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Relative Topological Properties

Katedra matematické analýzy

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Abstrakt: V práci je studován koncept relativních topologických vlastností a je uvedeno několik základních faktů a vztahů. Nejvíce jsem se zaměřil na různé verze relativní normality, relativní regularity a relativní kompaktnosti. Je rovněž zodpovězeno několik dosud otevřených otázek z literatury. Teorie relativních topologických vlastností byla poprvé systematicky představena A. V. Archangelským a H. M. M. Genedim v roce 1989.

Hlavní výsledky práce jsou (1) příklad, který popisuje způsob jak upravit jakýkoli Dowkerův prostor a získat prostor X takový, že $X \times [0, 1]$ není κ-normalní (Example 4.2.12). (2) Věta, jež implikuje existenci prostoru, který je vnitřně kompaktní ve větším regularním prostoru a není Tichonovův (Theorem 5.2.8) a (3) věta charakterizující dvojice uzavřených neoddělitelných podmnožin Niemytzkého roviny (Theorem 4.3.4).
Klíčová slova: relativní normalita, relativní kompaktnost

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Abstract: In this thesis we study the concepts of relative topological properties and present some basic facts and relations between them. Our main focus is on various versions of relative normality, relative regularity and relative compactness. We give examples which answer some open questions and contradict some conjectures in the literature. The theory of relative topological properties was introduced by A. V. Arhangel'skii and H. M. M. Genedi in 1989.

Our main results are (1) an example which presents a way to modify any Dowker space to get a normal space X such that $X \times [0, 1]$ is not κ-normal (Example 4.2.12). (2) A theorem which implies the existence of a non-Tychonoff space that is internally compact in a larger regular space

(Theorem 5.2.8), and (3) a theorem that characterizes couples of closed subsets of the Niemytzki plane which cannot be separated by open sets (Theorem 4.3.4).

Keywords: relative normality, relative compactness

Chapter 1

Introduction

In general topology we often encounter the question of how a certain space Y is located in a larger space X. Examples of pairs of spaces, where the smaller space is specially located in a larger one, are a Tychonoff space in its Čech-Stone extension, a Hausdorff space in its Katětov extension or a T_1 space in its hyperspace. In situations like this it is natural to examine what are properties of Y with respect to X. A particular class of such properties is called relative topological properties of a space Y in a superspace X and we will study it in this thesis.

The systematic study of relative topological properties was begun by A. V. Arhangel'skii and H. M. M. Genedi in a paper published in Russian in 1989 [4]. In 1996 Arhangel'skii wrote a survey article on this topic [2]. Parts of this thesis are based on this article and all theorems in Sections 3.1, 4.1, 4.2 and 5.1 without any citation are originated there.

Relative topological properties often generalize a global property in the sense that if the smaller space Y coincides with the larger space X, then the relative topological property should be the same as the global one. For example of some global properties we mention Hausdorffness, regularity, normality, metrizability, and compactness like properties. In this text we will mainly study various version of relative separation axioms and relative compactness. We will also see that some global properties can be generalized in several ways yielding several relative versions of the global property.

In Chapter 3 we give a short survey of relative topological properties obtained from regularity, and we also show how various relative versions of regularity arise.

In Chapter 4 we discuss relative normality and the "normality on", which has a close relation to κ-normality. The notion of κ-normality was introduced in 1972 by E. V. Schepin in [17]. In Chapter 4 we give answers to two questions from Arhangel'skii's article [3] by constructing a normal space X, such that its product with the closed unit interval $X \times I$ is not κ-normal.This example is a joint work with Eva Murtinová. In the last part of Chapter 4 we study relative normality of subspaces of the Niemytzki plane and we will derive a general condition for such a subspace to be relatively normal. From this condition we easily obtain a negative answer to a question of M. G. Tkachenko et al. in [18] by showing that the Niemytzki plane is normal on a certain type of its dense countable subspaces.

Finally, in Chapter 5 we consider relative compactness, and answer two questions of Arhangel'skii from [3] by proving that there exists a non-Tychonoff space that is internally compact in a larger regular space.

At many places of this text, propositions and theorems are trivial or straightforward to prove or the proof is similar to some well known argument. In such situations proofs will be only sketched or completely skipped.

Chapter 2

Prerequisites

In this chapter we will give a short review of basic topological and set-theoretic notions and principles. Examples and theorems presented here will be used for various constructions in later chapters.

Our basic topological reference is [9]. For set theory e.g. [12] can be used.

2.1 Basic Topological Notions

We use the standard notation: the set of all natural numbers is denoted by ω, the set of all nonzero natural numbers $\mathbb{N}$, the set of real numbers $\mathbb{R}$, the set of positive real numbers $\mathbb{R}^+$, the set of all rational numbers $\mathbb{Q}$ and the set of irrational numbers $\mathbb{P}$. These sets are considered with the Eauclidean topology.

All topological spaces in this thesis are assumed to be T_1 and are usually denoted X or Y. For a subset A of a topological space (X, τ) the closure of A in (X, τ) is denoted by $\overline{A}$. If we want to emphasize the space or the topology we use the notation $\overline{A}^X$ or $\overline{A}^\tau$. The interior of the set A in the space X is denoted $int_X A$ or just $int A$.

Definition 2.1.1. Let X and Y be topological spaces. A mapping $f : X \to Y$ is *closed* if f is a continuous mapping and for each closed subset A of X the image $f[A]$ is a closed subset of Y.

Theorem 2.1.2. *Let $f : X \to Y$ be a closed mapping onto Y. Then:*

1. *If X is T_1, then Y is T_1.*
2. *If X is normal, then Y is normal.*

On the other hand other basic separation axioms are not preserved by closed mappings [7]. A stronger preservation property has the class of perfect mappings.

Definition 2.1.3. A mapping $f : X \to Y$ is a *perfect mapping* if X is a Hausdorff space, f is a continuous closed mapping and for each $y \in Y$ the preimage $f^{-1}[\{y\}]$ is a compact subset of X.

Theorem 2.1.4. *Let $f : X \to Y$ be a perfect mapping onto Y. Then:*

1. *If X is Hausdorff, then Y is Hausdorff.*
2. *X is regular if and only if Y is regular.*
3. *If X is normal, then Y is normal.*
4. *X is compact if and only if Y is compact.*
5. *X is locally compact if and only if Y is locally compact.*

It easy to construct examples showing that implications in 1 and 3 cannot be reversed.

In this thesis, we will often use relative versions of some topological properties. So if we need to emphasize that we are dealing with a general version of some property, we will e.g. use the notation "X is normal in itself" instead of just "X is normal".

Definition 2.1.5. Let Y be a subspace of a topological space X. The space Y is *C^0-embedded in X* if each continuous function $f : Y \to [0, 1]$ can be extended to a continuous function $F : X \to \mathbb{R}$.

The next construction will be used in a counterexample in section 4.2.

Example 2.1.6. Let X be any topological space. Let us recall a construction of a space $X^\star$ which contains X and is called its *Alexandroff double*. Put $X^\star = X \times 2$ (where $2 = \{0, 1\}$) and topologize $X^\star$ as follows. All points of $X \times \{1\}$ are isolated and a basic open neighborhood of a point $x \in X \times \{0\}$ is the set $(O \times 2) \setminus \{(x, 1)\}$ where O is an open subset of X containing x.

The following general construction is the most common way to construct a regular non-Tychonoff space. It will be used in section 5.2.

Example 2.1.7. Let X be a T_1 regular non-normal topological space. We will sketch the construction of a canonical T_1 regular non-Tychonoff space $J(X)$. The construction of the space $J(X)$ uses a method called *Jones machine.*

Pick two closed disjoint subsets A_0 and A_1 of X such that A_0 and A_1 cannot be separated by disjoint open neighborhoods. Add one new point z to the product $X \times \omega$. Let the base of topology at z consist of the sets of the form

$$\{z\} \cup \big(X \times (\omega \setminus 2n+i)\big) \cup \big((X \setminus A_i) \times \{2n-1+i\}\big)$$

for $n \in \mathbb{N}$ and $i \in \{0,1\}$. The resulting space $(X \times \omega) \cup \{z\}$ will be denoted $P(X)$. Finally identify each point $(a, 2n)$ in the set $A_0 \times \{2n\}$ with the corresponding point $(a, 2n+1)$ in $A_0 \times \{2n+1\}$ and each point $(a, 2n+1) \in A_1 \times \{2n+1\}$ with $(a, 2n+2) \in A_1 \times \{2n+2\}$ for every $n \in \omega$. This quotient space is the Jones space $J(X)$ and the quotient mapping will be denoted $q : P(X) \to J(X)$. Note that q is a perfect mapping.

