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Testing the Effects of Parameter Changes in the Bornholdt's Model

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Abstract:
In this work we thoroughly analyze Bornholdt's version of Ising model of ferromagnetism, with emphasis on its ability to mimic some basic stylized facts of financial series. Initially, we provide a breakdown of model definition and analysis of underlying dynamics. Subsequently, we examine and confirm model's ability to mimic stylized facts of financial series. To examine robustness of this ability to parameter change, we conduct simulations over a set of parameter combinations. We conclude that there is a wide set of combinations that yields acceptable simulation results. We also note that the seemingly best results are obtained at parameter values close to border of this set.

Keywords: Ising model, financial markets, stylized facts.

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Abstrakt:
Tato práce se zabývá zevrubnou analýzou Bornholdtovy verze Isingova modelu feromagnetu se zaměřením na schopnost modelu imitovat vlastnosti finančních časových řad. Model nejprve podrobujeme analýze jak z hlediska definice, tak z hlediska vnitřní dynamiky. Následně zkoumáme a potvrzujeme schopnost modelu imitovat vlastnosti finančních časových řad. Abychom otestovali robustnost této schopnosti vůči změně ve vstupních parametrech, provádíme simulace přes různé jejich kombinace. Docházíme k závěru, že existuje široká množina kombinací, pro něž dostáváme simulace uspokojivých vlastností. Závěrem poznamenáváme, že zdánlivě nejlepších výsledků dosahuje model na hranici zmíněné množiny.

Klíčová slova: Isingův model, finanční trhy, stylizovaná fakta.

Declaration of authorship:

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature.

The author hereby declares that this thesis has not been used to obtain another university degree.

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Prague, July 31, 2014 Štěpán Chrz

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1 Introduction

Various financial series, such as market indices, exchange rates, commodities, and their respective derivatives have been a subject of numerous studies for at least half a century. It was only in the last decade or two, however, that widespread availability of financial data together with unprecedented computational power at hand made it possible to adopt a systematic data-based approach (Cont, 2001). Independent studies reported some general properties that seemed to apply to a wide range of financial series, regardless of the type, time period or market of origin. These properties became known as "stylized (empirical) facts".

The same circumstances that led to new analytical approaches in economics made it possible to apply methods developed in physics to study the financial data. Statistical mechanics is a branch of physics widely used to study economic phenomena. Using probability theory, it infers macroscopic behavior of systems from interaction of large number of elements governed by simple mechanical laws at microscopic level. The parallel with financial markets is straightforward; price movements are macroscopic phenomena driven by interaction of countless microscopic individuals. Even though each of these individuals might exhibit complex behavior, it is plausible that few simple rules approximate their actions on the market closely enough for macroscopic patterns to emerge.

One distinct group of models that aim to mimic behavior of financial series are those based on a model of ferromagnet introduced by Ising [1925]. Among those, modification by Bornholdt [2001] is widely used as a reference model that is further augmented (e.g. by Sieczka and Holyst, 2008, Sornette and Zhou, 2006, Denys et al., 2013).

This model will be a primary subject of our analysis for several reasons. Firstly, its similarity to the original two-dimensional Ising model makes it possible to draw parallels between the two and examine to what extent the original model is relevant to the study of financial series.

Secondly, model's relative simplicity allows to describe inner dynamics in detail and examine the roots of the patterns it exhibits, which is far from common in the relevant literature. Typically, only model definition together with rationale behind it are presented, followed by analysis of resulting series. Even though the dynamics can be inferred from the definitions, the stochasticity makes it very hard to do so without actually constructing and running the simulation of the model.

Thirdly, with only two tunable parameters, we can simulate the model for relatively exhaustive set of parameters and observe the changes in resulting series. We can subsequently determine what combination yields the results most alike to financial data and whether the model requires a fine-tuning to generate series with required characteristics.

The thesis is structured as follows: In Section 2 we present overview of commonly accepted stylized facts. A brief rationale behind each is provided together with measures employed to assess its presence in data. In Section 3 we analyze three different types of financial series to get a representative picture of the market. Section 4 introduces the Bornholdt's model in the context of its physical predecessor. It also provides descriptions of model's rules with detailed discussion of their implications and behavioral patterns that they are supposed to represent. In Section 5 we present results of our simulations. After description of model's dynamics, the presence of stylized facts in a simulated series is assessed. Impact of a change in input parameters is then examined via simulation over a wide range of parameter combinations. Section 6 provides summary of our results and concludes.

The simulations as well as analysis presented in this work are conducted using Wolfram Mathematica. Unless stated otherwise, own calculations are sources of data in tables and figures.

2 Stylized facts about financial series

Financial time series studied in economics comprise a wide range of assets - company stocks, various commodities, foreign exchange rates and many more. Most of them even have several versions because of related financial derivatives. But despite all the differences these series share a number of nontrivial statistical properties as was shown in numerous empirical studies (e.g. Lux [2008] and Cont [2001]). This common denominator of properties is called *stylized (empirical) facts.* Before we discuss them, we need to define a logarithmic return, that we will further work with. For price $p(t)$ and time-scale Δt the log return at scale Δt is

$$r(t,\,\Delta t) = ln\,p(t+\Delta t) - ln\,p(t).$$

When we work with returns, we will implicitly assume $\Delta t = 1$ in appropriate units, i.e. days in our daily data, unless stated otherwise.

2.1 Dependence in returns

One empirical feature of financial series is absolutely crucial for a vast majority of theoretical models and is a cornerstone of Efficient market hypothesis : the martingale property of financial prices. It states that currently observed value is the best estimate of the next value while knowledge of past observations does not allow for better prediction.

Consequently, correlation between values $\rho(\tau) = \mathrm{corr}(r(t),\, r(t+\tau))$ of the returns series at different time lags τ is assumed to be zero for most economic purposes. The reasoning behind this assumption is very simple. Possibility to identify a significant correlation exploitable through a trading strategy would allow for a so called statistical arbitrage. As Mandelbrot [1971] puts it, "arbitrage tends to whiten the spectrum of price changes."

It should be noted that in financial series one can often find some marginally significant autocorrelations at the first few lags. Moreover, for fine scales (for τ representing minutes or even smaller timeframe) one might find first few lags to be significant. These autocorrelations are believed to result from microstructure noise and, though statistically significant, they cannot be exploited through a pertinent trading strategy. They are thus not considered to pose a strong evidence against efficient market hypothesis (Lux, 2008). On the contrary, an overall absence of autocorrelation is often used to support the hypothesis (Fama and Malkiel, 1970).

2.1.1 Autocorrelation of returns

To assess whether there is a correlation in returns, we run a Ljung-Box test and examine the autocorrelation function (ACF). Under the null of the Ljung-Box test (for details, see Ljung and Box [1978]), returns are independently distributed and no autocorrelation is present. The autocorrelation function is defined as

$$\rho(\tau) = \frac{\sum_{t=1}^{T-\tau}(r_t - \bar{r})(r_{t+\tau} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2} \tag{2.1}$$

where r is return, τ is a time lag, T is a number of observations and $\bar{r} = \frac{1}{T}\sum_1^T r_t$ is an average value of returns.

2.1.2 Autocorrelation of absolute returns

Absolute returns (possibly of higher powers than 1) exhibit significant positive autocorrelation as a result of volatility clustering. This phenomenon was first described by Mandelbrot [1963] as "large changes tend to be followed by large changes – of either sign – and small changes tend to be followed by small changes." To quantify persistence in volatility, we use autocorrelation of absolute returns

$$\rho_{Abs}(\tau) = \frac{\sum_{t=1}^{T-\tau}(|r_t| - |\bar{r}|)(|r_{t+\tau}| - |\bar{r}|)}{\sum_{t=1}^{T}(|r_t| - |\bar{r}|)^2} \tag{2.2}$$

where variables correspond to those of Equation 2.1. Various higher powers of absolute returns are often used, but we will employ simple absolute return of power 1 as these should be relatively more predictable (Granger and Ding, 1994).

It has been found by number of works (e.g. Harvey, 2002, Muller et al., 1990) that decay in autocorrelation of absolute returns can be fitted with a power-law decay

$$\rho_{Abs}(\tau) \backsim \frac{A}{\tau^{\alpha}} \tag{2.3}$$

with exponent α ranging from 0.2 to 0.4 (Cont, 2001) or 0.1 to 0.4 (Chakraborti et al., 2011). The exponent captures a long-range dependence and the low values might thus indicate a slow decay in volatility.

Short-range correlation, on the other hand, decays exponentially (Meyers, 2009) and ACF function with autoregressive process should thus decay with

$$\rho_{Abs}(\tau) \backsim \frac{B}{e^{\tau\beta}}. \tag{2.4}$$

2.1.3 Heteroscedasticity of returns

To further capture dependence in volatility we employ the General Autoregressive Conditionally Heteroscedasticity (GARCH) model by Bollerslev [1986]. This tool accommodates stylized facts such as leptokurtic distribution and persistence of volatility.

If the return process r_t has zero mean and no trend, it can be modeled as:

$$r_t = e_t \sqrt{h_t}$$

where e_t is a white noise with N(0,1) and $h_t = \mathrm{Var}(\varepsilon_t | \mathcal{F}_{t-1})$ is conditional variance dependent on information set $\mathcal{F}_{t-1}$. We use GARCH (1,1), under which the variance is given by

$$h_t = \gamma_0 + \gamma_1 r_{t-1}^2 + \delta_1 h_{t-1}$$

where γ_0, γ_1, and δ_1 are parameters that satisfy $\gamma_0 > 0$; $\gamma_1 \geq 0$; $\delta_1 \geq 0$ and $\gamma_1 + \delta_1 < 1$.

Parameter γ_1 is a measure of an extent to which current shock feeds into the next period (Campbell et al., 1997). Its values typically vary between 0.05 for stable market and 0.1 for volatile market (Alexander, 2008). Persistence parameter δ_1 usually ranges between 0.85 and 0.98 and, in combination with γ_1, measures how fast the shock dies out. If our restriction $\gamma_1 + \delta_1 < 1$ did not hold, the volatility would be persistent and the series of r_t^2 non-stationary (Chan, 2011).

2.1.4 ARFIMA model parameters

To further examine dependence in a way that is easy to compare across many time series, we determine some parameters of Autoregressive fractionally integrated moving average (ARFIMA) model for both absolute and normal returns. To capture a short-range dependence we estimate a first-order autocorrelation, i.e. a simple AR(1) model

$$r_t = \phi r_{t-1} + \varepsilon_t$$

where ϕ is estimated parameter and ε_t is a white noise series.

The long-range dependence will be examined via estimating fractional integration parameter d of ARFIMA model using a local Whittle estimator. This semi-parametric maximum likelihood estimator focuses on only a part of a series' spectrum near the origin. This means that only low frequencies are taken into consideration which makes the estimator resistant to short-term memory bias (Kristoufek and Vosvrda, 2014). To estimate the spectrum of series $\{r_t\}$, we use a periodogram defined as

$$I(\lambda_j) = \frac{1}{T} \sum_{t=1}^{T} \exp(-2\pi i t \lambda_j)\, r_t$$

where $\lambda_j = \frac{2\pi j}{T}$ are Furrier frequencies with $j = 1, 2, \ldots, m$. Parameter m reduces the number of considered frequencies and must satisfy $m \leq \frac{T}{2}$; in this work, we use $m = 0.2T$. The local Whittle estimator is defined according to Shao and Wu [2007] as a minimizer of local objective function $R(d)$, i.e.

$$\hat{d} = \arg \min_{-0.5<d<\infty} R(d)$$

with

$$R(d) = log \left(\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j) \right) - \frac{2d}{m} \sum_{j=1}^{m} \log \lambda_j.$$

Fractional integration parameter d can take real values ranging from -0.5 to infinity. When $-\frac{1}{2} < d < 0$, the ARFIMA $(0, d, 0)$ process has a short memory and is antipersistent, i.e. it reverses itself more often than a random series would. For $d = 0$ the process is a white noise and thus is mean-reverting in a short time and shock effects diminish quickly. When $0 < d < \frac{1}{2}$, the process is stationary with long memory. Border values $d = \pm\frac{1}{2}$ as well as $d > \frac{1}{2}$ are also possible but are of marginal interest here. For further information on the parameter, see Hosking [1981].

2.2 Distribution of returns

Given the stochasticity of returns, it is quite natural to assume that aggregate returns obey the Central Limit Law. Mandelbrot [1963] has shown, however, that normal distribution fits financial data rather poorly. Though only marginally correlated, asset returns are not independent and identically distributed stochastic processes. Independence implies that any nonlinear transformation of returns should not exhibit autocorrelation either (Cont, 2001). This property, however, is violated by absolute or squared returns.

2.2.1 Heavy tails and peakedness

Compared to normal distribution, the empirical ones have more probability mass in their center and tails. Fluctuations in empirical series are thus predominantly smaller than under normal distribution, though with higher occurrence of extreme events. For instance, an event exceeding the sample standard deviation fivefold would be a rare phenomenon in Gaussian market but can be observed regularly in the real data. Tail weight and peakedness (width of peak) is measured by kurtosis - the standardized fourth moment. Throughout this work, we will use *excess* kurtosis defined as

$$\kappa = \frac{1}{N} \sum_{t=1}^{N} \left(\frac{r_t - \bar{r}}{\sigma} \right)^4 - 3 \tag{2.5}$$

where $\bar{r}$ is the mean value of returns and σ the standard deviation of the sample. For normal distribution $\kappa = 0$, while for empirical data it virtually always holds that $\kappa > 0$, i.e. the distribution is leptokurtic or fat-tailed. The tails tend to lose some but not all of its heaviness even after correcting returns for volatility clustering (e.g. by GARCH family models).

2.2.2 Asymmetry of distribution

Distribution of financial data is often found to be asymmetric. Because of positive trend in data, normal every-day returns tend to be positive which shifts the distribution's peak to the right. Mainly due to leverage effect (Bouchaud and Potters, 2001), reaction to negative price movement is found to be stronger than to positive one and large negative changes are more likely to occur. This results in relatively more mass under the left tail of the distribution. Both of these effects account for negative skewness which is the third standardized moment

$$\gamma = \frac{1}{N} \sum_{t=1}^{N} \left(\frac{r_t - \bar{r}}{\sigma} \right)^3 . \tag{2.6}$$

2.2.3 Aggregate normality

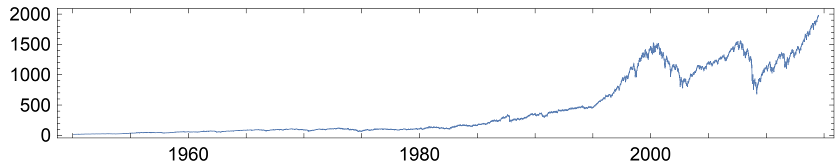
It has been shown by Kullmann et al. [1999] that as one increases the time scale Δt of returns the heavy tails and sharp peak become less pronounced. For very large scales (e.g. a month) the distribution of returns closely resembles the normal distribution (Chakraborti et al., 2011). The change of distribution with Δt suggests that the underlying structure of prices must be of non-trivial nature.

3 Analysis of financial series

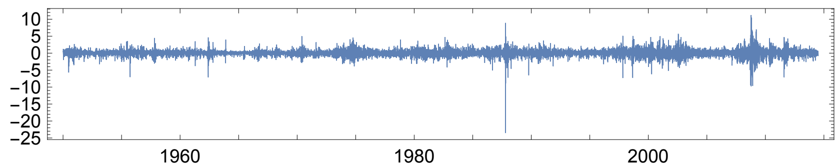
To assess how Bornholdt's model mimics stylized facts, we analyze three financial series of different types. S&P 500 is chosen to represent stock market since it covers large portion of U.S. market and is very diverse in constituency. Exchange rates will be represented by British Pound Sterling per U.S. Dollar rate for which long data series are available, as opposed to for example Euro rates. Finally, gold is chosen to represents commodities since it should by highly liquid as a consequence of being used as an investment vehicle for hedging against inflation (Narayan et al., 2010).

3.1 S&P 500

S&P 500 is a market-capitalization weighted index of 500 stocks listed on the American Stock Exchange (AMEX), New York Stock Exchange (NYSE) and National Association of Security Dealers Automated Quotation system (NASDAQ). The index is updated with every new recorded transaction in any of the 500 underlying stocks. Although the index is only updated every 15 seconds and thus might not entirely capture a microstructure noise, at a scale of days it can be considered extremely liquid (Huang et al., 2007). This, combined with the fact that it covers about 75% of American equity market, makes it a popular choice for financial researches.



(a) Daily closing prices



(b) Daily standardized log-returns

Figure 3.1: S&P 500: Daily closing prices and standardized returns from January 4, 1950 to July 8, 2014

We use daily data obtained from Wolfram Mathematica database[1] that span from January 4, 1950 to July 8, 2014 counting 16232 observations in total. Figure 3.1a shows price development over the period which clearly reflects all major events that took place on financial markets. Following a steep rise during the dot-com bubble period in the late 1990s, the prices plummeted at an unprecedented pace. Subsequent recovery ends in an even more spectacular crash of the 2008 crisis followed by a steady increase until recent days.

3.1.1 Distribution of returns

Standardized log returns in Figure 3.1b, with obvious volatility clusters and occasional spikes of 5σ to 10σ (standard deviations), likely deviate from the normal distribution. In our analysis of distribution, we will firstly focus on "usual" returns of lag 1 and subsequently examine whether the statistics change with increasing lag.

Table (3.1) presents several descriptive statistics of the series along with result of Jarque-Bera test (Jarque and Bera, 1980). It reveals that returns are strongly leptokurtic and skewed[2] to the left. This means that compared to normal distribution, the empirical distribution has more mass around the central part and fatter tails. The middle peak is inclined to the right and the left tail is heavier than the right one. The peakedness is well observable in a plot of empirical probability density function (PDF) based on a smooth kernel density estimate (Figure 3.2), while the fat tails are well illustrated by a quantile plot (Figure 3.3). In practical financial terms, usual everyday returns tend to be positive but extreme losses are more likely than extreme gains. As expected,

[1]Wolfram uses finance.yahoo.com and Xignite as their main financial data sources.

[2]There is no definitive threshold value regarding skewness but as a general rule of thumb, skewness of absolute value greater than 1 means a substantially asymmetric distribution.

Lag	Mean	St. D.	Min.	Max.	Skew.	Kurt.	JBT (p-val.)	N
1	0.000	0.010	-23.55	11.22	-1.03	27.72	522867.5 (0.00)	16231
4	0.001	0.019	-8.37	6.43	-0.47	5.13	4615.1 (0.00)	4057
8	0.002	0.027	-7.27	5.54	-0.51	3.83	1337.4 (0.00)	2028
16	0.005	0.037	-8.68	3.19	-1.14	6.48	2024.8 (0.00)	1014
64	0.019	0.073	-4.22	2.51	-0.69	1.91	62.2 (0.00)	253
128	0.037	0.104	-3.46	2.10	-0.49	0.57	7.6 (0.04)	126

Table 3.1: Descriptive statistics and Jarque-Bera test results for S&P 500 returns at different lags

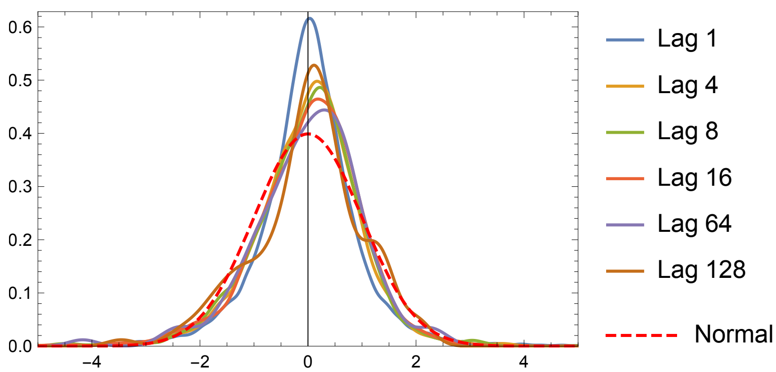


Figure 3.2: S&P 500: Empirical distribution of returns for different values of lag. Normal distribution (dashed line) is added for comparison

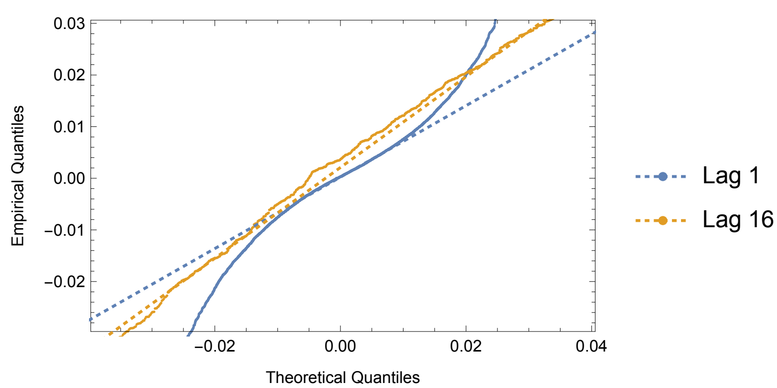
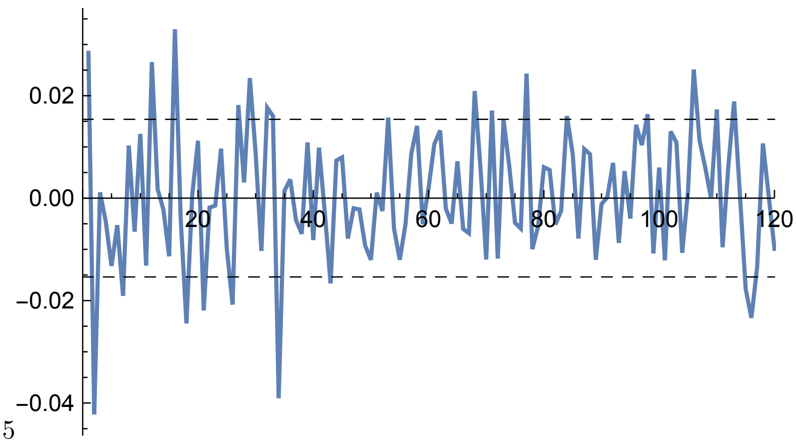


Figure 3.3: S&P 500: Quantile plot return distribution for two different values of lag



3.5

Figure 3.4: S&P 500: Autocorrelation function of daily returns

Jarque-Bera (JB) test strongly rejects its null hypothesis of normality of the data.[3]

We will now see whether the empirical distribution does approach normal with increasing lags or not. Table 3.1 shows that while there is no clear downward trend in skewness, the kurtosis rapidly decreases with lag. The change in peakedness is again well noticeable in Figure 3.2 whilst Figure 3.3 shows at lag 16 tails very much alike those of normal distribution. This is reflected also in lower scores of JB test even though its null is rejected in all cases on at least 5% significance level. Nonetheless, our data seem to support aggregate normality, though only at a scale of several months.

3.1.2 Dependence in returns

To assess autocorrelation of returns, we run a Ljung-Box test and examine an autocorrelation function. With a test statistic of 56.4, the Ljung-Box rejects its null at any reasonable level of significance. The ACF along with a 95% confidence interval band is plotted in Figure 3.4. It is marginally significant especially in the region of the first 40 lags. A fairly low yet highly significant AR(1) parameter $\phi = 0.028$ indicating a weak autoregressive process. Local Whittle estimate of fractional integration parameter is $d = -0.019$, indicating no long-range dependence or weak antipersistence in the series.

3.1.3 Dependence in absolute returns

Autocorrelation function of absolute returns (Figure 3.5) reveals a correlation that remains fairly high and positive throughout the whole studied region of the first 120 lags. This indicates presence of memory in the process, i.e. current

[3] To be more precise, JB tests null is a joint hypothesis of the skewness and excess kurtosis both being zero.

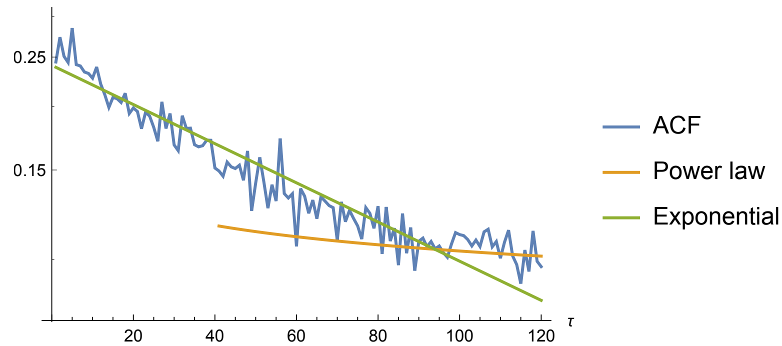


Figure 3.5: S&P 500: Log-normal plot of autocorrelation function of absolute returns with power and exponential functions fitted

value of volatility is largely dependent on the past values and volatility clusters are likely to form. A very gentle decline of the ACF at high τ could hint at presence of a long-range dependence in the process.

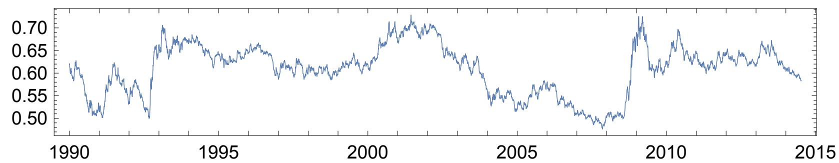
Furthermore, Figure 3.5 features fitted exponential and power-law functions. Since the power law is supposed to fit the ACF only asymptotically, values for $\tau < 40$ were omitted in the estimation. The estimated parameters are $A = 0.186$, $\alpha = 0.127$ and $B = 0.241$, $\beta = 0.009$ for power-law and exponential functions, respectively. As expected, exponential function seems to approximate the ACF better for lower values of τ, until around $\tau = 90$. For higher τ the power-law function fits more accurately. Neither of the functions provides a particularly good fit overall, however.

First-order autocorrelation parameter of 0.254 is highly significant and confirms an autoregressive process in the series. Local Whittle estimate of fractional integration parameter $d = 0.269$ indicates presence of long-range dependence.

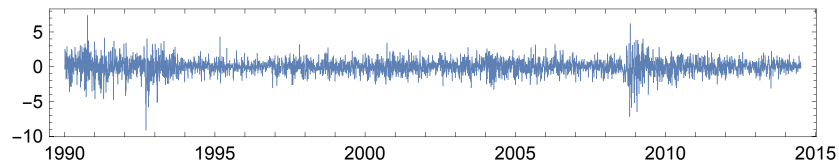
Parameters of GARCH model, $\gamma_1 = 0.088$ and $\delta_1 = 0.867$, are both highly significant and indicate that the series is rather volatile. Nevertheless, both are well within range of values usually observed in finance.

3.2 British Pound Sterling per U.S. Dollar exchange rate

For analysis of GBP/USD exchange rate we use daily data obtained from Quandle database that span from January 1, 1990 to July 8, 2014 with 6396 observations in total. Figure 3.6a shows the development of value of 1 U.S. Dollar in UK Pound Sterlings over the said period, with two occurrences of GBP abrupt depreciation standing out. First is a Black Wednesday on which George Soros famously earned about 1 billion GBP by shorting the currency. This event forced British government to withdraw from European Exchange Rate Mechanism; it was not until a decade later that the rate reached its pre-Black Wednesday values. Another dent in the value of GBP followed the 2008 financial crisis with currency's depreciation reaching one of the highest rates to-date. An all-time



(a) Daily closing prices



(b) Daily standardized log-returns

Figure 3.6: GBP/USD: Daily closing prices and standardized returns from January 1, 1990 to July 8, 2014

Lag	Mean	St. D.	Min.	Max.	Skew.	Kurt.	JBT (p-val.)	N
1	0.000	0.004	-7.39	9.15	-0.47	5.12	7252.1 (0.00)	6396
4	0.000	0.011	-4.44	7.00	-0.61	4.03	1190.3 (0.00)	1599
8	0.000	0.016	-3.07	6.14	-0.64	2.67	298.6 (0.00)	799
16	0.000	0.024	-3.46	5.44	-0.95	3.40	262.4 (0.00)	399
64	0.001	0.051	-1.76	4.65	-1.63	6.13	230.5 (0.00)	99
128	0.000	0.071	-1.87	3.49	-0.99	2.24	24.1 (0.01)	49

Table 3.2: Descriptive statistics and Jarque-Bera test results for GBP/USD exchange rate returns at different lags

low exchange rate with U.S. Dollar occurred on January 23, 2009 at \$0.72 per £1.

3.2.1 Distribution of returns

Standardized log returns in Figure 3.6b again exhibit clear volatility clusters that are, however, lower in number than in the case of S&P 500. Spikes of extreme values rarely exceed 5σ and while these would be very unlikely to occur under normal distribution, the series is again very moderate compared to the market index. As in previous case, we will begin with analysis of returns of lag 1 and subsequently examine whether the statistics change with its higher value.

Table (3.2) presents descriptive statistics and Jarque-Bera test results of the series. Returns are leptokurtic and skewed to the left. The histogram in Figure 3.7 shows that the empirical distribution has higher peak with no apparent inclination and only the left tail is clearly heavier than the ones of normal distribution. Quantile plot in Figure 3.8 reveals that even right tail is heavier but not as much as the opposite one. Because of the lack of peak inclination,

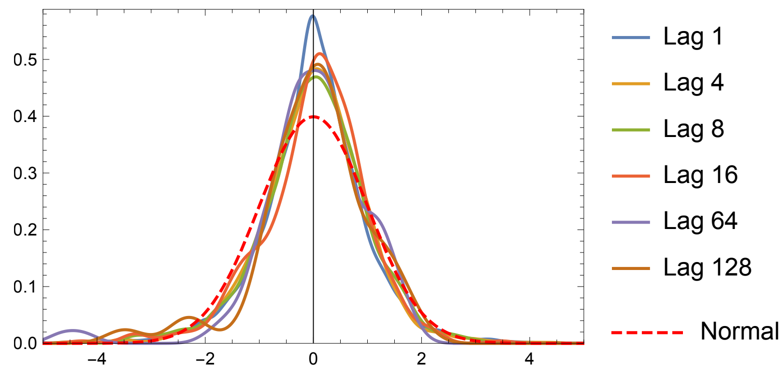


Figure 3.7: GBP/USD: Empirical distribution of returns for different values of lag. Normal distribution (dashed line) is added for comparison

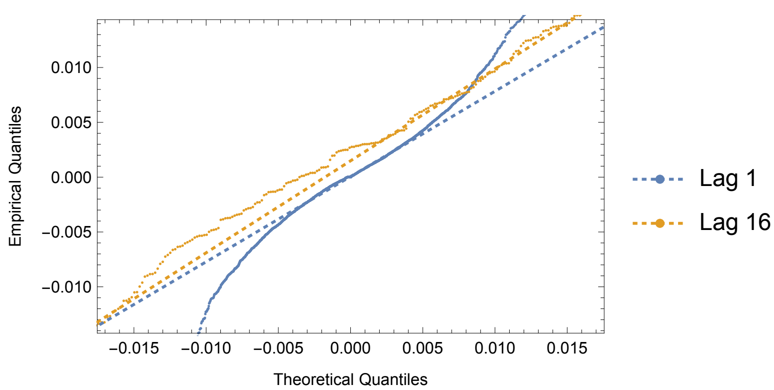


Figure 3.8: GBP/USD: Quantile plot return distribution for two different values of lag

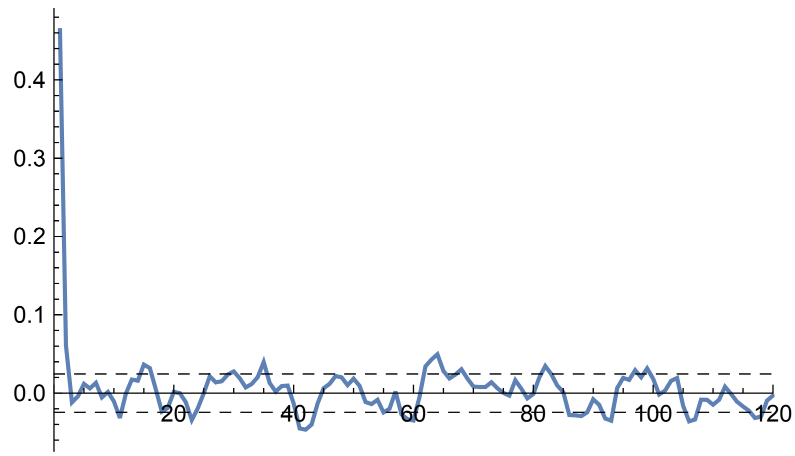


Figure 3.9: GBP/USD: Autocorrelation function of daily returns

the left skewness seems to come predominantly from the left tail being fatter than the right one. The normality of the data overall is strongly rejected by Jarque-Bera test.

The distribution characteristics of the exchange rate returns resemble those of S&P 500, albeit the non-normality is slightly less pronounced. The peakedness and heaviness of the tails reflect high occurrence of average low returns with real possibility of extreme events which is quite in line with expectations. Heavier left tail suggesting higher probability of extreme negative losses, is however far from natural. In case of asset returns, higher probability of negative extreme events can be explained by the leverage effect (Bouchaud and Potters, 2001). But due to positive return on GBP/USD rate being negative on the inverse rate, a more or less symmetrical distribution could be expected. What we observe means that traders with a long position in Pound can expect some extreme gains more often than extreme losses.

As we increase the lag of returns, the series approaches normality as shown by decreasing JB test scores in Table 3.2. The score is driven down mainly by decreasing kurtosis as there is not a clear downward trend in skewness. Figure 3.7 shows a clear difference between peakedness at lag 1 and at higher lags but hardly any among higher lags as a group. Despite approaching normal distribution at a scale of months, the series still bears some distinct features of financial data.

3.2.2 Dependence in returns

With a test statistic of 1401 (25 times higher than in the case of S&P 500), the Ljung-Box strongly rejects its null of no autocorrelation. The ACF along with a 95% confidence interval band is plotted in Figure 3.9. Some marginally significant lags in the whole plotted region resemble ACF of the S&P 500. What differs

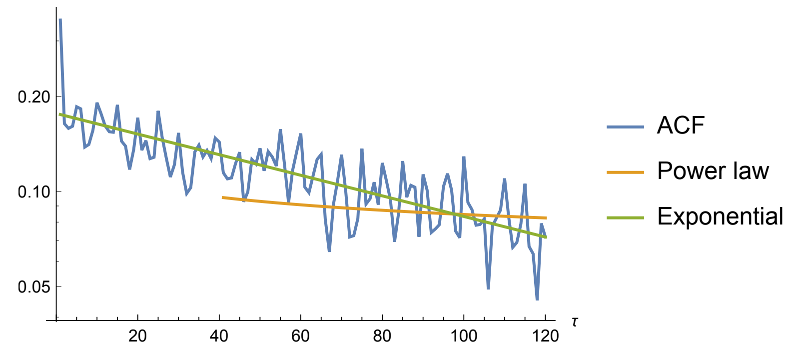


Figure 3.10: GBP/USD: Log-normal plot of autocorrelation function of absolute returns with power and exponential functions fitted

and truly stands out, however, is an extremely significant value at the first lag. This is reflected by fairly high and significant AR(1) parameter $\phi = 0.463$. Local Whittle estimate of fractional integration parameter is $d = 0.061$, indicating a possibility of weak long-range correlation even in the series of returns.

