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Arrow-Debreu Model of General Equilibrium

Bachelor thesis

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Author: Filip Juřena
Supervisor: RNDr. Michal Červinka, Ph.D.
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Abstract

In this thesis, we deal with the Arrow-Debreu model of general equilibrium, which is an integrated model of production, exchange and consumption.

At the beginning, we present and discuss the original assumptions of the Arrow-Debreu model, i.e. the assumptions introduced by Kenneth J. Arrow and Gerard Debreu in 1954. Under these assumptions, Arrow and Debreu proved the existence of a general equilibrium.

As a part of the proof, Arrow and Debreu showed that the equilibria of their model are the same as the equilibria of an abstract economy, or a generalized Nash equilibrium problem (GNEP). We describe the GNEP and look at whether there is a connection which allows to apply results developed by researchers from other disciplines to the Arrow-Debreu model.

A part of the thesis is dedicated to a two-factor, two-commodity, two-consumer model, which is based on the original assumptions of Arrow and Debreu. In order to find the solution, we use a method called applied general equilibrium modelling and a software called GAMS. We examine the impact of better technology and taxes on consumers and producers.

We have brief remarks on applications of the model at the end.

Abstrakt

V této práci se zabýváme Arrowovým-Debreuovým modelem všeobecné rovnováhy, což je model integrující výrobu, směnu a spotřebu.

Na začátku uvádíme a diskutujeme původní předpoklady Arrowova-Debreuova modelu, tj. předpoklady představené Kennethem J. Arrowem a Gerardem Debreuem roku 1954. Za těchto předpokladů dokázali Arrow a Debreu existenci všeobecné rovnováhy.

V jedné části tohoto důkazu Arrow a Debreu ukázali, že rovnováhy v jejich modelu jsou zároveň rovnováhami v jisté abstraktní ekonomice, neboli v problému zobecněné Nashovy rovnováhy. Vysvětlujeme, co problém zobecněné Nashovy rovnováhy je, a díváme se, zda existuje spojitost, která dovoluje výsledky dosažené výzkumníky z jiných disciplín aplikovat na Arrowův-Debreuův model.

Část práce je věnována modelu se dvěma výrobními faktory, dvěma komoditami a dvěma spotřebiteli, založeném na původních předpokladech Arrowa a Debreua. Abychom nalezli řešení, používáme metodu aplikovaného modelování všeobecné rovnováhy a software nazvaný GAMS. Zkoumáme vliv lepší technologie a daní na spotřebitele a výrobce.

Na konci máme stručné poznámky k aplikacím modelu.

Keywords

Arrow-Debreu model; general equilibrium; existence; generalized Nash equilibrium problem; uniqueness; stability; $2 \times 2 \times 2$ model; applied general equilibrium modelling

Klíčová slova

Arrowův-Debreuův model; všeobecná rovnováha; existence; problém zobecněné Nashovy rovnováhy; jednoznačnost; stabilita; $2 \times 2 \times 2$ model; aplikované modelování všeobecné rovnováhy

Declaration of Authorship

I hereby proclaim that I wrote my bachelor thesis on my own under the leadership of my supervisor and that the references include all resources and literature I have used.

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Prague, May 13, 2015

Signature

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The Arrow-Debreu model of general equilibrium is an integrated model of production, exchange and consumption. The model can be viewed as the basis of the economic theory of price determination and resource allocation. Since the seminal paper by K.J. Arrow and G. Debreu published in 1954, remarkable progress has been made in the issues of existence and uniqueness of general equilibrium. There have also been developed new numerical methods suitable for finding solutions to generalized Nash equilibrium problems. The student will summarize the scientific advances in the theory of the Arrow-Debreu model and comment on its relation to generalized Nash equilibrium problems. Illustrating the theory on a special case of the model, e.g. a two-commodity two-household two-firm model, the student will apply a modern solution method and examine the theoretical properties of the solution. Discussion about applications and shortcomings of the Arrow-Debreu model will follow.

References:

Kenneth J. Arrow, Gerard Debreu (1954): Existence of an Equilibrium for a Competitive Economy, Econometrica 22, 265-290.

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Francisco Facchinei, Christian Kanzow (2010): Generalized Nash Equilibrium Problems, Annals of Operations Research 175, 177-211.

Preliminary scope of work:

Introduction

Assumptions of the Model

Existence and Uniqueness of General Equilibrium

Special Case of the Model

Applications of the Model

Conclusion

Hypothesis:

There is a connection which allows to apply results developed for generalized Nash equilibrium problems to the theory of the Arrow-Debreu model.

Contents

1 Introduction

In 1954, Kenneth J. Arrow and Gerard Debreu, both Nobel Prize laureates, proved the existence of an equilibrium for an integrated model of production, exchange and consumption. New advances have been made in the theory of the model since 1954, but the core of the model has remained unchanged and is the basis of what is now known as the Arrow-Debreu model of general equilibrium.

At the beginning of the thesis, in chapter 2, we present the original assumptions of the Arrow-Debreu model, i.e. those assumptions introduced by Arrow and Debreu in 1954. As some of these assumptions may appear unrealistic, we provide further discussions and acquaint the reader with how Arrow, Debreu and other authors tried to weaken such assumptions. Also in chapter 2, the existence theorem of Arrow and Debreu is stated.

The proof of this theorem works with the concept of an abstract economy. Generalized Nash equilibrium problem (GNEP) is another name for the abstract economy. GNEPs have found many applications in various fields recently. In chapter 3, we try to look at whether there is a connection which allows to apply results developed for GNEPs to the Arrow-Debreu model.

In chapter 4, we examine uniqueness and stability of general equilibria in the Arrow-Debreu model.

We deal with a two-factor, two-commodity, two-consumer model (a $2\times2\times2$ model) in chapter 5. This $2 \times 2 \times 2$ model is based on the original assumptions of the Arrow-Debreu model. In order to find the solution, we use a method called applied general equilibrium modelling or computable general equilibrium modelling and a software called General Algebraic Modeling System (GAMS). Among other things, we address the following question: What is the effect of innovations in the technology of a producer on the consumers in our $2 \times 2 \times 2$ model? Then we add the government to the model, to illustrate one of the most typical applications of applied general equilibrium modelling: an analysis of the impact of tax reforms.

In chapter 6, we provide comments on finding solutions to the Arrow-Debreu model.

We have brief remarks on applications of the model in chapter 7.

The aims of the thesis are primarily

- to provide a good description of the Arrow-Debreu model which would reflect modern scientific advances and applications;
- and to draw attention to the fact that there still might be room for improvement of the model.

2 Original Assumptions of the Arrow-Debreu Model

A model of general equilibrium was first formulated by Walras in 1874. Walras set the problem and research agenda perhaps for all of twentieth-century mathematical general equilibrium theory by formulating his model (Starr 2011, p. 8). Walras did not, however, prove the existence of general equilibrium.

The existence of general equilibrium was proved in 1954, almost simultaneously by Arrow and Debreu and by McKenzie.

This section includes nine assumptions, denoted by Roman numerals, which were introduced by Arrow and Debreu (1954) in their seminal paper *Existence of an Equilibrium for a Competitive Economy* and can be considered the original assumptions of the Arrow-Debreu model of general equilibrium.

The assumptions are divided into two groups: assumptions on the production process and assumptions on the consumption process.

After discussing the assumptions, we present a note on Arrow-Debreu commodities and then we state the original existence theorem for the Arrow-Debreu model.

2.1 Original Assumptions of Arrow and Debreu on the Production Process

Let l be the number of commodities. The index h, which runs from 1 to l, will designate different commodities.

Let n be the number of producers. The index j, which runs from 1 to n, will designate different producers. Let y_j be a particular production plan of the j-th producer. Let y_{jh} be the amount of produced commodity h according to the particular production plan y_j. Commodities which serve as inputs in the production plan y_j will be treated as negative components. Let Y_j be the set of all possible production plans for the j-th producer, $j = 1, \dots, n$.

I For all $j \in 1, \dots, n$, Y_j is a closed convex subset of $\mathbb{R}^l$ containing $\mathbf{0}$ (which is a vector all of whose components are 0).

Arrow and Debreu (1954, p. 267) argue that the assumption I implies non-increasing returns to scale. They implicitly suppose that producers can be output inefficient, i.e. that they can use a particular combination of inputs to produce less output than how much output it is actually possible to produce from those inputs. Nevertheless, a profit-maximizing producer will, of course, never be output inefficient. By assuming that Y_j contains $\mathbf{0}$ it is meant that a producer does not have to produce.

II $Y \cap W = \mathbf{0}$, where

$$Y = \{y \mid y = \sum_{j=1}^{n} y_j, y_j \in Y_j\}$$

and

$$W = \{w \mid w \in \mathbb{R}^l_+\}.$$

This assumption, called weak essentiality, means that there is no practicable input-output schedule for the production sector as a whole such that some output is produced without using any input. It follows that no producer is able to produce anything from nothing.

III $Y \cap -Y = \mathbf{0}$.

The assumption, sometimes called irreversibility, means that a producer can never use their output to obtain back all of the inputs used to produce the output. The assumption reflects the fact that some labour or energy always have to be used in the production process.

2.2 Original Assumptions of Arrow and Debreu on the Consumption Process

Let m be the number of consumers. The index i, which runs from 1 to m, will designate different consumers. Let x_i be a particular vector in $\mathbb{R}^l$ representing consumption of the i-th consumer. This vector will be called the consumption bundle of the i-th consumer. Let x_{ih} be the amount of consumed commodity h according to the consumption bundle x_i. If h denotes a labour service, then $x_{ih} \leq 0$. Let X_i be the set of all consumption bundles for the i-th consumer, $i = 1, \ldots, m$.

IV X_i, $i = 1, \ldots, m$, is a closed convex subset of $\mathbb{R}^l$ bounded from below, i.e. there is a vector ξ_i such that $\xi_i \leq x_i$ for all $x_i \in X_i$.

Arrow and Debreu (1954, pp. 268-269) interpret the set X_i as a set including all consumption bundles x_i among which the i-th consumer could conceivably choose if there were no budgetary restraints. The justification of why X_i should be bounded from below is that consumers cannot supply more than 24 hours of labour services per day and they cannot consume less than nothing.

Consumers are supposed to be capable of comparing their consumption bundles between each other based on their preferences. Arrow and Debreu (1954) present utility function u_i to describe preferences of the i-th consumer. For u_i, $i = 1, \ldots, m$, it holds:

- $u_i(x_i') = u_i(x_i'')$ if and only if the i-th consumer is indifferent between x_i' and x_i'';
- $u_i(x_i') > u_i(x_i'')$ if and only if the i-th consumer prefers x_i' to x_i''.

V $u_i(x_i)$ is a continuous function on X_i.

Debreu (1954) showed that this assumption is equivalent to the assumption that for all x_i' the better set for x_i', i.e. $\{x_i \mid x_i \in X_i$ and x_i' is preferred or indifferent to $x_i\}$, as well as the worse set for x_i', i.e. $\{x_i \mid x_i \in X_i$ and x_i is preferred or indifferent to $x_i'\}$, are closed in X_i. It means that the better set and the worse set will have a part of their boundaries in common. This part will be composed of points x_i for which $u_i(x_i) = u_i(x_i')$. The set of such points x_i is called the indifference set. Gravelle and Rees (2004, p. 14) tell us that, thanks to such an assumption, given two goods in the consumption bundle of a consumer, we can reduce the amount of one good, and however small this reduction is, we can always find an increase in the other good which will leave the consumer with a consumption bundle indifferent to the first.[1] But, if the two goods in his consumption bundle were, for example, race horses and chocolate, it is open to doubt whether there exists a part of a race horse which would exactly offset one bar of chocolate (since one race horse would be typically worth of many bars of chocolate, whereas a part of a horse would be typically worth nothing, or even a negative amount).