It follows from the construction that $J(X)$ is a T_1 regular space and the closed set $A_1 \times \{0\}$ and the point z cannot be separated by a continuous real valued function, hence $J(X)$ is not Tychonoff. The space $J(X)$ inherits many properties from the original space X. For details see the original paper by F. B. Jones [13].

2.2 H-closed Spaces

A closed connection with compact spaces has the class of H-closed spaces. In chapter 5 we will see that this notion has also a close relation with relative compactness. We will remind just a few facts about H-closed spaces in this section, more can be found in [14].

Definition 2.2.1. A Hausdorff space X is called *H-closed* if X is a closed subspace of each of its Hausdorff superspaces.

Theorem 2.2.2 ([14])**.** *If X is a Hausdorff space, the following conditions are equivalent:*

1. *X is H-closed.*

2. *For every centered family $\mathcal{V}$ of open subsets of X the intersection $\bigcap\{\overline{V} : v \in \mathcal{V}\}$ is non-empty.*

3. *Every open cover $\mathcal{U}$ of X has a finite subset $\mathcal{U}'$ such that $\bigcup\{\overline{U} : U \in \mathcal{U}'\} = X$.*

Proposition 2.2.3. *A regular space X is H-closed if and only if X is compact.*

We will also use the notion of R-closed space.

Definition 2.2.4. A regular space X is called *R-closed* if X is a closed subspace of each of its regular superspaces.

2.3 Set Theory

Definition 2.3.1. A family $\mathcal{C}$ of sets is *centered* if for each finite set $\mathcal{A} \subset \mathcal{C}$ is $\bigcap \mathcal{A} \neq \emptyset$.

Definition 2.3.2. Two infinite sets A and B are *almost disjoint* if $A \cap B$ is finite. A system $\mathcal{A}$ consisting of infinite sets is called *almost disjoint system* (or *AD*) if each two members of $\mathcal{A}$ are almost disjoint. Let Z be an infinite set. The system $\mathcal{A}$ is a *maximal almost disjoint system* (or *MAD*) of subsets of Z if $\mathcal{A}$ consists of infinite subsets of Z, $\mathcal{A}$ is an AD system and $\mathcal{A}$ is a maximal such system (with respect to inclusion).

An ordinal number α is considered as the set of its predecessors. This for example means that the set $\{0, 1\}$ can be denoted 2.

Definition 2.3.3. Let κ be an infinite regular cardinal. The set A is a *closed unbounded set* (or *club*) if A is an unbounded subset of κ and contains all its limit points (κ is considered with the order topology and unbounded means cofinal).

Definition 2.3.4. Let κ be an infinite regular cardinal. A set $S \subset \kappa$ is *stationary* if for each closed unbounded set $A \subset \kappa$, $A \cap S$ is nonempty.

Theorem 2.3.5 (Fodor's Lemma)**.** *Let κ be an uncountable regular cardinal and S a stationary subset of κ. Then each function $f : S \to \kappa$, such that $f(\alpha) < \alpha$ for each $\alpha \in S$, $\alpha \neq 0$, is constant on some stationary subset of κ.*

Theorem 2.3.6 (Solovay)**.** *Let κ be an uncountable regular cardinal and S a stationary subset of κ. Then κ is a union of κ many pairwise disjoint stationary subsets of κ.*

Chapter 3

Lower Separation Axioms

3.1 Relative Regularity

Definition 3.1.1. A topological space Y *is regular in* X, if for each $y \in Y$ and for each subset A of X which is closed in X and such that $y \notin A$, there are two disjoint sets U and V, open in X, such that $y \in U$ and $A \cap Y \subset V$.

If the larger space X is regular, then clearly each subspace of X is regular in X.

Proposition 3.1.2. *If the space Y is regular in X, then Y is a regular (in itself) subspace of X.*

So in some cases the relative property implies an absolute property of the smaller space. The converse is sometimes also true, but in this case absolute regularity of the smaller space does not imply relative regularity. The next example is a version of a classical construction of a Hausdorff non-regular space (see, e.g. [9]).

Example 3.1.3. We will construct a regular space Y and a larger Hausdorff space X in which Y is not regular.

Let $P = \{1/n : n \in \mathbb{N}\}$ be a subset of the real line $\mathbb{R}$. Add one new element $\mathbb{R} \setminus P$ to the euclidean topology of the real line and denote the resulting topology τ. Let X be the space $(\mathbb{R}, \tau)$ and $Y = P \cup \{0\}$. Y is a discrete subset of X, thus it is regular, but Y is not regular in X.

The definition of relative regularity is not the only one natural generalization of regularity. There are more definitions arising from regularity and we will mention just one of them.

Definition 3.1.4. A topological space is Y *internally regular in* X, if for every $y \in Y$ and every subset A of Y which is closed in X and such that $y \notin A$, there are two disjoint sets U and V open in X such that $y \in U$ and $A \subset V$.

The phrase "internally" is usually used when some property is required for all subsets of the smaller space, which are closed in the larger one. Later we will define internal normality and internal compactness in this way.

It is easy to see that if Y is regular in X then Y is internally regular in X, but the converse is not true. A Hausdorff space X with a non-regular subspace Y and such that Y is internally regular in X, is constructed in Example 5.2.7.

Let us close this section by mentioning some conditions sufficient for Y to be regular in X.

Proposition 3.1.5. *If Y is a dense subspace of a space X, then Y is regular in X if and only if Y is regular.*

Theorem 3.1.6 ([5])**.** *For a Hausdorff space Y the following conditions are equivalent:*

1. *Y is regular in every larger Hausdorff space X.*

2. *Y is internally regular in every larger Hausdorff space.*

3. *Y is compact.*

Proof. 3⇒1: Let Y be a compact space and X a larger Hausdorff space. If A is a closed subset of X then $A \cap Y$ is a compact set and thus $A \cap Y$ and any point of $X \setminus A$ can be separated by disjoint open neighborhoods. Hence Y is regular in X.

Since regularity of Y in a Hausdorff space X implies internal regularity of Y in X, it is sufficient to prove 2⇒3. Let Y be a non-compact Hausdorff space. We will construct a larger Hausdorff space X in which Y is not internally regular.

Fix a centered family $\mathcal{C}$ of closed subsets of Y, which has an empty intersection. We may pick a $C_0 \in \mathcal{C}$ which is a proper subset of Y and assume that all the sets in $\mathcal{C}$ are subsets of C_0. Choose $y_0 \in Y \setminus C_0$. We aim to extend Y to a Hausdorff space X, so that C_0 will be closed in X but y_0 and C_0 cannot be separated by disjoint open subsets of X.

Let $X = Y \cup (C_0 \times \omega)$ and topologize X as follows. Let $Y \setminus (C_0 \cup y_0)$ be an open subspace and let all points of $C_0 \times \omega$ be isolated. A basic open neighborhood of y_0 has the form $U \cup (C \times \omega)$ for U an open neighborhood of y_0 in $Y \setminus C_0$ and $C \in \mathcal{C}$. A basic open neighborhood of $x \in C_0$ has the form $V \cup \big((V \cap C_0) \times (\omega \setminus n)\big)$ where V is some open neighborhood of x in Y such that $y_0 \notin V$ and $n \in \omega$.

To show that this is a correctly defined base of topology in X we need to check that finite intersections of basic sets are open sets. Let $W = U \cup (C \times \omega)$ be a basic neighborhood of y_0 and let $Z = V \cup \big((V \cap C_0) \times (\omega \setminus n)\big)$ be a basic open neighborhood of $x \in C_0$. Now $W \cap Z = (U \cap V) \cup \big((V \cap C) \times (\omega \setminus n)\big)$ where $U \cap V \subset Y \setminus (C_0 \cup \{y_0\})$ and so $W \cap Z$ is an open set.

