (0, 0, 0, 0, 2, 0, 0, 0) = \tfrac{1}{\sqrt{2}}\phi_{78} - \tfrac{1}{\sqrt{2}}\phi_{910},\\
45^{41}_{phys} &\equiv (0, 0, 0, 0, 2, 1, 0, 0) = -\tfrac{1}{2}\phi_{79} + \tfrac{1}{2}i\phi_{710} - \tfrac{1}{2}i\phi_{89} - \tfrac{1}{2}\phi_{810},\\
45^{42}_{phys} &\equiv (0, 0, 0, 0, 2, -1, 0, 0) = -\tfrac{1}{2}\phi_{79} - \tfrac{1}{2}i\phi_{710} + \tfrac{1}{2}i\phi_{89} - \tfrac{1}{2}\phi_{810},\\
45^{43}_{phys} &\equiv (0, 0, 0, 0, 0, 0, 2, 0) = \tfrac{1}{\sqrt{2}}\phi_{78} + \tfrac{1}{\sqrt{2}}\phi_{910},\\
45^{44}_{phys} &\equiv (0, 0, 0, 0, 0, 0, 2, 1) = \tfrac{1}{2}i\phi_{79} + \tfrac{1}{2}\phi_{710} + \tfrac{1}{2}\phi_{89} - \tfrac{1}{2}i\phi_{810},\\
45^{45}_{phys} &\equiv (0, 0, 0, 0, 0, 0, 2, -1) = -\tfrac{1}{2}i\phi_{79} + \tfrac{1}{2}\phi_{710} + \tfrac{1}{2}\phi_{89} + \tfrac{1}{2}i\phi_{810},
\end{aligned}
$$

where the numbers labelling these vectors correspond to the eigenvalues of the Cartan and Casimir operators in the adjoint representation. Hence, the physical vectors are labelled by these numbers in the following order

$$(C^{45}_C, T_{C3}, T_{C8}, \frac{1}{2}T_{BL}, C_L, T_{L3}, C^{45}_R, T_{R3}),$$

where T and C are just $10\otimes 10$ versions of the 10-dimensional operators presented in Appendix B. Moreover, we also expressed here the physical vectors in terms of the vectors from the defining basis.

D. Tree-level scalar spectrum of the $45 \oplus 16$ model

Within this appendix we present the explicit form of the tree-level scalar spectrum of the 45⊕16 model. In the following we will denote the vectors using the quantum numbers corresponding to the eight mutually commuting generators (printed in the Appendix A) in this order: $(C_C, T_{C3}, T_{C8}, \frac{1}{2}T_{BL}, C_{R3}, T_{R3}, C_{L3}, T_{L3})$. Since we are breaking the $SO(10)$ symmetry down to the Standard Model, we should find (and we will check this) altogether $45 - 12 = 33$ Goldstone bosons.

Let us start with the problematic pseudo-Goldstone states, whose masses we have already presented in the main text of the thesis. The submultiplet $(1, 3, 0)$ is spanned over the basis formed by vectors

$$\{(0,0,0,0,2,-1,0,0), (0,0,0,0,2,0,0,0), (0,0,0,0,2,1,0,0)\}$$

and the 3×3 mass submatrix contains three real degrees of freedom with mass

$$2a_2(\omega_R - \omega_Y)(2\omega_R + \omega_Y).$$

Likewise, the submultiplet $(8, 1, 0)$ is spanned over the basis formed by vectors

$$\begin{aligned}\{&(3,0,0,0,0,0,0,0), (3,-1,0,0,0,0,0,0), (3,0,0,0,0,0,0,0),\\ &(3,1,0,0,0,0,0,0), (3,-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2},0,0,0,0,0), (3,\tfrac{1}{2},-\tfrac{\sqrt{3}}{2},0,0,0,0,0),\\ &(3,-\tfrac{1}{2},\tfrac{\sqrt{3}}{2},0,0,0,0,0), (3,\tfrac{1}{2},\tfrac{\sqrt{3}}{2},0,0,0,0,0)\}\end{aligned}$$

and the 8×8 mass submatrix contains eight real degrees of freedom with mass

$$-2a_2(\omega_R - \omega_Y)(\omega_R + 2\omega_Y).$$

Next massive degrees of freedom are assigned to the multiplet $(1, 2, -\frac{1}{2}) + h.c.$ are spanned over the couple of vectors

