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(Conformal) Killing spinor valued forms on Riemannian manifolds

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: (Konformní) Killingovy spinor hodnotové formy na Riemannovských varietách

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Abstrakt: Cílem této práce je zavést na Riemannovské *Spin*-varietě soustavu parciálních diferenciálních rovnic pro spinor-hodnotové diferenciální formy, která se nazývá Killingovy rovnice. Zkoumáme základní vlastnosti různých druhů Killingových polí a vztahy mezi nimi. Uvádíme jednoduchou konstrukci Killingových spinor-hodnotových forem z Killingových spinorů a Killingových forem. Probíráme také konstrukci metrického konu a diskutujeme vztah mezi Killingovými spinor-hodnotovými formami na podkladové varietě a paralelními spinor-hodnotovými formami na metrickém konu.

Klíčová slova: spinor-hodnotové formy, Killingovy rovnice, invariantní diferenciální operátory, metrický konus

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Abstract: The goal of the present thesis is to introduce on a Riemannian *Spin*-manifold a system of partial differential equations for spinor-valued differential forms called Killing equations. We study basic properties of several types of Killing fields and relationships among them. We provide a simple construction of Killing spinor-valued forms from Killing spinors and Killing forms. We also review the construction of metric cone and discuss the relationship between Killing spinor-valued forms on the base manifold and parallel spinor-valued forms on the metric cone.

Keywords: spinor-valued forms, Killing equations, invariant differential operators, metric cone

Contents

Introduction

Spinor-valued differential forms, arising in the tensor products of differential forms and spinor fields, are fundamental objects in (pseudo-)Riemannian differential *Spin*-geometry. Many properties of spinors and differential forms carry over and generalize to spinor-valued forms. The subject of present thesis concentrates on Killing fields, which are solutions of particular invariant system of partial differential equations called Killing equations.

A prominent motivating example of the type just mentioned are Killing vector fields, the infinitesimal generators of isometries on a (pseudo-)Riemannian manifold. A direct generalization are then Killing differential forms. Killing spinor fields naturally appear in the study of the Dirac operator on *Spin*-manifolds of constant scalar curvature. In any case, the Killing equations are overdetermined and the existence of corresponding Killing fields on particular manifold is a rare phenomenon. However once they exist, they provide valuable information about geometry of the underlying manifold. In particular, the existence of Killing spinor fields imposes strong restrictions on curvature. In general, Killing tensors-spinors give integrals of motion for the geodesic equation and contribute to its integrability, and from a broader perspective can be regarded as hidden symmetries of the underlying pseudo-Riemannian manifold.

The main goal of the present thesis is to introduce the Killing equations for the case of spinor-valued forms. The definition is quite straightforward generalization of Killing differential forms and Killing spinor fields. To our best knowledge, the present definition is in its full generality new, although some authors produced a special case of ours. Subsequently we deduce several properties of Killing spinor-valued forms. In particular, we prove that the tensor product of a Killing form and a Killing spinor is a Killing spinor-valued form according to our definition, which is the main justification of our approach.

We also discuss two additional variants of Killing fields. The first variant are the conformal Killing fields, defined by weaker conditions than Killing fields. The general concept of conformal Killing fields is already well established and our definition of conformal Killing spinor-valued forms simply covers this case. It can be shown that the conformal Killing equations are conformally covariant and thus apply in the more general framework of conformal geometry.

The second variant are the special Killing fields, defined by stronger conditions than Killing fields. Their main application is the so called cone construction. It starts with the construction of metric cone over the original base manifold. Then it is shown that special Killing fields on the base manifold correspond to certain parallel fields on the metric cone. The cone construction for special Killing spinor-valued forms is our main result. It is again analogous to the results for spinor fields and differential forms. Though in the case of spinor fields there is no stronger notion of special Killing spinors, the cone construction applies to Killing spinors as well. The cone construction has further applications in the holonomy classification of manifolds admitting special Killing fields. However, such a classification in the case of spinor-valued forms is beyond the scope of this thesis.

The thesis is divided into three chapters. The first chapter has purely algebraic character. Here we closely examine several representations of spin group and deduce several formulas needed for later computations with spinor-valued forms. The representation theory also serves as a ground for the construction of invariant differential operators. In the second chapter, we introduce several types of Killing fields and discuss some of their basic properties. We describe the Killing equations in terms of invariant differential operators. This perspective offers a better insight into the definitions and reveals general pattern behind the different types of Killing fields. The third, final chapter, is devoted to the cone construction and our main results.

Chapter 1

Spinor-valued forms

This chapter is devoted to algebraic preliminaries needed for our study of spinor-valued forms. After the introduction of basic setting and notions we deal mostly with representations of the spin group. Our primary objective is the decomposition of several such representations. The most important part of the decomposition are the so called twistor modules which will later give rise to twistor operators. In the course of our exposition we introduce an effective algebraic calculus for spinor-valued forms.

1.1 Vectors and forms

First of all we establish the notation and briefly recall some basic properties of vectors and forms. We start with an oriented unitary real vector space. All subsequent notions and constructions in the present chapter will depend solely on this initial data. Note that we consider only positive-definite inner product though most results could be presumably generalized to arbitrary non-degenerate bilinear product.

For the sake of simplicity we consider the arithmetic space and denote:

- $\mathsf{V} = \mathbb{R}^n$ — the *real arithmetic vector space* of dimension n,
- $(e_1, \ldots, e_n)$ — its canonical basis,
- $g\colon \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ — the canonical *inner product* on V,
- the standard *orientation* of V,

such that the canonical basis is orthonormal and positively oriented.

We further denote:

- V^* — the *dual vector space* of V.

The inner product g induces mappings:

- ${}^*\colon \mathsf{V} \rightleftarrows \mathsf{V}^*$ — the *orthogonal dual* of a vector or 1-form,

which are mutually inverse, and hence yielding:

$$\mathsf{V} \cong \mathsf{V}^*. \tag{1.1}$$

We further denote:

- A^p — the space of *alternating p-forms* on V, where
- p is the *degree* of form.

Unless otherwise stated we always assume that

$$p \in \{0, \dots, n\}. \tag{1.2}$$

In particular, we can identify:

$$\mathsf{A}^0 = \mathbb{R}, \qquad \text{and} \qquad \mathsf{A}^1 = \mathsf{V}^*. \tag{1.3}$$

Altogether the spaces A^p form

- A — the *exterior algebra* over V^*,

$$\mathsf{A} = \mathsf{A}^0 \oplus \cdots \oplus \mathsf{A}^n. \tag{1.4}$$

It is a real $\mathbb{Z}$-graded associative algebra and we additionally define:

$$\mathsf{A}^p = 0, \qquad \forall p \in \mathbb{Z} \setminus \{0, \dots, n\}. \tag{1.5}$$

The basic operations with vectors and alternating forms are:

- $\wedge \colon \mathsf{A}^p \times \mathsf{A}^{p'} \to \mathsf{A}^{p+p'}$ — the *exterior product*,

which is just the multiplication in the exterior algebra A,

- $\lrcorner \colon \mathsf{V} \times \mathsf{A}^p \to \mathsf{A}^{p-1}$ — the *interior product*,

which generalizes the usual pairing of vectors and 1-forms. The exterior and interior products satisfy the well-known identities:

$$\begin{aligned} \alpha \wedge \alpha' &= (-1)^{pp'} \alpha' \wedge \alpha, \\ X \lrcorner (X' \lrcorner \alpha) &= -X' \lrcorner (X \lrcorner \alpha), \\ X \lrcorner (\alpha \wedge \alpha') &= (X \lrcorner \alpha) \wedge \alpha' + (-1)^p \alpha \wedge (X \lrcorner \alpha'), \end{aligned} \tag{1.6}$$

$\forall \alpha \in \mathsf{A}^p$, $\alpha' \in \mathsf{A}^{p'}$, $X, X' \in \mathsf{V}$.

Finally we recall

- $\omega \in \mathsf{A}^n$ — the *volume form* on V,

$$\omega = e_1^* \wedge \cdots \wedge e_n^*, \tag{1.7}$$

which is uniquely determined by the inner product and orientation. It further induces mappings:

- $\star \colon \mathsf{A}^p \leftrightarrows \mathsf{A}^{n-p}$ — the *Hodge dual* of alternating forms,

which are mutually inverse up to the factor of $(-1)^{p(n-p)}$, and hence yielding:

$$\mathsf{A}^p \cong \mathsf{A}^{n-p}. \tag{1.8}$$

1.2 Spinors

As a next step towards spinor-valued forms we recall the construction and basic properties of spinors. The topic of spinors and related Clifford algebras is quite extensive, so we restrict ourselves only to necessary details. For the comprehensive theory see, e.g., [5], [12], [7] or [3].

There are both real and complex Clifford algebras, however, we restrict ourselves just to the complex case. We denote:

- $\mathsf{Cl}(n)$ — the *complex Clifford algebra* of V, that is, the Clifford algebra of the complexified space $\mathsf{V} \otimes \mathbb{C}$,
- $\cdot\colon \mathsf{Cl}(n) \times \mathsf{Cl}(n) \to \mathsf{Cl}(n)$ — the multiplication in $\mathsf{Cl}(n)$.

It is a complex associative algebra with unit, which is generated by V and is universal with respect to the main property:

$$X \cdot X = -2g(X, X), \qquad \forall X \in \mathsf{V}. \tag{1.9}$$

By the polarization identity, the equality (1.9) is equivalent to

$$X \cdot X' + X' \cdot X = -2g(X, X'), \qquad \forall X, X' \in \mathsf{V}. \tag{1.10}$$

This identity is fundamental for all computations with spinors and we shall use it frequently without further reference. In terms of the canonical basis, (1.10) can be expressed also as

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}, \qquad \forall i, j \in \{1, \dots, n\}, \tag{1.11}$$

where δ_{ij} is the *Kronecker delta*.

The Clifford algebra is always a $\mathbb{Z}_2$-graded algebra and we denote:

- $\mathsf{Cl}_+(n)$ — the *even part* of $\mathsf{Cl}(n)$,
- $\mathsf{Cl}_-(n)$ — the *odd part* of $\mathsf{Cl}(n)$.

The grading is completely determined by:

$$\mathbb{C} \subseteq \mathsf{Cl}_+(n), \qquad \text{and} \qquad \mathsf{V} \subseteq \mathsf{Cl}_-(n), \tag{1.12}$$

because V generates $\mathsf{Cl}(n)$.

There appear substantial differences between even and odd dimension. As usual, we determine these two cases by taking

- k, such that

$$n = 2k \text{ or } 2k + 1. \tag{1.13}$$

Now we can introduce:

- S — the *complex spinor space* corresponding to $\mathsf{Cl}(n)$,

which is an irreducible $\mathsf{Cl}(n)$-module and, in particular, a complex vector space. It arises from the isomorphisms which describe the structure of $\mathsf{Cl}(n)$:

$$\mathsf{Cl}(2k) \cong \mathsf{End}_{\mathbb{C}}(\mathsf{S}), \qquad \text{and} \qquad \mathsf{Cl}(2k+1) \cong \mathsf{End}_{\mathbb{C}}(\mathsf{S}) \oplus \mathsf{End}_{\mathbb{C}}(\mathsf{S}), \tag{1.14}$$

where we denote:

- $\mathsf{End}_{\mathbb{C}}(\mathsf{S})$ — the *full algebra of complex linear endomorphisms* of S.

In case $n = 2k + 1$ there are clearly two possible $\mathsf{Cl}(n)$-module structures on the vector space S; we just permanently choose one of them.

The choice can be made explicit by considering the element

- $\widetilde{\omega} \in \mathsf{Cl}(n)$,

$$\widetilde{\omega} = \mathsf{i}^{\frac{n(n+1)}{2}}\, e_1 \cdots e_n. \tag{1.15}$$

This element is uniquely determined by the inner product and the orientation. In fact, up to the scalar factor it directly corresponds to the volume form ω. Now the two possible projections in case $n = 2k+1$,

- $\pi_+^{\mathsf{Cl}}, \pi_-^{\mathsf{Cl}} \colon \mathsf{Cl}(2k+1) \to \mathsf{End}_{\mathbb{C}}(\mathsf{S})$,

are explicitly distinguished because it holds:

$$\pi_\pm^{\mathsf{Cl}}(\widetilde{\omega}) = \pm 1. \tag{1.16}$$

Also note that the two projections coincide when restricted to the even part $\mathsf{Cl}_+(2k+1)$.

Since $\mathsf{V} \subseteq \mathsf{Cl}(n)$, the $\mathsf{Cl}(n)$-module structure on the spinor space S restricts to a bilinear mapping:

- $\cdot \colon \mathsf{V} \times \mathsf{S} \to \mathsf{S}$ — the *Clifford multiplication* of spinors by vectors.

On the other hand, the Clifford multiplication completely determines the $\mathsf{Cl}(n)$-module structure on S because V generates $\mathsf{Cl}(n)$. Note that we use the same symbol $\cdot$ for both the product in $\mathsf{Cl}(n)$ and Clifford multiplication. This notation can hardly cause any confusion and is commonly used.

1.3 Representations

Before proceeding to spinor-valued forms we outline the representation theory of orthogonal and spin groups. The representations are essential for the later passage to geometry, in particular, the construction of associated vector bundles. Like in the previous section we are brief and focus only on necessary results. For more details see any textbook on the representation theory of Lie groups and algebras, e.g., [6] or [8].

As a general rule, when referring to representations we always consider a unique standard group action and thus identify representations with their underlying spaces.

Firstly, we consider the classical Lie groups:

- $\mathsf{O}(n)$ — the *orthogonal group* of V,

defined as the group of all automorphisms of V which preserve the inner product g; and its subgroup

- $\mathsf{SO}(n)$ — the *special orthogonal group* of V,

defined as the group of all automorphisms of V which in addition preserve the orientation. Secondly, we consider groups which arise by taking particular invertible elements in $\mathsf{Cl}(n)$:

- $\mathsf{Pin}(n)$ — the *pin group* of V, generated by unit vectors in V; and its subgroup

- $\mathsf{Spin}(n)$ — the *spin group* of V,

which is generated by products of two unit vectors in V. The latter two groups are closed submanifolds of $\mathsf{Cl}(n)$ and hence Lie groups.

The two pairs of Lie groups are related via

- the *covering homomorphism* $\lambda\colon \mathsf{Pin}(n) \to \mathsf{O}(n)$.

It comes from a modification of the adjoint action of $\mathsf{Pin}(n)$ on $\mathsf{Cl}(n)$, which leaves the subspace V invariant. Moreover the action on V satisfies:

1) $\mathsf{Pin}(n)$ preserves the inner product g.

2) $\mathsf{Spin}(n)$ preserves in addition to g also the orientation.

Eventually λ fits into the commutative diagram:

$$\begin{array}{ccc} \mathsf{Spin}(n) & \hookrightarrow & \mathsf{Pin}(n) \\ \lambda\downarrow & & \downarrow\lambda \\ \mathsf{SO}(n) & \hookrightarrow & \mathsf{O}(n) \end{array}, \tag{1.17}$$

where the kernel of λ is:

$$\mathrm{Ker}(\lambda) = \{\pm 1\}. \tag{1.18}$$

The λ is also a smooth two-fold covering map, hence it is indeed a covering homomorphism of topological groups.

Given the defining action of $\mathsf{O}(n)$ and $\mathsf{SO}(n)$ and the covering homomorphism λ, the vector space V becomes a representation of all the four groups. Consequently, the vector spaces V^* and A^p constructed from V become representations in the usual way. Note that the actions of distinct groups are always related through the commutative diagram (1.17).

The groups $\mathsf{Pin}(n)$ and $\mathsf{Spin}(n)$ have also the *spin representation* on the spinor space S. The action on S is given simply by restriction of the $\mathsf{Cl}(n)$-module structure. The spin representation of $\mathsf{Spin}(n)$ is faithful and thus cannot be induced from a representation of $\mathsf{SO}(n)$. Consequently neither the spin representation of $\mathsf{Pin}(n)$ can be induced from a representation of $\mathsf{O}(n)$.

Note that the spin representation of $\mathsf{Pin}(n)$ depends on the choice of projection π_+^{Cl} or π_-^{Cl} from section 1.2 and in fact there are two non-equivalent representations. However, since $\mathsf{Spin}(n) \subseteq \mathsf{Cl}_+(n)$ the spin representation of $\mathsf{Spin}(n)$ is independent of this choice and hence unique.

All the representations introduced so far are irreducible with the following exceptions in case $n = 2k$:

1) The complexification of real representation A^k of $\mathsf{SO}(2k)$ and $\mathsf{Spin}(2k)$ decomposes as

$$\mathsf{A}^k \otimes \mathbb{C} = \mathsf{A}^k_+ \oplus \mathsf{A}^k_-. \tag{1.19}$$

This decomposition is induced by the involutive action of the volume form ω, in particular the Hodge dual. The real representation A^k itself decomposes similarly only in the case $n = 4l$.

2) The spin representation S of $\mathsf{Spin}(2k)$ decomposes as

$$\mathsf{S} = \mathsf{S}_+ \oplus \mathsf{S}_-, \tag{1.20}$$

where

- S_+ and S_- are the *half-spinor* spaces.

This decomposition is induced by the element $\widetilde{\omega}$ from (1.15).

Next we examine several mappings introduced before from the perspective of representation theory. In order to obtain proper linear maps we first extend the bilinear products to the corresponding tensor products:

- the inner product $g\colon \mathsf{V} \otimes \mathsf{V} \to \mathbb{R}$,
- the exterior product $\wedge\colon \mathsf{A}^p \otimes \mathsf{A}^{p'} \to \mathsf{A}^{p+p'}$,
- the interior product $\lrcorner\colon \mathsf{V} \otimes \mathsf{A}^p \to \mathsf{A}^{p-1}$,
- the Clifford multiplication $\cdot\colon \mathsf{V} \otimes \mathsf{S} \to \mathsf{S}$.

It turns out that the mappings mostly commute with the Lie group actions, i.e., they are *intertwining mappings* of the representations. In particular:

1) The inner, exterior and interior products and the orthogonal dual are intertwining with respect to $\mathsf{O}(n)$ and thus also $\mathsf{SO}(n)$, $\mathsf{Pin}(n)$ and $\mathsf{Spin}(n)$.

2) The Clifford multiplication is intertwining with respect to $\mathsf{Pin}(n)$ and thus also $\mathsf{Spin}(n)$.

3) The Hodge dual is intertwining only with respect to $\mathsf{SO}(n)$ and thus also $\mathsf{Spin}(n)$ because it depends on the orientation.

We conclude the present section by recalling the highest weight theory for the Lie algebra $\mathfrak{so}(n)$. But the theory applies only to simple Lie algebras, so we shall assume $n \geq 3$.

However, for our study of spinor-valued forms this theory is not at all essential and we reveal it just for the sake of reference. Hence the assumption $n \geq 3$ applies only to the statements about highest weights, which we deem just as auxiliary. We do not provide proofs of such statements; they can be carried out, for instance, by directly determining the highest weight vectors.

We consider the Lie algebras:

- $\mathfrak{so}(n)$ — the *special orthogonal Lie algebra* of V,

which is the Lie algebra $\mathsf{O}(n)$ and $\mathsf{SO}(n)$,

- $\mathfrak{spin}(n)$ — the *spin Lie algebra* of V,

which is the Lie algebra of $\mathsf{Pin}(n)$ and $\mathsf{Spin}(n)$.

The differential of the covering homomorphism,

- $\mathrm{d}\lambda\colon \mathfrak{spin}(n) \to \mathfrak{so}(n)$,

turns out to be an isomorphism of Lie algebras:

$$\mathfrak{spin}(n) \cong \mathfrak{so}(n). \tag{1.21}$$

We shall identify the Lie algebras via this isomorphism and subsequently deal only with $\mathfrak{so}(n)$.

To each representation of $\mathsf{O}(n)$, $\mathsf{SO}(n)$, $\mathsf{Pin}(n)$ or $\mathsf{Spin}(n)$ corresponds a representation of $\mathfrak{so}(n)$ defined by taking its differential. Note that the resulting representation of $\mathfrak{so}(n)$ is the same regardless of the Lie group, as long as the actions of distinct groups are related by the homomorphisms in diagram (1.17). On the other hand, not every representation of $\mathfrak{so}(n)$ comes from a representation of $\mathsf{O}(n)$, $\mathsf{SO}(n)$ or $\mathsf{Pin}(n)$. This is generally possible only for $\mathsf{Spin}(n)$ which is the *simply connected compact real form* of $\mathfrak{so}(n)$.

A *dominant weight* of the Lie algebra $\mathfrak{so}(n)$ is a k-tuple consisting entirely of integers or entirely of half-integers,

$$(\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k \cup (\mathbb{Z} + \tfrac{1}{2})^k, \tag{1.22}$$

such that:

a) if $n = 2k$ then

$$\mu_1 \geq \cdots \geq \mu_{k-1} \geq |\mu_k|, \tag{1.23}$$

b) if $n = 2k + 1$ then

$$\mu_1 \geq \cdots \geq \mu_k \geq 0. \tag{1.24}$$

Subsequently we can assign a unique dominant weight:

- $\mu(\mathsf{U})$ — the *highest weight* of U,

to each finite-dimensional irreducible representation U of $\mathfrak{so}(n)$. This way, all such representations are classified and enumerated up to an equivalence.

In order to simplify matters we also introduce a special notation to handle decompositions like (1.19) and (1.20),

$$\mu(\mathsf{U}) = (\mu_1, \ldots, \mu_{k-1}, \pm\, \mu_k), \tag{1.25}$$

meaning that:

a) either $n = 2k$ and U is a direct sum of two irreducible representations U_+ and U_- such that

$$\mu(\mathsf{U}_+) = (\mu_1, \ldots, \mu_{k-1}, \mu_k),$$
$$\mu(\mathsf{U}_-) = (\mu_1, \ldots, \mu_{k-1}, -\, \mu_k), \tag{1.26}$$

b) or $n = 2k + 1$ and U is irreducible such that

$$\mu(\mathsf{U}) = (\mu_1, \ldots, \mu_k). \tag{1.27}$$

The highest weights of the representations introduced so far are:

$$\begin{aligned}
\mu(\mathbb{R}) &= (0, \ldots, 0), \\
\mu(\mathsf{V}) &= \mu(\mathsf{V}^*) = (1, 0, \ldots, 0), \\
\mu(\mathsf{A}^p) &= (\underbrace{1, \ldots, 1}_{p\times}, 0, \ldots, 0), && \forall p \in \{0, \ldots, k-1\}, \\
\mu(\mathsf{A}^k) &= (1, \ldots, 1, \pm 1), \\
\mu(\mathsf{S}) &= (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \pm\tfrac{1}{2}).
\end{aligned} \tag{1.28}$$

In general, any representation of $\mathfrak{so}(n)$ which comes from a representation of $\mathsf{SO}(n)$ or $\mathsf{O}(n)$ has highest weight consisting of integers.

1.4 Spinor-valued forms

In this section we introduce and decompose the spaces of spinor-valued forms. Unless otherwise stated, we always work with representations of the spin group $\mathsf{Spin}(n)$. Accordingly, we assume all mappings intertwining and subspaces invariant with respect to $\mathsf{Spin}(n)$. We do not necessarily come down to irreducible summands. For instance, we do not pursue the decompositions (1.19) and (1.20). In fact, our decompositions could be proved irreducible just up to a similar sum of two half-spaces.

