Charles University in Prague
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DOCTORAL THESIS

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Inhomogeneous cosmology and averaging methods

Institute of Theoretical Physics

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Study programme:	F–1
Specialization:	Theoretical physics

Prague 2014

I would like to thank RNDr. Otakar Svítek, Ph.D. for giving me opportunity to work on this topic and for his guidance during my studies. I would like also to thank prof. A. A. Coley, prof. M. Bradley, prof. R. A. Sussman, Jakub Hruška, Robert Švarc and David Vrba for the discussions, prof. J. E. Åman and prof. M. Bradley for providing us the algebraic program SHEEP. I would like also to mention Radim Kusák and Jiří Eliášek for the help with computer programs.

I would like also to thank to my parents. Without their support my studies would have been more difficult. I can not forget to mention my friends who encouraged me during my studies.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Nehomogenní kosmologie a středovací metody

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Katedra: Ústav teoretické fyziky

Vedoucí disertační práce: RNDr. Otakar Svítek, Ph.D., ÚTF

Abstrakt: V této práci jsme prozkoumali různé středovací metody v obecné teorii relativity a kosmologii. Vyvinuli jsme metodu založenou na středování Cartanových skalárů. Vypočetli jsme backreakci pro plochý LTB model se speciální volbou radiální funkce, která má stejné chování jako kladná kosmologická konstanta. V další části této disertační práce jsme zkoumali středování prostoročasů LRS třídy II se zdrojem ve formě prachu. Pro tuto třídu prostoročasů jsme středovali všechny Einsteinovy rovnice a výsledný systém rovnic zobecňuje Buchertovy rovnice. Numericky jsme zkoumali dva modely, ve kterých decelerační parametr mění své znaménko z kladného na záporné.

Klíčová slova: nehomogenní kosmologie, středovací metody, temná hmota, temná energie

Title: Inhomogeneous cosmology and averaging methods

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Abstract: In this work we have examined different methods of averaging in general relativity and cosmology. We developed the method based on Cartan scalars. We computed the backreaction term for a flat LTB model with a special ansatz for the radial function. We found out that it behaves as a positive cosmological constant. In the next part of this thesis we were interested in averaging inside LRS class II dust model. For this family we averaged all the Einstein equations and the resulting system generalizes the Buchert equations. We numerically worked out two concrete examples where deceleration parameter changes its sign from positive to negative.

Keywords: inhomogeneous cosmology, averaging methods, dark matter, dark energy

Contents

Introduction

The averaging problem in general relativity and cosmology is of fundamental importance. Observational facts about our universe tell us that on large enough scales the universe appears to be homogeneous and isotropic. However, gravitational physics is well tested on smaller scales, e.g. within our solar system or for the dynamics of the galaxies. In order to obtain equations which describe our universe on the largest scales it is reasonable to use some averaging procedure.

The problem is that we do not have any simple consistent and universal averaging procedure. Our task is to average not only the Einstein equations but also spacetime geometry. These are represented by tensors and it is not clear how to integrate tensor field on a curved manifold. This is due to the fact that during integration we are summing tensors defined in different spaces and by integrating tensor fields we do not obtain tensor object. The way how to overcome this step is for example to integrate bilocally extended objects or to restrict on integrating scalars.

The task is similar to the problem of averaging in electrodynamics. There we have microscopic Lorentz equations describing fluctuating electromagnetic field and by averaging it is possible to obtain "macroscopic"' Maxwell equations. In general relativity the problem is more difficult due to the fact that the Einstein equations are nonlinear so we can expect that these equations will be modified by the so called correlation term. This term is zero in usual consideration in cosmology, but for the correct treatment of inhomogeneities we should take this term into account. The question is under which conditions we can use FRW metric with the additional correlation terms in our approximation and when we are forced to consider exact inhomogeneous cosmological models.

Averaging in cosmology gained its popularity after the discovery of the acceleration of the universe. This phenomenon is usually explained by introducing elusive dark energy component, cosmological constant or by modifying gravitational dynamics. In the averaged Einstein equations we can recognize correlation term which does not need to satisfy usual energy conditions and there exist speculations that this term can at least partly explain observed acceleration. Another motivation comes from the so called coincidence problem. The question is why the omega factors representing matter and dark energy are today comparable despite the fact that they were in the past very different. They became comparable at the time when the first structures formed. From this fact we can speculate that dark energy is only effective term in the Einstein equations created because we did not take into account inhomogeneous structures [19].

In this thesis we will first introduce several averaging methods with an emphasis on the latest developments. We keep separate chapter for one of the most promising averaging method, so called Macroscopic Gravity. Then after short introduction to anisotropic cosmological models we briefly review exact inhomogeneous cosmological models. In the next chapter we include the articles about averaging by Cartan scalars, averaging within LRS class II family and a review article about averaging in general relativity and cosmology. In the attachments we present text which was not included in the article about Cartan scalars. These include fixed frame formalism and the algorithm how to compute minimal set of

Cartan scalars in NP formalism. Finally we attach the Cartan scalars computed for a flat LTB spacetime by the algebraic program SHEEP.

1. Averaging methods

1.1 Early times of the averaging methods

In this chapter we will review different methods dealing with averaging problem. The list of the relevant work on this topic up to year 1997 can be found in Krasiński's book Inhomogenous Cosmological Models [61]. We will omit one particular approach to averaging - so called Macroscopic Gravity (MG) developed by Zalaletdinov, which uses bilocal operators for defining averages and not only Einstein equations but also geometrical Cartan structure equations are averaged. We will consider this method in a separate chapter.

The history of averaging goes into early sixties by the work of Shirokov and Fisher [85]. They considered highly fluctuating metric tensor $g_{\mu\nu}$ and proposed the way of averaging by

$$\langle g_{\mu\nu}(x)\rangle = \frac{\int\limits_{\Omega} g\,(x+x')\sqrt{-g}\,d^4x'}{\int\limits_{\Omega}\sqrt{-g}\,d^4x'}, \tag{1.1}$$

where the domain Ω is large enough to contain nontrivial portion of the mass density. The problem of this definition is that the scheme is not covariant. Averaging of a tensor field does not result in a new tensor field. They considered on average homogeneous and isotropic model with a dust source and obtained macroscopic Einstein equations corrected by an additional terms which represent repulsive gravitational terms.

In a similar way Noonan [73], [74] extended this definition for a general tensor object

$$\langle Q\,(x)\rangle = \frac{\int\limits_{\Omega} Q'\,(x+x')\sqrt{-g}\,d^4x'}{\int\limits_{\Omega}\sqrt{-g}\,d^4x'}. \tag{1.2}$$

In the approximation of a weak field and a small velocity he found an extra contribution to the energy-momentum tensor. However, $g(x')$ here denotes determinant of the macroscopic metric and it is not clear how this macroscopic metric is constructed.

In the previous case authors were integrating tensors defined above different points. One way how to deal with this problem is to introduce some bilocal operator. In general relativity, there is a preferred bilocal operator associated with a parallel transport along geodesic. Tensor originally defined at a point x' is parallel transported to a point x (assuming that the points are close to each other so that geodesic is unique). Then, we have tensors defined in one particular point and we can proceed with averaging. The average value of the tensor field $A_{\mu\nu}$ is defined by

$$\langle A_{\mu\nu}(x)\rangle_{BH} = \frac{1}{V_\Omega}\int\limits_{\Omega} g^{\alpha'}_{\mu}(x,x')\,g^{\beta'}_{\nu}(x,x')\,A_{\alpha'\beta'}(x')\sqrt{-g(x')}\,d^4x', \tag{1.3}$$

where $g^{\alpha'}_{\mu}$ is the bilocal propagator. This method was used by Brill and Hartle [14] for the analysis of gravitational geons. Then, Isaacson applied this averaging scheme for computing backreaction of the high frequency gravitational waves [52]. Considering tensor perturbations $h_{\mu\nu}$, he obtained the following effective stress energy tensor for the high frequency gravitational wave.

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi}\left\langle h^{\rho\tau}{}_{;\mu}h_{\rho\tau;\upsilon}\right\rangle_{BH}. \tag{1.4}$$

It can be shown that the following properties hold:

- One can ignore the terms $\left\langle A_{\mu\nu}{}^{\rho}{}_{;\rho}\right\rangle_{BH}$.
- One can integrate by parts.
- Covariant derivatives commute.

An important step for understanding of the averaging problem was the article by Ellis [33]. He stressed the importance in defining different scales in cosmology and he also showed that in the macroscopic gravitational equations, there should be an additional correlation term.

Futamase [38], [39] used 3+1 splitting of spacetime for computations of averages. After using approximation scheme he utilizes Isaacson approach for averaging local inhomogeneities. He found that the resulting effective correlation term is in the form of a negative curvature.

1.2 Buchert equations

Nowadays the most popular approach for averaging is the one initiated by Buchert, which utilizes 3+1 splitting of spacetime and for a given domain defines the spatial averages. Nevertheless, the method is used for averaging of only scalar part of the Einstein equations. The averaged equations resemble Friedmann equation with an additional term. Here we will review the averaged model describing an irrotational dust [17] (see generalization for a perfect fluid in [18] and for a more general stress energy tensor in [15])

Let us suppose we have a metric tensor in the form $ds^2 = -dt^2 + g_{ij}dX^i dX^j$. For a given spacelike domain $\mathcal{D}$ we can define average value of a scalar Ψ by the prescription

$$\left\langle \Psi(t, X^i)\right\rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} Jd^3X\Psi(t, X^i), \tag{1.5}$$

$$V_{\mathcal{D}} = \int_{\mathcal{D}} Jd^3X, \tag{1.6}$$

where $J :=\det\sqrt{g_{ij}}$, g_{ij} is the metric of the spacelike hypersurface after the 3+1 splitting of spacetime and X^i are comoving coordinates following geodetic motion of a dust. As we can see from the form of the equation (1.5) the time derivative and averaging do not commute. We have the following simple relation

$$\partial_t \left\langle \Psi(t, X^i)\right\rangle_{\mathcal{D}} - \left\langle \partial_t\Psi(t, X^i)\right\rangle_{\mathcal{D}} = \left\langle \Psi(t, X^i)\right\rangle_{\mathcal{D}}\left\langle \Theta\right\rangle_{\mathcal{D}} - \left\langle \Psi(t, X^i)\Theta\right\rangle_{\mathcal{D}}. \tag{1.7}$$

Expansion Θ is related to a four velocity of the fluid u_μ by the relation $\Theta = u^\mu{}_{;\mu}$. With this expression we can, in a similar way as in cosmology, define effective Hubble parameter $H_\mathcal{D}$

$$\langle\Theta\rangle_\mathcal{D} = \frac{\dot{V_\mathcal{D}}}{V_\mathcal{D}} = 3\frac{\dot{a_\mathcal{D}}}{a_\mathcal{D}} =: 3H_\mathcal{D}, \tag{1.8}$$

and an effective scale factor $a_\mathcal{D}$

$$a_\mathcal{D} = \left(\frac{V_\mathcal{D}}{V_{\mathcal{D}i}}\right)^{\frac{1}{3}}. \tag{1.9}$$

The dot here denotes partial derivative with respect to time and $V_{\mathcal{D}i}$ is the initial volume which evolved geodetically to the volume $V_\mathcal{D}$. Now we have the rules how to average scalar equations. The problem is that Einstein equations are expressed by tensor fields. In this moment we are able to average only the scalar part of the equations. To obtain scalar equations we have to contract Einstein equations with an available geometrical objects - $g^{\mu\nu}, u^\mu$ and ∇^μ. After averaging and using commutation formula (1.7) we obtain modified Raychaudhuri equation, Hamiltonian constraint and the mass conservation equation.

$$3\frac{\ddot{a_\mathcal{D}}}{a_\mathcal{D}} + 4\pi G\,\langle\rho\rangle_\mathcal{D} - \Lambda = \mathcal{Q}_\mathcal{D}, \tag{1.10}$$

$$\left(\frac{\dot{a}_\mathcal{D}}{a_\mathcal{D}}\right)^2 - \frac{8\pi G}{3}\langle\rho\rangle_\mathcal{D} + \frac{\langle\mathcal{R}\rangle_\mathcal{D}}{6} - \frac{\Lambda}{3} = -\frac{\mathcal{Q}_\mathcal{D}}{6}, \tag{1.11}$$

$$\partial_t\langle\rho\rangle_\mathcal{D} + 3\frac{\dot{a_\mathcal{D}}}{a_\mathcal{D}}\langle\rho\rangle_\mathcal{D} = 0. \tag{1.12}$$