$$\{(0,0,0,1,\tfrac{3}{4},\tfrac{1}{2},0,0), (0,0,0,1,\tfrac{3}{4},-\tfrac{1}{2},0,0)\}$$

and the corresponding mass submatrix contains the mass

$$-\frac{(\omega_R + 3\omega_Y)\left(\tau\chi_R\chi_R^*\omega_R + 3\tau\chi_R\chi_R^*\omega_Y + 4a_2\omega_R^3\omega_Y + 4a_2\omega_R^2\omega_Y^2\right)}{\chi_R\chi_R^*(2\omega_R + 3\omega_Y)}$$

Obviously, this submatrix stands for four real degrees of freedom in spinors.

Another massive fields are accomodated within the submultiplet $(3, 1, +\frac{1}{3}) + h.c.$, which is spanned over the basis

$$\{(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},0,0,\tfrac{3}{4},-\tfrac{1}{2}), (\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},0,0,\tfrac{3}{4},-\tfrac{1}{2}), (\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},\tfrac{1}{3},0,0,\tfrac{3}{4},-\tfrac{1}{2})\}$$

and the mass submatrix contains the mass

$$-\frac{4(\omega_R + \omega_Y)^2\left(\tau\chi_R\chi_R^* + 2a_2\omega_R\omega_Y^2\right)}{\chi_R\chi_R^*(2\omega_R + 3\omega_Y)}$$

In this case there are six real degrees of freedom in spinors.

From now on we will deal with Goldstone bosons. The submultiplet $(3, 2, +\frac{5}{6})+$ $h.c.$ is spanned over the set of vectors

$$\begin{gathered}\{(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},\tfrac{3}{4},-\tfrac{1}{2}),(\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},\tfrac{3}{4},-\tfrac{1}{2}),\\ (\tfrac{4}{3},0,-\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},\tfrac{3}{4},-\tfrac{1}{2}),(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},\tfrac{3}{4},-\tfrac{1}{2}),\\ (\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},\tfrac{4}{3},-\tfrac{1}{2}),(\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},\tfrac{4}{3},-\tfrac{1}{2})\}\end{gathered}$$

and the corresponding mass submatrix is trivial 6×6 zero matrix, which means that all the 12 degrees of freedom hidden in this sector are massless and thus they represent the Goldstone bosons of the theory.

The sector $(1, 1, +1) + h.c.$ spans over the couple of vectors

$$\{(0,0,0,0,0,0,2,1),(0,0,0,1,0,0,\tfrac{3}{4},\tfrac{1}{2})\}.$$

The relevant mass submatrix then reads

$$\begin{pmatrix} -\frac{\tau\chi_R\chi_R^*+6a_2\omega_Y^2(\omega_R+\omega_Y)}{2\omega_R+3\omega_Y} & \frac{2\omega_R\left(\tau\chi_R\chi_R^*+6a_2\omega_Y^2(\omega_R+\omega_Y)\right)}{\chi_R^*(2\omega_R+3\omega_Y)} \\ \frac{2\omega_R\left(\tau\chi_R\chi_R^*+6a_2\omega_Y^2(\omega_R+\omega_Y)\right)}{\chi_R(2\omega_R+3\omega_Y)} & -\frac{4\omega_R^2\left(\tau\chi_R\chi_R^*+6a_2\omega_Y^2(\omega_R+\omega_Y)\right)}{\chi_R\chi_R^*(2\omega_R+3\omega_Y)} \end{pmatrix}.$$

This sector covers four degrees of freedom and it is obvious that every other of them is massless.