The other cases are trivial to check so the definition of the topology works. Moreover, the topology of X coincides with the topology of Y on Y and C_0 is closed in X.

We will prove that Y is not internally regular in X. Let $W = U \cup (C \times \omega)$ be a basic open neighborhood of y_0. Take arbitrary $x \in C$ and let $Z = V \cup \big((V \cap C_0) \times (\omega \setminus n)\big)$ be any basic open neighborhood of x. Now $\{x\} \times (\omega \setminus n) \subset W \cap Z$ and thus $W \cap Z$ is nonempty and $x \in \overline{W} \cap C_0$.

It only remains to show that X is Hausdorff. The only nontrivial case is again y_0 and $x \in C_0$. The space Y is Hausdorff, hence there exist disjoint open subsets U, V of Y such that $y_0 \in U$ and $x \in V$. Since the intersection of $\mathcal{C}$ is empty, there exists a $C \in \mathcal{C}$ such that $x \notin C$. For such C is $Z = V \cap (Y \setminus C)$ an open neighborhood of x in Y. Then $x \in A = Z \cup (Z \cap C_0) \times \omega$ and $y_0 \in B = U \cup (C \times \omega)$, so A and B are disjoint open sets in X separating y_0 and x. □

The first proof of equivalence 1⇔3 in Theorem 3.1.6 was given in [5]. The authors used a different construction, which for each non-compact space Y gives a larger space X in which Y is not regular. The proof of 2⇒3 uses a simplified construction from [10].

Chapter 4

Relative Normality

4.1 Relative Normality

In this section, properties obtained by generalization of normality will be studied. There are again many natural ways how to define such a location property. We will consider just three of them: the normality in, normality on and internal normality. Let us start with the most common one.

Definition 4.1.1. Let Y be a subspace of a topological space X. The space Y is said to be *normal in* X if for every A and B which are disjoint closed subsets of X, there are two disjoint open sets U and V in X such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

Proposition 4.1.2. *If a space Y is normal in some larger space X, then Y is a regular space.*

Proof. If Y is normal in X then Y is obviously regular in X and due to Proposition 3.1.2, Y is a regular space. □

It is not known yet if the smaller space Y which is normal in a larger regular space X has to be Tychonoff. There is only a consistent result; MA + $\aleph_2 < 2^{\aleph_0}$ implies that there is a non-Tychonoff space normal in a larger regular space [10].

Theorem 4.1.3. *If Y is regular in X and the space Y is Lindelöf, then Y is normal in X.*

Normality of Y in X cannot be guaranteed by normality of Y, even if Y is closed in X. Pick a non-normal space X and two closed subsets A and B

of X which cannot be separated by disjoint open neighborhoods in X and such that $Y = A \cup B$ is discrete. Then Y is a discrete closed subset of X but Y is not normal in X.

The definition of internal normality follows the same pattern as the definition of internal regularity. Internal normality is again weaker than normality in.

Definition 4.1.4. A topological space Y is *internally normal in* X, if for every two disjoint subsets A and B of Y which are closed in X, there are disjoint sets U and V, open in X, such that $A \subset U$ and $B \subset V$.

Here we can mention some easy-to-prove propositions.

Proposition 4.1.5. *Let Y be a dense subspace of a space X and Z be internally normal in Y. Then Z is internally normal in X.*

Corollary 4.1.6. *Every normal subspace Y of X which is dense in X, is internally normal in X.*

The next example shows that internal normality does not coincide with relative normality.

Example 4.1.7 ([3])**.** There is a Tychonoff space X with a dense subspace Y such that Y is internally normal in X and not normal in X.

Let L be the set of all limit ordinals in ω_1 and S, T two disjoint subsets of L stationary in ω_1. Put

$$M = (\omega_1 + 1) \setminus S,$$

$$X' = \{(\alpha, \beta) : \beta \leq \alpha \leq \omega_1\},$$

$$X = X' \setminus \{(\omega_1, \omega_1)\} \text{ and } Y = (M \times M) \cap X$$

and let π be the projection from X' to the second coordinate. The topology on X, Y and X' is inherited from $(\omega_1 + 1) \times (\omega_1 + 1)$. It is easy to see that X' is compact, X is locally compact and Y is dense in X since S contains only limit ordinals in ω_1.

Put $A = \{(\alpha, \alpha) : \alpha \in T\}$ and $B = \{(\omega_1, \alpha) : \alpha \in T\}$. Obviously, A and B are subsets of Y with disjoint closures in X. We will show that A and B cannot be separated by disjoint open sets in X so Y is not normal in X.

Let U be an open neighborhood of A in X. For each $\alpha \in T$ fix some $\delta(\alpha) < \alpha$ such that $V_\alpha = (\delta(\alpha), \alpha]^2 \cap X$ is a subset of U. By Fodor's

Lemma (2.3.5) there exist $\beta < \omega_1$ and a stationary subset E of T such that $\delta(\alpha) = \beta$ for each $\alpha \in E$. This implies that $(\omega_1, \alpha) \in \overline{\bigcup\{V_\alpha : \alpha \in E\}} \subset \overline{U}$ for each $\alpha \in E$. Since E is a subset of T, $\overline{U}$ intersects B and A and B cannot be separated in X.

On the other hand, we will show that Y is internally normal in X. We will prove that each subset of Y, which is closed in X, is compact. This property will be defined in Chapter 5 as internal compactness and Lemma 5.2.3 then implies internal normality of Y in X.

Let P be a non-compact closed subset of X. The space X' is compact so P cannot be closed in X' and $P' = \overline{P}^{X'} = P \cup \{(\omega_1, \omega_1)\}$. Now $\pi[P']$ is a closed subset of $\omega_1 + 1$ since P' is compact, and $\omega_1 \in \pi[P']$. Thus $\pi[P] = \pi[P'] \setminus \{\omega_1\}$ is a closed unbounded set in ω_1 and $\pi[P]$ has nonempty intersection with the stationary set S. But this shows that P is not a subset of Y.

Proposition 4.1.2 states, that each space normal in some larger space is regular. Generally we can ask the question, whether a relative property implies any absolute property of the smaller space. Arhangel'skii stated this question in the following way [3, Question 10]: Let Y be a subspace of a regular space X such that Y is internally normal in X. Is then Y Tychonoff? We will give a negative answer to this question in Corollary 5.2.9.

For the definition of normality on, the following notion is used.

Definition 4.1.8. A subset A of X *is concentrated on* $Y \subset X$, if $A \subset \overline{A \cap Y}^X$.

Hence closed subsets of X concentrated on Y are closures of subsets of Y.

Definition 4.1.9. A space X is *normal on* its subspace Y, if for every two disjoint closed subsets A and B of X concentrated on Y there are disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

If X is normal, then X is normal on every subspace Y of X.

Proposition 4.1.10. *If X is normal on Y, then Y is normal in X.*

"Normality on" is stronger that "normality in". The following example shows that it is strictly stronger.

Example 4.1.11. There exists a countable dense subspace Y of the space $X = \mathbb{R}^{\mathfrak{c}}$ such that X is not normal on Y. But Y is Lindelöf and so Y is normal in X. For details see [1].

If Y is a non-normal subspace of a normal space X then X is normal on Y. On the other hand, Proposition 4.1.10 and 4.1.2 imply that normality on is sufficient for regularity of the smaller space. We will see that if a regular space X is normal on Y then Y has to be Tychonoff (Theorem 4.2.7). The prove this, the notion of κ-normality appears to be a useful tool.

4.2 On κ-normality

Definition 4.2.1. A space X is *densely normal* if there exists a dense subspace Y of X such that X is normal on Y.

Definition 4.2.2. A set A is a *regular closed* set if A is a closure of an open set.

Definition 4.2.3. A space X is *κ-normal* if every two disjoint regular closed sets in X can be separated by disjoint open neighborhoods.