Facchinei and Kanzow (2010, p. 192) show that there is a possibility of replacing the continuity assumption with a weaker assumption of pseudo-continuity in generalized Nash equilibrium problems. As we will see later, the Arrow-Debreu model can be considered an example of generalized Nash equilibrium problems.

VI For every x_i' there exists x_i'' such that $u_i(x_i'') > u_i(x_i')$.

So, there is no point of saturation, i.e. no consumption bundle which would be preferred to all other consumption bundles.

VII If $u_i(x_i'') > u_i(x_i')$ and $0 < t < 1$, then $u_i[tx_i' + (1-t)x_i''] > u_i(x_i')$.

Arrow and Debreu (1954, pp. 269-270) show that this assumption is stronger than the assumption of quasi-concavity[2] but weaker than the assumption of strict quasi-concavity.[3]

VIII $\alpha_{ij} \geq 0$ for all i, j and $\sum_{i=1}^{m} \alpha_{ij} = 1$ for all j, where α_{ij} is the contractual claim of the i-th consumer to the share of the profit of the j-th producer.

All profits are paid out to shareholders (who are consumers at the same time), i.e. no earnings are retained in firms.

IX $x_i < \zeta_i$ for some $x_i \in X_i$, where $\zeta_i \in \mathbb{R}^l$ is the vector of initial endowment for the i-th consumer.

This assumption is by far the most problematic one. What the assumption says is that every consumer owns such an amount of each commodity that after consuming out of his initial endowment in some feasible way, they will still have a positive amount of each commodity available for trading in the market. Necessity of an assumption like this results from the fact that it must be secured that every consumer owns at least one commodity which is valuable at the market. Nevertheless, Arrow and

[1] Gravelle and Rees do not include labour services in the consumption bundles at this part of their book.

[2] Let $M \subset \mathbb{R}^k$, $k \in \mathbb{N}$, be a convex set and let f be a function defined on M. We say that f is quasi-concave on M if

$$\forall a, b \in M \; \forall t \in [0; 1] : f(ta + (1-t)b) \geq \min\{f(a); f(b)\}.$$

[3] Let $M \subset \mathbb{R}^k$, $k \in \mathbb{N}$, be a convex set and let f be a function defined on M. We say that f is strictly quasi-concave on M if

$$\forall a, b \in M, a \neq b \; \forall t \in (0; 1) : f(ta + (1-t)b) > \min\{f(a); f(b)\}.$$

Debreu (1954, pp. 279-280) were able to come up with a weakening of the assumption – namely, they showed that it is sufficient to assume instead that a commodity exists which is desired by everyone and that each consumer is endowed with a commodity which cannot be produced and which is always productive.[4] Labour could be considered such a commodity. However, it is important to remember that labour of a doctor and labour of an ice-cream vendor will be considered different commodities in a typical economy.

Since then, other authors managed to weaken the assumption IX. McKenzie (1959, 1961) shows that the condition of irreducibility can be assumed instead of the assumption IX so that the existence of a general equilibrium still can be proved. The condition of irreducibility means that no such a partition of economic agents (consumers and producers) into two groups exists that one group is not capable of supplying any commodities to the other group while the former group wants some commodities that the latter group has. Maxfield (1997) shows that the assumption IX can be relaxed when utility functions and sets of all production plans are restricted to a special class of forms. According to Maxfield, e.g. Cobb-Douglas and constant elasticity of substitution (CES) utility functions can be used for representing preferences of consumers, while Cobb-Douglas and constant elasticity of substitution (CES) production functions can be used for representing technology of producers.[5]

2.3 Arrow-Debreu Commodities

Let us focus on the notion of commodity for a while. Commodities are goods and services transferable in the market. Arrow and Debreu suppose that there is a finite number of distinct commodities. They distinguish commodities according to

1. their attributes;
2. the location at which they are made available;
3. and the date at which they are made available.

[4] A productive commodity can be described as a commodity such that, if no restriction is imposed on the amount of this commodity, it is possible to increase the output of at least one commodity that is desired by every consumer without decreasing the output or increasing the input of any commodity other than the productive commodity under consideration. A precise mathematical definition is given by Arrow and Debreu (1954, p. 280).

[5] The Cobb-Douglas utility function for the i-th consumer is a function of the form

$$u_i(x_i) = \prod_{h=1}^{l} x_{ih}^{\beta_{ih}},$$

where $\beta_{ih} \geq 0, h = 1, \ldots, l$.

The CES utility function for the i-th consumer is a function of the form

$$u_i(x_i) = (\sum_{h=1}^{l} a_{ih} x_{ih}^{\beta_i})^{1/\beta_i},$$

where $a_{ih} \geq 0, h = 1, \ldots, l$ and $\beta_i < 1$.

The Cobb-Douglas and CES production functions are defined similarly, see Maxfield (1997, p. 31).

So, for instance,

1. wheat available today in Prague and barley available today in Prague are deemed to be different commodities;
2. wheat available today in Prague and wheat available today in Bratislava are deemed to be different commodities;
3. wheat available today in Prague and wheat available in Prague in twelve months are deemed to be different commodities.

Clearly, the set of possible bundles of attributes, the set of possible locations and the set of possible dates must be finite, should the number of distinct commodities be finite.

If there are two commodities which differ so little that the difference influences neither the decisions of consumers nor the decisions of producers (they are perfect substitutes for everyone), we can effectively consider these two commodities to be the same commodity. Such a commodity is sometimes referred to as the Arrow-Debreu commodity.

> When the descriptions [of commodities] are so precise that further refinements cannot yield imaginable allocations which increase the satisfaction of the agents in the economy, then the commodities are called Arrow-Debreu commodities.
>
> (Geanakoplos 1989)

This definition is illustrated on the following example:

Suppose that there are three squares in a city and that these squares are identical perfectly competitive ice-cream markets (there are many ice-cream vendors in each of these squares).

The first two of these squares are very close to each other. Ice cream in the first square and ice cream in the second square are perfect substitutes for all consumers in the city, irrespective of where the consumers are. Hence, ice cream in the first square and ice cream in the second square are a single Arrow-Debreu commodity.

In contrast, the third of these squares is quite far from the other squares. Some consumers who are close to the third square would purchase the same amount of ice cream in the third square even if the price was slightly higher than the price of the same ice cream in the first two squares. Similarly, some consumers who are close to the first two squares would purchase the same amount of ice cream in one of the first two squares even if the price was slightly higher than the price of ice cream in the third square. Hence, it is necessary to regard ice cream in the third square as another Arrow-Debreu commodity.

2.4 Existence of an Equilibrium for the Arrow-Debreu Model

The notion of equilibrium for an economy satisfying the assumptions given above needs to be defined, before we start to occupy ourselves with the existence of an equilibrium. We should point out that Arrow and Debreu denote general equilibrium in their model as competitive equilibrium. Other names for the competitive equilibrium appear in the literature, such as the Arrow-Debreu equilibrium or the Walrasian equilibrium. Let p_h be the price of the h-th commodity, $h = 1, \dots, l$, and let p be the vector of prices of those l commodities.

Definition 1. A set of vectors $(x_1^*, \dots, x_m^*, y_1^*, \dots, y_n^*, p^*)$ is said to be a *competitive equilibrium* if the following conditions are satisfied:

1. y_j^* maximizes $(p^*)^T y_j$ over the set Y_j for each j;
2. x_i^* maximizes $u_i(x_i)$ over the set
$$\{x_i \mid x_i \in X_i, (p^*)^T x_i \leq (p^*)^T \zeta_i + \sum_{j=1}^{n} \alpha_{ij} (p^*)^T y_j^*\};$$
3. $p^* \in \{p \mid p \in \mathbb{R}^l, p \geq \mathbf{0}, \sum_{h=1}^{l} p_h = 1\}$;
4. $z^* \leq \mathbf{0}, (p^*)^T z^* = 0$, where
$$z = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j - \sum_{i=1}^{m} \zeta_i.$$

The first condition means that producers maximize their profits,[6] while they take prices as given (which is a usual assumption of perfect competition). The second condition means that consumers maximize their utility functions while being constrained by their budgets. The third condition means that prices must be non-negative and at least one of them must be positive. The fourth condition means that there may be some free goods, so that supply may exceed demand, but it can never happen that demand exceeds supply in an equilibrium (since prices would rise in such a case).

Now we are ready to state the existence theorem.

Theorem 1. *For any economic system that satisfies assumptions I-IX there exists a competitive equilibrium.*

The proof is provided by Arrow and Debreu (1954, pp. 274-279). Arrow and Debreu made use of the Kakutani's fixed-point theorem, which is a generalization of the Brouwer's fixed-point theorem. The use of a fixed-point theorem for demonstrating the existence of an equilibrium of a game was pioneered by Nash in 1950 (Starr 2011, p. 10). It should be noted that McKenzie (1954) used the Kakutani's fixed point theorem as well to show the existence of a general equilibrium. McKenzie's paper was published one month earlier than the 1954 paper of Arrow and Debreu (which was written independently of the McKenzie's paper, though). That is why the Arrow-Debreu model is sometimes called the Arrow-Debreu-McKenzie model. Unlike Arrow and Debreu, McKenzie never received the Nobel Prize. Geanakoplos (1989) writes that the disadvantage of the McKenzie's theorem, compared to the Arrow-Debreu theorem, was that the assumptions were made on demand functions, rather

[6]Because $(p^*)^T y_j$ includes revenues as well as costs.

than on preferences. The roles of these three authors as well as some issues concerning the Nobel Prizes are discussed by Dueppe and Weintraub (2014).

The proof by Arrow and Debreu works with the concept of an abstract economy. As a part of the proof, Arrow and Debreu showed that the equilibria of their model are the same as the equilibria of their abstract economy. It emerged that the abstract economy is a really useful concept not only in economics, but also in many other fields. Another name for the abstract economy is frequently used: the generalized Nash equilibrium problem.

3 Generalized Nash Equilibrium Problems

We find it useful to look at the other properties of the Arrow-Debreu equilibrium, such as uniqueness or stability, from a more general perspective of generalized Nash equilibrium problems. The generalized Nash equilibrium problems are presented in this chapter.

The Nash equilibrium problem is a multi-player non-cooperative game where the goal is to find a solution in which no player has any motivation to change their own strategy unilaterally (Kubota and Fukushima, 2009). The generalized Nash equilibrium problem (GNEP) is a generalization of the Nash equilibrium problem in which each player's strategy set depends on the chosen strategies of other players. The following formal description of the GNEP is based on a description of the GNEP by Facchinei and Kanzow (2010).

Let there be k players, each player ν controlling decision variables $d^\nu \in \mathbb{R}^{q_\nu}$. We denote by d the vector formed by all these decision variables:

$$d := (d^1, \ldots, d^\nu, \ldots, d^k)^T,$$

which has the dimension

$$q := q_1 + \ldots + q_\nu + \ldots + q_k.$$

Furthermore, we denote by $d^{-\nu}$ the vector formed by all the decision variables but for the decision variables controlled by player ν:

$$d^{-\nu} := (d^1, \ldots, d^{\nu-1}, d^{\nu+1}, \ldots, d^k)^T.$$

Each player has an objective function $\theta_\nu : \mathbb{R}^q \to \mathbb{R}$ which depends on his own decision variables d^ν as well as on the decision variables of all the other players $d^{-\nu}$.

In addition, each player's strategy must belong to a strategy set

$$D_\nu(d^{-\nu}) \subset \mathbb{R}^{q_\nu}.$$

The aim of player ν, given the other players' strategies $d^{-\nu}$, is to choose a strategy d^ν, which solves the following maximization problem:

$$\begin{aligned} \underset{d^\nu}{\text{maximize}} \quad & \theta_\nu(d) \\ \text{subject to} \quad & d^\nu \in D_\nu(d^{-\nu}). \end{aligned}$$

Let us denote the solution set of this maximization problem as $S_\nu(d^{-\nu})$.