We denote:

- SA^p the space of *spinor-valued alternating p-forms*,

which is defined as the tensor product

$$\mathsf{SA}^p = \mathsf{A}^p \otimes \mathsf{S}. \tag{1.29}$$

Note that, in particular,

$$\mathsf{SA}^0 = \mathbb{R} \otimes \mathsf{S} = \mathsf{S} \tag{1.30}$$

is just the spinor space. Like with ordinary forms we use the convention

$$\mathsf{SA}^p = 0, \qquad \forall p \in \mathbb{Z} \setminus \{0, \ldots, n\}, \tag{1.31}$$

which will simplify some subsequent definitions and arguments by induction.

The Clifford multiplication of spinors by vectors can be treated as an $\mathsf{End}_{\mathbb{C}}(\mathsf{S})$-valued 1-form. We denote this 1-form and its orthogonal dual by

- $\gamma\cdot \in \mathsf{V}^* \otimes \mathsf{End}_{\mathbb{C}}(\mathsf{S})$ — the *Clifford multiplication form*,
- $\gamma^*\!\cdot \in \mathsf{V} \otimes \mathsf{End}_{\mathbb{C}}(\mathsf{S})$.

They can be described in terms of an orthonormal basis by equations:

$$\gamma\cdot = \sum_{i=1}^{n} e_i^* \otimes (e_i\cdot), \qquad \gamma^*\!\cdot = \sum_{i=1}^{n} e_i \otimes (e_i\cdot). \tag{1.32}$$

Recall that the Clifford multiplication is an intertwining mapping and so will be mappings constructed using $\gamma\cdot$ and $\gamma^*\!\cdot$. In fact, the $\gamma\cdot$ and $\gamma^*\!\cdot$ are invariant elements of the respective representations.

Subsequently we can build complex expressions from spinor-valued forms and $\gamma\cdot$ or $\gamma^*\cdot$ using:

- the exterior and interior product operating on the form part,
- the Clifford multiplication operating on the spinor part.

As an example, the Clifford multiplication can be expressed as:

$$X \cdot \Psi = (X \lrcorner \gamma\cdot)\Psi = (\gamma^*\cdot \lrcorner X^*)\Psi, \qquad \forall X \in \mathsf{V},\ \Psi \in \mathsf{S}. \tag{1.33}$$

From (1.6) and (1.10) we can further deduce several identities:

$$\begin{aligned} X \cdot (\gamma\cdot \wedge \Phi) + \gamma\cdot \wedge (X \cdot \Phi) &= -2X^* \wedge \Phi, \\ X \cdot (\gamma^*\cdot \lrcorner \Phi) + \gamma^*\cdot \lrcorner (X \cdot \Phi) &= -2X \lrcorner \Phi, \\ X \lrcorner (\gamma\cdot \wedge \Phi) + \gamma\cdot \wedge (X \lrcorner \Phi) &= X \cdot \Phi, \\ X^* \wedge (\gamma^*\cdot \lrcorner \Phi) + \gamma^*\cdot \lrcorner (X^* \wedge \Phi) &= X \cdot \Phi, \end{aligned} \tag{1.34}$$

$\forall \Phi \in \mathsf{SA}^p$, $X \in \mathsf{V}$. These formulas are fundamental for our computations with spinor-valued forms and we shall use them without further notice. Another important identity is subject of the next lemma.

Lemma 1. *$\forall \Phi \in \mathsf{SA}^p$,*

$$\gamma^*\cdot \lrcorner (\gamma\cdot \wedge \Phi) - \gamma\cdot \wedge (\gamma^*\cdot \lrcorner \Phi) = (2p - n)\Phi. \tag{1.35}$$

Proof. Using (1.6), (1.11) and (1.32) we compute:

$$\begin{aligned} &\gamma^*\cdot \lrcorner (\gamma\cdot \wedge \Phi) = \sum_{i,j=1}^{n} e_i \lrcorner (e_j^* \wedge (e_i \cdot e_j \cdot \Phi)) = \\ &= \sum_{i=1}^{n} e_i \cdot e_i \cdot \Phi - \sum_{i,j=1}^{n} e_j^* \wedge (e_i \lrcorner (e_i \cdot e_j \cdot \Phi)) = \\ &= -n\Phi + 2\sum_{i=1}^{n} e_i^* \wedge (e_i \lrcorner \Phi) + \sum_{i,j=1}^{n} e_j^* \wedge (e_i \lrcorner (e_j \cdot e_i \cdot \Phi)) = \\ &= (2p - n)\Phi + \gamma\cdot \wedge (\gamma^*\cdot \lrcorner \Phi). \end{aligned}$$

□

We can also put together the spaces SA^p yielding:

- SA — the space of all *spinor-valued forms*,

$$\mathsf{SA} = \mathsf{A} \otimes \mathsf{S} = \mathsf{SA}^0 \oplus \cdots \oplus \mathsf{SA}^n. \tag{1.36}$$

Note that SA is clearly a representation of $\mathsf{Spin}(n)$ but it is no longer an algebra, contrary to the exterior algebra A. Now $(\gamma\cdot\wedge)$ and $(\gamma^*\cdot\lrcorner)$ become linear algebraic operators:

- $(\gamma\cdot\wedge)\colon \mathsf{SA} \to \mathsf{SA}$,

$$(\gamma\cdot\wedge)(\Phi) = \gamma\cdot \wedge \Phi, \tag{1.37}$$

- $(\gamma^*\cdot\lrcorner)\colon \mathsf{SA} \to \mathsf{SA}$,

$$(\gamma^*\cdot\lrcorner)(\Phi) = \gamma^*\cdot \lrcorner\, \Phi, \tag{1.38}$$

$\forall \Phi \in \mathsf{SA}$. In order to get more compact formulas, we shall often work with these operators alone without the argument Φ. We can also take the iterations of these operators. Restricting back to p-forms we get mappings for $p' > p$,

- $(\gamma\cdot\wedge)^{p'-p}|_{\mathsf{SA}^p}\colon \mathsf{SA}^p \to \mathsf{SA}^{p'}$,

$$(\gamma\cdot\wedge)^{p'-p}(\Phi) = \underbrace{\gamma\cdot \wedge (\dots(\gamma\cdot\wedge}_{(p'-p)\times}\Phi)\dots), \tag{1.39}$$

$\forall \Phi \in \mathsf{SA}^p$,

- $(\gamma^*\cdot\lrcorner)^{p'-p}|_{\mathsf{SA}^{p'}}\colon \mathsf{SA}^{p'} \to \mathsf{SA}^p$,

$$(\gamma^*\cdot\lrcorner)^{p'-p}(\Phi) = \underbrace{\gamma^*\cdot \lrcorner (\dots(\gamma^*\cdot\lrcorner}_{(p'-p)\times}\Phi)\dots), \tag{1.40}$$

$\forall \Phi \in \mathsf{SA}^{p'}$.

As usual we additionally define the zero power of an operator to be the identity mapping,

$$(\gamma\cdot\wedge)^0 = (\gamma^*\cdot\lrcorner)^0 = 1_{\mathsf{SA}}. \tag{1.41}$$

Unfortunately, the mappings $(\gamma\cdot\wedge)^{p-p'}$ and $(\gamma^*\cdot\lrcorner)^{p-p'}$ are generally not even one-sided inverses of each other, but they come close to being so.

Decomposition of the space SA and, in particular, its subspaces SA^p can be obtained using the technique of *Howe dual pairs*. If we denote the following intertwining operators on SA

$$X = (\gamma\cdot\wedge), \qquad Y = -(\gamma^*\cdot\lrcorner), \qquad H = [X, Y], \tag{1.42}$$

it turns out that they span a Lie algebra isomorphic to

- $\mathfrak{sl}(2)$ — the *special linear Lie algebra* on 2-dimensional vector space.

Consequently, the well-known structure of representations of $\mathfrak{sl}(2)$ can be utilized to decompose SA with respect to $\mathsf{Pin}(n)$ and $\mathsf{Spin}(n)$. This approach was first carried out by M. Slupinski in [15] and later employed by P. Somberg in [16]; for the general technique see also [9].

Here we follow the approach in a rather elementary way yielding the decomposition without explicitly utilizing the representations of $\mathfrak{sl}(2)$. First note that (1.35) can be written as

$$H|_{\mathsf{SA}^p} = [(\gamma^*\cdot\lrcorner), (\gamma\cdot\wedge)]|_{\mathsf{SA}^p} = (2p - n). \tag{1.43}$$

Hence the decomposition of SA to SA^p is in fact the eigenvalue decomposition with respect to H and the spaces SA^p are *weight spaces* of $\mathfrak{sl}(2)$.

To further decompose SA^p we introduce:

- the invariant subspace of *primitive spinor-valued q-forms* $\mathsf{U}^q \subseteq \mathsf{SA}^q$,

$$\mathsf{U}^q = \mathrm{Ker}((\gamma^*\!\cdot\lrcorner)|_{\mathsf{SA}^q}) = \{\Phi \in \mathsf{SA}^q \mid \gamma^*\!\cdot \lrcorner\, \Phi = 0\}. \tag{1.44}$$

Recall the number k defined in (1.13) and unless otherwise stated we always assume that

$$q \in \{0, \dots, k\}. \tag{1.45}$$

Comparing the dimensions implies that all the U^q must be non-zero. Note that, in particular,

$$\mathsf{U}^0 = \mathsf{SA}^0 = \mathsf{S}. \tag{1.46}$$

The highest weight of U^q with respect to $\mathfrak{so}(n)$ is given by

$$\mu(\mathsf{U}^q) = (\underbrace{\tfrac{3}{2}, \dots, \tfrac{3}{2}}_{q\times}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \pm\tfrac{1}{2}), \qquad \forall q \in \{0, \dots, k-1\},$$
$$\mu(\mathsf{U}^k) = (\tfrac{3}{2}, \dots, \tfrac{3}{2}, \pm\tfrac{3}{2}). \tag{1.47}$$

On the other hand, U^q consists of *lowest weight vectors* with respect to $\mathfrak{sl}(2)$ from the Howe duality.

We further introduce

- invariant subspaces $\mathsf{SA}^p[q] \subseteq \mathsf{SA}^p$,

$$\mathsf{SA}^p[q] = (\gamma\cdot\wedge)^{p-q}(\mathsf{U}^q), \qquad \forall q \in \{0, \dots, l(p)\}, \tag{1.48}$$

- where we denote $l(p) \in \{0, \dots, k\}$,

$$l(p) = \min\{p, n-p\}. \tag{1.49}$$

Note that, in particular,

$$\mathsf{SA}^q[q] = \mathsf{U}^q. \tag{1.50}$$

Also note that

$$q \in \{0, \dots, l(p)\} \qquad \text{is equivalent to} \qquad p \in \{q, \dots, n-q\}. \tag{1.51}$$

These subspaces will turn out to form an eigenvalue decomposition of SA^p.

Lemma 2. $\forall q \in \{0, \dots, l(p)\}$,

$$\gamma^*\!\cdot \lrcorner\, (\gamma\cdot\wedge)|_{\mathsf{SA}^p[q]} = -(p-q+1)(n-p-q). \tag{1.52}$$

Proof. We proceed by induction on p.

1) If $p = q$ then (1.52) follows directly from (1.35).

2) Next suppose $p > q$. So let $\Phi \in \mathsf{SA}^p[q]$ and by (1.48) there exists $\Phi' \in \mathsf{U}^q$ such that

$$\Phi = (\gamma\cdot\wedge)^{p-q}(\Phi').$$

Using (1.35) and the induction hypothesis, we compute:

$$\begin{aligned}
&\gamma^*\cdot \lrcorner\, (\gamma\cdot \wedge \Phi) = \gamma^*\cdot \lrcorner\, (\gamma\cdot\wedge)^{p-q+1}(\Phi') = \\
&= (2p-n)(\gamma\cdot\wedge)^{p-q}(\Phi') + \gamma\cdot \wedge (\gamma^*\cdot \lrcorner\, (\gamma\cdot\wedge)^{p-q}(\Phi')) = \\
&= (2p-n-(p-q)(n+1-p-q))(\gamma\cdot\wedge)^{p-q}(\Phi') = \\
&= -(p-q+1)(n-p-q)\Phi.
\end{aligned}$$

□

Combining (1.52) with (1.35) we get the following dual identity.

Corollary 3. $\forall q \in \{0,\dots,l(p)\}$,

$$\gamma\cdot \wedge (\gamma^*\cdot\lrcorner)|_{\mathsf{SA}^p[q]} = -(p-q)(n-p-q+1). \tag{1.53}$$

We denote the scalar factor from (1.52) by

$$c(q,p) = -(p-q+1)(n-p-q). \tag{1.54}$$

Clearly $c(q,p)$ is non-zero unless $p = n-q$.

Corollary 4. *Let $q \in \{0,\dots,k\}$ and $p, p' \in \{q,\dots,n-q\}$ such that $p < p'$. Then the restricted mappings*

- $(\gamma\cdot\wedge)^{p'-p}|_{\mathsf{SA}^p[q]} \colon \mathsf{SA}^p[q] \to \mathsf{SA}^{p'}[q]$, *and*
- $(\gamma^*\cdot\lrcorner)^{p'-p}|_{\mathsf{SA}^{p'}[q]} \colon \mathsf{SA}^{p'}[q] \to \mathsf{SA}^p[q]$

are mutually inverse isomorphisms up to a non-zero constant. In particular,

$$\mathsf{U}^q = \mathsf{SA}^q[q] \cong \dots \cong \mathsf{SA}^p[q] \cong \dots \cong \mathsf{SA}^{p'}[q] \cong \dots \cong \mathsf{SA}^{n-q}[q]. \tag{1.55}$$

Proof. At first we prove the case when $p' = p+1$. By (1.52) we have

$$(\gamma^*\cdot\lrcorner) \circ (\gamma\cdot\wedge)|_{\mathsf{SA}^p[q]} = \gamma^*\cdot \lrcorner\, (\gamma\cdot\wedge)|_{\mathsf{SA}^p[q]} = c(q,p),$$

with $c(q,p)$ non-zero since $p+1 \le n-q$. By (1.48) the image of $(\gamma\cdot\wedge)|_{\mathsf{SA}^p[q]}$ is indeed $\mathsf{SA}^{p+1}[q]$ and the claim follows.

Now the general case follows easily by induction. □

Corollary 5. *The subspaces $\mathsf{SA}^p[q]$ are linearly independent, that is, they form a direct sum*

$$\mathsf{SA}^p[0] \oplus \dots \oplus \mathsf{SA}^p[l(p)] \subseteq \mathsf{SA}^p. \tag{1.56}$$

Proof. According to (1.52), the subspaces $\mathsf{SA}^p[q]$ are eigenvalue subspaces of the mapping $\gamma^*\cdot \lrcorner\, (\gamma\cdot\wedge)$. Hence it suffices to show that the eigenvalues $c(q,p)$ are mutually different. Now by (1.54) if

$$c(q_1,p) = c(q_2,p),$$

then either

$$q_1 = q_2 \qquad \text{or} \qquad q_1 + q_2 = n+1.$$

But the second case cannot occur since $q_1, q_2 \in \{0,\dots,k\}$. □

Proposition 6. *The space* SA^p *decomposes as:*

$$\mathsf{SA}^p = \mathsf{SA}^p[0] \oplus \cdots \oplus \mathsf{SA}^p[l(p)] \cong \mathsf{U}^0 \oplus \cdots \oplus \mathsf{U}^{l(p)}. \tag{1.57}$$

Proof. We shall proceed by induction on p.

1) First if $p = 0$, then

$$\mathsf{SA}^0 = \mathsf{U}^0 = \mathsf{SA}^0[0],$$

so there is nothing to prove.

2) Next let $0 < p \leq k$. By the induction hypothesis the space SA^{p-1} decomposes as

$$\mathsf{SA}^{p-1} = \mathsf{SA}^{p-1}[0] \oplus \cdots \oplus \mathsf{SA}^{p-1}[p-1].$$

Hence, according to corollaries 4 and 5, the mapping $(\gamma^* \cdot \lrcorner)$ maps the subspace

$$\mathsf{SA}^p[0] \oplus \cdots \oplus \mathsf{SA}^p[p-1] \subseteq \mathsf{SA}^p$$

isomorphically onto SA^{p-1}. Moreover, we have

$$\operatorname{Ker}((\gamma^* \cdot \lrcorner)|_{\mathsf{SA}^p}) = \mathsf{U}^p = \mathsf{SA}^p[p]$$

and (1.57) follows by the isomorphism theorem.

3) Finally let $k < p \leq n$. From corollary 5 we already have the inclusion

$$\mathsf{SA}^p[0] \oplus \cdots \oplus \mathsf{SA}^p[n-p] \subseteq \mathsf{SA}^p.$$

From the induction hypothesis for $n - p$ we have

$$\mathsf{SA}^{n-p} = \mathsf{SA}^{n-p}[0] \oplus \cdots \oplus \mathsf{SA}^{n-p}[n-p].$$

The spaces SA^p and SA^{n-p} have equal dimension and by corollary 4

$$\mathsf{SA}^p[q] \cong \mathsf{SA}^{n-p}[q],$$

hence the inclusion must be equality. □

We denote the projections corresponding to the decomposition (1.57) by

- $\pi^{\mathsf{SA}}_{p,q} \colon \mathsf{SA}^p \to \mathsf{SA}^p[q]$ — the qth *primitive part*,

$\forall q \in \{0, \ldots, l(p)\}$. We additionally define

$$\pi^{\mathsf{SA}}_{p,q} = 0, \tag{1.58}$$

$\forall q \notin \{0, \ldots, l(p)\}$. From corollary 4 and proposition 6 follows that these projections commute with the mappings $(\gamma \cdot \wedge)$ and $(\gamma^* \cdot \lrcorner)$,

$$\begin{aligned} \gamma \cdot \wedge \pi^{\mathsf{SA}}_{p,q} &= \pi^{\mathsf{SA}}_{p+1,q} \circ (\gamma \cdot \wedge)|_{\mathsf{SA}^p}, \\ \gamma^* \cdot \lrcorner \, \pi^{\mathsf{SA}}_{p,q} &= \pi^{\mathsf{SA}}_{p-1,q} \circ (\gamma^* \cdot \lrcorner)|_{\mathsf{SA}^p}. \end{aligned} \tag{1.59}$$

Corollary 7. *In particular,* $\forall \Phi \in \mathsf{U}^k$*:*

a) either $n = 2k+1$ *and then*

$$\gamma\cdot \wedge (\gamma\cdot \wedge \Phi) = 0. \tag{1.60}$$

b) or $n = 2k$ *and then*

$$\gamma\cdot \wedge \Phi = 0, \tag{1.61}$$

Proof. Again a direct consequence of corollary 4 and proposition 6. □

Having the decomposition (1.57), we can denote

- invariant subspace $\mathsf{W}^p \subseteq \mathsf{SA}^p$,

$$\mathsf{W}^p = \mathsf{SA}^p[1] \oplus \cdots \oplus \mathsf{SA}^p[l(p)], \tag{1.62}$$

with the lowest summand isomorphic to $\mathsf{U}^0 = \mathsf{S}$ omitted. In particular,

$$\mathsf{W}^0 = \mathsf{W}^n = 0. \tag{1.63}$$

This space and the following technical lemma will play an important role in the next section. In fact, the lemma is the main reason why we examined the decomposition so closely.

Lemma 8. *Let* $p < n$. *The mapping*

- $\eta_p \colon \mathsf{SA}^p \to \mathsf{SA}^p$,

$$\eta_p = 1 + \tfrac{1}{(p+1)(n-p)}\, \gamma^*\!\cdot \lrcorner\, (\gamma\cdot\wedge), \tag{1.64}$$

maps SA^p *onto* W^p *and its kernel is*

$$\mathrm{Ker}(\eta_p) = \mathsf{SA}^p[0]. \tag{1.65}$$

The restriction of η_p *to* W^p *has an inverse* $\eta_p^{-1} \colon \mathsf{W}^p \to \mathsf{W}^p$ *determined by:*

$$\eta_p^{-1}|_{\mathsf{SA}^p[q]} = \tfrac{(p+1)(n-p)}{q(n-q+1)}. \tag{1.66}$$

Proof. Using (1.52) we evaluate η_p on $\mathsf{SA}^p[q]$:

$$\eta_p|_{\mathsf{SA}^p[q]} = 1 + \tfrac{1}{(p+1)(n-p)}\, \gamma^*\!\cdot \lrcorner\, (\gamma\cdot\wedge) = 1 - \tfrac{(p-q+1)(n-p-q)}{(p+1)(n-p)} = \tfrac{q(n-q+1)}{(p+1)(n-p)}.$$

The numerator is clearly zero if and only if $q = 0$ and the claim follows. □

1.5 Twistor module

In the present section we decompose

- the tensor product $\mathsf{DSA}^p = \mathsf{V}^* \otimes \mathsf{SA}^p$,

with respect to $\mathsf{Spin}(n)$. We first deal with the cases $p = 0$ or n for which the decomposition degenerates. As a matter of fact, the decomposition of DSA^0 is just a special case of (1.57):

$$\mathsf{V}^* \otimes \mathsf{S} = \mathsf{SA}^1 \cong \mathsf{S} \oplus \mathsf{U}^1. \tag{1.67}$$

Recall from (1.44) that U^1 is defined as kernel of the mapping $(\gamma^* \lrcorner)|_{\mathsf{SA}^1}$ which realizes the Clifford multiplication. The corresponding projections and injection can be easily computed to be:

- $\pi_0^{\mathsf{DS}} \colon \mathsf{V}^* \otimes \mathsf{S} \to \mathsf{S}$,

$$\pi_0^{\mathsf{DS}}(\xi \otimes \Psi) = \xi^* \cdot \Psi, \tag{1.68}$$

- $\iota_0^{\mathsf{DS}} \colon \mathsf{S} \to \mathsf{V}^* \otimes \mathsf{S}$,

$$\iota_0^{\mathsf{DS}}(\Psi) = -\tfrac{1}{n} \sum_{i=1}^{n} e_i^* \otimes (e_i \cdot \Psi), \tag{1.69}$$

- $\pi_1^{\mathsf{DS}} \colon \mathsf{V}^* \otimes \mathsf{S} \to \mathsf{U}^1$,

$$\begin{aligned} \pi_1^{\mathsf{DS}}(\xi \otimes \Psi) &= (1_{\mathsf{V}^* \otimes \mathsf{S}} - \iota_0^{\mathsf{DS}} \circ \pi_0^{\mathsf{DS}})(\xi \otimes \Psi) = \\ &= \xi \otimes \Psi + \tfrac{1}{n} \sum_{i=1}^{n} e_i^* \otimes (e_i \cdot \xi^* \cdot \Psi), \end{aligned} \tag{1.70}$$

$\forall \xi \in \mathsf{V}^*$, $\Psi \in \mathsf{S}$. The projection π_1^{DS} gives rise to the twistor operator and therefore we call the space

- U^1 — *twistor module* of S.

From (1.47) we can deduce that it contains the highest weight component,

$$\mu(\mathsf{U}^1) = (\tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \pm\tfrac{1}{2}). \tag{1.71}$$

We have also $\mathsf{SA}^n \cong \mathsf{SA}^0$ and hence DSA^n decomposes analogously.

Before proceeding further we turn to ordinary p-forms for a while. There is a decomposition invariant with respect to $\mathsf{O}(n)$:

$$\mathsf{V}^* \otimes \mathsf{A}^p \cong \mathsf{A}^{p-1} \oplus \mathsf{A}^{p+1} \oplus \mathsf{A}^{p,1}. \tag{1.72}$$

We call the space

- $\mathsf{A}^{p,1}$ — *twistor module* of A^p.