$\langle\mathcal{R}\rangle_\mathcal{D}$ here denotes average value of the spatial Ricci scalar, $\langle\rho\rangle_\mathcal{D}$ average value of the dust density and the term $\mathcal{Q}_\mathcal{D}$, so called kinematic backreaction, describes the deviation from homogeneity and isotropy and is defined by the relation

$$\mathcal{Q}_\mathcal{D} := \frac{2}{3}\left\langle(\Theta - \langle\Theta\rangle_\mathcal{D})^2\right\rangle_\mathcal{D} - 2\left\langle\sigma^2\right\rangle_\mathcal{D}. \tag{1.13}$$

A shear scalar $\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}$ is built from a shear tensor σ_{ij}. If we perform time derivative of the Hamiltonian constraint, the resulting equation agrees with the Raychaudhuri equation if the following integrability condition holds

$$\partial_t\mathcal{Q}_\mathcal{D} + 6\frac{\dot{a}_\mathcal{D}}{a_\mathcal{D}}\mathcal{Q}_\mathcal{D} + \partial_t\langle\mathcal{R}\rangle_\mathcal{D} + 2\frac{\dot{a}_\mathcal{D}}{a_\mathcal{D}}\langle\mathcal{R}\rangle_\mathcal{D} = 0. \tag{1.14}$$

As we can see, these equations do not close. Later we will review some models where the equations are closed. The previous equations can be rewritten in a form which resembles FRW cosmological equations. If we introduce an effective density and an effective pressure

$$\rho^\mathcal{D}_{eff} := \langle\rho\rangle_\mathcal{D} - \frac{1}{16\pi G}\mathcal{Q}_\mathcal{D} - \frac{1}{16\pi G}\langle\mathcal{R}\rangle_\mathcal{D}, \tag{1.15}$$

$$p^\mathcal{D}_{eff} := -\frac{1}{16\pi G}\mathcal{Q}_\mathcal{D} + \frac{1}{48\pi G}\langle\mathcal{R}\rangle_\mathcal{D}, \tag{1.16}$$

we can rewrite equations into the form

$$3\frac{\ddot{a_{\mathcal{D}}}}{a_{\mathcal{D}}} + 4\pi G\left(\rho^{\mathcal{D}}_{eff} + 3p^{\mathcal{D}}_{eff}\right) - \Lambda = 0, \tag{1.17}$$

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - \frac{8\pi G}{3}\rho^{\mathcal{D}}_{eff} - \frac{\Lambda}{3} = 0, \tag{1.18}$$

$$\dot{\rho}^{\mathcal{D}}_{eff} + 3\frac{\dot{a_{\mathcal{D}}}}{a_{\mathcal{D}}}\left(\rho^{\mathcal{D}}_{eff} + p^{\mathcal{D}}_{eff}\right) = 0. \tag{1.19}$$

To close the equations, we have to characterize an equation of state $p^{\mathcal{D}}_{eff} = p^{\mathcal{D}}_{eff}\left(\rho^{\mathcal{D}}_{eff}, a_{\mathcal{D}}\right)$. The form of the effective density and the pressure shows similarity with results in the theory of scalar fields. New sources $\mathcal{Q}_{\mathcal{D}}$ and $\langle\mathcal{R}\rangle_{\mathcal{D}}$ can be interpreted as a result of an additional scalar field, so called morphon field [21], whose Klein-Gordon equation is the integrability equation (1.14).

In a similar way as in FRW cosmology we can define dimensionless parameters (omega factors), which represent functionals for a domain $\mathcal{D}$.

$$\Omega^{\mathcal{D}}_m := \frac{8\pi G}{3H^2_{\mathcal{D}}}\langle\rho\rangle_{\mathcal{D}}; \quad \Omega^{\mathcal{D}}_{\Lambda} := \frac{\Lambda}{3H^2_{\mathcal{D}}}; \quad \Omega^{\mathcal{D}}_{\mathcal{R}} := -\frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{6H^2_{\mathcal{D}}}; \quad \Omega^{\mathcal{D}}_{\mathcal{Q}} := -\frac{\mathcal{Q}_{\mathcal{D}}}{6H^2_{\mathcal{D}}} \tag{1.20}$$

and the Hamiltonian constraint can be rewritten into the standard form.

Buchert's approach consists of selecting preferred spatial hypersuface and performing the averaging of scalars in the rest frame of the fluid. Larena [64] generalized the method into an arbitrary reference frame. Beside four-velocity of the fluid u^μ he introduced also four-velocity of an arbitrary observer n^μ and Buchert equations contain apart from kinematic backreaction (and dynamical backreaction for the perfect fluid with nonzero pressure) other backreaction terms.

The problem remains how to close the system of equations. In [21] the so called scaling solutions were considered. From the continuity equation the averaged dust matter density evolves as

$$\langle\rho\rangle_{\mathcal{D}} = \langle\rho\rangle_{\mathcal{D}_i} {a_{\mathcal{D}}}^{-3} \tag{1.21}$$

To close the equations they considered an ansatz for the averaged curvature term and the kinematic backreaction term. The scaling solutions read

$$\mathcal{Q}_{\mathcal{D}} = \mathcal{Q}_{\mathcal{D}_i}{a_{\mathcal{D}}}^n, \tag{1.22}$$

$$\langle\mathcal{R}\rangle_{\mathcal{D}} = \mathcal{R}_{\mathcal{D}_i}{a_{\mathcal{D}}}^p. \tag{1.23}$$

$\mathcal{Q}_{\mathcal{D}_i}$ and $\mathcal{R}_{\mathcal{D}_i}$ here denotes initial value of the $\mathcal{Q}_{\mathcal{D}}$ and $\langle\mathcal{R}_{\mathcal{D}}\rangle$. We can plug this form into the integrability equation (1.14) and try to find the suitable indices n and p. It can be shown that one of the possible solutions reads

$$\mathcal{Q}_{\mathcal{D}} = \mathcal{Q}_{\mathcal{D}_i}{a_{\mathcal{D}}}^{-6}, \langle\mathcal{R}\rangle_{\mathcal{D}} = \mathcal{R}_{\mathcal{D}_i}{a_{\mathcal{D}}}^{-2}. \tag{1.24}$$

This is the only solution with $n \neq p$. In this solution averaged curvature and backreaction evolve independently. Scaling of the averaged curvature is similar to a constant curvature term in the Friedmann model. For consideration of the scaling solution of the second type $n = p$ a new backreaction parameter r is introduced

$$\mathcal{Q}_{\mathcal{D}} = r\langle\mathcal{R}\rangle_{\mathcal{D}} = r\mathcal{R}_{\mathcal{D}_i}{a_{\mathcal{D}}}^n, \tag{1.25}$$

where

$$n = -2\frac{(1+3r)}{1+r}. \tag{1.26}$$

Here we consider the case with $r \neq -1$ ($r = -1$ represents Friedman model with the vanishing curvature and backreaction). If we interpret the backreaction terms as an effective one resulting from the morphon field, this scalar field is characterized by an effective equation of state

$$w_{\Phi}^{\mathcal{D}} = -\frac{1}{3}\frac{(1-3r)}{(1+r)} = -\frac{1}{3}(n+3). \tag{1.27}$$

Scaling solutions for $n = p$ were further investigated in [84]. The phase portrait of the solutions were examined. They discussed instability of the averaged scaling solutions, which drives averaged model away from the FRW. The equations which were investigated were rewritten for the effective omega factors (1.20). They used instead of $\Omega_{\mathcal{Q}}^{\mathcal{D}}$ a new omega factor $\Omega_X^{\mathcal{D}} = -\frac{X_{\mathcal{D}}}{6H_{\mathcal{D}}^2}$, where for scaling solution with $n = p$ $X_{\mathcal{D}}$ can be computed by

$$(n+2)X_{\mathcal{D}} = -4\mathcal{Q}_{\mathcal{D}} \tag{1.28}$$

The resulting Buchert equations for the scaling solutions $n = p$ are

$$\Omega_m^{\mathcal{D}} - (n+2)\Omega_X^{\mathcal{D}} = 2q_{\mathcal{D}}, \tag{1.29}$$

$$\Omega_m^{\mathcal{D}} + \Omega_k^{\mathcal{D}} + \Omega_X^{\mathcal{D}} = 1, \tag{1.30}$$

$$\Omega_m^{\mathcal{D}\prime} = \Omega_m^{\mathcal{D}}(\Omega_m^{\mathcal{D}} - (n+2)\Omega_X^{\mathcal{D}} - 1), \tag{1.31}$$

$$\Omega_X^{\mathcal{D}\prime} = \Omega_X^{\mathcal{D}}(\Omega_m^{\mathcal{D}} - (n+2)\Omega_X^{\mathcal{D}} + n + 2), \tag{1.32}$$

where $\Omega_k^{\mathcal{D}}$ is the omega factor representing constant curvature term defined by $\Omega_k^{\mathcal{D}} := -\frac{k_{\mathcal{D}i}}{a_{\mathcal{D}}^2 H_{\mathcal{D}}^2}$. The prime here denotes derivative with respect to the evolution parameter $N_{\mathcal{D}} := \ln a_{\mathcal{D}}$. In [84] the phase portrait of these autonomous equations is investigated. They determined fixed points of the system and its stability.

The next possible way how to close the system of equations is to consider an exact solution. In this case we have an exact solution of the unaveraged equations. We can compute averages according to the rule (1.5), investigate backreaction terms and consider if they can lead to a negative deceleration parameter or examine its effective equation of state.

In [79] Paranjape and Singh investigated backreaction inside the LTB model. This is an exact spherically symmetric solution of the Einstein equations with a dust source. They found that for a vanishing spatial curvature kinematic backreaction term is equal to zero. On the other hand they constructed curvature dominated model, numerically integrated backreaction term and they showed that the solution exhibits acceleration.

LTB spacetimes are often criticized for its simple spherically symmetric configuration and for putting observer near to center of the large void, which is in contrast with the Copernican principle. Generalization of this model is quasispherical Szekeres metric, where spherical shells are not concentric and a mass distribution is a dipole superposed on a monopole. Bolejko investigated [8] backreaction inside quasispherical Szekeres model and showed that in computation of the averaged deceleration parameter dipole distribution does not contribute and the resulting expression behaves in a similar way as in the LTB model.

Another interesting example to explore is to look at the backreaction within a perturbed FRW model. Li and Schwarz [66] closed the system of equations by considering second order perturbation around a flat FRW dust model in a synchronous gauge. They computed a kinematic backrection $\mathcal{Q}_{\mathcal{D}}$, an averaged curvature term $\mathcal{R}_{\mathcal{D}}$, an averaged density $\rho_{\mathcal{D}}$, an averaged expansion $\langle\theta\rangle_{\mathcal{D}}$ and an effective equation of state w_{eff}. In [67] Li and Schwarz considered scale dependence of these quantities. They found that backreaction effects leads to the observational effects up to the scales of $\approx 200Mpc$, but these effects are not able to explain observed acceleration.

The work on perturbation around a flat FRW spacetime continued in an article of Behrend et al [4]. They considered linearly perturbed metric in a Newtonian gauge, where also dynamical backreaction is present. They found that contribution of backreaction is small of the order of 10^{-5} for both ΛCDM and Einstein de-Sitter and the slowly varying effective equation of state $w \approx -1/19$. In [15] authors considered situation with a more general fluid. They included dust, radiation and dark energy. They estimated omega factor representing backreaction and the effective equation of state of a backreaction fluid. For ΛCDM model they found that the backreaction is of order 4×10^{-6}. They found that the effective equation of state for ΛCDM, EdS and quintessence model is positive, i.e. describing dust-like source. The exception is strongly phantom model where the backreaction has the effective equation of state $w_{eff} < -1/3$, i.e. causing acceleration. The discussion of gauge issues with a more general coordinate system can be found in [16]. They wrote backreaction terms in unspecified gauge. Then they computed backreaction terms in uniform curvature and Newtonian gauges.

Finally, fully consistent treatment of the second order perturbation in a Poisson gauge can be found in [25]. They found that there exists homogeneity scale. Above this scale averaged Hubble parameter becomes independent on the averaging scale with a negligible variance and is corrected by the value 10^{-5}. They also computed variance and the mean value of a deccelaration parameter. They found that variance can be large bellow homogeneity scale.