3.2.3 Dependence in absolute returns

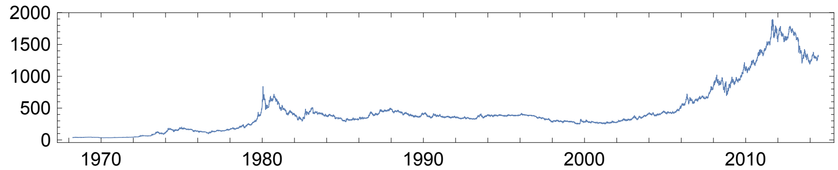
Autocorrelation function of absolute returns (Figure 3.10) again shows a high correlation at the first lag and slow linear decrease on a log-normal scale. This indicates that current value of volatility is dependent on a few previous values and does not provide a hint of long-range dependence. Due to linear decline in autocorrelation throughout the plotted range, the exponential function provides a very good fit as opposed to power-law function, which again indicates that short-range process dominates here. The estimated parameters are $A = 0.160$, $\alpha = 0.138$ and $B = 0.177$, $\beta = 0.008$ for power-law and exponential functions, respectively.

Parameter of AR(1) at $\phi = 0.349$ is significant and fairly high and confirms presence of autoregressive process in the series. Although fractional integration parameter $d = 0.200$ provides evidence for some long-range dependence, it is weaker than in case of S&P's absolute returns.

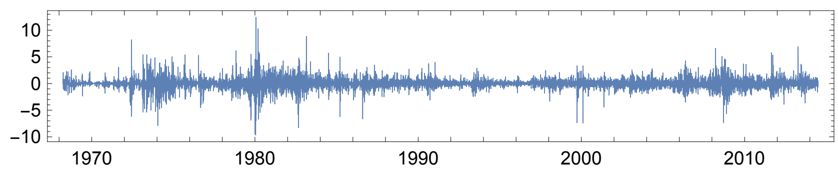
Low persistence is also shown by GARCH model parameters, $\gamma_1 = 0.300$ and $\delta_1 = 0.112$, that are far from usual values. Volatility feeds strongly from one period to the next one as already indicated by high AR parameter ϕ and first lag of ACF. Conversely, low δ_1 confirms that long-range persistence is very weak.

3.3 Gold

We analyze daily data of prices of gold obtained from Quandl database spanning from January 4, 1968 to July 7, 2014 with 11701 observations in total. Figure 3.11a shows closing prices and logarithmic returns over the period.



(a) Daily closing prices



(b) Daily standardized log-returns

Figure 3.11: Gold: Daily closing prices and standardized returns from January 4, 1968 to July 8, 2014

Lag	Mean	St. D.	Min.	Max.	Skew.	Kurt.	JBT (p-val.)	N
1	0.000	0.013	-9.69	12.44	-0.07	13.29	86301.7 (0.00)	11701
4	-0.001	0.026	-8.32	7.69	-0.37	11.35	15845.0 (0.00)	2925
8	-0.002	0.035	-7.12	6.43	-0.41	7.81	3796.4 (0.00)	1462
16	-0.005	0.051	-7.48	5.14	-0.83	6.98	1604.4 (0.00)	731
64	-0.019	0.112	-4.98	2.96	-0.87	3.79	144.1 (0.00)	182
128	-0.038	0.163	-4.84	1.99	-1.8	6.53	246.8 (0.00)	91

Table 3.3: Descriptive statistics and Jarque-Bera test results for gold returns at different lags

3.3.1 Distribution of returns

Standardized log returns in Figure 3.11b show number of clusters and a large number of spikes often higher than 5σ with some exceeding 10σ. Table 3.3 presents descriptive statistics and Jarque-Bera test results of the series. Returns are leptokurtic with $\kappa = 13.2$ being between the values of the exchange rate and S&P 500. High peakedness with hardly any skew is well observable in histogram (Figure 3.12) while quantile plot (Figure 3.13) reveals heavy, yet symmetric tails.

JB test score's decrease in higher lags can be again primarily attributed to lowering kurtosis as the trend in skewness is quite the opposite.

3.3.2 Dependence in returns

Ljung-Box test again strongly rejects its null but the score of 34.3 indicates that autocorrelation is not as strong as in previous cases. The ACF in Figure 3.14 shows some marginally significant lags in the whole plotted region. Negative first lag of ACF and AR parameter $\phi = -0.039$ both indicate negative correlation

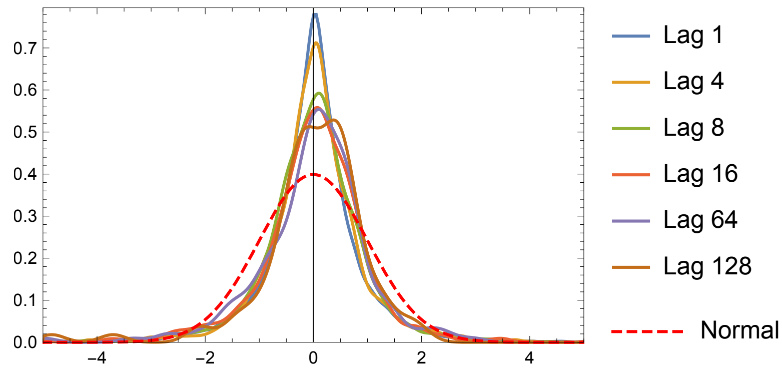


Figure 3.12: Gold: Empirical distribution of returns for different values of lag. Normal distribution (dashed line) is added for comparison.

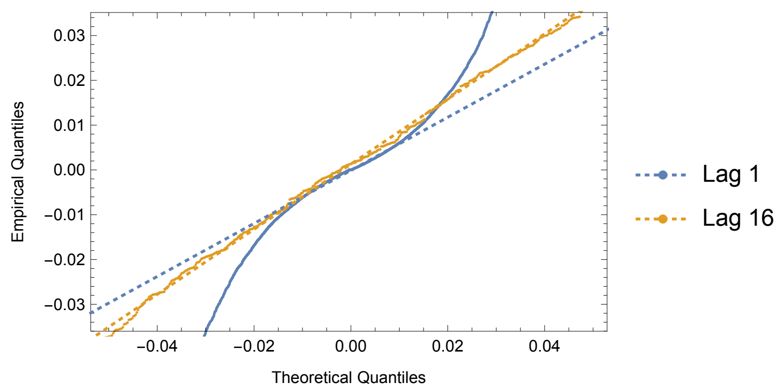


Figure 3.13: Gold: Quantile plot return distribution for two different values of lag

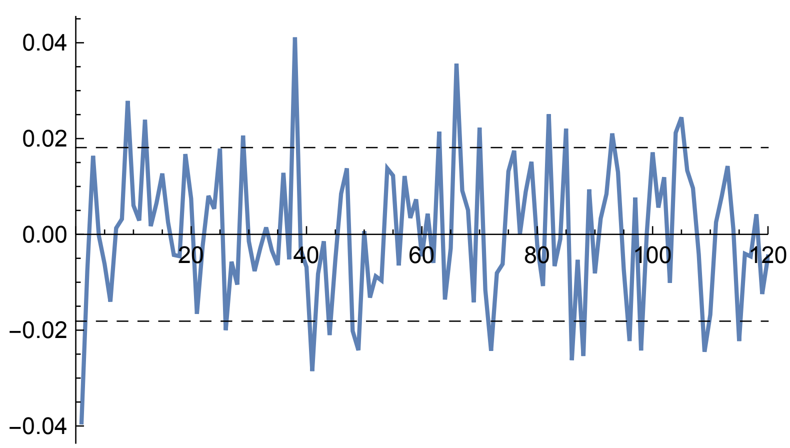


Figure 3.14: Gold: Autocorrelation function of daily returns

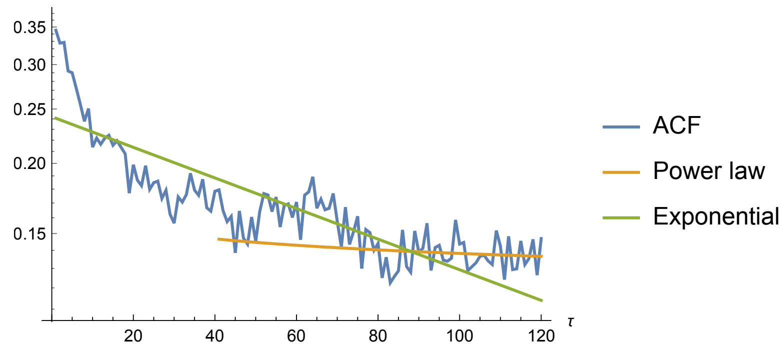


Figure 3.15: Gold: Log-normal plot of autocorrelation function of absolute returns with power and exponential functions fitted

in the series. Fractional integration parameter is $d = 0.014$, indicating that long-range dependence is very weak, if present at all.

3.3.3 Dependence in absolute returns

Autocorrelation function of absolute returns in Figure 3.15 shows a steep decrease within the first 20 or 30 lags with flatter decrease thereafter. The autoregressive process can thus be strong yet limited to first few lags whilst at higher lags a long-range persistence prevails. A poor fit of exponential function ($A = 0.187$, $\alpha = 0.065$) and a very well fitting power-law function ($B = 0.242$, $\beta = 0.006$) support this.

A significant parameter of AR(1) at $\phi = 0.3495$ and fractional integration parameter $d = 0.312$ confirm presence of autoregressive process and strong long-range dependence, respectively.

GARCH model parameters, $\gamma_1 = 0.173$ and $\delta_1 = 0.753$, indicate above all a strong feed in volatility from last value to the current. This is in line with unusually high number of spikes in return series.

4 Model overview

In this work, we primarily focus on model of Bornholdt [2001] and it's ability to mimic behavior of financial series that we examined above. Bornholdt's model is based on a physical model of ferromagnetism first described in Ising [1925]. We first briefly introduce econophysics in general to show that looking for parallels between economics and physics is a meaningful effort that has a long history. We then define the physical model, describe the phenomena that it mimics, examine the dynamics using simulated data and show how the model workings transcend into the study of financial markets.

This will bring us to Bornholdt's model itself. While defining the model we will look into the behavioral patterns that it is based on and will try to assess whether they primarily reflect human nature or rather help achieve desired behavior of the model.

Most of the works on this and related topics limit themselves to definition of a model and presentation of resulting series. This makes understanding of inner dynamics of the model very difficult, especially for those without a background statistical mechanics or similar fields. Throughout this and subsequent section we will try to provide detailed description of model's components and their effect on model's dynamics.

4.1 A brief primer on econophysics

Econophysics is a very young discipline. Even though the first attempts to examine parallels between statistical laws in physics and social sciences can be traced as far as early 1940's, it was not until 1990's (Chakraborti et al., 2011) that the effort became truly systematic. With immense computational power at hand, physicists started to examine one of the largest records of human activity - financial time series. According to Roehner [2002], the first article on financial series analysis that was published in physics journal was Mantegna [1991] and the first conference on the topic was held six years later. Ever since, the number of articles has grown rapidly with most physics journals publishing works on finance on a regular basis.

Still, almost two decades after the term "Econophysics" was first used by H. E. Stanley only few economists have heard of it and even fewer of the concepts that it comprises. Most of those who have would probably doubt that physical theories developed to describe world in terms of particles can be applied to study complex man-made systems such as financial markets.

One can object that physical models provide precise description and prediction of physical phenomena based on a few simple universal properties. As such, these models seem unable to grasp the unpredictable nature of man. It is possible, though, that there are a few simple properties that can be universally attributed to human beings and that under certain conditions infer more complex patterns.

The branch of physics that is typically used to study social phenomena is called statistical mechanics and it "*aims to predict and explain the measurable properties of macroscopic systems on the basis of the properties and behavior of the microscopic constituents of those systems.*"[4]

We can take the Brownian motion as an example. It is possible to set up a simple stochastic model of ideal balls that are initially assigned some random motion vectors and then let to bounce off each other. Even if we disregard some properties of real particles (e.g. friction or elasticity) we obtain a model that on macroscopic level exhibits most empirically observed properties.[5]

[4] Encyclopædia Britannica. Retrieved September 11. 2013. from britannica.com

[5] Despite having an important role in finance, Brownian motion is still primarily perceived as physical model of motion of particles. Interestingly, when the concept was first proposed, the Paris Bourse price fluctuations were used to demonstrate its properties (Bachelier [1900], Chakraborti et al. [2011]).

Similarly, we can construct a stochastic model with a large number of traders that follow a few simple rules and subsequently observe complex phenomena at macroscopic level. The question of crucial importance is, however, what level of microscopic detail we have to maintain to still infer desired macroscopic patterns.

Due to econophysics being still such a young discipline, it provides a great opportunity for original research. There are also certain drawbacks, however. Most importantly, some authors argue that the universal principles that should apply to both natural and financial phenomena often result from lack of rigor in statistical analysis.

4.2 Ising model of ferromagnet

Ising model was first described by Ernst Ising in his doctoral thesis (Ising, 1925). This work primarily focuses on a solution of one-dimensional version which, however, lacks most of the properties we are interested in. The dynamics of two-dimensional version which is typically used today was analytically solved almost two decades later by Onsager [1944]. Before we define the model and describe its dynamics, let us briefly review real-world (physical) phenomena that it attempts to mimic.

4.2.1 Magnetism, ferromagnetism, paramagnetism

Following paragraphs are based on Hook and Hall [2013] and Guinier et al. [1989].

Magnetism origins from electric current and fundamental magnetic moments of elementary particles also called *spins*. If we disregard the current induced magnetism, we can study magnetism of materials as a sum of their spins. Nevertheless, this is far from trivial as values of the spins are not necessarily aligned. Depending on material, they can order themselves through a variety of mechanisms but we will only consider the two most typical ones - ferromagnetism and paramagnetism.

Through ferromagnetism, permanent magnets are formed by certain materials such as iron. The spins align themselves with each other, thus generating a significant total magnetic field. Under paramagnetism, however, order is not self-organized but only takes place if external magnetic field is applied. In everyday terms, ferromagnet is what we usually call magnet while paramagnet could be a piece of iron that is not magnetic by itself but becomes magnetic if attached to a magnet.

Interestingly, ferromagnetism is not innate to a material but is subject to material's temperature. It is only present when the temperature is below certain value called critical or Curie temperature (T_C). Above this point, high energy leads to fundamental particles becoming magnetically disorganized and the material becoming paramagnetic.

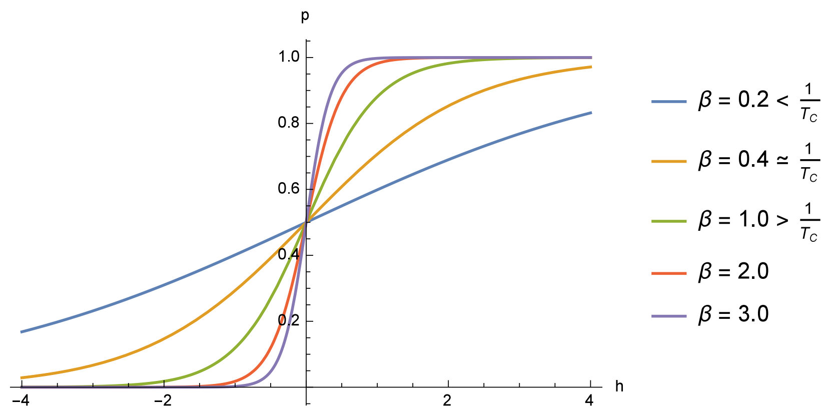


Figure 4.1: Heat-bath dynamics

4.2.2 Model definition

The two-dimensional Ising model of ferromagnet is constructed as a lattice of atoms which act as simple magnets (or spins) and can interact with each other. The lattice is of dimensions $k \times k$ where each node constitutes a spin. We denote $S_i(t)$ the spin value of i-th position at time t, where $i \in (1, \dots, k^2)$ and $t \in (1, \dots, T)$. The force exerted on spin $S_i(t)$ is given by its local field

$$h_i(t) = \sum_{<i,j>} J_{ij} S_j(t) \tag{4.1}$$

where $\sum_{<i,j>}$ denotes[6] a sum over the set of agent i's neighbors. J_{ij} is a measure of neighboring spin $S_j(t)$'s influence on spin $S_i(t)$. Typically, neighbors of $S_i(t)$ are the four spins adjacent from each side but other configurations are also possible. The actual spin value is updated according to heat-bath dynamics. In physics, Boltzman distribution or Metropolis algorithm are often used, but for our purposes a heat-bath dynamics used by Bornholdt [2001] is fully sufficient. The spin value is defined as

$$\begin{aligned} S_i(t+1) &= +1 \text{ with probability } p = \frac{1}{1 + exp\left(-2\beta h_i(t)\right)} \\ S_i(t+1) &= -1 \text{ with probability } 1 - p \end{aligned} \tag{4.2}$$

and is in effect a probability density function over $h_i(t)$. Figure 4.1 shows shapes of the function for different values of $\beta = \frac{1}{T}$. This parameter is called inverse temperature and is of crucial importance since it controls responsiveness

[6] The names and notation of variables are not that used by Ising [1925]. Instead, in all Ising-type models throughout this work we follow a notation of Bornholdt [2001]. This is so because Bornholdt's model is a cornerstone of our further analysis and with common notation, parallels between models will be easier to comprehend.

of the probability of spin value $S_i(t)$ to local field $h_i(t)$. Because the temperature enters the model via β, it determines whether general regime of the system is paramagnetic or ferromagnetic. For under-critical $\beta < \beta_C = \frac{1}{T_C}$ (i.e. over-critical $T > T_C$), the model behaves erratically as paramagnet while for over-critical $\beta > \beta_C = \frac{1}{T_C}$ (i.e. under-critical $T < T_C$) it behaves as a ferromagnet and converges to stable state over time. For two-dimensional Ising model, Kramers and Wannier [1941] analytically found the critical temperature to be $T_c = \frac{2}{\ln(1+\sqrt{2})} \approx 2.269$ which translates to $\beta_C = \frac{1}{T_c} \approx 0.441$. For the sake of clarity, we will further refer to under- and over-critical values solely in terms of β and not T. For instance, some simulations at high temperature yielding paramagnetic behavior will be referred to as under-critical since $\beta < \beta_C$.

In physics, Ising model can be used to study a wide variety of problems including movement of gas molecules or even neuron activity. Number of properties, that can be studied within each problem, is large as well but most often the model is used to study dynamics of a convergence to stable state. We will define a stable state as a state of no or little change in total magnetisation which is defined as

$$M = \frac{1}{k^2} \sum_{i=1}^{k^2} S_i \tag{4.3}$$

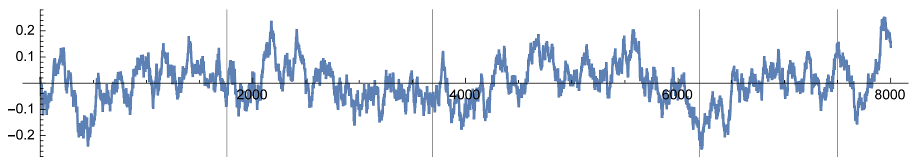
where $M(t) \in [-1, 1]$. When $M(t) = 0$ there is the same amount of positive and negative spins and their individual magnetisations cancel each other out. This typically happens as a result of disorganization, which does not necessarily hold true for other (economic) versions. Conversely, when $M(t)$ is close to either side of the interval, the system is highly ordered with spins forming large blocks of identical magnetisation.

4.2.3 Model dynamics

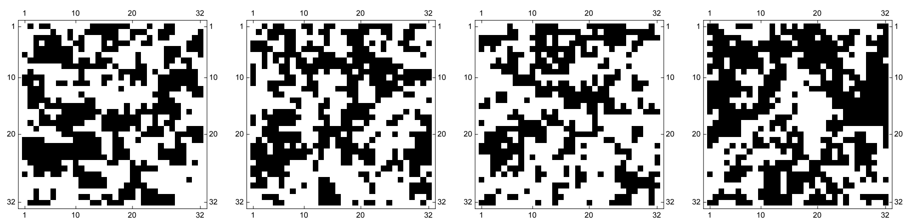
As mentioned above, the model has two main regimes: disordered paramagnetic when $\beta < \beta_C = \frac{1}{T_C}$ and ordered paramagnetic when $\beta > \beta_C = \frac{1}{T_C}$.

The paramagnetic regime comes as a result of relatively flat heat-bath dynamics function as shown in Figure 4.1. The probability of spin value $S_i(t)$ depends only weakly on the local field $h_i(t)$ which carries the information of neighboring spins. Local interactions therefore lose their influence with decreasing β. For $\beta \to 0$ (note that this implies infinite temperature T) the probability of each state of $S_i(t)$ is the same regardless of $h_i(t)$, thus making the spin value random. This is illustrated by Figure 4.2 that presents results of Ising model simulation at under-critical $\beta = 0.3$. Even though some short-range spatial order is present, neither of the spin values dominates at any time and the system fluctuates around $M(t) = 0$.

Ferromagnetic regime starts to dominate as β grows beyond β_C. Changes in the local field can now influence probability of a spin value and local interaction thus becomes stronger. This leads to formation of large clusters of the same spins. One of the clusters eventually dominates over the other, system becomes ordered and total magnetisation reaches border-value ($|M(t)| \to 1$).



(a) Magnetisation. Vertical grid-lines represent time instances of snap-shots.



(b) Lattice state at $t = 1850$
(c) Lattice state at $t = 3800$
(d) Lattice state at $t = 6100$
(e) Lattice state at $t = 7500$

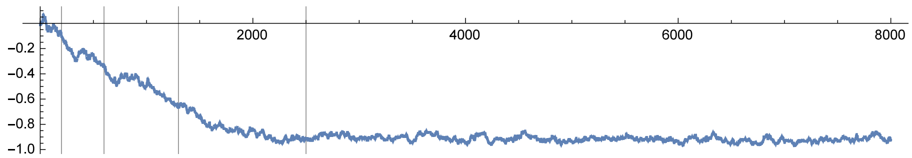
Figure 4.2: Underlying dynamics of Ising model at under-critical β. Four snapshots of lattice state at different times are shown. Spin values $S_i(t) = 1$ and $S_i(t) = -1$ are represented by black and white, respectively.

This is illustrated by Figure 4.3 that presents results of Ising model simulation at over-critical $\beta = 0.5$. From an initial state of randomized lattice (not shown), the system develops evident clusters of the same spins (Figure 4.3b). In this particular simulation, negative spins dominate and the cluster of positive values gradually diminishes (Figures 4.3c,d). The model eventually converges to a stable state at $M(t) = -1$ where only local and short-lasting clusters of positive spins occur (Figure 4.3e). Note that convergence to $M(t) = 1$ is just as likely and its dynamics are identical.

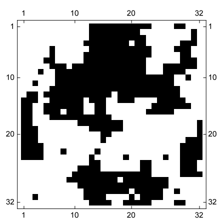
4.3 Economic interpretation of Ising model

The construct can be converted into economic terms - the whole lattice $k \times k$ can be interpreted as a market, individual spins $S_i(t)$ as traders (or agents) who can either buy ($S_i(t) = 1$) or sell ($S_i(t) = -1$) some asset. The local field in Equation 4.1 forces agent to adjust their value to their neighbors' thus inducing herding behavior. Local fields of most financial versions contain other terms that often have opposite impact than the basic Ising term and are designed to simulate other behavioral patterns than herding and, as we shall see, to prevent model from convergence.

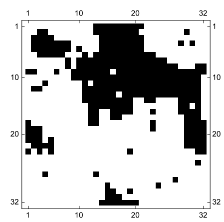
Total magnetisation of the system can be interpreted as a deviation of market price $p(t)$ from fundamental price $p^*(t)$ or simply as a price $p(t)$ itself (if we assume $p^*(t) = 0$, $\forall t$ and allow for $p(t) < 0$).



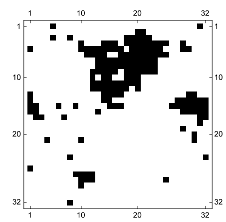
(a) Magnetisation. Vertical grid-lines represent time instances of snap-shots.



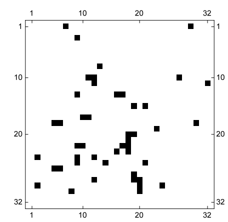
(b) Lattice state at $t = 200$



(c) Lattice state at $t = 600$



(d) Lattice state at $t = 1300$



(e) Lattice state at $t = 2500$

Figure 4.3: Underlying dynamics of Ising model at over-critical β. Four snap-shots of lattice state at different times are shown. Spin values $S_i(t) = 1$ and $S_i(t) = -1$ are represented by black and white, respectively.

When the size of buyer and seller groups is equal and $M(t) = 0$, the market is in its equilibrium. However, if large areas of the lattice become identically magnetised and $|M(t)| \to 1$, there is a speculative bubble on the market.

Variable of interest from economic perspective is predominantly return $r(t)$ which is derived from total magnetisation of the system (Equation 4.3). Given that $M(t)$ will take negative values just as likely as positive, the returns of such a series is calculated as

$$r(t)_\Delta = M(t) - M(t - \Delta) \tag{4.4}$$

where Δ is a value of lag. Kaizoji et al. [2002] demonstrate that this is, under certain assumptions, equivalent to a logarithmic return of financial series. For details, see Section 4.4.2.

4.4 Bornholdt's model

For our research we opted for a version of the model as defined by Bornholdt [2001] for which we have two main reasons:

Firstly, Bornholdt's model is relatively similar to the physical 2D Ising. For certain parameter values ($\alpha = 0$ in Equation 4.6) these two models are equivalent. We can therefore test to what extent some basic characteristics of the physical model are carried over to our model. For instance, we will be interested in the effect of critical temperature for $\alpha > 0$.

Secondly, the simplicity of the model allows for analytical tractability that is far from common in complex agent simulations. With only two tunable parameters, we can run simulations over relatively exhaustive set of parameter combinations. We can then assess whether the model requires fine-tuning or generates acceptable results over a wide range of parameter values. This is important as necessity of fine tuning indicates low general applicability to real phenomena. Even though this does not render a concept wrong, it is a sign that its explanatory power is limited (de Carlos and Casas, 1993). If similarity between generated and real-world data is obtained in the large, we can then assess how certain statistics change with respective parameters and what parameter combination yields results most alike to financial time series.

In the following paragraphs we first present and discuss original paper Bornholdt [2001] and then subsequent paper Kaizoji et al. [2002] that provides a different derivation of simplified version of Bornholdt's model. While the former work is crucially important for understanding of the subsequent sections, the later is not.

For the sake of clarity, other literature not directly related to Bornholdt's model is omitted this section. To provide context and comparison, and to show possibilities of Ising-based models, review of three other works can be found in Section A.1 of Appendix.

4.4.1 Model definition

Bornholdt [2001] proposes a relatively simple model that is much alike to the physical Ising model. It works with $k \times k$ square lattice of k^2 spins with orientations $S_i(t) = \pm 1$. The value of a spin depends on a local field h_i and is updated with a heat-bath dynamics as we defined it in Ising model:

$$\begin{aligned} S_i(t+1) &= +1 \, \text{with probability} \, p = \frac{1}{1 + exp\left(-2\beta h_i(t)\right)}. \\ S_i(t+1) &= -1 \, \text{with probability} \, 1 - p \end{aligned} \tag{4.5}$$

The local field h_i is different in that it has a second term added. It is specified as

$$h_i(t) = \sum_{<i,j>} J_{ij} S_j(t) - \alpha C_i(t) M(t). \tag{4.6}$$

The first term (which comes from the original Ising model) induces ferromagnetic order (herding behavior, in economic interpretation) by aligning agent i with its neighbors. The second term represents a global coupling to the system magnetisation where α is coupling constant. $C_i = \pm 1$ is the strategy spin of agent i and $M(t) = \frac{1}{k^2} \sum_{i=1}^{k^2} S_i(t)$ is the total magnetisation.

The value and dynamics of C_i greatly influences the dynamics of the whole system. $C_i = -1$ induces ferromagnetic order and aligns the agents with the total magnetisation. Agents with such strategy are called chartists and tend to follow a trend given by a sign of $M(t)$. In contrast, agents with $C_i = 1$ or fundamentalists tend to join the minority and thus oppose the sign of $M(t)$.[7]

[7] This is a motivation similar to that found in a minority game (see Galla et al. [2006]) where agent tends to align with minority of the spins.

Still, the first term of the equation introduces a certain level of ferromagnetic noise trading even in case of fundamentalist.

Let us first assume that $C_i(t) = 1$, $\forall i, t$. The second term then induces an anti-ferromagnetic coupling to the total magnetisation and if the second term is not outweighed by the first then the global magnetisation is brought to a near-vanishing point. For $C_i(t) = -1$, $\forall i, t$, the second term becomes a global equivalent of the first term that induces local herding. In such case, either positive or negative spin quickly dominates the whole system and $|M(t)| \to 1$.

If we allow the agents to choose their strategy, the dynamics of the system become non-trivial and interesting. Bornholdt [2001] defines a rule for strategy spin change as

$$C_i(t+1) = -C_i(t) \text{ if } \alpha S_i(t) C_i(t) M(t) < 0. \tag{4.7}$$

Therefore, agent will always choose $C_i(t) = 1$ if he is in the majority and $C_i(t) = 1$ if he is in the minority.[8]

To fully grasp the effect of this rule, let us imagine that all agents comply with the rule before each round of trading (as we shall see later, this is not far from reality and is the case in the simplified model). Then the second term of local field (Equation 4.6) of all majority agents (who are $C_i(t) = 1$) has sign opposite to that of total magnetisation and prompts them to swap. Inversely, second term of minority agents' (who are $C_i(t) = -1$) local field will have the same sign as total magnetisation and and thus prompts them to swap as well. Because $M(t)$ is a part of the second term, the general tendency to swap will grow stronger with $M(t)$'s deviation from zero. The further $M(t)$ deviates the more agents will tend to oppose it and the stronger will be their tendency to oppose. Such behavior has above all a very practical implication for the model; it eliminates inherent tendency of original Ising model to converge to either of the border states.

Author provides a rationale behind this behavior: An agent in majority will often switch in order to avoid loss resulting from future reversion of current trend. This tendency grows stronger with absolute magnetisation as space for further growth in current direction becomes smaller. Conversely an agent who is in minority and possibly expects future returns, might become unsatisfied with current returns the more so, the larger the majority group. As a consequence, he also tends to switch at higher magnetisation.

While majority agent acts rationally, the minority agent clearly does not. This does not render the model wrong; after all, bounded rationality is a frequent subject of econophysical models. Still, we cannot help the feeling that avoiding model's convergence is the primary motivation of this rule, whilst the proposed behavioral patterns are an ex-post rationalization.

[8] To illustrate, let us consider a chartist ($C_i(t) = -1$) buyer ($S_i(t) = 1$) in a market with positive total magnetisation ($M(t) > 0$). As a chartist in a majority, he does not satisfy the rule. But because for positive α the expression $\alpha S_i(t) C_i(t) M(t)$ is negative ($\alpha \times 1 \times (-1) \times 1 < 0$), the agent will change his strategy ($C_i(t+1) = -C_i(t)$) and comply with the rule.

If the assumptions we made for our illustration are stated explicitly as instant strategy adjustment, the strategy spin drops out and we obtain following simplified definition of local field:

$$h_i(t) = \sum_{j=1}^{N} J_{ij} S_j(t) - \alpha S_i(t) \left| M(t) \right| \tag{4.8}$$

In this case, the second term always prompts an agent to change its spin value and this tendency grows stronger with $|M(t)|$.

4.4.2 Kaizoji's alternative derivation of simplified model

Kaizoji et al. [2002] focus on the simplified version of Bornholdt's model and significantly expand the derivation and the interpretation of parameters that is rather brief in the original Bornholdt [2001] paper. Authors again define two types of traders - fundamentalists and chartists (here called interacting traders).

The fundamentalists are assumed to know the fundamental price of a stock $p^{\star}(t)$. If the actual price $p(t)$ drops below its fundamental price, the fundamentalist tends to buy the undervalued stock and vice versa. Fundamentalists' buying/selling order is then defined as

$$x^F(t) = a\, m\, \left(ln\, p^{\star}(t) - ln\, p(t) \right) \tag{4.9}$$

where a is a parameter denoting the strength of a reaction to price deviation and m is a number of fundamentalists. Note that this definition of fundamentalists has no apparent link to above-mentioned lattice model. We will see, however, that the concept is important for derivation of price and volume parameters.

The chartists are organized in a lattice and represented by their investment attitude $S_i(t)$ as defined by Bornholdt [2001]. $S_i(t)$ changes according to heat-bath dynamics shown in Equation 4.5.

We assume that chartists base their decisions upon two sources of information - local and global. While local information depends on the actions of close neighbors, global information depends on the total magnetisation $M(t)$. It is universally known, regardless of agent's position within lattice and affiliation with buyers/sellers or minority/majority group. The difference in size of majority and minority is measured by the absolute value of magnetisation $|M(t)|$.

Let us now examine the profit-maximizing strategies of the chartist agents. To gain profit an agent must be in majority group which, in addition, must grow in size during the next trading period. The majority group cannot grow indefinitely, however, and the larger is the absolute magnetisation $|M(t)|$, the smaller is the space available for further expansion. In order to avoid capital loss as a result of crash the chartists in the majority group tend to switch to minority. A chartist in the minority also tends to switch groups, in this case in order to gain capital. Therefore, with increasing magnetisation the chartists in the majority become more risk averse while those in minority become more risk seeking. This behavior is reminiscent of that of Bornholdt, 2001 and also seems to be a justification of a convenient mathematical rule, rather than a real-world pattern to be reflected by the model.

These interactions are contained in Bornholdt's simple version of local field $h_i(t)$ as stated in Equation 4.8. The first term induces the imitation of close neighbors and the second produces the increase in propensity to switch with rising magnetisation.

It is further assumed that chartists' excess demand is approximated by

$$x^C(t) = b\,n\,M(t) \tag{4.10}$$

where n is the number of chartists and b is a fixed amount of stock they are able to trade in each trading period.

Derivation of price and volume It is assumed that price is adjusted by a market maker to its market clearing price. We can write the balance of supply and demand

$$x^F(t) + x^C(t) = a\,m\,(ln\,p^\star(t) - ln\,p(t)) + b\,n\,M(t) = 0. \tag{4.11}$$

Hence the price is

$$ln\,p(t) = ln\,p^\star(t) + \lambda M(t), \quad \lambda = \frac{b\,n}{a\,m} \tag{4.12}$$

and volume

$$V(t) = bn\frac{1 + |M(t)|}{2}. \tag{4.13}$$

There are three possible situations at the market as follows from Equation 4.12:

- $M(t) = 0$: the market price is equal to the fundamental price.
- $M(t) > 0$: the market price is above the fundamental price (bull regime).
- $M(t) < 0$: the market price is bellow the fundamental price (bear regime).