The notion of κ-normality was introduced by E.V. Schepin in [17]. As we will see, κ-normality is an absolute property, but it has interesting relations with some versions of relative normality.

Lemma 4.2.4. *If Y is dense in X then each regular closed subset of X is concentrated on Y.*

Theorem 4.2.5. *Every densely normal space is κ-normal.*

Proof. This follows immediately from Lemma 4.2.4. □

The converse is not true; there exists a κ-normal Tychonoff space which is not densely normal. This space is constructed in [1].

An important property of κ-normal spaces was proved in [17]. It can be proved in a similar way to the Urysohn's Lemma.

Theorem 4.2.6 ([17])**.** *If X is a κ-normal space, then every two disjoint regular closed sets A and B in X are functionally separated (there exists a continuous function $f : X \to \mathbb{R}$ such that $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$).*

Theorem 4.2.6 is a natural tool for proving that certain spaces need to be Tychonoff.

Theorem 4.2.7. *If a regular space X is normal on Y, then the space Y is Tychonoff.*

Proof. Pick any nonempty closed subset A of Y and a point $b \in Y \setminus A$. Since the space $\overline{Y}$ is regular, we can take two disjoint regular closed sets G and H in $\overline{Y}$ such that $A \subset G$ and $b \in H$. Note that $\overline{Y}$ is normal on Y, thus densely normal and κ-normal. The result follows from Theorem 4.2.6. □

In the class of regular spaces normality on is sufficient for the smaller space being Tychonoff and internal normality is not sufficient. The situation for normality in is not known yet in ZFC and the general conjecture is that it is again not sufficient [2].

Another way to recognize that a space is normal on its subspace is given by Proposition 4.2.8.

Proposition 4.2.8. *If Y is a normal subspace of X and Y is C^0-embedded in X then X is normal on Y.*

Corollary 4.2.9. *If X has a dense normal subspace Y and Y is C^0-embedded in X, then the space X is densely normal (and hence κ-normal).*

Example 4.2.10. Let Y be any normal space and X any space such that

$$Y \subset X \subset \beta Y$$

Then X is normal on Y by Corollary 4.2.9.

Example 4.2.11. Let X be a non-normal topological space which is dense in itself and let A and B be two closed subsets of X each of which is dense in itself and which cannot be separated by open neighborhoods. We can get such a space by setting $X = X' \times \mathbb{R}$ for any non-normal space X'. Let $X^\star = X \times 2$ be the Alexandroff double of X (see 2.1.6). Now $A \times \{1\}$ and $B \times \{1\}$ are open sets and thus $A \times 2$ and $B \times 2$ are two disjoint regular closed sets in $X^\star$. These two sets cannot be separated by disjoint open sets in $X^\star$ since A and B cannot be separated in X. This shows that $X^\star$ is not κ-normal and not normal on $Y = X \times \{1\}$. The space Y is discrete, hence this example shows that no separation property (even discreteness) of the smaller space Y can be strong enough to guarantee that X has to be normal on Y.

One of the famous questions in General Topology was the existence of a normal space X whose product with the closed unit interval I is not normal. Such spaces X are called Dowker spaces, and Dowker and Katětov proved that X is a Dowker space if and only if X is normal and not countably

paracompact ([15], [8]). The existence of such space in ZFC was proved in [16]. Arhangel'skii gave two related questions.
[3, Question 7,8]: Is the product of a normal space X and a compact Hausdorff space (the closed interval I) always κ-normal?

Our next example gives a negative answer to these questions. It is a modification of any Dowker space.

Example 4.2.12. Let Y be any Dowker space. Put $X' = (\omega+1) \times Y$ and refine the product topology by declaring all points in $\omega \times Y$ to be isolated. The resulting space will be denoted X. As a subspace, the top level $\{\omega\} \times Y$ is isomorphic to Y and will be denoted Y'.

The space X is normal. Indeed, let A and B be two disjoint closed subsets of X. Then $A \cap Y'$ and $B \cap Y'$ are two disjoint closed subsets of Y' and there exist disjoint open subsets U and V of Y' separating $A \cap Y'$ and $B \cap Y'$, since Y' is normal. The sets $(A \setminus Y') \cup ((\omega+1) \times U) \setminus B$ and $(B \setminus Y') \cup ((\omega+1) \times V) \setminus A$ are disjoint open sets in X separating A and B.

The construction of canonically closed subsets of $X \times I$ is analogous to the classical one (see, e.g., [9, Chapter 5.2]). Since Y' is not countably paracompact, there exists a sequence $\{F_n : n \in \omega\}$ of closed subsets of Y such that $F_{n+1} \subset F_n$, $\bigcap\{F_n : n \in \omega\} = \emptyset$ and for each sequence $\{G_n : n \in \omega\}$ of open sets in Y, such that $F_n \subset G_n$, $\bigcap\{G_n : n \in \omega\}$ is nonempty.

For each $n \in \omega$, put

$$B_n = (\omega \setminus n) \times F_n \times \left(\frac{1}{2(n+1)}, \min\left\{ \frac{3}{2(n+1)}, 1 \right\} \right)$$

and

$$S_n = n \times Y \times \left[0, \frac{1}{2(n+2)} \right).$$

Note that B_n and S_n are open subsets of $X \times I$ and $B_n \cap S_m = \emptyset$ for each $n, m \in \omega$.

We will define regular closed subsets of $X \times I$:

$$F = \overline{\bigcup\{B_n : n \in \omega\}}$$

and

$$E = \overline{\bigcup\{S_n : n \in \omega\}}.$$

To prove that E and F are disjoint it is only necessary to show that $(Y' \times \{0\}) \cap F = \emptyset$. Pick any $x \in Y'$, fix $n \in \omega$ such that $x \notin \{\omega\} \times F_n$ and let O be an open neighborhood of $(x, 0)$, where

$$O = (\omega + 1) \times (Y \setminus F_n) \times \left[0, \frac{1}{2(n+1)}\right).$$

We will show that O is disjoint from B_m for each $m \in \omega$ and thus disjoint from F. If $m \leq n$, then

$$O \subset (\omega + 1) \times Y \times \left[0, \frac{1}{2(n+1)}\right)$$

and

$$B_m \subset (\omega + 1) \times Y \times \left(\frac{1}{2(n+1)}, 1\right]$$

so O and B_m are disjoint. If $n < m$, then $F_m \subset F_n$ so $B_m \subset (\omega+1) \times F_n \times I$ and this set is disjoint from O.

Now it is clear that

$$E = (Y' \times \{0\}) \cup \bigcup\{\overline{S_n} : n \in \omega\}$$

and

$$F = \bigcup\{\overline{B_n} : n \in \omega\}$$

where

$$\overline{B_n} = ((\omega + 1) \setminus n) \times F_n \times \left[\frac{1}{2(n+1)}, \min\left\{\frac{3}{2(n+1)}, 1\right\}\right].$$

The sets E and F cannot be separated by disjoint open neighborhoods. If $F \subset U$ and U is open then $\{\omega\} \times F_n \times \{1/(n+1)\} \subset U$ for each n and thus $\{G_n : n \in \omega\}$, where $G_n = \pi_Y[U \cap (Y' \times \{1/(n+1)\})]$, is a sequence of open sets in Y such that $F_n \subset G_n$ (π_Y is the projection from $\{\omega\} \times Y \times I$ onto Y). This implies that there exists some $x \in \bigcap\{G_n : n \in \omega\}$. For this x we have $(\omega, x, 0) \in \overline{U} \cap E$ and therefore E and F cannot be separated. This shows that $X \times I$ is not κ-normal.