Räsänen showed in his simple model [83] that it is possible to obtain negative averaged deceleration parameter despite the fact that deccelation parameter is localy non-negative. He considered a simple toy model of a gravitational collapse. It consisted of two disjoint regions: The first one was an empty space representing a void with a scale factor a_1. The second one simulated collapsing structure with a scale factor a_2. The overall scale factor was given by $a^3 = a_1^3 + a_2^3$. Corresponding deceleration parameter started from possivive value, crossed the zero and finally became negative.

1.3 Ricci flow

In the last section where we introduced Buchert's approach the average quantities were defined on a real inhomogeneous manifold $\mathcal{M}$. We have seen that in a perturbed FRW model the contribution from the kinematic backreaction is not sufficiently large. On the other hand, cosmological data are interpreted in a symmetric FRW model. We need to relate geometrical objects defined on a real inhomogeneous manifold and on an averaged manifold. One possible way how to achieve this is to consider Ricci flow. If we perform smoothing of spacetime geometry the omega factors change. We can speculate that this change of omega factors can be important and that the omega factor responsible for backreaction can play a nontrivial role due to the smoothing of geometry.

Besides averaging of the Einstein equations we should also perform averaging of the spacetime geometry. The theory of MG deals with this problem by averaging of the Cartan structure equations. Alternatively we could average scalar invariants created by Riemann tensor and the final number of its covariant derivatives or we could average Cartan scalars. There exists mathematically interesting alternative how to smooth out the geometry of spacetime and how to obtain spaces of constant curvature. It utilizes the technique of the Ricci deformational flow [47]. An important contribution to this topic was given by Carfora and Piotrkowska [23] who showed the relation between a method of a renormalization group and a critical phenomenons in cosmology.

Let us have a given metric g_{ab} on a closed three manifold without boundary, which depends on a parameter β (typically cosmological time) and let it evolve in the direction of a Ricci tensor

$$\frac{\partial}{\partial \beta} g_{ab}(\beta) = -2R_{ab}(\beta),$$

$$g_{ab}(\beta = 0) = g_{ab}, \quad 0 \leq \beta \leq T_0. \tag{1.33}$$

It can be shown that the local solution on a compact three manifold always exists. Moreover, if the initial three metric has a positive Ricci scalar then the solution exists for all β and converge exponentially quickly to the space of a constant curvature (technical details ad references can be found in [20]).

This procedure changes also other cosmological parameters. For example averaged mass density will change from $\langle\rho\rangle_{\mathcal{D}_0}$ to $\langle\rho\rangle_{\overline{\mathcal{D}}} = M_{\overline{\mathcal{D}}}/V_{\overline{\mathcal{D}}}$ after smoothing of the region $\mathcal{D}_0$ to $\overline{\mathcal{D}}$. Due to the mass conservation during the deformation of geometry these densities are not the same

$$\langle\rho\rangle_{\mathcal{D}_0} = \langle\rho\rangle_{\overline{\mathcal{D}}} \frac{V_{\overline{\mathcal{D}}}}{V_{\mathcal{D}_0}}. \tag{1.34}$$

In the same way we could obtain the whole set of renormalized omega factors which can be very different from the initial one. We come to the conclusion that despite the low value of the omega factor $\Omega_{\mathcal{Q}}^{\mathcal{D}}$ the proportion between omega factors can change due to the metric deformation.

Finally we can separate 3+1 averaging in two steps: The time evolution (deformation in the direction of an exterior curvature) and the scale evolution (deformation in the direction of spatial Ricci curvature).

1.4 Gauge invariant averages

In this section we will review the construction of a covariant and gauge invariant averaging. For the definitions and notations we will follow an article by Gasperini, Marozzi and Veneziano [41].

If we have a scalar function $S(x)$ it changes under the general coordinate transformation $x \to \tilde{x} = f(x)$ as $S(x) \to \tilde{S}(\tilde{x})$, where

$$\tilde{S}(\tilde{x}) = S(x) \tag{1.35}$$

Under the gauge transformation old and new function is evaluated in the same point, but the function changes its form. To be more concrete under gauge transformation $S(x)$ transforms as $S(x) \to \tilde{S}(x)$, where

$$\tilde{S}(x) = S(f^{-1}x). \tag{1.36}$$

From this expression we can see that the scalar function $S(x)$ is not gauge invariant. In order to define covariant and gauge invariant average value of a scalar function we introduce a window function $W_\Omega(x)$ which we consider as a dynamical field (Ω is four dimensional region in spacetime). The window function $W_\Omega(x)$ behaves as a scalar and changes the transformation properties of the integrand so that the integration over a scalar $S(x)$ is gauge invariant and covariant and is defined by the following formula.

$$F(S,\Omega) = \int\limits_{\mathcal{M}_4} d^4x \sqrt{-g(x)} S(x) W_\Omega(x). \tag{1.37}$$

We will now review covariant and gauge invariant version of the Buchert-Ehlers commutation rule and the corresponding generalization of the Buchert equations[42]. We choose a domain determined by spacelike hypersurface $\Sigma(A)$ over which a scalar field $A(x)$ takes a constant value A_0. Boundary is defined by the condition $B(x) < r_0$, where $B(x)$ is a scalar function with a spacelike gradient and r_0 is a positive constant. The appropriate window function reads

$$W_\Omega(x) = n^\mu \nabla_\mu \theta[A(x) - A_0]\theta[r_0 - B(x)], \tag{1.38}$$

where A_0 is a constant that determines spacelike hypersuface over which the integral is given and n^μ is the future-directed unit normal to $\Sigma(A)$. The gauge invariant and covariant integral (1.37) reads

$$F(S, A_0) = \int\limits_{\mathcal{M}_4} d^4x \sqrt{-g(x)} \delta[A(x) - A_0](-\partial_\mu A \partial^\mu A)^{1/2} \theta[r_0 - B(x)] S(x). \tag{1.39}$$

Then it is possible to define average value of the scalar field $S(x)$ over the hypersurface of constant $A(x)$ as

$$\langle S \rangle_{A_0} = \frac{F(S, A_0)}{F(1, A_0)} \tag{1.40}$$

If we perform "time" derivative of this integral we receive gauge invariant and covariant generalization of the Buchert-Ehlers commutation rule

$$\frac{\partial \langle S \rangle_{A_0}}{\partial A_0} = \left\langle \frac{\partial_\mu A \partial^\mu S}{\partial_\mu A \partial^\mu A} \right\rangle_{A_0} + \left\langle \frac{S\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0} - \langle S \rangle_{A_0} \left\langle \frac{\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0}. \tag{1.41}$$

Θ denotes expansion scalar defined by $\nabla_\mu n^\mu$. Partial derivative with respect to A_0 is the analog of the time derivative. As in the Buchert formalism we can define effective scale factor $\tilde{a}$ by

$$\frac{1}{\tilde{a}}\frac{\partial \tilde{a}}{\partial A_0} = \frac{1}{3}\left\langle \frac{\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0} . \tag{1.42}$$

The generalization of the Buchert equations (1.10) and (1.11) read

$$\begin{aligned} \left(\frac{1}{\tilde{a}}\frac{\partial \tilde{a}}{\partial A_0}\right)^2 &= \frac{8\pi G}{3}\left\langle \frac{\epsilon}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} - \frac{1}{6}\left\langle \frac{R_s}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} \\ &- \frac{1}{9}\left[\left\langle \frac{\Theta^2}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} - \left\langle \frac{\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0}^2\right] \\ &+ \frac{1}{3}\left\langle \frac{\sigma^2}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} , \end{aligned} \tag{1.43}$$

$$\begin{aligned} -\frac{1}{\tilde{a}}\frac{\partial^2 \tilde{a}}{\partial A_0^2} &= \frac{4\pi G}{3}\left\langle \frac{\epsilon + 3\pi}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} - \frac{1}{3}\left\langle \frac{\nabla^\nu (n^\mu \nabla_\mu n_\nu)}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} \\ &+ \frac{1}{6}\left\langle \frac{\partial_\mu A \partial^\mu (\partial_\nu A \partial^\nu A)}{(-\partial_\mu A \partial^\mu A)^{5/2}}\Theta \right\rangle_{A_0} + \frac{2}{3}\left\langle \frac{\sigma^2}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} \\ &- \frac{9}{2}\left[\left\langle \frac{\Theta^2}{(-\partial_\mu A \partial^\mu A)} \right\rangle_{A_0} - \left\langle \frac{\Theta}{(-\partial_\mu A \partial^\mu A)^{1/2}} \right\rangle_{A_0}^2\right] , \end{aligned} \tag{1.44}$$

where ϵ is ADM energy density defined by $\epsilon = T_{\mu\nu} n^\mu n^\nu$, π is ADM pressure defined by $\pi = T_{\mu\nu} h^{\mu\rho} h^\nu_\rho / 3$ and σ is shear scalar. The advantage of these equations is that the averaged quantities can be computed in an arbitrary coordinate system and that they are gauge invariant.

1.5 Timescale cosmology

An interesting approach to averaging is the method based on Buchert formalism and developed by Wiltshire [96], [97], [98]. He introduced two scale model and the Buchert equations are explicitly solved. An observer is situated in a bound system which effectively decouples from the ambient cosmological expansion. From the observational cosmology we know that the large scale structure consists of voids surrounded by filaments. The metric near the center of the void reads.

$$ds^2_{D_c} = -d\tau_v^2 + a_v^2(\tau_v)\left[d\eta_v^2 + \sinh^2(\eta_v) d\Omega^2\right] . \tag{1.45}$$

Next, he considered, following work of Ellis [33], the notion of finite infinity. It is a timelike surface within which the dynamics of an isolated system such as the solar system can be treated without reference to the rest of the universe. The finite infinity is defined in terms of a scale over which the average expansion is zero, while it is positive outside [96]. Region inside finite infinity resembles universe at the time of last scattering where the metric can be described by a flat FRW metric

$$ds^2_{fi} = -d\tau_w^2 + a_w^2(\tau_w)\left[d\eta_w^2 + (\eta_w^2) d\Omega^2\right] . \tag{1.46}$$

In Buchert equations the scale factor $\bar{a}(t)$ is not described by $a_v(\tau_v)$ or $a_w(\tau_w)$ but by the combination $\bar{a}^3 = f_{vi}^3 a_v^3 + f_{wi}^3 a_w^3$. Here, t denotes averaged time parameter appearing in Buchert equations, f_{vi} and $f_{wi} = 1 - f_{vi}$ denotes initial fractions of void and wall regions. The metric which represents averaged geometry reads [97]

$$ds^2 = -dt^2 + \bar{a}^2(t)d\bar{\eta}^2 + A(\bar{\eta}, t)d\Omega^2. \tag{1.47}$$

We can see that the time t is different from τ_w by $dt = \bar{\gamma}(t)d\tau_w$, where $\bar{\gamma}(t)$ is the mean lapse function. In order to relate the finite infinite geometry (1.46) with the averaged geometry (1.47), these two are connected by conformal matching of radial null geodesics. The bare cosmological parameters are defined with respect to variables in (1.47). Despite the fact that the bare deceleration parameter is positive, the dressed deceleration parameter (related to (1.46)) can be found to be negative [96].

1.6 Averaging by scalar invariants

On a given manifold $(\mathcal{M}, g_{\alpha\beta})$ we are able to average scalar functions. The question is how we can characterize manifold by the set of scalars. Coley, Hervik and Pelavas showed [29] that the class of four dimensional Lorentzian manifolds which can not be completely characterized by invariants constructed from Riemann tensor and the finite number of its covariant derivatives is necessarily of Kundt type (i.e. spacetime admitting geodetic null vector field with zero expansion rotation and shear).

For a given spacetime $(\mathcal{M}, g_{\alpha\beta})$ they defined the set of scalar invariants

$$\mathcal{I} \equiv \left\{ R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}, R_{\mu\nu\alpha\beta;\gamma}R^{\mu\nu\alpha\beta;\gamma}, R_{\mu\nu\alpha\beta;\gamma\delta}R^{\mu\nu\alpha\beta;\gamma\delta}, ... \right\}. \tag{1.48}$$

If we integrate these scalar functions over a given domain Ω we receive a set $\bar{\mathcal{I}}$ which characterizes new (macroscopic) geometry. The problem is that relations of the type $\overline{R_{\mu\nu}R^{\mu\nu}} = \bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$ do not hold and a metric tensor $\bar{g}_{\mu\nu}$ associated with the scalar functions $\bar{\mathcal{I}}$ does not necessarily exist. Coley dealt with this problem in the following way [28]: He removed from the set $\mathcal{I}$ algebraically independent functions and received the set $\mathcal{I}_A \subseteq \mathcal{I}$. Then he removed the functions which can be derived from the equations characterizing given spacetime ('syzygies') to receive another subset $\mathcal{I}_{SA} \subseteq \mathcal{I}_A$. Next he continued with averaging of the set $\mathcal{I}_{SA}$ and received a new set $\bar{\mathcal{I}}_{SA}$. He assumed that the constraints remain the same in the average manifold as in the unaveraged manifold and by inverse procedure, it is possible to receive the set $\bar{\mathcal{I}}$ which characterize completely the averaged manifold.