Using again the Equation 4.12 the logarithmic return of a share is

$$ln\,p(t) - ln\,p(t-1) = ln\,p^\star(t) - ln\,p^\star(t-1) + \lambda\,(M(t) - M(t-1))\,. \tag{4.14}$$

Most works, including Bornholdt [2001] and Kaizoji et al. [2002], use a constant fundamental price $p^\star(t)$ in which case only the last term of Equation 4.14 remains and the log-return thus depends solely on magnetisation.

5 Simulations of the model

This section presents results of simulations and subsequent analysis of simulated series. In all of our simulations, we run Bornholdt's model (with or without strategy spin) on a 32×32 lattice with 10^6 rounds. Because in each round only one randomly chosen agent can act and only very limited change in magnetisation can thus occur, state of the model is not reported for all calculated instances. Instead, variables are recorded after every 100 rounds to allow for more complex patterns to emerge. This reflects real-world situation where a number of transactions takes place during each time period. The simulated series of total magnetisation $M(t)$ therefore have observations at 10^4 time instances t. Due to the fact that initially the lattice values are randomized and as such are not determined by the system itself, we allow for a warm-up period at $t \in \{1, \ldots, 2000\}$ and use only the remaining data further. As a result, we report and analyze series consisting of 8000 observations.

We will define neighbors of agent S_i as sites directly adjacent to S_i both vertically and horizontally. If S_i is next to one of the borders of the lattice, spin opposite to the given borderline segment will serve as a neighbor as illustrated in Figure 5.1. This concept is called *periodic boundary condition* and helps to approximate a very large system by relatively small lattice.

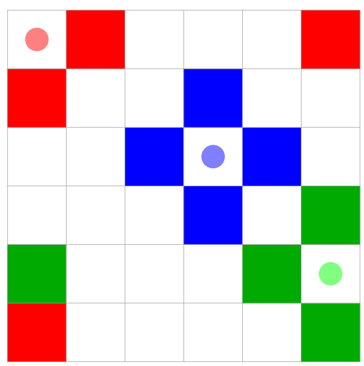


Figure 5.1: Periodic boundary conditions. Source: Dvorak [2012]

We will begin with description of dynamics of Bornholdt's simplified model because it well illustrates the top-level dynamics without a need to delve into complex inner workings. The complex inner dynamics of model with strategy will be described subsequently. We will then assess how well our models mimic the real-world data. Firstly, one simulated series will be analyzed in detail in manner similar to Section 3. Secondly, we will analyze multiple series simulated for various combinations of model's parameters to see how model's behavior changes with its parameters.

5.1 Inner dynamics of simplified model

Because of its relative simplicity we first simulate Bornholdt's simplified model (without a strategy spin), meaning that we use local field $h_i(t) = \sum_{<i,j>} J_{ij}S_j(t) - \alpha S_i(t)\,|M(t)|$. We choose $\alpha = 10$ and $\beta = 1.5$. Note that β is above the critical value β_C of the Ising model.

Figure 5.2a shows the development of key variables. On the top there is a plot of total magnetisation $M(t)$, followed by a plot of relative sizes of buyers ($S_i(t) = 1$) and sellers ($S_i(t) = -1$). Returns $r(t)$, shown in the third plot, do not influence the system per se but we include them because they show different levels of volatility more clearly than the $M(t)$ series. Moreover, they are primary subject in further analysis.

Figures 5.2b to 5.2e depict snapshots of the lattice at four different instances t with pie-charts showing relative sizes of each group at those times. These values of t are also indicated by four vertical grid lines in the plots above. In the snapshots as well as in the plot of agent groups, red and blue color represent buyer and seller, respectively.

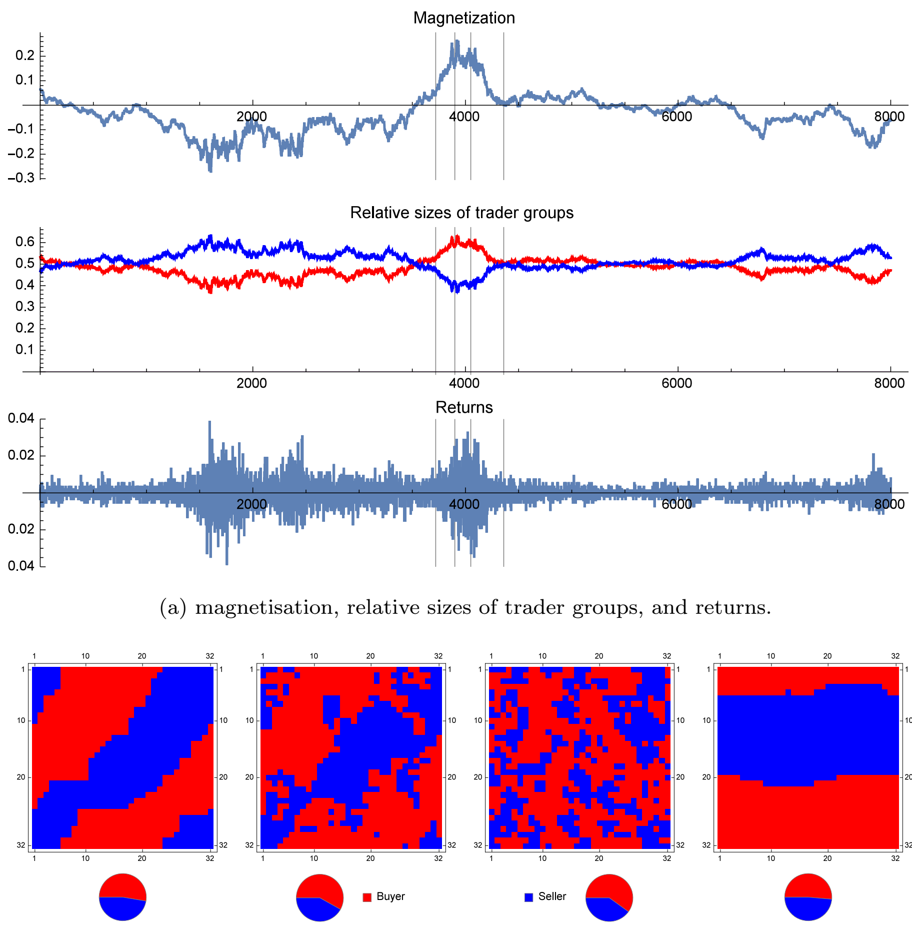
The model has two basic modes - stable organized phase and intermittent phase of high turbulence.

The stable phase occurs in the periods of $M(t) \approx 0$, when the second term of agent's local field is negligible and thus the herding behavior induced by the first term prevails (local ferromagnetic behavior). During the stable phase neighbor interactions play a substantial role and agents organize themselves into large clusters of identical spin. This is the case in snapshots 5.2b, e. Note how separate areas of single color connect through the periodic boundary condition and in fact form one continuous strip.

Then, observing the basic mechanism of physical Ising model, one of the clusters dominates and the total magnetisation deviates from zero. The further it deviates the higher is the tendency to change spin value (global antiferromagnetic behavior) induced by the second term of agent's local field. At some point the antiferromagnetic tendency dominates and some agents change their spins regardless of the neighbors (Figure 5.2c). This, in turn, decreases uniformity within a cluster and further reduces the influence of the first term. These two effects might eventually lead to a complete disintegration of the clusters and an intermittent phase (such as one shown in Figure 5.2d) takes place. The behavior at that time is highly erratic and large price changes occur until $M(t)$ reaches vicinity of 0 and stable agent clusters appear again. The most stable cluster configuration, that is typical for long periods of low absolute magnetisation, is naturally one with shortest possible border such as one depicted in Figure 5.2e.

5.2 Inner dynamics of model with strategy

Our second simulation employs Bornholdt's model with strategy spin. That is, we use local field $h_i(t) = \sum_{<i,j>} J_{ij}S_j(t) - \alpha C_i(t)M(t)$ where the strategy of an agent is updated according to the rule $C_i(t+1) = -C_i(t)$ if $\alpha S_i(t)C_i(t)M(t) < 0$. We choose under-critical setting $\alpha = 10$ and $\beta = 1.5$ as in previous simulation.



(a) magnetisation, relative sizes of trader groups, and returns.

(b) Lattice state at $t = 3720$ (c) Lattice state at $t = 3900$ (d) Lattice state at $t = 4630$ (e) Lattice state at $t = 4760$

Figure 5.2: Underlying dynamics of Bornholdt's model with strategy spin

Figure 5.3a presents development of key variables in the same manner as above. This time, however, we need to distinguish agents both according to the value of their spin (buyer: $S_i(t) = 1$ and seller: $S_i(t) = -1$) and according to their strategy (fundamentalist: $C_i(t) = 1$ and chartist: $C_i(t) = -1$).

There are again two basic modes - stable, where local interactions lead to large clusters, and intermittent, where clusters disintegrate as a result of high magnetisation. Although at a high level the dynamics is the same as in the simplified model, the underlying mechanism is more complex. To show the difference, we need to examine the role of strategy changes in the model.

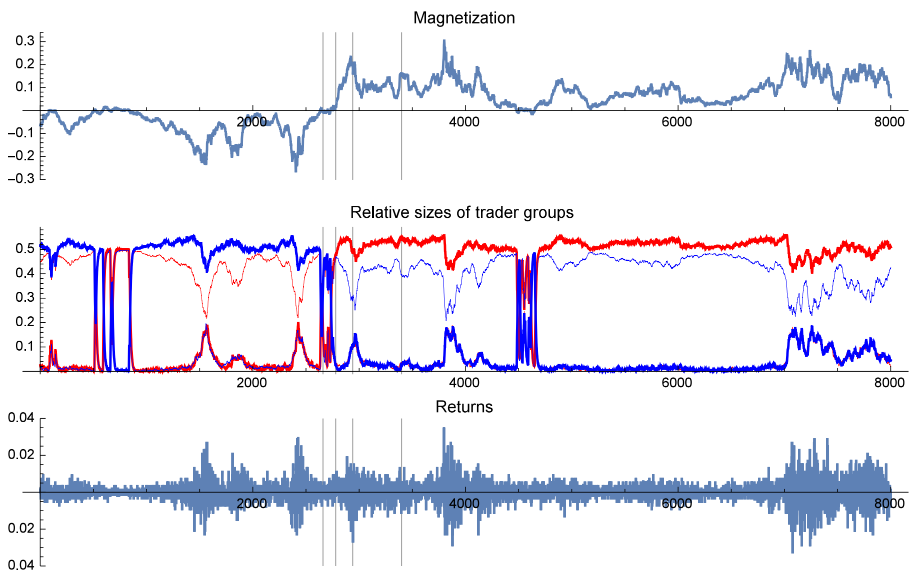
Whereas the spin value $S_i(t)$ is driven by a probability density function, there is a strict deterministic rule for a change of agent's strategy $C_i(t)$. Let us observe a theoretical agent who by some system development has recently become a minority trader and who is supposed to take action at this time instant. As an ex-majority trader, he is likely to be fundamentalist and as such would tend to oppose magnetisation by keeping his current spin. This is, however, far from certain as neighbors can also influence his local field h_i which in turn is only a parameter of heat-bath probability density function (Equation 4.5) that drives the actual change of spin. In contrast, there is no such ambiguity in case of agent's strategy $C_i(t)$. As a part of minority, he will be a chartist for purpose of the next round.

This is well illustrated by the difference between Figures 5.3b and 5.3c that correspond to the first two vertical grid lines in the plots in 5.3a. Plot of magnetisation shows that a change in sign of $M(t)$ has occurred between these two snapshots. Notwithstanding the sign change, the difference in value of $M(t)$ is negligible. The two snapshots exhibit hardly any change in terms of spin values $S_i(t)$ (red/blue areas). Conversely, there has been almost a complete swap of strategy values $C(i)$ (light/dark areas). At a time-scale of t (i.e. 100 rounds per 1 period) the swap happens almost instantaneously.[9] This change will certainly leave a mark in the plot of group sizes, but does not affect returns in any way.

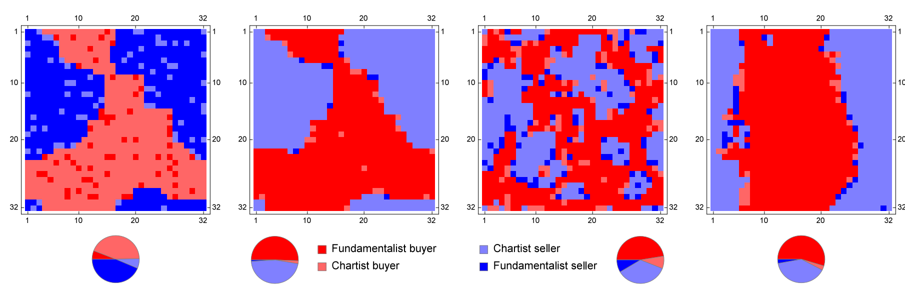
We have shown in Section 4.4.1 that with sufficiently fast strategy adjustment, the further $M(t)$ deviates, the more agents will tend to oppose it and the stronger will be their tendency to oppose. In previous paragraphs we have illustrated that change in strategy is in fact "quite" fast and both versions of the model should thus behave "very" similarly. And indeed, looking at Figure 5.3, the dynamics would be hard to distinguish from the simplified version. But because the difference is subtle if any at all, it is problematic to assess it on the grounds of single realization of each version. The difference will, therefore, be examined in Section 5.4 where results of multiple realizations are taken into account.

Note that stable and intermittent periods differ substantially for Ising's and for Bornholdt's model. As we described above, the stable phases of Bornholdt's model occur around $M(t) = 0$ when two opposing clusters of similar size form.

[9] Obviously with number of agents being 32^2 a theoretical minimum is a little more than $\Delta t = 10$, i.e. 1000 rounds.



(a) magnetisation, relative sizes of trader groups, and returns.



(b) Lattice state at $t = 2720$

(c) Lattice state at $t = 2800$

(d) Lattice state at $t = 2960$

(e) Lattice state at $t = 3400$

Figure 5.3: Underlying dynamics of Bornholdt's model with strategy spin

The intermittent phase occurs only after one of the clusters dominates, $|M(t)|$ grows higher, the second term becomes relevant, and induces switching of spins.

In the Ising model discussed in Section 4.2.3, stable phases occur as a result of convergence to one of the border states. In such case, most spins are of the same value and $|M(t)| \rightarrow 1$. The intermittent phases last only temporarily around $M(t) = 0$ before formation of large clusters and consequent convergence to one of the border states. These clusters around $M(t) = 0$ are similar in both nature and origin to those of Bornholdt's stable phase. They, however, tend not to last long as $|M(t)|$ is not bound by the second term of local field.

At under-critical $\beta < \beta_C$ the influence of local field $h_i(t)$ is equally diminished for both versions of Bornholdt's model as well as for Ising's model. Because the models differ only in definition of the local field, the dynamics of all three models at under-critical setting are quite similar.

5.3 Detailed characteristics of simulated time series

Analogical analysis was conducted for both versions of Bornholdt's model (simulated over the same parameters) but only results for the strategy model are presented. The reason is that the series exhibited very similar behavior. The only exceptions were some of the correlation indicators but the differences were still too subtle to infer a conclusion based on comparison of only two realizations.

Simulated series of the strategy model are shown in Figure 5.4 along with their real-world equivalents. Without a doubt, the two pairs resemble each other especially in case of returns. The simulated series exhibits volatility clusters similar to those of S&P 500, even though the distinction between periods of high and low volatility is somewhat clearer than in the real-world data.

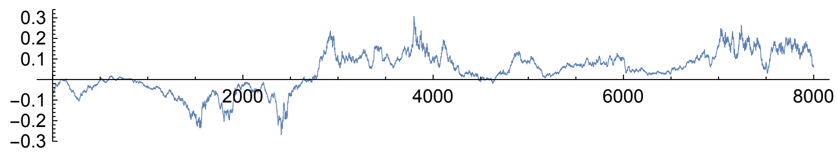
5.3.1 Distribution of returns

Standardized returns in Figure 5.4b with clear volatility clusters will likely deviate from the normal distribution even though the extreme spikes are less pronounced than in case of the market index. The spikes rarely exceed 5σ which is a value unlikely to appear under normal distribution but commonly encountered in financial series.

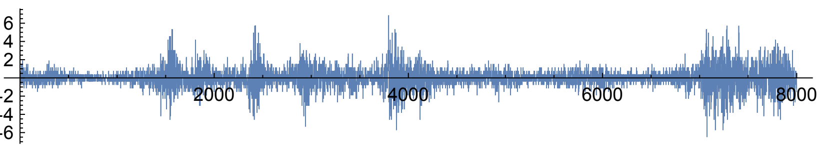
With skewness of 0.07 and excess kurtosis of 5.00, Jarque-Berra test rejects its null of normal distribution at any reasonable level of significance. The right skew is only marginal and as we shall see in Section 5.4.1 is a result of a chance rather than a rule. In contrast, the kurtosis is quite substantial, roughly at a level of GBP/USD exchange rate or S&P 500 returns at lag 4. The peakedness as well as heavy tails are well observable in a plot of empirical probability density function in Figure 5.5.

5.3.2 Dependence in returns

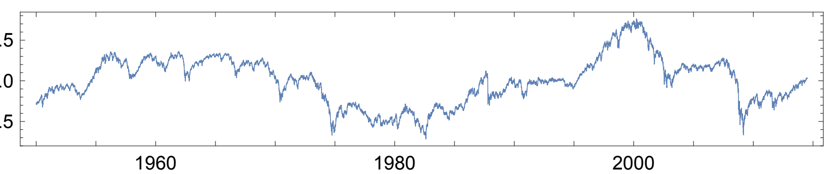
To assess autocorrelation of returns, we run a Ljung-Box test and examine an autocorrelation function. Test statistic of the Ljung-Box at 46.4 is similar to that



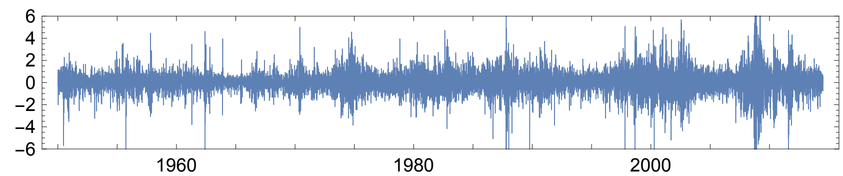
(a) Total magnetisation



(b) Returns (first differences) of magnetisation



(c) Log values of detrended S&P 500



(d) Daily standardized log-returns of S&P 500

Figure 5.4: Magnetisation and returns from Bornholdt's model with strategy spin. S&P 500 equivalents added for comparison.

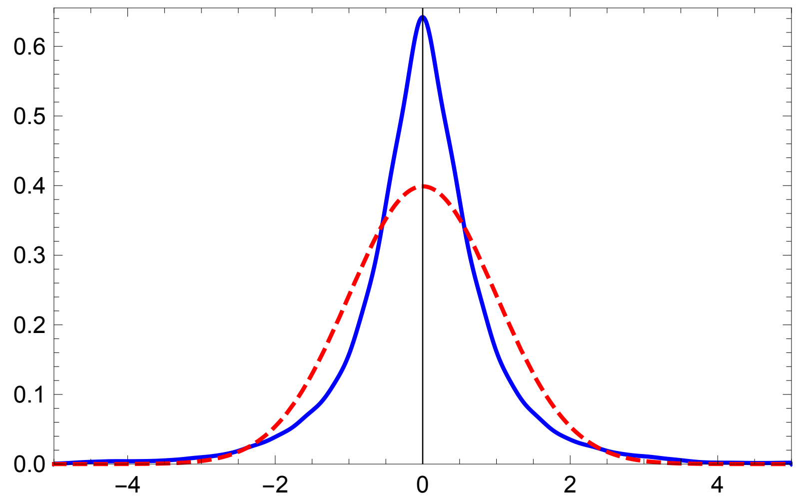


Figure 5.5: Empirical distribution of returns. Normal distribution (dashed line) is added for comparison.

of S&P 500 and the test strongly rejects its null of no autocorrelation. The ACF along with a 95% confidence interval band is plotted in Figure 5.6. Marginally significant correlations are infrequent but present until about $\tau = 100$. Low yet highly significant AR(1) parameter $\phi = 0.029$ is almost identical to that of S&P 500. Local Whittle estimate of fractional integration parameter is $d = -0.028$, indicating some antipersistence in the series.

5.3.3 Dependence in absolute returns

Autocorrelation function of absolute returns (Figure 5.7) is fairly high for low τ but follows a clear linear downward trend on a log-normal scale. Some short-range correlation therefore seems to be present in the process and current value of volatility thus depends on the past values. At high τ the ACF still seems to steadily decline and thus does not indicate presence of long memory.

The estimated parameters for fitted power-law and exponential functions are $A = 0.186$, $\alpha = 0.127$ and $B = 0.241$, $\beta = 0.009$, respectively. In this case exponential function undoubtedly provides a better fit for the whole examined range of τ.

First-order autocorrelation parameter $\phi = 0.254$ is highly significant and confirms an autoregressive process in the series. Despite little evidence of long-range dependence so far, local Whittle estimate of fractional integration parameter $d = 0.276$ indicates its presence in the series. These values are again almost identical to those of S&P500 ($\phi = 0.245$, $d = 0.269$).

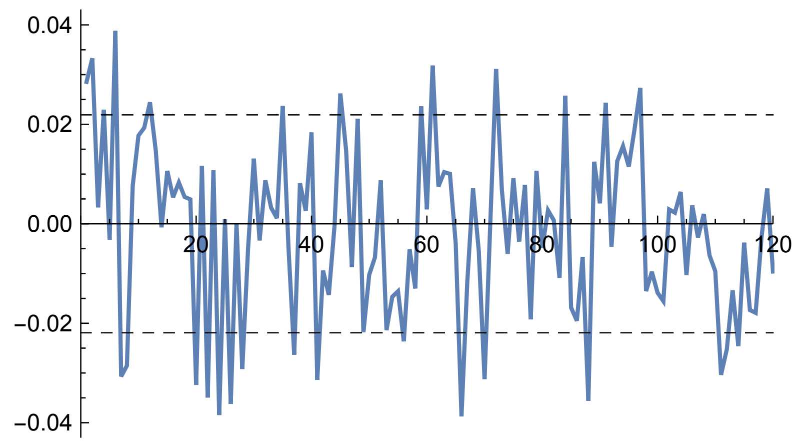


Figure 5.6: Autocorrelation function of daily returns

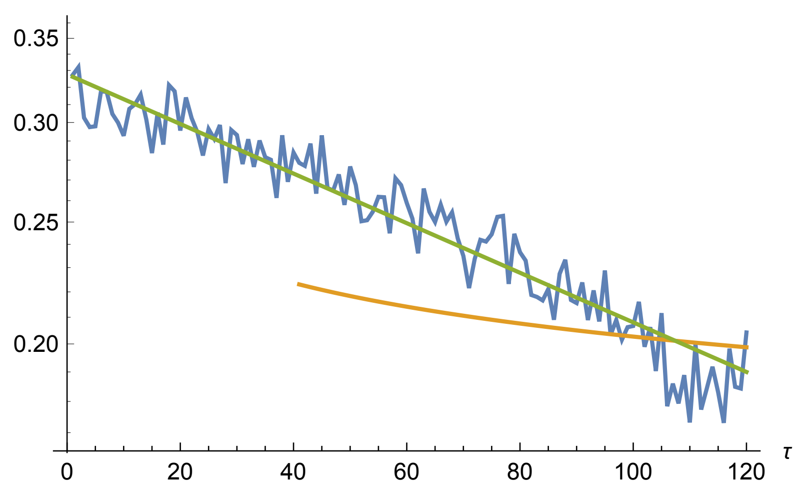


Figure 5.7: Log-normal plot of autocorrelation function of absolute returns with power and exponential functions fitted

5.3.4 Overall similarity to financial data

As far as distribution of returns is concerned, simulated series clearly deviate from normality in a manner similar to financial series. The deviation is usually not that pronounced, especially in case of skewness and occurrence of extreme values. The kurtosis, however, is similar to financial data; it is close to that of GBP/USD or S&P 500 at lag 4.

In terms of dependence in returns series, the model mimics financial data well, exhibiting some marginal correlation, which leads to Ljung-Box test rejecting its null. In this particular simulation, there appears to be a weak positive autoregressive process and weak long-range antipersistence This is a pattern commonly described in literature (Meyers, 2009) and encountered for example in S&P 500. The values are, however, close to zero and need to be interpreted carefully because, as we shall see in Section 5.4, are not robust to certain changes in model set-up.

For absolute return series, both AR parameter and difference parameter are virtually identical to S&P 500. The other two financial series do not dramatically differ from S&P 500 in this regard and are, therefore, also mimicked well by the model. ACF and two fitted functions are reminiscent of GBP/USD exchange rate in that the decay seems to be predominantly exponential. The values of fitted functions are in the general vicinity of values encountered in all financial series. This, however, might not be a good indicator of similarity since neither of the functions provided a good fit of ACF decay for all three financial series.

5.4 Simulation of model with strategy for various (α, β)

Here we present results of Bornholdt's model simulations over a set of parameter combinations (α, β). We examine how the parameters influence characteristics of standardized returns from the simulated series. Simulations are conducted over 31 values of parameter β ranging from 0 to 3 with a step of 0.1 and 4 values of parameter α ranging from 0 to 30 with a step of 10. For every parameter combination, 32 simulations are carried out and only series that do not converge are considered for further analysis. Parameter combinations with less than a quarter of series being non-convergent are dropped from analysis altogether to avoid bias from low number of observations.

Given characteristic is calculated for each individual non-convergent simulation and an average of results along with its 95% confidence level margin of error is reported in Tables A.2 to A.36 of Appendix. For more convenient and illustrative in-text reference, the average results are also plotted in contour plots. Note that while the plots have the same color scale, the values it represents are unique to each plot.

Figure 5.8 shows number of non-convergent series within each parameter combination. Some of the combinations are dropped altogether, which is the case of $\beta \in [1.5, 3]$ for $\alpha = 0$ and $\beta \in [2.5, 3]$ for $\alpha = 10$. The corresponding regions in plots are left white and in tables marked "$\times$". Note that there are parameter combinations that are considered for analysis despite a non-negligible

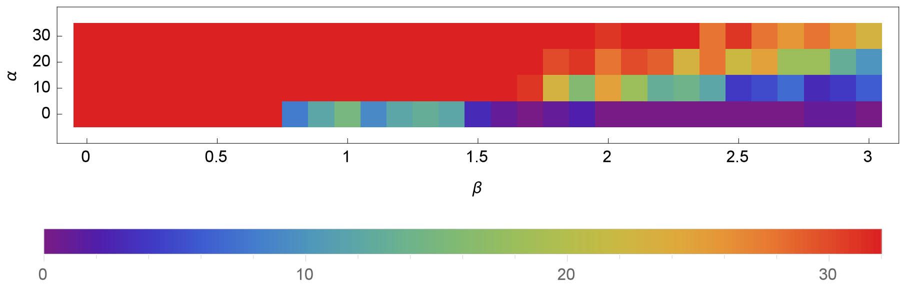


Figure 5.8: Number of non-convergent series. For numerical values refer to Table A.1.

number of convergent series. This is especially of concern in case of $\beta \in [0.8, 1.4]$ for $\alpha = 0$ and $\beta \in [2.2, 2.4]$ for $\alpha = 1$. Whatever results are obtained for those (α, β), they need to be interpreted with the convergence rate in mind.

To make comparison with financial series easier, their values of examined characteristics are presented in Table 5.1. Before we delve into examination of individual characteristics, let us observe some common high-level patterns in Figures 5.9 to 5.17d. In a large number of plots a distinctive region is apparent for all values of α and values of β lower than 0.4 or 0.5. We will refer to this general region as under-critical since $\beta < \beta_C \approx 0.44$. Naturally, the rest will be referred to as over-critical with possible distinction between over-critical Ising ($\alpha = 0$) and over-critical Bornholdt ($\alpha \geq 10$) region.

5.4.1 Distributon of returns

Bornholdt's model is symmetric in terms of magnetisation sign and thus both mean and skewness should be negligible. Moreover, maxima and minima should be of about equal absolute value for a given parameter combination. Even before standardization, mean was indeed effectively zero in all cases with no apparent pattern and the values are thus not presented. While skewness is also close to zero in all cases, it deviates (to either side) slightly more with increasing β as Table A.2 shows.

The maxima and minima are mostly symmetrical as expected, and for this reason only absolute maxima are presented in Figure 5.9. Given that we work with standardized returns, the plot presents the absolute maxima as multiples of standard deviations (σ). The under-critical region of $\beta < 0.5$ shows maxima around just over 4 standard deviations. Under normal distribution, a 3.5σ event (one outside $\mu \pm 3.5\sigma$ range) happens with frequency of about 1 per 2100 while a 4σevent with frequency of about 1 per 15800. If we assumed a normal distribution, with 8000 observations slightly lower maxima would be expected. Nonetheless, observed values do not differ dramatically from the normal distribution. As we move to $\beta > 0.5$, an area of about 6σ maxima starts at $(\alpha, \beta) = (0, 0.7)$ and seems to continue linearly upwards to about

	S&P 500	GBP/USD	Gold
Max of $\|r_t\|$	23.55	9.15	12.44
Kurtosis	27.72	5.12	13.30
Skewness	-1.03	-0.47	-0.07
JB test (p-val)	522867.5 (0.00)	7252.1 (0.00)	86301.7 (0.00)
log of JB test	13.17	8.89	11.36
LB test (p-val)	56.50 (0.00)	1401.88 (0.00)	34.28 (0.00)
AR(1) (p-val)	0.028 (0.00)	0.463 (0.00)	-0.039 (0.00)
Whittle of r_t	-0.019	0.061	0.014
AR(1) of $\|r_t\|$ (p-val)	0.245 (0.00)	0.349 (0.00)	0.345 (0.00)
Whittle $\|r_t\|$	0.269	0.200	0.312
GARCH α (p-val)	0.088 (0.00)	0.300 (0.00)	0.174 (0.00)
GARCH β (p-val)	0.867 (0.00)	0.112 (0.00)	0.752 (0.00)

Table 5.1: Overview of characteristics to be compared

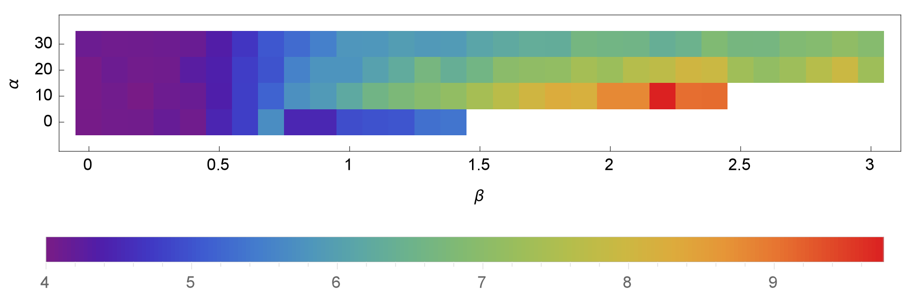


Figure 5.9: Maximum of absolute magnetisation $|M(t)|$. For numerical values refer to Table A.3.

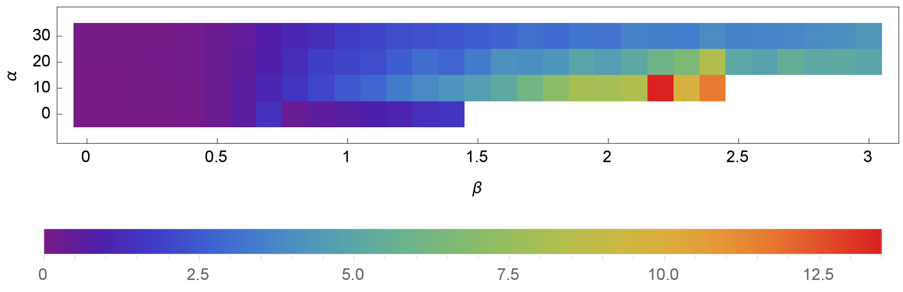


Figure 5.10: Kurtosis of return series. For numerical values refer to Table A.4.

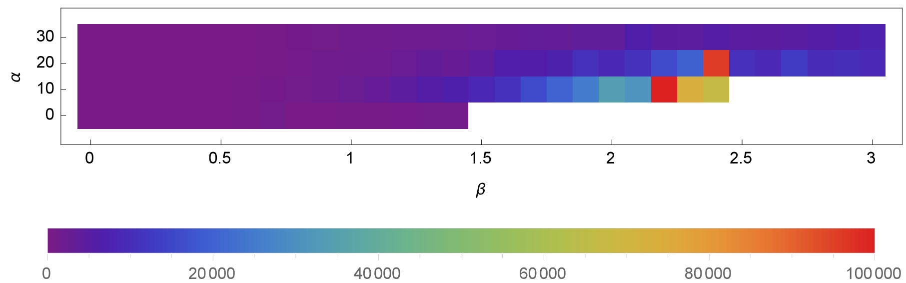
(30, 1.1). For $\alpha = 30$ similar values can then be found for all $\beta > 1.1$. Values that are highest and thus closest to real-world data, can be found in the series that exhibit some tendency to converge. Values around 9 that are comparable to GBP/USD exchange rate are in the region of $\alpha = 10$ and $\beta \in [2.0, \ldots, 2.4]$ where, however, more than half of simulated series converged.

In terms of excess kurtosis κ, Figure 5.10 suggests a relatively normal-like behavior for an under-critical region of about $\beta \leq 0.6$ for which $\kappa = 0$ falls within κ's 95% confidence interval. In the Bornholdt's ($\alpha > 0$) over-critical region κ grows with β to about 4 or 5 which is a value encountered in GBP/USD series. Albeit values as high as 13 can be found at $\alpha = 10$ and $\beta \in [1.9, 2.4]$, if we take into account their margins of error, we cannot say there is a value higher than 7 at 95% confidence level. None of the parameter combinations thus comes close to kurtosis values of S&P 500 (27.7) and gold (13.3).