Then the GNEP is the problem of finding a vector $\bar{d}$ such that

$$\bar{d}^\nu \in S_\nu(\bar{d}^{-\nu}) \text{ for all players } \nu = 1, \ldots, k.$$

Such a point $\bar{d}$ is called a generalized Nash equilibrium. If the strategy sets $D_\nu(d^{-\nu})$ do not depend on the strategies of the other players, the GNEP reduces to the Nash equilibrium problem.

The GNEP was first formally introduced by Debreu (1952), who used the term social equilibrium. This paper was intended to be a preparation for the Arrow and Debreu 1954 paper, where the term abstract economy was used.

Their abstract economy is a GNEP where there are $k = n + m + 1$ players: n producers, m consumers and 1 fictitious market participant.

The problem of the j-th producer, who controls the variables y_j, is:

$$\begin{aligned} \underset{y_j}{\text{maximize}} \quad & p^T y_j \\ \text{subject to} \quad & y_j \in Y_j. \end{aligned}$$

The problem of the i-th consumer, who controls the variables x_i, is:

$$\begin{aligned} \underset{x_i}{\text{maximize}} \quad & u_i(x_i) \\ \text{subject to} \quad & x_i \in X_i, \\ & p^T x_i \leq p^T \zeta_i + \max\{0; \sum_{j=1}^{n} \alpha_{ij} p^T y_j\}. \end{aligned}$$

The problem of the fictitious market participant is:

$$\begin{aligned} \underset{p}{\text{maximize}} \quad & p^T z \\ \text{subject to} \quad & p \in \mathbb{R}^l, p \geq \mathbf{0}, \\ & \sum_{h=1}^{l} p_h = 1, \end{aligned}$$

where

$$z = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j - \sum_{i=1}^{m} \zeta_i.$$

The problem of the fictitious market participant corresponds to the classical law of supply and demand. Indeed, for a given z, the fictitious market participant's objective function can be increased by increasing p_h for those commodities for which $z_h > 0$ or by decreasing p_h for those commodities for which $z_h < 0$.

The need for formulating the economic system as a GNEP in the Arrow-Debreu model arises from the fact that consumers are subject to the restriction that the cost of the commodities chosen at current prices does not exceed their income. But the prices and possibly some or all of the components of their income are determined by the choices of the other consumers and by the choices of producers as well.

So, the Arrow-Debreu model can be classified as a GNEP.

Generalized Nash equilibrium problems tend to occur where the players share a common resource (for example a communication link or an electrical transmission line) or where the players share a common limitation, such as a common limit on the total pollution in an area (Facchinei and Kanzow 2010, p. 181).

GNEPs are being frequently used in many different fields. Facchinei and Kanzow write that GNEPs lie at the intersection of disciplines such as economics, engineering, mathematics, computer science and operations research. Quite naturally, researchers from some disciplines have sometimes worked independently of researchers from other disciplines. This means that it might be really beneficial for economists to check the results obtained in other disciplines, as well as for researchers from other disciplines to check the results obtained in economics.

4 Uniqueness and Stability of General Equilibrium

4.1 Uniqueness of General Equilibrium

Facchinei and Kanzow (2010, p. 196) say that global uniqueness results, as far as the GNEPs are concerned, can be obtained, but it is usually possible only in the context of some specific problems. We will see that some results were actually obtained in the context of the Arrow-Debreu model, but only under restrictive assumptions. Furthermore, Facchinei and Kanzow state that local uniqueness might be also of interest.

Two notions need to be explained: a null set (or a set of measure zero) and a pure exchange economy.

We say that a subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, is null if it has Lebesgue measure zero in $\mathbb{R}^n$ (Debreu 1970, p. 388). In a simplified way, if we say that something is true outside a null subset, we mean it is true almost everywhere (Bartoszynski and Niewiadomska-Bugaj 2008, p. 145). Examples of null subsets in $\mathbb{R}^n$ include finite sets or sets of all elements of a countable sequence in $\mathbb{R}^n$.

A pure exchange economy is such an economy in which economic agents have given endowments of commodities and exchange the commodities among themselves to achieve preferred consumption patterns (Gravelle and Rees 2004, p. 92). Such an economy does not contain a production sector. In the pure exchange economy, each consumer can transform their endowed bundle of commodities into some other bundle through exchange, but the total amount consumed of each commodity cannot exceed the total initial amount of it. This implies that a pure exchange economy can be cast as a GNEP (von Heusinger 2009, p. 13).

According to Facchinei and Kanzow (2010), a significant paper in terms of local uniqueness is *Economies with a Finite Set of Equilibria* by Debreu (1970). Debreu showed that outside a null closed subset of the space of pure exchange economies, every economy has a finite set of equilibria under the assumptions that there are l commodities and m consumers whose needs and preferences are fixed and whose resources vary and that there is a consumer by whom every commodity is desired. In addition, it is assumed that excess-demand functions (functions expressing the excess of quantity demanded over quantity supplied) are differentiable functions of both prices and the distribution of endowments. Arrow and Hahn (1971, p. 244) believe that the result achieved by Debreu is the best possible result, in terms of uniqueness of competitive equilibrium, that is short of too restrictive assumptions.

Local uniqueness is a prerequisite for comparative statics to be well defined in the Arrow-Debreu model (Geanakoplos 1989, p. 121). The method of comparative statics is used, for example, in applied general equilibrium modelling, when it is being examined what the effect of a policy could be (Cardenete *et al.* 2012, p. 15). Later on, when dealing with the $2 \times 2 \times 2$ model, we will incorporate the government to the Arrow-Debreu model and look at how it is possible to study potential effects of a tax levied by the government.

In addition, local uniqueness provides a satisfactory foundation for conducting research on the stability of competitive equilibria (Debreu 1970, p. 387).

However, local uniqueness is not enough to secure that comparative statics can be used without problems (Kehoe 1998, p. 38).

Under some further assumptions, the equilibrium of the Arrow-Debreu model can be proved to be even globally unique. For this purpose, we define gross substitutes (according to Arrow and Hurwicz 1960).

Definition 2. Commodities are all said to be *gross substitutes*, if $\frac{\partial z_{h_1}}{\partial p_{h_2}} > 0$ for all $h_1 \neq h_2$. Here, z_{h_k}, $k = 1, 2$, is the excess demand for commodity h_k, i.e.

$$z_{h_k} = \sum_{i=1}^{m} x_{ih_k} - \sum_{j=1}^{n} y_{jh_k} - \sum_{i=1}^{m} \zeta_{ih_k}.$$

Arrow and Hahn (1971, p. 222) present the following theorem:

Theorem 2. *If all commodities are gross substitutes for every set of equilibrium prices and if there exists a numeraire, then the equilibrium is globally unique.*

A numeraire is a commodity in terms of which all prices in the economy are expressed.

The condition that there exists a numeraire replaces the condition that $\sum_{h=1}^{l} p_h = 1$. Since the price of the numeraire will be 1, it is guaranteed that at least one price will be positive.

The condition $\frac{\partial z_{h_1}}{\partial p_{h_2}} > 0$ for all $h_1 \neq h_2$ is a sufficient condition for global uniqueness, while $\frac{\partial z_{h_1}}{\partial p_{h_2}} \geq 0$ for all $h_1 \neq h_2$ is a necessary condition (Kehoe 1998, p. 43).

Gross substitutability is quite a restrictive assumption. Under gross substitutability, demands for all commodities must be elastic, because if a rise in the price of the commodity h raises the demand for every commodity other than h, then the total expenditure on the commodity h must diminish. However, Fisher (1972) shows that there exist some common utility functions for which gross substitutability holds – for instance the Cobb-Douglas utility function. We will make use of this fact later.

Instead of gross substitutability, the weak axiom of revealed preference can be assumed to hold (Kehoe 1998, pp. 44-47). However, the weak axiom of revealed preference has a big disadvantage compared to the condition of gross substitutability – even though all excess demand functions satisfy the axiom, their sum may not and there may be non-unique equilibria (Kehoe 1998, p. 45).

4.2 Stability of General Equilibrium

The condition of gross substitutability plays a key role also as far as the stability of general equilibrium is concerned. Gravelle and Rees (2004, p. 206) give the following definition of globally stable systems:

Definition 3. Assume that there exists at least one equilibrium price vector $p^* = (p_1^*, p_2^*, \ldots, p_l^*)$, and at an initial moment of time $t = 0$, there exists a price vector $p(0) \neq p^*$. Furthermore, assume that time varies continuously, and the price vector is a vector-valued function of time, $p(t) = (p_1(t), p_2(t), \ldots, p_l(t))$. Then a system is said to be *globally stable* if

$$\lim_{t\to\infty} p(t) = p^*$$

given any initial price vector $p(0)$.

Let the time path of prices, $p(t)$, be determined by the tâtonnement adjustment process. Then it can be shown (Gravelle and Rees 2004, pp. 262-265) that a system is globally stable, if all goods in the economy are gross substitutes.

Again, the condition of gross substitutability can be replaced by the weak axiom of revealed preference in proving the stability of general equilibrium (Kehoe *et al.* 2005, p. 4).

The results regarding the stability of general equilibrium presented here were achieved by Arrow *et al.* (1959) and by Uzawa (1960).

An interesting general result regarding stability was obtained by Morgan and Scalzo (2008).

Definition 4. Let $f : M \subset \mathbb{R}^n \to \mathbb{R}$ be a function.

- f is said to be *upper pseudo-continous at* $x \in M$, if for all $y \in M$ such that $f(x) < f(y)$, we have

  $$\limsup_{z\to x} f(z) < f(y),$$

  and f is said to be *upper pseudo-continuous on* M if it is upper pseudo-continuous at every point $x \in M$.

- f is said to be *lower pseudo-continuous* at $x \in M$ (on M) if $-f$ is upper pseudo-continous at x (on M).

- f is said to be *pseudo-continuous* at $x \in M$ (on M) if it is both upper and lower pseudo-continuous at x (on M).

Morgan and Scalzo investigate the stability of GNEPs using an alternative name for GNEPs: social Nash equilibrium. They show that it is possible to get stability results for GNEPs even in such cases where the objective functions are not continuous but only pseudo-continuous. Again, since the Arrow-Debreu model can be represented as a GNEP, the results achieved by Morgan and Scalzo could be applied to the Arrow-Debreu model.

5 The $2 \times 2 \times 2$ Model

In what follows a special case of the Arrow-Debreu model will be discussed – a two-factor, two-commodity, two-consumer model (a $2 \times 2 \times 2$ model). We will show a method to obtain solutions to $2 \times 2 \times 2$ models.

To learn the way of obtaining solutions to $2 \times 2 \times 2$ models may be considered the first step in learning how to apply the general equilibrium theory. As we will see later, the Arrow-Debreu model does have real-world applications. One class of these applications is called the applied general equilibrium (AGE), or sometimes the computable general equilibrium (CGE).

The very first researcher to come up with a numerical application of the general equilibrium theory was Johansen (1960) who tried to reveal the sources of economic growth in Norway.

Scarf (1967) came up with an algorithm which guaranteed that equilibria could be found to any desired degree of approximation. The ideas contained in his 1967 paper were further developed in the monograph *The Computation of Economic Equilibria*, which was written by Scarf and Hansen (1973).