Next submultiplet can be denoted as $(3, 1, +\frac{2}{3}) + h.c.$. It spans over the basis

$$\begin{gathered}\{(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{2}{3},0,0,0,0),(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},0,0,\tfrac{3}{4},\tfrac{1}{2}),\\ (\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{2}{3},0,0,0,0),(\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},0,0,\tfrac{3}{4},\tfrac{1}{2}),\\ (\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},\tfrac{2}{3},0,0,0,0),(\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},\tfrac{1}{3},0,0,\tfrac{3}{4},\tfrac{1}{2})\}\end{gathered}$$

and the respective mass submatrix has three blocks of the form

$$\begin{pmatrix} -\frac{\tau\chi_R\chi_R^*+2a_2\omega_R\left(2\omega_R^2+3\omega_R\omega_Y+\omega_Y^2\right)}{2\omega_R+3\omega_Y} & \frac{2i\omega_Y\left(\tau\chi_R\chi_R^*+2a_2\omega_R\left(2\omega_R^2+3\omega_R\omega_Y+\omega_Y^2\right)\right)}{\chi_R(2\omega_R+3\omega_Y)} \\ -\frac{2i\omega_Y\left(\tau\chi_R\chi_R^*+2a_2\omega_R\left(2\omega_R^2+3\omega_R\omega_Y+\omega_Y^2\right)\right)}{\chi_R^*(2\omega_R+3\omega_Y)} & -\frac{4\omega_Y^2\left(\tau\chi_R\chi_R^*+2a_2\omega_R\left(2\omega_R^2+3\omega_R\omega_Y+\omega_Y^2\right)\right)}{\chi_R\chi_R^*(2\omega_R+3\omega_Y)} \end{pmatrix}.$$

Obviously, every other of these four degrees of freedom is massless.

The submultiplet $(3, 2, +\frac{1}{6}) + h.c.$ is spanned over the set of vectors

$$\begin{gathered}\{(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2}),(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},0,0),\\ (\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2}),(\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},0,0),\\ (\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2}),(\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},\tfrac{1}{3},\tfrac{3}{4},\tfrac{1}{2},0,0),\\ (\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2}),(\tfrac{4}{3},\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},0,0),\\ (\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2}),(\tfrac{4}{3},-\tfrac{1}{2},\tfrac{1}{2\sqrt{3}},\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},0,0),\\ (\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},-\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},\tfrac{3}{4},\tfrac{1}{2}),(\tfrac{4}{3},0,-\tfrac{1}{\sqrt{3}},\tfrac{1}{3},\tfrac{3}{4},-\tfrac{1}{2},0,0)\}\end{gathered}$$

and the mass submatrix of this fields possesses six blocks of the form

$$\begin{pmatrix} -\frac{\tau\chi_R\chi_R^*+4a_2\omega_R\omega_Y(\omega_R+2\omega_Y)}{2\omega_R+3\omega_Y} & \frac{(\omega_R+\omega_Y)(\tau\chi_R\chi_R^*+4a_2\omega_R\omega_Y(\omega_R+2\omega_Y))}{\chi_R^*(2\omega_R+3\omega_Y)} \\ \frac{(\omega_R+\omega_Y)(\tau\chi_R\chi_R^*+4a_2\omega_R\omega_Y(\omega_R+2\omega_Y))}{\chi_R(2\omega_R+3\omega_Y)} & -\frac{(\omega_R+\omega_Y)^2(\tau\chi_R\chi_R^*+4a_2\omega_R\omega_Y(\omega_R+2\omega_Y))}{\chi_R\chi_R^*(2\omega_R+3\omega_Y)} \end{pmatrix}.$$

Again, on grounds of the form of the mass submatrix, every other of the degrees of freedom is massless.

Lastly, the sector $(1, 1, 0)$ spans over the basis

$$\{(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,2,0),(0,0,0,1,0,0,\tfrac{3}{4},-\tfrac{1}{2}),\\ (0,0,0,-1,0,0,\tfrac{3}{4},\tfrac{1}{2})\}$$