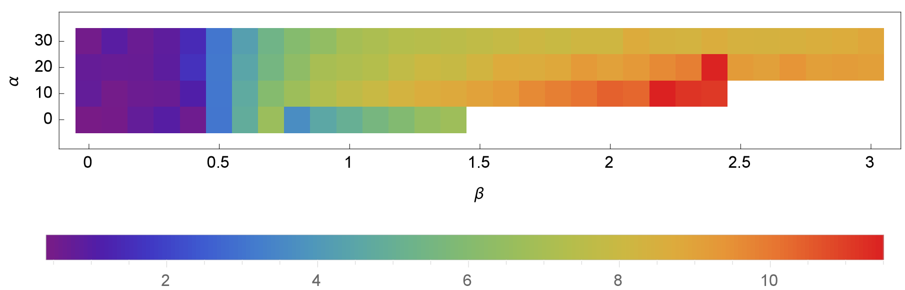
Jarque-Bera test statistics are shown in Figure 5.11a but since it does not provide a clear picture for some lower values of β, logarithms of the statistics are added below. JBT tests a joint hypothesis of no skewness and no excess kurtosis, where the first one is likely to hold for our data. Therefore, information that both plots of test statistics provide is very similar to that given by excess kurtosis in Figure 5.10. P-value shown in Figure 5.11c makes the distinction between under-critical and over-critical regions very clear. Regardless of α, the null hypothesis of normality is by far not rejected up to $\beta = 0.4$ which is just below $\beta_C \approx 0.44$. Just above, at $\beta = 0.5$, the null is rejected generally at 5% but not 1% level of significance. For $\beta \geq 0.6$ the null is rejected at any reasonable level with few marginal exceptions in Ising case.

5.4.2 Dependence in returns

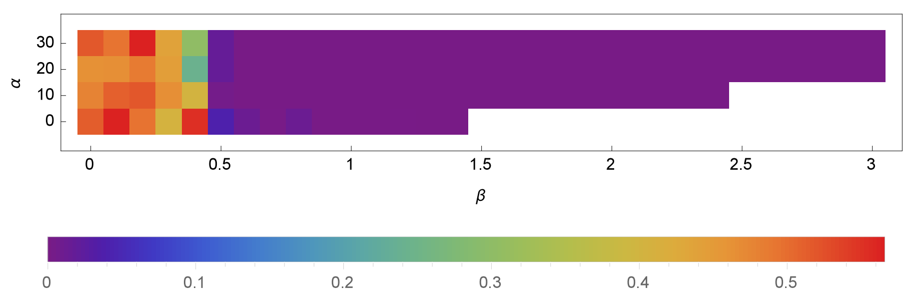
Figure 5.12, which presents test statistics and p-value of Ljung-Box test applied to returns, does not follow the same pattern as previously examined statistics. Characteristics change primarily with β but not simply along the critical value of β_C. Instead, the series with autocorrelation are (in terms of β) on both sides of the normal-like region of little autocorrelation.



(a) Test statistic

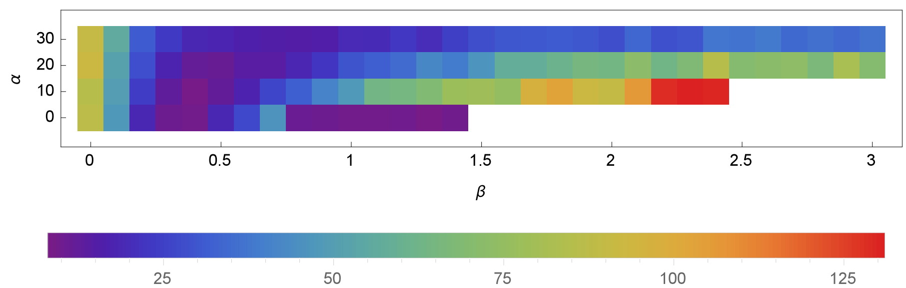


(b) Logarithm of test statistic

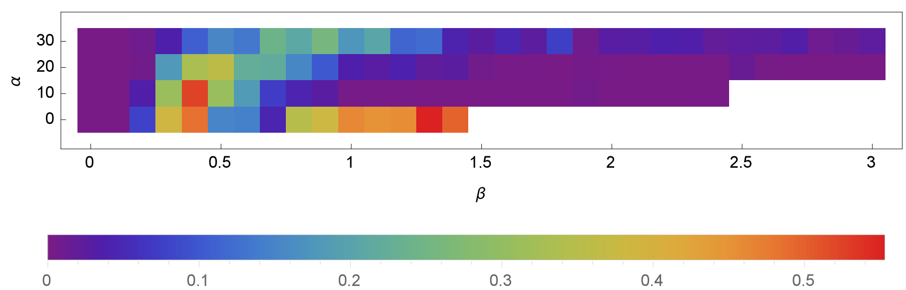


(c) P-value

Figure 5.11: Result of Jarque-Berra test of returns series. For numerical values of test statistics and p-values refer to Tables A.5 and A.6 respectively.

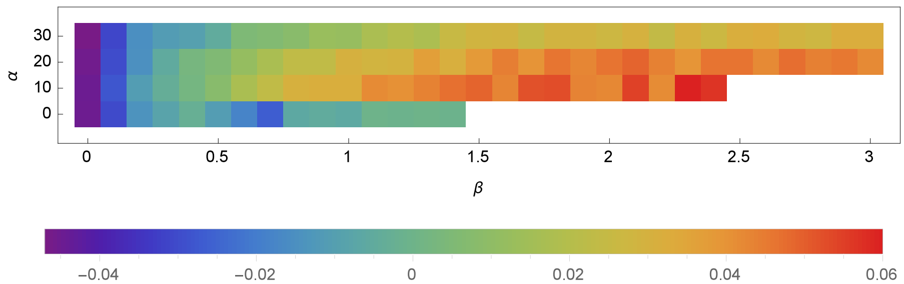


(a) Test statistics

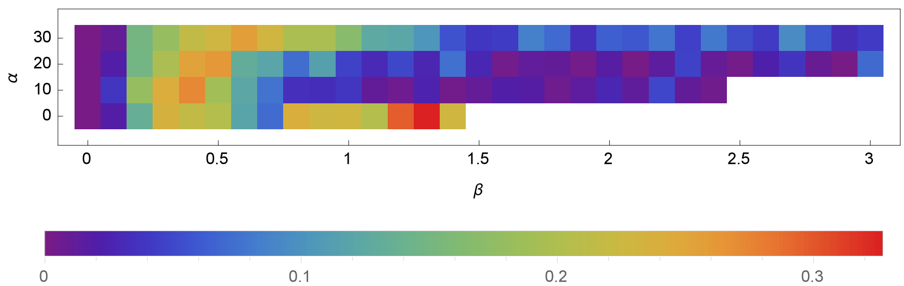


(b) P-values

Figure 5.12: Results of Ljung-Box test of returns series. For numerical values of test statistics and p-values refer to Tables A.7 and A.8 respectively.



(a) Parameter values



(b) P-value of parameter

Figure 5.13: AR(1) model parameter and its p-value for returns series. For numerical values of the parameter and p-value refer to Tables A.9 and A.10 respectively.

For $\alpha = 0$ the null of no autocorrelation is not rejected at 10% significance level in case of $\beta \in [0.2, 0.6]$. For every increment of 10 in α the β interval of no autocorrelation shifts roughly 0.2 towards higher values and the upper limit becomes less distinct. This results into two distinct groups of combinations (α, β) that exhibit autocorrelation - one with higher β ($\beta > 1.1$ for $\alpha = 10$ and $\beta > 1.4$ for $\alpha \geq 20$) and one with very low β ($\beta \leq 0.2$ for $\forall \alpha$).

The autocorrelation in the region of higher β seems to stem from short-range as well as long-range dependence, both of which have positive values that are low but significant at least at 10% level in most cases (Figures 5.13 and 5.14). These parameter combinations exhibited some features of financial series previously and some marginal autocorrelation therefore comes as no surprise.

Because at $\beta \leq 0.2$ spin updates are random or close to random and the resulting series seemingly resemble white noise, it is rather peculiar to detect significant autocorrelation. Both AR parameter and Whittle estimate of d are significant and show negative correlation with magnitude more than comparable to the series of high β. Long-range dependence is strongest at $\beta = 0$, where the value of d indicates a strong anti-persistence; this might come as a result of the small size of our lattice that allows only limited deviation in either direction.

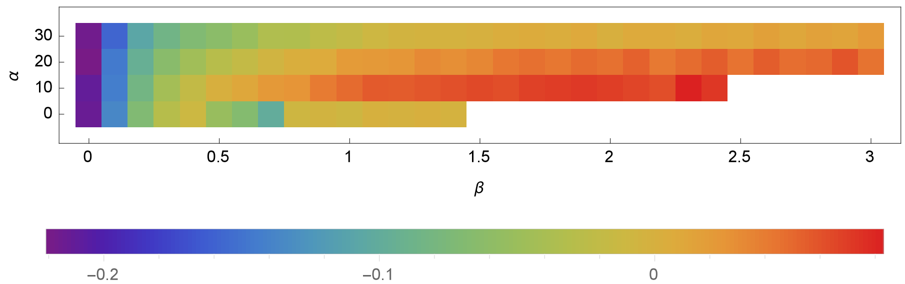


Figure 5.14: Local Whittle estimate of fractional difference parameter d of returns. For numerical values refer to Table A.11.

5.4.3 Dependence in absolute returns

For absolute returns, detailed results of Ljung-Box test are omitted as the correlation is more palpable in plots of AR parameter ϕ and Whittle estimate of d (Figures 5.15 and 5.16 respectively).

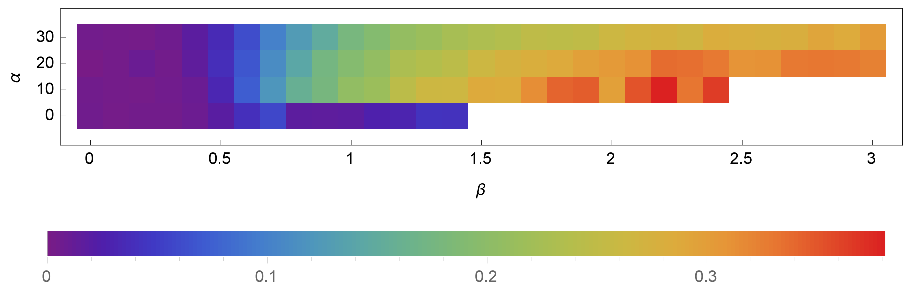
AR parameters of under-critical region ($\beta \leq 0.4$) are insignificant without an exception, while over-critical Ising results ($\alpha = 0$) are mixed but generally also insignificant. Bornholdt's over-critical combinations ($\alpha \geq 10$) are significant at 5% level for $\beta = 0.5$ and at 1% level for higher β. Values of the significant parameters can be as low as 0.05 for some lower β but in most cases range from 0.2 to slightly above 0.3, which is exactly what we observed in financial series (Table 5.1).

Whittle estimate of d for over-critical Bornholdt's model grows with β and ranges from 0.1 to 0.3. Values encountered in financial series were between 0.2 and 0.3 which are values found at $\beta \geq 0.7$. In terms of both short- and long-range memory, financial series are thus mimicked well by a wide set of parameter combinations of Bornholdt's model.

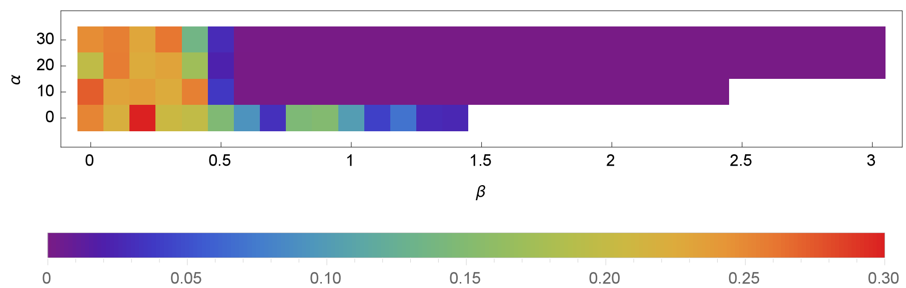
Significance of both GARCH parameters, γ_1 and δ_1, shown in Figure 5.17 is almost identical to that of AR parameters discussed just above, i.e. they are significant for over-critical Bornholdt's combinations. For these, value of γ_1 ranges from 0.05 for lower αs and βs, and grows in both α and β to values just over 0.9. Parameter δ_1 complements γ_1 in that the sum $\gamma_1 + \delta_1$ lies between 0.97 and 0.99 for all the significant combinations. The volatility in the series is thus fairly persistent but not to an extent that would render r_t^2 non-stationary. This behavior is perfectly in-line with theoretical values suggested by literature (see Subsection 2.1.3 for details). Out of three analyzed financial series, however, only S&P 500 behaves according to the literature and thus is the only series well mimicked by Bornholdt's model.

5.4.4 Overall similarity to financial data

Before we assess the ability of the model to mimic financial series in terms of individual characteristics, let us observe some high-level patterns.



(a) Parameter values



(b) P-value of parameter

Figure 5.15: AR(1) model parameter and its p-value for absolute returns series. For numerical values of the parameter and p-value refer to Tables A.12 and A.13 respectively.

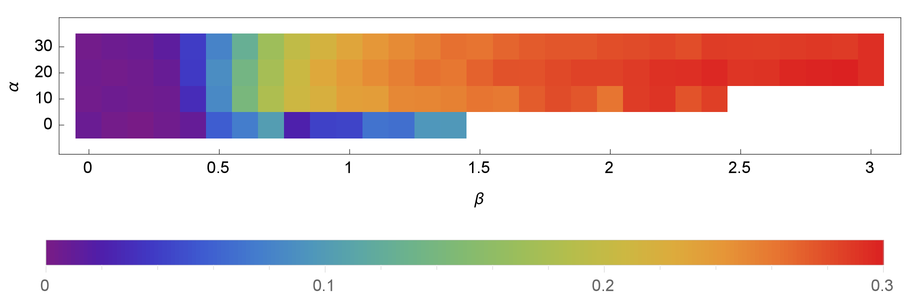
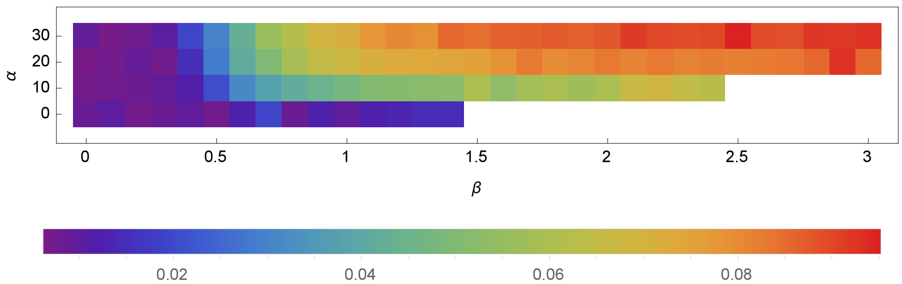
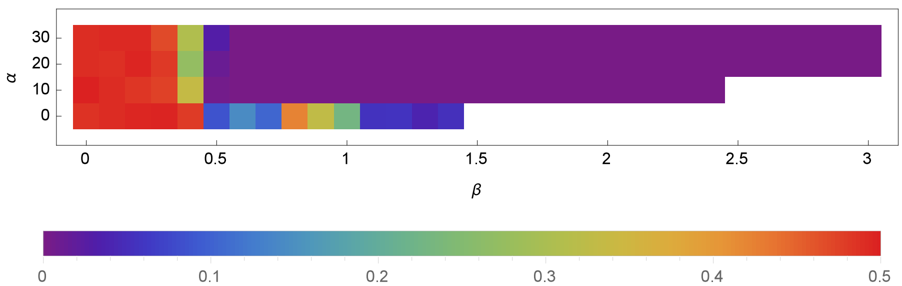


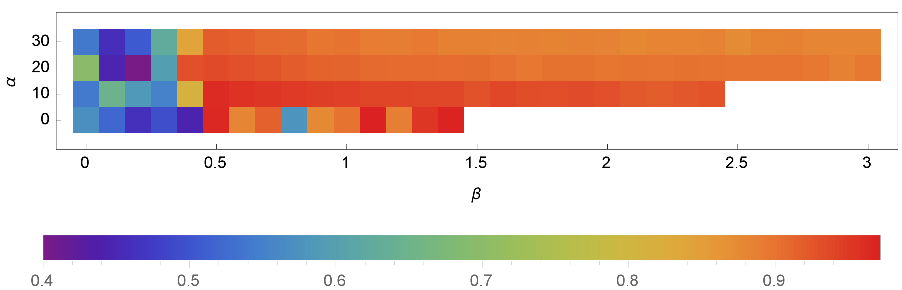
Figure 5.16: Local Whittle estimate of fractional difference parameter d for absolute returns. For numerical values refer to Table A.14.



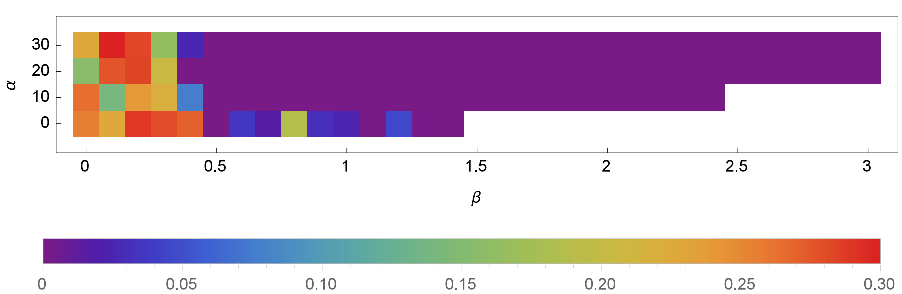
(a) Values of parameter γ_1



(b) P-values of parameter γ_1



(c) Values of parameter δ_1



(d) P-values of parameter δ_1

Figure 5.17: GARCH (1,1) model parameters and their p-values. For numerical values of the parameter γ_1 and corresponding p-values refer to Tables A.15 and A.16 respectively. For numerical values of the parameter δ_1 and corresponding p-values refer to Tables A.15 and A.16 respectively.

At under-critical setting (i.e. $\beta \leq 0.4$) resulting returns series bear hardly any resemblance to their financial counterparts; their characteristics are similar to those of white noise with exception of slightly higher extreme deviations and in case of $\beta \leq 0.1$ also faster mean-reversion.

Results do not show any significant difference between under-critical simulations of Bornholdt's and Ising models. For $\beta > \beta_C$, however, the behavior differs substantially. The Ising model converges in virtually all cases to one of the border values of $|M(t)|$. For lower over-critical βs there is some white noise fluctuation but the magnetisation generally does not change its sign as is well illustrated by Figure A.2.[10] Ising model in its original physical form is therefore of no use in mimicking financial data.

In contrast, Bornholdt's model exhibits at least some of the features observed in financial series for most over-critical parameter combinations, especially for $\beta \geq 1.0$. For some of the higher βs, however, the simulated series tend to converge. Moreover, these series usually vary more than non-convergent ones and thus provide inconsistent results. This is the case of $\beta \geq 1.6$ for $\alpha = 10$, $\beta \geq 1.7$ for $\alpha = 20$, and $\beta \geq 2.3$ for $\alpha = 30$. Let us now examine the calculated characteristics of Bornholdt's over-critical parameter combinations in more detail. Extreme values of studied characteristics are often found in parameter combinations that exhibit convergence. We will disregard those in the following summary and only consider combinations providing consistent results.

As discussed previously, due to the symmetry of the model, the simulated series are generally unskewed, and negative and positive extremes are equal in size. Maximum of $|M(t)|$ grows in β with 7σ events being the highest in non-convergent combinations. Such values are only slightly lower than those of GBP/USD (9.1) and gold (12.4). Kurtosis also grows with β to values just over 5 which is exactly what was observed in GBP/USD rate.

Both short- and long-range correlation of returns are mostly positive, both increase with β and decrease with α. Highest values of AR parameter ϕ are around 0.04 which is similar to S&P 500, whilst weak positive long-range dependence encountered here was present in series of GBP/USD rate and Gold.

Both indicators of autocorrelation of absolute returns grow in β from about 0.2 to about 0.3, which perfectly fits all of our data. Parameters γ_1 and δ_1 of GARCH model are perfectly in line with literature. There is a persistence in volatility close to unity in most series and first order spillover represented by $\gamma_1$grows with β from 0.05 (low volatility) to 0.09 (high volatility).

5.5 Simulation of simplified model for various (α, β)

Simulations and subsequent analysis of simplified model are conducted for the same set of parameter combinations (α, β) to allow comparison with strategy model. The results are reported in similar way as in previous case, but only important contour plots are shown within these paragraphs not to congest text with figures. The remaining plots can be found in Section A.3.1 of Appendix.

[10] For high over-critical β other equilibria than $M(t) = \pm 1$ might occasionally occur if two clusters of regular shapes form.

Number of non-convergent series within each parameter combination (Figure A.9) is similar to the model with strategy for $\beta \leq 2.0$ but tends to be lower thereafter, especially for $\alpha \geq 20$. Four more combinations had to be dropped from analysis ($\beta = \{2.6, 2.7, 2.9, 3.0\}$ for $\alpha = 20$) and one was added ($(\alpha, \beta) = (10, 2.4)$).

In cases of most characteristics, we can again distinguish an under-critical region approximately for $\beta \leq 0.5$, over-critical Ising region ($\beta > 0.5$, $\alpha = 0$) and over-critical Bornholdt's region ($\beta > 0.5$, $\alpha \geq 10$). Throughout this section, we will disregard the Ising region as it is not influenced by local field $h_i(t)$ and the results are thus the same as in case of model with strategy. Hence, any statement will be valid only for $\alpha \geq 10$.

5.5.1 Distributon of returns

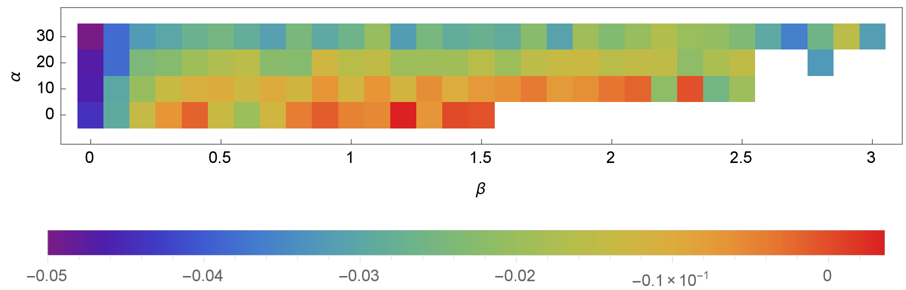
As in previous model, symmetry in terms of sign leads to negligible mean and skewness. Also, absolute values of maxima and minima are similar which is why only maxima of $|M(t)|$ are presented in Figure A.10. The range of maximum values is similar to the strategy model but in the over-critical region exhibits more randomness with respect to parameters α and β. Whereas in case of strategy model, the maximum grows with β, in this case the top values are found anywhere in the region of $\beta \geq 2$. Identical pattern occurs in kurtosis values (Figure A.11) where it is even more pronounced. Since skewness is far from substantial, the pattern naturally carries over to results of JB test (Figure A.12).

5.5.2 Dependence in returns

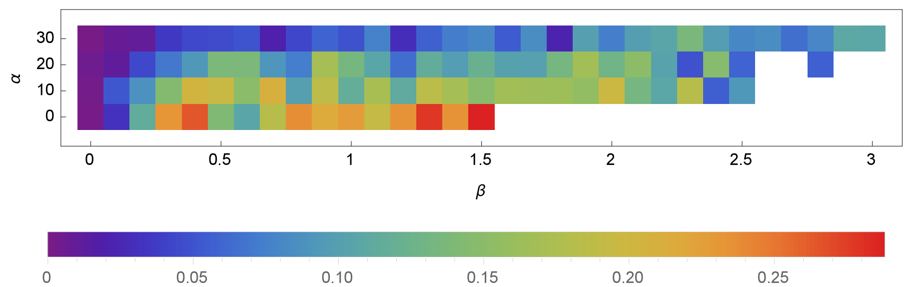
Results of Ljung-Box test shown in Figure suggest a non-trivial difference between the models. While both exhibit significant correlation for $\beta \leq 0.2$, at higher values the patterns are hardly the same. The simplified model exhibits weak or no autocorrelation at $\alpha = 10$ but becomes significant for higher α; for $\alpha = 30$ the null is rejected at 1% level in virtually all cases. Autocorrelation in the strategy model is, on the other hand, strongest at $\alpha = 10$ and weakens with increasing α. To explain why this is, we need to examine short- and long-range components of correlation in each model.

In both models the AR(1) parameter ϕ as well as the fractional parameter d are significantly negative at low β, which, as discussed earlier, can be attributed to negative correlation stemming from limited size of our lattice. In the over-critical region, however, the models differ in sign of both short- and long-range dependence.

Figures 5.18 and 5.19 show that both autoregressive and fractional parameters are close to zero but generally gravitate to negative values, especially for high α. This indicates both short-term negative correlation and weak antipersistence in the series. Conversely, in the strategy model, we could find generally positive values that were highest and most significant at $\alpha = 10$.



(a) Parameter values



(b) P-value of parameter

Figure 5.18: AR(1) model parameter and its p-value for returns series. For numerical values of the parameter and p-value refer to Tables A.27 and A.28 respectively.

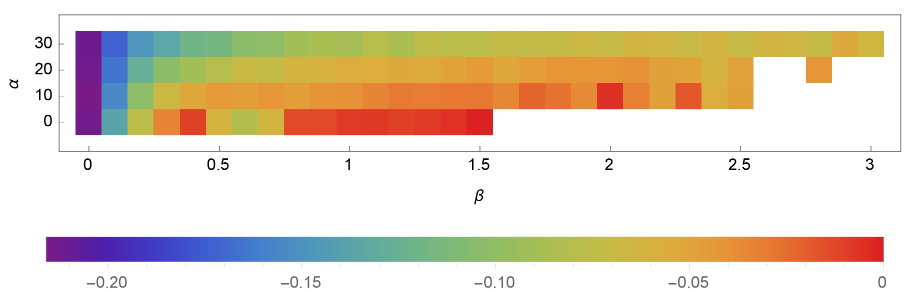


Figure 5.19: Local Whittle estimate of fractional difference parameter d of returns. For numerical values refer to Table A.29.

5.5.3 Dependence in absolute returns

Differences in autocorrelation of returns translate also to autocorrelation of absolute returns but are less pronounced. Both models exhibit high AR(1) and difference parameters for the whole over-critical region with values slightly growing with β. The simple model has higher values for $\alpha = 10$, while the strategy model for $\alpha = 30$ which are the regions where the respective models exhibit more significant correlation in returns. Nevertheless, the values of both correlation parameters for both models are in the vicinity of 0.3, which is a value observed in the financial series.

Lastly, the GARCH parameter values (Figure A.16) are very much alike in both cases and do not indicate any difference between the model versions.

5.5.4 Discussion of differences between model versions

The most profound difference between the two versions of Bornholdt's model is the opposite sign of both short- and long-range correlation in returns. It elegantly illustrates functioning of the second term of local field $h_i(t)$ and also shows that it is significantly stronger in case of simplified model. As discussed in Section 4.4.1, second term prompts an agent to swap. In simplified model this is always true, whereas in the strategy model only after strategy adjustment. The second term thus induces behavior that opposes any deviation from zero, therefore, inducing mean reversion. For higher α (measure of second term's strength) the generated process will thus become less positively or more negatively correlated, which is exactly what we observe in correlation indicators (ϕ and d alike) of both models. In strategy model, the positive correlation induced by the first term prevails, but weakens with increasing α. Values in the simplified model gravitate towards negative values, more so with higher α. Although the strategy adjustment seems to be fast at least at time-scales that we use, it introduces an inertia into the model that influences short- and long-range correlation.

6 Conclusion

Purpose of this work was to thoroughly analyze Bornholdt's model and to provide a detailed and accessible description not only of the results but also of the underlying dynamics.

We have presented a breakdown of definitions of both versions of Bornholdt's model as well as the original Ising model. We have discussed parallels and concluded that while Bornholdt's augmentation might reflect some behavioral patterns, it more likely serves primarily to eliminate Ising's inherent tendency to converge.

With an aid of simulated series, we subsequently described and compared inner dynamics of the strategy model and the simplified model. This comparison indicated that models' dynamics as well as the resulting series are the same which would deem the whole strategy concept somewhat redundant. Later simulations over wide range of parameters, however, proved that the lag in strategy adjustment introduces an inertia in magnetisation change. This allows for simulation of positive short- and long-range dependence in returns series.

To assess how well Bornholdt's model mimics real-world financial data, we first tested presence of stylized facts in three series of different types - S&P 500 market index, GBP/USD exchange rate and gold as a commodity. Having more than just one referential financial series proved to be useful; even though all the series generally followed stylized facts, they turned out to be diverse in number of aspects.

Deviation from normal distribution was noticeable in all series, with exhibits of volatility clustering and frequent occurrence of at least 5σ events. All distributions were highly leptokurtic, with S&P 500 being by far the most. Interestingly, skewness exceeding -1 was observed only in case of the market index, whereas gold proved to be unskewed.

Both short- and long-range correlation proved to be low but significant in all three series except for the exchange rate where remarkably strong AR(1) parameter was observed.

Autocorrelation of absolute returns was similar for all three series. First order autocorrelation of about 0.3 indicated presence of short-range dependence, whilst Whittle estimate of difference parameter indicated long-range dependence in the series. Fitting ACF of absolute returns yielded mixed results as neither function provided a good approximation of decay in all three cases.

GARCH results in case of S&P 500, exhibiting relatively high volatility with usual level of persistence, were in line with literature. Persistence in the other two series was somewhat lower, whereas influence of the last observation was two to three times higher than literature suggests.

After obtaining our reference statistics, we simulated Bornholdt's model with strategy and analyzed the resulting return series. The distribution deviated from normality in a manner similar to financial series, albeit with the deviation being not as pronounced. In terms of autocorrelation of returns series, the model exhibited weak positive autoregressive process and weak long-range antipersistence, similarly to S&P 500. Regarding autocorrelation of absolute returns, the model exhibited very similar behavior; the values of AR(1) parameter and differencing parameter were almost the same as those of the market index. The decay of absolute returns' ACF was, unlike that of the index, well captured by exponential function. Exponential decay was encountered in case of the exchange rate which, however, was rather specific due to its high first lag and a fact that both functions provided a good fit at the same time.

Our goal in the next part was to test whether Bornholdt's model generates data with desired characteristics for other input parameter combinations (α, β) as well and to assess how the characteristics change with those parameters. We ran simulations over a range of parameter combinations chosen such that they comprise both Bornholdt's and original Ising model, and both under- and over-critical setting. We concluded that at under-critical setting, resulting returns series are basically a white noise for both types of models and are of little interest. So is Ising model at over-critical setting since it tends to converge to one of the border values of magnetisation, producing rather flat return series.

In contrast, both versions of Bornholdt's model at over-critical setting generate series that in many aspects resemble the financial series.

Because of the symmetry of the model, the distribution is also symmetrical with maxima and minima of equal size. The extremes are not as pronounced as in case of financial series; 7σ events, occurring generally at higher β, are the highest. Kurtosis also grows with β to values just over 5 which is a value encountered in GBP/USD rate.

Short- and long-range correlation of returns are dependent on α as well as version of the model; both positive and negative significant values can be obtained.

Autocorrelation of absolute returns grows in β from about 0.2 to about 0.3, which is exactly the range of values observed in the financial data. The parameters of GARCH span the whole range suggested by literature; γ_1 grows with β from 0.05 to 0.09 and δ_1 decreasing accordingly.

In case of distribution parameters, results closest to the values observed in financial data are often found at higher β. Unfortunately, we cannot increase β arbitrarily since above a certain point this leads to increased variance in results and convergence of simulated series. There is not a universal value of this maximum β; instead it increases with α. As a result, region of parameter combinations that yield the very best results does not seem to be very compact. Instead it might be spread along the border between convergent and non-convergent parameter combinations.

To determine more general or more subtle patterns, a simulation over wider range or with lower step in parameter value would be beneficial. This would however require much more efficient code for simulations or availability of server-level computational capacity for a period of one or two weeks.

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A Apendix

A.1 Other Ising based models

Following Ising-based models are provided for two reasons. First is to give reader an opportunity to appreciate simplicity of Bornhodt's model. Second is to illustrate the range of possible augmentations. Each of these models, as well as Bornholdt's and many others, can further be modified by subtle changes such as those proposed in a remarkable work of Dvorak [2012].

Sieczka and Holyst [2008] introduces a model which is based on Bornholdt [2001] but in addition has certain features reminiscent of those used in Iori [1999]. Authors use identical lattice set-up as Bornholdt [2001] but the spin $S_i(t)$ is updated with a different dynamics:

$$S_i(t) = sign_{\lambda|M(t-1)|}\left[\sum_{<i,j>} J_{ij}S_j(t-1) + \sigma\eta_i(t)\right] \tag{A.1}$$

where $sign_q$ is a threshold function

$$sign_q(x) = \begin{cases} 1 & if\ x > q \\ 0 & if\ -q < x < q. \\ -1 & if\ x < -q \end{cases} \tag{A.2}$$

Lastly, $\eta_i(t)$ is a random Gaussian function with 0 mean and variance of 1 simulating i-th trader's individual erratic opinion with parameter σ as a measure of its influence.

The threshold q of the $sign_q$ function is dependent on magnetisation and for parameter $\lambda = 0$ it is identical to the 2-valued spin model.

There are three factors that influence the i-th spin. The influence of neighboring spin S_j (identical to Bornholdt's model), the erratic opinion $\eta_i(t)$ and a threshold $q = \lambda\,|M(t-1)|$. While in the simple version of Bornholdt's model (Equation 4.8) the higher magnetisation forces the chartist to switch group, in Sieczka and Holyst's model it is quite the opposite. Higher absolute magnetisation means higher threshold that, in order to trade, needs to be exceeded by a combination of neighbors' influence and own opinion. Authors' rationale is that in times of high absolute magnetisation (i.e. deviation from fundamental price) the agents are afraid of trading and need stronger incentives to do so.

Iori [1999] proposes a model where each agent i owns capital $K_i(t)$ consisting of cash $C_i(t)$ and $N_i(t)$ units of stock at price $p(t)$. Capital of agent i at time t is then given by $K_i(t) = C_i(t) + p(t)N_i(t)$. For each agent i at time step t, there are three possible values of his spin $S_i(t)$: 0 if he remains inactive, +1 if he decides to buy one piece of stock and −1 if he decides to sell one. At each time step each agent is influenced by a local field h_i :

$$h_i(t) = \sum_{<i,j>} J_{ij} S_j(t) + A\nu_i(t) + B\varepsilon(t) \tag{A.3}$$

where once again $\sum_{<i,j>}$ denotes a sum over the set of agent i's neighbors, J_{ij} is a measure of neighboring agent j's influence on agent i. While $\varepsilon(t)$ is a signal accessible to all traders, $\nu_i(t)$ is agent specific and is analogous to temperature. To induce an agent to trade his local field $h_i(t)$ must exceed his specific threshold $\xi_i(t)$.[11] The decision rule is

$$\begin{aligned} S_i(t) =& 1 \quad \text{if } h_i(t) \geq \xi_i(t) \\ S_i(t) =& 0 \quad \text{if } -\xi_i(t) < h_i(t) < \xi_i(t) \\ S_i(t) =& -1 \text{ if } h_i(t) \leq -\xi_i(t) \end{aligned} \tag{A.4}$$

Unlike most others, Iori [1999] then uses a series of consultation rounds before a trade is conducted. Agents make an initial decision depending on their local fields and thresholds according to Equation A.4. This influences their neighbors' local fields and possibly makes them change their initial decision. When the system converges to a stable state, the orders are placed simultaneously.