5.1 Assumptions of the $2 \times 2 \times 2$ Model

In order to obtain the solution to our $2 \times 2 \times 2$ model, we will adopt an approach descended from the input-output models, which were pioneered by Nobel laureate Wassily Leontief (see, for instance, Leontief 1966). Such an approach was used for example in the monograph *Applied General Equilibrium: An Introduction* by Cardenete *et al.* (2012) and we adopt their approach in this thesis. We will see that this approach is in accordance with the assumptions of the Arrow-Debreu model. The methodology that will be used is applied general equilibrium modelling and the software that will be used is GAMS.

Hosoe *et al.* (2010) adopt a similar approach and use GAMS as well (and advise the readers how they can use GAMS on their own). Their computable general equilibrium model contains even only one household.

The $2 \times 2 \times 2$ model will be further elaborated afterwards, as we would like to show how the government can be incorporated. Including the government gives rise to one of the most important application of computable general equilibrium modelling: evaluation of taxes.

We assume there are two producers, $j = 1, 2$, each of whom produces a distinct commodity $h = 1, 2$, respectively. In spite of the small number of agents, we assume that the producers act as price takers. The same is assumed in the case of consumers $i = 1, 2$. Alternatively, we can imagine that there are two perfectly competitive industries in the economy, $j = 1, 2$, each of the industries consisting of identical firms, and two groups of consumers, $i = 1, 2$, each group containing consumers who are similar to each other, as far as their utility functions and initial endowments are concerned. Then it would be plausible to assume that the producers and consumers act as price takers. For clarity, however, we will keep saying just producer 1, producer 2, consumer 1 and consumer 2 in what follows.

In the process of production, the producers use two factors of production, say labour and land, which are commodities initially owned by the consumers.

These factors of production are therefore the initial endowments of the consumers.

Preferences of our consumers are represented by the Cobb-Douglas utility functions of the following form:

$$u_1(x_{11}; x_{12}) = x_{11}^{\beta_{11}} x_{12}^{\beta_{12}},$$

$$u_2(x_{21}; x_{22}) = x_{21}^{\beta_{21}} x_{22}^{\beta_{22}},$$

where $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22} > 0$, $\beta_{12} = 1 - \beta_{11}$, $\beta_{22} = 1 - \beta_{21}$.

Recall please that x_{ih} refers to the amount of commodity h consumed by the i-th consumer (where $i = 1, 2$ and $h = 1, 2$ in this case). However, we do not stick to the convention that labour (or land) are negative components of x_i, hoping that it will not lead to any confusion. Cobb-Douglas utility functions are strictly quasi-concave so that the assumption VII of chapter 2 holds, according to the note below the assumption. Clearly, the assumptions V and VI hold too.

The endowment vectors are of the form

$$\zeta_1 = (\zeta_{11}, \zeta_{12}),$$

$$\zeta_2 = (\zeta_{21}, \zeta_{22}),$$

where ζ_{ig}, $i = 1, 2$ and $g = 1, 2$, is the initial endowment of the i-th consumer with the g-th factor of production ($g = 1$ refers to labour and $g = 2$ refers to land). We assume that $\zeta_{ig} > 0$, $i = 1, 2$ and $g = 1, 2$.

The assumption IV is fulfilled because our consumers cannot consume negative components of commodities $h = 1, 2$ and, at the same time, they cannot offer more labour than ζ_{i1} and more land than ζ_{i2}. Unfortunately, the assumption IX is violated in this setting. However, as is mentioned below the assumption IX, the purpose of the assumption is to secure that every consumer owns at least one commodity which is valuable at the market. This requirement will be met in our setting, as labour and land are always needed for production. Thus, the price of labour and land will be positive if something is produced in the economy, which can be taken for granted.

The technology of the producers will be described by the Leontief production functions

$$y_j = \min\left\{\frac{\phi_j}{v_j}; \frac{\psi_{j1}}{a_{j1}}; \frac{\psi_{j2}}{a_{j2}}\right\}, j = 1, 2,$$

where ϕ_j are auxiliary production functions which are used to represent the substitution possibilities of both producers between labour and land; $\psi_{j1} = a_{j1}y_j$, $\psi_{j2} = a_{j2}y_j$ are the amounts of commodities 1 and 2 (those produced by producers 1 and 2) needed to produce commodity j (intermediary inputs for commodity j); a_{j1}, a_{j2} are input-output coefficients that describe the relation between the output and intermediary inputs in the production process conducted by the j-th producer.

In our $2 \times 2 \times 2$ model,

$$\phi_j = \mu_j \chi_{j1}^{\gamma_{j1}} \chi_{j2}^{\gamma_{j2}},$$

where $\gamma_{j1}, \gamma_{j2} > 0$, $\gamma_{j2} = 1 - \gamma_{j1}$. Furthermore, χ_{j1}, χ_{j2} are the amounts of factors of production, i.e. labour and land, respectively. Again, we decided not to stick to the convention that inputs are negative components of y_j.

The coefficients a_{jh} can be either positive, or zero; but if $a_{jh} = 0$, we have to drop the expression $\frac{\psi_{jh}}{a_{jh}}$ from the production function $y_j = \min\left\{\frac{\phi_j}{v_j}; \frac{\psi_{j1}}{a_{j1}}; \frac{\psi_{j2}}{a_{j2}}\right\}$.

We can arrange the coefficients a_{jh} into the input-output matrix

$$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Hosoe *et al.* (2010, p. 3) write that, for practical purposes, the world-wide input-output tables, as well as other data needed for computable general equilibrium modelling, are prepared by the Global Trade Analysis Project.

In what follows we assume that $a_{jh} > 0$, $j = 1, 2$ and $h = 1, 2$. In such a case, we can, according to Arrow and Debreu (1954, p. 270), use another assumption instead of the assumption IX that will guarantee that a solution to our $2 \times 2 \times 2$ model exists: Each commodity enters into every production process as an input or as an output.

Our production functions exhibit constant returns to scale, which means that the assumption I is fulfilled. The assumptions II and III are fulfilled as well, thanks to the form of the Leontief production functions.

There is one more assumption we have not discussed yet – the assumption VIII. We will see that this assumption is not violated, as one of our equilibrium conditions will require that the price of the commodity produced by the j-th producer equals to the average costs (costs per unit of the commodity) of the j-th producer; and the profits are therefore zero. Of course, we could include the income from dividends into the budget constraints of our consumers, but this would not enrich our analysis very much.

Hence, all of the Arrow-Debreu assumptions can be considered fulfilled. This implies that an equilibrium should exist.

As for the uniqueness of general equilibrium, we have already mentioned that if the assumption of gross substitutability holds, the solution is unique. Fisher (1971) conducted research on which utility functions are in accord with the assumption of gross substitutability. He found out that, if all initial endowments are strictly positive (they are in our case), then the Cobb-Douglas utility function yields individual demand functions with the gross substitute property. Hence, we can expect that our solution will be unique.

Our task now is to find

1. the demand for commodity h by consumer i (i.e. x_{ih}), $h = 1, 2$ and $i = 1, 2$,
2. the demand for factor of production g by producer j (i.e. χ_{jg}), $g = 1, 2$ and $j = 1, 2$, and the demand for commodity h (intermediary input h) by producer j (i.e. ψ_{jh}), $h = 1, 2$ and $j = 1, 2$.

5.2 The Consumers' Demands

The budget constraint of the i-th consumer is

$$p_1 x_{i1} + p_2 x_{i2} = \omega_1 \zeta_{i1} + \omega_2 \zeta_{i2},$$

where ω_1 denotes the wage rate, while ω_2 denotes the rent for land. So, the maximization problem of the i-th consumer is as follows:

$$\begin{aligned} \underset{x_i}{\text{maximize}} \quad & u_i(x_{i1}; x_{i2}) \\ \text{subject to} \quad & p_1 x_{i1} + p_2 x_{i2} = \omega_1 \zeta_{i1} + \omega_2 \zeta_{i2}, \\ & x_{i1}, x_{i2} \geq 0. \end{aligned}$$

Let us suppose that $x_{i1}, x_{i2} > 0$. The marginal rate of substitution is

$$MRS_{21}^i = \frac{\frac{\partial u_i}{\partial x_{i1}}}{\frac{\partial u_i}{\partial x_{i2}}}(x_{i1}; x_{i2}) = \frac{x_{i2}^{\beta_{i2}} \beta_{i1} x_{i1}^{(\beta_{i1}-1)}}{x_{i1}^{\beta_{i1}} \beta_{i2} x_{i2}^{(\beta_{i2}-1)}} = \frac{\beta_{i1} x_{i2}}{\beta_{i2} x_{i1}}.$$

The marginal rate of substitution must equal the ratio of the price of the commodity 1 to the price of the commodity 2, so that

$$\frac{\beta_{i1} x_{i2}}{\beta_{i2} x_{i1}} = \frac{p_1}{p_2}.$$

So,

$$x_{i2} = \frac{\beta_{i2} p_1 x_{i1}}{\beta_{i1} p_2}.$$

This expression can be plugged into the budget constraint:

$$p_1 x_{i1} + p_2 \frac{\beta_{i2} p_1 x_{i1}}{\beta_{i1} p_2} = \omega_1 \zeta_{i1} + \omega_2 \zeta_{i2}.$$

Thus,

$$x_{i1} = \frac{\omega_1 \zeta_{i1} + \omega_2 \zeta_{i2}}{p_1 + p_1 \frac{\beta_{i2}}{\beta_{i1}}} = \frac{\beta_{i1}(\omega_1 \zeta_{i1} + \omega_2 \zeta_{i2})}{p_1},$$

since $\beta_{i1} + \beta_{i2} = 1$ according to our assumption.

It is then clear that

$$x_{i2} = \frac{\beta_{i2}(\omega_1 \zeta_{i1} + \omega_2 \zeta_{i2})}{p_2}.$$

5.3 The Producers' Demands

We will determine the conditional demand for factor of production g by producer j, where the demand will depend on the value of the function

$$\phi_j = \mu_j \chi_{j1}^{\gamma_{j1}} \chi_{j2}^{\gamma_{j2}}.$$

To be able to do so, we can make the j-th producer solve the cost minimization problem

$$\begin{aligned} \underset{\chi_j}{\text{minimize}} \quad & \omega_1 \chi_{j1} + \omega_2 \chi_{j2} \\ \text{subject to} \quad & \phi_j = \mu_j \chi_{j1}^{\gamma_{j1}} \chi_{j2}^{\gamma_{j2}}, \\ & \chi_{j1}, \chi_{j2} \geq 0. \end{aligned}$$

Let us suppose that $\chi_{j1}, \chi_{j2} > 0$. The marginal rate of technical substitution is

$$MRTS_{21}^{j} = \frac{\frac{\partial \phi_j}{\partial \chi_{j1}}}{\frac{\partial \phi_j}{\partial \chi_{j2}}}(\chi_{j1}; \chi_{j2}) = \frac{\mu_j \chi_{j2}^{\gamma_{j2}} \gamma_{j1} \chi_{j1}{}^{\gamma_{j1}-1}}{\mu_j \chi_{j1}^{\gamma_{j1}} \gamma_{j2} \chi_{j2}{}^{\gamma_{j2}-1}} = \frac{\gamma_{j1}\chi_{j2}}{\gamma_{j2}\chi_{j1}}.$$

The marginal rate of technical substitution must equal the ratio of the wage rate to the rent for land, so that

$$\frac{\gamma_{j1}\chi_{j2}}{\gamma_{j2}\chi_{j1}} = \frac{\omega_1}{\omega_2}.$$

So,

$$\chi_{j2} = \frac{\gamma_{j2}\omega_1\chi_{j1}}{\gamma_{j1}\omega_2}.$$

This expression can be plugged into $\phi_j = \mu_j \chi_{j1}^{\gamma_{j1}} \chi_{j2}^{\gamma_{j2}}$, where ϕ_j is a fixed value now:

$$\phi_j = \mu_j \chi_{j1}^{\gamma_{j1}} \left(\frac{\gamma_{j2}\omega_1\chi_{j1}}{\gamma_{j1}\omega_2}\right)^{\gamma_{j2}}.$$

Thus,

$$\chi_{j1} = \frac{\phi_j}{\mu_j}\left(\frac{\gamma_{j1}\omega_2}{\gamma_{j2}\omega_1}\right)^{\gamma_{j2}}.$$

Apparently,

$$\chi_{j2} = \frac{\phi_j}{\mu_j}\left(\frac{\gamma_{j2}\omega_1}{\gamma_{j1}\omega_2}\right)^{\gamma_{j1}},$$

since $\gamma_{j2} = 1 - \gamma_{j1}$ according to our assumption.