In the end of [28] there is a concrete example of a static spherically symmetric model with a stress energy tensor of a perfect fluid.

$$ds^2 = -e^{f(r)}dt^2 + e^{f(r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{1.49}$$

The set $\mathcal{I}_{SA}$ here consists of two terms. After averaging it is possible to build $\bar{\mathcal{I}}_{SA}$ from the averaged Ricci tensor $\bar{R}_{\mu\nu}$ and a small correction which can be interpreted as an additional spatial curvature.

1.7 Further averaging methods

An interesting approach for averaging was developed by Brannlund et al [13]. This method is used for averaging tensor fields. Similarly as in Macroscopic Gravity tensors are parallelly transported into one point before integration. However, here they did not used Levi-Cevita connection, but the Weitzenböck connection. This is a connection with non-zero torsion and zero curvature. In order to construct averaged geometry they started by averaging spin connection. They speculated that this kind of averaging method could play an important role in averaging modified gravity, namely Einstein-Cartan theory. The averaging process is fully covariant and mathematically well defined.

Korzyński defined coarse-graining of inhomogeneous dust flow using isometric embedding theorem for S^2 [60], [59]. This theorem reads: *Given a compact, orientable surface S homeomorphic to S^2, with positive metric q whose scalar curvature $R > 0$. Then there exists an isometric embedding $f : S \to E^3$ into the 3–dimensional Euclidean space and the embedding is unique up to rigid rotations, translations and reflexions.* The geometrical object which is averaged is the gradient of velocity. Isometric embedding theorem enables us to identify the boundary of the averaging domain with a domain in three dimensional Euclidean space. Inspired by divergence theorem which holds in Newtonian cosmology, the average of the velocity gradient is rewritten by the integral over the boundary homeomorphic to S^2 which can be realized as a domain in E^3, where the integration can be performed.

Hellaby modeled inhomogeneities of a gravitational field by gluing together different spacetimes which creates regular lattice [48]. In his work he considered different Kasner regions and its non-vacuum generalizations. These belong to flat anisotropic Bianchi models of type I. The Kasner-type metric reads

$$ds^2 = -dt^2 + t^{2\alpha}dx^2 + t^{2\beta}dy^2 + t^{2\gamma}dz^2. \tag{1.50}$$

In order to have an exact solution of the Einstein equations, these regions must fulfill Darmois junction conditions. This requirement forces two Kasner indices on the tangent plane of the crossing surfaces to be the same. The third index perpendicular to the crossing surface can be different for two joining surfaces. Hellaby gave an example of a cubic lattice built from 2x2x2 blocks. He considered three different examples of the building Kasner blocks. He computed volume for each of eight component regions and its time derivatives and defined an averaged expansion rate and an averaged deceleration parameter constructed by a volume-weighted average and he showed how these functions evolve in time (despite different time evolution of each block).

Sussman used quasi-local averaging which is valid for LTB spacetime [87]. Quasi-local average is defined as an ordinary average but weighted by the function of curvature. The resulting average behaves as in flat LTB model. As a result backreaction term is equal to zero and the equations describing LTB model resemble those of FRW model (i.e. they are without additional terms). The LTB model can be completely described in terms of the quasi-local functions. The theory was applied to investigation of the evolution of radial profiles in regular LTB spacetime [88], backreaction and effective acceleration in LTB model [89] or investigation of invariant characterization of the growing and decaying density modes in LTB models.

1.8 Backreaction problem

Averaging problem is connected with the so called backreaction. If we have a particular averaging method, we can average Einstein equations and spacetime geometry. Then, we can identify correlation term which describes influence of inhomogeneities on the scale factor in the case of the averaged FRW universe. This impact of inhomogeneities on the scale factor is called backreaction. The correlations term can be created as follows. If we have some unspecified rule for averaging tensors, we could average both sides of the Einstein equations. In the same time it would be possible to average metric tensor. In most of the applications of general relativity we use averaged metric tensor. The correct averaged Einstein equation should contain correlation term and the equations will be modified

$$E_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi \langle T_{\mu\nu}\rangle + C_{\mu\nu}, \tag{1.51}$$

where correlation term is defined by the construction

$$C_{\mu\nu} = E_{\mu\nu}(\langle g_{\mu\nu}\rangle) - \langle E_{\mu\nu}(g_{\mu\nu})\rangle . \tag{1.52}$$

In this sections we will consider different approaches for backreaction problem without particular reference to the averaging method.

In recent years there existed speculations that the backreaction from the fluctuations created during inflation with the wavelengths bigger than Hubble scale were responsible for the effect of an acceleration of the universe. In [1] Barausse et al. computed luminosity distance-redshift relation in a perturbed flat matter-dominated Universe. They came with a conclusion that the effects of backreaction can mimic dark energy term. The same conclusion was found in the work of Kolb et al. [57]. They found that super-horizon modes were responsible for a large variance of the deceleration parameter and were able to account for the accelerated expansion. However Geshnizjani et al. found [43] that the term suggested to cause acceleration only leads to a renormalization of local spatial curvature, and thus cannot account for the negative deceleration. Nowadays it is believed that this term can not be responsible for the effect of acceleration of the universe.

Usual estimation of the impact of backreaction in inhomogeneous cosmology is that this effect is too small to lead to the observed acceleration of the universe. This assumptions were confirmed in the perturbed FRW models in Buchert approach where the effective omega factor responsible for backreaction led only to small correction. This assumption was made precise in the work of Ishibashi and Wald [53]. They assumed that they have a given Newtonianly perturbed FRW spacetime

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1+2\Phi)\gamma_{ij}dx^i dx^j, \tag{1.53}$$

with the function Φ and its derivatives satisfying conditions

$$|\Phi| << 1, \left|\frac{\partial \Phi}{\partial t}\right|^2 << \frac{1}{a^2} D^i\Phi D_i\Phi, \left(D^i\Phi D_i\Phi\right)^2 << \left(D^i D^j \Phi\right)\left(D_i D_j \Phi\right), \tag{1.54}$$

where D_i denotes covariant derivative with respect to spatial metric γ_{ij}. Authors of the article showed that the corrections to the Einstein equations are negligible. They argued that this metric describes our universe on all scales except in the vicinity of black holes or neutron stars. They claimed that despite the large density contrast, the conditions (1.54) are satisfied for the solar system, galaxies or clusters of galaxies.

In [58] Kolb et al. considered perturbed FRW metric. They divided perturbation into two parts: Long-wavelength and short-wavelength. They confirmed that long-wavelength mode can not lead to the observed acceleration and it contributes to local curvature. Considering short-wavelength mode (shorter than Hubble scale) they used renormalization group technique and found that an instability occurs in the perturbative expansion. Because the perturbation theory for sub-horizon modes breaks down, we can not say definite conclusion whether these terms can cause observed acceleration of the universe.

Baumann et al. used effective field theory [3] in order to compute influence of short-scale physics on large-scale physics. In nonlinear regime, different scales couple together. They integrated out short wavelength contribution and they found that on very large scales, influence of short-scale physics can be represented by an effective fluid. Its density and pressure renormalize the background. Its effective pressure is always positive and too small to affect background evolution.

Green and Wald derived mathematically precise way how to compute the form of backreaction effect of small scale inhomogeneities [44]. They assumed one-parameter family of metrics $g_{ab}(\lambda, x)$ on a manifold $\mathcal{M}$. They considered the following conditions [45]:

1. The one parameter family of metrics $g_{ab}(\lambda, x)$ satisfies Einstein equations with a stress energy tensor obeying weak energy condition.

2. There exists a smooth positive function C_1 on $\mathcal{M}$ such that
$$h_{ab}(\lambda, x) \leq \lambda C_1(x), \tag{1.55}$$
where $h_{ab}(\lambda, x) = g_{ab}(\lambda, x) - g_{ab}(0, x)$.

3. There exists a smooth positive function C_2 on $\mathcal{M}$ such that
$$\nabla_c h_{ab}(\lambda, x) \leq \lambda C_2(x). \tag{1.56}$$

4. There exists a smooth tensor field μ_{abcdef} on $\mathcal{M}$ such that
$$\text{w-lim}_{\lambda\to 0}\left[\nabla_a h_{cd}(\lambda)\nabla_b h_{ef}(\lambda)\right] = \mu_{abcdef} \tag{1.57}$$

The weak limit is defined as follows [45]: $A_{a_1,\ldots,a_n}(\lambda)$ converges weakly to $A^{(0)}_{a_1,\ldots,a_n}$ as $\lambda \to 0$ if and only if, for all smooth tensor fields $f^{a_1\ldots a_n}$ of compact support
$$\lim_{\lambda\to 0}\int f^{a_1\ldots a_n} A_{a_1,\ldots,a_n}(\lambda) = \int f^{a_1\ldots a_n} A^{(0)}_{a_1,\ldots,a_n}. \tag{1.58}$$

Given these assumptions it can be shown that the background metric tensor $g^{(0)}_{ab}$ satisfies Einstein equations modified by an additional term.
$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}. \tag{1.59}$$

Then Green and Wald proved (given the above assumptions) two theorems which state:

- the effective stress-energy tensor $t_{ab}^{(0)}$ is traceless,

$$t^{(0)a}_{\ \ a} = 0. \tag{1.60}$$

- the effective stress-energy tensor $t_{ab}^{(0)}$ satisfies the weak energy condition, i.e.

$$t_{ab}^{(0)} t^a t^b \geq 0, \tag{1.61}$$

  for all t^a that are timelike with respect to $g_{ab}^{(0)}$.

The form of the effective stress-energy tensor suggests that it describes gravitational radiation and can not lead to an observed acceleration. In subsequent paper [45] they considered an example of a family of polarized vacuum Gowdy spacetimes with toroidal geometry, which satisfy required conditions and they found that the effective stress energy tensor is traceless.

Nowadays there exist several methods for estimating backreaction. They use different averaging methods or different schemes for computation of backreaction. Most of them suggest that the impact of inhomogeneities is not large enough to explain observed acceleration of the universe. However, we still do not have unambiguous averaging scheme for computation of backreaction so we can not tell a definite conclusion.

2. Macroscopic Gravity

2.1 Averaging scheme

In this chapter we will review Zalaletdinov's theory of Macroscopic Gravity [99], [100], [76]. The theory is based on the introduction of the bilocal operators for transportation of tensors and for Lie dragging of the regions. This approach is very promising since not only Einstein equations but also geometrical Cartan structure equations are averaged. Unfortunately the resulting equations of MG are very complex and the exact solutions are known only for a few models with high symmetry.