Because each agent can only buy one piece of stock at a time, supply $Z(t)$ and demand $D(t)$ are given by the number of sellers and buyers, respectively and the trading volume $V(t)$ by their sum. After each trading round, the stock price changes according to the rule

$$P(t+1) = P(t) \left(\frac{D(t)}{Z(t)} \right)^{\alpha}, \quad \alpha = a \frac{V(t)}{L^2}$$

where L^2 is the total number of traders and therefore also the maximum number of stock that can be traded each round. This is designed to reflect the overreaction of market to imbalanced orders in times of high activity.

Sornette and Zhou [2006] introduce somewhat more complex model that deviates in many ways from the basic Ising model. It is designed to simulate more behavioral patterns and is mentioned here as last in order to show other possible extensions to our model.

Authors again use a square lattice described above with possible spin values of $S_i = \pm 1$ and dynamics of update

$$S_i(t) = sign \left[\sum_{<i,j>} J_{ij}(t) E[S_j](t) + \sigma_i(t) G(t) + \varepsilon_i(t) \right] \tag{A.5}$$

where $E[S_j](t)$ is agent i's expectation of agent j's decision at time t. The expectation is what distinguishes the first term from that of Bornholdt's and

[11] The thresholds are normally distributed and change in time proportionally to price of the stock.

most others. In the second term G is a random ± 1 function that represents a universally accessible global information and σ_i measures its impact on agent i's action. The last term represents a private information similar to that of Sieczka and Holyst [2008].

Unlike in previous models the market price and the influence of neighbors are not constant. The market price is updated according to

$$p(t) = p(t-1)exp[r(t)], \quad r(t) = \frac{M(t)}{\lambda} \tag{A.6}$$

where λ is a measure of liquidity and is constant. The ability of agents to learn is accounted for by adaptive coefficient of influence of neighbors:

$$J_{ij}(t) = b_{ij} + \alpha_i J_{ij}(t-1) + \beta r(t-1)G(t-1) \tag{A.7}$$

where b_{ij} measures the intrinsic influence of neighbors and $\alpha_i > 0$ measures the loss of memory of past influences. The last term $\beta \neq 0$ measures how the influence changes in relation to the global news G. This parameter is of high importance since its sign decides whether the agent acts rationally or not. If global information G known at time $t-1$ has the same sign as the return $r(t-1)$, a rational agent should follow the news rather than behavior of others, in which case should be $\beta < 0$. The magnitude of return also plays a role as high return of either sign has higher financial as well as psychological effect.

Positive β would imply limited rationality where agent might for example wrongly attribute the origin of correct impulses. Such behavior might result from several mechanisms well described by behavioral economics literature (e.g. Heath and Gonzalez [1995] and Wyart and Bouchaud [2007]) such as mutually-reinforcing optimism and overconfidence.

A.2 Bornholdt's model with strategy spin

A.2.1 Tables of statistics for different (α, β)

30	32	32	32	32	32	32	32	32	32	32	
20	32	32	32	32	32	32	32	32	32	32	
10	32	32	32	32	32	32	32	32	32	32	
0	32	32	32	32	32	32	32	32	8	12	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	32	32	32	32	32	32	32	32	32	32	
20	32	32	32	32	32	32	32	32	30	31	
10	32	32	32	32	32	32	32	31	23	16	
0	15	9	12	13	12	3	1	0	1	2	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	31	32	32	32	28	31	28	26	28	26	23
20	28	30	29	23	28	22	25	18	18	13	10
10	25	18	13	14	12	4	5	7	3	4	6
0	0	0	0	0	0	0	0	0	1	1	0
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.1: Bornholdt's model with strategy: Number of non-convergent series

30	-0.0079 (0.0080)	-0.0053 (0.0102)	-0.0030 (0.0102)	0.0037 (0.0079)	-0.0003 (0.0114)	-0.0011 (0.0104)	0.0110 (0.0137)	-0.0006 (0.0156)	-0.0066 (0.0183)	0.0106 (0.0220)	
20	-0.0066 (0.0098)	-0.0080 (0.0102)	0.0049 (0.0078)	-0.0045 (0.0104)	0.0004 (0.0123)	0.0019 (0.0099)	0.0091 (0.0122)	0.0108 (0.0128)	0.0042 (0.0181)	-0.0020 (0.0196)	
10	-0.0007 (0.0086)	-0.0053 (0.0079)	-0.0035 (0.0104)	-0.0029 (0.0095)	-0.0063 (0.0085)	-0.0031 (0.0115)	-0.0022 (0.0134)	0.0143 (0.0210)	-0.0051 (0.0245)	-0.0099 (0.0297)	
0	-0.0012 (0.0069)	-0.0021 (0.0087)	0.0030 (0.0105)	-0.0023 (0.0103)	0.0023 (0.0080)	-0.0060 (0.0109)	-0.0130 (0.0139)	-0.0321 (0.0206)	0.0047 (0.0271)	-0.0069 (0.0189)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	-0.0052 (0.0214)	-0.0108 (0.0233)	-0.0050 (0.0195)	-0.0146 (0.0225)	0.0085 (0.0237)	0.0064 (0.0305)	-0.0219 (0.0348)	0.0020 (0.0299)	0.0200 (0.0297)	-0.0080 (0.0289)	
20	0.0100 (0.0229)	-0.0060 (0.0253)	-0.0280 (0.0281)	0.0274 (0.0325)	0.0125 (0.0321)	0.0074 (0.0314)	0.0103 (0.0334)	-0.0313 (0.0521)	-0.0068 (0.0472)	-0.0357 (0.0460)	
10	0.0006 (0.0284)	-0.0002 (0.0354)	0.0174 (0.0463)	-0.0030 (0.0496)	0.0102 (0.0453)	0.0087 (0.0541)	-0.0089 (0.0504)	0.0690 (0.0717)	-0.0494 (0.0987)	-0.0816 (0.1199)	
0	-0.0116 (0.0167)	-0.0167 (0.0351)	0.0084 (0.0134)	-0.0106 (0.0288)	-0.0034 (0.0326)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0225 (0.0292)	-0.0024 (0.0335)	0.0244 (0.0273)	0.0134 (0.0343)	-0.0337 (0.0401)	0.0057 (0.0323)	-0.0018 (0.0359)	0.0011 (0.0444)	0.0213 (0.0405)	-0.0087 (0.0378)	0.0116 (0.0392)
20	-0.0195 (0.0452)	-0.0466 (0.0612)	-0.0479 (0.0663)	0.0098 (0.0705)	0.0757 (0.1520)	0.0223 (0.0496)	-0.0247 (0.0432)	-0.0283 (0.0837)	-0.0280 (0.0609)	-0.0358 (0.0945)	-0.0189 (0.0851)
10	0.0890 (0.1501)	0.0572 (0.1065)	0.1192 (0.1233)	0.1005 (0.2613)	0.0877 (0.1797)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.2: Bornholdt's model with strategy: Skewness

30	4.0131 (0.1023)	3.77 (0.0897)	3.18 (0.0687)	4.0005 (0.1073)	4.0379 (0.1119)	4.93 (0.1269)	4.06 (0.1528)	4.15 (0.2082)	5.1685 (0.1817)	5.43 (0.2194)	
20	3.60 (0.0975)	4.0128 (0.1048)	3.67 (0.0908)	3.39 (0.0963)	4.1839 (0.1212)	4.10 (0.1072)	4.16 (0.1739)	4.11 (0.1648)	5.80 (0.1924)	5.35 (0.1920)	
10	3.49 (0.0989)	3.89 (0.0980)	3.29 (0.0605)	3.84 (0.1294)	4.0385 (0.1306)	4.85 (0.1198)	4.00 (0.1120)	5.0793 (0.1618)	5.77 (0.2591)	5.56 (0.2843)	
0	3.87 (0.0726)	3.98 (0.1031)	3.43 (0.1074)	4.0690 (0.1340)	3.37 (0.1167)	4.36 (0.1641)	4.61 (0.1864)	5.16 (0.3881)	4.75 (0.2711)	4.90 (0.1828)	
$\alpha \backslash \beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	5.79 (0.2501)	5.87 (0.2178)	5.32 (0.1915)	5.09 (0.2125)	5.00 (0.1851)	6.0420 (0.2934)	6.1568 (0.2479)	6.75 (0.2069)	6.77 (0.2156)	6.45 (0.3601)	
20	5.57 (0.1406)	5.82 (0.2353)	6.63 (0.2884)	6.93 (0.3282)	6.68 (0.2597)	6.87 (0.2084)	6.91 (0.3494)	7.0170 (0.3195)	7.0519 (0.4027)	7.34 (0.4398)	
10	6.1397 (0.2274)	6.59 (0.3927)	6.14 (0.3666)	6.12 (0.4363)	7.0567 (0.3976)	7.69 (0.5575)	7.75 (0.4975)	8.0046 (0.5091)	8.14 (0.5697)	8.1627 (0.8386)	
0	4.29 (0.2404)	4.55 (0.2720)	4.81 (0.2748)	5.27 (0.4725)	5.12 (0.3944)	× ×	× ×	× ×	× ×	× ×	
$\alpha \backslash \beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	6.12 (0.3355)	6.47 (0.3852)	6.09 (0.3014)	6.12 (0.3182)	6.24 (0.4076)	6.15 (0.3221)	6.26 (0.2612)	6.64 (0.3451)	6.05 (0.3095)	7.0215 (0.3642)	6.95 (0.4463)
20	7.30 (0.4906)	7.53 (0.4267)	7.26 (0.4996)	8.0057 (0.6976)	7.75 (1.1547)	7.26 (0.6772)	7.0570 (0.3265)	7.03 (0.6768)	7.95 (0.4515)	7.47 (0.6255)	7.39 (0.5240)
10	8.24 (1.0221)	8.47 (0.9909)	9.55 (1.1705)	9.0615 (1.4320)	9.0820 (1.2466)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha \backslash \beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.3: Bornholdt's model with strategy: Maximum of absolute magnetisation $|M(t)|$

30	0.0037 (0.0183)	0.0014 (0.0232)	0.0117 (0.0169)	0.0359 (0.0217)	0.0783 (0.0164)	0.2303 (0.0311)	0.4489 (0.0293)	0.7756 (0.0527)	1.0907 (0.0682)	1.59 (0.0665)	
20	-0.0082 (0.0195)	0.0024 (0.0175)	0.0141 (0.0215)	0.0270 (0.0193)	0.0806 (0.0202)	0.2378 (0.0300)	0.4976 (0.0379)	0.8432 (0.0506)	1.65 (0.0774)	1.74 (0.1446)	
10	-0.0145 (0.0223)	-0.0252 (0.0166)	-0.0024 (0.0162)	0.0143 (0.0184)	0.0529 (0.0212)	0.2391 (0.0202)	0.5537 (0.0473)	1.0853 (0.0844)	1.17 (0.1292)	1.57 (0.2114)	
0	-0.0265 (0.0172)	-0.0082 (0.0169)	-0.0109 (0.0189)	0.0153 (0.0235)	0.0240 (0.0185)	0.2252 (0.0334)	0.5533 (0.0878)	1.52 (0.1843)	0.3193 (0.0691)	0.5247 (0.0576)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	1.02 (0.1160)	1.85 (0.1229)	2.1365 (0.1213)	2.27 (0.1464)	2.76 (0.1820)	2.75 (0.2068)	2.91 (0.1884)	3.0098 (0.2618)	2.49 (0.1969)	3.1600 (0.2617)	
20	1.52 (0.1262)	2.1267 (0.1154)	2.20 (0.1876)	2.03 (0.2479)	2.36 (0.2087)	3.90 (0.2604)	4.1031 (0.4659)	3.14 (0.4764)	4.84 (0.5205)	4.52 (0.9169)	
10	2.29 (0.2121)	2.73 (0.3008)	3.66 (0.4007)	3.39 (0.5302)	4.1676 (0.5335)	4.23 (0.6600)	5.1645 (0.7088)	6.1113 (1.0544)	7.0661 (1.2402)	8.0030 (1.5936)	
0	0.6480 (0.0727)	0.8741 (0.0581)	1.0049 (0.2155)	1.40 (0.2086)	1.23 (0.2410)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	3.1285 (0.2952)	3.01 (0.8514)	3.39 (0.4216)	3.28 (0.3743)	3.38 (0.4596)	3.79 (0.3650)	3.83 (0.2769)	3.24 (0.3691)	3.69 (0.3858)	4.0138 (0.5316)	4.41 (0.6937)
20	4.08 (0.7665)	5.1306 (0.8705)	5.95 (1.2172)	6.77 (1.6681)	8.01 (5.5266)	5.0817 (0.9162)	4.73 (0.6798)	5.77 (1.2595)	5.0871 (0.5858)	5.80 (0.8552)	4.93 (1.0559)
10	7.16 (2.4620)	8.09 (2.3075)	13.5128 (6.0290)	9.12 (5.9783)	11.33 (4.6587)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.4: Bornholdt's model with strategy: Excess kurtosis

30	1.7 (0.5)	2.6 (1.0)	2.0 (0.9)	2.4 (0.8)	4.2 (1.3)	21.6 (5.2)	72.0 (9.0)	211.4 (28.0)	414.0 (50.4)	605.0 (61.8)	
20	2.1 (0.6)	2.0 (0.6)	2.0 (0.6)	2.5 (0.8)	4.9 (1.4)	22.5 (5.0)	88.5 (11.9)	246.7 (30.5)	556.1 (70.0)	1116.4 (189.8)	
10	2.2 (0.8)	1.7 (0.5)	1.9 (0.6)	2.0 (0.6)	3.0 (1.0)	21.7 (3.3)	110.7 (17.6)	418.2 (64.1)	815.4 (140.1)	1434.7 (304.1)	
0	1.5 (0.4)	1.6 (0.6)	2.2 (0.9)	2.7 (0.9)	1.8 (0.9)	21.4 (5.2)	125.6 (31.3)	767.1 (164.6)	38.9 (12.1)	96.8 (19.5)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	1018.9 (135.4)	1251.3 (167.0)	1569.2 (187.0)	1714.5 (227.5)	1937.2 (312.2)	2259.9 (399.2)	2598.0 (364.4)	3221.1 (656.3)	2857.1 (380.1)	3530.1 (636.0)	
20	1313.4 (175.5)	1554.4 (172.6)	2297.7 (338.4)	3165.8 (531.3)	2662.3 (425.9)	3896.0 (648.0)	6221.5 (1539.4)	5832.2 (1504.9)	6849.3 (1573.5)	10419.0 (4865.5)	
10	2091.7 (351.5)	2816.2 (700.9)	4198.2 (1092.9)	5542.2 (1947.2)	6591.7 (1609.3)	8342.8 (2409.5)	10291.4 (2451.6)	15434.9 (5909.3)	19697.2 (7353.5)	24788.8 (9402.6)	
0	148.5 (30.3)	261.7 (34.0)	382.9 (115.2)	652.9 (161.3)	834.3 (201.3)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	3507.1 (675.6)	6152.4 (5095.9)	4450.5 (1287.7)	4191.3 (1054.0)	5603.8 (1458.3)	4611.6 (989.2)	4494.0 (697.5)	4868.2 (1062.8)	5386.9 (1035.4)	6008.8 (1759.6)	7207.7 (2530.5)
20	8469.8 (2980.5)	10744.1 (4600.5)	15555.4 (7491.0)	19462.6 (10850.9)	95091.8 (161495.3)	10177.6 (4108.9)	8733.0 (2694.3)	12446.8 (6940.6)	9175.0 (2077.2)	9827.7 (3313.0)	8979.5 (3846.4)
10	33858.7 (25285.0)	30679.2 (17102.7)	98986.7 (90039.3)	72537.8 (100208.1)	66089.5 (46205.6)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.5: Bornholdt's model with strategy: Jarque-Berra test - test statistic

α\β	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
30	0.5193 (0.0857)	0.4953 (0.1218)	0.5665 (0.1103)	0.4394 (0.1003)	0.3023 (0.0983)	0.0164 (0.0182)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.4643 (0.1038)	0.4667 (0.0894)	0.4875 (0.0994)	0.4452 (0.1027)	0.2420 (0.0964)	0.0167 (0.0128)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.4796 (0.1121)	0.5128 (0.0829)	0.5203 (0.1027)	0.4663 (0.0868)	0.4028 (0.1065)	0.0039 (0.0041)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
0	0.5153 (0.0733)	0.5658 (0.1018)	0.4961 (0.0974)	0.4070 (0.1032)	0.5543 (0.0892)	0.0384 (0.0480)	0.0117 (0.0197)	0.0002 (0.0003)	0.0113 (0.0220)	0.0000 (0.0000)
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**

α\β	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
0	0.0000 (0.0000)	0.0000 (0.0000)	0.0002 (0.0004)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**

α\β	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.6: Bornholdt's model with strategy: Jarque-Berra test - p-value

30	90.66 (5.24)	56.12 (4.91)	31.00 (3.22)	23.41 (2.42)	18.42 (2.26)	17.86 (2.51)	16.22 (1.85)	15.59 (2.79)	15.11 (2.33)	15.69 (3.23)	
20	93.07 (4.11)	51.84 (3.40)	28.13 (2.29)	17.45 (2.71)	11.71 (1.69)	11.12 (1.65)	14.20 (2.19)	14.20 (1.91)	18.23 (2.79)	22.45 (3.40)	
10	85.79 (4.34)	50.10 (4.23)	24.12 (2.51)	12.68 (1.97)	8.58 (1.42)	12.9 (1.69)	17.17 (3.02)	26.8 (4.27)	31.87 (4.62)	41.06 (8.46)	
0	87.46 (3.55)	47.84 (3.44)	18.55 (2.02)	10.21 (1.30)	9.30 (1.58)	18.61 (2.96)	26.89 (5.41)	46.00 (6.82)	10.71 (2.36)	10.27 (2.19)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	19.39 (3.78)	19.70 (4.32)	23.7 (4.16)	20.6 (3.42)	24.90 (3.41)	28.46 (3.87)	30.20 (5.02)	30.58 (4.94)	31.1 (5.54)	30.11 (3.66)	
20	29.39 (4.51)	32.09 (4.48)	34.80 (5.30)	42.34 (7.97)	39.06 (7.65)	46.30 (6.75)	58.00 (8.15)	58.13 (9.14)	61.07 (9.38)	65.48 (11.76)	
10	48.27 (8.61)	63.91 (12.21)	64.68 (8.15)	69.11 (6.84)	77.52 (14.52)	78.76 (9.63)	75.32 (11.24)	97.81 (15.99)	103.43 (21.34)	93.00 (20.93)	
0	9.29 (2.27)	9.26 (2.21)	9.70 (2.83)	8.15 (2.30)	9.43 (3.27)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	28.31 (4.92)	33.65 (6.48)	28.63 (4.41)	29.72 (5.66)	37.61 (6.83)	36.87 (6.11)	38.48 (6.60)	34.36 (6.61)	36.03 (5.48)	34.15 (7.01)	36.55 (8.75)
20	64.21 (11.05)	72.65 (14.61)	63.49 (12.61)	68.49 (12.62)	85.73 (26.40)	70.36 (15.66)	71.72 (15.58)	73.39 (14.53)	67.25 (14.25)	81.80 (26.39)	70.41 (19.94)
10	90.17 (23.50)	106.99 (21.21)	129.44 (44.32)	131.44 (43.31)	130.52 (30.98)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.7: Bornholdt's model with strategy: Ljung-Box test - test statistic

30	0.0000 (0.0000)	0.0000 (0.0000)	0.0075 (0.0065)	0.0361 (0.0214)	0.1074 (0.0621)	0.1521 (0.0818)	0.1336 (0.0565)	0.2411 (0.0970)	0.2080 (0.0900)	0.2571 (0.0985)	
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0055 (0.0034)	0.1804 (0.0841)	0.3324 (0.0927)	0.3595 (0.1003)	0.2239 (0.0781)	0.2172 (0.0782)	0.1531 (0.0764)	0.0996 (0.0652)	
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0337 (0.0238)	0.3098 (0.0955)	0.5249 (0.1025)	0.3079 (0.0951)	0.1830 (0.0795)	0.0716 (0.0552)	0.0422 (0.0527)	0.0263 (0.0336)	
0	0.0000 (0.0000)	0.0001 (0.0002)	0.0782 (0.0322)	0.3880 (0.0977)	0.4859 (0.1005)	0.1498 (0.0884)	0.1439 (0.0856)	0.0430 (0.0347)	0.3497 (0.1587)	0.3816 (0.1575)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.1743 (0.0817)	0.2021 (0.1001)	0.1139 (0.0631)	0.1200 (0.0568)	0.0395 (0.0305)	0.0249 (0.0191)	0.0449 (0.0342)	0.0240 (0.0195)	0.0746 (0.0734)	0.0057 (0.0042)	
20	0.0348 (0.0301)	0.0270 (0.0336)	0.0392 (0.0497)	0.0209 (0.0326)	0.0244 (0.0204)	0.0053 (0.0064)	0.0011 (0.0015)	0.0015 (0.0014)	0.0003 (0.0006)	0.0024 (0.0046)	
10	0.0013 (0.0016)	0.0017 (0.0030)	0.0014 (0.0026)	0.0000 (0.0000)	0.0001 (0.0001)	0.0000 (0.0000)	0.0004 (0.0008)	0.0000 (0.0001)	0.0000 (0.0001)	0.0049 (0.0097)	
0	0.4623 (0.1662)	0.4511 (0.1587)	0.4575 (0.1959)	0.5530 (0.1489)	0.4978 (0.2163)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0270 (0.0156)	0.0276 (0.0225)	0.0369 (0.0276)	0.0341 (0.0205)	0.0180 (0.0166)	0.0233 (0.0305)	0.0236 (0.0230)	0.0318 (0.0322)	0.0087 (0.0085)	0.0133 (0.0111)	0.0232 (0.0215)
20	0.0018 (0.0034)	0.0006 (0.0011)	0.0004 (0.0007)	0.0001 (0.0001)	0.0000 (0.0000)	0.0133 (0.0258)	0.0010 (0.0020)	0.0000 (0.0000)	0.0002 (0.0002)	0.0002 (0.0003)	0.0000 (0.0000)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0003 (0.0006)	0.0005 (0.0009)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.8: Bornholdt's model with strategy: Ljung-Box test - p-value

30	-0.0471 (0.0039)	-0.0313 (0.0027)	-0.0153 (0.0034)	-0.0112 (0.0044)	-0.0096 (0.0038)	-0.0049 (0.0033)	0.0050 (0.0037)	0.0062 (0.0033)	0.0082 (0.0040)	0.0126 (0.0042)
20	-0.0459 (0.0038)	-0.0300 (0.0039)	-0.0151 (0.0037)	-0.0052 (0.0044)	0.0027 (0.0042)	0.0057 (0.0037)	0.0117 (0.0053)	0.0172 (0.0040)	0.0235 (0.0052)	0.0235 (0.0053)
10	-0.0451 (0.0033)	-0.0282 (0.0038)	-0.0109 (0.0040)	-0.0034 (0.0039)	0.0022 (0.0032)	0.0081 (0.0036)	0.0172 (0.0036)	0.0234 (0.0054)	0.0310 (0.0042)	0.0322 (0.0054)
0	-0.0453 (0.0037)	-0.0304 (0.0035)	-0.0141 (0.0034)	-0.0083 (0.0036)	-0.0026 (0.0048)	-0.0108 (0.0038)	-0.0182 (0.0047)	-0.0268 (0.0050)	-0.0062 (0.0062)	-0.0050 (0.0055)
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**

30	0.0121 (0.0043)	0.0175 (0.0038)	0.0201 (0.0053)	0.0179 (0.0048)	0.0257 (0.0046)	0.0288 (0.0052)	0.0292 (0.0062)	0.0249 (0.0055)	0.0291 (0.0074)	0.0291 (0.0043)
20	0.0307 (0.0057)	0.0290 (0.0050)	0.0299 (0.0050)	0.0369 (0.0058)	0.0315 (0.0064)	0.0384 (0.0059)	0.0448 (0.0049)	0.0403 (0.0055)	0.0466 (0.0077)	0.0430 (0.0068)
10	0.0322 (0.0058)	0.0422 (0.0064)	0.0409 (0.0054)	0.0439 (0.0061)	0.0474 (0.0055)	0.0495 (0.0065)	0.0433 (0.0064)	0.0524 (0.0070)	0.0532 (0.0091)	0.0437 (0.0102)
0	-0.0062 (0.0048)	-0.0002 (0.0083)	-0.0006 (0.0066)	0.0000 (0.0041)	-0.0007 (0.0077)	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**

30	0.0271 (0.0059)	0.0305 (0.0062)	0.0240 (0.0058)	0.0310 (0.0058)	0.0269 (0.0059)	0.0321 (0.0060)	0.0331 (0.0063)	0.0290 (0.0085)	0.0274 (0.0054)	0.0324 (0.0063)	0.0323 (0.0065)
20	0.0467 (0.0068)	0.0495 (0.0045)	0.0442 (0.0076)	0.0400 (0.0086)	0.0468 (0.0072)	0.0469 (0.0079)	0.0422 (0.0075)	0.0477 (0.0119)	0.0443 (0.0090)	0.0460 (0.0070)	0.0423 (0.0154)
10	0.0427 (0.0072)	0.0551 (0.0098)	0.0419 (0.0148)	0.0603 (0.0141)	0.0573 (0.0126)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.9: Bornholdt's model with strategy: AR(1) parameter

30	0.0008 (0.0009)	0.0102 (0.0059)	0.1500 (0.0504)	0.1801 (0.0520)	0.2176 (0.0558)	0.2304 (0.0452)	0.2571 (0.0544)	0.2306 (0.0476)	0.1977 (0.0515)	0.1974 (0.0577)	
20	0.0014 (0.0012)	0.0196 (0.0105)	0.1492 (0.0555)	0.1979 (0.0487)	0.2539 (0.0581)	0.2628 (0.0521)	0.1310 (0.0481)	0.1199 (0.0542)	0.0727 (0.0436)	0.1153 (0.0611)	
10	0.0007 (0.0006)	0.0385 (0.0250)	0.1806 (0.0540)	0.2425 (0.0522)	0.2738 (0.0475)	0.1900 (0.0434)	0.1233 (0.0500)	0.0768 (0.0441)	0.0338 (0.0273)	0.0327 (0.0268)	
0	0.0007 (0.0005)	0.0187 (0.0111)	0.1351 (0.0441)	0.2359 (0.0560)	0.2172 (0.0535)	0.2049 (0.0582)	0.1197 (0.0457)	0.0715 (0.0413)	0.2417 (0.0956)	0.2277 (0.0712)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.1682 (0.0510)	0.1260 (0.0515)	0.1221 (0.0480)	0.1035 (0.0458)	0.0562 (0.0318)	0.0392 (0.0299)	0.0425 (0.0270)	0.0851 (0.0488)	0.0709 (0.0370)	0.0343 (0.0201)	
20	0.0461 (0.0339)	0.0306 (0.0202)	0.0497 (0.0347)	0.0270 (0.0317)	0.0745 (0.0457)	0.0278 (0.0230)	0.0034 (0.0035)	0.0124 (0.0116)	0.0110 (0.0109)	0.0028 (0.0032)	
10	0.0389 (0.0246)	0.0111 (0.0095)	0.0056 (0.0054)	0.0243 (0.0306)	0.0035 (0.0053)	0.0104 (0.0150)	0.0204 (0.0264)	0.0179 (0.0225)	0.0051 (0.0063)	0.0144 (0.0138)	
0	0.2292 (0.0725)	0.2043 (0.0901)	0.2974 (0.0995)	0.3274 (0.0699)	0.2285 (0.0803)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0638 (0.0387)	0.0586 (0.0361)	0.0778 (0.0399)	0.0457 (0.0244)	0.0804 (0.0507)	0.0532 (0.0389)	0.0420 (0.0359)	0.0943 (0.0562)	0.0593 (0.0472)	0.0330 (0.0213)	0.0421 (0.0380)
20	0.0172 (0.0245)	0.0012 (0.0019)	0.0147 (0.0128)	0.0463 (0.0511)	0.0097 (0.0117)	0.0029 (0.0022)	0.0223 (0.0204)	0.0376 (0.0521)	0.0088 (0.0108)	0.0012 (0.0012)	0.0710 (0.0993)
10	0.0283 (0.0340)	0.0125 (0.0229)	0.0491 (0.0548)	0.0111 (0.0211)	0.0044 (0.0085)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.10: Bornholdt's model with strategy: P-value of AR(1) parameter

30	-0.2171 (0.0067)	-0.1581 (0.0064)	-0.1076 (0.0055)	-0.0845 (0.0055)	-0.0705 (0.0054)	-0.0622 (0.0040)	-0.0510 (0.0044)	-0.0339 (0.0059)	-0.0334 (0.0037)	-0.0234 (0.0051)	
20	-0.2206 (0.0059)	-0.1460 (0.0053)	-0.0948 (0.0048)	-0.0634 (0.0052)	-0.0453 (0.0044)	-0.0283 (0.0049)	-0.0180 (0.0051)	-0.0062 (0.0044)	0.0033 (0.0043)	0.0079 (0.0057)	
10	-0.2107 (0.0058)	-0.1435 (0.0060)	-0.0829 (0.0044)	-0.0427 (0.0046)	-0.0189 (0.0040)	0.0012 (0.0051)	0.0106 (0.0059)	0.0229 (0.0065)	0.0259 (0.0066)	0.0410 (0.0065)	
0	-0.2134 (0.0041)	-0.1361 (0.0049)	-0.0706 (0.0050)	-0.0287 (0.0039)	-0.0108 (0.0053)	-0.0517 (0.0084)	-0.0675 (0.0152)	-0.0994 (0.0169)	-0.0088 (0.0084)	-0.0046 (0.0097)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	-0.0185 (0.0055)	-0.0096 (0.0054)	-0.0046 (0.0058)	-0.0031 (0.0055)	-0.0023 (0.0058)	-0.0001 (0.0063)	0.0049 (0.0059)	0.0071 (0.0068)	0.0048 (0.0059)	0.0094 (0.0052)	
20	0.0212 (0.0048)	0.0229 (0.0068)	0.0249 (0.0069)	0.0330 (0.0061)	0.0299 (0.0063)	0.0343 (0.0073)	0.0433 (0.0071)	0.0469 (0.0072)	0.0437 (0.0062)	0.0492 (0.0080)	
10	0.0493 (0.0063)	0.0574 (0.0067)	0.0562 (0.0062)	0.0590 (0.0057)	0.0617 (0.0084)	0.0651 (0.0057)	0.0630 (0.0069)	0.0688 (0.0073)	0.0707 (0.0106)	0.0723 (0.0137)	
0	-0.0090 (0.0068)	0.0000 (0.0082)	-0.0016 (0.0064)	0.0019 (0.0074)	-0.0020 (0.0080)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0018 (0.0061)	0.0083 (0.0067)	0.0084 (0.0070)	0.0049 (0.0064)	0.0114 (0.0105)	0.0105 (0.0079)	0.0169 (0.0071)	0.0110 (0.0062)	0.0158 (0.0069)	0.0135 (0.0061)	0.0215 (0.0094)
20	0.0468 (0.0076)	0.0542 (0.0094)	0.0422 (0.0076)	0.0489 (0.0096)	0.0561 (0.0117)	0.0456 (0.0103)	0.0560 (0.0109)	0.0481 (0.0083)	0.0503 (0.0117)	0.0601 (0.0180)	0.0458 (0.0144)
10	0.0703 (0.0119)	0.0670 (0.0127)	0.0628 (0.0227)	0.0834 (0.0172)	0.0735 (0.0204)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.11: Bornholdt's model with strategy: Local Whittle estimate of Fractional integration parameter d

30	0.0013 (0.0041)	0.0001 (0.0038)	-0.0004 (0.0036)	0.0029 (0.0037)	0.0144 (0.0049)	0.0320 (0.0038)	0.0591 (0.0044)	0.0977 (0.0067)	0.1252 (0.0070)	0.1476 (0.0055)	
20	-0.0024 (0.0044)	-0.0001 (0.0037)	0.0059 (0.0032)	0.0009 (0.0037)	0.0122 (0.0034)	0.0356 (0.0049)	0.0656 (0.0045)	0.1082 (0.0062)	0.1403 (0.0068)	0.1739 (0.0089)	
10	0.0017 (0.0033)	-0.0001 (0.0039)	-0.0004 (0.0047)	0.0019 (0.0043)	0.0061 (0.0038)	0.0286 (0.0041)	0.0705 (0.0041)	0.1177 (0.0081)	0.1567 (0.0106)	0.1758 (0.0105)	
0	0.0023 (0.0037)	-0.0003 (0.0043)	0.0010 (0.0027)	0.0009 (0.0046)	0.0048 (0.0042)	0.0156 (0.0038)	0.0367 (0.0114)	0.0547 (0.0115)	0.0139 (0.0074)	0.0130 (0.0040)	
$\alpha \backslash \beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.1765 (0.0072)	0.1904 (0.0088)	0.2063 (0.0084)	0.2129 (0.0094)	0.2248 (0.0075)	0.2313 (0.0090)	0.2363 (0.0089)	0.2478 (0.0090)	0.2458 (0.0122)	0.2489 (0.0114)	
20	0.1909 (0.0085)	0.2055 (0.0067)	0.2289 (0.0094)	0.2364 (0.0088)	0.2444 (0.0105)	0.2631 (0.0106)	0.2733 (0.0164)	0.2805 (0.0163)	0.2863 (0.0166)	0.2972 (0.0165)	
10	0.2043 (0.0116)	0.2135 (0.0156)	0.2446 (0.0134)	0.2622 (0.0163)	0.2635 (0.0185)	0.2859 (0.0178)	0.2837 (0.0280)	0.3140 (0.0215)	0.3417 (0.0213)	0.3462 (0.0242)	
0	0.0142 (0.0057)	0.0239 (0.0057)	0.0267 (0.0105)	0.0400 (0.0104)	0.0390 (0.0117)	× ×	× ×	× ×	× ×	× ×	
$\alpha \backslash \beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.2625 (0.0139)	0.2687 (0.0222)	0.2719 (0.0176)	0.2668 (0.0131)	0.2816 (0.0155)	0.2787 (0.0164)	0.2761 (0.0126)	0.2788 (0.0145)	0.2914 (0.0157)	0.2853 (0.0165)	0.3019 (0.0202)
20	0.3035 (0.0213)	0.3105 (0.0181)	0.3373 (0.0230)	0.3356 (0.0294)	0.3287 (0.0280)	0.3095 (0.0187)	0.3106 (0.0144)	0.3290 (0.0222)	0.3303 (0.0197)	0.3292 (0.0295)	0.3229 (0.0336)
10	0.2972 (0.0411)	0.3519 (0.0268)	0.3807 (0.0535)	0.3313 (0.0644)	0.3650 (0.0721)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha \backslash \beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.12: Bornholdt's model with strategy: AR(1) parameter for $|r_t|$