The form of the production function $y_j = \min\left\{\frac{\phi_j}{v_j}; \frac{\psi_{j1}}{a_{j1}}; \frac{\psi_{j2}}{a_{j2}}\right\}$ implies, along with the principle of cost minimization, that

- $\phi_j = v_j y_j$,
- $\psi_{j1} = a_{j1} y_j$,
- $\psi_{j2} = a_{j2} y_j$,

where y_j is a fixed value now – it will be determined based on the principle of profit maximization.[7]

From the fact that $\phi_j = v_j y_j$, we can see that

$$\chi_{j1} = \frac{v_j y_j}{\mu_j}\left(\frac{\gamma_{j1}\omega_2}{\gamma_{j2}\omega_1}\right)^{\gamma_{j2}},$$

and

$$\chi_{j2} = \frac{v_j y_j}{\mu_j}\left(\frac{\gamma_{j2}\omega_1}{\gamma_{j1}\omega_2}\right)^{\gamma_{j1}}.$$

So, we have got all demands for inputs, be it factors of production or intermediary inputs, but these demands are conditional on the output y_j. What

[7] This principle is in harmony with the principle of cost minimization; or rather, cost minimization is a necessary condition for profit maximization.

we would like to get is demands for inputs expressed as functions of prices – the producers need to know how much to produce. We can try to find it out through profit maximization, i.e. through maximizing the difference between the value of the revenue function and the value of the cost function.

The revenue function $R_j(y_j)$ is of the form

$$R_j(y_j) = p_j y_j$$

(p_j does not depend on y_j in the conditions of perfect competition), while the cost function is of the form

$$C_j(y_j) = \omega_1\chi_{j1} + \omega_2\chi_{j2} + p_1\psi_{j1} + p_2\psi_{j2},$$

where $\chi_{j1}, \chi_{j2}, \psi_{j1}$ and ψ_{j2} are the conditional demands we have obtained when solving the problem of cost minimization.

Hence,

$$\begin{aligned}
C_j(y_j) &= \omega_1\frac{v_j y_j}{\mu_j}\left(\frac{\gamma_{j1}\omega_2}{\gamma_{j2}\omega_1}\right)^{\gamma_{j2}} + \omega_2\frac{v_j y_j}{\mu_j}\left(\frac{\gamma_{j2}\omega_1}{\gamma_{j1}\omega_2}\right)^{\gamma_{j1}} + p_1 a_{j1} y_j + p_2 a_{j2} y_j = \\
&= \frac{v_j y_j}{\mu_j}\left(\frac{\gamma_{j1}^{\gamma_{j2}}\omega_2^{\gamma_{j2}}\omega_1^{\gamma_{j1}}}{\gamma_{j2}^{\gamma_{j2}}} + \frac{\gamma_{j2}^{\gamma_{j1}}\omega_1^{\gamma_{j1}}\omega_2^{\gamma_{j2}}}{\gamma_{j1}^{\gamma_{j1}}}\right) + p_1 a_{j1} y_j + p_2 a_{j2} y_j = \\
&= \frac{v_j y_j}{\mu_j}\frac{\gamma_{j1}\omega_2^{\gamma_{j2}}\omega_1^{\gamma_{j1}} + \gamma_{j2}\omega_1^{\gamma_{j1}}\omega_2^{\gamma_{j2}}}{\gamma_{j1}^{\gamma_{j1}}\gamma_{j2}^{\gamma_{j2}}} + p_1 a_{j1} y_j + p_2 a_{j2} y_j = \\
&= \frac{v_j\omega_1^{\gamma_{j1}}\omega_2^{\gamma_{j2}} y_j}{\mu_j\gamma_{j1}^{\gamma_{j1}}\gamma_{j2}^{\gamma_{j2}}} + p_1 a_{j1} y_j + p_2 a_{j2} y_j.
\end{aligned}$$

The function that is the difference between the revenue function $R_j(y_j)$ and the cost function $C_j(y_j)$ is often called the profit function $\Pi_j(y_j)$.

The final problem of the j-th producer is the problem of profit maximization:

$$\begin{aligned}
\underset{y_j}{\text{maximize}} \quad & \Pi_j(y_j) \\
\text{subject to} \quad & y_j \geq 0.
\end{aligned}$$

It can be seen that $\Pi_j(y_j)$ is linear in our case:

$$\Pi_j(y_j) = \left(p_j - \frac{v_j\omega_1^{\gamma_{j1}}\omega_2^{\gamma_{j2}}}{\mu_j\gamma_{j1}^{\gamma_{j1}}\gamma_{j2}^{\gamma_{j2}}} - p_1 a_{j1} - p_2 a_{j2}\right) y_j.$$

If p_j was higher than

$$\frac{v_j\omega_1^{\gamma_{j1}}\omega_2^{\gamma_{j2}}}{\mu_j\gamma_{j1}^{\gamma_{j1}}\gamma_{j2}^{\gamma_{j2}}} + p_1 a_{j1} + p_2 a_{j2},$$

which is the marginal and average cost of the j-th producer, then the j-th producer would like to produce an infinite amount of the output, which would mean infinite profits for them. However, the consumers have only limited endowments. At the same time, the j-th producer cannot produce an infinite amount of the output from finite amounts of the inputs – because the j-th producer is restricted by the technology. They cannot also gain infinite profits. Rather, the more the j-th producer would produce, the more p_j would decrease. But then the j-th producer would stop being a price taker, which is our assumption. Thus, we can conclude that it makes no sense for p_j to be higher than the marginal and average cost of the j-th producer in our simplified model of perfect competition.

On the other hand, if p_j was lower than the marginal and average cost of the j-th producer, then the profit-maximizing strategy of the producer would be not to produce. As there is no one else in our economy who would produce the same commodity as the j-th producer, the commodity would not be produced at all. However, the utility functions of the consumers suggest that if one of the commodities was not produced, then even the other commodity could not bring any utility to the consumers. Hence, neither of the commodities would be produced and there would be no economy to analyse. That is why it is sensible to conclude that p_j will be equal to the marginal and average cost of the j-th producer.

If prices equal average costs, our producers will have zero profits. One may ask why the producers would waste their time doing business when their profits are zero. It is necessary to realize, though, that the owners of the producers (i.e. firms) must be our consumers. At the same time, the employees of the firms are our consumers too. So, one can take it so, that the owners receive wages from their firms (they are owners as well as employees) and these wages are subsequently involved in the costs of the firms. So, if the owner of a firm decided to stop doing their business, they would lose their job and wages and would not be able to consume as much as now. As long as price equals marginal cost, every firm will adjust its output level to match the demand.

An interesting thing is that the amounts of produced outputs y_1 and y_2 are not virtually determined by the producers in our case – the output levels are not determined independently of demand. It is a consequence of the fact that the production functions we chose exhibit constant returns to scale (Cardenete *et al.* 2012, p. 27).

5.4 The Equilibrium Equations

In our setting, the equilibrium (we expect there will be only one equilibrium) will be characterized by the following conditions:

- Supply equals demand in both markets for produced commodities;
- supply equals demand in both markets for factors of production;
- both producers maximize their profits.

The equilibrium system therefore consists of the following six equations:

1)
$$y_1 = x_{11}(p;\omega) + x_{21}(p;\omega) + \psi_{11}(y) + \psi_{21}(y),$$

i.e.

$$y_1 = \frac{\beta_{11}(\omega_1\zeta_{11} + \omega_2\zeta_{12})}{p_1} + \frac{\beta_{21}(\omega_1\zeta_{21} + \omega_2\zeta_{22})}{p_1} + a_{11}y_1 + a_{21}y_2;$$

2)

$$y_2 = x_{12}(p;\omega) + x_{22}(p;\omega) + \psi_{12}(y) + \psi_{22}(y),$$

i.e.

$$y_2 = \frac{\beta_{12}(\omega_1\zeta_{11} + \omega_2\zeta_{12})}{p_2} + \frac{\beta_{22}(\omega_1\zeta_{21} + \omega_2\zeta_{22})}{p_2} + a_{12}y_1 + a_{22}y_2;$$

3)

$$\zeta_{11} + \zeta_{21} = \chi_{11}(\omega;y) + \chi_{21}(\omega;y),$$

i.e.

$$\zeta_{11} + \zeta_{21} = \frac{v_1 y_1}{\mu_1}\left(\frac{\gamma_{11}\omega_2}{\gamma_{12}\omega_1}\right)^{\gamma_{12}} + \frac{v_2 y_2}{\mu_2}\left(\frac{\gamma_{21}\omega_2}{\gamma_{22}\omega_1}\right)^{\gamma_{22}};$$

4)

$$\zeta_{12} + \zeta_{22} = \chi_{12}(\omega;y) + \chi_{22}(\omega;y),$$

i.e.

$$\zeta_{12} + \zeta_{22} = \frac{v_1 y_1}{\mu_1}\left(\frac{\gamma_{12}\omega_1}{\gamma_{11}\omega_2}\right)^{\gamma_{11}} + \frac{v_2 y_2}{\mu_2}\left(\frac{\gamma_{22}\omega_1}{\gamma_{21}\omega_2}\right)^{\gamma_{21}};$$

5)

$$0 = p_1 - \frac{v_1\omega_1^{\gamma_{11}}\omega_2^{\gamma_{12}}}{\mu_1\gamma_{11}^{\gamma_{11}}\gamma_{12}^{\gamma_{12}}} - p_1 a_{11} - p_2 a_{12};$$

6)

$$0 = p_2 - \frac{v_2\omega_1^{\gamma_{21}}\omega_2^{\gamma_{22}}}{\mu_2\gamma_{21}^{\gamma_{21}}\gamma_{22}^{\gamma_{22}}} - p_1 a_{21} - p_2 a_{22}.$$

It can be seen that we have a system of 6 equations and 6 unknowns: p_1, p_2, ω_1, ω_2, y_1 and y_2. But this alone does not mean we will get a unique solution.[8] Since all the demand functions are homogeneous of degree zero in prices, it holds that if the price vector (p; ω) is an equilibrium price vector, then the price vector (kp; $k\omega$), $k > 0$, will be another equilibrium price vector. We can easily fix it by assuming that

$$p_1 + p_2 + \omega_1 + \omega_2 = 1.$$

Such an assumption is in harmony with the definition of competitive equilibrium above.

In order to have only 6 equations again, we can get rid of one of the original equations[9] according to the Walras's law. The Walras's law tells us that at

[8] Or even any solution at all (Cardenete *et al.* 2012, p. 28).

[9] For instance the equation 1).

any price vector the total value of excess demands equals zero, whether or not the prices are equilibrium prices. Mathematically,

$$\sum_{h=1}^{l} p_h z_h = 0,$$

where

$$z_h = \sum_{i=1}^{m} x_{ih} - \sum_{j=1}^{n} y_{jh} - \sum_{i=1}^{m} \zeta_{ih}.$$

Then it follows directly from the Walras's law that in an economy with l commodities, whenever there is market equilibrium for $l-1$ goods, the l-th market clears too (Starr 2011, p. 20). Now we have 6 equations and 6 unknowns and the solution should be unique, by the Arrow-Debreu theorem and by the fact that the utility functions have the property of gross substitutability.