It is defined as the intersection of kernels of the exterior and interior product maps. In fact, it is again the highest weight component,

$$\begin{aligned} \mu(\mathsf{A}^{p,1}) &= (2, \underbrace{1, \dots, 1}_{(p-1)\times}, 0, \dots, 0), \qquad \forall p \in \{1, \dots, k-1\}, \\ \mu(\mathsf{A}^{k,1}) &= (2, 1, \dots, 1, \pm 1). \end{aligned} \tag{1.73}$$

This decomposition was proved by Stein and Weiss in [17] and later employed by Semmelmann in [14].

Now we combine the cases of spinors and ordinary forms. As a result, we carry the notion of twistor module over to spinor-valued forms. We take the following projections and injections:

- $\pi^{\mathsf{DSA}}_{p,p-1}\colon \mathsf{V}^* \otimes \mathsf{SA}^p \to \mathsf{SA}^{p-1}$,

  $$\pi^{\mathsf{DSA}}_{p,p-1}(\xi \otimes \Phi) = \xi^* \lrcorner \Phi, \tag{1.74}$$

- $\iota^{\mathsf{DSA}}_{p,p-1}\colon \mathsf{SA}^{p-1} \to \mathsf{V}^* \otimes \mathsf{SA}^p$,

  $$\iota^{\mathsf{DSA}}_{p,p-1}(\Xi') = \sum_{i=1}^{n} e_i^* \otimes (e_i^* \wedge \Xi'), \tag{1.75}$$

- $\pi^{\mathsf{DSA}}_{p,p}\colon \mathsf{V}^* \otimes \mathsf{SA}^p \to \mathsf{SA}^p$,

  $$\pi^{\mathsf{DSA}}_{p,p}(\xi \otimes \Phi) = \xi^* \cdot \Phi, \tag{1.76}$$

- $\iota^{\mathsf{DSA}}_{p,p}\colon \mathsf{SA}^p \to \mathsf{V}^* \otimes \mathsf{SA}^p$,

  $$\iota^{\mathsf{DSA}}_{p,p}(\Phi) = \sum_{i=1}^{n} e_i^* \otimes (e_i \cdot \Phi), \tag{1.77}$$

- $\pi^{\mathsf{DSA}}_{p,p+1}\colon \mathsf{V}^* \otimes \mathsf{SA}^p \to \mathsf{SA}^{p+1}$,

  $$\pi^{\mathsf{DSA}}_{p,p+1}(\xi \otimes \Phi) = \xi \wedge \Phi, \tag{1.78}$$

- $\iota^{\mathsf{DSA}}_{p,p+1}\colon \mathsf{SA}^{p+1} \to \mathsf{V}^* \otimes \mathsf{SA}^p$,

  $$\iota^{\mathsf{DSA}}_{p,p+1}(\Xi) = \sum_{i=1}^{n} e_i^* \otimes (e_i \lrcorner \Xi), \tag{1.79}$$

$\forall \xi \in \mathsf{V}^*$, $\Xi' \in \mathsf{SA}^{p-1}$, $\Phi \in \mathsf{SA}^p$, $\Xi \in \mathsf{SA}^{p+1}$. Up to a non-zero constant the injections are indeed right inverses of the projections. We are omitting the normalization for now because we shall need to modify those projections and injections first.

Lemma 9. *The projections and injections from* (1.74)–(1.79) *satisfy:*

$$\begin{aligned}
&\pi^{\mathsf{DSA}}_{p,p-1} \circ \iota^{\mathsf{DSA}}_{p,p-1} = n - p + 1, && \pi^{\mathsf{DSA}}_{p,p-1} \circ \iota^{\mathsf{DSA}}_{p,p+1} = 0,\\
&\pi^{\mathsf{DSA}}_{p,p+1} \circ \iota^{\mathsf{DSA}}_{p,p+1} = p + 1, && \pi^{\mathsf{DSA}}_{p,p+1} \circ \iota^{\mathsf{DSA}}_{p,p-1} = 0,
\end{aligned} \tag{1.80}$$

$$\begin{aligned}
&\pi^{\mathsf{DSA}}_{p,p} \circ \iota^{\mathsf{DSA}}_{p,p-1} = (\gamma\cdot\wedge)|_{\mathsf{SA}^{p-1}}, && \pi^{\mathsf{DSA}}_{p,p} \circ \iota^{\mathsf{DSA}}_{p,p+1} = (\gamma^*\cdot\lrcorner)|_{\mathsf{SA}^{p+1}},\\
&\pi^{\mathsf{DSA}}_{p,p-1} \circ \iota^{\mathsf{DSA}}_{p,p} = (\gamma^*\cdot\lrcorner)|_{\mathsf{SA}^{p}}, && \pi^{\mathsf{DSA}}_{p,p+1} \circ \iota^{\mathsf{DSA}}_{p,p} = (\gamma\cdot\wedge)|_{\mathsf{SA}^{p}},
\end{aligned} \tag{1.81}$$

$$\pi^{\mathsf{DSA}}_{p,p} \circ \iota^{\mathsf{DSA}}_{p,p} = -n. \tag{1.82}$$

Proof. The first four equations (1.80) follow easily from the basic properties of the exterior and interior product (1.6). The next four equations (1.81) are in turn a direct consequence of the definition (1.32) of $\gamma\cdot$ and $\gamma^*\cdot$. For the last equation (1.82) we have:

$$\pi^{\mathsf{DSA}}_{p,p} \circ \iota^{\mathsf{DSA}}_{p,p} = \sum_{i=1}^{n} (e_i \cdot e_i \cdot) = -n. \qquad \square$$

However, decomposition of DSA^p is not as simple as it may appear now. Firstly, lemma 9 shows that $\iota^{\mathsf{DSA}}_{p,p}(\mathsf{SA}^p)$ does not lie in the kernels of $\pi^{\mathsf{DSA}}_{p,p+1}$ and $\pi^{\mathsf{DSA}}_{p,p-1}$. But there is also a more substantial difficulty. As we shall see in a moment, the spaces SA^p, SA^{p+1} and SA^{p-1} cannot be embedded in DSA^p independently; in fact, DSA^p contains just two copies of the spinor space S. A key observation is the subject of the following lemma.

Lemma 10. *The projections of $\pi^{\mathsf{DSA}}_{p,p-1}$, $\pi^{\mathsf{DSA}}_{p,p}$ and $\pi^{\mathsf{DSA}}_{p,p+1}$ are not independent, namely the zeroth primitive parts are related by:*

$$\pi^{\mathsf{SA}}_{p,0} \circ \left(\tfrac{1}{n-p+1}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{p,p-1} - \pi^{\mathsf{DSA}}_{p,p} + \tfrac{1}{p+1}\, \gamma^*\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{p,p+1} \right) = 0. \tag{1.83}$$

Proof. We take $\xi \in \mathsf{V}^*$ and $\Phi \in \mathsf{SA}^p$ and proceed by induction on p.

1) Let $p = 0$ and we directly compute:

$$\left(\tfrac{1}{n+1}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{0,-1} - \pi^{\mathsf{DSA}}_{0,0} + \gamma^*\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{0,1} \right)(\xi \otimes \Phi) =$$
$$= -\xi^* \cdot \Phi + \gamma^*\cdot \lrcorner\, (\xi \wedge \Phi) = -\xi^* \cdot \Phi + \xi^* \cdot \Phi - \xi \wedge (\gamma^*\cdot \lrcorner\, \Phi) = 0.$$

2) Let $p > 0$. We first compute using (1.53) and (1.59):

$$\gamma\cdot \wedge \pi^{\mathsf{SA}}_{p-1,0}(\gamma\cdot \wedge (\xi^* \lrcorner\, (\gamma^*\cdot \lrcorner\, \Phi))) = \pi^{\mathsf{SA}}_{p,0}((p-1)(n-p+2)\gamma\cdot \wedge (\xi^* \lrcorner\, \Phi)),$$

$$\gamma\cdot \wedge \pi^{\mathsf{SA}}_{p-1,0}(\xi^* \cdot (\gamma^*\cdot \lrcorner\, \Phi)) = \pi^{\mathsf{SA}}_{p,0}(-2\gamma\cdot \wedge (\xi^* \lrcorner\, \Phi) + p(n-p+1)\xi^* \cdot \Phi),$$

$$\gamma\cdot \wedge \pi^{\mathsf{SA}}_{p-1,0}(\gamma^*\cdot \lrcorner\, (\xi \wedge (\gamma^*\cdot \lrcorner\, \Phi))) =$$
$$= \pi^{\mathsf{SA}}_{p,0}(p(n-p+1)(-\xi^* \cdot \Phi + \gamma^*\cdot \lrcorner\, (\xi \wedge \Phi))).$$

Now we use the induction hypothesis and substitute the above equalities:

$$0 = \gamma\cdot \wedge \pi^{\mathsf{SA}}_{p-1,0} \circ \left(\tfrac{1}{n-p+2}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{p-1,p-2} - \pi^{\mathsf{DSA}}_{p-1,p-1} + \right.$$
$$\left. + \tfrac{1}{p}\, \gamma^*\cdot \lrcorner\, \pi^{\mathsf{SA}}_{p-1,p} \right)(\xi \otimes (\gamma^*\cdot \lrcorner\, \Phi)) =$$
$$= \pi^{\mathsf{SA}}_{p,0}((p-1)\gamma\cdot \wedge (\xi^* \lrcorner\, \Phi) + 2\gamma\cdot \wedge (\xi^* \lrcorner\, \Phi) - p(n-p+1)\xi^* \cdot \Phi -$$
$$- (n-p+1)\xi^* \cdot \Phi + (n-p+1)\gamma^*\cdot \lrcorner\, (\xi \wedge \Phi)) =$$
$$= (p+1)(n-p+1)\pi^{\mathsf{SA}}_{p,0} \circ \left(\tfrac{1}{n-p+1}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{p,p-1} - \xi^* \cdot \pi^{\mathsf{DSA}}_{p,p} + \right.$$
$$\left. + \tfrac{1}{p+1}\, \gamma^*\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{p,p+1} \right)(\xi \otimes \Phi). \qquad \square$$

For the rest of the section we exclude the degenerate cases and assume

$$p \in \{1, \ldots, n-1\}. \tag{1.84}$$

Definition 11. We call

- $\mathsf{SA}^{p,1}$ — *twistor module* of SA^p,

the intersection of kernels of the projections defined above,

$$\mathsf{SA}^{p,1} = \mathrm{Ker}(\pi^{\mathsf{DSA}}_{p,p-1}) \cap \mathrm{Ker}(\pi^{\mathsf{DSA}}_{p,p}) \cap \mathrm{Ker}(\pi^{\mathsf{DSA}}_{p,p+1}). \tag{1.85}$$

Now we shall modify the embeddings given by the projections and injections from (1.74)–(1.79) in order to obtain decomposition of DSA^p. Since the spaces SA^p are highly reducible as shown in proposition 6, we have certain freedom in choosing the modified embeddings. Perhaps the most natural choice would be to modify just the embedding of SA^p since the embeddings of SA^{p-1} and SA^{p+1} are already independent. However, for our purposes we make a different choice:

1) We preserve the embedding of SA^{p+1} as it is.

2) We modify the embedding of SA^p to be independent of SA^{p+1}.

3) Finally, we modify the embedding of SA^{p-1} to be independent of the first two. As already suggested and implied by lemma 10, we cannot obtain an embedding of the whole space SA^{p-1} but only of its subspace which turns out to be W^{p-1}.

Even within these constraints we still have certain freedom of choice and we choose particular formulas that are preferably simple. By now we also include appropriate normalization of the injections. So we define:

- $\widetilde{\pi}^{\mathsf{DSA}}_{p,p+1} \colon \mathsf{DSA}^p \to \mathsf{SA}^{p+1}$,

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p+1} = \pi^{\mathsf{DSA}}_{p,p+1}, \tag{1.86}$$

- $\widetilde{\iota}^{\mathsf{DSA}}_{p,p+1} \colon \mathsf{SA}^{p+1} \to \mathsf{DSA}^p$,

$$\widetilde{\iota}^{\mathsf{DSA}}_{p,p+1} = \tfrac{1}{p+1}\, \iota^{\mathsf{DSA}}_{p,p+1}, \tag{1.87}$$

- $\widetilde{\pi}^{\mathsf{DSA}}_{p,p} \colon \mathsf{DSA}^p \to \mathsf{SA}^p$,

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p} = \pi^{\mathsf{DSA}}_{p,p} - \tfrac{1}{p+1}\, \gamma^{*}\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{p,p+1} + \tfrac{1}{p+1}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{p,p-1}, \tag{1.88}$$

- $\widetilde{\iota}^{\mathsf{DSA}}_{p,p} \colon \mathsf{SA}^p \to \mathsf{DSA}^p$,

$$\widetilde{\iota}^{\mathsf{DSA}}_{p,p} = -\tfrac{p+1}{p(n+2)} \Big(\iota^{\mathsf{DSA}}_{p,p} - \tfrac{1}{p+1}\, \iota^{\mathsf{DSA}}_{p,p+1} \circ (\gamma\cdot\wedge) \Big), \tag{1.89}$$

- $\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} \colon \mathsf{DSA}^p \to \mathsf{W}^{p-1}$,

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} = \pi^{\mathsf{DSA}}_{p,p-1} + \tfrac{p+1}{p(n+2)}\, \gamma^{*}\!\cdot \lrcorner\, \widetilde{\pi}^{\mathsf{DSA}}_{p,p}, \tag{1.90}$$

- $\widetilde{\iota}^{\mathsf{DSA}}_{p,p-1} \colon \mathsf{W}^{p-1} \to \mathsf{DSA}^p$,

$$\widetilde{\iota}^{\mathsf{DSA}}_{p,p-1} = \tfrac{1}{n-p+1} \Big(\iota^{\mathsf{DSA}}_{p,p-1} - \tfrac{n+2}{p+1}\, \widetilde{\iota}^{\mathsf{DSA}}_{p,p} \circ (\gamma\cdot\wedge) \Big) \circ \eta^{-1}_{p-1}, \tag{1.91}$$

$\forall \xi \in \mathsf{V}^*$, $\Xi \in \mathsf{SA}^{p+1}$, $\Phi \in \mathsf{SA}^p$, $\Xi' \in \mathsf{W}^{p-1}$ and where η^{-1}_{p-1} is the inverse mapping from lemma 8.

Lemma 12. *The zeroth primitive part of* $\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1}$ *vanishes,*

$$\pi^{\mathsf{SA}}_{p-1,0} \circ \widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} = 0, \tag{1.92}$$

so the image of $\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1}$ *is indeed contained in* W^{p-1}.

Proof. Using (1.52) and (1.59) we compute:

$$\begin{aligned}
\pi^{\mathsf{SA}}_{p-1,0} \circ \widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} &= \gamma^{*}\!\cdot \lrcorner\, \pi^{\mathsf{SA}}_{p-1,0} \circ \Big(- \tfrac{1}{p(n-p+1)}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{p,p-1} + \\
&\quad + \tfrac{p+1}{p(n+2)} \Big(\pi^{\mathsf{DSA}}_{p,p} - \tfrac{1}{p+1}\, \gamma^{*}\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{p,p+1} + \tfrac{1}{p+1}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{p,p-1} \Big) \Big) = \\
&= - \tfrac{p+1}{p(n+2)}\, \gamma^{*}\!\cdot \lrcorner\, \pi^{\mathsf{SA}}_{p-1,0} \Big(\tfrac{1}{n-p+1}\, \gamma\cdot \wedge \pi^{\mathsf{DSA}}_{p,p-1} - \pi^{\mathsf{DSA}}_{p,p} + \tfrac{1}{p+1}\, \gamma^{*}\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{p,p+1} \Big),
\end{aligned}$$

and (1.92) follows by (1.83) of lemma 10. □

Lemma 13. $\forall r \in \{p, p+1\}$, $s \in \{p-1, p, p+1\}$,

$$\begin{aligned}
\widetilde{\pi}^{\mathsf{DSA}}_{p,r} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,s} &= \delta_{rs} 1_{\mathsf{SA}^{r}}, \\
\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,s} &= \delta_{(p-1)s} 1_{\mathsf{W}^{p-1}},
\end{aligned} \tag{1.93}$$

Proof. Using the defining equations (1.86)–(1.91) together with (1.80)–(1.82), (1.35) and (1.64) we compute:

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p+1} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p+1} = 1,$$

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p+1} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p} = -\tfrac{p+1}{p(n+2)} ((\gamma\cdot\wedge) - (\gamma\cdot\wedge)) = 0,$$

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p+1} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p-1} = \tfrac{1}{n-p+1} (0 - 0) \circ \eta^{-1}_{p-1} = 0,$$

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p+1} = \tfrac{1}{p+1} ((\gamma^{*}\!\cdot\lrcorner) - (\gamma^{*}\!\cdot\lrcorner) + 0) = 0,$$

$$\begin{aligned}
\widetilde{\pi}^{\mathsf{DSA}}_{p,p} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p} &= -\tfrac{p+1}{p(n+2)} \Big(-n - \tfrac{1}{p+1} (\gamma^{*}\!\cdot \lrcorner\, (\gamma\cdot\wedge) - \gamma\cdot \wedge (\gamma^{*}\!\cdot\lrcorner)) - 0 \Big) = \\
&= -\tfrac{p+1}{p(n+2)} \Big(-n - \tfrac{2p-n}{p+1} \Big) = 1,
\end{aligned}$$

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p-1} = \tfrac{1}{n-p+1} \Big((\gamma\cdot\wedge) - 0 + \tfrac{n-p+1}{p+1} (\gamma\cdot\wedge) - \tfrac{n+2}{p+1} (\gamma\cdot\wedge) \Big) \circ \eta^{-1}_{p-1} = 0,$$

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p+1} = \tfrac{1}{p+1} (0 + 0) = 0,$$

$$\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p} = -\tfrac{p+1}{p(n+2)} ((\gamma^{*}\!\cdot\lrcorner) - 0) + \tfrac{p+1}{p(n+2)} (\gamma^{*}\!\cdot\lrcorner) = 0,$$

$$\begin{aligned}
\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} \circ \widetilde{\iota}^{\mathsf{DSA}}_{p,p-1} &= \tfrac{1}{n-p+1} \Big((n-p+1) - 0 + 0 - \tfrac{1}{p}\, \gamma^{*}\!\cdot \lrcorner\, (\gamma\cdot\wedge) \Big) \circ \eta^{-1}_{p-1} = \\
&= \eta_{p-1} \circ \eta^{-1}_{p-1} = 1.
\end{aligned}$$

□

Lemma 14. *The simultaneous kernel of the modified projections coincides with the twistor module* $\mathsf{SA}^{p,1}$,

$$\begin{aligned}&\mathrm{Ker}(\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1}) \cap \mathrm{Ker}(\widetilde{\pi}^{\mathsf{DSA}}_{p,p}) \cap \mathrm{Ker}(\widetilde{\pi}^{\mathsf{DSA}}_{p,p+1}) = \\ &= \mathrm{Ker}(\pi^{\mathsf{DSA}}_{p,p-1}) \cap \mathrm{Ker}(\pi^{\mathsf{DSA}}_{p,p}) \cap \mathrm{Ker}(\pi^{\mathsf{DSA}}_{p,p+1}) = \mathsf{SA}^{p,1}. \end{aligned} \tag{1.94}$$

Proof. From the defining equations (1.86), (1.88) and (1.90) follows that $\widetilde{\pi}^{\mathsf{DSA}}_{p,p+1}$, $\widetilde{\pi}^{\mathsf{DSA}}_{p,p}$, $\widetilde{\pi}^{\mathsf{DSA}}_{p,p-1}$ are linearly dependent on $\pi^{\mathsf{DSA}}_{p,p+1}$, $\pi^{\mathsf{DSA}}_{p,p}$, $\pi^{\mathsf{DSA}}_{p,p-1}$ and also vice versa. Hence the simultaneous kernels must coincide. □

We further denote

- the invariant subspaces $\mathsf{DSA}^p[r] \subseteq \mathsf{DSA}^p$ given as

$$\begin{aligned}\mathsf{DSA}^p[r] &= \widetilde{\iota}^{\mathsf{DSA}}_{p,r}(\mathsf{SA}^r), \qquad \forall r \in \{p, p+1\},\\ \mathsf{DSA}^p[p-1] &= \widetilde{\iota}^{\mathsf{DSA}}_{p,p-1}(\mathsf{W}^{p-1}).\end{aligned} \tag{1.95}$$

Proposition 15. *The tensor product* $\mathsf{DSA}^p = \mathsf{V}^* \otimes \mathsf{SA}^p$ *decomposes as*

$$\begin{aligned}\mathsf{DSA}^p &= \mathsf{DSA}^p[p-1] \oplus \mathsf{DSA}^p[p] \oplus \mathsf{DSA}^p[p+1] \oplus \mathsf{SA}^{p,1} \cong \\ &\cong \mathsf{W}^{p-1} \oplus \mathsf{SA}^p \oplus \mathsf{SA}^{p+1} \oplus \mathsf{SA}^{p,1}.\end{aligned} \tag{1.96}$$

Proof. Follows from lemmas 13 and 14 by the isomorphism theorem. □

The remaining projection onto the twistor module

- $\pi^{\mathsf{DSA}}_{p,(p,1)} \colon \mathsf{DSA}^p \to \mathsf{SA}^{p,1}$,

is now given by:

$$\pi^{\mathsf{DSA}}_{p,(p,1)} = 1_{\mathsf{DSA}^p} - \widetilde{\iota}^{\mathsf{DSA}}_{p,p-1} \circ \widetilde{\pi}^{\mathsf{DSA}}_{p,p-1} - \widetilde{\iota}^{\mathsf{DSA}}_{p,p} \circ \widetilde{\pi}^{\mathsf{DSA}}_{p,p} - \widetilde{\iota}^{\mathsf{DSA}}_{p,p+1} \circ \widetilde{\pi}^{\mathsf{DSA}}_{p,p+1}. \tag{1.97}$$

1.6 Primitive twistor module

In this section we decompose

- the tensor product $\mathsf{DU}^q = \mathsf{V}^* \otimes \mathsf{U}^q$,

with respect to $\mathsf{Spin}(n)$. We again exclude the already resolved case $q = 0$ and assume

$$q \in \{1, \dots, k\}. \tag{1.98}$$

Definition 16. We call

- $\mathsf{U}^{q,1}$ — *twistor module* of U^q,

the corresponding subspace of $\mathsf{SA}^{q,1}$,

$$\begin{aligned}\mathsf{U}^{q,1} &= \mathsf{DU}^q \cap \mathsf{SA}^{q,1} = \\ &= \mathsf{DU}^q \cap \mathrm{Ker}(\pi^{\mathsf{DSA}}_{q,q-1}) \cap \mathrm{Ker}(\pi^{\mathsf{DSA}}_{q,q}) \cap \mathrm{Ker}(\pi^{\mathsf{DSA}}_{q,q+1}).\end{aligned} \tag{1.99}$$

It can be shown, that the highest weight of $\mathsf{U}^{q,1}$ is given by

$$\begin{aligned}\mu(\mathsf{U}^{1,1}) &= (\tfrac{5}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \pm\tfrac{1}{2}),\\ \mu(\mathsf{U}^{q,1}) &= (\tfrac{5}{2}, \underbrace{\tfrac{3}{2}, \dots, \tfrac{3}{2}}_{(q-1)\times}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \pm\tfrac{1}{2}), \qquad \forall q \in \{2, \dots, k-1\},\\ \mu(\mathsf{U}^{k,1}) &= (\tfrac{5}{2}, \tfrac{3}{2}, \dots, \tfrac{3}{2}, \pm\tfrac{3}{2}),\end{aligned} \tag{1.100}$$

so it again contains the highest weight component.