If we have a tensor field defined on a given manifold, there is no straightforward way how to define its average value. One way to deal with this problem is to introduce bilocal operator for transporting tensor from the point x' to the point x. In the theory of MG, there are constraints which select bilocal operator $\mathcal{W}^{\alpha'}_{\beta}(x',x)$ later used for averaging. They read

$$\lim_{x'\to x} \mathcal{W}^{\alpha'}_{\beta}(x',x) = \delta^{\alpha}_{\beta}, \tag{2.1}$$

$$\mathcal{W}^{\alpha'}_{\gamma''}(x',x'')\mathcal{W}^{\gamma''}_{\beta}(x'',x) = \mathcal{W}^{\alpha'}_{\beta}(x',x). \tag{2.2}$$

If we apply limit (2.1) to the equation (2.2) we can derive the inverse operator $[\mathcal{W}^{\alpha'}_{\gamma''}(x',x'')]^{-1} = \mathcal{W}^{\gamma''}_{\alpha'}(x'',x')$. It can also be shown that the properties (2.1) and (2.2) are equivalent to the following form of the bilocal operator [70]

$$\mathcal{W}^{\alpha'}_{\beta}(x,x') = F^{\alpha'}_{\gamma}(x')F^{-1\gamma}_{\beta}(x). \tag{2.3}$$

Now it is possible to define an average value of the tensor $t^{\alpha\cdots}_{\beta\ldots}(x)$ for a given domain $\Omega \subset \mathcal{M}$ on an n-dimensional manifold $(\mathcal{M}, g_{\alpha\beta})$ with a volume n-form according to the rule

$$\bar{t}^{\alpha\cdots}_{\beta\ldots}(x) = \frac{1}{V_\Omega}\int\limits_{\Omega} \tilde{t}^{\alpha\cdots}_{\beta\ldots}(x,x')\sqrt{-g'}d^n x', \tag{2.4}$$

g'=det$(g'_{\alpha\beta})$, V_Ω is the volume of the domain Ω,

$$V_\Omega = \int\limits_{\Omega} \sqrt{-g'}d^n x'. \tag{2.5}$$

$\tilde{t}^{\alpha\cdots}_{\beta\ldots}(x,x')$ here represents bilocally extended general tensor object using the bilocal operator

$$\tilde{t}^{\alpha\cdots}_{\beta\ldots}(x,x') = \mathcal{W}^{\alpha}_{\alpha'}(x',x)\ldots\mathcal{W}^{\beta'}_{\beta}(x',x)\ldots t^{\alpha'\cdots}_{\beta'\ldots}(x'). \tag{2.6}$$

It follows from the definition of the bilocally extended tensor that it transforms at the point x as a tensor, but at the point x' as a scalar. This can be seen from the fact that the prime indices are contracted. This property allows the correct definition of the average value (2.4).

In the next part of the chapter the geometrical equations will be bilocally extended and averaged. For this to be done, it is convenient to compare the average value of the derivative of the tensor and the derivative of the average value. The definition of the derivative reads

$$\frac{d}{d\lambda}\bar{t}^{\alpha\dots}_{\beta\dots}(x) = \lim_{\Delta\lambda\to 0}\frac{1}{\Delta\lambda}\left(\bar{t}^{\alpha\dots}_{\beta\dots}(x+\xi\Delta\lambda) - \bar{t}^{\alpha\dots}_{\beta\dots}(x)\right), \tag{2.7}$$

where x and $x+\xi\Delta\lambda$ are coordinates of the points close to each other and $\Delta\lambda$ is a small displacement of the parameter along the integral curve of the vector field ξ. For computation of the derivative it is necessary to introduce a new averaging region - the points from $x \in \Omega$ are Lie dragged along a new bilocal vector field $S^{\alpha'}$. Situation will simplify if we use the same bivector $\mathcal{W}^{\alpha'}_{\beta}(x',x)$ for the definition of $S^{\alpha'}$

$$S^{\alpha'}(x,x') = \mathcal{W}^{\alpha'}_{\beta}(x',x)\xi^{\beta}(x). \tag{2.8}$$

The resulting relation reads (for simpler notation angular brackets denote the same type of averaging as in definition (2.4))

$$\frac{d}{d\lambda}\bar{t}^{\alpha\dots}_{\beta\dots} = \xi^{\rho}(x)\left[\left\langle \eth_{\rho}\tilde{t}^{\alpha\dots}_{\beta\dots}\right\rangle + \left\langle \mathcal{W}^{\sigma'}_{\rho;\sigma'}\tilde{t}^{\alpha\dots}_{\beta\dots}\right\rangle - \left\langle \mathcal{W}^{\sigma'}_{\rho;\sigma'}\right\rangle \bar{t}^{\alpha\dots}_{\beta\dots}\right]. \tag{2.9}$$

Symbol $\eth$ here denotes bilocal covariant derivative

$$\eth_{\rho} := \partial_{\rho} + \mathcal{W}^{\sigma'}_{\rho}\partial_{\sigma'}. \tag{2.10}$$

The definition of the average value (2.4) is not unique. If we consider the equation (2.9), we can see that it is convenient to restrict to bilocal operators fulfilling

$$\mathcal{W}^{\sigma'}_{\rho;\sigma'} = 0. \tag{2.11}$$

Its physical interpretation is clear - it keeps the volume during the Lie dragging unchanged, i.e. the vector field $S^{\alpha'}(x,x')$ has a zero divergence. We would also like to have the partial derivatives commuting. The necessary and sufficient conditions are

$$\mathcal{W}^{\alpha'}_{[\beta,\gamma]} + \mathcal{W}^{\alpha'}_{[\beta,\delta'}\mathcal{W}^{\delta'}_{\gamma]} = 0. \tag{2.12}$$

Under which conditions it is possible to fulfill these relations is analyzed in the article by Mars and Zalaletdinov [70], where the following theorem is proven: *In an arbitrary n-dimensional manifold $(\mathcal{M}, g_{\alpha\beta})$ with a volume n-form there locally exists bivector $\mathcal{W}^{\alpha'}_{\beta}(x',x)$ fulfilling (2.11) and (2.12).*

2.2 Geometry formulated in p-form formalism

Now we would like to proceed with averaging equations describing (pseudo) Riemannian geometry. In order to see clearly the covariance of the procedure and also to decrease the number of indices in tensor expressions we will use the formalism of p-forms [71], [72]. We will pick a basis $\mathbf{e}_{\mu}$ with its dual $\mathbf{d}x^{\mu}$. Connection 1-forms $\boldsymbol{\omega}^{\mu}{}_{\nu}$ are defined with a help of exterior derivative extended for tensor-valued p-form by

$$\mathbf{d}\mathbf{e}_{\mu} = \boldsymbol{\omega}^{\rho}{}_{\mu}\mathbf{e}_{\rho}. \tag{2.13}$$

Cartan structure equations for a metric with zero torsion read

$$\boldsymbol{\omega}^{\mu}{}_{\rho}\wedge\mathbf{d}x^{\rho} = 0, \tag{2.14}$$

$$\mathbf{d}\boldsymbol{\omega}^{\mu}{}_{\nu} + \boldsymbol{\omega}^{\mu}{}_{\rho} \wedge \boldsymbol{\omega}^{\rho}{}_{\nu} = \mathbf{r}^{\mu}{}_{\nu}. \tag{2.15}$$

In the last equation curvature 1-forms $\mathbf{r}^{\mu}{}_{\nu}$ can be expressed by the components of the Riemann tensor by $\mathbf{r}^{\mu}{}_{\nu} = 1/2 R^{\mu}{}_{\nu\rho\sigma}\mathbf{d}x^{\rho} \wedge \mathbf{d}x^{\sigma}$. In order to have a simpler notation we define covariant exterior derivative $\mathbf{D}_{\omega}$ (associated with the connection $\boldsymbol{\omega}^{\mu}{}_{\nu}$) by effect on tensor valued p-form $\mathbf{t}^{\alpha\cdots}_{\beta\ldots}(x)$ (form indices are not explicitly written).

$$\mathbf{D}_{\omega}\mathbf{t}^{\alpha\ldots}_{\beta\ldots} = \mathbf{d}\mathbf{t}^{\alpha\ldots}_{\beta\ldots} - \boldsymbol{\omega}^{\rho}{}_{\beta} \wedge \mathbf{t}^{\alpha\ldots}_{\rho\ldots} + \ldots + \boldsymbol{\omega}^{\alpha}{}_{\rho} \wedge \mathbf{t}^{\rho\ldots}_{\beta\ldots} + \ldots \tag{2.16}$$

With the help of the covariant exterior derivative $\mathbf{D}_{\omega}$ we can write compatibility equation between metric and connection as

$$\mathbf{D}_{\omega}g_{\mu\nu} = \mathbf{d}g_{\mu\nu} - g_{\mu\rho}\boldsymbol{\omega}^{\rho}{}_{\nu} - g_{\rho\nu}\boldsymbol{\omega}^{\rho}{}_{\mu} = 0. \tag{2.17}$$

For completeness we will write integrability equations which we obtain by applying exterior derivative on equations (2.14), (2.15) and (2.17).

$$\boldsymbol{r}^{\mu}{}_{\rho} \wedge \mathbf{d}x^{\rho} = 0, \tag{2.18}$$

$$\mathbf{d}\boldsymbol{r}^{\mu}{}_{\nu} - \boldsymbol{\omega}^{\rho}{}_{\nu} \wedge \boldsymbol{r}^{\mu}{}_{\rho} + \boldsymbol{\omega}^{\mu}{}_{\rho} \wedge \boldsymbol{r}^{\rho}{}_{\nu} = 0, \tag{2.19}$$

$$g_{\mu\rho}\mathbf{r}^{\rho}{}_{\nu} + g_{\rho\nu}\mathbf{r}^{\rho}{}_{\mu} = 0. \tag{2.20}$$

In order to be able to average geometrical equations, we will utilize the theory of bilocal calculus [100]. For bilocal (p,k') form (i.e. p-form at the point x and k-form at the point x')

$$\boldsymbol{\alpha}(x, x') = \frac{1}{p!k!}\alpha_{\rho\ldots\sigma'\ldots}\mathbf{d}x^{\rho} \wedge \ldots \wedge \mathbf{d}x^{\sigma'} \wedge \ldots, \tag{2.21}$$

we introduce shifted exterior derivative $\mathbf{d}'_{\mathcal{W}}$ according to the rule (derivative at the point x', antisymmetrization at the point x)

$$\mathbf{d}'_{\mathcal{W}}\boldsymbol{\alpha}(x, x') = \frac{1}{p!k!}\alpha_{\rho\ldots\sigma'\ldots,\tau'}\mathcal{W}^{\tau'}{}_{\lambda}\mathbf{d}x^{\lambda} \wedge \mathbf{d}x^{\rho} \wedge \ldots \wedge \mathbf{d}x^{\sigma'} \wedge \ldots \tag{2.22}$$

We will replace ordinary exterior derivative $\mathbf{d}$ by bilocal exterior derivative $\boldsymbol{\text{đ}} = \mathbf{d} + \mathbf{d}'_{\mathcal{W}}$. Conditions (2.11) and (2.12) required for bivector $\mathcal{W}$ transform into

$$\mathrm{div}_{\epsilon}\mathcal{W} = \mathcal{W}^{\rho'}{}_{\alpha;\rho'}\mathbf{d}x^{\alpha} = 0, \tag{2.23}$$

$$\boldsymbol{\text{đ}}\mathcal{W}^{\alpha'} = \boldsymbol{\text{đ}}(\mathcal{W}^{\alpha'}{}_{\rho}\mathbf{d}x^{\rho}) = 0. \tag{2.24}$$

The second of the conditions is equivalent to the nilpotence of the bilocal exterior derivative $\boldsymbol{\text{đ}}\boldsymbol{\text{đ}} = 0$. Under these circumstances we will find simple commutation relation

$$\mathbf{d}\bar{t}^{\alpha\ldots}_{\beta\ldots} = \left\langle \boldsymbol{\text{đ}}\tilde{t}^{\alpha\ldots}_{\beta\ldots} \right\rangle. \tag{2.25}$$

Now we can continue with a bilocal extension of the equations describing (pseudo) Riemannian geometry. For a given p-form $\boldsymbol{\alpha}(x)$ we denote bilocally extended objects by a tilde.

$$\tilde{\boldsymbol{\alpha}}(x, x') = \frac{1}{p!}\alpha_{\rho'\ldots\sigma'}\mathcal{W}^{\rho'}{}_{\alpha}\ldots\mathcal{W}^{\sigma'}{}_{\beta}\mathbf{d}x^{\alpha} \wedge \ldots \wedge \mathbf{d}x^{\beta}, \tag{2.26}$$

We define bilocal connection 1-form $\mathbf{\Omega}^{\mu}{}_{\nu}$ with a help of the bilocaly extended basis vector $\mathcal{W}^{\rho'}{}_{\mu}\mathbf{e}_{\rho'}$.

$$\mathbf{đ}(\mathcal{W}^{\rho'}{}_{\mu}\mathbf{e}_{\rho'}) = \mathbf{\Omega}^{\sigma}_{\mu}(\mathcal{W}^{\rho'}{}_{\sigma}\mathbf{e}_{\rho'}). \tag{2.27}$$