5.5 Solving the Equilibrium Equations

We will not try to solve the system of 6 equations manually. Instead, we will assume some particular values for the coefficients involved in the 6 equations, after which we will use the General Algebraic Modeling System (GAMS).

Let $\zeta_{11} = 30$, $\zeta_{12} = 20$, $\zeta_{21} = 20$, $\zeta_{22} = 5$; $\beta_{11} = 0.3$, $\beta_{12} = 0.7$, $\beta_{21} = 0.6$, $\beta_{22} = 0.4$; $a_{11} = 0.2$, $a_{12} = 0.3$, $a_{21} = 0.5$, $a_{22} = 0.25$; $\gamma_{11} = 0.8$, $\gamma_{12} = 0.2$, $\gamma_{21} = 0.4$, $\gamma_{22} = 0.6$; $v_1 = 0.5$, $v_2 = 0.25$; $\mu_1 = \gamma_{11}^{-\gamma_{11}}\gamma_{12}^{-\gamma_{12}}$, $\mu_2 = \gamma_{21}^{-\gamma_{21}}\gamma_{22}^{-\gamma_{22}}$. These are values used by Cardenete *et al.* (2012). As we will see, this combination of values has the nice property that all prices, i.e. p_1, p_2, ω_1, ω_2, are the same. Then it will be easier to perceive the price changes when some of the coefficients are changed.

We report the GAMS results in Figure 1 in the Appendix.

By plugging the computed values of p_1, p_2, ω_1, ω_2, y_1 and y_2 into the original equilibrium equations, we can verify that p_1, p_2, ω_1 and ω_2 are indeed equilibrium prices and y_1 and y_2 are indeed equilibrium outputs. In Figure 2 in the Appendix, the verification is done. Besides, it is straightforward to verify that the sum of all prices is 1.

In addition, it may be interesting to compute the GDP. This can be done by means of the formula

$$\text{GDP} = p_1 \cdot x_{11} + p_2 \cdot x_{12} + p_1 \cdot x_{21} + p_2 \cdot x_{22}.$$

It can be calculated that $x_{11} = 15$, $x_{12} = 35$, $x_{21} = 15$ and $x_{22} = 10$. Hence, the GDP is equal to

$$0.25 \cdot 15 + 0.25 \cdot 35 + 0.25 \cdot 15 + 0.25 \cdot 10 = 18.75.$$

Let us look at how the equilibrium prices, outputs, consumptions and GDP change when some of the coefficients change. Imagine, for example, that the producer 2 comes up with an innovation that reduces the dependence on the output of the producer 1, so that $a_{21} = 0.25$ now.

It can be seen in Figure 3 in the Appendix that p_1 and p_2 decreased. Price p_2 could have decreased because it is cheaper now to produce a unit of the commodity 2, while the condition of zero profits still holds. The reason why p_1 slightly decreased may be that the total demand for the commodity 1 (which is as large as the total supply of commodity 1) decreased. Then, given that the sum of all prices must be 1, the other prices must have increased (or at least one of them).

As for the outputs, the output of the producer 1 increased, whereas the output of the producer 2 decreased. This might have been expected, essentially because of the same reasons that were given when we were arguing what the reasons for the changes in p_1 and p_2 might have been.

A really interesting thing is that all the equilibrium consumptions, x_{11}, x_{12}, x_{21} and x_{22} increased – now, $x_{11} \doteq 18.1$, $x_{12} \doteq 56.3$, $x_{21} \doteq 17.3$ and $x_{22} \doteq 15.4$. Thus, the innovation has had a positive impact on consumers.

The nominal GDP increased. We can be sure that the real GDP increased as well, since x_{11}, x_{12}, x_{21} and x_{22} increased.

In Figure 4 in the Appendix, it can be checked that the left side and the right side of the equilibrium equations still equal (there is some rounding error, as GAMS does not return exact values).

5.6 Including the Government Sector

We will continue with adding the government sector into our model. This will enable us to analyse the influence of introducing taxes. We will occupy ourselves with an *ad valorem* tax, which is a tax based on the value of transactions. An example of such an *ad valorem* tax is a value-added tax.

Let τ_j be the *ad valorem* tax rate on the output of producer j. Then the equilibrium equations 5) and 6) will change as follows:

5')
$$0 = p_1 - (1+\tau_1)\left(\frac{v_1 \omega_1^{\gamma_{11}} \omega_2^{\gamma_{12}}}{\mu_1 \gamma_{11}^{\gamma_{11}} \gamma_{12}^{\gamma_{12}}} + p_1 a_{11} + p_2 a_{12}\right);$$

6')
$$0 = p_2 - (1+\tau_2)\left(\frac{v_2 \omega_1^{\gamma_{21}} \omega_2^{\gamma_{22}}}{\mu_2 \gamma_{21}^{\gamma_{21}} \gamma_{22}^{\gamma_{22}}} + p_1 a_{21} + p_2 a_{22}\right),$$

where p_1 and p_2 are the gross-of-tax prices of commodities 1 and 2, respectively (i.e. prices that are paid by consumers). Thus, with respect to what the prices mean now, the equations 5') and 6') do not differ much from the equations 5) and 6). It must hold that the marginal revenues of the j-th producer (i.e. the price p_j) equal the marginal cost of the j-th producer, which in turn equals the average cost of the j-th producer in our model.

If the total amount of taxes collected by the government is denoted as T and the weights according to which the amount T is distributed between the two consumers are denoted as δ_1 and δ_2, where $\delta_1 + \delta_2 = 1$, then the budget constraint of the i-th consumer, $i = 1, 2$, changes to

$$p_1 x_{i1} + p_2 x_{i2} = \omega_1 \zeta_{i1} + \omega_2 \zeta_{i2} + \delta_i T.$$

So, the new equilibrium equations 1') and 2') will be

1')

$$y_1 = \frac{\beta_{11}(\omega_1\zeta_{11} + \omega_2\zeta_{12} + \delta_1 T)}{p_1} + \frac{\beta_{21}(\omega_1\zeta_{21} + \omega_2\zeta_{22} + \delta_2 T)}{p_1} + a_{11}y_1 + \\ + a_{21}y_2;$$

2')

$$y_2 = \frac{\beta_{12}(\omega_1\zeta_{11} + \omega_2\zeta_{12} + \delta_1 T)}{p_2} + \frac{\beta_{22}(\omega_1\zeta_{21} + \omega_2\zeta_{22} + \delta_2 T)}{p_2} + a_{12}y_1 + \\ + a_{22}y_2.$$

As we will assume that expenditures of the government equal its revenues, a new equilibrium equation is

7)

$$T = \tau_1 y_1 \left(\frac{v_1 \omega_1^{\gamma_{11}} \omega_2^{\gamma_{12}}}{\mu_1 \gamma_{11}^{\gamma_{11}} \gamma_{12}^{\gamma_{12}}} + p_1 a_{11} + p_2 a_{12} \right) + \\ + \tau_2 y_2 \left(\frac{v_2 \omega_1^{\gamma_{21}} \omega_2^{\gamma_{22}}}{\mu_2 \gamma_{21}^{\gamma_{21}} \gamma_{22}^{\gamma_{22}}} + p_1 a_{21} + p_2 a_{22} \right).$$

Suppose that $\tau_1 = \tau_2 = 0.1$, $\delta_1 = \delta_2 = 0.5$. Moreover, we return to the case that $a_{21} = 0.5$ as at the beginning. The other coefficients stay the same. Again, we apply the condition that the sum of p_1, p_2, ω_1 and ω_2 is equal to 1.

Then the results can be found in Figure 5 in the Appendix.

It can be seen that the prices p_1 and p_2 are now higher and the prices ω_1 and ω_2 relatively lower than in the case with no taxes (when all prices were equal to 0.250). This is because the producers need to offset the losses from taxes they have to pay now.

As for the outputs, y_1 rose a little bit, while y_2 dropped a little bit. One may have expected some large dead-weight losses because of the tax. However, this is not the case because we assume full employment of labour as well as land. Cardenete *et al.* (2012) show how it is possible to incorporate unemployment into the model.

The government collected 5.14 on taxes which is approximately 24 % of GDP. A government could use a model like this to see how much money could be collected from tax-payers, after a tax is levied.

One can notice that the nominal GDP increased but, as for now, we cannot be sure whether the real GDP increased as well. In order to find it out, one could use the Laspeyres index or the Paasche index.

According to the Laspeyres method, we use the old prices to calculate the real GDP. The real GDP before the *ad valorem* tax was levied had equalled the nominal GDP, i.e. 18.75. The new real GDP (the real GDP after the *ad valorem* tax was levied) can be calculated as the inner product of old prices and new consumptions, that is,

$$\text{GDP}_{real}^{new} \doteq 0.25 \cdot 14.523 + 0.25 \cdot 32.147 + 0.25 \cdot 17.339 + 0.25 \cdot 10.966 \doteq 18.744.$$

The Laspeyres index (LI) is then computed as

$$LI \doteq 18.744/18.75 \doteq 0,9997,$$

which implies that the real GDP decreased by approximately 0.03 %.

By contrast, according to the Paasche method, we use the new prices to calculate the real GDP. The new real GDP is therefore 21.415, while the old real GDP was

$$\text{GDP}_{real}^{old} \doteq 0.277 \cdot 15 + 0.292 \cdot 35 + 0.277 \cdot 15 + 0.292 \cdot 10 = 21.45.$$

The Paasche index (PI) is then computed as

$$PI \doteq 21.415/21.45 \doteq 0.9984,$$

which implies that the real GDP decreased by approximately 0.16 %.

We can say that there was virtually no real GDP growth or decline.

Furthermore, one can notice (Figure 6 in the Appendix) that the consumer 2 is now better off while the consumer 1 is worse off. Since the consumer 1 could have been considered richer (because his or her consumption of each commodity was greater than or equal to the consumption by the consumer 2 before the tax was levied), the tax probably did not deepen inequality.

Again, the reader can check that all (7, in this case) equilibrium equations hold as equalities in the equilibrium.

In a similar way, the model can be extended to include income taxes (Cardenete *et al.* 2012).

As for other possible extensions of the model, the government can be allowed to operate a non-balanced budget, the external sector can be included and the model can be even allowed for unemployment (Cardenete *et al.* 2012).

Applied general equilibrium modelling is a vivid field and much more complex than we were able to show here. More on this topic can be found in *Notes and Problems in Applied General Equilibrium Economics* (Dixon *et al.* 2014).

6 Solution Techniques for the Arrow-Debreu Model

The $2 \times 2 \times 2$ model we have just occupied ourselves with is a special type of the Arrow-Debreu model which in turn is a generalized Nash equilibrium problem (GNEP). We were able, by means of GAMS, to find the solution to our simple $2 \times 2 \times 2$ model. But recall that we have made some special assumptions regarding the utility and production functions – we assumed Cobb-Douglas utility functions and Leontief production functions. In full generality, where no specific functional forms are assumed, it is really difficult to come up with a method capable of finding solutions to the Arrow-Debreu model or to GNEPs.

The proof of existence of an equilibrium for a competitive economy by Arrow and Debreu was non-constructive and an algorithm for finding prices is therefore needed, should the prices be determined.

Facchinei and Kanzow (2010) summarize general results that were obtained in terms of GNEPs and they enumerate and describe methods suitable for finding solutions to GNEPs. They write (p. 207) that what is typical for dealing with generalized Nash equilibrium problems is that the problems are transformed into other problems that are understood better, such as variational inequalities or quasi-variational inequalities. However, in the same breath, Facchinei and Kanzow add that this approach has had only a limited success. The reasons are that the conditions under which the transformations are done may be very demanding or of difficult interpretation.