4.3 Bubble Spaces

Let us now recall the definition of the Niemytzki plane $\mathbf{N}$ (also known as the bubble space) and establish some notation. Let $\mathbf{L} = \{(t, 0) : t \in \mathbb{R}\}$,

$\mathbf{E} = \{(r,s) : r \in \mathbb{R}, s \in \mathbb{R}^+\}$, $\mathbf{N} = \mathbf{L} \cup \mathbf{E}$. For $x = (r,s) \in \mathbf{E}$ and $0 < \varepsilon < s$ let

$$B_\varepsilon(x) = \{(r_1,s_1) \in \mathbf{E} : (r_1 - r)^2 + (s_1 - s)^2 < \varepsilon^2\}$$

and for $x = (t,0) \in \mathbf{L}$ and $\varepsilon \in \mathbb{R}^+$ let

$$B_\varepsilon(x) = B_\varepsilon(t,\varepsilon) \cup \{x\}.$$

The Niemytzki plane is the set $\mathbf{N}$ with the topology generated by the sets $B_\varepsilon(x)$ for $x \in \mathbf{N}$ and $\varepsilon \in \mathbb{R}^+$. On the set $\mathbf{L}$ we will also use the Euclidean topology of the real line denoted by $\mathcal{R}$.

The next lemma formulates a well known property of the Niemytzki plane. It is an application of the Baire Category Theorem.

Lemma 4.3.1. *The sets $Q = \{(t,0) : t \in \mathbb{Q}\}$ and $P = \{(t,0) : t \in \mathbb{P}\}$ are closed subsets of $\mathbf{N}$ which cannot be separated by disjoint open neighborhoods in $\mathbf{N}$.*

Since each countable space is normal *in* every larger regular space (Theorem 4.1.3) it is natural to study normality of a topological space X *on* its countable subspaces. This topic is investigated in the article of Tkachenko, Tkachuk, Wilson and Yaschenko [18]. A special case when X is the Niemytzki plane was mainly considered there.

Example 4.3.2 ([18])**.** In this example a countable dense subset C of $\mathbf{N}$, such that $\mathbf{N}$ is not normal on C, was constructed. Let

$$A = \{(x,y) \in \mathbf{E} : x, y \in \mathbb{Q}\} \text{ and } Q = \{(x,0) : x \in \mathbb{Q}\}.$$

Then $\mathbf{N}$ is not normal on $C = A \cup Q$. Details can be found in the original article.

Example 4.3.3 ([18])**.** There is a separable Tychonoff space which is not normal on any countable dense subspace. This space is constructed by a modification of the Niemytzki plane. It is again a kind of a "bubble" space but this space is not first countable.

In the light of the previous examples, the authors of [18] raised the following problem ([18, Problem 3.4]): Is it true that the Niemytzki plane is not normal on any of its countable dense subspaces?

We answer this question in the negative by describing certain type of countable dense subspaces of $\mathbf{N}$ on which $\mathbf{N}$ is normal.

Theorem 4.3.4. *Let G, H be disjoint closed subsets of $\mathbf{N}$. Then G and H can be separated by disjoint open sets if and only if there exist sets G_i and H_i for $i \in \mathbb{N}$ such that $G \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} G_i$, $H \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} H_i$ and*

$$\overline{G}_i^{\mathcal{R}} \cap H = \emptyset = \overline{H}_i^{\mathcal{R}} \cap G$$

for every $i \in \mathbb{N}$.

We will use the following technical Lemma in the proof of Theorem 4.3.4.

Lemma 4.3.5. *For each $x \in \mathbf{E}$ there exists some $\iota \in \mathbb{R}^+$ such that $x \notin B_\varepsilon(y)$ implies $B_{\varepsilon/2}(y) \cap B_\iota(x) = \emptyset$ for each $y \in \mathbf{L}$ and each $\varepsilon \in \mathbb{R}^+$, $\varepsilon \leq 1$.*

Proof of Lemma 4.3.5. Without loss of generality we may assume $x = (0, a)$. Take any ι such that $\iota + \iota^2 \leq a^2/2$ and $\iota \leq a/2$. We will prove that this ι works. Let $y = (b, 0) \in \mathbf{L}$ and $\varepsilon \in \mathbb{R}^+$, $\varepsilon \leq 1$, be such that $x \notin B_\varepsilon(y)$ (and thus $\varepsilon^2 \leq b^2 + (a-\varepsilon)^2$). We have to prove that $B_{\varepsilon/2}(y) \cap B_\iota(x) = \emptyset$. This fact can be reformulated as $(\iota + \varepsilon/2)^2 \leq b^2 + (a - \varepsilon/2)^2$.

Case 1: $a/2 \leq \varepsilon \leq 1$

$$\begin{aligned}(\iota + \varepsilon/2)^2 = \varepsilon^2/4 + \varepsilon\iota + \iota^2 \leq \varepsilon^2/4 + \iota + \iota^2 &\leq a^2/2 + \varepsilon^2/4 \\ &\leq a\varepsilon + \varepsilon^2/4\end{aligned}$$

and here we can use $0 \leq b^2 + (a-\varepsilon)^2 - \varepsilon^2$:

$$a\varepsilon + \varepsilon^2/4 \leq b^2 + (a-\varepsilon)^2 + a\varepsilon + \varepsilon^2/4 - \varepsilon^2 = b^2 + (a - \varepsilon/2)^2.$$

Case 2: $0 < \varepsilon < a/2$

$$(\iota + \varepsilon/2)^2 = \iota^2 + \varepsilon\iota + \varepsilon^2/4 \leq \iota^2 + \iota + \varepsilon^2/4 \leq a^2/2 + \varepsilon^2/4,$$

now use $0 \leq a(a/2 - \varepsilon) = a^2/2 - a\varepsilon$:

$$a^2/2 + \varepsilon^2/4 \leq a^2 - a\varepsilon + \varepsilon^2/4 \leq b^2 + (a - \varepsilon/2)^2.$$

□

Proof of Theorem 4.3.4. We will denote $G' = G \cap \mathbf{L}$, $H' = H \cap \mathbf{L}$.

First, let us show that if the condition is not fulfilled, then the sets G and H cannot be separated. Suppose U and V are open sets, such that $G \subset U$ and $H \subset V$. To each $x \in G'$ ($x \in H'$) assign $\varepsilon(x) \in \mathbb{R}^+$, for which $B_{\varepsilon(x)}(x) \subset U$ ($B_{\varepsilon(x)}(x) \subset V$, respectively). Now if $G_i = \{x \in G' : \varepsilon(x) > \frac{1}{i}\}$

and $H_i = \{x \in H' : \varepsilon(x) > \frac{1}{i}\}$ for $i \in \mathbb{N}$, then without lost of generality $(\exists j \in \mathbb{N})(\exists h \in \overline{G}_j^{\mathcal{R}})(h \in H')$. Otherwise G_i, H_i satisfy the given condition. This implies for such j and h,

$$\emptyset \neq \bigcup_{y \in G_j} B_{\varepsilon(y)}(y) \cap B_{\varepsilon(h)}(h) \subset U \cap V$$

and U and V are not disjoint.

Now let us fix sets G and H, which satisfy the condition given in the theorem, and construct the disjoint sets U and V. In the first (and crucial) step we will separate G' and H'. For $x = (t, 0) \in \mathbf{L}$ let $P_\varepsilon(x)$ be "the area between a horizontal line and a parabola":

$$P_\varepsilon(x) = \{(r, s) \in \mathbf{E} : \varepsilon > s > (t - r)^2\} \cup \{x\}.$$

Now for $x \in G_1$ fix any $\varepsilon(x) \in (0, 1)$. For each $x = (t, 0) \in H_1$ fix an $\varepsilon(x) \in (0, 1)$ such that $\{(t', 0) \in \mathbf{L} : |t' - t| < 2\sqrt{\varepsilon(x)}\} \cap G_1 = \emptyset$. That is possible since $\overline{G}_1^{\mathcal{R}} \cap H_1 = \emptyset$. Thus

$$P_{\varepsilon(x)}(x) \cap \bigcup_{y \in G_1} P_{\varepsilon(y)}(y) = \emptyset$$

for every $x \in H_1$.