Averaged bilocal connection 1-form will play an important role in the construction of the averaged geometry. Bilocal Cartan structure equations read

$$\mathbf{\Omega}^{\mu}{}_{\rho} \wedge \mathbf{d}x^{\rho} = 0, \tag{2.28}$$

$$\mathbf{đ}\mathbf{\Omega}^{\mu}{}_{\nu} + \mathbf{\Omega}^{\mu}{}_{\rho} \wedge \mathbf{\Omega}^{\rho}{}_{\nu} = \tilde{\mathbf{r}}^{\mu}{}_{\nu}, \tag{2.29}$$

where $\tilde{\mathbf{r}}^{\mu}{}_{\nu}$ is bilocally extended curvature 2-form. Similarly with a help of a bilocally extended covariant exterior derivative $\mathbf{Đ}_{\Omega}$ condition for a covariant constancy of the metric (2.17) can be rewritten as

$$\mathbf{Đ}_{\Omega} g_{\mu\nu} = \mathbf{đ}\tilde{g}_{\mu\nu} - \tilde{g}_{\mu\rho}\mathbf{\Omega}^{\rho}{}_{\nu} - \tilde{g}_{\rho\nu}\mathbf{\Omega}^{\rho}{}_{\mu} = 0. \tag{2.30}$$

In the same way the integrability conditions (2.18) - (2.20) read

$$\tilde{\boldsymbol{r}}^{\mu}{}_{\rho} \wedge \mathbf{d}x^{\rho} = 0, \tag{2.31}$$

$$\mathbf{đ}\tilde{\boldsymbol{r}}^{\mu}{}_{\nu} - \mathbf{\Omega}^{\rho}{}_{\nu} \wedge \tilde{\boldsymbol{r}}^{\mu}{}_{\rho} + \mathbf{\Omega}^{\mu}{}_{\rho} \wedge \tilde{\boldsymbol{r}}^{\rho}{}_{\nu} = 0, \tag{2.32}$$

$$\tilde{g}_{\mu\rho}\tilde{\mathbf{r}}^{\rho}{}_{\nu} + \tilde{g}_{\rho\nu}\tilde{\mathbf{r}}^{\rho}{}_{\mu} = 0. \tag{2.33}$$

2.3 Averaging Cartan structure equations

The next step is to construct geometrical objects on the averaged manifold $\bar{\mathcal{M}}$ with a help of the conditions (2.28) - (2.33). It is straightforward to average equations (2.28) and (2.31) (with a notation $\boldsymbol{R}^{\mu}{}_{\nu} = \langle\tilde{\boldsymbol{r}}^{\mu}{}_{\nu}\rangle$).

$$\bar{\mathbf{\Omega}}^{\mu}{}_{\rho} \wedge \mathbf{d}x^{\rho} = 0, \tag{2.34}$$

$$\boldsymbol{R}^{\mu}{}_{\rho} \wedge \mathbf{d}x^{\rho} = 0. \tag{2.35}$$

The main geometrical structure from which we construct another tensor field on the averaged manifold $\bar{\mathcal{M}}$ is the averaged connection 1-form $\bar{\mathbf{\Omega}}^{\mu}{}_{\nu}$. Departure from the trivial averaging is measured by the correlation 2-form

$$\boldsymbol{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = \left\langle \mathbf{\Omega}^{\alpha}{}_{\beta} \wedge \mathbf{\Omega}^{\gamma}{}_{\delta}\right\rangle - \bar{\mathbf{\Omega}}^{\alpha}{}_{\beta} \wedge \bar{\mathbf{\Omega}}^{\gamma}{}_{\delta}, \tag{2.36}$$

whose components will appear on the right hand side of the averaged Einstein equations. Curvature 2-form $\mathbf{M}^{\mu}{}_{\nu}$ (constructed from $\bar{\mathbf{\Omega}}^{\mu}{}_{\nu}$) on the manifold $\bar{\mathcal{M}}$ is defined by the structure equation

$$\mathbf{d}\bar{\mathbf{\Omega}}^{\mu}{}_{\nu} + \bar{\mathbf{\Omega}}^{\mu}{}_{\rho} \wedge \bar{\mathbf{\Omega}}^{\rho}{}_{\nu} = \mathbf{M}^{\mu}{}_{\nu}. \tag{2.37}$$

If we average equation (2.29) we receive after consideration of the definition of the correlation 2-form $\boldsymbol{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}$ the equation

$$\mathbf{M}^{\mu}{}_{\nu} = \mathbf{R}^{\mu}{}_{\nu} - \mathbf{Z}^{\mu}{}_{\rho}{}^{\rho}{}_{\nu}. \tag{2.38}$$

Similar form of the equation can be found in electrodynamics where by averaging of the microscopic (linear) Lorentz equations of the electromagnetic field Maxwell equations can be derived. Analog of an electromagnetic induction is played by the curvature 2-form $\mathbf{M}^{\mu}{}_{\nu}$ and the term which behaves as a polarization is the correlation term $\boldsymbol{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}$.

By applying the wedge product on the previous relation with a help of the equations (2.34) - (2.36) we obtain an integrability equation for (2.34).

$$\mathbf{M}^{\mu}{}_{\rho} \wedge \mathbf{d}x^{\rho} = 0. \tag{2.39}$$

Next steps of averaging are more complicated. The question is how to rewrite expressions of the type $\left\langle \boldsymbol{\Omega}^{\mu}{}_{\rho} \wedge \tilde{\boldsymbol{r}}^{\rho}{}_{\nu} \right\rangle$, $\langle \tilde{g}_{\mu\rho} \boldsymbol{\Omega}^{\rho}{}_{\nu} \rangle$ and $\langle \tilde{g}_{\mu\rho} \tilde{\mathbf{r}}^{\rho}{}_{\nu} \rangle$ with a help of the variables on the averaged manifold $\bar{\mathcal{M}}$. If we apply covariant exterior derivative for the relation defining correlation 2-form (2.36), we obtain one of the rules we are looking for.

$$\mathbf{D}_{\bar{\Omega}} \mathbf{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = -2\mathbb{P}\mathbf{Y}^{\alpha}{}_{\rho}{}^{\rho}{}_{\beta}{}^{\gamma}{}_{\delta} + 2\mathbb{P}(\left\langle \tilde{\boldsymbol{r}}^{\alpha}{}_{\beta} \wedge \boldsymbol{\Omega}^{\gamma}{}_{\delta} \right\rangle - \mathbf{R}^{\alpha}{}_{\beta} \wedge \bar{\boldsymbol{\Omega}}^{\gamma}{}_{\delta}). \tag{2.40}$$

Symbol $\mathbb{P}$ here permutes only free indices in pairs - for example $\mathbb{P}\mathbf{M}^{\alpha\gamma\epsilon}_{\beta\delta\zeta} = 1/3!(\mathbf{M}^{\alpha\gamma\epsilon}_{\beta\delta\zeta} - \mathbf{M}^{\gamma\alpha\epsilon}_{\delta\beta\zeta} + \mathbf{M}^{\alpha\epsilon\gamma}_{\beta\zeta\delta})$. The next term which makes averaging more difficult is the correlation 3-form.

$$\mathbf{Y}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}{}^{\epsilon}{}_{\zeta} = \left\langle \boldsymbol{\Omega}^{\alpha}{}_{\beta} \wedge \boldsymbol{\Omega}^{\gamma}{}_{\delta} \wedge \boldsymbol{\Omega}^{\epsilon}{}_{\zeta} \right\rangle - 3\mathbb{P}(\mathbf{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} \wedge \bar{\boldsymbol{\Omega}}^{\epsilon}{}_{\zeta}) - \bar{\boldsymbol{\Omega}}^{\alpha}{}_{\beta} \wedge \bar{\boldsymbol{\Omega}}^{\gamma}{}_{\delta} \wedge \bar{\boldsymbol{\Omega}}^{\epsilon}{}_{\zeta}, \tag{2.41}$$

which fixes the differential properties of the correlation 2-form. Analogically if we apply covariant exterior derivative on the definition of the correlation 3-form, we can identify another nontrivial expression - correlation 4-form. Higher correlation terms do not appear on the 4-dimensional manifold (spacetime). Although only correlation 2-form appears in the averaged structure equations, for completeness we need to know also higher order correlation terms, because these fix the differential properties of the correlation 2-form. Fortunately, there exists a way how to put the higher order correlation terms equal to zero. Zalaletdinov showed [100] that to successful annihilation of higher order correlation terms leads the following ansatz

$$\mathbf{D}_{\bar{\Omega}} \mathbf{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = 0 = \mathbf{D}_{\bar{\Omega}} \mathbf{R}^{\alpha}{}_{\beta}, \tag{2.42}$$

with an integrability equation

$$\mathbb{P}(\mathbf{R}^{\alpha}{}_{\rho} \wedge \mathbf{Z}^{\rho}{}_{\beta}{}^{\gamma}{}_{\delta} - \mathbf{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\rho} \wedge \mathbf{R}^{\rho}{}_{\delta}) = 0 \tag{2.43}$$

and with a requirement

$$\mathbb{P}(\mathbf{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} \wedge \mathbf{Z}^{\epsilon}{}_{\zeta}{}^{\eta}{}_{\theta}) = 0. \tag{2.44}$$

If we contract equation (2.40) with indices β and γ, the result can be used for averaging of the equation (2.32), which gives required identity for the curvature 2-form $\mathbf{M}^{\mu}{}_{\nu}$

$$\mathbf{d}\boldsymbol{M}^{\mu}{}_{\nu} - \bar{\boldsymbol{\Omega}}^{\rho}{}_{\nu} \wedge \boldsymbol{M}^{\mu}{}_{\rho} + \bar{\boldsymbol{\Omega}}^{\mu}{}_{\rho} \wedge \boldsymbol{M}^{\rho}{}_{\nu} = 0. \tag{2.45}$$

Two equations (2.17) and (2.33) only remain to be averaged. Let us suppose ([99], [100]) that for a given class of slowly changing tensor fields (tensor valued p-forms) $\mathbf{c}^{\mu\ldots}_{\nu\ldots}$, including covariantly constant tensors and Killing tensors (symmetry should not be broken by averaging), the following assumptions hold

$$\left\langle \boldsymbol{\Omega}^{\alpha}{}_{\beta} \wedge \tilde{\mathbf{c}}^{\mu\ldots}_{\nu\ldots} \right\rangle = \bar{\boldsymbol{\Omega}}^{\alpha}{}_{\beta} \wedge \bar{\mathbf{c}}^{\mu\ldots}_{\nu\ldots}, \tag{2.46}$$

$$\left\langle \mathbf{\Omega}^{\alpha}{}_{\beta} \wedge \mathbf{\Omega}^{\gamma}{}_{\delta} \wedge \tilde{\mathbf{c}}^{\mu\ldots}_{\nu\ldots} \right\rangle = \left\langle \mathbf{\Omega}^{\alpha}{}_{\beta} \wedge \mathbf{\Omega}^{\gamma}{}_{\delta} \right\rangle \wedge \bar{\mathbf{c}}^{\mu\ldots}_{\nu\ldots}. \tag{2.47}$$

Then, equation (2.17) and its equivalent for $\bar{g}^{\mu\nu}$ gives

$$\mathbf{D}_{\bar{\Omega}} \bar{g}_{\mu\nu} = 0; \mathbf{D}_{\bar{\Omega}} \bar{g}^{\mu\nu} = 0. \tag{2.48}$$

These equations enable us to choose $\bar{g}_{\mu\nu} = G_{\mu\nu}$ ($G_{\mu\nu}$ is the metric on the averaged manifold $\bar{\mathcal{M}}$). An analogy of this relation with contravariant indices does not hold, $\bar{g}^{\mu\nu} \neq G^{\mu\nu}$, $\bar{g}_{\mu\rho}\bar{g}^{\rho\nu} \neq \delta^{\mu}_{\nu}$ and this inequality can be characterized by the tensor $U^{\mu\nu} = \bar{g}^{\mu\nu} - G^{\mu\nu}$. The last equation on the manifold $\bar{\mathcal{M}}$ we are looking for can be found if we apply exterior derivative of the assumption (2.46).

$$- \left\langle \mathbf{\Omega}^{\alpha}{}_{\beta} \wedge \mathbf{Đ}_{\bar{\Omega}} \tilde{\mathbf{c}}^{\mu\ldots}_{\nu\ldots} \right\rangle + \bar{\mathbf{\Omega}}^{\alpha}{}_{\beta} \wedge \mathbf{D}_{\bar{\Omega}} \bar{\mathbf{c}}^{\mu\ldots}_{\nu\ldots} + \left\langle \tilde{\boldsymbol{R}}^{\alpha}{}_{\beta} \wedge \tilde{\mathbf{c}}^{\mu\ldots}_{\nu\ldots} \right\rangle - \mathbf{R}^{\alpha}{}_{\beta} \wedge \bar{\mathbf{c}}^{\mu\ldots}_{\nu\ldots} =$$
$$= -\mathbf{Z}^{\alpha}{}_{\beta}{}^{\mu}{}_{\rho} \wedge \bar{\mathbf{c}}^{\rho\ldots}_{\nu\ldots} - \ldots + \mathbf{Z}^{\alpha}{}_{\beta}{}^{\rho}{}_{\nu} \wedge \bar{\mathbf{c}}^{\mu\ldots}_{\rho\ldots} + \ldots \tag{2.49}$$

and substituting into the last remaining equation (2.33) gives

$$\bar{g}_{\mu\rho}\mathbf{M}^{\rho}{}_{\nu} + \bar{g}_{\rho\nu}\mathbf{M}^{\rho}{}_{\mu} = 0; \qquad \mathbf{M}^{\mu}{}_{\rho}\bar{g}^{\rho\nu} + \mathbf{M}^{\nu}{}_{\rho}\bar{g}^{\mu\rho} = 0. \tag{2.50}$$

Finally we obtained averaged structure equations and its integrability equations on the averaged manifold $\bar{\mathcal{M}}$. The equations are valid for a general n-dimensional manifold with a volume n-form.