This time we modify the projections and injections from (1.74)–(1.79) so that we get independent embeddings of U^{q-1}, U^q and U^{q+1} in DU^q. Yet we have to be careful in case $q = k$, at least because the space U^{k+1} does not even exist. In the primitive case no multiplicities occur in the decomposition and hence the embeddings are unique. So we define:

- $\pi^{\mathsf{DU}}_{q,q-1} \colon \mathsf{DU}^q \to \mathsf{U}^{q-1}$,

$$\pi^{\mathsf{DU}}_{q,q-1} = \pi^{\mathsf{DSA}}_{q,q-1}|_{\mathsf{DU}^q}, \tag{1.101}$$

- $\iota^{\mathsf{DU}}_{q,q-1} \colon \mathsf{U}^{q-1} \to \mathsf{DU}^q$,

$$\begin{aligned}\iota^{\mathsf{DU}}_{q,q-1} &= \tfrac{1}{n-q+2}\Big(\iota^{\mathsf{DSA}}_{q,q-1} - \tfrac{1}{n-2q+2}\Big(\iota^{\mathsf{DSA}}_{q,q} \circ (\gamma\cdot\wedge) + \\ &\qquad + \tfrac{1}{n-2q+1}\,\iota^{\mathsf{DSA}}_{q,q+1} \circ (\gamma\cdot\wedge)^2\Big)\Big)\Big|_{\mathsf{U}^{q-1}} = \\ &= \sum_{i=1}^{n} e_i^* \otimes \iota^{\mathsf{DU}}_{q,q-1}[i],\end{aligned} \tag{1.102}$$

where

$$\begin{aligned}\iota^{\mathsf{DU}}_{q,q-1}[i] &= \tfrac{1}{n-q+2}\Big((e_i^*\wedge) - \tfrac{1}{n-2q+2}\Big(e_i \cdot (\gamma\cdot\wedge) + \\ &\qquad + \tfrac{1}{n-2q+1}\, e_i \lrcorner (\gamma\cdot\wedge)^2\Big)\Big)\Big|_{\mathsf{U}^{q-1}},\end{aligned} \tag{1.103}$$

- $\pi^{\mathsf{DU}}_{q,q} \colon \mathsf{DU}^q \to \mathsf{U}^q$, defined only when $2q < n$,

$$\pi^{\mathsf{DU}}_{q,q} = \Big(\pi^{\mathsf{DSA}}_{q,q} - \tfrac{2}{n-2q+2}\,\gamma\cdot \wedge \pi^{\mathsf{DSA}}_{q,q-1}\Big)\Big|_{\mathsf{DU}^q}, \tag{1.104}$$

- $\iota^{\mathsf{DU}}_{q,q} \colon \mathsf{U}^q \to \mathsf{DU}^q$, defined only when $2q < n$,

$$\iota^{\mathsf{DU}}_{q,q} = -\tfrac{1}{n+2}\Big(\iota^{\mathsf{DSA}}_{q,q} + \tfrac{2}{n-2q}\,\iota^{\mathsf{DSA}}_{q,q+1} \circ (\gamma\cdot\wedge)\Big)\Big|_{\mathsf{U}^q} = \sum_{i=1}^{n} e_i^* \otimes \iota^{\mathsf{DU}}_{q,q}[i], \tag{1.105}$$

where

$$\iota^{\mathsf{DU}}_{q,q}[i] = -\tfrac{1}{n+2}\Big((e_i\cdot) + \tfrac{2}{n-2q}\, e_i \lrcorner (\gamma\cdot\wedge)\Big)\Big|_{\mathsf{U}^q}, \tag{1.106}$$

- $\pi^{\mathsf{DU}}_{q,q+1} \colon \mathsf{DU}^q \to \mathsf{U}^{q+1}$, defined only when $2q+1 < n$,

$$\begin{aligned}\pi^{\mathsf{DU}}_{q,q+1} &= \Big(\pi^{\mathsf{DSA}}_{q,q+1} + \tfrac{1}{n-2q}\Big(\gamma\cdot \wedge \pi^{\mathsf{DSA}}_{q,q} - \\ &\qquad - \tfrac{1}{n-2q+1}\,(\gamma\cdot\wedge)^2 \circ \pi^{\mathsf{DSA}}_{q,q-1}\Big)\Big)\Big|_{\mathsf{DU}^q},\end{aligned} \tag{1.107}$$

- $\iota^{\mathsf{DU}}_{q,q+1} \colon \mathsf{U}^{q+1} \to \mathsf{DU}^q$, defined only when $2q+1 < n$,

$$\iota^{\mathsf{DU}}_{q,q+1} = \tfrac{1}{q+1}\,\iota^{\mathsf{DSA}}_{q,q+1}|_{\mathsf{U}^{q+1}} = \sum_{i=1}^{n} e_i^* \otimes \iota^{\mathsf{DU}}_{q,q+1}[i], \tag{1.108}$$

where

$$\iota^{\mathsf{DU}}_{q,q+1}[i] = \tfrac{1}{q+1}\,(e_i\lrcorner)|_{\mathsf{U}^{q+1}}. \tag{1.109}$$

Recall the symbol $l(p)$ from (1.49). Consequently note that $\pi^{\mathsf{DU}}_{q,r}$ and $\iota^{\mathsf{DU}}_{q,r}$ are defined if and only if:

$$r \in \{q-1,\dots,l(q+1)\} = \begin{cases} \{q-1,q,q+1\}, & \text{when } 2q+1<n,\\ \{q-1,q\}, & \text{when } 2q+1=n,\\ \{q-1\}, & \text{when } 2q=n. \end{cases} \tag{1.110}$$

We employ this fact in the following in order to treat at once all the cases including the degenerate ones when $q=k$.

Lemma 17. $\forall r \in \{q-1,\dots,l(q+1)\}$, $i\in\{1,\dots,n\}$,

$$(\gamma^*\!\cdot\lrcorner)\circ\pi^{\mathsf{DU}}_{q,r} = 0, \qquad \textit{and} \qquad (\gamma^*\!\cdot\lrcorner)\circ\iota^{\mathsf{DU}}_{q,r}[i] = 0. \tag{1.111}$$

In other words, it indeed holds

$$\pi^{\mathsf{DU}}_{q,r}(\mathsf{DU}^q) \subseteq \mathsf{U}^r \qquad \textit{and} \qquad \iota^{\mathsf{DU}}_{q,r}(\mathsf{U}^r) \subseteq \mathsf{DU}^q. \tag{1.112}$$

Proof. Let $\xi\in\mathsf{V}^*$, $\Phi\in\mathsf{U}^q$ and using (1.35) we compute for the projections:

$$\gamma^*\!\cdot\lrcorner\,\pi^{\mathsf{DU}}_{q,q-1}(\xi\otimes\Phi) = \gamma^*\!\cdot\lrcorner\,(\xi^*\lrcorner\,\Phi) = -\xi^*\lrcorner\,(\gamma^*\!\cdot\lrcorner\,\Phi) = 0,$$

$$\gamma^*\!\cdot\lrcorner\,\pi^{\mathsf{DU}}_{q,q}(\xi\otimes\Phi) = \gamma^*\!\cdot\lrcorner\,\Big(\xi^*\cdot\Phi - \tfrac{2}{n-2q}\,\gamma\cdot\wedge(\xi^*\lrcorner\,\Phi)\Big) =$$
$$= -\xi^*\cdot(\gamma^*\!\cdot\lrcorner\,\Phi) - 2\xi^*\lrcorner\,\Phi + \tfrac{2(n-2q)}{n-2q}\,\xi^*\lrcorner\,\Phi = 0.$$

$$\gamma^*\!\cdot\lrcorner\,\pi^{\mathsf{DU}}_{q,q+1}(\xi\otimes\Phi) =$$
$$= \gamma^*\!\cdot\lrcorner\,\Big(\xi\wedge\Phi + \tfrac{1}{n-2q}\Big(\gamma\cdot\wedge(\xi^*\cdot\Phi) - \tfrac{1}{n-2q+1}\,(\gamma\cdot\wedge)^2(\xi^*\lrcorner\,\Phi)\Big)\Big) =$$
$$= -\xi\wedge(\gamma^*\!\cdot\lrcorner\,\Phi) + \xi^*\cdot\Phi - \tfrac{n-2q}{n-2q}\,\xi^*\cdot\Phi +$$
$$+\tfrac{1}{n-2q}\Big(-2\gamma\cdot\wedge(\xi^*\lrcorner\,\Phi) + \tfrac{(n-2q)+(n-2q+2)}{n-2q+1}\,\gamma\cdot\wedge(\xi^*\lrcorner\,\Phi)\Big) = 0.$$

And for the injections:

$$\gamma^*\!\cdot\lrcorner\,\iota^{\mathsf{DU}}_{q,q+1}[i]|_{\mathsf{U}^{q+1}} = \gamma^*\!\cdot\lrcorner\,(e_i\lrcorner)|_{\mathsf{U}^{q+1}} = -e_i\lrcorner\,(\gamma^*\!\cdot\lrcorner)|_{\mathsf{U}^{q+1}} = 0,$$

$$\gamma^*\!\cdot\lrcorner\,\iota^{\mathsf{DU}}_{q,q}[i]|_{\mathsf{U}^q} = -\tfrac{1}{n+2}\,\gamma^*\!\cdot\lrcorner\,\Big((e_i\cdot) + \tfrac{2}{n-2q}\,e_i\lrcorner\,(\gamma\cdot\wedge)\Big)\Big|_{\mathsf{U}^q} =$$
$$= \tfrac{1}{n+2}\Big(e_i\cdot(\gamma^*\!\cdot\lrcorner) + 2(e_i\lrcorner) - \tfrac{2(n-2q)}{n-2q}\,(e_i\lrcorner)\Big)\Big|_{\mathsf{U}^q} = 0,$$

$$\gamma^*\!\cdot\lrcorner\,\iota^{\mathsf{DU}}_{q,q-1}[i]|_{\mathsf{U}^{q-1}} =$$
$$= \tfrac{1}{n-q+2}\,\gamma^*\!\cdot\lrcorner\,\Big((e_i^*\wedge) - \tfrac{1}{n-2q+2}\Big(e_i\cdot(\gamma\cdot\wedge) + \tfrac{1}{n-2q+1}\,e_i\lrcorner\,(\gamma\cdot\wedge)^2\Big)\Big)\Big|_{\mathsf{U}^{q-1}} =$$
$$= \tfrac{1}{n-q+2}\Big(-e_i^*\wedge(\gamma^*\!\cdot\lrcorner) + (e_i\cdot) - \tfrac{n-2q+2}{n-2q+2}\,(e_i\cdot) +$$
$$+\tfrac{1}{n-2q+2}\Big(2e_i\lrcorner\,(\gamma\cdot\wedge) - \tfrac{(n-2q)+(n-2q+2)}{n-2q+1}\,e_i\lrcorner\,(\gamma\cdot\wedge)\Big)\Big)\Big|_{\mathsf{U}^{q-1}} = 0. \qquad \square$$

We further denote

- the invariant subspaces $\mathsf{DU}^q[r]\subseteq\mathsf{DU}^q$ given as

$$\mathsf{DU}^q[r] = \iota^{\mathsf{DU}}_{q,r}(\mathsf{U}^r), \qquad \forall r\in\{q-1,\dots,l(q+1)\}. \tag{1.113}$$

Lemma 18. $\forall r \in \{q-1, \dots, l(q+1)\}$,

$$\pi^{\mathsf{DU}}_{q,r} \circ \iota^{\mathsf{DU}}_{q,s} = \delta_{rs} 1_{\mathsf{U}^r}. \tag{1.114}$$

Proof. Using the defining equations (1.101)–(1.108) together with (1.80)–(1.82), (1.52) and (1.53) we compute:

$$\pi^{\mathsf{DU}}_{q,q-1} \circ \iota^{\mathsf{DU}}_{q,q+1} = \pi^{\mathsf{DSA}}_{q,q-1} \circ \iota^{\mathsf{DU}}_{q,q+1} = 0,$$

$$\pi^{\mathsf{DSA}}_{q,q} \circ \iota^{\mathsf{DU}}_{q,q+1} = \tfrac{1}{q+1}(\gamma^*\cdot\lrcorner)|_{\mathsf{U}^{q+1}} = 0,$$

$$\pi^{\mathsf{DU}}_{q,q} \circ \iota^{\mathsf{DU}}_{q,q+1} = 0 - 0 = 0,$$

$$\pi^{\mathsf{DSA}}_{q,q+1} \circ \iota^{\mathsf{DU}}_{q,q+1} = 1,$$

$$\pi^{\mathsf{DU}}_{q,q+1} \circ \iota^{\mathsf{DU}}_{q,q+1} = 1 + 0 - 0 = 1,$$

$$\pi^{\mathsf{DU}}_{q,q-1} \circ \iota^{\mathsf{DU}}_{q,q} = \pi^{\mathsf{DSA}}_{q,q-1} \circ \iota^{\mathsf{DU}}_{q,q} = -\tfrac{1}{n+2}((\gamma^*\cdot\lrcorner) + 0)|_{\mathsf{U}^q} = 0,$$

$$\begin{aligned}\pi^{\mathsf{DSA}}_{q,q} \circ \iota^{\mathsf{DU}}_{q,q} &= -\tfrac{1}{n+2}\Big(-n + \tfrac{2}{n-2q}\gamma^*\cdot \lrcorner\,(\gamma\cdot\wedge)\Big)\Big|_{\mathsf{U}^q} = \\ &= -\tfrac{1}{n+2}\Big(-n - \tfrac{2(n-2q)}{n-2q}\Big) = 1,\end{aligned}$$

$$\pi^{\mathsf{DU}}_{q,q} \circ \iota^{\mathsf{DU}}_{q,q} = 1 - 0 = 1,$$

$$\pi^{\mathsf{DSA}}_{q,q+1} \circ \iota^{\mathsf{DU}}_{q,q} = -\tfrac{1}{n+2}\Big((\gamma\cdot\wedge) + \tfrac{2(q+1)}{n-2q}(\gamma\cdot\wedge)\Big)\Big|_{\mathsf{U}^q} = -\tfrac{1}{n-2q}\,(\gamma\cdot\wedge)|_{\mathsf{U}^q},$$

$$\pi^{\mathsf{DU}}_{q,q+1} \circ \iota^{\mathsf{DU}}_{q,q} = \Big(-\tfrac{1}{n-2q}(\gamma\cdot\wedge) + \tfrac{1}{n-2q}((\gamma\cdot\wedge) - 0)\Big)\Big|_{\mathsf{U}^q} = 0,$$

$$\begin{aligned}\pi^{\mathsf{DU}}_{q,q-1} \circ \iota^{\mathsf{DU}}_{q,q-1} &= \pi^{\mathsf{SA}}_{q,q-1} \circ \iota^{\mathsf{DU}}_{q,q-1} = \\ &= \tfrac{1}{n-q+2}\Big((n-q+1) - \tfrac{1}{n-2q+2}(\gamma^*\cdot \lrcorner\,(\gamma\cdot\wedge) + 0)\Big)\Big|_{\mathsf{U}^{q-1}} = \\ &= \tfrac{1}{n-q+2}((n-q+1)+1) = 1,\end{aligned}$$

$$\begin{aligned}\pi^{\mathsf{DSA}}_{q,q} &\circ \iota^{\mathsf{DU}}_{q,q-1} = \\ &= \tfrac{1}{n-q+2}\Big((\gamma\cdot\wedge) - \tfrac{1}{n-2q+2}\Big(-n(\gamma\cdot\wedge) + \tfrac{1}{n-2q+1}\,\gamma^*\cdot \lrcorner\,(\gamma\cdot\wedge)^2\Big)\Big)\Big|_{\mathsf{U}^{q-1}} = \\ &= \tfrac{1}{n-q+2}\Big(1 + \tfrac{n+2}{n-2q+2}\Big)(\gamma\cdot\wedge)|_{\mathsf{U}^{q-1}} = \tfrac{2}{n-2q+2}(\gamma\cdot\wedge)|_{\mathsf{U}^{q-1}},\end{aligned}$$

$$\pi^{\mathsf{DU}}_{q,q} \circ \iota^{\mathsf{DU}}_{q,q-1} = \left(\tfrac{2}{n-2q+2}(\gamma\cdot\wedge) - \tfrac{2}{n-2q+2}(\gamma\cdot\wedge)\right)\Big|_{\mathsf{U}^{q-1}} = 0,$$

$$\pi^{\mathsf{DSA}}_{q,q+1} \circ \iota^{\mathsf{DU}}_{q,q-1} = \tfrac{1}{n-q+2}\left(0 - \tfrac{1}{n-2q+2}\left((\gamma\cdot\wedge)^2 + \tfrac{q+1}{n-2q+1}(\gamma\cdot\wedge)^2\right)\right)\Big|_{\mathsf{U}^{q-1}} =$$
$$= -\tfrac{1}{(n-2q+2)(n-2q+1)}(\gamma\cdot\wedge)^2|_{\mathsf{U}^{q-1}},$$

$$\pi^{\mathsf{DU}}_{q,q+1} \circ \iota^{\mathsf{DU}}_{q,q-1} = \Big(-\tfrac{1}{(n-2q+2)(n-2q+1)}(\gamma\cdot\wedge)^2$$
$$+ \tfrac{1}{n-2q}\left(\tfrac{2}{n-2q+2}(\gamma\cdot\wedge)^2 - \tfrac{1}{n-2q+1}(\gamma\cdot\wedge)^2\right)\Big)\Big|_{\mathsf{U}^{q-1}} = 0. \qquad \square$$

Next we express the original projections by formulas inverse to (1.101), (1.104) and (1.107). But first we prove a technical lemma which will help us handle the degenerate cases.

Lemma 19. *$\forall \Phi \in \mathsf{U}^q$ and $\xi \in \mathsf{V}^*$:*

a) if $2q+1 = n$, then

$$\xi \wedge \Phi = -\gamma\cdot \wedge (\xi^* \cdot \Phi) + \tfrac{1}{2}\,\gamma\cdot \wedge (\gamma\cdot \wedge (\xi^* \lrcorner)); \tag{1.115}$$

b) if $2q = n$, then

$$\xi^* \cdot \Phi = \gamma\cdot \wedge (\xi^* \lrcorner \Phi), \tag{1.116}$$
$$\xi \wedge \Phi = -\tfrac{1}{2}\,\gamma\cdot \wedge (\gamma\cdot \wedge (\xi^* \lrcorner \Phi)). \tag{1.117}$$

Proof. a) When $2q+1 = n$ we have by corollary 7

$$\gamma\cdot \wedge (\gamma\cdot \wedge \Phi) = 0.$$

Hence we can compute using (1.34):

$$\xi^* \cdot (\gamma\cdot \wedge \Phi) = \xi^* \lrcorner (\gamma\cdot \wedge (\gamma\cdot \wedge \Phi)) + \gamma\cdot \wedge (\xi^* \lrcorner (\gamma\cdot \wedge \Phi)) =$$
$$= \gamma\cdot \wedge (\xi^* \lrcorner (\gamma\cdot \wedge \Phi)) = \gamma\cdot \wedge (\xi^* \cdot \Phi) - \gamma\cdot \wedge (\gamma\cdot \wedge (\xi^* \lrcorner \Phi)),$$
$$\xi \wedge \Phi = -\tfrac{1}{2}(\gamma\cdot \wedge (\xi^* \cdot \Phi) + \xi^* \cdot (\gamma\cdot \wedge \Phi)) =$$
$$= -\gamma\cdot \wedge (\xi^* \cdot \Phi) + \tfrac{1}{2}\,\gamma\cdot \wedge (\gamma\cdot \wedge \lrcorner(\xi^* \lrcorner \Phi)).$$

b) When $2q = n$ we have by corollary 7

$$\gamma\cdot \wedge \Phi = 0.$$

Hence we can again compute using (1.34):

$$\xi^* \cdot \Phi = \xi^* \lrcorner (\gamma\cdot \wedge \Phi) + \gamma\cdot \wedge (\xi^* \lrcorner \Phi) = \gamma\cdot \wedge (\xi^* \lrcorner \Phi),$$
$$\xi \wedge \Phi = -\tfrac{1}{2}(\gamma\cdot \wedge (\xi^* \cdot \Phi) + \xi^* \cdot (\gamma\cdot \wedge \Phi)) =$$
$$= -\tfrac{1}{2}\,\gamma\cdot \wedge (\xi^* \cdot \Phi) = -\tfrac{1}{2}\,\gamma\cdot \wedge (\gamma\cdot \wedge (\xi^* \lrcorner \Phi)). \qquad \square$$

Lemma 20. *The restrictions of the original projections to* DU^q *are given by:*

a) if $2q+1<n$*, then*

$$\pi^{\mathsf{DSA}}_{q,q-1}|_{\mathsf{DU}^q} = \pi^{\mathsf{DU}}_{q,q-1}, \tag{1.118}$$

$$\pi^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q} = \pi^{\mathsf{DU}}_{q,q} + \tfrac{2}{n-2q+2}\,\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q-1}, \tag{1.119}$$

$$\begin{aligned}\pi^{\mathsf{DSA}}_{q,q+1}|_{\mathsf{DU}^q} &= \pi^{\mathsf{DU}}_{q,q+1} - \tfrac{1}{n-2q}\,\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q} - \\ &\quad - \tfrac{1}{(n-2q+2)(n-2q+1)}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q-1};\end{aligned} \tag{1.120}$$

b) if $2q+1=n$*, then* (1.118) *and* (1.119) *remain unchanged and instead of* (1.120) *we have*

$$\begin{aligned}\pi^{\mathsf{DSA}}_{q,q+1}|_{\mathsf{DU}^q} &= -\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q} + \tfrac{n-2q-2}{2(n-2q+2)}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q-1} = \\ &= -\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q} - \tfrac{1}{6}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q-1};\end{aligned} \tag{1.121}$$

c) if $2q=n$*, then* (1.118) *remains unchanged and instead of* (1.119) *and* (1.120) *we have*

$$\pi^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q} = \gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q-1}, \tag{1.122}$$

$$\pi^{\mathsf{DSA}}_{q,q+1}|_{\mathsf{DU}^q} = -\tfrac{1}{2}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q-1}. \tag{1.123}$$

Proof. a) When $2q+1<n$, all the three projections $\pi^{\mathsf{DU}}_{q,q-1}$, $\pi^{\mathsf{DU}}_{q,q}$ and $\pi^{\mathsf{DU}}_{q,q+1}$ are available. The equations (1.118) and (1.119) follow directly from (1.101) and (1.104). As for (1.120), we substitute (1.118) and (1.119) into (1.107):

$$\begin{aligned}\pi^{\mathsf{DSA}}_{q,q+1}|_{\mathsf{DU}^q} &= \pi^{\mathsf{DU}}_{q,q+1} - \tfrac{1}{n-2q}\Big(\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q} + \tfrac{2}{n-2q+2}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q} - \\ &\qquad - \tfrac{1}{n-2q+1}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q}\Big) = \\ &= \pi^{\mathsf{DU}}_{q,q+1} - \tfrac{1}{n-2q}\,\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q} - \tfrac{1}{(n-2q+2)(n-2q+1)}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q-1}.\end{aligned}$$

b) When $2q+1=n$, only the projections $\pi^{\mathsf{DU}}_{q,q-1}$ and $\pi^{\mathsf{DU}}_{q,q}$ are available. The equations (1.118) and (1.119) follow as in the previous case. As for (1.121), we use (1.115) of lemma 19 and substitute (1.118) and (1.119):

$$\begin{aligned}\pi^{\mathsf{DSA}}_{q,q+1} &= -\gamma\cdot\wedge\left(\pi^{\mathsf{DU}}_{q,q} + \tfrac{2}{n-2q+2}\,\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q-1}\right) + \tfrac{1}{2}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q-1} = \\ &= -\gamma\cdot\wedge\,\pi^{\mathsf{DU}}_{q,q} + \tfrac{n-2q-2}{2(n-2q+2)}(\gamma\cdot\wedge)^2\circ\pi^{\mathsf{DU}}_{q,q-1}.\end{aligned}$$

c) When $2q=n$, only the projection $\pi^{\mathsf{DU}}_{q,q-1}$ is available. The equation (1.118) follows as in the previous cases. As for (1.122) and (1.123) we just use (1.116) and (1.117) of lemma 19 and substitute (1.118). □

Corollary 21. *The simultaneous kernel of the modified projections coincides with the twistor module* $\mathsf{U}^{q,1}$,

$$\operatorname{Ker}(\pi^{\mathsf{DU}}_{q,q-1}) \cap \cdots \cap \operatorname{Ker}(\pi^{\mathsf{DU}}_{q,l(q+1)}) =$$
$$= \mathsf{DU}^q \cap \operatorname{Ker}(\pi^{\mathsf{DSA}}_{q,q-1}) \cap \operatorname{Ker}(\pi^{\mathsf{DSA}}_{q,q}) \cap \operatorname{Ker}(\pi^{\mathsf{DSA}}_{q,q+1}) = \mathsf{U}^{q,1}. \tag{1.124}$$

Proof. From the defining equations (1.107), (1.104) and (1.101) follows that $\pi^{\mathsf{DU}}_{q,q+1}$, $\pi^{\mathsf{DU}}_{q,q}$, $\pi^{\mathsf{DU}}_{q,q-1}$ are linearly dependent on $\pi^{\mathsf{DSA}}_{q,q+1}|_{\mathsf{DU}^q}$, $\pi^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q}$, $\pi^{\mathsf{DSA}}_{p,p-1}|_{\mathsf{DU}^q}$ and also vice versa including the degenerate cases, as shown in lemma 20. Hence the simultaneous kernels must coincide. □

For later use we also express the modified projections from the previous section. Again we first prove an auxiliary lemma.