Further, we may assume that the sets G_i (H_i, respectively) are pairwise disjoint and we will continue inductively: to $x \in G_n$ (H_n, respectively) we assign $\varepsilon(x)$ in the same way: for $x = (t, 0) \in G_n$ let $\varepsilon(x) \in (0, 1)$ be such that

$$\{(t', 0) \in \mathbf{L} : |t - t'| < 2\sqrt{\varepsilon(x)}\} \cap \bigcup_{i<n} H_i = \emptyset.$$

Such $\varepsilon(x)$ exists since $\overline{\bigcup_{i<n} H_i}^{\mathcal{R}} \cap G_n = \emptyset$. For x and $\varepsilon(x)$ chosen in this way

$$P_{\varepsilon(x)}(x) \cap \bigcup_{i<n} \bigcup_{y \in H_i} P_{\varepsilon(y)}(y) = \emptyset.$$

For $x \in H_n$ the construction (and also the resulting property) is similar. From the construction it follows that

$$\bigcup_{y \in G'} P_{\varepsilon(y)}(y) \cap \bigcup_{y \in H'} P_{\varepsilon(y)}(y) = \emptyset.$$

Since $B_{\varepsilon/2}(x) \subset P_\varepsilon(x)$ for $x \in \mathbf{L}$ and $\varepsilon \in (0,1)$,

$$U_1 = \bigcup_{x \in G'} B_{\varepsilon(x)/2}(x)$$

and

$$V_1 = \bigcup_{x \in H'} B_{\varepsilon(x)/2}(x)$$

are disjoint open sets in $\mathbf{N}$ and $G' \subset U_1$, $H' \subset V_1$.

In the second step we will separate G' from H: for each $x \in G'$ fix $\delta'(x) \in (0,1)$ such that $B_{\delta'(x)}(x) \cap H = \emptyset$. For $x \in G'$ let

$$\delta(x) = \min\{\delta'(x)/2, \varepsilon(x)/2\}.$$

The set

$$U_2 = \bigcup_{x \in G'} B_{\delta(x)}(x)$$

is open and contains G'. We will prove that $\overline{U}_2 \cap H = \emptyset$. Let us show that $h \in H \Rightarrow h \notin \overline{U}_2$.

If $h \in H'$, then $U_1 \cap V_1 = \emptyset$ and $U_2 \subset U_1$, V_1 is open and $H' \subset V_1$. Thus $h \notin \overline{U}_2$. If $h \in H \cap \mathbf{E}$, then $h \notin B_{\delta'(x)}(x)$ for each $x \in G'$. From this and Lemma 4.3.5 it follows that there exists $\iota \in \mathbb{R}$ such that $B_\iota(h) \cap B_{\delta(x)}(x) = \emptyset$ for all $x \in G'$, so $B_\iota(h) \cap U_2 = \emptyset$ and $h \notin \overline{U}_2$. Similarly we can construct an open set V_2 such that $H' \subset V_2$, $\overline{V}_2 \cap G = \emptyset$ and $V_2 \subset V_1$, which implies $U_2 \cap V_2 = \emptyset$.

Finally, let us separate the whole sets. $\mathbf{E}$ is an open normal subspace of $\mathbf{N}$, $G \cap \mathbf{E}$ and $H \cap \mathbf{E}$ are disjoint closed subsets of $\mathbf{E}$, so there exist disjoint open subsets U_3,V_3 of $\mathbf{E}$ (and thus open in $\mathbf{N}$) such that $G \cap \mathbf{E} \subset U_3$, $H \cap \mathbf{E} \subset V_3$. Hence $U = (U_2 \cup U_3) \setminus \overline{V}_2$ and $V = (V_2 \cup V_3) \setminus \overline{U}_2$ are the desired disjoint open sets separating G and H. □

Lemma 4.3.6. $\mathbf{N}$ *is normal on* $\mathbf{E}$*.*

Proof. Consider G, H subsets of $\mathbf{E}$, $\overline{G} \cap \overline{H} = \emptyset$. We will show, that $\overline{G}$ and $\overline{H}$ fulfill the condition of Theorem 4.3.4 and thus they can be separated. Put

$$G_i = \{x \in \overline{G} \cap \mathbf{L} : B_{1/i}(x) \cap \overline{H} = \emptyset\}$$

and

$$H_i = \{x \in \overline{H} \cap \mathbf{L} : B_{1/i}(x) \cap \overline{G} = \emptyset\}$$

for $i \in \mathbb{N}$. It is obvious that $\overline{G} \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} G_i$ and $\overline{H} \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} H_i$, so it remains to show that $\overline{G_i}^{\mathcal{R}} \cap \overline{H} = \emptyset$ ($\overline{H_i}^{\mathcal{R}} \cap \overline{G} = \emptyset$, respectively).

For contradiction assume that there is some $n \in \mathbb{N}$ and $h \in \overline{G_n}^{\mathcal{R}}$ such that $h \in \overline{H}$. Since $h \in \overline{H}$, there exists $h' \in H \cap B_{1/n}(h)$. Now $h \in \overline{G_n}^{\mathcal{R}}$,

$$B_{1/n}(h) \subset \bigcup_{x\in G_n} B_{1/n}(x)$$

and this implies that $h' \in B_{1/n}(g)$ for some $g \in G_n$. This is a contradiction. The case $(\exists n \in \mathbb{N})(\exists g \in \overline{H_n}^{\mathcal{R}})(h \in \overline{G})$ is similar. □

Corollary 4.3.7. **N** *is normal on each subset of* **E**. □

So each countable dense subset of **E** (and such clearly exists) gives us an example of a countable dense subspace of **N** on which **N** is normal.

Chapter 5

Relative Compactness

5.1 Relative Compactness

Definition 5.1.1. A topological space Y is *compact in* its superspace X, if every open cover of X has a finite subsystem which covers Y.

Observe that if Y is compact in X and Z is any subset of Y, then Z is compact in X. It is also easy to see that Y is compact in X if and only if Y is compact in $\overline{Y}^X$. These two facts together give the following Proposition.

Proposition 5.1.2. *If Y is compact in X and $Z \subset Y$ is closed in X, then Z is compact.*

It is also easy to prove that relative compactness can be characterized in a similar way to absolute compactness.

Lemma 5.1.3. *The space Y is compact in X if and only if for each centered family $\mathcal{C}$ of subsets of Y the intersection $\bigcap\{\overline{P}^X : P \in \mathcal{C}\}$ is nonempty.*

In the case of regular spaces relative compactness appears to be quite close to compactness. In particular a space compact in a larger regular space has to be Tychonoff.

Theorem 5.1.4. *If X is a regular space then Y is compact in X if and only if $\overline{Y}^X$ is compact.*

Proof. If $\overline{Y}^X$ is compact then Y is compact in $\overline{Y}^X$ and thus compact in X.

Let Y be compact and dense in a regular space $X = \overline{Y}$. We will prove that X is compact. Let $\mathcal{U}$ be an open cover of X. Since X is regular, there exists an open cover $\mathcal{O}$ of X such that for each $O \in \mathcal{O}$ there is some $U \in \mathcal{U}$ such that $\overline{O} \subset U$. The subspace Y is compact in X so there is a finite $\mathcal{O}' \subset \mathcal{O}$ which covers Y. Thus there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that

$$\bigcup\{\overline{O} : O \in \mathcal{O}'\} \subset \bigcup \mathcal{U}'.$$

Now

$$X = \overline{Y} \subset \overline{\bigcup \mathcal{O}'} = \bigcup\{\overline{O} : O \in \mathcal{O}'\} \subset \bigcup \mathcal{U}'$$

and X is compact. □

Proposition 5.1.5. *If X is a Hausdorff space, and Y is compact in X and dense in X, then X is an H-closed space.*

Proof. Let $\mathcal{C}$ be a centered family of open subsets of X. The family $\{C \cap Y : C \in \mathcal{C}\}$ is a centered family of subsets of Y since Y is dense in X. Lemma 5.1.3 and Theorem 2.2.2 now imply that X is H-closed. □

A folklore argument shows that compact spaces are normal. A similar argument can be used to prove the following theorem.

Theorem 5.1.6. *If X is Hausdorff and Y is compact in X, then Y is normal in X.*

Corollary 5.1.7. *If X is Hausdorff and Y is compact in X, then Y is a regular space.*

Proof. This follows from Theorem 5.1.6 and Proposition 4.1.2. □

This leads to the question whether a space compact in a larger Hausdorff space also needs to be Tychonoff. Example 5.1.11 shows that this is not the case.