and the obtained mass submatrix has the following form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= \frac{-3\tau\chi_R\chi_R^* - 192a_1\omega_Y^2(2\omega_R+3\omega_Y)}{4\omega_R+6\omega_Y} \\
&\quad - \frac{4a_2\left(2\omega_R^3+2\omega_R^2\omega_Y+14\omega_R\omega_Y^2+21\omega_Y^3\right)}{4\omega_R+6\omega_Y}, \\
a_{12} = a_{21} &= \frac{\sqrt{6}(\tau\chi_R\chi_R^* + 4\omega_R\omega_Y(32a_1\omega_R+5a_2\omega_R+48a_1\omega_Y+8a_2\omega_Y))}{4\omega_R+6\omega_Y}, \\
a_{13} = a_{31} &= a_-\frac{2\sqrt{3}\omega_Y(-4\alpha\chi_R\chi_R^* + a_2\omega_R(\omega_R+\omega_Y))}{\chi_R^*}, \\
a_{14} = a_{41} &= -\frac{2\sqrt{3}\omega_Y(-4\alpha\chi_R\chi_R^* + a_2\omega_R(\omega_R+\omega_Y))}{\chi_R}, \\
a_{22} &= -\frac{\tau\chi_R\chi_R^* + 64a_1\omega_R^2(2\omega_R+3\omega_Y)}{2\omega_R+3\omega_Y} \\
&\quad - \frac{2a_2\left(8\omega_R^3+12\omega_R^2\omega_Y+3\omega_R\omega_Y^2+3\omega_Y^3\right)}{2\omega_R+3\omega_Y}, \\
a_{23} = a_{32} &= \frac{2\sqrt{2}\omega_R(-4\alpha\chi_R\chi_R^* + a_2\omega_Y(\omega_R+\omega_Y))}{\chi_R^*}, \\
a_{24} = a_{42} &\frac{2\sqrt{2}\omega_R(-4\alpha\chi_R\chi_R^* + a_2\omega_Y(\omega_R+\omega_Y))}{\chi_R}, \\
a_{33} &= \frac{\lambda_1\chi_R\chi_R^*}{2}, \\
a_{34} = a_{43} &= \frac{\lambda_1(\chi_R)^2}{2}, \\
a_{44} &= \frac{\lambda_1\chi_R\chi_R^*}{2}.
\end{aligned}$$

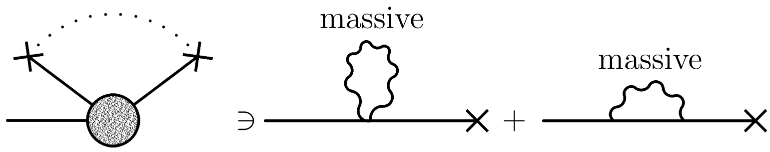
In case we assume that χ_R is very small, then $a_{33} = a_{34} = a_{43} = a_{44} = 0$. Hence, it is not difficult to see that there are four degrees of freedom in this sector and just one of them is massless.

It ca be easily verified, the total number of Goldstone bosons (massless degrees of freedom) is indeed 33 as we supposed before our calculation.

E. Leading polynomial gauge corrections in the $45 \oplus 16$ model

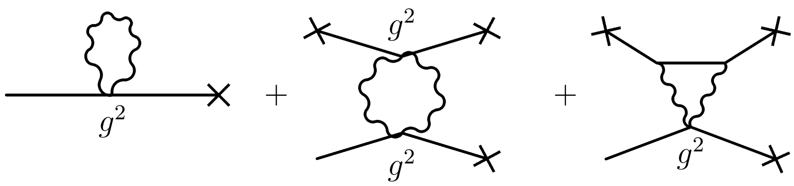
In the first section of the third chapter we mentioned that using the effective potential approach it is possible to calculate also the gauge corrections corresponding to the submultiplets $(8, 1, 0)$ and $(1, 3, 0)$. Therefore, one could suggest that similarly as we computed the scalar mass corrections of these submultiplets using the standard diagrammatic methods we could diagrammatically determine also the gauge mass corrections. However, the calculation of these contributions turns out to be much more complicated since there are many more diagrams contributing to the mass corrections.

Naturally, all the contributing one-point functions are tadpoles; therefore, each of them now contributes as in the case of scalar corrections. However, there are still many types of diagrams, which have to be taken into account. Particularly, there are two basic independent series of graphs, which can be schematically depicted as

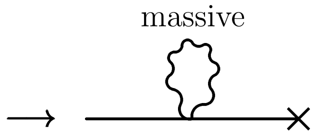


(E.1)

The concrete types of diagrams forming these two series can be illustrated in the following way: the first diagram on the right hand side of the equation (E.1) represents the sum of the series

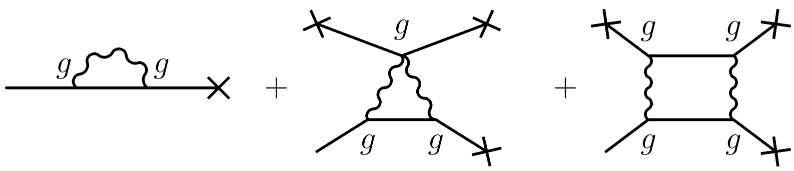


(E.2)