Lemma 22. *It holds:*

$$\gamma^*\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{q,q+1}|_{\mathsf{DU}^q} = \pi^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q}, \tag{1.125}$$
$$\gamma^*\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q} = -2\,\pi^{\mathsf{DSA}}_{q,q-1}|_{\mathsf{DU}^q}. \tag{1.126}$$

Proof. Let $\Phi \in \mathsf{U}^q$ and $\xi \in \mathsf{V}^*$. Using (1.34) and (1.44) we compute:

$$\begin{aligned}\gamma^*\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{q,q+1}(\xi \otimes \Phi) &= \gamma^*\!\cdot \lrcorner\,(\xi \wedge \Phi) = \xi^* \cdot \Phi - \xi \wedge (\gamma^*\!\cdot \lrcorner\, \Phi) = \\ &= \xi^* \cdot \Phi = \pi^{\mathsf{DSA}}_{q,q}(\xi \otimes \Phi),\\ \gamma^*\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{q,q}(\xi \otimes \Phi) &= \gamma^*\!\cdot \lrcorner\,(\xi^* \cdot \Phi) = -2\,\xi^* \lrcorner\, \Phi - \xi^* \cdot (\gamma^*\!\cdot \lrcorner\, \Phi) = \\ &= -2\,\xi^* \lrcorner\, \Phi = -2\,\pi^{\mathsf{DSA}}_{q,q-1}(\xi \otimes \Phi).\end{aligned}$$ □

Lemma 23. *The restrictions of the modified projections from* (1.88) *and* (1.90) *to* DU^q *are given by:*

a) if $2q < n$, *then*

$$\widetilde{\pi}^{\mathsf{DSA}}_{q,q-1}|_{\mathsf{DU}^q} = \tfrac{q-1}{q}\,\pi^{\mathsf{DU}}_{q,q-1}, \tag{1.127}$$
$$\widetilde{\pi}^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q} = \tfrac{q}{q+1}\,\pi^{\mathsf{DU}}_{q,q} + \tfrac{n+2}{(q+1)(n-2q+2)}\,\gamma\cdot \wedge\, \pi^{\mathsf{DU}}_{q,q-1}; \tag{1.128}$$

b) if $2q = n$, *then the equation* (1.127) *remains unchanged and instead of* (1.128) *we have*

$$\widetilde{\pi}^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q} = \gamma\cdot \wedge\, \pi^{\mathsf{DU}}_{q,q-1}. \tag{1.129}$$

Proof. We first compute in both cases using (1.125) and (1.126) of lemma 22 and also (1.52):

$$\widetilde{\pi}^{\mathsf{DSA}}_{q,q} = \left(\tfrac{q}{q+1}\,\pi^{\mathsf{DSA}}_{q,q} + \tfrac{1}{q+1}\,\gamma\cdot \wedge\, \pi^{\mathsf{DSA}}_{q,q-1}\right)\Big|_{\mathsf{DU}^q},$$
$$\begin{aligned}\widetilde{\pi}^{\mathsf{DSA}}_{q,q-1} &= \left(\pi^{\mathsf{DSA}}_{q,q-1} + \tfrac{1}{q(n+2)}(q\,\gamma^*\!\cdot \lrcorner\, \pi^{\mathsf{DSA}}_{q,q} + \gamma^*\!\cdot \lrcorner\,(\gamma\cdot \wedge\, \pi^{\mathsf{DSA}}_{q,q-1}))\right)\Big|_{\mathsf{DU}^q} = \\ &= \tfrac{q-1}{q}\,\pi^{\mathsf{DSA}}_{q,q}|_{\mathsf{DU}^q}.\end{aligned}$$

Substituting (1.118) into the second equation, we immediately get (1.127). As for the first equation, we need to discuss the two cases separately.

a) When $2q < n$, we substitute (1.119) yielding (1.128).

b) When $2q = n$, we substitute (1.122) yielding (1.129). □

Now we finally prove the decomposition of DU^q.

Proposition 24. *The space* DU^q *decomposes as*

$$\begin{aligned}\mathsf{DU}^q &= \mathsf{DU}^q[q-1] \oplus \cdots \oplus \mathsf{DU}^q[l(q+1)] \oplus \mathsf{U}^{q,1} \cong \\ &\cong \mathsf{U}^{q-1} \oplus \cdots \oplus \mathsf{U}^{l(q+1)} \oplus \mathsf{U}^{q,1}.\end{aligned} \tag{1.130}$$

In more detail:

a) if $2q+1 < n$*, then*

$$\begin{aligned}\mathsf{DU}^q &= \mathsf{DU}^q[q-1] \oplus \mathsf{DU}^q[q] \oplus \mathsf{DU}^q[q+1] \oplus \mathsf{U}^{q,1} \cong \\ &\cong \mathsf{U}^{q-1} \oplus \mathsf{U}^{q} \oplus \mathsf{U}^{q+1} \oplus \mathsf{U}^{q,1},\end{aligned} \tag{1.131}$$

b) if $2q+1 = n$*, then*

$$\mathsf{DU}^q = \mathsf{DU}^q[q-1] \oplus \mathsf{DU}^q[q] \oplus \mathsf{U}^{q,1} \cong \mathsf{U}^{q-1} \oplus \mathsf{U}^{q} \oplus \mathsf{U}^{q,1}, \tag{1.132}$$

c) if $2q = n$*, then*

$$\mathsf{DU}^q = \mathsf{DU}^q[q-1] \oplus \mathsf{U}^{q,1} \cong \mathsf{U}^{q-1} \oplus \mathsf{U}^{q,1}. \tag{1.133}$$

Proof. Follows from lemmas 18 and 21 by the isomorphism theorem. □

The remaining projection onto the twistor module

- $\pi^{\mathsf{DU}}_{q,(q,1)} \colon \mathsf{DU}^q \to \mathsf{U}^{q,1}$,

is now given by:

$$\pi^{\mathsf{DU}}_{q,(q,1)} = 1_{\mathsf{DU}^q} - \iota^{\mathsf{DU}}_{q,q-1} \circ \pi^{\mathsf{DU}}_{q,q-1} - \cdots - \iota^{\mathsf{DU}}_{q,l(q+1)} \circ \pi^{\mathsf{DU}}_{q,l(q+1)}. \tag{1.134}$$

Lemma 25. *The projection onto the primitive twistor module* $\mathsf{U}^{q,1}$ *coincides with restriction of the projection onto the twistor module* $\mathsf{SA}^{q,1}$*,*

$$\pi^{\mathsf{DU}}_{q,(q,1)} = \pi^{\mathsf{DSA}}_{q,(q,1)}|_{\mathsf{DU}^q}. \tag{1.135}$$

Proof. In lemma 14 and corollary 21 we deduced that the simultaneous kernels of two mutually linearly dependent sets of projections must coincide. In particular, we have

$$\begin{aligned}\pi^{\mathsf{DU}}_{q,(q,1)}(\mathsf{DU}^q) &= \operatorname{Ker}(\pi^{\mathsf{DU}}_{q,q-1}) \cap \cdots \cap \operatorname{Ker}(\pi^{\mathsf{DU}}_{q,l(q+1)}) = \\ &= \mathsf{DU}^q \cap \operatorname{Ker}(\widetilde{\pi}^{\mathsf{DSA}}_{q,q-1}) \cap \operatorname{Ker}(\widetilde{\pi}^{\mathsf{DSA}}_{q,q}) \cap \operatorname{Ker}(\widetilde{\pi}^{\mathsf{DSA}}_{q,q+1}) = \\ &= \pi^{\mathsf{DSA}}_{q,(q,1)}(\mathsf{DU}^q).\end{aligned}$$

Similarly, the set of injections $\iota^{\mathsf{DU}}_{q,q-1}, \ldots, \iota^{\mathsf{DU}}_{q,l(q+1)}$ is linearly dependent on the set of injections $\widetilde{\iota}^{\mathsf{DSA}}_{q,q-1}$, $\widetilde{\iota}^{\mathsf{DSA}}_{q,q}$, $\widetilde{\iota}^{\mathsf{DSA}}_{q,q+1}$. Hence we have the inclusion

$$\begin{aligned}\operatorname{Ker}(\pi^{\mathsf{DU}}_{q,(q,1)}) &= \iota^{\mathsf{DU}}_{q,q-1}(\mathsf{U}^{q-1}) + \cdots + \iota^{\mathsf{DU}}_{q,l(q+1)}(\mathsf{U}^{l(q+1)}) \subseteq \\ &\subseteq \mathsf{DU}^q \cap (\widetilde{\iota}^{\mathsf{DSA}}_{q,q-1}(\mathsf{SA}^{q-1}) + \widetilde{\iota}^{\mathsf{DSA}}_{q,q}(\mathsf{SA}^{q}) + \widetilde{\iota}^{\mathsf{DSA}}_{q,q+1}(\mathsf{SA}^{q+1})) = \\ &= \mathsf{DU}^q \cap \operatorname{Ker}(\pi^{\mathsf{DSA}}_{q,(q,1)}).\end{aligned}$$

Moreover, both the projections $\pi^{\mathsf{DU}}_{q,(q,1)}$ and $\pi^{\mathsf{DSA}}_{q,(q,1)}$ are idempotent and thus

$$\pi^{\mathsf{DU}}_{q,(q,1)}(\mathsf{DU}^q) \oplus \operatorname{Ker}(\pi^{\mathsf{DU}}_{q,(q,1)}) = \mathsf{DU}^q = \widetilde{\pi}^{\mathsf{DSA}}_{q,(q,1)}(\mathsf{DU}^q) \oplus (\mathsf{DU}^q \cap \operatorname{Ker}(\widetilde{\pi}^{\mathsf{DSA}}_{q,(q,1)})).$$

Consequently, the projections must coincide on DU^q. □

Comparing the decompositions (1.96) and (1.130) we can deduce also decomposition of the twistor module $\mathsf{SA}^{p,1}$.

Proposition 26. *The twistor module* $\mathsf{SA}^{p,1}$ *decomposes as:*

$$\mathsf{SA}^{p,1} \cong \mathsf{U}^{1,1} \oplus \cdots \oplus \mathsf{U}^{l(p),1}. \tag{1.136}$$

Proof. First denote the numbers $q = l(p)$ and

$$(r,s) = \begin{cases} (q, q+1), & \text{when } 2q+1 < n, \\ (q,q), & \text{when } 2q+1 = n, \\ (q-1, q), & \text{when } 2q = n. \end{cases}$$

Substituting (1.57) into DSA^p and using (1.67) and (1.130) we get:

$$\begin{aligned} &\mathsf{DSA}^p \cong \mathsf{U}^0 \oplus \cdots \oplus \mathsf{U}^{q-1} \oplus \mathsf{U}^0 \cdots \oplus \mathsf{U}^r \oplus \mathsf{U}^1 \cdots \oplus \mathsf{U}^s \oplus \mathsf{U}^{1,1} \oplus \ldots \mathsf{U}^{q,1} \cong \\ &\cong \mathsf{W}^{p-1} \oplus \mathsf{SA}^p \oplus \mathsf{SA}^{p+1} \oplus \mathsf{U}^{1,1} \oplus \ldots \mathsf{U}^{l(p),1}. \end{aligned}$$

Now comparing with (1.96) follows (1.136). □

In particular, note that the twistor module $\mathsf{SA}^{p,1}$ does not contain a copy of the twistor module U^1 corresponding to the zeroth primitive part of SA^p.

Chapter 2

Killing equations

In the present chapter we introduce the notion of so called Killing spinor-valued differential forms. We start by recalling basic notions and results in Riemannian *Spin*-geometry, and then proceed to the study of several types of fields defined by Killing equations, namely the Killing forms, the Killing spinors and their generalization called Killing spinor-valued forms. Finally, we pass to examine several basic properties and relations among the different types of Killing fields.

We present both the Riemannian and conformal variants of Killing fields, although we focus primarily on the Riemannian case. In addition, we introduce the so called special Killing fields, which play an essential role in the next chapter. We describe the several types of Killing equations in terms of invariant differential operators. This perspective offers a better insight into the definitions and reveals the general pattern behind the different types of Killing fields.

2.1 Riemannian manifolds

In this section we briefly review basics of Riemannian geometry. For more details we recommend the classical textbook on differential geometry, [11].

Let $\mathcal{M}$ be a *Riemannian manifold* of dimension n and denote by

- g — the *Riemannian metric* on $\mathcal{M}$.

Note that we only consider positive-definite metric and so exclude the pseudo-Riemannian manifolds.

As usual, we introduce the following natural vector bundles:

- $\mathcal{T}$ — the *tangent bundle* of $\mathcal{M}$,
- $\mathcal{T}^*$ — the *cotangent bundle* of $\mathcal{M}$,
- $\mathcal{A}^p$ — the *p-th exterior form bundle* of $\mathcal{M}$,

and the corresponding spaces of smooth sections:

- $\mathcal{X}$ — the Lie algebra of *vector fields* on $\mathcal{M}$,
- Ω^p — the space of *differential p-forms* on $\mathcal{M}$.

In general, if $\mathcal{B}$ is a smooth *fibre bundle* on $\mathcal{M}$, we denote by

- $\Gamma(\mathcal{B})$ — the space of smooth *sections* of $\mathcal{B}$.

Recall the notion of *principal* G-*bundle* on $\mathcal{M}$ and the construction of

- the *associated fiber bundle* $\mathcal{P} \times_{\mathsf{G}} \mathsf{F}$,

where G is a Lie group, $\mathcal{P}$ a principal G-bundle and F a manifold on which G acts from the left. Now if b is a local section of $\mathcal{P}$, then every local section a of $\mathcal{P} \times_{\mathsf{G}} \mathsf{F}$ can be expressed with respect to b as

$$a = [b, f], \tag{2.1}$$

where $f \colon \mathcal{M} \to \mathsf{F}$ is a smooth locally defined mapping. In case F is a vector space and thus a representation of G we speak of

- the *associated vector bundle* $\mathcal{P} \times_{\mathsf{G}} \mathsf{F}$.

The Riemannian metric on the manifold $\mathcal{M}$ defines

- $\mathcal{P}_{\mathsf{O}}$ — the principal $\mathsf{O}(n)$-bundle of *orthonormal frames.*

If $\mathcal{M}$ is *oriented*, we have also

- $\mathcal{P}_{\mathsf{SO}}$ — the principal $\mathsf{SO}(n)$-bundle of *positively oriented orthonormal frames.*

The natural vector bundles on $\mathcal{M}$ are now canonically equivalent to the associated bundles:

$$\begin{aligned} &\mathcal{T} \cong \mathcal{P}_{\mathsf{O}} \times_{\mathsf{O}(n)} \mathsf{V}, \qquad\qquad && \mathcal{T}^* \cong \mathcal{P}_{\mathsf{O}} \times_{\mathsf{O}(n)} \mathsf{V}^*, \\ &\mathcal{A}^p \cong \mathcal{P}_{\mathsf{O}} \times_{\mathsf{O}(n)} \mathsf{A}^p. \end{aligned} \tag{2.2}$$

Note that we always consider only the standard representation structures as in the first chapter. These bundle equivalences can be thought of as a *reduction of the structure group* of the natural vector bundles to $\mathsf{O}(n)$. In the case $\mathcal{M}$ is oriented, a further reduction to $\mathsf{SO}(n)$ is also possible.

2.2 Riemannian *Spin*-manifolds

In this section we proceed to the introduction of Riemannian *Spin*-geometry. For more details we refer to, e.g., [5] or [12].

Suppose that the manifold $\mathcal{M}$ is oriented and recall that

- a *Spin-structure* $\mathcal{P}_{\mathsf{Spin}}$ on $\mathcal{M}$

is a principal $\mathsf{Spin}(n)$-bundle which is a lift of the principal bundle $\mathcal{P}_{\mathsf{SO}}$ via the covering homomorphism λ from (1.17). That is, there exists

- a two-fold covering bundle map $\Lambda \colon \mathcal{P}_{\mathsf{Spin}} \to \mathcal{P}_{\mathsf{SO}}$, such that

$$\Lambda(sA) = \Lambda(s)\,\lambda(A), \tag{2.3}$$

$\forall s \in \mathcal{P}_{\mathsf{Spin}}$ and $A \in \mathsf{Spin}(n)$.

A Riemannian *spin manifold* is an oriented Riemannian manifold together with a chosen *Spin*-structure. Note that the *Spin*-structure does not need to exist and is generally not unique. From now on we assume that $\mathcal{M}$ is a spin manifold with chosen *Spin*-structure $\mathcal{P}_{\mathsf{Spin}}$.

In the case of spin manifolds is the structure group of natural vector bundles of $\mathcal{M}$ reduced to $\mathsf{Spin}(n)$, so we get canonical bundle equivalences:

$$\mathcal{T} \cong \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{V}, \qquad \mathcal{T}^* \cong \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{V}^*,$$
$$\mathcal{A}^p \cong \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{A}^p. \tag{2.4}$$

There are additional natural vector bundles associated to a *Spin*-structure:

- $\mathcal{S}$ — the *complex spinor bundle* of $\mathcal{M}$,
  $$\mathcal{S} = \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{S}, \tag{2.5}$$
- $\mathcal{SA}^p$ — the *p-th spinor-valued differential (or, exterior) form bundle* of $\mathcal{M}$,
  $$\mathcal{SA}^p = \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{SA}^p, \tag{2.6}$$

and the corresponding spaces of smooth sections

- Σ — the space of *spinor fields* on $\mathcal{M}$,
- $\Sigma\Omega^p$ — the space of *spinor-valued differential forms* on $\mathcal{M}$.

All algebraic notions of the representation theory naturally carry over to corresponding associated bundles. Firstly, invariant elements give rise to distinguished global sections. For instance, the Riemannian metric g on $\mathcal{M}$ corresponds to the inner product g on V. There is also an $\mathsf{End}_{\mathbb{C}}(\mathcal{S})$-valued differential 1-form and its orthogonal dual,

- $\gamma\cdot \in \Gamma(\mathcal{T}^* \otimes \mathsf{End}_{\mathbb{C}}(\mathcal{S}))$ — the *Clifford multiplication form*,
- $\gamma^*\cdot \in \Gamma(\mathcal{T} \otimes \mathsf{End}_{\mathbb{C}}(\mathcal{S}))$,

which realize the Clifford multiplication in vector bundles on $\mathcal{M}$ in the sense of (1.33). Moreover, all algebraic identities which are invariant with respect to the structure group remain valid on the level of sections of the associated vector bundles. In particular, we can employ (1.34) and its consequences in computations with spinor-valued differential forms.

Secondly, intertwining mappings give rise to bundle maps. Consequently, a decomposition of the fibre as a representation of the structure group carries over to decomposition of the whole associated bundle. So we can utilize the decompositions from chapter 1 and introduce further vector bundles on $\mathcal{M}$:

- $\mathcal{U}^q$ — the *primitive spinor-valued exterior form bundle* of $\mathcal{M}$,
  $$\mathcal{U}^q = \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{U}^q, \tag{2.7}$$
- $\mathcal{A}^{p,1}$ — the *twistor module bundle* of $\mathcal{A}^p$,
  $$\mathcal{A}^{p,1} = \mathcal{P}_{\mathsf{O}} \times_{\mathsf{O}(n)} \mathsf{A}^{p,1} = \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{A}^{p,1}, \tag{2.8}$$
- $\mathcal{SA}^{p,1}$ — the *twistor module bundle* of $\mathcal{SA}^p$,
  $$\mathcal{SA}^{p,1} = \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{SA}^{p,1}, \tag{2.9}$$
- $\mathcal{U}^{q,1}$ — the *twistor module bundle* of $\mathcal{U}^q$,
  $$\mathcal{U}^{q,1} = \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{U}^{q,1}, \tag{2.10}$$
- $\mathcal{W}^{p-1}$,
  $$\mathcal{W}^{p-1} = \mathcal{P}_{\mathsf{Spin}} \times_{\mathsf{Spin}(n)} \mathsf{W}^{p-1}. \tag{2.11}$$

2.3 Levi-Civita connection

We shall start with the definition of Levi-Civita connection induced by the Riemannian metric and then lift it to the *Spin*-structure, thereby producing the spin connection. For more details see again [11], [5] or [12].

On the Riemannian manifold $\mathcal{M}$ we consider

- $\nabla\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ — the unique *covariant derivative*,

which is induced by the *Levi-Civita connection* on the principal bundle $\mathcal{P}_{\mathsf{O}}$. This covariant derivative is determined by

$$2g(\nabla_X(Y), Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + \\ + g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X), \qquad (2.12)$$

$\forall X, Y, Z \in \mathcal{X}$, where g is the metric. In general, the connection induces a unique covariant derivative on any vector bundle $\mathcal{B}$ associated to $\mathcal{P}_{\mathsf{O}}$,

- $\nabla\colon \mathcal{X} \times \Gamma(\mathcal{B}) \to \Gamma(\mathcal{B})$.

For instance, we have the covariant derivative on differential forms. Another example is the tensor product of such vector bundles, where the covariant derivative extends by the Leibniz rule.