30	0.2486 (0.0513)	0.2561 (0.0503)	0.2299 (0.0404)	0.2608 (0.0519)	0.1353 (0.0477)	0.0280 (0.0314)	0.0000 (0.0001)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.1968 (0.0432)	0.2573 (0.0466)	0.2249 (0.0453)	0.2324 (0.0424)	0.1720 (0.0501)	0.0211 (0.0272)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.2733 (0.0522)	0.2322 (0.0447)	0.2359 (0.0519)	0.2251 (0.0545)	0.2559 (0.0511)	0.0360 (0.0238)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.2531 (0.0499)	0.2193 (0.0526)	0.3004 (0.0405)	0.2061 (0.0502)	0.1982 (0.0504)	0.1478 (0.0503)	0.0920 (0.0491)	0.0314 (0.0342)	0.1457 (0.1058)	0.1502 (0.0764)	
$\alpha \backslash \beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.1016 (0.0536)	0.0412 (0.0394)	0.0697 (0.0573)	0.0275 (0.0484)	0.0253 (0.0391)	× ×	× ×	× ×	× ×	× ×	
$\alpha \backslash \beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha \backslash \beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.13: Bornholdt's model with strategy: P-value of AR(1) parameter for $|r_t|$

30	0.0008 (0.0044)	0.0035 (0.0042)	0.0051 (0.0041)	0.0111 (0.0048)	0.0379 (0.0049)	0.0796 (0.0046)	0.1240 (0.0040)	0.1710 (0.0039)	0.1975 (0.0056)	0.2157 (0.0043)	
20	0.0023 (0.0038)	0.0009 (0.0041)	0.0026 (0.0049)	0.0062 (0.0041)	0.0364 (0.0041)	0.0871 (0.0054)	0.1362 (0.0051)	0.1772 (0.0045)	0.2062 (0.0037)	0.2284 (0.0051)	
10	0.0013 (0.0034)	0.0039 (0.0048)	0.0022 (0.0044)	0.0037 (0.0043)	0.0276 (0.0040)	0.0836 (0.0045)	0.1409 (0.0044)	0.1833 (0.0056)	0.2077 (0.0042)	0.2197 (0.0036)	
0	0.0052 (0.0039)	0.0006 (0.0048)	-0.0015 (0.0043)	0.0023 (0.0044)	0.0075 (0.0048)	0.0557 (0.0081)	0.0750 (0.0161)	0.1015 (0.0168)	0.0191 (0.0099)	0.0405 (0.0063)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.2310 (0.0044)	0.2422 (0.0036)	0.2501 (0.0052)	0.2562 (0.0047)	0.2649 (0.0045)	0.2631 (0.0045)	0.2702 (0.0043)	0.2747 (0.0052)	0.2770 (0.0055)	0.2768 (0.0044)	
20	0.2402 (0.0038)	0.2497 (0.0042)	0.2567 (0.0043)	0.2644 (0.0044)	0.2614 (0.0048)	0.2717 (0.0045)	0.2787 (0.0061)	0.2786 (0.0047)	0.2829 (0.0052)	0.2863 (0.0054)	
10	0.2356 (0.0057)	0.2376 (0.0066)	0.2514 (0.0051)	0.2528 (0.0072)	0.2555 (0.0085)	0.2625 (0.0077)	0.2600 (0.0128)	0.2742 (0.0077)	0.2816 (0.0073)	0.2772 (0.0079)	
0	0.0419 (0.0076)	0.0687 (0.0069)	0.0667 (0.0224)	0.0948 (0.0152)	0.0959 (0.0195)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.2805 (0.0055)	0.2820 (0.0064)	0.2847 (0.0062)	0.2813 (0.0041)	0.2888 (0.0056)	0.2882 (0.0056)	0.2878 (0.0053)	0.2892 (0.0053)	0.2908 (0.0055)	0.2893 (0.0064)	0.2946 (0.0065)
20	0.2860 (0.0053)	0.2894 (0.0061)	0.2938 (0.0063)	0.2946 (0.0085)	0.2979 (0.0083)	0.2919 (0.0067)	0.2932 (0.0043)	0.2983 (0.0066)	0.2996 (0.0076)	0.3012 (0.0090)	0.2951 (0.0074)
10	0.2623 (0.0183)	0.2888 (0.0075)	0.2927 (0.0173)	0.2786 (0.0225)	0.2877 (0.0258)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.14: Bornholdt's model with strategy: Local Whittle estimate of Fractional integration parameter d for $|r_t|$

30	0.0091 (0.0035)	0.0064 (0.0022)	0.0073 (0.0026)	0.0100 (0.0029)	0.0192 (0.0035)	0.0297 (0.0019)	0.0429 (0.0022)	0.0560 (0.0025)	0.0621 (0.0023)	0.0698 (0.0020)
20	0.0062 (0.0027)	0.0072 (0.0020)	0.0089 (0.0024)	0.0069 (0.0021)	0.0149 (0.0021)	0.0283 (0.0017)	0.0410 (0.0021)	0.0500 (0.0020)	0.0586 (0.0019)	0.0653 (0.0027)
10	0.0069 (0.0023)	0.0073 (0.0023)	0.0078 (0.0047)	0.0092 (0.0028)	0.0113 (0.0030)	0.0204 (0.0012)	0.0310 (0.0014)	0.0374 (0.0012)	0.0416 (0.0018)	0.0446 (0.0018)
0	0.0085 (0.0027)	0.0100 (0.0039)	0.0068 (0.0022)	0.0081 (0.0032)	0.0095 (0.0024)	0.0072 (0.0014)	0.0125 (0.0025)	0.0193 (0.0023)	0.0081 (0.0039)	0.0122 (0.0029)
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**

30	0.0715 (0.0032)	0.0785 (0.0027)	0.0806 (0.0024)	0.0792 (0.0027)	0.0855 (0.0033)	0.0845 (0.0031)	0.0867 (0.0032)	0.0868 (0.0037)	0.0877 (0.0029)	0.0867 (0.0041)
20	0.0671 (0.0023)	0.0715 (0.0039)	0.0733 (0.0037)	0.0738 (0.0026)	0.0746 (0.0029)	0.0758 (0.0037)	0.0792 (0.0040)	0.0831 (0.0045)	0.0800 (0.0028)	0.0811 (0.0037)
10	0.0477 (0.0029)	0.0509 (0.0026)	0.0514 (0.0031)	0.0523 (0.0037)	0.0526 (0.0033)	0.0603 (0.0055)	0.0532 (0.0048)	0.0579 (0.0040)	0.0598 (0.0049)	0.0571 (0.0061)
0	0.0095 (0.0028)	0.0120 (0.0022)	0.0131 (0.0042)	0.0143 (0.0011)	0.0144 (0.0033)	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**

30	0.0880 (0.0032)	0.0917 (0.0044)	0.0903 (0.0042)	0.0894 (0.0027)	0.0899 (0.0035)	0.0952 (0.0043)	0.0894 (0.0028)	0.0889 (0.0031)	0.0922 (0.0033)	0.0919 (0.0035)	0.0930 (0.0034)
20	0.0833 (0.0039)	0.0813 (0.0034)	0.0832 (0.0051)	0.0817 (0.0044)	0.0826 (0.0040)	0.0818 (0.0036)	0.0828 (0.0038)	0.0837 (0.0042)	0.0855 (0.0043)	0.0929 (0.0087)	0.0855 (0.0047)
10	0.0596 (0.0112)	0.0667 (0.0080)	0.0691 (0.0230)	0.0649 (0.0086)	0.0633 (0.0142)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.15: Bornholdt's model with strategy: Parameter γ_1 of GARCH (1,1) model

30	0.4825 (0.0095)	0.4853 (0.0160)	0.4851 (0.0088)	0.4607 (0.0178)	0.3026 (0.0318)	0.0281 (0.0102)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.4826 (0.0100)	0.4799 (0.0159)	0.4880 (0.0065)	0.4733 (0.0140)	0.2670 (0.0310)	0.0111 (0.0069)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.4911 (0.0050)	0.4830 (0.0084)	0.4755 (0.0193)	0.4677 (0.0267)	0.3290 (0.0417)	0.0041 (0.0048)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.4783 (0.0120)	0.4834 (0.0085)	0.4874 (0.0081)	0.4892 (0.0043)	0.4719 (0.0144)	0.0840 (0.0550)	0.1411 (0.0636)	0.1020 (0.0527)	0.4157 (0.0845)	0.3226 (0.0719)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.2237 (0.0737)	0.0539 (0.0451)	0.0548 (0.0824)	0.0370 (0.0722)	0.0512 (0.0721)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.16: Bornholdt's model with strategy: P-value of parameter γ_1 of GARCH (1,1) model

30	0.5389 (0.1308)	0.4560 (0.1369)	0.5034 (0.1489)	0.6247 (0.1454)	0.8409 (0.0703)	0.9213 (0.0068)	0.9172 (0.0049)	0.9093 (0.0046)	0.9072 (0.0036)	0.8979 (0.0033)	
20	0.6967 (0.1416)	0.4426 (0.1591)	0.3984 (0.1534)	0.5941 (0.1484)	0.9315 (0.0153)	0.9371 (0.0049)	0.9315 (0.0042)	0.9280 (0.0036)	0.9202 (0.0034)	0.9152 (0.0035)	
10	0.5391 (0.1176)	0.6455 (0.1328)	0.5848 (0.1373)	0.5506 (0.1509)	0.8127 (0.1217)	0.9639 (0.0024)	0.9578 (0.0024)	0.9542 (0.0018)	0.9508 (0.0024)	0.9476 (0.0024)	
0	0.5689 (0.1362)	0.5179 (0.1486)	0.4584 (0.1508)	0.4923 (0.1312)	0.4413 (0.1524)	0.9659 (0.0242)	0.8803 (0.0899)	0.9174 (0.0583)	0.5761 (0.3278)	0.8779 (0.1075)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.9016 (0.0050)	0.8916 (0.0045)	0.8906 (0.0040)	0.8948 (0.0038)	0.8859 (0.0049)	0.8869 (0.0050)	0.8846 (0.0050)	0.8863 (0.0057)	0.8844 (0.0045)	0.8863 (0.0065)	
20	0.9141 (0.0031)	0.9088 (0.0056)	0.9084 (0.0053)	0.9080 (0.0036)	0.9068 (0.0042)	0.9069 (0.0052)	0.9033 (0.0056)	0.8972 (0.0063)	0.9021 (0.0033)	0.9016 (0.0048)	
10	0.9452 (0.0035)	0.9413 (0.0034)	0.9416 (0.0037)	0.9406 (0.0045)	0.9404 (0.0039)	0.9311 (0.0067)	0.9400 (0.0054)	0.9346 (0.0050)	0.9330 (0.0060)	0.9362 (0.0074)	
0	0.9014 (0.1001)	0.9722 (0.0084)	0.8880 (0.1743)	0.9537 (0.0471)	0.9708 (0.0111)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.8855 (0.0046)	0.8798 (0.0064)	0.8832 (0.0063)	0.8831 (0.0042)	0.8847 (0.0054)	0.8756 (0.0060)	0.8843 (0.0043)	0.8858 (0.0047)	0.8803 (0.0052)	0.8821 (0.0047)	0.8814 (0.0053)
20	0.8981 (0.0053)	0.9011 (0.0046)	0.8999 (0.0065)	0.9016 (0.0060)	0.9008 (0.0049)	0.9006 (0.0048)	0.8988 (0.0054)	0.8989 (0.0058)	0.8959 (0.0051)	0.8859 (0.0104)	0.8954 (0.0071)
10	0.9315 (0.0140)	0.9241 (0.0096)	0.9206 (0.0308)	0.9256 (0.0097)	0.9286 (0.0164)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.17: Bornholdt's model with strategy: Parameter δ_1 of GARCH (1,1) model

30	0.2276 (0.0766)	0.2987 (0.0858)	0.2823 (0.0882)	0.1624 (0.0827)	0.0253 (0.0316)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.1540 (0.0876)	0.2752 (0.0902)	0.2829 (0.0861)	0.2028 (0.0871)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.2637 (0.0780)	0.1395 (0.0776)	0.2381 (0.0827)	0.2188 (0.0840)	0.0766 (0.0627)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
0	0.2550 (0.0851)	0.2268 (0.0914)	0.2889 (0.0846)	0.2800 (0.0818)	0.2699 (0.0839)	0.0001 (0.0001)	0.0346 (0.0439)	0.0155 (0.0305)	0.1856 (0.1776)	0.0302 (0.0592)
$\alpha \backslash \beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**

30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
0	0.0232 (0.0345)	0.0000 (0.0000)	0.0455 (0.0891)	0.0010 (0.0019)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×
$\alpha \backslash \beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**

30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha \backslash \beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.18: Bornholdt's model with strategy: P-value of parameter δ_1 of GARCH (1,1) model

A.2.2 Plots of realizations for different (α, β)

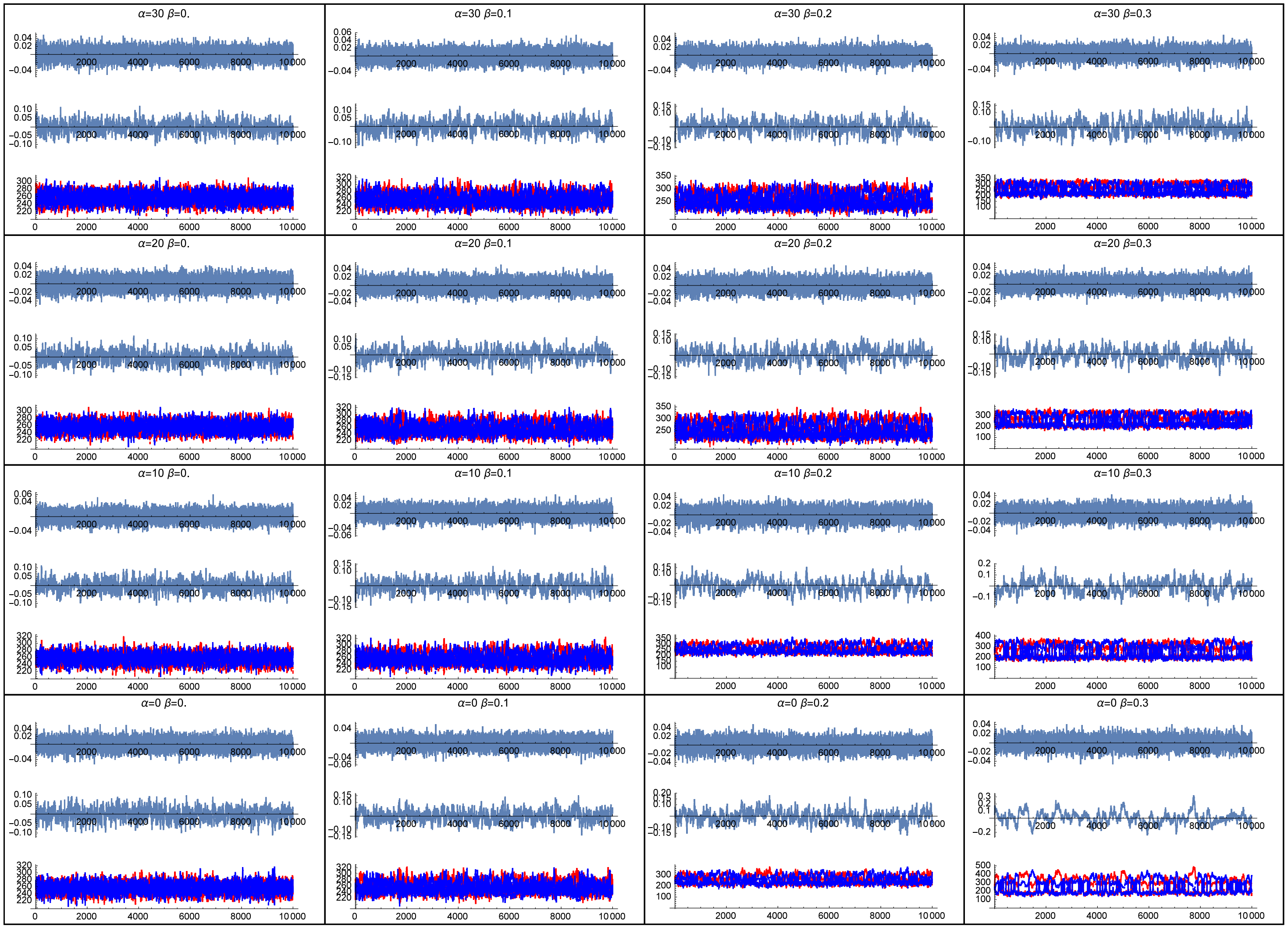


Figure A.1: Bornholdt's model with strategy simulated at $\beta = \{0.0,\ 0.1,\ 0.2,\ 0.3\}$.

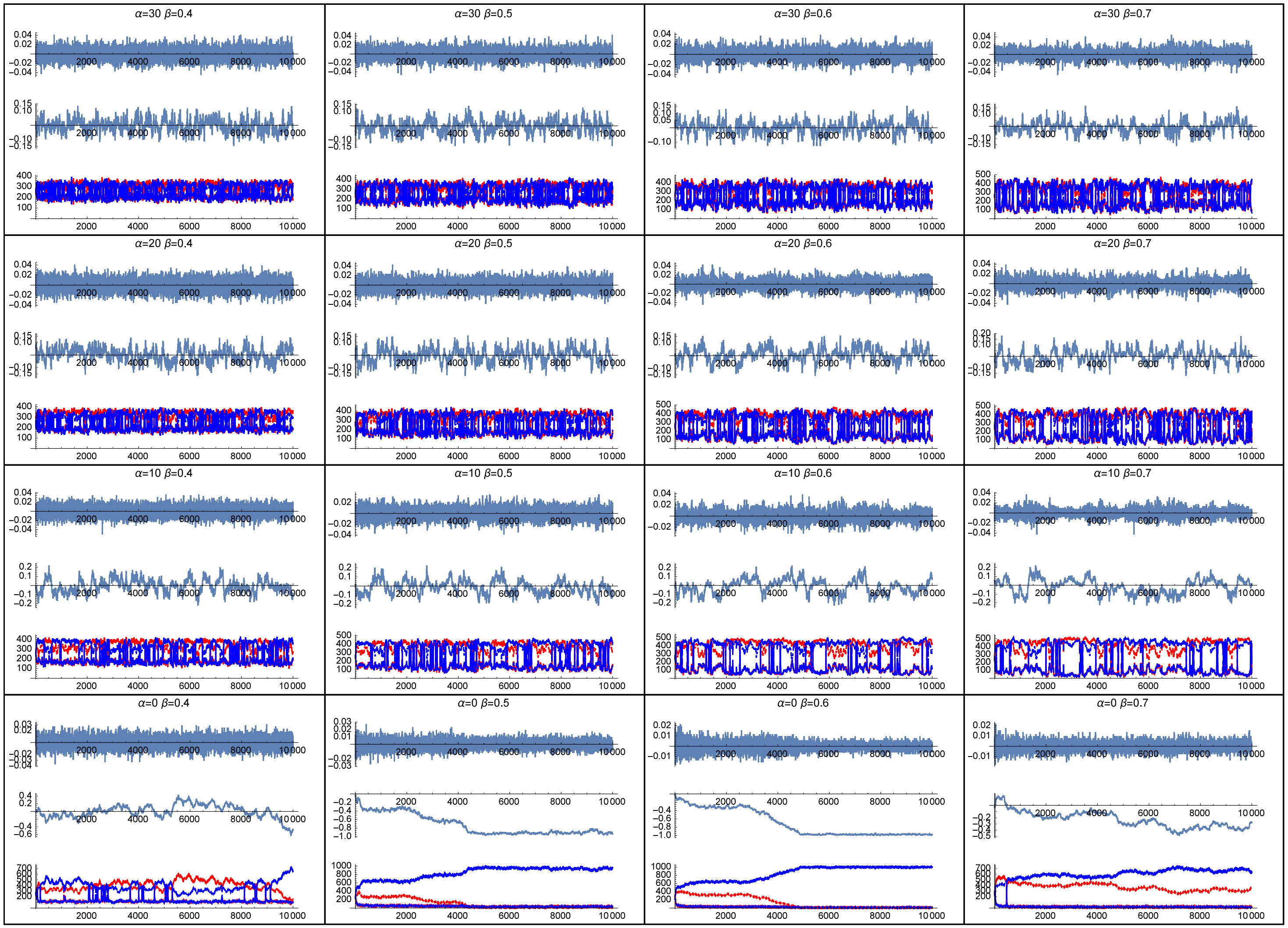


Figure A.2: Bornholdt's model with strategy simulated at $\beta = \{0.4,\ 0.5,\ 0.6,\ 0.7\}$.

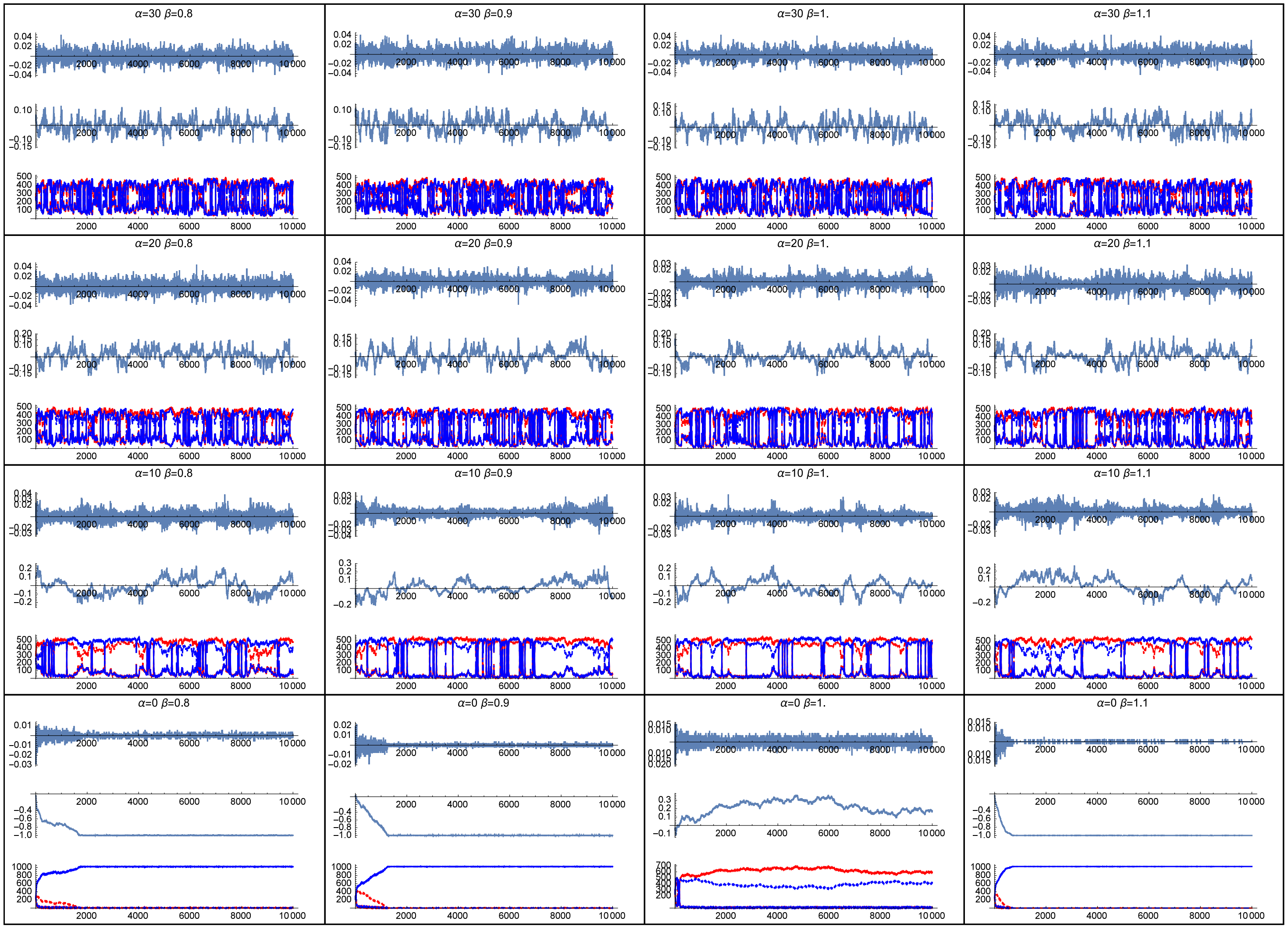


Figure A.3: Bornholdt's model with strategy simulated at $\beta = \{0.8,\ 0.9,\ 1.0,\ 1.1\}$.

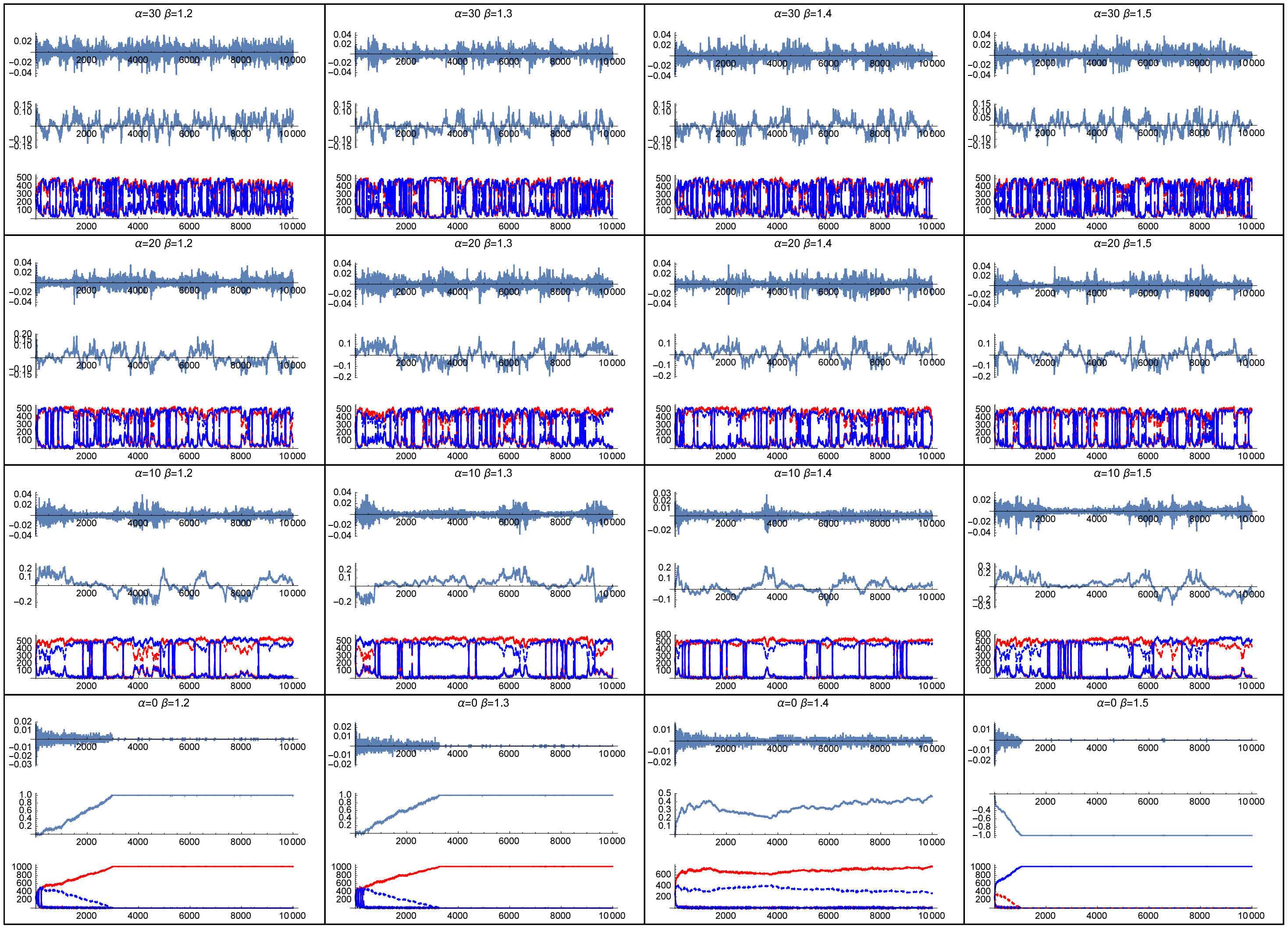


Figure A.4: Bornholdt's model with strategy simulated at $\beta = \{1.2,\ 1.3,\ 1.4,\ 1.5\}$.

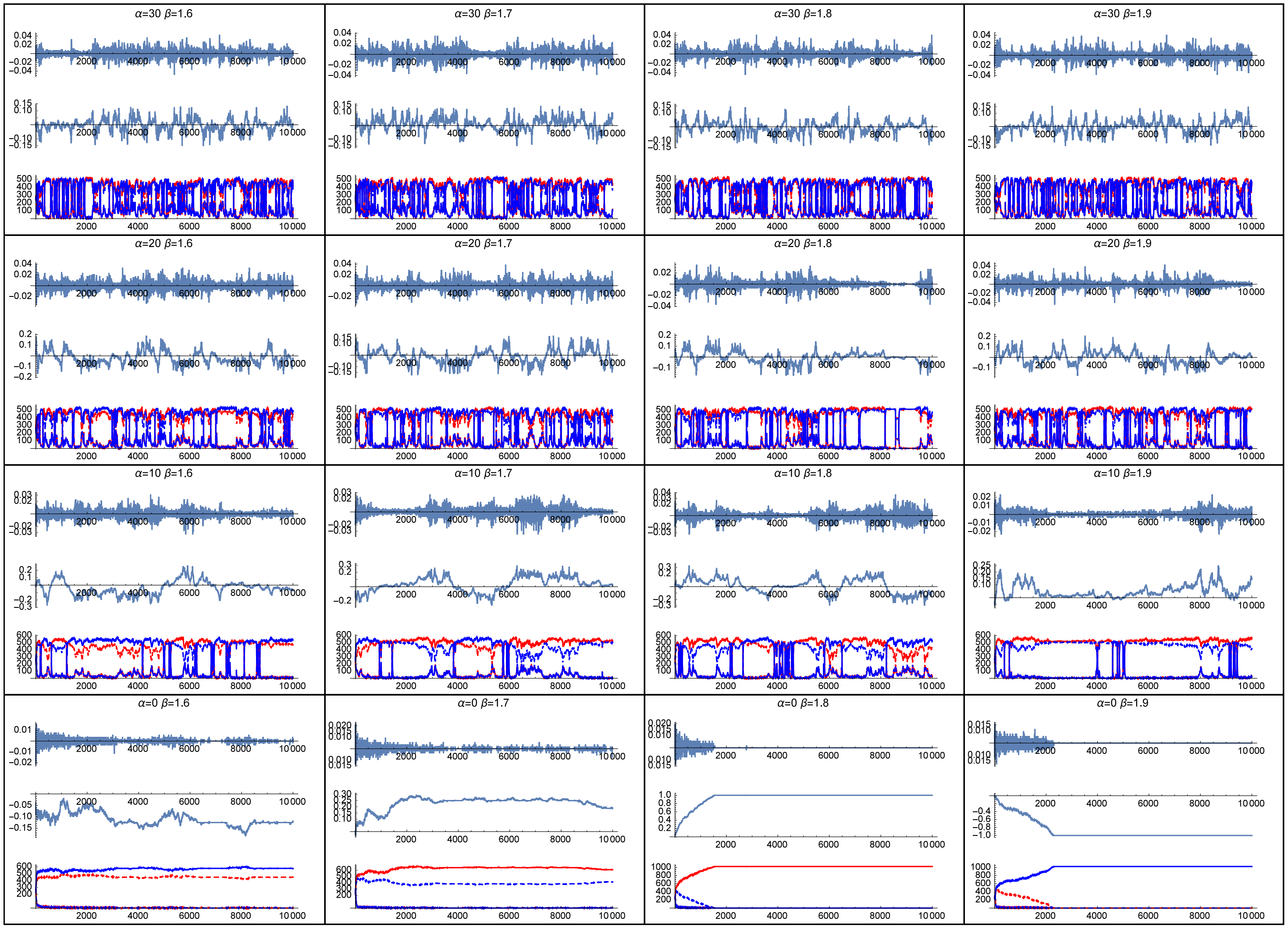


Figure A.5: Bornholdt's model with strategy simulated at $\beta = \{1.6, 1.7, 1.8, 1.9\}$.

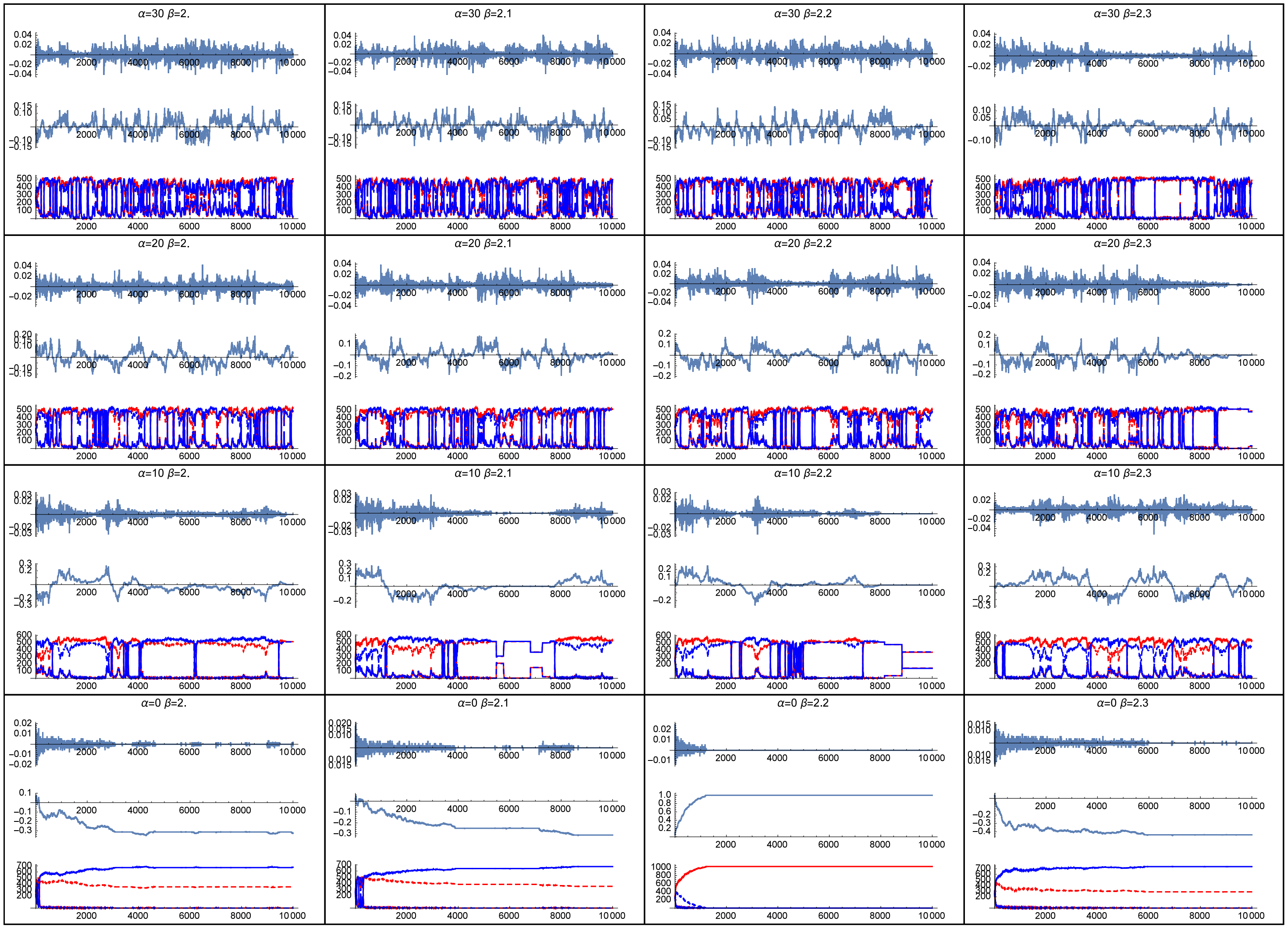


Figure A.6: Bornholdt's model with strategy simulated at $\beta = \{2.0,\ 2.1,\ 2.2,\ 2.3\}$.