According to Facchinei and Kanzow (2010, p. 207), there are currently two possibilities how to cope with these difficulties. One of them is studying problems with special structures, such as web or telecommunication applications, or the Arrow-Debreu model. The other possibility is to study classes of GNEPs special from the mathematical point of view, such as jointly convex GNEPs.

In her PhD. thesis elaborated under the advice of Kanzow, von Heusinger (2009, p. 13) points out an article written by Codenotti and Varadarajan (2007). The article introduces convex programming techniques to compute market equilibria in the pure exchange economy, which is such a case of the Arrow-Debreu model where no production sector exists, and afterwards in the Arrow-Debreu model as such. The pure exchange economy as well as the Arrow-Debreu model can be cast as a GNEP.

Important papers dealing with computing market equilibria were those by Scarf (1967) and Hansen and Scarf (1973), authors who have already been mentioned. Other algorithms were provided by Jain (2007) or by Ye (2008). The algorithms by Jain and Ye make use of the concept of the pure exchange economy.

7 Applications of the Arrow-Debreu Model

Geanakoplos (1989) devotes an entire section of his paper to what the Arrow-Debreu model in its full generality does not explain. He mentions that in Arrow-Debreu equilibrium, there is no trade in shares of firms – shares of firms are not taken as Arrow-Debreu commodities in the Arrow-Debreu model.[10] In addition, if there were a market for firm shares, there would not be any trade anyway, since ownership of the firm and the income necessary to purchase it would be perfect substitutes (Geanakoplos 1989). Besides, money does not appear in the Arrow-Debreu model. Another thing is that all trade takes place at the beginning of time – that is, time does not appear in the Arrow-Debreu model as well. It is because an equilibrium is reached at the beginning of time and the economic agents have no incentive to trade afterwards.

Nevertheless, the Arrow-Debreu model is a crucial part of general equilibrium analysis. Starr (2011, p. 5) believes that general equilibrium analysis has proved fundamental in modern economics in describing the efficiency and stability of the market mechanism, in providing the logical foundations of microeconomics, and even in macroeconomic analysis.

Starr (2011, p. 6) continues that general equilibrium theory provides the basis for major innovations in modern economic theory and for the full mathematically rigorous confirmation of long-held traditional views in economics.

Another important fact is that the research that has been conducted in terms of the Arrow-Debreu model has had large impact on other fields. We have shortly discussed generalized Nash equilibrium problems. These GNEPs were introduced by Debreu (1952) and Arrow and Debreu (1954). Besides, many important properties of GNEPs were revealed through elaborating the Arrow-Debreu model. We have seen that for example the proofs of existence or local uniqueness of solutions to GNEPs were obtained thanks to elaborating the Arrow-Debreu model. Since GNEPs are now being used in other fields as well, the Arrow-Debreu model can be said to have influenced these fields. This should be always remembered, even by people who are sceptical of the Arrow-Debreu model (e.g. because of too much mathematics used by researchers who have worked on the Arrow-Debreu model).

As for the real-world applications, we have paid close attention to applied general equilibrium modelling, which is a discipline based on the Arrow-Debreu model.

According to Kehoe *et al.* (2005, p. 5), applied general equilibrium modelling was adopted by many governments all around the world (e.g. the United States, Australia, the United Kingdom, the Netherlands) or by international organizations such as the World Bank, the World Trade Organization or the International Monetary Fund.

After a government, an external sector or investments and savings are included into the model, applied general equilibrium modelling can be used for analysing issues such as the impact of tax reforms, global warming problems, assistance for developing countries or deregulation of electric power industry (Hosoe *et al.* 2010).

[10]The Arrow-Debreu commodities were discussed in subsection 1.3.

Kehoe *et al.* state that the great strength of applied general equilibrium modelling has been its ability to provide numerical assessments of the equity and efficiency implications of microeconomic policy changes – which is hard to do with conventional econometric models. Further, they write that in some situations of simultaneous changes in several policies, there is no alternative to applied general equilibrium modelling.

Hosoe *et al.* (2010, p. 5) notice that a great advantage of applied general equilibrium models over econometric models is that usually only data for one year is needed in case of applied general equilibrium models, while data for several years may be needed in case of econometric models so that the models have a sufficient amount of degrees of freedom. Thus, the applied general equilibrium models can be highly preferred for economies that experienced drastic changes or where the data is not available.

8 Conclusion

In this thesis, we dealt with the Arrow-Debreu model of general equilibrium. The original assumptions of the Arrow-Debreu model, i.e. the assumptions introduced by Kenneth J. Arrow and Gerard Debreu in their seminal paper *Existence of an Equilibrium for a Competitive Economy*, were presented and discussed at the beginning. Particularly one of these assumptions may be considered rather problematic: the assumption IX says that each consumer is endowed with a positive amount of every commodity. Arrow and Debreu were aware of the fact that the assumption was unrealistic and tried to address the problem already in their paper just mentioned. Several years later, McKenzie (1959, 1961) came up with an alternative assumption, called irreducibility. Another author who tried to address the problem was Maxfield (1997).

Under the original assumptions, Arrow and Debreu (1954) managed to prove the existence of a competitive equilibrium, using the Kakutani's fixed-point theorem. McKenzie (1954) achieved the same success in the same year, albeit under slightly different assumptions. That is why the model is sometimes called the Arrow-Debreu-McKenzie model.

It is really interesting to think of the Arrow-Debreu model in terms of generalized Nash equilibrium problems (GNEPs). As a part of their proof of the existence of a competitive equilibrium, Arrow and Debreu (1954) showed that the equilibria of their model are the same as the equilibria of a certain GNEP. Thus, there is a connection which allows to apply results developed for GNEPs to the theory of the Arrow-Debreu model. At the beginning, GNEPs were considered mainly by economists working on the Arrow-Debreu model. Recently, however, researchers from the areas of mathematics, engineering, computer science or operations research have increasingly begun to explore and develop the theory of GNEPs (Facchinei and Kanzow 2010). Economists who are aware of this fact might bring some new insights to their discipline.

After the introduction of GNEPs, it was easier to comment on the progress made in exploring the properties of solutions, such as the uniqueness, local uniqueness or stability of Arrow-Debreu equilibria.

A part of the thesis was dedicated to a two-factor, two-commodity, two-consumer model, which was based on the original assumptions of Arrow and Debreu. In order to find the solution, we used a method called applied general equilibrium modelling or computable general equilibrium modelling (Cardenete *et al.* 2012). We chose some specific forms of the consumers' utility functions (Cobb-Douglas utility functions) and of the functions representing the producers' technology (Leontief production functions). Then we derived the equilibrium equations, chose the values of coefficients and solved the equations by means of GAMS. Afterwards, we tried to change the values of some coefficients and found out that a better technology of a producer had a positive impact on both consumers in our setting (while the profits of both producers remained zero). Thereafter, to illustrate a real-world application of applied general equilibrium modelling, we added the government sector, which enabled us to analyse the influence of introducing taxes on consumers and producers. We dealt with an *ad valorem* tax. The same method, i.e. applied general equilibrium modelling, has been used by governments and international organizations all around the world (Kehoe *et al.* 2005). After a government, an external sector or investments and savings are included into the model and coefficients of utility and production functions are estimated, applied general equilibrium modelling can be used not only for analysing the impact of tax reforms, but also issues as various as global warming problems, assistance for developing countries or deregulation of electric power industry (Hosoe *et al.* 2010).

Besides the fact that the Arrow-Debreu model serves as a framework for applied general equilibrium modelling, its role was crucial in developing microeconomic theories (Starr 2011). In addition, the Arrow-Debreu model has had a positive impact on disciplines dealing with GNEPs.

As for our recommendations on future follow-ups, we believe that there is a possibility to enrich the theory of the Arrow-Debreu model by using new advances in the theory of generalized Nash equilibrium problems. In this respect, it could be even possible to make some contributions to applied general equilibrium modelling. For example, algorithms developed by researchers in the area of generalized Nash equilibrium problems could perhaps be used by applied general equilibrium modellers.

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Appendix

Figure 1: GAMS Results – Initial Values of Coefficients

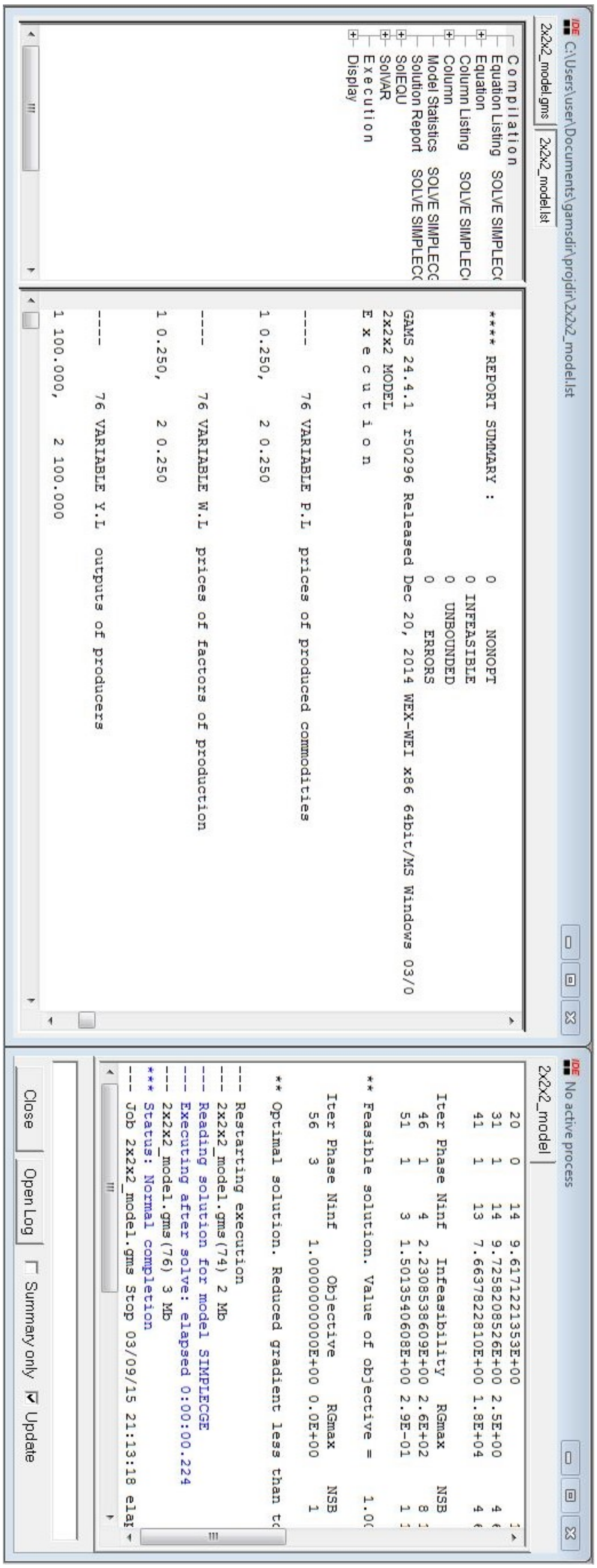

Figure 2: Testing the GAMS Results – Initial Values of Coefficients

	A	B	C	D	E	F	G	H	I	J	K
1	coefficient	value		unknown	computed		equation	left side		right side	
2	zeta11	30		p1	0,25		1)	=E6	100	=B6*(E4*B2+E5*B3)/E2+B8*(E4*B4+E5*B5)/E2+B10*E6+B12*E7	100
3	zeta12	20		p2	0,25		2)	=E7	100	=B7*(E4*B2+E5*B3)/E3+B9*(E4*B4+E5*B5)/E3+B11*E6+B13*E7	100
4	zeta21	20		omega1	0,25		3)	=B2+B4	50	=(B18*E6/B20)*(B14*E5/(B15*E4))^B15+(B19*E7/B21)*(B16*E5/(B17*E4))^B17	50
5	zeta22	5		omega2	0,25		4)	=B3+B5	25	=(B18*E6/B20)*(B15*E4/(B14*E5))^B14+(B19*E7/B21)*(B17*E4/(B16*E5))^B16	25
6	beta11	0,3		y1	100		5)	=0	0	=E2-B18*E4^B14*E5^B15/(B20*B14^B14*B15^B15)-E2*B10-E3*B11	0
7	beta12	0,7		y2	100		6)	=0	0	=E3-B19*E4^B16*E5^B17/(B21*B16^B16*B17^B17)-E2*B12-E3*B13	0
8	beta21	0,6									
9	beta22	0,4		x11	15						
10	a11	0,2		x12	35						
11	a12	0,3		x21	15						
12	a21	0,5		x22	10						
13	a22	0,25									
14	gamma11	0,8		GDP	18,75						
15	gamma12	0,2									
16	gamma21	0,4									
17	gamma22	0,6									
18	v1	0,5									
19	v2	0,25									
20	mu1	1,649385									
21	mu2	1,960132									