An important consequence of (2.12) is that the metric g regarded as a symmetric covariant 2-tensor is *parallel*,

$$\nabla(g) = 0. \qquad (2.13)$$

The connection itself can be identified with

- ω — the *connection 1-form*.

Though it is a slightly more complicated object, it can be represented locally as a skew-symmetric matrix ω_{jm} of ordinary 1-forms on $\mathcal{M}$ with respect to an orthonormal frame. Now if

$$b = (X_1, \ldots, X_n) \qquad (2.14)$$

is a local orthonormal frame field, then the covariant derivative ∇ is uniquely determined by

$$\nabla_{X_i}(X_j) = \sum_{m=1}^{n} \omega_{jm}(X_i)\, X_m, \qquad \forall i, j \in \{1, \ldots, n\}. \qquad (2.15)$$

In our case when $\mathcal{M}$ is a spin manifold, the Levi-Civita connection on $\mathcal{P}_{\mathsf{O}}$ lifts to a *spin connection* on the *Spin*-structure $\mathcal{P}_{\mathsf{Spin}}$. Thus we get a unique covariant derivative on any vector bundle associated to $\mathcal{P}_{\mathsf{Spin}}$. Note that for vector bundles which are associated to both $\mathcal{P}_{\mathsf{O}}$ and $\mathcal{P}_{\mathsf{Spin}}$ the two induced covariant derivatives agree.

We can compute covariant derivative of spinor fields,

- $\nabla\colon \mathcal{X} \times \Sigma \to \Sigma$,

locally using (2.1) and (2.15). First, let b be a local orthonormal frame field as in (2.14) and ω_{jm} the respective local expression of the connection 1-form. Next, let s be a local section of $\mathcal{P}_{\mathsf{Spin}}$ which is a *lift* of b, that is,

$$\Lambda(s) = b, \tag{2.16}$$

and $\psi\colon \mathcal{M} \to \mathsf{S}$ a locally defined spinor-valued function. Then the covariant derivative of the corresponding spinor field is given by

$$\nabla_X([s,\psi]) = \Big[s,\ X(\psi) + \tfrac{1}{4}\sum_{j,m=1}^{n} \omega_{jm}(X)\, e_j \cdot e_m \cdot \psi\Big], \tag{2.17}$$

$\forall X \in \mathcal{T}$. Combining (2.15) and (2.17) we can compute the covariant derivative for all vector bundles of interest.

Finally, the Clifford multiplication form $\gamma\cdot$ and its dual $\gamma^{*}\cdot$ are parallel,

$$\nabla(\gamma\cdot) = 0, \qquad\qquad \nabla(\gamma^{*}\cdot) = 0, \tag{2.18}$$

analogous to (2.13) for the metric.

In general, covariant derivative on the vector bundle $\mathcal{B}$ on M can be regarded as a mapping:

- $\nabla\colon \Gamma(\mathcal{B}) \to \Gamma(\mathcal{T}^{*} \otimes \mathcal{B})$.

Given an invariant decomposition of the tensor product $\mathcal{T}^{*} \otimes \mathcal{B}$, we can construct first-order invariant differential operators on $\Gamma(\mathcal{B})$ given by projections of ∇ onto summands in the decomposition. When applied to the decompositions in chapter 1, we shall obtain invariant operators which are more or less explicitly encompassed in the equations defining several types of Killing fields.

2.4 Killing forms

Killing forms were introduced by Yano in [21] as a ground for the construction of invariants along geodesics. Recently, Killing forms were studied by Semmelmann in [14].

We start by introducing invariant differential operators, which arose in the decomposition (1.72):

- the *exterior derivative* $\mathrm{d}\colon \Omega^p \to \Omega^{p+1}$,

$$\mathrm{d}(\alpha) = \sum_{i=1}^{n} X_i^{*} \wedge \nabla_{X_i}(\alpha), \tag{2.19}$$

- the *codifferential* $\mathrm{d}^{*}\colon \Omega^p \to \Omega^{p-1}$,

$$\mathrm{d}^{*}(\alpha) = \sum_{i=1}^{n} X_i \lrcorner \nabla_{X_i}(\alpha), \tag{2.20}$$

- the *twistor operator* $\mathrm{T}\colon \Omega^p \to \Gamma(\mathcal{A}^{p,1})$,

$$\begin{aligned}\mathrm{T}(\alpha) = \nabla(\alpha) - \tfrac{1}{p+1} \sum_{i=1}^{n} X_i^* \otimes X_i \lrcorner \mathrm{d}(\alpha) - \\ - \tfrac{1}{n-p+1} \sum_{i=1}^{n} X_i^* \otimes X_i^* \wedge \mathrm{d}^*(\alpha),\end{aligned} \tag{2.21}$$

$\forall \alpha \in \Omega^p$. For the local expressions we always assume that $(X_1, \ldots, X_n)$ is an orthonormal frame as in (2.14). Unless otherwise stated , we also assume

$$p \in \{1, \ldots, n-1\} \tag{2.22}$$

excluding the trivial cases $p = 0$ and n.

Definition 27. A *Killing p-form* is a p-form α such that

$$\nabla_X(\alpha) = \tfrac{1}{p+1} X \lrcorner \mathrm{d}(\alpha), \qquad \forall X \in \mathcal{T}. \tag{2.23}$$

Taking orthogonal dual of a Killing 1-form α we can deduce that (2.23) is equivalent to

$$g(\nabla_X(\alpha^*), Y) + g(\nabla_Y(\alpha^*), X) = 0, \qquad \forall X, Y \in \mathcal{T}. \tag{2.24}$$

Killing 1-forms are thus just orthogonal duals of Killing vectors, justifying the terminology (cf. also [21]).

Proposition 28. *A p-form α is a Killing form, if and only if it satisfies the following two conditions:*

1) α is coclosed, that is, it belongs to the kernel of the codifferential,

$$\mathrm{d}^*(\alpha) = 0, \tag{2.25}$$

2) α belongs to the kernel of the twistor operator,

$$\mathrm{T}(\alpha) = 0. \tag{2.26}$$

Proof. A direct consequence of the decomposition (1.72). □

Proposition 28 manifests the relationship between Killing forms and the invariant operators, which applies to all types of Killing fields.

As we have suggested in the beginning, the basic property which supports the generalization of Killing vectors to forms is that they yield invariants along geodesics. This property boils down to the following lemma. Recall that if X is the tangent vector field of a geodesic in $\mathcal{M}$ then it satisfies

$$\nabla_X(X) = 0 \tag{2.27}$$

along the geodesic.

Lemma 29. *Let α be a Killing p-form and X the tangent vector field of a geodesic in $\mathcal{M}$. Then $X \lrcorner \alpha$ is covariantly constant,*

$$\nabla_X(X \lrcorner \alpha) = 0, \tag{2.28}$$

along the geodesic.

Proof. Using (2.27) and (2.23) we compute:

$$\nabla_X(X \lrcorner \alpha) = X \lrcorner \nabla_X(\alpha) = \tfrac{1}{p+1} X \lrcorner (X \lrcorner \mathrm{d}(\alpha)) = 0. \qquad \square$$

The conformal Killing forms were introduced by Tachibana in [18] for the case of 2-forms and by Kashiwada in [10] for the general case. They are defined by a weaker equation which turns out to impose condition just on the twistor operator.

Definition 30. A *conformal Killing p-form* is a p-form α such that

$$\nabla_X(\alpha) = \tfrac{1}{p+1} X \lrcorner \mathrm{d}(\alpha) - \tfrac{1}{n-p+1} X^* \wedge \mathrm{d}^*(\alpha), \qquad \forall X \in \mathcal{T}. \tag{2.29}$$

Proposition 31. *A p-form α is a conformal Killing form, if and only if it belongs to the kernel of the twistor operator,*

$$\mathrm{T}(\Psi) = 0. \tag{2.30}$$

Proof. Again a direct consequence of the decomposition (1.72). $\square$

On the other hand, special Killing forms introduced by Tachibana in [19] are defined with an additional second order condition on the differential of the form.

Definition 32. A *special Killing p-form* is a Killing p-form α, for which there additionally exists $a \in \mathbb{R}$ such that

$$\nabla_X(\mathrm{d}(\alpha)) = aX^* \wedge \alpha, \qquad \forall X \in \mathcal{T}. \tag{2.31}$$

2.5 Killing spinors

Killing spinors are objects with many interesting applications in geometry, but here we present only few basic properties focusing on the transition to other types of Killing fields. For more details and omitted proofs see, e.g., [5] or [2].

Again we start by introducing invariant differential operators, this time given by the decomposition (1.67):

- the *Dirac operator* $\mathrm{D}\colon \Sigma \to \Sigma$,

$$\mathrm{D}(\Psi) = \pi_0^{\mathsf{DS}}(\nabla(\Psi)) = \sum_{i=1}^{n} X_i \cdot \nabla_{X_i}(\Psi), \tag{2.32}$$

- the *twistor operator* $\mathrm{T}\colon \Sigma \to \Gamma(\mathcal{U}^1)$,

$$\mathrm{T}(\Psi) = \pi_1^{\mathsf{DS}}(\nabla(\Psi)) = \nabla(\Psi) + \tfrac{1}{n} \sum_{i=1}^{n} X_i^* \otimes X_i \cdot \mathrm{D}(\Psi), \tag{2.33}$$

$\forall \Psi \in \Gamma(\mathcal{S})$. The Dirac operator plays a central role in the subject of Riemannian *Spin*-geometry and many of its properties are well known. In particular, the question of eigenvalue estimates has been discussed to a great extent, leading to the introduction of the notion of Killing spinor fields.

Definition 33. A *Killing spinor field*, or *Killing spinor* for short, is a spinor field Ψ for which there exists $a \in \mathbb{C}$ such that

$$\nabla_X(\Psi) = aX \cdot \Psi, \tag{2.34}$$

$\forall X \in \mathcal{T}$. The number a is called *Killing number* of Ψ.

Proposition 34. *A spinor field Ψ is a Killing spinor with Killing number a, if and only if it satisfies the following two conditions:*

1) Ψ is an eigenvector of the Dirac operator,

$$\mathrm{D}(\Psi) = -na\,\Psi, \tag{2.35}$$

2) Ψ belongs to the kernel of the twistor operator,

$$\mathrm{T}(\Psi) = 0. \tag{2.36}$$

Proof. First suppose that (2.34) holds. We compute using (2.32) and (2.33):

$$\mathrm{D}(\Psi) = \sum_{i=1}^{n} aX_i \cdot X_i \cdot \Psi = -na\,\Psi,$$
$$\mathrm{T}(\Psi) = \sum_{i=1}^{n} X_i^* \otimes (aX_i \cdot \Psi - aX_i \cdot \Psi) = 0.$$

On the other hand suppose that (2.35) and (2.36) hold. Again using (2.32) and (2.33) we compute:

$$\nabla(\Psi) = -\tfrac{1}{n}\sum_{i=1}^{n} X_i^* \otimes X_i \cdot \mathrm{D}(\Psi) = \sum_{i=1}^{n} X_i^* \otimes aX_i \cdot \Psi. \qquad \square$$

However, there is much stronger relationship between Killing spinors, the Dirac operator and the geometry of the underlying manifold. As we already noticed, the Killing spinors are directly related to the eigenvalue estimates. This relationship is unique for the Killing spinors and does not show up for the other types of Killing fields.

Theorem 35. *Let $\mathcal{M}$ be a compact spin manifold. Then any eigenvalue c of the Dirac operator on $\mathcal{M}$ satisfies the inequality*

$$c^2 \geq \tfrac{n}{4(n-1)}\, R_0, \tag{2.37}$$

where R_0 is the minimum of the scalar curvature of $\mathcal{M}$. Moreover, if the eigenvalue c attains equality in (2.37), then the corresponding eigenvector Ψ is a Killing spinor with Killing number

$$a = -\tfrac{c}{n} = \mp\sqrt{\tfrac{1}{4n(n-1)}\, R_0}. \tag{2.38}$$

Theorem 36. *Let $\mathcal{M}$ be a connected spin manifold and Ψ a Killing spinor on $\mathcal{M}$ with Killing number a. Then $\mathcal{M}$ has constant scalar curvature R given by the equation*

$$a^2 = \tfrac{1}{4n(n-1)}\, R. \tag{2.39}$$

In particular, the Killing number a is always real or purely imaginary.

There is also a weaker notion of conformal Killing spinors which omits the eigenvalue condition for the Dirac operator.

Definition 37. A *conformal Killing spinor field*, or *twistor spinor* for short, is a spinor field Ψ such that

$$\nabla_X(\Psi) = -\tfrac{1}{n}\, X \cdot \mathrm{D}(\Psi), \tag{2.40}$$

$\forall X \in \mathcal{T}$.

Proposition 38. *A spinor field Ψ is a conformal Killing spinor, if and only if it belongs to the kernel of the twistor operator,*

$$\mathrm{T}(\Psi) = 0. \tag{2.41}$$

Proof. A direct consequence of (2.33). □

2.6 Killing spinor-valued forms

In the present section we introduce Killing spinor-valued differential forms. This definition is quite straightforward generalization of Killing forms and Killing spinors, which will become apparent in the expressions highlighting invariant differential operators.

To our best knowledge, the present definition is in its full generality new. In particular, the introduction of Killing number in case of spinor-valued forms has several precedents, some authors produced a special case of ours. Much of this work was published in the field of mathematical physics, and is considered for pseudo-Riemannian manifolds in low dimensions.

Walker and Penrose in [20] suggested a general definition of what they call Killing spinors, given by equation in abstract index notation

$$\nabla^{(A'_0}_{(B_0}\, \chi^{A'_1 \dots A'_r)}_{B_1 \dots B_s)} = 0, \tag{2.42}$$

for the case of Lorentzian 4-manifold. Depending on the valence (r, s) this definition covers the cases of spinors, forms, spinor-valued forms and other spinor-tensors. In particular, the simplest cases

$$(r, s) = (0, 1) \text{ and } (1, 0);\ (1, 1);\ (0, 2) \text{ and } (2, 0)$$

apply to spinors, complexified 1-forms (or vectors) and complexified 2-forms respectively. In fact, the left-hand side of (2.42) is just the appropriate twistor operator. So the equation defines rather the conformal Killing fields.

Later Duff and Pope in [4] introduced Killing spinor-vectors defined as

$$\bar{D}_{(\mu}\eta_{\nu)} = 0, \qquad \text{where} \qquad \bar{D}_\mu = \nabla_\mu \mp c\,\gamma\cdot_\mu, \tag{2.43}$$

which involves the indeterminate constant c. This definition is equivalent to ours given by (2.50) for the case of spinor-valued 1-forms.

Subsequently Nieuwenhuizen in [13] analyses (2.42) and (2.43) in more details. He confirms our previous statement concluding that the definition of Walker and Penrose indeed defines only the conformal Killing vectors and spinor-vectors. He somewhat obscures the case of spinors when he imprecisely claims that (2.42) defines Killing spinors. The argument relies on the assumption of Einstein space (cf. [2], theorem 5 of chapter 2) and a closer look confirms our statement also for spinors.

Recently Somberg in [16] introduced Killing spinor-valued p-forms for arbitrary degree p. Similarly to our approach he employs the Howe duality to decompose representations, but considers only primitive spinor-valued forms. More importantly, his definition lacks the indeterminate Killing number a and is equivalent to ours given by (2.50) for the case $a = 0$. Subsequently he deduces the property analogous to lemma 29 that a Killing spinor-valued form yields invariants along geodesics. We repeat this result in lemma 42.

Before we state the definition, we introduce invariant differential operators given by the decomposition (1.96). We include also the more common operators corresponding to the unmodified projections (1.74) and (1.76):

- the *covariant exterior derivative* $\mathrm{d}\colon \Sigma\Omega^p \to \Sigma\Omega^{p+1}$,

$$\mathrm{d}(\Phi) = \widetilde{\pi}^{\mathsf{DSA}}_{p,p+1}(\nabla(\Phi)) = \pi^{\mathsf{DSA}}_{p,p+1}(\nabla(\Phi)) = \sum_{i=1}^{n} X_i^* \wedge \nabla_{X_i}(\Phi), \tag{2.44}$$

- the *twisted Dirac operator* $\mathrm{D}\colon \Sigma\Omega^p \to \Sigma\Omega^p$,

$$\mathrm{D}(\Phi) = \pi^{\mathsf{DSA}}_{p,p}(\nabla(\Phi)) = \sum_{i=1}^{n} X_i \cdot \nabla_{X_i}(\Phi), \tag{2.45}$$

- the *codifferential* $\mathrm{d}^*\colon \Sigma\Omega^p \to \Sigma\Omega^{p-1}$,

$$\mathrm{d}^*(\Phi) = \pi^{\mathsf{DSA}}_{p,p-1}(\nabla(\Phi)) = \sum_{i=1}^{n} X_i \lrcorner \nabla_{X_i}(\Phi), \tag{2.46}$$

- the *modified twisted Dirac operator* $\widetilde{\mathrm{D}}\colon \Sigma\Omega^p \to \Sigma\Omega^p$,

$$\widetilde{\mathrm{D}}(\Phi) = \widetilde{\pi}^{\mathsf{DSA}}_{p,p}(\nabla(\Phi)) = \mathrm{D}(\Phi) - \tfrac{1}{p+1}\,\gamma^*\!\cdot \lrcorner\, \mathrm{d}(\Phi) + \tfrac{1}{p+1}\,\gamma\cdot \wedge \mathrm{d}^*(\Phi), \tag{2.47}$$

- the *modified codifferential* $\widetilde{\mathrm{d}^*}\colon \Sigma\Omega^p \to \Gamma(\mathcal{W}^{p-1})$,

$$\widetilde{\mathrm{d}^*}(\Phi) = \widetilde{\pi}^{\mathsf{DSA}}_{p,p-1}(\nabla(\Phi)) = \mathrm{d}^*(\Phi) + \tfrac{p+1}{p(n+2)}\,\gamma^*\!\cdot \lrcorner\, \widetilde{\mathrm{D}}(\Phi), \tag{2.48}$$

- and the *twistor operator* $\mathrm{T}\colon \Sigma\Omega^p \to \Gamma(\mathcal{SA}^{p,1})$,

$$\begin{aligned}\mathrm{T}(\Phi) &= \pi^{\mathsf{DSA}}_{p,p,1}(\nabla(\Phi)) = \\ &= \nabla(\Phi) - \widetilde{\iota}^{\mathsf{DSA}}_{p,p+1}(\mathrm{d}(\Phi)) - \widetilde{\iota}^{\mathsf{DSA}}_{p,p}(\widetilde{\mathrm{D}}(\Phi)) - \widetilde{\iota}^{\mathsf{DSA}}_{p,p-1}(\widetilde{\mathrm{d}^*}(\Phi)),\end{aligned} \tag{2.49}$$

$\forall \Phi \in \Sigma\Omega^p$.

Definition 39. A *Killing spinor-valued p-form* is a spinor-valued p-form for which there exists $a \in \mathbb{C}$ such that

$$\nabla_X(\Phi) = a\Big(X \cdot \Phi - \tfrac{1}{p+1}\, X \lrcorner (\gamma\cdot \wedge \Phi)\Big) + \tfrac{1}{p+1}\, X \lrcorner \, \mathrm{d}(\Phi), \tag{2.50}$$

$\forall X \in \mathcal{T}$. The number a is called *Killing number* of Φ.

Proposition 40. *A spinor-valued p-form Φ is a Killing spinor-valued form with Killing number a, if and only if it satisfies the following three conditions:*

1) Φ is an eigenvector of the modified Dirac operator,

$$\widetilde{\mathrm{D}}(\Phi) = -\tfrac{p(n+2)a}{p+1}\,\Phi, \tag{2.51}$$

2) Φ belongs to the kernel of the modified codifferential,

$$\widetilde{\mathrm{d}}^*(\Phi) = 0, \tag{2.52}$$

3) Φ belongs to the kernel of the twistor operator,

$$\mathrm{T}(\Phi) = 0. \tag{2.53}$$

Proof. We can rewrite the equation (2.50) using (1.89) and (1.87) in form

$$\nabla(\Phi) = -\tfrac{p(n+2)a}{p+1}\,\widetilde{\iota}^{\,\mathsf{DSA}}_{p,p}(\Phi) + \widetilde{\iota}^{\,\mathsf{DSA}}_{p,p+1}(\mathrm{d}(\Phi)).$$

Now the claim follows from the decomposition (1.96). □

Comparing the three types of Killing fields we can deduce the general pattern in the definition of Killing field:

1) When defined, the (covariant) exterior derivative is the only component of the covariant derivative, which is not prescribed, it is not even restricted.

2) When defined, the (possibly modified) Dirac operator is prescribed by eigenvalue condition.

3) All the other components of the covariant derivative, in particular, the twistor operator, are prescribed to vanish.

The modification of the standard Dirac operator was necessary since otherwise we would impose a restricting condition on the covariant exterior derivative. On the other hand, our particular modification is not the only possible one; recall that in section 1.5 we observed a freedom in choosing the projections. Moreover, there can possibly be modifications which are still independent of the exterior covariant derivative and lead to non-equivalent definitions of Killing spinor-valued forms. This possibility is a subject of further research with a perspective of family of admissible modifications parametrized by a product of projective spaces.

Also note that all ambiguities and the problem of a modification of the standard Dirac operator disappears in the case of primitive spinor-valued forms. However, defining the Killing fields in a highly reducible case like this one has its own value. Indeed, the equation (2.50) does not simply reduce to requiring that the individual primitive parts of Φ are Killing spinor-valued forms, it is more general than that. On the other hand, if the primitive parts $\Phi[q]$ are Killing spinor-valued forms with Killing numbers a_q, then Φ does not necessarily need to satisfy (2.50). The point is that the equation (2.50) explicitly prescribes the ratios $a_q : a_{q'}$.

The main justification of our definition 39 is subject of the next proposition, which allows us to construct Killing spinor-valued forms out of Killing spinors and Killing forms. Note that the construction yields in general a non-primitive Killing spinor-valued form. Hence a definition restricted to primitive Killing spinor-valued forms would not be sufficient.

Proposition 41. *Let Ψ be a Killing spinor with Killing number a and α be a Killing p-form. Then the tensor product*

$$\Phi = \alpha \otimes \Psi \tag{2.54}$$

is a Killing spinor-valued form with Killing number a.

Proof. First we compute the covariant exterior derivative of Φ using (2.19), (2.44) an (2.34):

$$\begin{aligned} \mathrm{d}(\Phi) &= \sum_{i=1}^{n} X_i^* \wedge \nabla_{X_i}(\Phi) = \sum_{i=1}^{n} X_i^* \wedge (\nabla_{X_i}(\alpha) \otimes \Psi + \alpha \otimes \nabla_{X_i}(\Psi)) = \\ &= \mathrm{d}(\alpha) \otimes \Psi + a\,\gamma\cdot \wedge \Phi. \end{aligned}$$

Now using (2.23) and (2.34) again we get:

$$\begin{aligned} \nabla_X(\Phi) &= \nabla_X(\alpha) \otimes \Psi + \alpha \otimes \nabla_X(\Psi) = \tfrac{1}{p+1}\, X \lrcorner\, (\mathrm{d}(\alpha) \otimes \Psi) + a\, X \cdot \Phi = \\ &= a\Big(X \cdot \Phi - \tfrac{1}{p+1}\, X \lrcorner\, (\gamma\cdot \wedge \Phi)\Big) + \tfrac{1}{p}\, X \lrcorner\, \mathrm{d}(\Phi). \end{aligned}$$

□

Thus the proposition yields many examples of Killing spinor-valued forms on manifolds which admit enough Killing spinors and forms, for instance, the Riemannian spheres. Unfortunately, no other examples which cannot be reduced to Killing spinors and Killing forms are known to us so far.