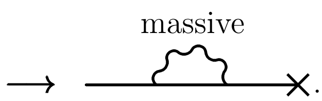


(E.3)

and the second diagram on the right hand side of the equation (E.1) can be obtained as

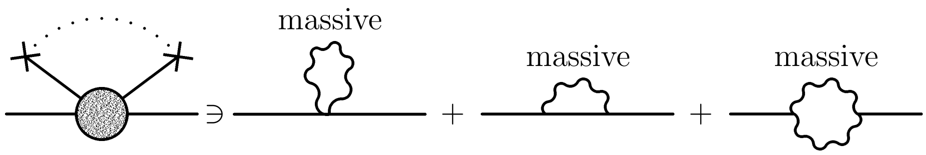


(E.4)



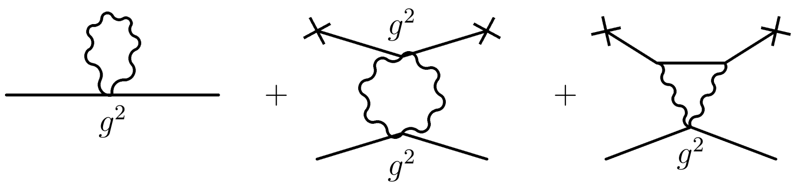
(E.5)

Likewise, we can now draw the series contributing to the two-point functions. While in the scalar case just the seagull type of the two-point functions yielded the polynomial terms, in the gauge case there much more options how to produce a nonlogarithmic correction by a two-point function, namely, we have to deal with three series of diagrams, which can be schematically expressed as

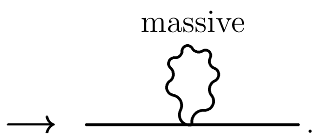


(E.6)

Again, the concrete types of diagrams forming the three series can be illustrated diagrammatically: the first diagram on the right hand side of the equation (E.6) we can get as

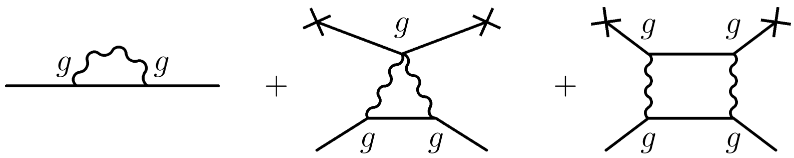


(E.7)

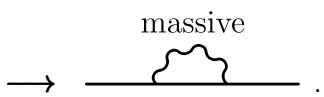


(E.8)

The second diagram on the right hand side of the equation (E.6) can be calculated as the sum of the series

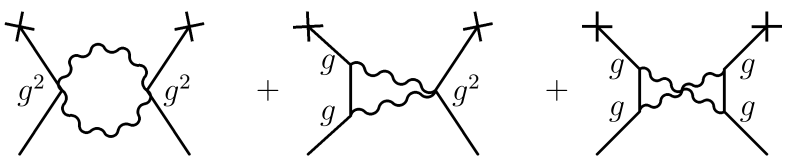


(E.9)

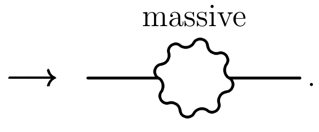


(E.10)

Eventually, the last diagram on the right hand side of the equation (E.6) represents the resulting contribution of the following series



(E.11)



(E.12)

Moreover, we would have to consider also other diagrams containing Goldstone bosons and ghosts. In consequence, to compute the gauge corrections of the desired masses we would obviously have to calculate a big number of diagrams. The best way would be to calculate directly the "massive" graphs, i.e., to sum the series. Therefore, the gauge corrections to the masses of $(8, 1, 0)$ and $(1, 3, 0)$ turn out to be too demanding to be calculated easily using the standard diagrammatic methods and, as such, they are beyond the scope of this thesis. Anyway, they offer an interesting direction of further research.

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List of Abbreviations

GUT	Grand Unified Theories
LEP	Large Electron Collider
LHC	Large Hadron Collider
MSSM	Minimal Supersymmetric Standard Model
SC	stationarity condition
SUSY	supersymmetry/supersymmetric
VEV	vacuum expectation value