Definition 5.1.8. A topological space Y is *potentially compact* if there is a Hausdorff space X such that Y is compact in X.

Proposition 5.1.9. *Every potentially compact space is regular and every Tychonoff space is potentially compact.*

Proof. If Y is a Tychonoff space then Y is compact in βY. The rest of the Proposition is Corollary 5.1.7. □

We will see that potential compactness is a new absolute separation property between regularity and $T_{3\frac{1}{2}}$. An inner characterization of this property is still unknown.

Theorem 5.1.10. *If a Hausdorff space A is a preimage of a potentially compact space under a perfect mapping, then A is a potentially compact space.*

Proof. Let Y be compact in X and let $f : A \to Y$ be a perfect mapping onto Y. We need to construct a Hausdorff space Z in which A is compact. Put $S = X \setminus Y$ and $Z = A \cup S$. A base $\mathscr{B}$ of the topology on Z will be defined as follows

$$\mathscr{B} = \{U : U \text{ open in } A\} \cup \{O(s, U) : U \text{ open in } X, s \in S \cap U\}$$

where $O(s, U) = \{s\} \cup f^{-1}[U \cap Y]$.

Claim 1. *Z is a Hausdorff space.* □

Claim 2. *For $E \subset A$ and $s \in S$; $s \in \overline{E}^Z$ if and only if $s \in \overline{f[E]}^X$.* □

Let $\mathcal{C}$ be a centered family of closed subsets of A which is closed under finite intersections. Put $\mathcal{C}_X = \{f[C] : C \in \mathcal{C}\}$ and $\overline{\mathcal{C}}_X = \{\overline{f[C]}^X : C \in \mathcal{C}\}$. The system $\mathcal{C}_X$ is a centered family of closed subsets of Y, thus the set $M = \bigcap \overline{\mathcal{C}}_X$ is nonempty. If there is some $c \in S \cap M$ then Claim 2 implies $\bigcap \mathcal{C} \neq \emptyset$.

Assume $M \subset Y$ and pick some $c \in M$. Then $B = f^{-1}[\{c\}]$ is a compact subset of A and $\mathcal{C} \restriction B = \{C \cap B : C \in \mathcal{C}\}$ is a centered family of closed subsets of B which is closed under finite intersections and which does not contain the empty set (because $c \in \overline{f[C]}^X \cap Y = f[C]$ for each $C \in \mathcal{C}$). Hence $\emptyset \neq \bigcap \mathcal{C} \restriction B \subset \bigcap \mathcal{C}$ and A is compact in Z. □

The next example was constructed in [11] and it shows that being potentially compact is strictly weaker than being Tychonoff.

Example 5.1.11. Let $f : X \to Y$ be a perfect mapping of a non-Tychonoff space X onto a Tychonoff space Y. Such an example of f, X and Y was constructed in [7]. Theorem 5.1.10 now implies that X is a non-Tychonoff potentially compact space.

There also exists an infinite potentially compact space on which every continuous real valued function is constant. Such a space was constructed in [6].

And now it only remains to show that being potentially compact is a stronger property than being regular.

Proposition 5.1.12. *Let Y be an R-closed space. Then Y is potentially compact if and only if Y is compact.*

Proof. Let Y be a R-closed space compact in some space X. Choose any $x \in X \setminus Y$ and put $Y' = Y \cup \{x\}$. The space Y' is also compact in X and so Y' is regular (Corollary 5.1.7) and thus Y is closed in Y'. That means that for each $x \in X \setminus Y$ is $x \notin \overline{Y}^X$, i. e. Y is closed in X. Proposition 5.1.2 now implies that Y is compact. □

The last proposition offers a way to recognize that certain space is not potentially compact. Construction of a regular not potentially compact space which uses this fact is given in example 5.1.13.

Example 5.1.13. The Jones space (the space obtained by the Jones machine) over $(\omega_1+1)\times(\omega_1+1)\setminus\{(\omega_1,\omega_1)\}$ is a non-Tychonoff regular R-closed space, so Proposition 5.1.12 implies that it is an example of a regular non-potentially compact space. For details see [6].

5.2 Internal Compactness

We will start this section with a well known characterization of compact spaces.

Theorem 5.2.1 ([14])**.** *A Hausdorff space X is compact if and only if all closed subsets of X are H-closed.*

In this theorem H-closed can be obviously replaced by compact. This gives a motivation for another definition of relative compactness.

Definition 5.2.2. A topological space Y is *internally compact in X* if every subspace of Y which is closed in X, is compact.

This version of relative compactness is clearly weaker than the previous one. The next theorem is a usual type of "compactness implies normality" fact and the proof is standard.

Theorem 5.2.3. *If the space Y is internally compact in a Hausdorff space X, then Y is internally normal in X.*

Internal compactness has an equivalent definition similar to lemma 5.1.3.

Lemma 5.2.4. *The space Y is internally compact in X if and only if for each centered family $\mathcal{C}$ of subsets of Y which are closed in X the intersection $\bigcap \mathcal{C}$ is nonempty.*

Theorem 5.2.5 states that the Jones machine introduced in Example 2.1.7 preserves internal compactness in the following sense. Let Y be a non-normal subspace of a regular space X. Suppose, moreover, that A_0 and A_1 are two disjoint closed subsets of Y which cannot be separated by disjoint open neighborhoods in Y and such that $\overline{A_0}^X \cap \overline{A_1}^X = \emptyset$. In this situation $J(Y)$ can be considered as a subspace of $J(X)$ in a natural way; the new point (in Example 2.1.7 denoted by z) is the same for both $J(Y)$ and $J(X)$. The two sets whose points are being identified are A_0 and A_1 for $J(Y)$ and $\overline{A_0}^X$ and $\overline{A_1}^X$ for $J(X)$.

Theorem 5.2.5. *Let Y be a non-normal subspace of a topological space X and suppose that the sets A_0, A_1 are as in the previous paragraph. Then if Y is internally compact in X, then $J(Y)$ is internally compact in $J(X)$.*

Proof. We will use the notation established in Example 2.1.7. Pick any centered system $\mathcal{C}$ of subsets of $J(Y)$ such that all sets in $\mathcal{C}$ are closed in $J(X)$. We have to prove that the intersection $\bigcap \mathcal{C}$ is nonempty.

Assume that $z \notin Z$ for some $Z \in \mathcal{C}$. Otherwise we are done. Then $q^{-1}[Z] \subset Y \times n$ for some $n \in \omega$. Since $q^{-1}[Z] \cap (X \times \{j\})$ is a subset of j-th copy of Y and it is closed in j-th copy of X for each $j \in n$ and since Y is internally compact in X, the set $q^{-1}[Z]$ is a finite sum of compact sets and thus compact. Hence $\emptyset \neq \bigcap\{q^{-1}[C] : C \in \mathcal{C}\} = q^{-1}[\bigcap \mathcal{C}]$ and $J(Y)$ is internally compact in $J(X)$. □

Closely related to Theorem 5.1.4, Theorem 5.1.6 and Corollary 5.1.7, is the question, whether a certain version of relative compactness does imply any absolute separation axiom for the smaller space Y. Arhangel'skii formulated one version of this problem in [3, Question 9] : Let Y be a subspace of a Hausdorff space X such that Y is internally compact in X. Is then true that Y is Tychonoff? What if we assume X to be regular?

A closely related Question 10 was also given in article [3]: Let Y be a subspace of a regular space X such that Y is internally normal in X. Is

Y Tychonoff? We will construct examples that provide negative answers to these questions.

Example 5.2.7 was constructed by Eva Murtinová and gives a negative answer to the first part of Question 9 from [3]. The construction uses objects described in the following lemma.