In case the Killing number a is zero, the equation (2.50) reduces to a simple analogy of (2.23). Subsequently, we get also the next result analogous to lemma 29.

Lemma 42. *Let Φ be a Killing p-form with Killing number $a = 0$ and X the tangent vector field to a geodesic in $\mathcal{M}$. Then $X \lrcorner\, \Phi$ is covariantly constant,*

$$\nabla_X(X \lrcorner\, \Phi) = 0 \tag{2.55}$$

along the geodesic.

Proof. Using (2.27) and (2.50) we compute:

$$
\begin{aligned}
&\nabla_X(X \lrcorner \Phi) = X \lrcorner \nabla_X(\Phi) = \\
&= a\Big(X \lrcorner (X \cdot \Phi) - \tfrac{1}{p+1} X \lrcorner (X \lrcorner (\gamma\cdot \wedge \Phi))\Big) + \tfrac{1}{p+1} X \lrcorner (X \lrcorner \mathrm{d}(\Phi)) = \\
&= a\, X \lrcorner (X \cdot \Phi) = 0.
\end{aligned}
$$

□

Next we turn to conformal Killing spinor-valued forms. We can already observe much simpler pattern in the definition of the conformal Killing field:

1) Only the twistor operator is prescribed to vanish.

Indeed, this is the generally accepted definition for all types of conformal Killing fields. However, this definition applies well only in case we start from an irreducible representation. In such a case the twistor operator is defined to be the unique highest weight component of the covariant derivative.

In our highly reducible case of spinor-valued forms we defined the twistor operator by (2.49). From (1.136) we know that the twistor module decomposes on the twistor modules corresponding to the individual primitive parts. Hence the equation

$$\mathrm{T}(\Phi) = 0 \tag{2.56}$$

does not bring anything really new. Moreover, the decomposition (1.136) does not contain a summand corresponding to the zeroth primitive part. Accordingly, the equation (2.56) leaves the covariant derivative of the zeroth primitive part $\Phi[0]$ completely unprescribed. For these reasons we avoid to introduce the definition of conformal Killing spinor-valued forms in general and reserve it only to the case of primitive spinor-valued differential forms.

Similarly to ordinary forms, we define special Killing spinor-valued forms by imposing an additional second order condition on its differential.

Definition 43. A *special Killing spinor-valued p-form* is a Killing spinor-valued p-form Φ, which in addition satisfies

$$
\begin{aligned}
\nabla_X(\mathrm{d}(\Phi)) = \tfrac{1}{2} X \cdot \mathrm{d}(\Phi) &+ \tfrac{1}{2(p+1)} \gamma\cdot \wedge (X \lrcorner \mathrm{d}(\Phi)) - \\
&- \Big(p + \tfrac{1}{2}\Big) X^* \wedge \Phi + \Big(\tfrac{1}{4} + \tfrac{ap}{2(p+1)}\Big) \gamma\cdot \wedge (X \cdot \Phi) + \\
&+ \tfrac{a}{2(p+1)} \gamma\cdot \wedge (\gamma\cdot \wedge (X \lrcorner \Phi)),
\end{aligned}
\tag{2.57}
$$

$\forall X \in \mathcal{T}$.

This definition is directly motivated by the cone construction which will be discussed in chapter 3. As such it is perhaps too restrictive and a more general definition involving another indeterminate constant would be worth to study on its own.

2.7 Primitive Killing spinor-valued forms

Again we start by introducing invariant differential operators, this time given by the decomposition (1.130):

- the *codifferential* $\mathrm{d}^*_{\mathsf{U}} \colon \Gamma(\mathcal{U}^q) \to \Gamma(\mathcal{U}^{q-1})$,

$$\mathrm{d}^*_{\mathsf{U}}(\Phi) = \pi^{\mathsf{DU}}_{q,q-1}(\nabla(\Phi)) = \pi^{\mathsf{DSA}}_{q,q-1}(\nabla(\Phi)) = \mathrm{d}^*(\Phi), \tag{2.58}$$

- the *Dirac operator* $\mathrm{D}_{\mathsf{U}} \colon \Gamma(\mathcal{U}^q) \to \Gamma(\mathcal{U}^q)$, defined only when $2q+1<n$,

$$\mathrm{D}_{\mathsf{U}}(\Phi) = \pi^{\mathsf{DU}}_{q,q}(\nabla(\Phi)) = \mathrm{D}(\Phi) - \tfrac{2}{n-2q+2}\,\gamma\cdot \wedge \mathrm{d}^*(\Phi), \tag{2.59}$$

- the *covariant exterior derivative* $\mathrm{d}_{\mathsf{U}} \colon \Gamma(\mathcal{U}^q) \to \Gamma(\mathcal{U}^{q+1})$,

defined only when $2q<n$,

$$\begin{aligned}\mathrm{d}_{\mathsf{U}}(\Phi) &= \pi^{\mathsf{DU}}_{q,q+1}(\nabla(\Phi)) = \\ &= \mathrm{d}(\Phi) + \tfrac{1}{n-2q}\Big(\gamma\cdot \wedge \mathrm{D}(\Phi) - \tfrac{1}{n-2q+1}\,\gamma\cdot \wedge (\gamma\cdot \wedge \mathrm{d}^*(\Phi))\Big),\end{aligned} \tag{2.60}$$

- the *twistor operator* $\mathrm{T}_{\mathsf{U}} \colon \Gamma(\mathcal{U}^q) \to \Gamma(\mathcal{U}^{q,1})$,

$$\mathrm{T}_{\mathsf{U}}(\Phi) = \pi^{\mathsf{DU}}_{q,q,1}(\nabla(\Phi)), \tag{2.61}$$

in more detail:

a) if $2q+1<n$, then

$$\mathrm{T}_{\mathsf{U}}(\Phi) = \nabla(\Phi) - \iota^{\mathsf{DU}}_{q,q-1}(\mathrm{d}^*_{\mathsf{U}}(\Phi)) - \iota^{\mathsf{DU}}_{q,q}(\mathrm{D}_{\mathsf{U}}(\Phi)) - \iota^{\mathsf{DU}}_{q,q+1}(\mathrm{d}_{\mathsf{U}}(\Phi)), \tag{2.62}$$

b) if $2q+1=n$, then

$$\mathrm{T}_{\mathsf{U}}(\Phi) = \nabla(\Phi) - \iota^{\mathsf{DU}}_{q,q-1}(\mathrm{d}^*_{\mathsf{U}}(\Phi)) - \iota^{\mathsf{DU}}_{q,q}(\mathrm{D}_{\mathsf{U}}(\Phi)), \tag{2.63}$$

c) if $2q=n$, then

$$\mathrm{T}_{\mathsf{U}}(\Phi) = \nabla(\Phi) - \iota^{\mathsf{DU}}_{q,q-1}(\mathrm{d}^*_{\mathsf{U}}(\Phi)), \tag{2.64}$$

$\forall \Phi \in \Gamma(\mathcal{U}^q)$. Unless otherwise stated we assume

$$q \in \{1,\ldots,k\}. \tag{2.65}$$

The definition 39 of Killing spinor-valued forms applies unchanged also to primitive spinor-valued forms. We just need to carefully handle the degenerate cases when $q=k$.

Proposition 44. *If $2q < n$, then a primitive spinor-valued q-form Φ is a Killing spinor-valued form with Killing number a, if and only if it satisfies the following three conditions:*

1) Φ is an eigenvector of the Dirac operator,

$$\mathrm{D}_{\mathsf{U}}(\Phi) = -(n+2)a\,\Phi, \tag{2.66}$$

2) Φ belongs to the kernel of the codifferential,

$$\mathrm{d}^*_{\mathsf{U}}(\Phi) = 0, \tag{2.67}$$

3) Φ belongs to the kernel of the twistor operator,

$$\mathrm{T}_{\mathsf{U}}(\Phi) = 0. \tag{2.68}$$

If $2q = n$, then a primitive spinor-valued differential q-form Φ is a Killing spinor-valued form, if and only if it is parallel,

$$\nabla(\Phi) = 0. \tag{2.69}$$

In particular, the Killing number a is necessarily zero in this case, unless Φ is zero.

Proof. Let $\Phi \in \Gamma(\mathcal{U}^q)$. We consider the case $2q < n$ first. We will show that the conditions (2.66)–(2.68) here are together equivalent to conditions (2.51)–(2.53) of proposition 40. From lemmas 23 and 25 follows

$$\begin{aligned}
\widetilde{\mathrm{D}}|_{\Gamma(\mathcal{U}^q)} &= \tfrac{q}{q+1}\,\mathrm{D}_{\mathsf{U}} + \tfrac{n+2}{(q+1)(n-2q+2)}\,\gamma\cdot\wedge\,\mathrm{d}^*_{\mathsf{U}},\\
\widetilde{\mathrm{d}^*}|_{\Gamma(\mathcal{U}^q)} &= \tfrac{q-1}{q}\,\mathrm{d}^*_{\mathsf{U}},\\
\mathrm{T}|_{\Gamma(\mathcal{U}^q)} &= \mathrm{T}_{\mathsf{U}},
\end{aligned}$$

and the conditions of proposition 40 can be thus written as

$$\mathrm{D}_{\mathsf{U}}(\Phi) + \tfrac{n+2}{q(n-2q+2)}\,\gamma\cdot\wedge\,\mathrm{d}^*_{\mathsf{U}}(\Phi) = -(n+2)a\,\Phi, \tag{2.70}$$

$$(q-1)\,\mathrm{d}^*_{\mathsf{U}}(\Phi) = 0, \tag{2.71}$$

$$\mathrm{T}_{\mathsf{U}}(\Phi) = 0. \tag{2.72}$$

Now note that the second term on the left-hand side of the first equation (2.70) belongs to a different primitive part of the space $\Gamma(\mathcal{SA}^q)$ than the other terms,

$$\Phi,\ \mathrm{D}_{\mathsf{U}}(\Phi) \in \Gamma(\mathcal{U}^q) = \Gamma(\mathcal{SA}^q[q]), \qquad \gamma\cdot\wedge\,\mathrm{d}^*_{\mathsf{U}}(\Phi) \in \Gamma(\mathcal{SA}^q[q-1]),$$

and hence the equation (2.70) is equivalent to (2.66) and (2.67) together. Hence the equations (2.70)–(2.72) are together equivalent to (2.66)–(2.68) and the proof for the case $2q < n$ is complete.

If $2q = n$, the equations (2.71) and (2.72) remain unchanged, but instead of (2.70) we have by lemma 23

$$\gamma\cdot \wedge \mathrm{d}^*_{\mathsf{U}}(\Phi) = -(n+2)a\,\Phi. \tag{2.73}$$

The term on the left-hand side again belongs to a different primitive part of the space $\Gamma(\mathcal{SA}^q)$ and hence the equation (2.73) is equivalent to

$$\mathrm{d}^*_{\mathsf{U}}(\Phi) = 0, \qquad \text{and} \qquad a = 0,$$

unless Φ is zero. Moreover, by the decomposition (1.133) the codifferential and twistor operator are the only components of the covariant derivative in this case. Hence the whole covariant derivative of Φ has to vanish and the proof is complete. □

For later use we also prove the following auxiliary lemma.

Lemma 45. *Let Φ be a primitive Killing spinor-valued q-form with Killing number a. Then it holds:*

$$\gamma^*\cdot \lrcorner\, \mathrm{d}(\Phi) = \mathrm{D}(\Phi) = \mathrm{D}_{\mathsf{U}}(\Phi) = -(n+2)a\,\Phi. \tag{2.74}$$

Proof. The first equality follows from (1.125) of lemma 22. The second equality follows from (2.59) and (2.67) of proposition 44. Finally, the third equality is just (2.66) of proposition 44. □

As promised in the previous section we introduce also the definition of primitive conformal Killing spinor-valued forms. For the sake of simplicity we state it directly in terms of the twistor operator.

Definition 46. A primitive spinor-valued p-form Φ is a *conformal Killing spinor-valued form* if it belongs to the kernel of the twistor operator,

$$\mathrm{T}_{\mathsf{U}}(\Phi) = 0. \tag{2.75}$$

Chapter 3

Cone construction

In this chapter we present our main result, the cone construction for special Killing spinor-valued forms. It is analogous to the result by Bär in [1] for Killing spinors and the result by Semmelmann in [14] for special Killing forms. First we introduce the metric cone $\overline{\mathcal{M}}$ over the manifold $\mathcal{M}$ and deduce formulas for the covariant derivative on $\overline{\mathcal{M}}$. In the course we also repeat the result by Bär as a prerequisite for our results.

Because we shall treat both an n-dimensional manifold $\mathcal{M}$ and its $(n+1)$-dimensional metric cone $\overline{\mathcal{M}}$, we need to extend our notation in order to distinguish them. We keep the notation from previous chapters for objects related to the base manifold $\mathcal{M}$ and use the bar to distinguish objects related to the cone $\overline{\mathcal{M}}$. Thus we have, for instance,

- $\overline{\mathsf{V}} = \mathbb{R}^{n+1}$ — the *real arithmetic vector space* of dimension $n+1$,
- $\overline{g}$ — the canonical *inner product* on $\overline{\mathsf{V}}$ or the Riemannian metric on $\overline{\mathcal{M}}$,
- $\overline{\mathcal{T}}$ — the *tangent bundle* of $\overline{\mathcal{M}}$,
- $\overline{\nabla}$ — the *covariant derivative* on $\overline{\mathcal{M}}$,

and so on. Moreover, we view V as a subspace of $\overline{\mathsf{V}}$. Hence we make an exception from the bar rule and denote the canonical basis of $\overline{\mathsf{V}}$ by

$$(e_1, \dots, e_n, e_{n+1}), \tag{3.1}$$

where $(e_1, \dots, e_n)$ is the canonical basis of V.

3.1 Metric cone

The *metric cone* over Riemannian manifold $\mathcal{M}$ with metric g is the product manifold $\overline{\mathcal{M}} = \mathcal{M} \times \mathbb{R}_+$ with metric $\overline{g}$ defined by

$$\overline{g} = r^2 g + \mathrm{d}r^2, \tag{3.2}$$

where r is the coordinate given by the projection onto $\mathbb{R}_+$. We denote the canonical projections and corresponding families of embeddings by

- $\pi_1 \colon \overline{\mathcal{M}} \to \mathcal{M}$ and $\pi_2 \colon \overline{\mathcal{M}} \to \mathbb{R}_+$,

$$\pi_1(x, r) = x, \qquad \pi_2(x, r) = r, \tag{3.3}$$

- $\iota_{1,r} \colon \mathcal{M} \to \overline{\mathcal{M}}$ and $\iota_{2,x} \colon \mathbb{R}_+ \to \overline{\mathcal{M}}$,

$$\iota_{1,r}(x) = (x,r), \qquad \iota_{2,x}(r) = (x,r), \tag{3.4}$$

$\forall x \in \mathcal{M},\ r \in \mathbb{R}_+$.

The tangent space of the cone $\overline{\mathcal{M}}$ naturally splits at any point (x,r) as

$$\overline{\mathcal{T}}_{(x,r)} = \iota_{1,r*}(\mathcal{T}_x) \oplus \iota_{2,x*}((\mathcal{T}\mathbb{R}_+)_r), \tag{3.5}$$

where $\iota_{1,r*}$ and $\iota_{2,r*}$ are differentials of the embeddings and $\mathcal{T}\mathbb{R}_+$ is the tangent bundle on the half-line $\mathbb{R}_+$. The splittings depend smoothly on (x,r) and hence yield a splitting of the whole tangent bundle,

$$\overline{\mathcal{T}} = \iota_{1*}(\mathcal{T}) \oplus \iota_{2*}(\mathcal{T}\mathbb{R}_+). \tag{3.6}$$

The subbundles $\iota_{1*}(\mathcal{T})$ and $\iota_{2*}(\mathcal{T}\mathbb{R}_+)$ are just distributions tangent to the closed submanifolds $\iota_{1,r}(\mathcal{M})$ and $\iota_{2,x}(\mathbb{R}_+)$ respectively.

To a tangent vector or a vector field X on $\mathcal{M}$ we associate a tangent vector or vector field $\overline{X}$ on $\overline{\mathcal{M}}$ by taking its image under $\iota_{1,r*}$ and rescaling,

$$\overline{X}_{(x,r)} = \tfrac{1}{r}\iota_{1,r*}(X_x). \tag{3.7}$$

We also denote

- ∂_r — the *radial vector field* on $\overline{\mathcal{M}}$,

$$(\partial_r)_{(x,r)} = \iota_{2,x*}\left(\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)_r\right), \tag{3.8}$$

where $\frac{\mathrm{d}}{\mathrm{d}r}$ is the canonical unit vector field on $\mathbb{R}_+$.

Thanks to the rescaling in (3.7) the metric $\overline{g}$ from (3.2) satisfies

$$\overline{g}(\overline{X},\overline{Y}) = g(X,Y), \tag{3.9}$$

and in addition

$$\overline{g}(\overline{X},\partial_r) = 0, \qquad \overline{g}(\partial_r,\partial_r) = 1, \tag{3.10}$$

$\forall X, Y \in \mathcal{T}$.

As for differential forms, we can simply take the pull-back of a form on $\mathcal{M}$ via the projection π_1. We again include appropriate rescaling and to a p-form α on $\mathcal{M}$ we associate a p-form $\overline{\alpha}$ on $\overline{\mathcal{M}}$ defined by

$$\overline{\alpha} = r^p\, {\pi_1}^*(\alpha). \tag{3.11}$$

We shall also often use the 1-form

- $\mathrm{d}r$ — the differential of the radial coordinate.

It is just the pull-back of the canonical 1-form on $\mathbb{R}_+$ via the projection π_2.

We also recall the general construction of a pull-back bundle, which we shall use later to relate the spinor bundles on $\mathcal{M}$ and $\overline{\mathcal{M}}$. Let

1) $F\colon \mathcal{M}' \to \mathcal{M}$ be a smooth mapping of manifolds, and

2) $\pi_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{M}$ be a smooth fiber bundle on $\mathcal{M}$.

Then the smooth *pull-back bundle* $F^*(\mathcal{B})$ on $\mathcal{M}'$ is defined as

$$F^*(\mathcal{B}) = \{(x', b) \mid F(x') = \pi_{\mathcal{B}}(b)\} \subseteq \mathcal{M}' \times \mathcal{B}, \tag{3.12}$$

with the projection given by

$$\pi'_{\mathcal{B}}(x', b) = x'. \tag{3.13}$$

As an example we can take the pull-backs of the tangent bundles $\mathcal{T}$ and $\mathcal{T}\mathbb{R}_+$ via the projections π_1 and π_2 respectively. Since we have

$$\pi_1 \circ \iota_{1,r} = 1_{\mathcal{M}}, \qquad \pi_2 \circ \iota_{2,x} = 1_{\mathbb{R}_+}, \qquad \forall x \in \mathcal{M},\ r \in \mathbb{R}_+, \tag{3.14}$$

we can identify the pull-back bundles with the subbundles from (3.6),

$$\pi_1^*(\mathcal{T}) \cong \iota_{1*}(\mathcal{T}), \qquad \pi_2^*(\mathcal{T}\mathbb{R}_+) \cong \iota_{2*}(\mathcal{T}\mathbb{R}_+). \tag{3.15}$$

Using this identification we can speak of the pull-back of vector field on $\mathcal{M}$ or $\mathbb{R}_+$ and express the equations (3.7) and (3.8) in the form

$$\overline{X} = \tfrac{1}{r}\,\pi_1^*(X), \qquad \partial_r = \pi_2^*\Big(\tfrac{\mathrm{d}}{\mathrm{d}r}\Big). \tag{3.16}$$

3.2 Spinors

The inclusion of vector spaces $\mathsf{V} \subseteq \overline{\mathsf{V}}$ induces a natural inclusion of the corresponding Clifford algebras, spin groups and spinor spaces,

$$\mathsf{Cl}(n) \subseteq \mathsf{Cl}(n+1), \qquad \mathsf{Spin}(n) \subseteq \mathsf{Spin}(n+1), \qquad \mathsf{S} \subseteq \overline{\mathsf{S}}. \tag{3.17}$$

A deeper analysis of the spinor spaces actually provides the following relations between the underlying vector spaces:

a) if $n = 2k$ then

$$\overline{\mathsf{S}} = \mathsf{S} = e_{n+1} \cdot \mathsf{S}, \tag{3.18}$$

b) if $n = 2k + 1$ then

$$\overline{\mathsf{S}} = \mathsf{S} \oplus e_{n+1} \cdot \mathsf{S}. \tag{3.19}$$

The subalgebra $\mathsf{Cl}_+(n)$ commutes with e_{n+1} and hence the relations remain valid also when viewing S and $\overline{\mathsf{S}}$ as $\mathsf{Cl}_+(n)$-modules. In particular, S is always an invariant subspace of $\overline{\mathsf{S}}$ with respect to $\mathsf{Spin}(n)$. However, for our purposes it is convenient to introduce also some other embeddings of S into $\overline{\mathsf{S}}$.

Lemma 47. *The mappings*

- $f_\pm \colon \mathsf{S} \to \overline{\mathsf{S}}$,

$$f_\pm(\psi) = (1 \mp e_{n+1}) \cdot \psi, \qquad \forall \psi \in \mathsf{S}, \tag{3.20}$$

are injective and $\mathsf{Spin}(n)$*-intertwining.*

Proof. The injectivity follows from the existence of inverse elements,

$$(1 \mp e_{n+1})^{-1} = \tfrac{1}{2}(1 \pm e_{n+1}).$$

The invariance with respect to $\mathsf{Spin}(n)$ follows again from the fact that the subalgebra $\mathsf{Cl}_+(n)$ commutes with e_{n+1}. □

Next we proceed towards the spinor bundles on $\mathcal{M}$ and $\overline{\mathcal{M}}$. Following the construction discussed in section 2.2 we start with the principal bundles $\mathcal{P}_{\mathsf{SO}}$ and $\overline{\mathcal{P}_{\mathsf{SO}}}$. To each positively oriented orthonormal frame

$$b = (X_1, \ldots, X_n) \tag{3.21}$$

at some point of $\mathcal{M}$, we can assign a positively oriented orthonormal frame

$$\bar{b} = (\overline{X_1}, \ldots, \overline{X_n}, \partial_r) \tag{3.22}$$

at the corresponding points of $\overline{\mathcal{M}}$. We can understand this assignment as an injective bundle map

- $I \colon \pi_1^*(\mathcal{P}_{\mathsf{SO}}) \to \overline{\mathcal{P}_{\mathsf{SO}}}$,

where $\pi_1^*(\mathcal{P}_{\mathsf{SO}})$ is the pull-back of $\mathcal{P}_{\mathsf{SO}}$.