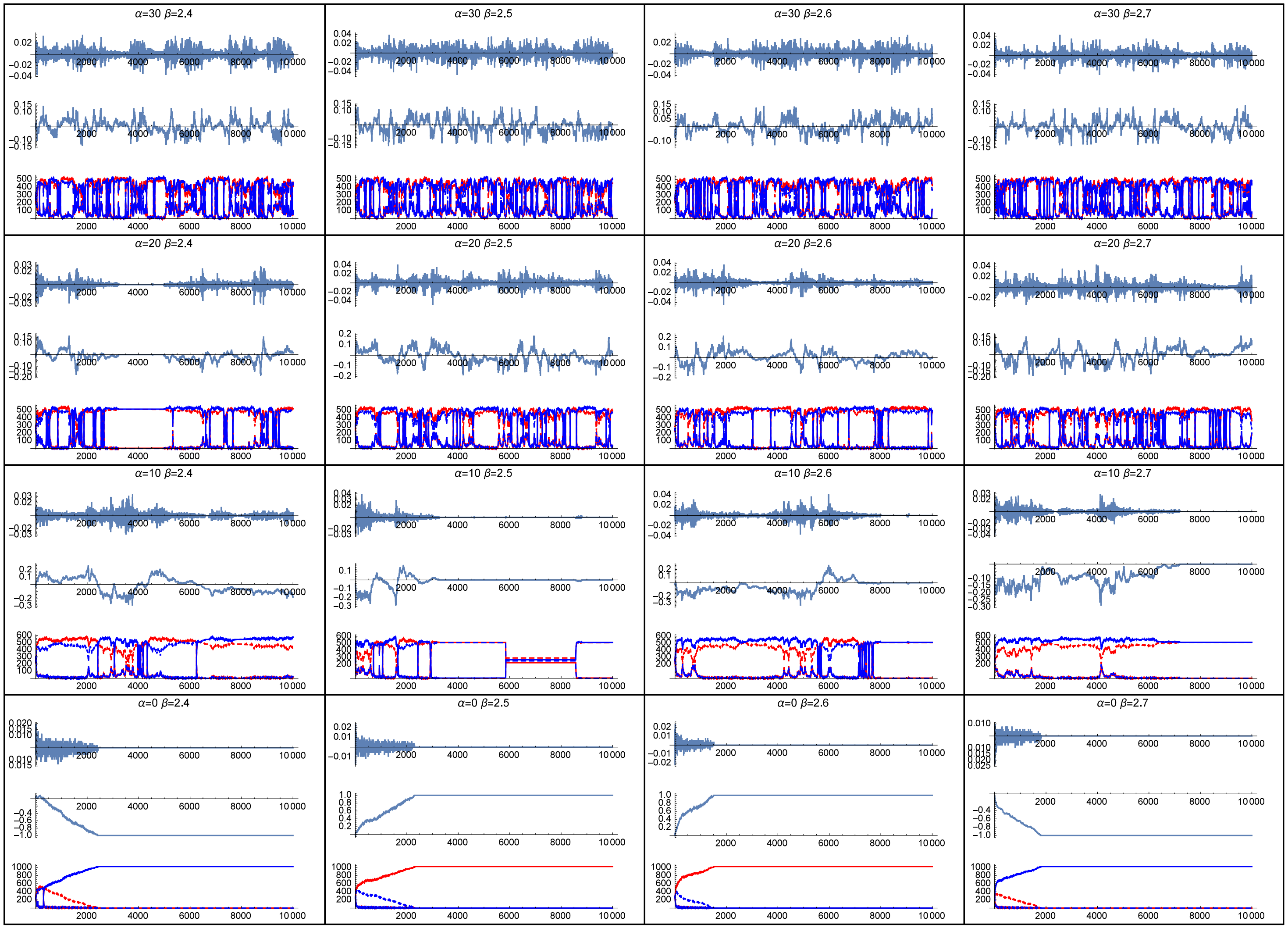


Figure A.7: Bornholdt's model with strategy simulated at $\beta = \{2.4,\ 2.5,\ 2.6,\ 2.7\}$.

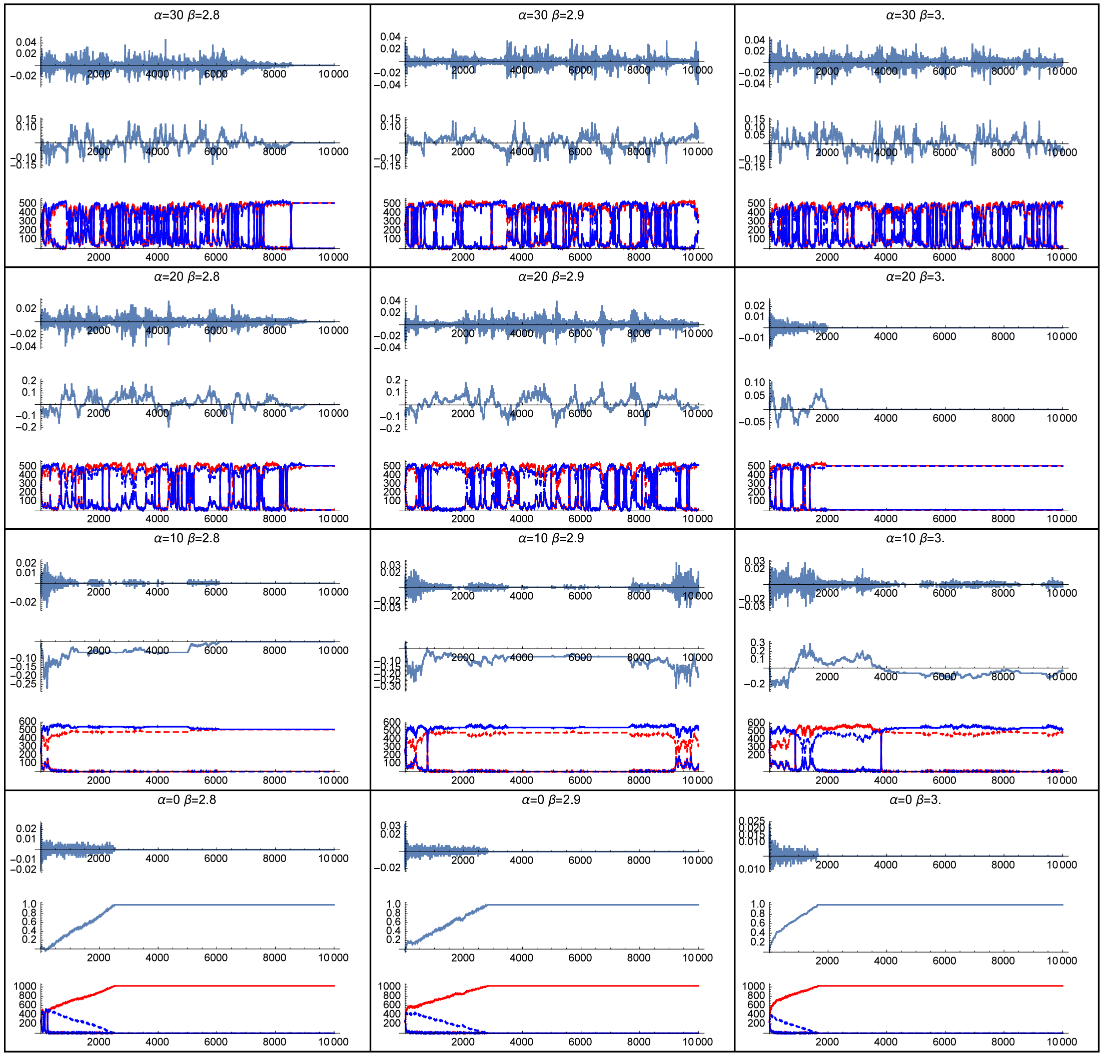


Figure A.8: Bornholdt's simplified model without strategy simulated at $\beta = \{2.8, 2.9, 3.0\}$.

A.3 Bornholdt's simplified model

A.3.1 Plots of statistics for different (α, β)

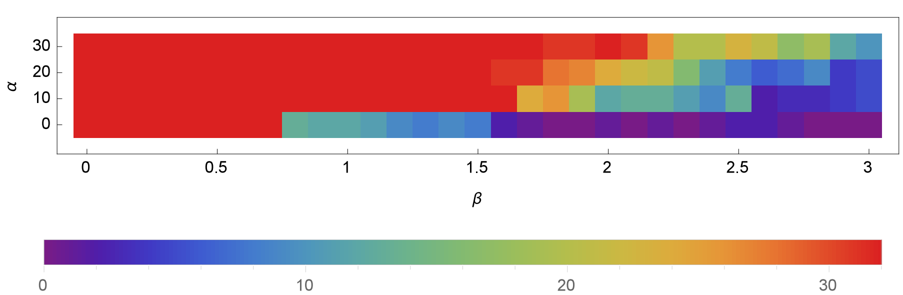


Figure A.9: Number of non-convergent series. For numerical values refer to Table A.19.

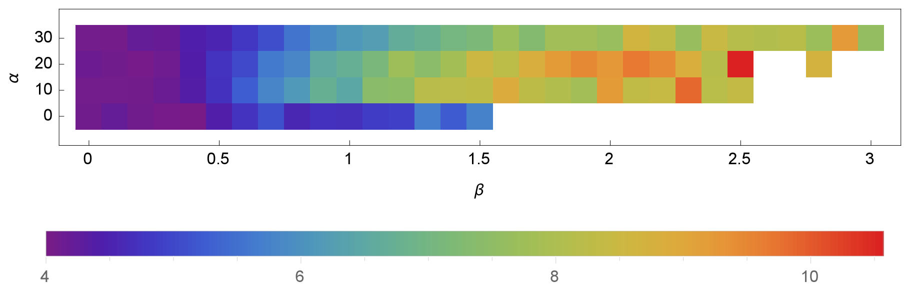


Figure A.10: Maximum of absolute magnetisation $|M(t)|$. For numerical values refer to Table A.21.

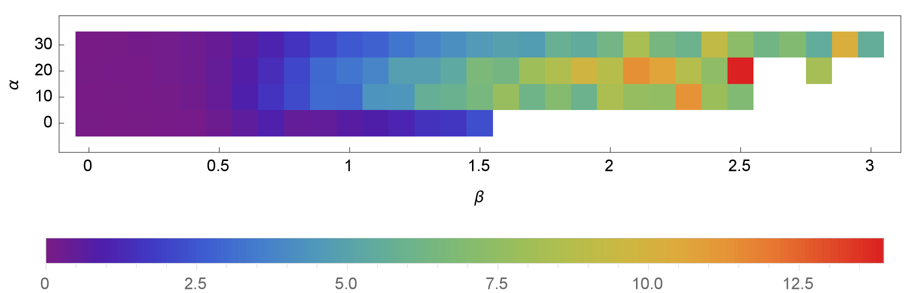
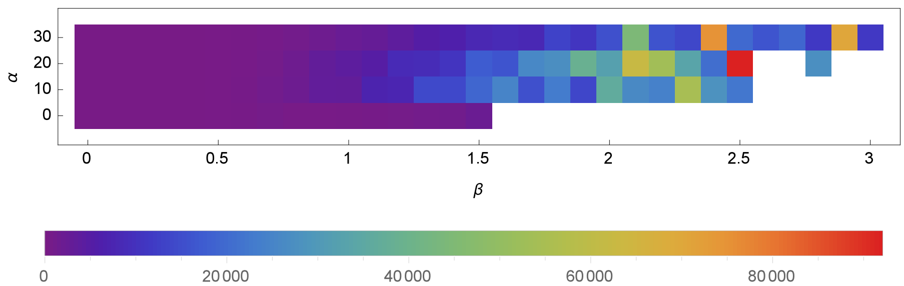
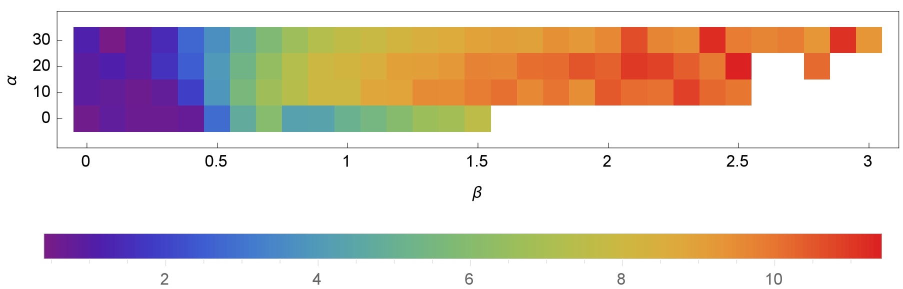


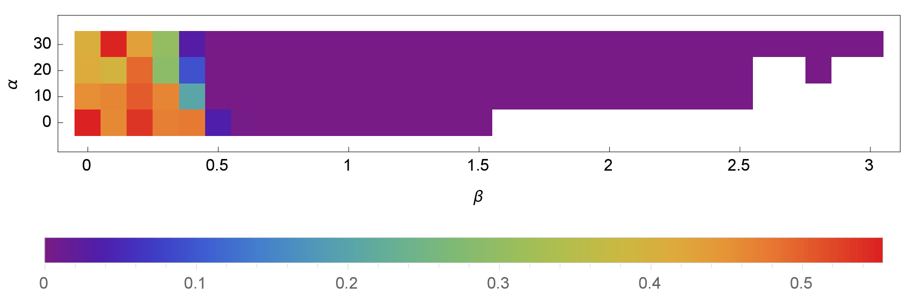
Figure A.11: Kurtosis of return series. For numerical values refer to Table A.22.



(a) Test statistic

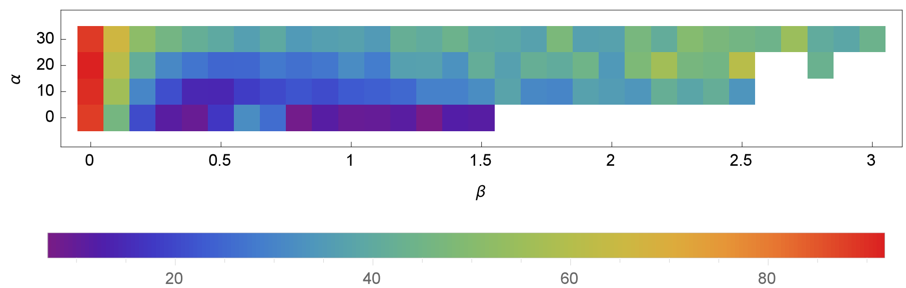


(b) Logarithm of test statistic

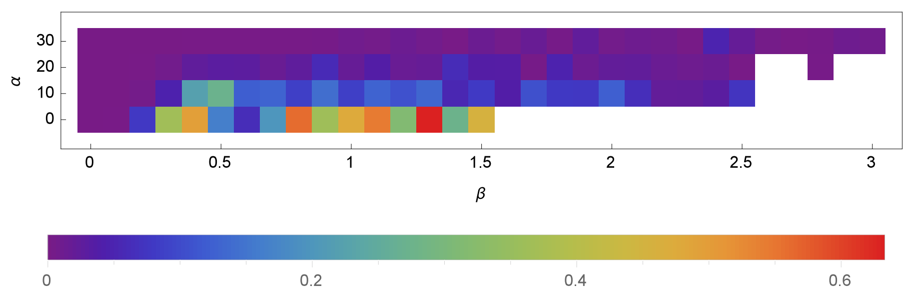


(c) P-value

Figure A.12: Result of Jarque-Berra test of returns series. For numerical values of test statistics and p-values refer to Tables A.23 and A.24 respectively.

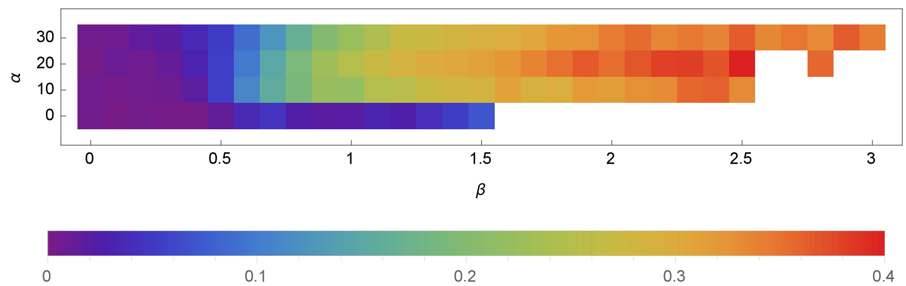


(a) Test statistics

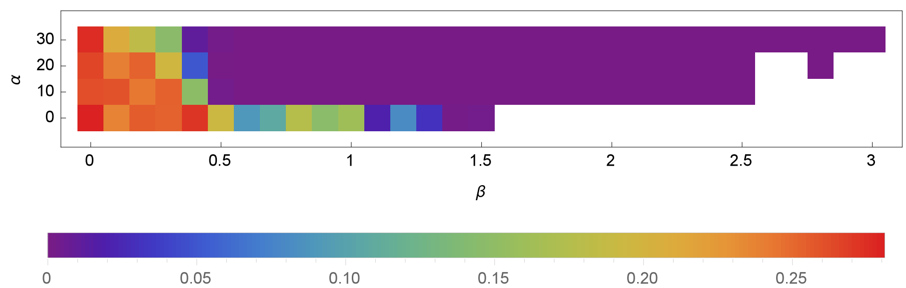


(b) P-values

Figure A.13: Results of Ljung-Box test of returns series. For numerical values of test statistics and p-values refer to Tables A.25 and A.26 respectively.



(a) Parameter values



(b) P-value of parameter

Figure A.14: AR(1) model parameter and its p-value for absolute returns series. For numerical values of the parameter and p-value refer to Tables A.30 and A.31 respectively.

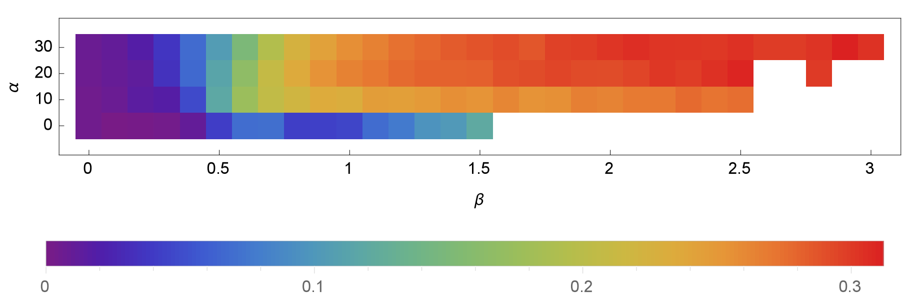
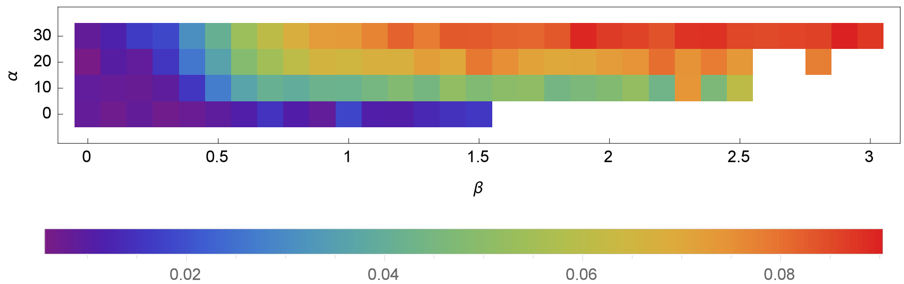
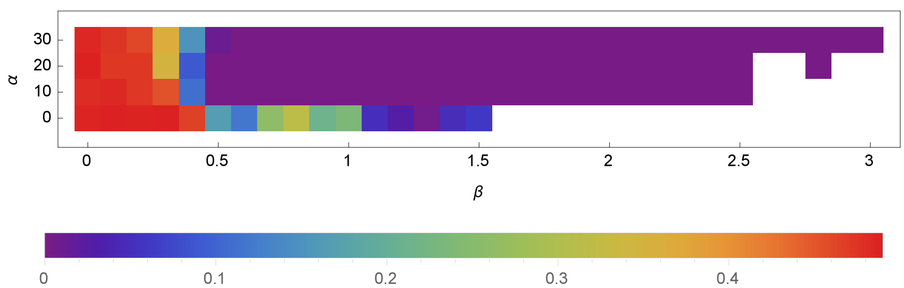


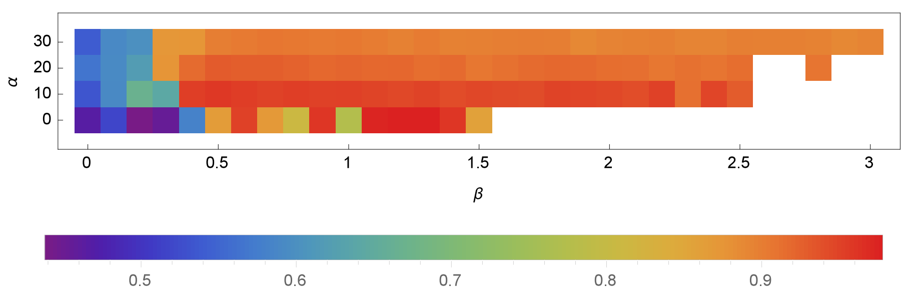
Figure A.15: Local Whittle estimate of fractional difference parameter d for absolute returns. For numerical values refer to Table A.32.



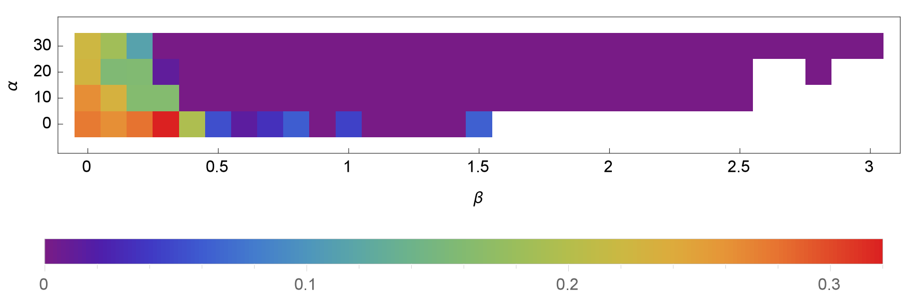
(a) Values of parameter γ_1



(b) P-values of parameter γ_1



(c) Values of parameter δ_1



(d) P-values of parameter δ_1

Figure A.16: GARCH (1,1) model parameters and their p-values. For numerical values of the parameter γ_1 and corresponding p-values refer to Tables A.33 and A.34 respectively. For numerical values of the parameter δ_1 and corresponding p-values refer to Tables A.33 and A.34 respectively.

A.3.2 Values of statistics for different (α, β)

30	32	32	32	32	32	32	32	32	32	32	
20	32	32	32	32	32	32	32	32	32	32	
10	32	32	32	32	32	32	32	32	32	32	
0	32	32	32	32	32	32	32	32	13	12	
$\alpha \backslash \beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	32	32	32	32	32	32	32	32	31	31	
20	32	32	32	32	32	32	31	31	28	27	
10	32	32	32	32	32	32	32	24	26	19	
0	12	11	9	8	9	8	2	1	0	0	
$\alpha \backslash \beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	32	31	26	20	20	23	21	17	19	12	10
20	24	22	21	16	11	8	6	7	9	4	5
10	12	13	13	11	9	13	2	3	3	4	5
0	1	0	1	0	1	2	2	1	0	0	0
$\alpha \backslash \beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.19: Bornholdt's simplified model without strategy: Number of non-convergent series

30	-0.0072 (0.0124)	-0.0107 (0.0086)	-0.0029 (0.0103)	0.0022 (0.0087)	0.0007 (0.0107)	-0.0015 (0.0107)	-0.0055 (0.0148)	0.0065 (0.0140)	0.0048 (0.0202)	-0.0031 (0.0216)	
20	0.0017 (0.0112)	-0.0028 (0.0120)	0.0116 (0.0101)	-0.0032 (0.0116)	-0.0019 (0.0097)	0.0065 (0.0112)	-0.0034 (0.0182)	-0.0002 (0.0199)	-0.0122 (0.0235)	0.0046 (0.0278)	
10	-0.0055 (0.0099)	-0.0017 (0.0090)	0.0060 (0.0094)	0.0047 (0.0105)	-0.0015 (0.0105)	-0.0047 (0.0118)	0.0021 (0.0128)	0.0112 (0.0198)	-0.0172 (0.0306)	0.0105 (0.0363)	
0	0.0011 (0.0088)	-0.0050 (0.0099)	-0.0024 (0.0102)	0.0016 (0.0089)	0.0029 (0.0084)	0.0030 (0.0112)	-0.0089 (0.0180)	0.0051 (0.0250)	-0.0102 (0.0213)	-0.0095 (0.0202)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0066 (0.0274)	0.0084 (0.0210)	0.0025 (0.0293)	-0.0177 (0.0322)	0.0159 (0.0430)	0.0325 (0.0350)	-0.0201 (0.0461)	0.0112 (0.0346)	0.0454 (0.0340)	0.0000 (0.0301)	
20	-0.0005 (0.0329)	-0.0123 (0.0409)	0.0289 (0.0494)	0.0179 (0.0493)	0.0191 (0.0384)	-0.0189 (0.0524)	-0.0042 (0.0633)	-0.0268 (0.0750)	-0.0457 (0.0661)	-0.0320 (0.0777)	
10	-0.0339 (0.0265)	0.0020 (0.0466)	-0.0139 (0.0460)	-0.0471 (0.0713)	-0.0051 (0.0627)	-0.0447 (0.0616)	0.0196 (0.0767)	-0.0219 (0.0767)	-0.0011 (0.0837)	0.0251 (0.0555)	
0	0.0021 (0.0243)	0.0221 (0.0184)	-0.0099 (0.0436)	0.0076 (0.0282)	-0.0147 (0.0364)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	-0.0379 (0.0452)	0.0029 (0.0780)	0.0042 (0.0446)	-0.0102 (0.0510)	0.0642 (0.0813)	0.0102 (0.0749)	-0.0485 (0.0580)	-0.0236 (0.0786)	0.0160 (0.0512)	0.1571 (0.2563)	0.0143 (0.0525)
20	-0.0388 (0.0813)	-0.0278 (0.1299)	0.0555 (0.1321)	0.1142 (0.1154)	0.0090 (0.1251)	-0.0450 (0.1483)	× ×	× ×	-0.0422 (0.1084)	× ×	× ×
10	-0.0982 (0.1445)	-0.0528 (0.0901)	-0.0079 (0.0919)	-0.0179 (0.1985)	0.0686 (0.0949)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.20: Bornholdt's simplified model without strategy: Skewness

30	3.22 (0.1068)	3.53 (0.0807)	4.1018 (0.1053)	4.0807 (0.0980)	4.26 (0.1460)	4.11 (0.1083)	4.69 (0.1624)	4.87 (0.1145)	5.39 (0.1668)	5.61 (0.1879)	
20	4.0232 (0.1394)	3.22 (0.0878)	3.96 (0.0996)	4.0027 (0.0879)	4.12 (0.1106)	4.71 (0.1285)	4.21 (0.1574)	5.35 (0.1870)	5.44 (0.2016)	6.02 (0.3398)	
10	3.91 (0.1149)	3.97 (0.1185)	3.99 (0.1197)	4.0506 (0.0865)	4.24 (0.1420)	4.24 (0.1605)	5.1709 (0.1823)	5.77 (0.2742)	6.0734 (0.2333)	6.48 (0.2677)	
0	3.63 (0.1269)	4.1036 (0.1209)	3.37 (0.1106)	3.99 (0.0918)	3.39 (0.0755)	4.87 (0.1640)	4.72 (0.1532)	5.0118 (0.2644)	4.31 (0.2362)	4.18 (0.1651)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	6.0542 (0.2326)	6.1606 (0.2783)	6.96 (0.2580)	6.44 (0.3264)	6.83 (0.3137)	7.1049 (0.4178)	7.01 (0.4014)	7.10 (0.4118)	7.20 (0.4337)	7.39 (0.4002)	
20	6.07 (0.3117)	7.0624 (0.3624)	7.65 (0.5848)	7.35 (0.4158)	7.55 (0.3804)	8.85 (0.5507)	8.78 (0.6146)	8.41 (0.6406)	9.80 (0.6101)	9.33 (0.7067)	
10	6.06 (0.2527)	7.01 (0.4105)	7.46 (0.4923)	8.1681 (0.6503)	8.23 (0.4778)	8.04 (0.6927)	8.73 (0.7412)	8.29 (0.8170)	8.0237 (0.8128)	7.15 (0.4799)	
0	4.06 (0.2810)	4.18 (0.3314)	4.65 (0.2943)	5.51 (0.2587)	5.1447 (0.3639)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	7.07 (0.4127)	8.69 (0.8772)	8.39 (0.4613)	7.70 (0.4694)	8.66 (0.8130)	8.1166 (0.4358)	8.0237 (0.5907)	8.1412 (0.5423)	7.44 (0.4297)	9.45 (1.7696)	7.45 (0.5015)
20	9.20 (0.8073)	9.26 (1.0265)	9.55 (0.9653)	8.29 (1.2184)	8.1323 (0.9001)	10.80 (1.9962)	× ×	× ×	8.57 (1.5319)	× ×	× ×
10	9.75 (1.4327)	8.05 (0.9987)	8.79 (1.2753)	9.49 (1.3913)	8.1525 (1.0932)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.21: Bornholdt's simplified model without strategy: Maximum of absolute magnetisation $|M(t)|$

30	-0.0277 (0.0213)	-0.0050 (0.0146)	0.0308 (0.0186)	0.0822 (0.0198)	0.1880 (0.0262)	0.3364 (0.0198)	0.6348 (0.0355)	1.0006 (0.0503)	1.44 (0.0888)	2.0400 (0.1357)	
20	-0.0105 (0.0201)	0.0126 (0.0234)	0.0204 (0.0190)	0.0748 (0.0239)	0.1651 (0.0283)	0.3834 (0.0339)	0.7656 (0.0528)	1.96 (0.0814)	2.0299 (0.1483)	2.19 (0.2815)	
10	-0.0168 (0.0215)	-0.0103 (0.0219)	0.0061 (0.0175)	0.0327 (0.0152)	0.1144 (0.0197)	0.3641 (0.0321)	0.8904 (0.0744)	1.18 (0.1410)	2.1267 (0.1990)	2.67 (0.2688)	
0	-0.0132 (0.0179)	-0.0171 (0.0204)	-0.0032 (0.0168)	0.0020 (0.0196)	0.0251 (0.0201)	0.1997 (0.0295)	0.5400 (0.0804)	0.9062 (0.2233)	0.4301 (0.0968)	0.4686 (0.0568)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	2.52 (0.1682)	2.53 (0.2321)	3.41 (0.2116)	3.22 (0.3949)	4.1011 (0.3477)	4.61 (0.5684)	4.00 (0.4649)	4.99 (0.5119)	5.78 (0.8424)	5.50 (0.5176)	
20	3.52 (0.3023)	3.36 (0.2756)	4.29 (0.5217)	4.52 (0.5987)	5.00 (0.6274)	6.65 (0.9910)	6.97 (0.9756)	7.26 (1.3350)	8.04 (1.0982)	9.40 (1.7134)	
10	2.93 (0.3351)	4.66 (0.5012)	4.85 (0.6234)	5.88 (1.0967)	5.08 (0.9251)	6.89 (1.3575)	7.34 (1.3926)	6.1326 (1.1679)	6.23 (1.7588)	5.73 (1.0487)	
0	0.6869 (0.0748)	0.8388 (0.0751)	1.0812 (0.1051)	1.12 (0.1461)	1.72 (0.3575)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	6.72 (0.8857)	8.15 (2.8723)	6.50 (0.9223)	5.62 (1.0361)	9.1713 (5.3319)	7.46 (0.9919)	6.67 (1.3097)	6.84 (1.5117)	5.24 (0.8105)	10.56 (6.0592)	5.17 (0.9500)
20	8.96 (1.8045)	11.40 (3.1864)	10.59 (2.9473)	8.23 (2.4818)	7.96 (1.6546)	13.8994 (6.7204)	× ×	× ×	8.1872 (2.6689)	× ×	× ×
10	8.21 (3.6333)	7.39 (2.5233)	7.33 (2.2376)	11.85 (3.6721)	7.86 (3.3123)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.22: Bornholdt's simplified model without strategy: Excess kurtosis