2.4 Macroscopic Einstein equations

Let us assume that the 'microscopic' equations are fully described by the set of the Einstein equations. These relate contracted Riemann tensor with a stress energy tensor.

$$g^{\mu\rho} r_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu} g^{\rho\sigma} r_{\rho\sigma} + \delta^{\mu}_{\nu}\Lambda = 8\pi T^{\mu(micro)}_{\nu}. \tag{2.51}$$

In order to obtain averaged equations of the Macroscopic Gravity we need to replace tensor variables on the manifold $\mathcal{M}$ by the tensors on $\bar{\mathcal{M}}$. From the form of the Einstein equations we can see that we need to use an identity (2.49), because it gives us the rule how to average the product of the Riemann tensor and the metric tensor. Macroscopic equations then have a form

$$G^{\mu\rho} M_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu} G^{\rho\sigma} M_{\rho\sigma} + \delta^{\mu}_{\nu}\Lambda = 8\pi T^{\mu(macro)}_{\nu}, \tag{2.52}$$

$$8\pi T^{\mu(macro)}_{\nu} = 8\pi T^{\mu(micro)}_{\nu} + \left(Z^{\mu}{}_{\rho\sigma\nu} - \frac{1}{2}\delta^{\mu}_{\nu} Q_{\rho\sigma} \right) \bar{g}^{\rho\sigma} - \left(U^{\mu\rho} M_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu} U^{\rho\sigma} M_{\rho\sigma} \right), \tag{2.53}$$

where we denoted $Z^{\alpha}{}_{\beta\gamma\delta} = 2Z^{\alpha}{}_{\beta\rho}{}^{\rho}{}_{\gamma\delta}$ and $Q_{\alpha\beta} = Z^{\rho}{}_{\alpha\rho\beta}$ the expressions derived from the correlation 2-form $\mathbf{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = Z^{\alpha}{}_{\beta\rho}{}^{\gamma}{}_{\sigma\delta}\mathbf{d}x^{\rho} \wedge \mathbf{d}x$.

The left hand side of the Einstein equations (Einstein tensor E^{μ}_{ν}) fulfills (due to the correct construction of the averaged equations on the manifold $\bar{\mathcal{M}}$) contracted Bianchi identities from which the local conservation law follow

$$E^{\rho}_{\mu;\rho} = (8\pi T^{\rho(micro)}_{\mu} + C^{\rho}_{\mu})_{;\rho} = 0. \tag{2.54}$$

Correlation term C^{μ}_{ν} which measures the difference between the standard form of the Einstein equations and the equations of MG reads

$$C^{\mu}_{\nu} = \left(Z^{\mu}{}_{\rho\sigma\nu} - \frac{1}{2}\delta^{\mu}_{\nu} Q_{\rho\sigma} \right) \bar{g}^{\rho\sigma} - \left(U^{\mu\rho} M_{\rho\nu} - \frac{1}{2}\delta^{\mu}_{\nu} U^{\rho\sigma} M_{\rho\sigma} \right). \tag{2.55}$$

2.5 Exact solutions of MG

Solutions of the MG equations can give an exact expression for the correlation term included in the averaged Einstein equations. Correlation 2-form must fulfill complicated equations (2.42), (2.43) and (2.44). The first exact solution was published in 2005 by Coley, Pelavas and Zalaletdinov [27] (details and extensions of the results was shown in [49]). After the ansatz of the flat FRW macroscopic metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{2.56}$$

they assumed the simplest form of the correlation tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\delta}{}_{\epsilon\zeta} = const.$, macroscopic metric tensor $\bar{g}_{\mu\nu} = G_{\mu\nu}$ and the null electric part of the correlation tensor (after splitting of the correlation term into an electric and a magnetic part). Resulting correlation term can be interpreted as an additional spatial curvature, which is in accordance with the later articles [30] and [31]. In these later articles it was shown that by averaging spherically symmetric metric (with certain assumptions on the form of inhomogeneity of the gravitational field) correlation term can be written as a sum of the terms representing spatial curvature and imperfect fluid.

Clifton, Coley, and van den Hoogen [26] generalized the result for a macroscopic FRW spacetime. They obtained a new solution with less symmetry. It changed Einstein equations by spatial curvature term as in the previous work. The difference can be seen when they concentrated on observational issues - for example calculating distance measures on macroscopic FRW background can be modified compared to the previous approach.

Van den Hoogen in [50] investigated a model with different assumptions. He assumed static, spherically symmetric macroscopic metric (inhomogeneity is given by the correlation term)

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2.57}$$

The stress energy tensor here describes a perfect fluid with a radial size R with a constant density ρ_0. Solution of the MG equations gives us the correlation term which effectively models anisotropic fluid with a radial pressure $p_r^{cor} = -\rho^{cor}$ and a density $\rho^{cor} = \frac{1}{8\pi}\frac{4h_1}{r^2}$, h_1 is positive integration constant.

The result has an interesting interpretation. In a galactic dynamics for the flattening of the rotational curves of the spiral galaxies the standard explanation is the existence of a dark matter. In an ordinary spherically symmetric model density scales like $\rho \propto r^{-3}$, but the data are better fitted with a dependence r^{-2}, which is the case for the correlation term. The resulting correlation term suggest that we could alternatively interpret flattening of the rotational curves in spiral galaxies as a result of averaging of the spherically symmetric spacetime.

2.6 3+1 limit of Macroscopic Gravity

In the Buchert formalism the appropriate spacelike domain is chosen and by averaging a scalar part of the Einstein equations is modified. Paranjape and Singh used similar approach for averaging in the theory of MG for a macroscopic FRW metric [80]. They considered a particular choice of a coordinate system -

so called volume-preserving coordinates $\widehat{x}^{\mu}$ in which the averaging is trivial [70]. They considered inhomogeneous manifold with a zero lapse function N, then they performed transformation to the volume preserving coordinates. The metric reads

$$ {}^{(\mathcal{M})}ds^2 = -\frac{d\widehat{t}^2}{h(\widehat{t},\widehat{x})} + h_{AB}(\widehat{t},\widehat{x})d\widehat{x}^A d\widehat{x}^B, \tag{2.58} $$

where h:=deth_{AB}. This metric represents spacetime which after averaging gives us averaged FRW model. Because in the volume-preserving coordinates spacelike average value of the tensor field $t^{\alpha\cdots}_{\beta\ldots}$ is given by

$$ \left\langle t^{\alpha\cdots}_{\beta\ldots} \right\rangle_P = \lim_{T\to 0} \frac{1}{TV_{\Omega^{(3)}}} \int\limits_{t'-T/2}^{t'+T/2} dt' \int\limits_{\Omega^{(3)}} t^{\alpha\cdots}_{\beta\ldots}(t',x',y',z')dx'dy'dz' \tag{2.59} $$

and bilocal extension of the tensor object is trivial it is possible to compute independent elements of the correlation two form $\mathbf{Z}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}$ and from macroscopic equations (2.53) choose a scalar part. Then it is possible to show that the resulting equations have a similar form as the Buchert equations. In addition, scalar corrections are defined on an averaged FRW background so it is straightforward to compare results with an observational data (in contrast with the Buchert approach). In [77] the theory was further developed in the context of a cosmological perturbation and numerical results showed that it is possible to neglect scalar correction in the perturbed FRW model with radiation and dark matter. Similarly [78] shows negligible correction by averaging spherically symmetric collapse of a perfect fluid with zero pressure modeled by LTB model.

3. Anisotropic cosmological models

3.1 Bianchi cosmologies

3.1.1 Introduction

FRW models describe our universe well at least at the time of the last scattering. The main assumption for considering these models is the observed homogeneity and isotropy on the large scales. In order to consider more general models it is possible to relax the assumption of isotropy and investigate anisotropic cosmological models. The most studied representatives of this family of models are Bianchi universes. In this section we will review basic facts about these spacetimes. More details can be found in [94], [82] , [34], [95] or [35].

We will start with a definition of the Bianchi models: These are models in which there is a group of isometries G_3 acting simply transitively on spacelike surfaces $t = const$. From this definition we can see that the metric components do not depend on spatial coordinates and the resulting Einstein equations depend only on time variable and they are in the form of the ordinary differential equations. There are two possible ways in which the source relates to the surface of homogeneity:

- Orthogonal models: Here the fluid flow lines lie orthogonal to the surface of homogeneity.
- Tilted model: Here the fluid flow lines do not lie orthogonal to the surface of homogeneity. In this case peculiar velocity has to be considered as an additional degree of freedom.

3.1.2 Classification of the orthogonal Bianchi cosmologies

Despite the fact that the system of the Einstein equations simplifies to the ordinary differential equations, the system is nonlinear and can exhibit nontrivial behaviour. There exist three ways of classifying orthogonal Bianchi cosmologies [35]. Here we will consider approach based on tetrad basis vectors. The tetrad consists of one timelike vector orthogonal to the surface of homogeneity and the three spatial vectors tangent to the surface of homogeneity. They are collectively denoted as $\{\mathbf{e}_a\}$, where a runs form 0 to 3. This vector basis does not commute but has a nontrivial commutation functions

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c{}_{ab}\mathbf{e}_c. \tag{3.1}$$

These commutation functions $\gamma^c{}_{ab}(t)$ are considered as dynamical variables. In the classification scheme the spatial commutation functions $\gamma^\alpha{}_{\beta\gamma}(t)$ (where Greek indices run from 1 to 3) are considered. They can be decomposed into pair of time-dependent objects $n_{\alpha\beta}$ and a_α. They satisfy the condition

$$n_{\alpha\beta}a^\beta = 0 \tag{3.2}$$

Because we have a homogeneous model, commutation functions depend only on time variable and we can perform time dependent spatial rotation of the tetrad so that the object $n_{\alpha\beta}$ is diagonal, i.e. $n_{\alpha\beta}$ =diag(n_1, n_2, n_3) and it can be shown that we can at the same time have $a_\alpha = (a, 0, 0)$ and the above condition reduces to

$$n_1 a = 0. \tag{3.3}$$

It means that we can consider two classes of models. They are further classified by the signs of the diagonal elements of the matrix $n_{\alpha\beta}$

- class A Bianchi models where $a = 0$.
  - Bianchi I model, $n_1 = n_2 = n_3 = 0$
  - Bianchi II model, $n_1 > 1, n_2 = n_3 = 0$
  - Bianchi VI_0 model, $n_1 = 0, n_2 > 0, n_3 < 0$
  - Bianchi VII_0 model, $n_1 = 0, n_2 > 0, n_3 > 0$
  - Bianchi $VIII$ model, $n_1 < 0, n_2 > 0, n_3 > 0$
  - Bianchi IX model, $n_1 > 0, n_2 > 0, n_3 > 0$
- class B Bianchi models where $a \neq 0$.
  - Bianchi V model, $n_1 = n_2 = n_3 = 0$
  - Bianchi IV model, $n_1 = n_2 = 0, n_3 > 0$
  - Bianchi VI_h model, $n_1 = 0, n_2 > 0, n_3 < 0$
  - Bianchi VII_h model, $n_1 = 0, n_2 > 0, n_3 > 0$

We considered classification of the Bianchi models based on commutation functions of the tetrad field. Another equivalent procedure would be to consider Killing vector fields and decompose structure constant of the corresponding Lie algebra and perform classification in the same way as we did. From this second approach it can be seen that classification of the Bianchi models is equivalent to the classification of the isometry group G_3. By this procedure it is also possible to classify tilted Bianchi models.