Lemma 5.2.6. *For each ultrafilter $\mathcal{U}$ on ω there exists a MAD system $\mathcal{A}$ on ω such that $\mathcal{A} \cap \mathcal{U} = \emptyset$.*

Proof. Fix any ultrafilter $\mathcal{U}$ and consider the system of all AD systems on ω satisfying the condition given in the Lemma, ordered by inclusion. This system is nonempty since the empty set is such an AD system. Since this system is closed under the union of increasing chains, Zorn's Lemma implies that there is a maximal such AD system $\mathcal{A}$. We will show that $\mathcal{A}$ is a MAD system. If not, there is an infinite set $A \in \mathcal{P}(\omega) \setminus \mathcal{A}$ such that $\mathcal{A} \cup \{A\}$ is an AD system. Split A into two infinite. Since $\mathcal{U}$ is an ultrafilter, at least one of the sets A_0 and A_1 does not belong to $\mathcal{U}$. Denote this set by A_i. Now $\mathcal{A} \cup \{A_i\}$ is an AD system contradicting maximality of $\mathcal{A}$. □

Example 5.2.7. We will describe a non-regular space internally compact in a Hausdorff space. The idea is to construct a space $X = Y \cup Z$ with Y non-regular such that all "nontrivial" infinite subsets of Y have cluster points in Z. Then there are only few closed subsets of X contained in Y and these are arranged to be compact.

Fix a free ultrafilter $\mathcal{U}$ on ω and let $\mathcal{A}$ be a MAD system on ω given by Lemma 5.2.6. Put $Y = \{y\} \cup ((\omega+1) \times \omega)$, $F = \{\omega\} \times \omega \subset Y$. Let us endow the set $X = Y \cup \mathcal{A}$ with a topology by declaring each point of $\omega \times \omega$ isolated,

$$\{((\omega+1) \setminus n_0) \times \{n\} : n_0 \in \omega\}$$

an open base in $(\omega, n) \in F$,

$$\{\{y\} \cup (\omega \times U) : U \in \mathcal{U}\}$$

an open base at y and

$$\{\{A\} \cup ((\omega+1) \times (A \setminus n_0)) : n_0 \in \omega\}$$

an open base in $A \in \mathcal{A}$. This obviously defines a Hausdorff topology on X, while the closed subset F of Y cannot be separated from y, hence Y is not regular.

It remains to show that Y is internally compact in X. Consider a closed subset C of X, $C \subset Y$ and an infinite $B \subset C$ whose cluster point is to be found in C. Since C is closed, the set

$$\{n \in A : C \cap ((\omega + 1) \times \{n\}) \neq \emptyset\}$$

is finite for every $A \in \mathcal{A}$. Thus

$$N = \{n \in \omega : C \cap ((\omega + 1) \times \{n\}) \neq \emptyset\}$$

is almost disjoint from $\mathcal{A}$. It follows that N is finite. As B is infinite, there is an n_0 such that $B \cap (\omega \times \{n_0\})$ is infinite. Now (ω, n_0) is a cluster point of B.

And a corollary of the following theorem provides an answer to the second part of Question 9. From Proposition 5.2.3 we now also get that there exists a non-Tychonoff space Y which is internally normal in a larger space X and that gives a negative answer to Question 10.

Theorem 5.2.8. *There exists a non-normal space Y which is internally compact in a zero-dimensional space X.*

Proof. Throughout this proof, all points in the Čech-Stone compactification βD of a discrete space D will be identified with ultrafilters on D. For any discrete space D let us also define a subspace γD of βD as

$$\gamma D = \{p \in \beta D : (\exists P \in p)\, |P| \leq \omega\}.$$

Let A and B be two disjoint sets of size ω_2, put $C = A \times B$ and π_A, π_B will denote the natural projections of C onto A and B. The underlying sets for X and Y are

$$Y = A \cup B \cup C$$

and

$$X = \gamma A \cup \gamma B \cup \gamma C$$

and the topology is defined as follows: γC is an open subspace of X, other basic open sets of X are

$$O \cup \overline{\pi_A^{-1}[O \cap A] \setminus K}^{\gamma C}$$

for $|K| \leq \omega$, O open subset of γA and

$$O \cup \overline{\pi_B^{-1}[O \cap B] \setminus K}^{\gamma C}$$

for $|K| \leq \omega$, O open subset of γB. It is a routine to check, that we have defined a base for a topology on X correctly.

Claim 1. *X is a Hausdorff space.*

Proof. We need to show that each two distinct points a and b in X can be separated by disjoint open neighborhoods. If $a, b \in \gamma C$, then $\gamma C \subset \beta C$ implies that these two points can be separated. If $a, b \in \gamma A$, then there are disjoint open sets U and V separating a and b in γA thus

$$U \cup \overline{\pi_A^{-1}[U \cap A]}^{\gamma C}$$

and

$$V \cup \overline{\pi_A^{-1}[V \cap A]}^{\gamma C}$$

separate a and b in X. Case $a, b \in \gamma B$ is similar. If $a \in \gamma A$ and $b \in \gamma B$, then fix countable sets $U \subset A$ and $V \subset B$ such that $a \in \overline{U}^{\gamma A}$ and $b \in \overline{V}^{\gamma B}$. The sets

$$U \cup \overline{\pi_A^{-1}[U] \setminus (U \times V)}^{\gamma C}$$

and

$$V \cup \overline{\pi_B^{-1}[V] \setminus (U \times V)}^{\gamma C}$$

separate a and b in X. And if $a \in \gamma A$, $b \in \gamma C$, then fix countable sets $U \subset A$ and $V \subset C$ such that $a \in \overline{U}^{\gamma A}$ and $b \in \overline{V}^{\gamma C}$. The sets

$$U \cup \overline{\pi_A^{-1}[U] \setminus V}^{\gamma C} \text{ and } \overline{V}^{\gamma C}$$

separate a and b in X. □

Claim 2. *X is a zero-dimensional space.*

Proof. For each $x \in \gamma C$ there is an open base at x which consists of the sets of the form γK where $K \subset C$ is such that $|K| \leq \omega$, and for such K is $\gamma K = \overline{K}^X$. For $x \in \gamma A$ there is an open base at x which consists of the sets of the form

$$B = \gamma O \cup \overline{\pi_A^{-1}[O \cap A] \setminus K}^{\gamma C}$$

where $K \subset C$, $|K| \leq \omega$ and $O \subset A$ is such that $|O| \leq \omega$. For such O and K is B closed in X. The case $x \in \gamma B$ is similar. □

Claim 3. *A and B are closed subsets of Y which cannot be separated by disjoint open sets in Y. Moreover, $\overline{A}^X \cap \overline{B}^X = \emptyset$.*

Proof. Let U be open in Y and let $A' \subset U \cap A$ be some set of size ω_1. We will show that $\overline{U} \cap B$ is nonempty. For each $a \in A'$ fix a $K_a \in [C]^\omega$ such that

$$\pi_A^{-1}[\{a\}] \setminus K_a \subset U.$$

Hence

$$\pi_A^{-1}[A'] \setminus K \subset U$$

where

$$K = \bigcup\{K_a : a \in A'\}$$

and notice that $|K| \leq \omega_1$. Each

$$b \in B \setminus \pi_B[K]$$

(and such clearly exists) is an element of $\overline{U}$ because

$$\pi_B^{-1}[\{b\}] \cap U \supset A' \times \{b\}$$

and the product $A' \times \{b\}$ has cardinality ω_1.

$\overline{A}^X \cap \overline{B}^X = \emptyset$ is a consequence of $\overline{A}^X = \gamma A$ and $\overline{B}^X = \gamma B$. □

Claim 4. *If $G \subset Y$ is closed in X then $|G| < \omega$.*

Proof. Suppose $G \subset Y$, $\omega \leq |G|$. Then at least one of the sets $G \cap A$, $G \cap B$ and $G \cap C$ must be infinite. Assume that $\omega \leq |G \cap C|$. Then $\emptyset \neq \overline{G \cap C}^{\gamma C} \setminus (G \cap C) \subset \overline{G} \setminus Y$. Thus G is not closed. Cases $\omega \leq |G \cap A|$ and $\omega \leq |G \cap B|$ work similarly. □

The last claim implies that Y is internally compact in X and the Theorem is proved. □

Corollary 5.2.9. *There exists a non-Tychonoff space Y which is internally compact in a regular T_1 space X.*

Proof. Use Theorem 5.2.8 and Theorem 5.2.5. □

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