Since the cone $\overline{\mathcal{M}}$ is homotopy equivalent to $\mathcal{M}$ there is a one-to-one correspondence between the *Spin*-structures on $\overline{\mathcal{M}}$ and $\mathcal{M}$. We can make this correspondence explicit thanks to the bundle map I. So let $\mathcal{P}_{\mathsf{Spin}}$ be the chosen *Spin*-structure on $\mathcal{M}$. First we take the pull-back $\pi_1^*(\mathcal{P}_{\mathsf{Spin}})$ which is a principal $\mathsf{Spin}(n)$-bundle on $\overline{M}$ and then we extend its fiber by taking the associated bundle

$$\overline{\mathcal{P}_{\mathsf{Spin}}} = \pi_1^*(\mathcal{P}_{\mathsf{Spin}}) \times_{\mathsf{Spin}(n)} \mathsf{Spin}(n+1), \tag{3.23}$$

where $\mathsf{Spin}(n)$ acts on $\mathsf{Spin}(n+1)$ by the left translation. We identify $\pi_1^*(\mathcal{P}_{\mathsf{Spin}})$ with a subbundle of $\overline{\mathcal{P}_{\mathsf{Spin}}}$ and denote the inclusion by

- $I' \colon \pi_1^*(\mathcal{P}_{\mathsf{Spin}}) \to \overline{\mathcal{P}_{\mathsf{Spin}}}$.

Now $\overline{\mathcal{P}_{\mathsf{Spin}}}$ is clearly a principal $\mathsf{Spin}(n+1)$-bundle and we define

- the two-fold covering bundle map $\overline{\Lambda} \colon \overline{\mathcal{P}_{\mathsf{Spin}}} \to \overline{\mathcal{P}_{\mathsf{SO}}}$,

such that it satisfies the equation

$$\overline{\Lambda} \circ I' \circ \pi_1^* = I \circ \pi_1^* \circ \Lambda \tag{3.24}$$

together with (2.3). A routine calculation shows that $\overline{\Lambda}$ is well-defined and makes $\overline{\mathcal{P}_{\mathsf{Spin}}}$ into a spin structure on $\overline{\mathcal{M}}$.

Given the principal bundle inclusion I' we can further reduce the structure group of the natural vector bundles of $\overline{\mathcal{M}}$ to $\mathsf{Spin}(n)$. In particular, we get bundle equivalences:

$$\overline{\mathcal{T}} \cong \pi_1^*(\mathcal{P}_{\mathsf{Spin}}) \times_{\mathsf{Spin}(n)} \overline{\mathsf{V}}, \qquad \overline{\mathcal{T}}^* \cong \pi_1^*(\mathcal{P}_{\mathsf{Spin}}) \times_{\mathsf{Spin}(n)} \overline{\mathsf{V}}^*,$$
$$\overline{\mathcal{S}} \cong \pi_1^*(\mathcal{P}_{\mathsf{Spin}}) \times_{\mathsf{Spin}(n)} \overline{\mathsf{S}}. \tag{3.25}$$

Since the spinor space S is an invariant subspace of $\overline{\mathsf{S}}$, we can naturally identify the pull-back of the spinor bundle $\mathcal{S}$ on $\mathcal{M}$,

$$\pi_1^*(\mathcal{S}) \cong \pi_1^*(\mathcal{P}_{\mathsf{Spin}}) \times_{\mathsf{Spin}(n)} \mathsf{S}, \tag{3.26}$$

with a subbundle of the spinor bundle $\overline{\mathcal{S}}$ on $\overline{\mathcal{M}}$.

The $\mathsf{Spin}(n)$-intertwining mappings $f_\pm$ from (3.20) induce bundle maps

- $F_\pm \colon \pi_1^*(\mathcal{S}) \to \overline{\mathcal{S}}$,

$$F_\pm(\pi_1^*(\Psi)) = (1 \mp \partial_r) \cdot \pi_1^*(\Psi), \qquad \forall \Psi \in \mathcal{S}. \tag{3.27}$$

To a spinor or spinor field Ψ on $\mathcal{M}$ we finally associate a spinor or spinor field $\overline{\Psi}$ on $\overline{\mathcal{M}}$ defined by

$$\overline{\Psi} = F_+(\pi_1^*(\Psi)). \tag{3.28}$$

If s and $\overline{s}$ are lifts of local orthonormal frames in form (3.21) and (3.22) respectively, we can express (3.28) with respect to s and $\overline{s}$ by

$$\overline{[s, \psi]} = [\overline{s}, f(\pi_1{}^*(\psi))] = [\overline{s}, f(\psi \circ \pi_1)], \tag{3.29}$$

where ψ is a locally defined spinor-valued function on $\mathcal{M}$. The use of mapping f_+ already in (3.29) is just for convenience and will simplify subsequent formulas. Also note that we could have used the mapping f_- equally well, only with some sign changes in subsequent formulas.

3.3 Connection

We denote by $\overline{\nabla}$ the covariant derivative on $\overline{\mathcal{M}}$ induced by the Levi-Civita connection given by the metric $\overline{g}$. In order to express $\overline{\nabla}$ in terms of ∇ on $\mathcal{M}$ we first compute the commutator

$$[\overline{X}, \overline{Y}] = \tfrac{1}{r}\overline{[X, Y]}, \qquad [\overline{X}, \partial_r] = \tfrac{1}{r}\,\overline{X}, \tag{3.30}$$

$\forall X, Y \in \mathcal{X}$. Now using (2.12), (3.9), (3.10) and (3.30) we get

$$\overline{\nabla}_{\overline{X}}(\overline{Y}) = \tfrac{1}{r}(\overline{\nabla_X(Y)} - g(X, Y)\,\partial_r), \qquad \overline{\nabla}_{\partial_r}(\overline{X}) = 0,$$
$$\overline{\nabla}_{\overline{X}}(\partial_r) = \tfrac{1}{r}\,\overline{X}, \qquad \overline{\nabla}_{\partial_r}(\partial_r) = 0, \tag{3.31}$$

and dually

$$\overline{\nabla}_{\overline{X}}(\overline{\alpha}) = \tfrac{1}{r}(\overline{\nabla_X(\alpha)} - \mathrm{d}r \wedge \overline{(X \lrcorner\, \alpha)}), \qquad \overline{\nabla}_{\partial_r}(\overline{\alpha}) = 0,$$
$$\overline{\nabla}_{\overline{X}}(\mathrm{d}r) = \tfrac{1}{r}\,\overline{X}^*, \qquad \overline{\nabla}_{\partial_r}(\mathrm{d}r) = 0, \tag{3.32}$$

$\forall X \in \mathcal{T}$, $Y \in \mathcal{X}$, $\alpha \in \Omega^p$.

Remark 48. Our formulas (3.31) and (3.32) for covariant derivative on the metric cone differ from the more common formulas

$$\overline{\nabla}_{\widetilde{X}}(\widetilde{Y}) = \widetilde{\nabla_X(Y)} - r\, g(X,Y)\,\partial_r, \qquad \overline{\nabla}_{\partial_r}(\widetilde{X}) = -\tfrac{1}{r}\widetilde{X}, \tag{3.31'}$$

$$\overline{\nabla}_{\widetilde{X}}(\widetilde{\alpha}) = \widetilde{\nabla_X(\alpha)} - \tfrac{1}{r}\,\mathrm{d}r \wedge (\widetilde{X \lrcorner \alpha}), \qquad \overline{\nabla}_{\partial_r}(\widetilde{\alpha}) = -\tfrac{p}{r}\,\widetilde{\alpha}, \tag{3.32'}$$

found, e.g., in [14]. This is because we rescale the pull-back of a vector or form by the appropriate homogeneous scalar factor right from the beginning. Indeed, substituting

$$\widetilde{X} = {\pi_1}^*(X) = r\overline{X}, \qquad \widetilde{\alpha} = {\pi_1}^*(\alpha) = \tfrac{1}{r^p}\,\overline{\alpha},$$

we can easily prove that our formulas are equivalent to (3.31') and (3.32').

Let b be a local orthonormal frame field on $\mathcal{M}$ and ω_{jm} the connection form on $\mathcal{M}$ expressed with respect to b as in (2.14) and (2.15). Furthermore, let $\overline{b}$ be a local orthonormal frame field on $\overline{\mathcal{M}}$ associated to b as in (3.22). Using (2.15) and (3.31) we compute the connection form $\overline{\omega}$ on $\overline{\mathcal{M}}$ with respect to $\overline{b}$,

$$\begin{aligned} &\overline{\omega}_{jm}(\overline{X_i}) = \tfrac{1}{r}\,\omega_{jm}(X_i), && \overline{\omega}_{(n+1)j}(\overline{X_i}) = -\overline{\omega}_{j(n+1)}(\overline{X_i}) = \tfrac{1}{r}\,\delta_{ij}, \\ &\overline{\omega}_{jm}(\partial_r) = 0, && \overline{\omega}_{(n+1)j}(\partial_r) = -\overline{\omega}_{j(n+1)}(\partial_r) = 0, \end{aligned} \tag{3.33}$$

$\forall i,j,m \in \{1,\dots,n\}$. From this description of the connection form we can compute the covariant derivative of associated spinor fields.

Lemma 49. *Let Ψ be a spinor field on $\mathcal{M}$ and $\overline{\Psi}$ the associated spinor field on the cone $\overline{\mathcal{M}}$. The covariant derivative of $\overline{\Psi}$ is given by*

$$\overline{\nabla}_{\overline{X}}(\overline{\Psi}) = \tfrac{1}{r}\overline{\Big(\nabla_X(\Psi) - \tfrac{1}{2}\,X\cdot\Psi\Big)}, \qquad \overline{\nabla}_{\partial_r}(\overline{\Psi}) = 0, \tag{3.34}$$

$\forall X \in \mathcal{T}$.

Proof. Let b and associated $\overline{b}$ be local orthonormal frames as in (3.21) and (3.22), and let s and associated $\overline{s}$ be their respective lifts to the spin structure. We compute using (2.17), (3.29) and (3.33):

$$\begin{aligned} \overline{\nabla}_{\overline{X_i}}(\overline{\Psi}) &= \overline{\nabla}_{\overline{X_i}}(\overline{[s,\psi]}) = \overline{\nabla}_{\overline{X_i}}([\overline{s}, f({\pi_1}^*(\psi))]) = \\ &= \Big[\overline{s},\ \overline{X_i}(f({\pi_1}^*(\psi))) + \tfrac{1}{4}\sum_{j,m}\overline{\omega}_{jm}(\overline{X_i})\, e_j\cdot e_m\cdot f({\pi_1}^*(\psi)) + \\ &\qquad + \tfrac{1}{2}\sum_j \overline{\omega}_{j(n+1)}(\overline{X_i})\, e_j\cdot e_{n+1}\cdot f({\pi_1}^*(\psi))\Big] = \\ &= \tfrac{1}{r}\Big[\overline{s},\ (1-e_{n+1})\cdot\Big({\pi_1}^*(X_i(\psi)) + \tfrac{1}{4}\sum_{j,m}\omega_{jm}(X_i)\,e_j\cdot e_m\cdot {\pi_1}^*(\psi)\Big) - \\ &\qquad - \tfrac{1}{2}\sum_j \delta_{ij}\, e_j\cdot e_{n+1}\cdot(1-e_{n+1})\cdot{\pi_1}^*(\psi)\Big] = \\ &= \tfrac{1}{r}\Big[\overline{s},\ f\Big({\pi_1}^*\Big(X_i(\psi) + \tfrac{1}{4}\sum_{j,m}\omega_{jm}(X_i)\,e_j\cdot e_m\cdot\psi - \tfrac{1}{2}\,e_i\cdot\psi\Big)\Big)\Big] = \\ &= \tfrac{1}{r}\overline{\Big(\nabla_{X_i}(\Psi) - \tfrac{1}{2}\,X_i\cdot\Psi\Big)}, \end{aligned}$$

where the indices i,j,m run through $\{1,\dots,n\}$. Following the same approach we get the second equation and the proof is complete. □

Comparing lemma 49 with the definition 33 of a Killing spinor implies the result by Bär in [1].

Corollary 50. *The associated spinor field $\overline{\Psi}$ on the cone $\overline{M}$ is parallel, if and only if the spinor field Ψ on $\mathcal{M}$ is Killing with Killing number $a = \frac{1}{2}$.*

Note that Bär has also a parallel result for the Killing number $a = -\frac{1}{2}$. This can be obtained by using the mapping f_- instead of f_+ in (3.29).

3.4 Killing spinor-valued forms

The final section is devoted to the construction of a parallel spinor-valued $(p+1)$-form on the cone $\overline{M}$ from a special Killing spinor-valued p-form on $\mathcal{M}$. Our result is analogous to the result by Bär in [1] mentioned above and the result by Semmelmann in [14]. In our notation we can state the result by Semmelmann as follows.

Proposition 51. *Let α be a differential p-form on $\mathcal{M}$. Then the differential $(p+1)$-form β on the cone $\overline{\mathcal{M}}$ defined by*

$$\beta = \mathrm{d}r \wedge \overline{\alpha} + \tfrac{1}{p+1}\,\overline{\mathrm{d}(\alpha)} \tag{3.35}$$

is parallel, if and only if α is a special Killing form with $a = -(p+1)$.

Combining (3.11) and (3.28) we associate to a spinor-valued p-form Φ on $\mathcal{M}$ a spinor-valued p-form $\overline{\Phi}$ on the cone $\overline{\mathcal{M}}$ by linearly extending the formula

$$\overline{\alpha \otimes \Psi} = \overline{\alpha} \otimes \overline{\Psi}, \tag{3.36}$$

where $\alpha \in \Omega^p$, $\Psi \in \Sigma$. With respect to local sections s and associated $\overline{s}$ as in the previous sections we can express (3.36) as

$$\overline{[s,\phi]} = [\overline{s}, f_+(\pi_1^*(\phi))], \tag{3.37}$$

where ϕ is a locally defined SA^p-valued function on $\mathcal{M}$. Note that the homogeneous factor r^p from (3.11) is already hidden in the local section $\overline{s}$ since it is a lift of some local orthonormal frame $\overline{b}$ from (3.22).

Lemma 52. *Let Φ be a spinor-valued p-form on $\mathcal{M}$ and $\overline{\Phi}$ the associated spinor-valued p-form on the cone $\overline{\mathcal{M}}$. The covariant derivative of $\overline{\Phi}$ is given by the equations*

$$\begin{aligned} \overline{\nabla}_{\overline{X}}(\overline{\Phi}) &= \tfrac{1}{r}\Big(\overline{\nabla_X(\Phi) - \tfrac{1}{2}X \cdot \Phi} - \mathrm{d}r \wedge \overline{(X \lrcorner \Phi)}\Big), \\ \overline{\nabla}_{\partial_r}(\overline{\Phi}) &= 0, \end{aligned} \tag{3.38}$$

$\forall X \in \mathcal{T}$.

Proof. Follows immediately from (3.32) and (3.34). □

Proposition 53. *Let Φ be a spinor-valued p-form on $\mathcal{M}$. Then the spinor-valued $(p+1)$-form Ξ on the cone $\overline{\mathcal{M}}$ defined by*

$$\Xi = \mathrm{d}r \wedge \overline{\Phi} - \tfrac{1}{2(p+1)}\,\overline{\gamma\cdot \wedge \Phi} + \tfrac{1}{p+1}\,\overline{\mathrm{d}(\Phi)} \tag{3.39}$$

is parallel, if and only if Φ is a special Killing spinor-valued form with Killing number $a = \frac{1}{2}$.

Proof. We take $X \in \mathcal{T}$ and compute using (3.32), (3.38) and (1.34):

$$\begin{aligned}
\nabla_{\overline{X}}(\mathrm{d}r \wedge \overline{\Phi}) &= \tfrac{1}{r}\Big(\overline{X^* \wedge \Phi} + \mathrm{d}r \wedge \overline{\Big(\nabla_X(\Phi) - \tfrac{1}{2}\,X\cdot\Phi\Big)}\Big),\\
\nabla_{\overline{X}}(\overline{\gamma\cdot \wedge \Phi}) &= \tfrac{1}{r}\Big(\overline{\gamma\cdot \wedge \nabla_X(\Phi) + X^*\wedge\Phi + \tfrac{1}{2}\,\gamma\cdot\wedge(X\cdot\Phi)} -\\
&\qquad - \mathrm{d}r \wedge \overline{(X\cdot\Phi - \gamma\cdot\wedge(X\,\lrcorner\,\Phi))}\Big),\\
\nabla_{\overline{X}}(\overline{\mathrm{d}(\Phi)}) &= \tfrac{1}{r}\Big(\overline{\nabla_X(\mathrm{d}(\Phi)) - \tfrac{1}{2}\,X\cdot\mathrm{d}(\Phi)} - \mathrm{d}r \wedge \overline{(X\,\lrcorner\,\mathrm{d}(\Phi))}\Big).
\end{aligned}$$

Collecting the terms and using (3.39) we get:

$$\begin{aligned}
&\nabla_{\overline{X}}(\Xi) =\\
&= \tfrac{1}{r}\Big(\mathrm{d}r \wedge \overline{\Big(\nabla_X(\Phi) - \tfrac{1}{p+1}\Big(\tfrac{1}{2}(p\,X\cdot\Phi + \gamma\cdot\wedge(X\,\lrcorner\,\Phi)) + (X\,\lrcorner\,\mathrm{d}(\Phi))\Big)\Big)} +\\
&\quad + \tfrac{1}{p+1}\overline{\Big(\nabla_X(\mathrm{d}(\Phi)) - \tfrac{1}{2}\,X\cdot\mathrm{d}(\Phi) + \Big(p+\tfrac{1}{2}\Big)X^*\wedge\Phi -}\\
&\qquad \overline{- \tfrac{1}{4}\,\gamma\cdot\wedge(X\cdot\Phi) - \tfrac{1}{2}\,\gamma\cdot\wedge\nabla_X(\Phi)\Big)}\Big).
\end{aligned}$$

The term involving $\mathrm{d}r$ is linearly independent from the other one, hence $\nabla_{\overline{X}}(\Xi) = 0$ if and only if

$$\begin{aligned}
\nabla_X(\Phi) &= \tfrac{1}{p+1}\Big(\tfrac{1}{2}\,(p\,X\cdot\Phi + \gamma\cdot\wedge(X\,\lrcorner\,\Phi)) + X\,\lrcorner\,\mathrm{d}(\Phi)\Big) =\\
&= \tfrac{1}{2}\Big(X\cdot\Phi + \tfrac{1}{p+1}\,X\,\lrcorner\,(\gamma\cdot\wedge\Phi)\Big) + \tfrac{1}{p+1}\,X\,\lrcorner\,\mathrm{d}(\Phi),
\end{aligned} \tag{3.40}$$

rearranged using again (1.34), and

$$\begin{aligned}
\nabla_X(\mathrm{d}(\Phi)) &= \tfrac{1}{2}\,X\cdot\mathrm{d}(\Phi) - \Big(p+\tfrac{1}{2}\Big)X^*\wedge\Phi + \tfrac{1}{4}\,\gamma\cdot\wedge(X\cdot\Phi) +\\
&\quad + \tfrac{1}{2}\,\gamma\cdot\wedge\nabla_X(\Phi).
\end{aligned} \tag{3.41}$$

Substituting (3.40) into (3.41) we get:

$$\begin{aligned}
\nabla_X(\mathrm{d}(\Phi)) &= \tfrac{1}{2}\,X\cdot\mathrm{d}(\Phi) + \tfrac{1}{2(p+1)}\,\gamma\cdot\wedge(X\,\lrcorner\,\mathrm{d}(\Phi)) -\\
&\quad - \Big(p+\tfrac{1}{2}\Big)X^*\wedge\Phi + \Big(\tfrac{1}{4} + \tfrac{p}{4(p+1)}\Big)\gamma\cdot\wedge(X\cdot\Phi) +\\
&\quad + \tfrac{1}{4(p+1)}\,\gamma\cdot\wedge(\gamma\cdot\wedge(X\,\lrcorner\,\Phi)).
\end{aligned} \tag{3.42}$$

From (3.32) and (3.38) we also have $\nabla_{\partial_r}(\Xi) = 0$. Now the claim follows by comparing (3.40) and (3.42) with (2.50) and (2.57) respectively. □

We conclude the present section by briefly examining the case of primitive spinor-valued form Φ.

Lemma 54. *Let Φ be a spinor-valued p-form on $\mathcal{M}$ and $\overline{\Phi}$ the associated spinor-valued p-form on the cone $\overline{\mathcal{M}}$. Then it holds*

$$\gamma^{*}\cdot \lrcorner \, \overline{\Phi} = \partial_r \cdot \overline{\gamma^{*}\cdot \lrcorner \, \Phi}. \tag{3.43}$$

Proof. Let s and associated $\overline{s}$ be local sections of the spin structures on $\mathcal{M}$ and $\overline{\mathcal{M}}$ as in the previous. Further let ϕ be the locally defined SA^p-valued function on $\mathcal{M}$ which corresponds to Φ with respect to s. We compute using (3.20), (3.37) and (1.32):

$$\begin{aligned}
\gamma^{*}\cdot \lrcorner \, \overline{\Phi} &= \Big[\overline{s},\ \sum_{i=1}^{n+1} e_i \cdot (1 - e_{n+1}) \cdot (e_i \lrcorner \, {\pi_1}^*(\phi))\Big] = \\
&= \Big[\overline{s},\ \sum_{i=1}^{n} e_i \cdot (1 - e_{n+1}) \cdot (e_i \lrcorner \, {\pi_1}^*(\phi))\Big] = \\
&= \Big[\overline{s},\ \sum_{i=1}^{n} (1 + e_{n+1}) \cdot e_i \cdot (e_i \lrcorner \, {\pi_1}^*(\phi))\Big] = \\
&= \Big[\overline{s},\ (1 + e_{n+1}) \cdot {\pi_1}^*(\gamma^{*}\cdot \lrcorner \, \phi)\Big] = \\
&= \Big[\overline{s},\ e_{n+1} \cdot (1 - e_{n+1}) \cdot {\pi_1}^*(\gamma^{*}\cdot \lrcorner \, \phi)\Big] = \partial_r \cdot \overline{\gamma^{*}\cdot \lrcorner \, \Phi}.
\end{aligned}$$

□

Proposition 55. *Let Φ be a primitive Killing spinor-valued p-form on $\mathcal{M}$ with Killing number $a = \frac{1}{2}$. Then the spinor-valued $(p+1)$-form Ξ on the cone $\overline{\mathcal{M}}$ from (3.39) is also primitive.*

Proof. We compute using (3.39), (3.43), (2.74), (1.34), (1.35), (1.44) and the assumption $a = \frac{1}{2}$:

$$\begin{aligned}
\gamma^{*}\cdot \lrcorner \, \Xi &= \gamma^{*}\cdot \lrcorner \, (\mathrm{d}r \wedge \overline{\Phi}) - \tfrac{1}{2(p+1)} \gamma^{*}\cdot \lrcorner \, \overline{\gamma\cdot \wedge \Phi} + \tfrac{1}{p+1} \gamma^{*}\cdot \lrcorner \, \overline{\mathrm{d}(\Phi)} = \\
&= \partial_r \cdot \overline{\Phi} - \mathrm{d}r \wedge (\partial_r \cdot \overline{(\gamma^{*}\cdot \lrcorner \, \Phi)}) - \tfrac{1}{2(p+1)} \partial_r \cdot \overline{(\gamma^{*}\cdot \lrcorner \, (\gamma\cdot \wedge \Phi))} + \\
&\qquad + \tfrac{1}{p+1} \partial_r \cdot \overline{(\gamma^{*}\cdot \lrcorner \, \mathrm{d}(\Phi))} = \\
&= \partial_r \cdot \overline{\Phi} + \tfrac{n-2p}{2(p+1)} \partial_r \cdot \overline{\Phi} - \tfrac{n+2}{2(p+1)} \partial_r \cdot \overline{\Phi} = 0.
\end{aligned}$$

□

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