α\β	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	3.2 (1.1)	1.5 (0.5)	2.4 (0.8)	4.2 (1.2)	15.0 (4.1)	40.3 (4.4)	140.7 (15.5)	343.8 (33.4)	782.2 (88.6)	1445.5 (195.0)	
20	2.5 (0.7)	3.1 (1.0)	2.4 (1.1)	4.9 (1.6)	12.4 (3.5)	53.8 (9.4)	207.2 (28.2)	604.2 (73.8)	1442.5 (209.9)	3094.0 (676.1)	
10	2.4 (0.8)	2.2 (0.8)	1.8 (0.6)	2.2 (0.7)	6.7 (1.5)	48.7 (8.5)	281.9 (47.3)	842.3 (164.5)	1628.7 (305.7)	3130.0 (569.2)	
0	1.7 (0.6)	2.3 (0.8)	1.9 (0.8)	1.9 (0.5)	2.1 (0.7)	17.1 (4.0)	118.7 (30.0)	415.8 (167.2)	73.8 (44.7)	78.3 (17.2)	
α\β	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	2148.3 (289.0)	2631.8 (457.8)	3689.7 (462.8)	5010.9 (1199.9)	5964.8 (989.2)	7850.6 (2113.1)	8463.8 (1673.7)	8004.5 (1808.0)	13017.1 (4675.9)	10648.4 (2017.2)	
20	3820.1 (746.1)	4855.3 (724.4)	8403.1 (1988.5)	8706.2 (2323.2)	10425.3 (2462.2)	17255.8 (6471.6)	15907.3 (5881.8)	25673.3 (8819.8)	27123.3 (6818.4)	38730.3 (14617.3)	
10	3120.5 (709.1)	6781.8 (1476.7)	7358.3 (1920.8)	14185.9 (5346.3)	13979.9 (4578.3)	19118.0 (7491.7)	24906.4 (9040.5)	15337.6 (5534.2)	23021.7 (11438.1)	13556.3 (4660.2)	
0	165.5 (35.4)	242.1 (43.9)	404.0 (79.8)	748.4 (142.5)	945.7 (270.9)	× ×	× ×	× ×	× ×	× ×	
α\β	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**
30	15300.3 (4355.6)	44543.1 (49531.3)	15784.4 (5200.6)	13761.0 (4768.3)	75101.4 (114372.8)	19601.3 (5277.0)	16001.9 (7208.9)	19109.3 (9289.0)	11232.1 (3253.3)	70384.6 (99817.9)	11159.8 (4231.7)
20	32312.3 (15975.2)	62248.7 (38216.7)	53438.9 (30903.3)	33471.7 (19868.2)	20526.6 (8443.8)	92065.0 (85357.8)	× ×	× ×	27378.1 (17919.5)	× ×	× ×
10	36184.3 (32049.6)	26043.2 (20649.9)	24163.9 (13042.6)	54724.0 (32473.4)	28080.4 (22708.7)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×

Table A.23: Bornholdt's simplified model without strategy: Jarque-Berra test - test statistic

30	0.4074 (0.1205)	0.5518 (0.0934)	0.4320 (0.0989)	0.3011 (0.1017)	0.0299 (0.0187)	0.0002 (0.0003)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.4155 (0.0998)	0.3938 (0.1108)	0.4939 (0.1096)	0.2901 (0.1126)	0.0930 (0.0639)	0.0000 (0.0001)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.4554 (0.1089)	0.4656 (0.0949)	0.5047 (0.0964)	0.4647 (0.1018)	0.2023 (0.1017)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.5533 (0.1096)	0.4617 (0.1027)	0.5370 (0.1013)	0.4731 (0.0859)	0.4771 (0.1020)	0.0346 (0.0247)	0.0027 (0.0026)	0.0000 (0.0001)	0.0000 (0.0000)	0.0000 (0.0000)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	0.0000 (0.0000)	× ×	× ×
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.24: Bornholdt's simplified model without strategy: Jarque-Berra test - p-value

30	88.57 (4.89)	66.28 (3.38)	51.52 (3.55)	45.62 (3.31)	41.67 (2.89)	39.62 (3.42)	36.23 (3.17)	39.57 (3.81)	34.82 (2.71)	36.27 (3.92)	
20	91.77 (3.48)	61.16 (3.70)	41.25 (3.37)	30.70 (2.71)	27.14 (2.88)	24.54 (2.93)	24.84 (2.84)	27.79 (3.04)	26.25 (2.70)	27.41 (4.20)	
10	90.29 (3.88)	56.04 (3.73)	30.23 (2.67)	20.83 (2.02)	14.11 (1.98)	14.3 (2.53)	18.12 (2.78)	19.99 (3.15)	21.73 (3.21)	20.42 (3.79)	
0	88.21 (3.81)	46.06 (3.49)	20.32 (2.22)	11.4 (1.66)	9.12 (1.57)	17.9 (2.36)	31.72 (4.96)	25.76 (6.83)	7.75 (1.25)	11.32 (3.09)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	37.06 (4.27)	35.03 (3.78)	41.92 (6.13)	40.24 (5.48)	43.77 (4.08)	39.52 (4.87)	38.92 (5.23)	36.99 (5.12)	47.36 (5.76)	36.59 (5.73)	
20	31.49 (4.85)	28.33 (4.08)	36.65 (7.90)	36.89 (5.94)	32.73 (5.15)	41.38 (8.52)	36.52 (6.46)	41.97 (6.15)	40.46 (9.09)	44.49 (11.07)	
10	22.71 (3.28)	23.50 (4.33)	24.90 (5.03)	29.27 (6.84)	29.24 (5.87)	31.74 (6.16)	37.69 (10.92)	30.73 (7.68)	30.3 (5.74)	36.94 (8.82)	
0	9.25 (2.13)	9.50 (3.84)	11.3 (1.94)	7.6 (2.22)	11.89 (2.22)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	37.07 (3.53)	46.62 (9.42)	40.97 (7.08)	49.78 (10.15)	47.14 (11.53)	45.58 (6.21)	44.17 (7.11)	54.63 (11.54)	40.32 (7.20)	38.52 (7.20)	43.62 (11.05)
20	34.54 (6.45)	48.06 (12.33)	56.13 (11.78)	46.51 (10.44)	45.31 (16.76)	60.98 (32.00)	× ×	× ×	43.01 (8.31)	× ×	× ×
10	35.55 (25.15)	33.84 (16.33)	41.99 (9.05)	38.27 (14.07)	41.44 (23.89)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.25: Bornholdt's simplified model without strategy: Ljung-Box test - test statistic

30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0003 (0.0004)	0.0004 (0.0004)	0.0004 (0.0002)	0.0010 (0.0012)	0.0008 (0.0006)	0.0017 (0.0016)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0003 (0.0003)	0.0064 (0.0057)	0.0230 (0.0240)	0.0283 (0.0162)	0.0272 (0.0164)	0.0145 (0.0107)	0.0242 (0.0194)	0.0580 (0.0432)
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0043 (0.0036)	0.0445 (0.0228)	0.2219 (0.0707)	0.2723 (0.1022)	0.1218 (0.0508)	0.1272 (0.0738)	0.0856 (0.0493)	0.1407 (0.0755)
0	0.0000 (0.0000)	0.0002 (0.0004)	0.0769 (0.0499)	0.3649 (0.0848)	0.4963 (0.1065)	0.1628 (0.0808)	0.0613 (0.0608)	0.1984 (0.0852)	0.5613 (0.1122)	0.3626 (0.1804)
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**

30	0.0073 (0.0092)	0.0044 (0.0063)	0.0121 (0.0120)	0.0059 (0.0096)	0.0003 (0.0004)	0.0124 (0.0198)	0.0050 (0.0063)	0.0152 (0.0184)	0.0018 (0.0029)	0.0226 (0.0299)
20	0.0179 (0.0143)	0.0355 (0.0248)	0.0148 (0.0111)	0.0180 (0.0156)	0.0599 (0.0519)	0.0349 (0.0355)	0.0340 (0.0317)	0.0018 (0.0017)	0.0461 (0.0477)	0.0117 (0.0116)
10	0.0857 (0.0650)	0.1274 (0.0868)	0.1062 (0.0748)	0.1308 (0.0949)	0.0525 (0.0442)	0.0789 (0.0603)	0.0371 (0.0263)	0.1052 (0.0902)	0.0772 (0.0856)	0.0759 (0.0700)
0	0.4775 (0.1546)	0.5458 (0.1900)	0.3147 (0.1199)	0.6333 (0.1938)	0.2746 (0.1281)	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**

30	0.0052 (0.0091)	0.0100 (0.0131)	0.0077 (0.0119)	0.0007 (0.0009)	0.0477 (0.0915)	0.0186 (0.0354)	0.0029 (0.0028)	0.0005 (0.0007)	0.0018 (0.0016)	0.0098 (0.0190)	0.0065 (0.0104)
20	0.0212 (0.0196)	0.0225 (0.0272)	0.0137 (0.0226)	0.0099 (0.0185)	0.0135 (0.0253)	0.0002 (0.0003)	× ×	× ×	0.0015 (0.0028)	× ×	× ×
10	0.1228 (0.1161)	0.0634 (0.0647)	0.0206 (0.0344)	0.0213 (0.0252)	0.0287 (0.0367)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.26: Bornholdt's simplified model without strategy: Ljung-Box test - p-value

30	-0.0505 (0.0035)	-0.0388 (0.0040)	-0.0326 (0.0034)	-0.0304 (0.0043)	-0.0276 (0.0041)	-0.0258 (0.0039)	-0.0286 (0.0042)	-0.0317 (0.0040)	-0.0251 (0.0033)	-0.0299 (0.0064)	
20	-0.0476 (0.0048)	-0.0388 (0.0045)	-0.0247 (0.0035)	-0.0231 (0.0039)	-0.0196 (0.0045)	-0.0173 (0.0051)	-0.0155 (0.0051)	-0.0226 (0.0045)	-0.0230 (0.0053)	-0.0127 (0.0048)	
10	-0.0470 (0.0039)	-0.0304 (0.0051)	-0.0207 (0.0038)	-0.0140 (0.0047)	-0.0109 (0.0045)	-0.0102 (0.0039)	-0.0129 (0.0053)	-0.0098 (0.0047)	-0.0123 (0.0070)	-0.0067 (0.0059)	
0	-0.0448 (0.0034)	-0.0296 (0.0041)	-0.0143 (0.0040)	-0.0068 (0.0030)	-0.0017 (0.0035)	-0.0137 (0.0028)	-0.0200 (0.0042)	-0.0124 (0.0054)	-0.0041 (0.0064)	-0.0013 (0.0068)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	-0.0268 (0.0052)	-0.0205 (0.0047)	-0.0323 (0.0047)	-0.0252 (0.0055)	-0.0290 (0.0063)	-0.0278 (0.0056)	-0.0291 (0.0053)	-0.0228 (0.0052)	-0.0311 (0.0072)	-0.0196 (0.0060)	
20	-0.0161 (0.0055)	-0.0149 (0.0067)	-0.0199 (0.0065)	-0.0196 (0.0060)	-0.0195 (0.0077)	-0.0161 (0.0094)	-0.0201 (0.0060)	-0.0157 (0.0088)	-0.0141 (0.0068)	-0.0137 (0.0066)	
10	-0.0124 (0.0063)	-0.0066 (0.0055)	-0.0119 (0.0062)	-0.0059 (0.0055)	-0.0103 (0.0069)	-0.0074 (0.0065)	-0.0062 (0.0081)	-0.0038 (0.0086)	-0.0074 (0.0081)	-0.0061 (0.0108)	
0	-0.0047 (0.0058)	-0.0054 (0.0077)	0.0036 (0.0092)	-0.0067 (0.0041)	0.0003 (0.0083)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	-0.0236 (0.0057)	-0.0207 (0.0104)	-0.0171 (0.0066)	-0.0202 (0.0111)	-0.0210 (0.0099)	-0.0239 (0.0098)	-0.0299 (0.0084)	-0.0365 (0.0100)	-0.0266 (0.0078)	-0.0158 (0.0156)	-0.0319 (0.0139)
20	-0.0158 (0.0075)	-0.0146 (0.0094)	-0.0151 (0.0126)	-0.0219 (0.0122)	-0.0181 (0.0113)	-0.0151 (0.0295)	× ×	× ×	-0.0330 (0.0102)	× ×	× ×
10	-0.0033 (0.0089)	-0.0019 (0.0102)	-0.0218 (0.0106)	-0.0002 (0.0105)	-0.0259 (0.0080)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.27: Bornholdt's simplified model without strategy: AR(1) parameter

30	0.0001 (0.0001)	0.0075 (0.0082)	0.0095 (0.0068)	0.0352 (0.0293)	0.0440 (0.0309)	0.0454 (0.0229)	0.0500 (0.0394)	0.0195 (0.0144)	0.0429 (0.0308)	0.0583 (0.0398)	
20	0.0043 (0.0077)	0.0110 (0.0137)	0.0439 (0.0255)	0.0700 (0.0396)	0.0931 (0.0350)	0.1383 (0.0562)	0.1375 (0.0510)	0.0909 (0.0466)	0.0735 (0.0410)	0.1692 (0.0593)	
10	0.0019 (0.0026)	0.0517 (0.0403)	0.0870 (0.0434)	0.1445 (0.0514)	0.2042 (0.0556)	0.1957 (0.0560)	0.1500 (0.0508)	0.2102 (0.0558)	0.0995 (0.0433)	0.1855 (0.0571)	
0	0.0007 (0.0006)	0.0311 (0.0196)	0.1146 (0.0351)	0.2340 (0.0438)	0.2652 (0.0492)	0.1414 (0.0430)	0.1062 (0.0476)	0.1814 (0.0568)	0.2380 (0.0804)	0.2167 (0.0819)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0492 (0.0287)	0.0766 (0.0402)	0.0282 (0.0213)	0.0571 (0.0325)	0.0730 (0.0424)	0.0796 (0.0490)	0.0535 (0.0395)	0.0839 (0.0408)	0.0219 (0.0147)	0.0994 (0.0490)	
20	0.1334 (0.0560)	0.1056 (0.0426)	0.0637 (0.0358)	0.1152 (0.0516)	0.0979 (0.0473)	0.1211 (0.0489)	0.1017 (0.0459)	0.1006 (0.0468)	0.1327 (0.0603)	0.1668 (0.0619)	
10	0.1160 (0.0496)	0.1715 (0.0538)	0.1120 (0.0358)	0.1860 (0.0560)	0.1695 (0.0537)	0.1476 (0.0528)	0.1667 (0.0554)	0.1647 (0.0649)	0.1642 (0.0634)	0.1555 (0.0835)	
0	0.2280 (0.0687)	0.1924 (0.0846)	0.2351 (0.1158)	0.2788 (0.0961)	0.2350 (0.0961)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0744 (0.0423)	0.0987 (0.0529)	0.1061 (0.0602)	0.1365 (0.0788)	0.0975 (0.0676)	0.0802 (0.0530)	0.0839 (0.0624)	0.0645 (0.0666)	0.0791 (0.0647)	0.1090 (0.0945)	0.1070 (0.1239)
20	0.1283 (0.0579)	0.1509 (0.0586)	0.1030 (0.0596)	0.0485 (0.0344)	0.1472 (0.0940)	0.0579 (0.0735)	× ×	× ×	0.0564 (0.0997)	× ×	× ×
10	0.1987 (0.0966)	0.1331 (0.0867)	0.1071 (0.0681)	0.1822 (0.0995)	0.0558 (0.0500)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.28: Bornholdt's simplified model without strategy: P-value of AR(1) parameter

30	-0.2125 (0.0060)	-0.1724 (0.0054)	-0.1466 (0.0050)	-0.1348 (0.0045)	-0.1183 (0.0046)	-0.1134 (0.0042)	-0.1014 (0.0045)	-0.0996 (0.0046)	-0.0882 (0.0054)	-0.0831 (0.0059)	
20	-0.2140 (0.0053)	-0.1643 (0.0059)	-0.1254 (0.0053)	-0.1006 (0.0054)	-0.0885 (0.0057)	-0.0777 (0.0056)	-0.0668 (0.0036)	-0.0694 (0.0054)	-0.0590 (0.0056)	-0.0569 (0.0065)	
10	-0.2164 (0.0046)	-0.1536 (0.0059)	-0.0990 (0.0040)	-0.0648 (0.0047)	-0.0474 (0.0044)	-0.0401 (0.0044)	-0.0418 (0.0049)	-0.0379 (0.0070)	-0.0436 (0.0071)	-0.0364 (0.0071)	
0	-0.2145 (0.0043)	-0.1362 (0.0052)	-0.0733 (0.0051)	-0.0297 (0.0036)	-0.0061 (0.0045)	-0.0587 (0.0060)	-0.0801 (0.0132)	-0.0597 (0.0199)	-0.0091 (0.0078)	-0.0087 (0.0085)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	-0.0863 (0.0062)	-0.0766 (0.0057)	-0.0841 (0.0081)	-0.0719 (0.0064)	-0.0748 (0.0065)	-0.0747 (0.0069)	-0.0693 (0.0063)	-0.0697 (0.0076)	-0.0693 (0.0075)	-0.0641 (0.0088)	
20	-0.0531 (0.0081)	-0.0511 (0.0061)	-0.0492 (0.0096)	-0.0503 (0.0081)	-0.0456 (0.0083)	-0.0406 (0.0105)	-0.0480 (0.0079)	-0.0417 (0.0100)	-0.0374 (0.0116)	-0.0383 (0.0098)	
10	-0.0355 (0.0075)	-0.0309 (0.0083)	-0.0257 (0.0081)	-0.0266 (0.0083)	-0.0243 (0.0111)	-0.0247 (0.0116)	-0.0313 (0.0110)	-0.0178 (0.0134)	-0.0226 (0.0137)	-0.0329 (0.0188)	
0	-0.0037 (0.0060)	-0.0034 (0.0059)	-0.0061 (0.0097)	-0.0038 (0.0075)	-0.0011 (0.0111)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	-0.0678 (0.0094)	-0.0594 (0.0104)	-0.0615 (0.0096)	-0.0677 (0.0120)	-0.0597 (0.0136)	-0.0690 (0.0114)	-0.0601 (0.0109)	-0.0616 (0.0102)	-0.0695 (0.0100)	-0.0497 (0.0126)	-0.0621 (0.0192)
20	-0.0370 (0.0110)	-0.0353 (0.0125)	-0.0447 (0.0113)	-0.0448 (0.0127)	-0.0590 (0.0151)	-0.0451 (0.0309)	× ×	× ×	-0.0376 (0.0220)	× ×	× ×
10	-0.0006 (0.0121)	-0.0289 (0.0155)	-0.0444 (0.0207)	-0.0137 (0.0214)	-0.0522 (0.0180)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.29: Bornholdt's simplified model without strategy: Local Whittle estimate of Fractional integration parameter d

30	0.0025 (0.0031)	0.0042 (0.0049)	0.0123 (0.0043)	0.0164 (0.0042)	0.0325 (0.0034)	0.0491 (0.0044)	0.0853 (0.0048)	0.1208 (0.0060)	0.1621 (0.0080)	0.1976 (0.0052)	
20	-0.0001 (0.0033)	0.0066 (0.0031)	0.0033 (0.0042)	0.0099 (0.0034)	0.0269 (0.0045)	0.0509 (0.0057)	0.0957 (0.0065)	0.1461 (0.0064)	0.1871 (0.0076)	0.2260 (0.0093)	
10	0.0032 (0.0038)	0.0003 (0.0038)	0.0022 (0.0037)	0.0048 (0.0037)	0.0151 (0.0043)	0.0500 (0.0042)	0.1074 (0.0062)	0.1529 (0.0088)	0.1863 (0.0142)	0.2181 (0.0164)	
0	0.0036 (0.0033)	-0.0023 (0.0041)	0.0017 (0.0041)	-0.0009 (0.0041)	-0.0009 (0.0037)	0.0104 (0.0043)	0.0324 (0.0086)	0.0416 (0.0173)	0.0211 (0.0201)	0.0165 (0.0106)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.2198 (0.0105)	0.2408 (0.0092)	0.2675 (0.0092)	0.2694 (0.0108)	0.2794 (0.0128)	0.2892 (0.0119)	0.2937 (0.0098)	0.3025 (0.0117)	0.3189 (0.0195)	0.3218 (0.0116)	
20	0.2465 (0.0091)	0.2663 (0.0100)	0.2837 (0.0109)	0.2947 (0.0124)	0.2994 (0.0167)	0.3077 (0.0110)	0.3193 (0.0189)	0.3341 (0.0236)	0.3483 (0.0186)	0.3619 (0.0312)	
10	0.2149 (0.0192)	0.2451 (0.0222)	0.2483 (0.0239)	0.2681 (0.0270)	0.2778 (0.0219)	0.2814 (0.0312)	0.3070 (0.0273)	0.2875 (0.0326)	0.2933 (0.0377)	0.3155 (0.0378)	
0	0.0164 (0.0089)	0.0289 (0.0060)	0.0226 (0.0073)	0.0374 (0.0135)	0.0503 (0.0122)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.3398 (0.0178)	0.3506 (0.0189)	0.3328 (0.0156)	0.3424 (0.0297)	0.3342 (0.0330)	0.3606 (0.0212)	0.3317 (0.0241)	0.3442 (0.0177)	0.3291 (0.0235)	0.3612 (0.0330)	0.3388 (0.0240)
20	0.3487 (0.0246)	0.3643 (0.0245)	0.3775 (0.0307)	0.3787 (0.0406)	0.3660 (0.0296)	0.3958 (0.0397)	× ×	× ×	0.3544 (0.0306)	× ×	× ×
10	0.3109 (0.0541)	0.3245 (0.0457)	0.3276 (0.0543)	0.3540 (0.0647)	0.3588 (0.0656)	× ×	× ×	× ×	× ×	× ×	
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.30: Bornholdt's simplified model without strategy: AR(1) parameter for $|r_t|$

30	0.2766 (0.0504)	0.2096 (0.0552)	0.1839 (0.0546)	0.1463 (0.0573)	0.0105 (0.0065)	0.0016 (0.0023)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.2661 (0.0474)	0.2402 (0.0463)	0.2536 (0.0553)	0.1967 (0.0492)	0.0504 (0.0270)	0.0005 (0.0003)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.2624 (0.0537)	0.2608 (0.0532)	0.2437 (0.0483)	0.2540 (0.0517)	0.1483 (0.0565)	0.0021 (0.0029)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.2814 (0.0532)	0.2380 (0.0509)	0.2564 (0.0543)	0.2539 (0.0514)	0.2731 (0.0549)	0.1928 (0.0519)	0.0895 (0.0503)	0.1077 (0.0484)	0.1769 (0.1002)	0.1468 (0.0821)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.1616 (0.0896)	0.0194 (0.0159)	0.0809 (0.0938)	0.0301 (0.0300)	0.0014 (0.0016)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	0.0000 (0.0000)	× ×	× ×
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.31: Bornholdt's simplified model without strategy: P-value of AR(1) parameter for $|r_t|$

30	0.0040 (0.0049)	0.0073 (0.0046)	0.0149 (0.0041)	0.0340 (0.0034)	0.0659 (0.0044)	0.1028 (0.0047)	0.1488 (0.0043)	0.1928 (0.0047)	0.2245 (0.0044)	0.2427 (0.0041)	
20	0.0029 (0.0041)	0.0062 (0.0049)	0.0096 (0.0038)	0.0316 (0.0044)	0.0645 (0.0034)	0.1117 (0.0046)	0.1647 (0.0043)	0.2080 (0.0039)	0.2346 (0.0051)	0.2547 (0.0039)	
10	0.0012 (0.0040)	0.0043 (0.0040)	0.0115 (0.0033)	0.0144 (0.0046)	0.0461 (0.0036)	0.1164 (0.0037)	0.1744 (0.0051)	0.2054 (0.0035)	0.2182 (0.0064)	0.2341 (0.0087)	
0	0.0012 (0.0030)	-0.0026 (0.0046)	-0.0015 (0.0047)	-0.0002 (0.0045)	0.0075 (0.0042)	0.0388 (0.0067)	0.0671 (0.0153)	0.0681 (0.0222)	0.0397 (0.0233)	0.0407 (0.0168)	
$\alpha \backslash \beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.2566 (0.0058)	0.2653 (0.0039)	0.2744 (0.0040)	0.2794 (0.0042)	0.2849 (0.0043)	0.2887 (0.0052)	0.2920 (0.0048)	0.2872 (0.0044)	0.2972 (0.0058)	0.2986 (0.0048)	
20	0.2646 (0.0032)	0.2704 (0.0043)	0.2778 (0.0044)	0.2817 (0.0046)	0.2816 (0.0064)	0.2825 (0.0045)	0.2902 (0.0079)	0.2922 (0.0097)	0.2954 (0.0084)	0.2927 (0.0095)	
10	0.2313 (0.0091)	0.2458 (0.0108)	0.2446 (0.0117)	0.2489 (0.0117)	0.2567 (0.0088)	0.2538 (0.0121)	0.2623 (0.0108)	0.2541 (0.0122)	0.2560 (0.0147)	0.2663 (0.0136)	
0	0.0432 (0.0114)	0.0663 (0.0070)	0.0750 (0.0081)	0.0939 (0.0141)	0.0999 (0.0218)	× ×	× ×	× ×	× ×	× ×	
$\alpha \backslash \beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.3028 (0.0058)	0.3065 (0.0059)	0.3030 (0.0059)	0.3026 (0.0069)	0.3015 (0.0113)	0.3047 (0.0075)	0.2999 (0.0080)	0.2996 (0.0065)	0.3037 (0.0074)	0.3125 (0.0076)	0.3042 (0.0047)
20	0.2921 (0.0107)	0.2953 (0.0088)	0.3018 (0.0098)	0.2991 (0.0085)	0.3049 (0.0107)	0.3104 (0.0066)	× ×	× ×	0.3004 (0.0090)	× ×	× ×
10	0.2639 (0.0191)	0.2694 (0.0166)	0.2701 (0.0180)	0.2778 (0.0238)	0.2731 (0.0184)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha \backslash \beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.32: Bornholdt's simplified model without strategy: Local Whittle estimate of Fractional integration parameter d for $|r_t|$

30	0.0084 (0.0027)	0.0115 (0.0032)	0.0165 (0.0033)	0.0184 (0.0032)	0.0307 (0.0023)	0.0409 (0.0030)	0.0534 (0.0020)	0.0606 (0.0022)	0.0676 (0.0026)	0.0727 (0.0029)	
20	0.0057 (0.0016)	0.0101 (0.0026)	0.0087 (0.0024)	0.0155 (0.0020)	0.0259 (0.0019)	0.0358 (0.0015)	0.0483 (0.0022)	0.0546 (0.0018)	0.0606 (0.0033)	0.0660 (0.0032)	
10	0.0087 (0.0029)	0.0081 (0.0031)	0.0078 (0.0021)	0.0092 (0.0021)	0.0154 (0.0015)	0.0271 (0.0014)	0.0363 (0.0015)	0.0411 (0.0014)	0.0394 (0.0020)	0.0424 (0.0025)	
0	0.0083 (0.0029)	0.0069 (0.0027)	0.0084 (0.0036)	0.0067 (0.0033)	0.0076 (0.0026)	0.0091 (0.0021)	0.0108 (0.0020)	0.0150 (0.0036)	0.0107 (0.0059)	0.0086 (0.0035)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0731 (0.0032)	0.0771 (0.0036)	0.0820 (0.0031)	0.0784 (0.0029)	0.0832 (0.0043)	0.0830 (0.0037)	0.0819 (0.0028)	0.0814 (0.0036)	0.0832 (0.0046)	0.0896 (0.0046)	
20	0.0647 (0.0024)	0.0675 (0.0030)	0.0675 (0.0037)	0.0719 (0.0038)	0.0700 (0.0036)	0.0789 (0.0057)	0.0753 (0.0043)	0.0713 (0.0038)	0.0700 (0.0048)	0.0704 (0.0055)	
10	0.0419 (0.0034)	0.0448 (0.0034)	0.0477 (0.0039)	0.0446 (0.0038)	0.0512 (0.0046)	0.0476 (0.0042)	0.0500 (0.0057)	0.0508 (0.0070)	0.0442 (0.0057)	0.0461 (0.0071)	
0	0.0180 (0.0085)	0.0108 (0.0025)	0.0108 (0.0025)	0.0129 (0.0031)	0.0148 (0.0031)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0869 (0.0054)	0.0859 (0.0043)	0.0840 (0.0045)	0.0880 (0.0057)	0.0882 (0.0045)	0.0856 (0.0046)	0.0851 (0.0059)	0.0857 (0.0037)	0.0861 (0.0042)	0.0903 (0.0113)	0.0873 (0.0064)
20	0.0728 (0.0077)	0.0746 (0.0074)	0.0803 (0.0099)	0.0749 (0.0049)	0.0783 (0.0067)	0.0736 (0.0109)	× ×	× ×	0.0781 (0.0059)	× ×	× ×
10	0.0479 (0.0130)	0.0518 (0.0071)	0.0432 (0.0078)	0.0741 (0.0257)	0.0464 (0.0065)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.33: Bornholdt's simplified model without strategy: Parameter γ_1 of GARCH (1,1) model

30	0.4859 (0.0082)	0.4755 (0.0090)	0.4625 (0.0113)	0.3668 (0.0225)	0.1504 (0.0319)	0.0091 (0.0068)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.4894 (0.0084)	0.4731 (0.0107)	0.4727 (0.0141)	0.3497 (0.0317)	0.0890 (0.0278)	0.0001 (0.0001)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.4821 (0.0116)	0.4851 (0.0082)	0.4737 (0.0129)	0.4547 (0.0247)	0.1105 (0.0243)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.4869 (0.0081)	0.4900 (0.0055)	0.4876 (0.0075)	0.4899 (0.0067)	0.4664 (0.0323)	0.1669 (0.0622)	0.1179 (0.0585)	0.2589 (0.0722)	0.3175 (0.1090)	0.2129 (0.0914)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.2360 (0.0973)	0.0492 (0.0505)	0.0272 (0.0374)	0.0038 (0.0049)	0.0476 (0.0932)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	0.0000 (0.0000)	× ×	× ×
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.34: Bornholdt's simplified model without strategy: P-value of parameter γ_1 of GARCH (1,1) model

30	0.5395 (0.1392)	0.5903 (0.1168)	0.6000 (0.0997)	0.8733 (0.0243)	0.8751 (0.0171)	0.9002 (0.0081)	0.9041 (0.0044)	0.9085 (0.0044)	0.9065 (0.0041)	0.9041 (0.0047)
20	0.5658 (0.1234)	0.5911 (0.1310)	0.6202 (0.1313)	0.8759 (0.0429)	0.9177 (0.0087)	0.9311 (0.0039)	0.9284 (0.0038)	0.9285 (0.0026)	0.9251 (0.0046)	0.9204 (0.0044)
10	0.5330 (0.1344)	0.5909 (0.1152)	0.6677 (0.1213)	0.6420 (0.1302)	0.9551 (0.0060)	0.9598 (0.0027)	0.9554 (0.0021)	0.9520 (0.0018)	0.9552 (0.0025)	0.9527 (0.0027)
0	0.4657 (0.1341)	0.5172 (0.1433)	0.4380 (0.1635)	0.4530 (0.1536)	0.5812 (0.1467)	0.8663 (0.1051)	0.9528 (0.0294)	0.8703 (0.0539)	0.8103 (0.1924)	0.9612 (0.0279)
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**

30	0.9065 (0.0045)	0.9025 (0.0052)	0.8970 (0.0048)	0.9035 (0.0042)	0.8969 (0.0064)	0.8983 (0.0051)	0.9002 (0.0039)	0.9009 (0.0050)	0.9002 (0.0059)	0.8904 (0.0064)
20	0.9233 (0.0034)	0.9203 (0.0041)	0.9211 (0.0047)	0.9157 (0.0050)	0.9188 (0.0045)	0.9074 (0.0079)	0.9130 (0.0059)	0.9181 (0.0046)	0.9204 (0.0058)	0.9195 (0.0067)
10	0.9530 (0.0043)	0.9503 (0.0039)	0.9467 (0.0044)	0.9506 (0.0044)	0.9431 (0.0056)	0.9472 (0.0049)	0.9450 (0.0066)	0.9436 (0.0080)	0.9511 (0.0064)	0.9495 (0.0080)
0	0.7743 (0.1970)	0.9756 (0.0085)	0.9781 (0.0079)	0.9784 (0.0057)	0.9615 (0.0334)	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**

30	0.8954 (0.0067)	0.8972 (0.0058)	0.8994 (0.0062)	0.8938 (0.0075)	0.8941 (0.0064)	0.8989 (0.0059)	0.8979 (0.0073)	0.8978 (0.0044)	0.8960 (0.0060)	0.8902 (0.0155)	0.8942 (0.0093)
20	0.9158 (0.0097)	0.9146 (0.0095)	0.9071 (0.0124)	0.9130 (0.0066)	0.9100 (0.0084)	0.9158 (0.0138)	× ×	× ×	0.9099 (0.0072)	× ×	× ×
10	0.9468 (0.0151)	0.9425 (0.0084)	0.9527 (0.0086)	0.9141 (0.0336)	0.9488 (0.0074)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.35: Bornholdt's simplified model without strategy: Parameter δ_1 of GARCH (1,1) model

30	0.2226 (0.0866)	0.1850 (0.0783)	0.1137 (0.0592)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.2264 (0.0798)	0.1576 (0.0772)	0.1585 (0.0739)	0.0119 (0.0195)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.2640 (0.0771)	0.2311 (0.0866)	0.1610 (0.0787)	0.1613 (0.0791)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.2764 (0.0785)	0.2637 (0.0926)	0.2807 (0.0869)	0.3203 (0.0799)	0.1951 (0.0849)	0.0527 (0.0519)	0.0137 (0.0269)	0.0333 (0.0426)	0.0609 (0.0868)	0.0000 (0.0000)	
$\alpha\backslash\beta$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	
0	0.0448 (0.0515)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	
$\alpha\backslash\beta$	**1**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**	**1.8**	**1.9**	
30	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
20	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	0.0000 (0.0000)	× ×	× ×
10	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	× ×	× ×	× ×	× ×	× ×	× ×
0	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×	× ×
$\alpha\backslash\beta$	**2**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3**

Table A.36: Bornholdt's simplified model without strategy: P-value of parameter δ_1 of GARCH (1,1) model

A.4 Thesis Proposal

Diploma Thesis Proposal

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Title:	Temperature Dependent Ising Model: Simulating the Effects of Changing Money Stock
Defense:	February 2013

Topic characteristics

The main focus of this thesis is to construct a temperature-variant Ising model and examine its ability to mimic some financial stylized facts including those concerning a money stock.

The Ising model is commonly used in physics to represent magnetic spins of particles in ferromagnetic materials. It can also be understood as a heterogeneous agent model to simulate behavior of market participants, as shown in number of research papers. To our best knowledge, for economic purposes the model has always been employed with a fixed temperature, i.e., total energy level of the system.

Incorporating a temperature as a variable could simulate an impact of the money stock and therefore help to asses a role of central bank. In particular, we will examine how the energy level influences frequency and nature of phase transitions that correspond to changes in the general mood of the market participants.

Hypotheses

1. The temperature-dependent Ising model is able to mimic some financial stylized facts.
2. The inclusion of temperature dependence significantly improves the model's ability to mimic the stylized facts.
3. Through variable temperature the model is able to simulate frequency and nature of periods of shifts in general market mood, i.e., periods of high volatility.

Methodology

Firstly, a temperature-dependent Ising model suitable for economical simulation will be constructed in line with previous research in physics. Secondly, a number of numerical simulations will be carried out with an aim to mimic patterns observable in financial markets. Stock indices such as S&P 500 might be used as a proxy for financial markets behavior. Lastly, through standard statistical and econometric procedures it will be tested to what extent the simulated and the actual financial series share the same properties.

Outline

1. Introduction
2. Stylized facts and a role of the central banks
3. Ising model overview
4. Ising model simulations
5. Conclusion

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