Figure 3: GAMS Results – Better Technology

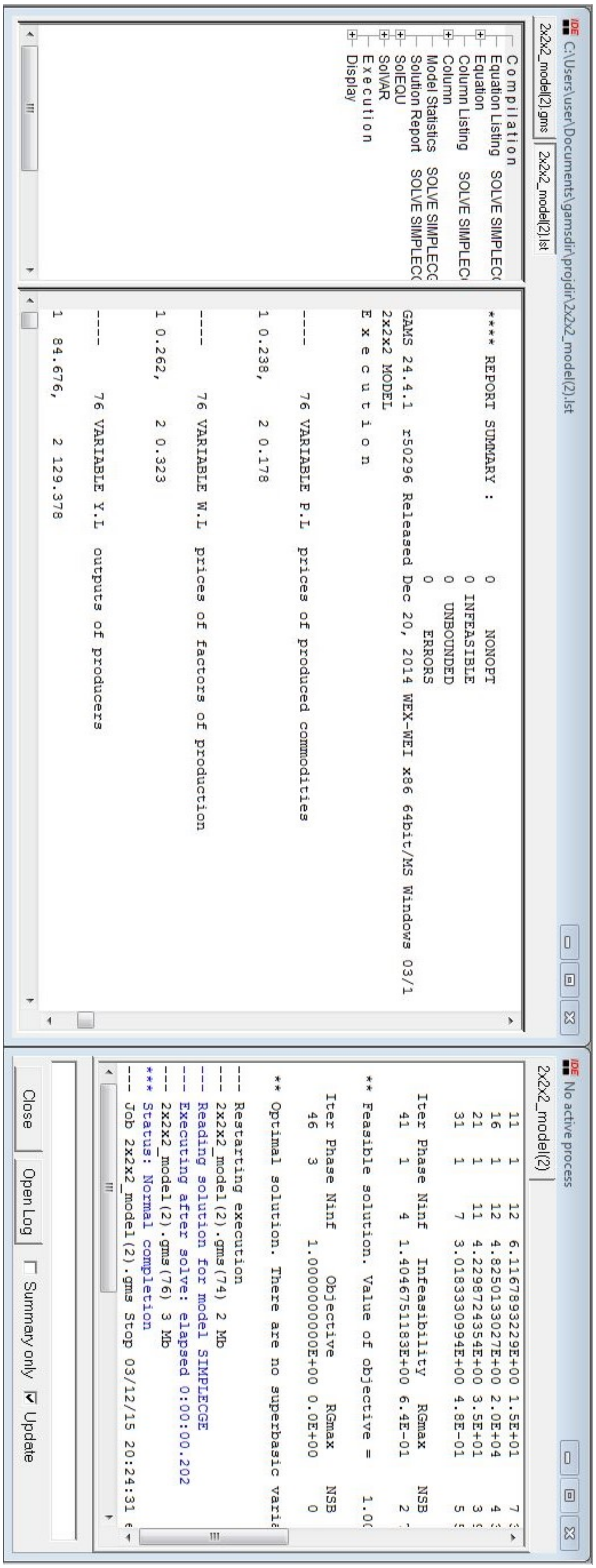

Figure 4: Testing the GAMS Results – Better Technology

	A	B	C	D	E	F	G	H	I	J	K
1	coefficient	value		unknown	computed		equation	left side		right side	
2	zeta11	30		p1	0,238		1)	=E6	84,676	=B6*(E4*B2+E5*B3)/E2+B8*(E4*B4+E5*B5)/E2+B10*E6+B12*E7	84,61163
3	zeta12	20		p2	0,178		2)	=E7	129,378	=B7*(E4*B2+E5*B3)/E3+B9*(E4*B4+E5*B5)/E3+B11*E6+B13*E7	129,4664
4	zeta21	20		omega1	0,262		3)	=B2+B4	50	=(B18*E6/B20)*(B14*E5/(B15*E4))^B15+(B19*E7/B21)*(B16*E5/(B17*E4))^B17	49,98739
5	zeta22	5		omega2	0,323		4)	=B3+B5	25	=(B18*E6/B20)*(B15*E4/(B14*E5))^B14+(B19*E7/B21)*(B17*E4/(B16*E5))^B16	25,01015
6	beta11	0,3		y1	84,676		5)	=0	0	=E2-B18*E4^B14*E5^B15/(B20*B14^B14*B15^B15)-E2*B10-E3*B11	0,0004
7	beta12	0,7		y2	129,378		6)	=0	0	=E3-B19*E4^B16*E5^B17/(B21*B16^B16*B17^B17)-E2*B12-E3*B13	-0,00026
8	beta21	0,6									
9	beta22	0,4		x11	18,05042						
10	a11	0,2		x12	56,314607						
11	a12	0,3		x21	17,281513						
12	a21	0,25		x22	15,404494						
13	a22	0,25									
14	gamma11	0,8		GDP	21,175						
15	gamma12	0,2									
16	gamma21	0,4									
17	gamma22	0,6									
18	v1	0,5									
19	v2	0,25									
20	mu1	1,649385									
21	mu2	1,960132									

Figure 5: GAMS Results – Including the Government Sector

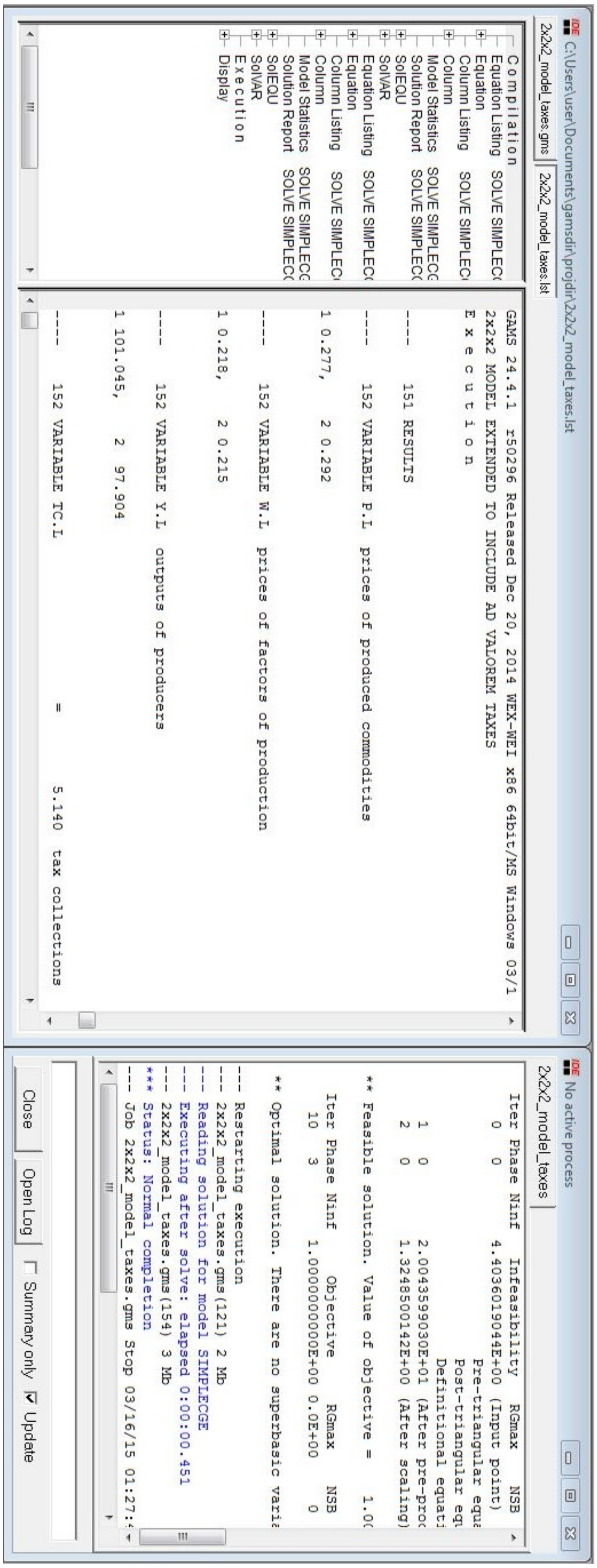

Figure 6: Testing the GAMS Results – Including the Government Sector

	A	B	C	D	E	F	G	H	I	J	K
1	coefficient	value		unknown	computed		equation	left side		right side	
2	zeta11	30		p1	0,277		1')	=E6	101,015	=B6*(E4*B2+E5*B3+E20*E8)/E2+B8*(E4*B4+E5*B5+E21*E8)/E2+B10*E6+B12*E7	101,0178
3	zeta12	20		p2	0,292		2')	=E7	97,904	=B7*(E4*B2+E5*B3)/E3+B9*(E4*B4+E5*B5)/E3+B11*E6+B13*E7	97,89351
4	zeta21	20		omega1	0,218		3)	=B2+B4	50	=B7*(E4*B2+E5*B3+E20*E8)/E3+B9*(E4*B4+E5*B5+E21*E8)/E3+B11*E6+B13*E7	50,00351
5	zeta22	5		omega2	0,215		4)	=B3+B5	25	=(B18*E6/B20)*(B15*E4/(B14*E5))^B14+(B19*E7/B21)*(B17*E4/(B16*E5))^B16	24,98133
6	beta11	0,3		y1	101,015		5')	=0	0	=E2-(1+E18)*(B18*E4^B14*E5^B15/(B20*B14^B14*B15^B15)+E2*B10+E3*B11)	0,000132
7	beta12	0,7		y2	97,904		6')	=0	0	=E3-(1+E19)*(B19*E4^B16*E5^B17/(B21*B16^B16*B17^B17)+E2*B12+E3*B13)	-0,0001
8	beta21	0,6		T	5,14		7)	=E8	5,14	=E18*E6*(B18*E4^B14*E5^B15/(B20*B14^B14*B15^B15)+E2*B10+E3*B11)+ +E19*E7*(B19*E4^B16*E5^B17/(B21*B16^B16*B17^B17)+E2*B12+E3*B13)	5,142359
9	beta22	0,4									
10	a11	0,2		x11	14,523466						
11	a12	0,3		x12	32,14726						
12	a21	0,5		x21	17,33935						
13	a22	0,25		x22	10,965753						
14	gamma11	0,8									
15	gamma12	0,2		GDP	21,415		GDP real	18,74396			
16	gamma21	0,4									
17	gamma22	0,6		coefficient	value						
18	v1	0,5		tau1	0,1						
19	v2	0,25		tau2	0,1						
20	mu1	1,649385		delta1	0,5						
21	mu2	1,960132		delta2	0,5						