3.1.3 Bianchi I cosmologies

As a concrete example of the Bianchi family we consider in more detail Bianchi I models. These are nontilted models that belong to type A in the classification scheme. These are the simplest generalization of the FRW spacetime. The line element reads in comoving coordinates.

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2. \tag{3.4}$$

In this model there are different expansion rates in different directions. In order to compare with the FRW model it is possible to define an effective scale factor $a(t) = \sqrt[3]{XYZ}$.

We can consider in a separate way the models with a perfect fluid equation of state $p = w\rho$. It can be shown that the shear will dominate the early expansion of the universe even in the case of very small anisotropy in the expansion presented today. This shear dominated epoch can be described by the well known vacuum Kasner solution. Its line element reads

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{3.5}$$

where $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$. These conditions ensure that one of the three parameters is negative or two of them are zero. In the first case the two directions are expanding and in the third direction spacetime is contracting. In this case the initial singularity is cigar-like. In the second case spacetime is expanding in only one direction and the initial singularity is pancake-type. It turns out that the sequence of different Kasner epochs plays an important role in the so called BKL approach to the singularity problem [68], [5], [6].

3.2 Kantowski-Sachs cosmologies

The special type of anisotropic cosmological models are Kantowski-Sachs cosmologies. These are invariant under four dimensional group of isometries for which a three-parameter subgroup acts on two-dimensional surfaces of constant positive curvature (without acting simply transitively on three spaces of homogeneity) [46]. The line element reads:

$$ds^2 = -dt^2 + A^2(t)dr^2 + B^2(t)(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{3.6}$$

where $A(t)$ and $B(t)$ denote two distinct scale factors. These models exhibit locally-rotational symmetry (LRS). This property also holds for some Bianchi models and LRS property will be discussed in later part of this thesis. Kantowski-Sachs spacetime was originally derived with a dust source, later were generalized for the more general perfect fluid source, in particular for radiation and stiff fluid [46]. The source term can also contain electromagnetic field and cosmological constant. The metric is spherically symmetric. There exist modifications where the function $\sin\theta$ in the metric tensor is replaced by θ or $\sinh\theta$. These are also solutions of the Einstein equations. However, these solutions exhibit symmetries of the Bianchi models and they belong according to the classification scheme to Bianchi models of the type I and III [94].

4. Inhomogeneous cosmological models

4.1 FRW limit

In the first part of this thesis we considered the problem how to average inhomogeneities. By certain averaging procedure we wanted to obtain FRW model in the last step of averaging process and interpret inhomogeneities as an additional correlation term. Averaging is used because FRW models are suitable to interpret cosmological data and there exist speculations that inhomogeneities can be interpreted as an extra source term. In this chapter we will consider different approach. We introduce certain exact inhomogeneous cosmological models. These models have more degrees of freedom and they are obtained if we break the assumption of homogeneity. Usually it is common to model inhomogeneities by the linear perturbation of the FRW model. If we want to consider nonlinear features of the theory it is possible to use exact solution of the Einstein equations.

There are more than 300 independently published solutions of the Einstein equations which can play the role of the inhomogeneous cosmological model. In [61] inhomogeneous cosmological models are defined by the rule that they have as a limit the FRW model if we take certain limiting values of arbitrary constants or functions parametrizing the given solution. The problem is how to define this limit. In order to perform limit of spacetime in a coordinate independent way, it is possible to consider Cartan scalars [54], [55], [56] and perform the limit covariantly [75].

Let us make the limiting procedure more precise. Let $g(\lambda)$ be a one parameter family of metrics on a manifold $\mathcal{M}(\lambda)$. Suppose we have a given metric h. The question is if there exists the limit $\lim_{\lambda\to\lambda_0} g(\lambda)$ and if this limit is equal to h. If we take a limit of spacetime the following holds:

1. The isometry group can be enlarged.

2. The Petrov and Segre type of spacetime can be more specialized but not more general.

3. A perfect fluid with nonzero shear can become shearfree in a limit, the opposite situation is not possible.

In the next sections we will consider certain cosmological models which have as a limit FRW model. It is helpful to have an invariant characterization of FRW model. The necessary and sufficient conditions for spacetime to be FRW are [61]

- The metric obeys the Einstein equations with a perfect fluid source.

- The velocity field of a perfect fluid source has a zero rotation, shear and acceleration.

There exists equivalent invariant definition of FRW model without referring to Einstein equations:

- The spacetime admits a foliation into spacelike hypersurface of constant curvature.
- The congruence of lines orthogonal to leaves of the foliation are shearfree geodesics.
- The expansion scalar of geodesic congruence has its gradient tangent to the geodesics.

These conditions are not easily imposed. Instead we should proceed in several steps. In each step we can try to reduce the solution to FRW or prove that this solution can not have FRW as a consistent limit. Here we will show the necessary conditions for FRW which need to hold and which can be enforced to hold in the limiting procedure [61].

1. The source must be a perfect fluid.
2. The acceleration must be zero.
3. The rotation must be zero.
4. The shear must be zero.
5. The gradient of pressure must be collinear with the velocity field.
6. The gradients of matter-density and of the expansion scalar must be collinear with velocity.
7. The barotropic equation of state must hold.
8. The Weyl tensor must vanish.
9. The hypersurfaces orthogonal to velocity field must have constant curvature.

4.2 LTB model

The most popular way to model inhomogeneity is to consider Lemaître-Tolman-Bondi (LTB) metric [65], [93], [11]. It is spherically symmetric exact solution of the Einstein equations with radial inhomogeneity. In this section we will briefly describe its basic properties. For the recent review of LTB metric see e.g. [9], [48] or [82]. This model exhibits LRS symmetry and its averaged equations will be investigated later. This spacetime belongs to Szekeres-Szafron family but because of its importance we will review this model first in a separate section.

LTB model has a source term which corresponds to an inhomogeneous dust with the stress energy tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu, \tag{4.1}$$

where u_μ is 4-velocity of a dust with a density ρ. The line element reads

$$ds^2 = -dt^2 + \frac{(R')^2}{1+2E(r)}dr^2 + R^2(t,r)(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{4.2}$$

where $E(r)$ is an arbitrary function and the prime denotes partial derivative with respect to r. Function $R(t,r)$ obeys Einstein equations if

$$R_{,t}^2 = 2E + \frac{2M}{R} + \frac{\Lambda}{3}R^2. \tag{4.3}$$

$M = M(r)$ is another arbitrary function of integration. The energy density ρ is determined by the equation

$$4\pi\rho = \frac{M'}{R'R^2}. \tag{4.4}$$

The function $E(r)$ determines a curvature of the space $t = const.$ (which is flat for $E(r) = 0$) and the function $M(r)$ is the gravitational mass contained within the comoving spherical shell at any given r. Equation (4.3) can be integrated to give the result

$$\int_0^R \frac{d\tilde{R}}{\sqrt{2E + \frac{2M}{\tilde{R}} + \frac{1}{3}\Lambda\tilde{R}^2}} = t - t_B(r). \tag{4.5}$$

$t_B(r)$ is the third free function of r (called the bang time function). In the LTB model, in general, the Big Bang is not simultaneous as in the FRW case, but it depends on the radial coordinate r. The given formulas are invariant under transformation $\tilde{r} = g(r)$. We can use this freedom to choose one of the functions $E(r), M(r)$ and $t_B(r)$. For $\Lambda = 0$ the above equation can be solved explicitly - when $E < 0$ (elliptic evolution)

$$R\,(t,r) = \frac{M}{(-2E)}\,(1 - \cos\eta)\,,$$

$$\eta - \sin\eta = \frac{(-2E)^{3/2}}{M}\,(t - t_B)\,. \tag{4.6}$$

If E $= 0$ (parabolic evolution)

$$R\,(t,r) = \left[\frac{9}{2}M\,(t - t_B)^2\right]^{1/3}, \tag{4.7}$$

when $E > 0$ (hyperbolic evolution)

$$R\,(t,r) = \frac{M}{2E}\,(\cosh\eta - 1)\,,$$

$$\sinh\eta - \eta = \frac{(2E)^{3/2}}{M}\,(t - t_B\,(r))\,. \tag{4.8}$$

For $\Lambda \neq 0$ the solution can be found in the form of the elliptic functions.

We can see that in certain points density field (4.4) diverges. The next indication of singularity is the blow up of the Kretschmann scalar which for LTB model reads

$$\mathcal{K} = R_{abcd}R^{abcd} = \frac{48M^2}{R^6} + \frac{32MM'}{R^5R'} + \frac{12(M')^2}{R^4(R')^2}. \tag{4.9}$$

The first kind of singularity appears when $R = 0$. This point correspond to initial big-bang singularity or final big-crunch singularity. The surfaces of both singularities are spacelike except possibly at the origin.

The second kind of singularity appears when two neighbouring shells collide. This singularity is called shell crossing and it shows up when $R' = 0$. The surface of shell crossing singularity is timelike. We could object that in more realistic model the pressure will prevent the shells to collide.

We will now look at the regularity conditions which have to hold. Firstly to retain Lorentzian signature we need $f \geq -1$. Next requirement would be to have a regular origin. An origin of spherical coordinates is a locus r_0 where $R(t, r_0) = 0, \forall t$ so that $\dot{R}(t, r_0) = 0$, $\ddot{R}(t, r_0) = 0$. We should demand that curvature and density have to be finite at the origin. This condition holds if we have the relations $M \propto r^3, f \propto r^2$.

LTB model was vastly used in the past. Among several application of this model we mention formation of cosmic voids, formation of black holes or formation of galaxy clusters [61]. LTB model became popular after the discovery of the acceleration of the universe. There appeared many papers which tried to explain this phenomenon using LTB model. We mention e.g. [51], [24], [40], [37] and [7]. Most of these studies placed observer near the center of the large void so that Copernican principle is violated. At the time of writing this thesis it seems that LTB models are ruled out. This is due to the fact that it is difficult to explain kinematic Sunyaev-Zel'dovich effect [22] within this model.

4.3 The Szekeres-Szafron family

In this section we will briefly introduce Szekeres-Szafron family of solutions. It is a general cosmological solution without any Killing vectors. From this section up to the end of the chapter we will use different sign convention in a definition of the metric tensor. This is because we will follow textbooks [61], [9], [82] and a review article [10]. This family is characterized by the requirement that there exist coordinates in which the metric reads

$$ds^2 = dt^2 - e^{2\alpha} dz^2 - e^{2\beta}(dx^2 + dy^2), \tag{4.10}$$

where α and β are functions of (t, x, y, z) which are determined from the Einstein equations. The form of the metric tensor was originally derived by Szekeres with a dust source [92] and later generalized by Szafron to arbitrary pressure [91]. Another generalization of the source term can be found in [61]. The invariant definition of the Szekeres-Szafron family is [82]:

1. The velocity field of the fluid is geodesic and irrotational.

2. The Weyl tensor is of type D, and the velocity vector of the fluid at every point of the spacetime lies in the 2-plane spanned by the two principal null directions.

3. Any vector orthogonal to both repeated principal null directions is an eigenvector of shear.

4. The 2-surfaces generated by the principal null directions admit orthogonal 2-surfaces.

There exist two subfamilies depending and the value of the function $\beta_{,r}$. The first subfamily is defined by the requirement $\beta_{,r} = 0$ and is a simultaneous generalization of the Friedmann and Kantowski–Sachs models. However, this subfamily has not been used in astrophysical applications so we will not consider this class. We will investigate in more detail $\beta_{,r} \neq 0$ model. The most important solution within this family is a Szekeres solution with a dust source. It contains LTB model as a limit. It is possible to solve the Einstein equations and the metric functions of a line element (4.10) read

$$\begin{aligned} e^{\beta} &= \Phi(t,r)e^{\nu(r,x,y)}, \\ e^{\alpha} &= h(r)\Phi(t,r)\beta_{,r} = h(r)(\Phi_{,r} + \Phi\nu_{,r}), \\ e^{-\nu} &= A(r)(x^2+y^2) + 2B_1(r)x + 2B_2(r)y + C(r), \end{aligned} \tag{4.11}$$

where the function $\Phi(t,r)$ solves the equation

$$\Phi_{,t}{}^2 = -k(r) + \frac{2M(r)}{\Phi} + \frac{1}{3}\Lambda\Phi^2. \tag{4.12}$$

Here we introduced arbitrary functions of radial coordinate $h(r)$, $k(r)$, $M(r)$, $A(r)$, $B_1(r)$,$B_2(r)$ and $C(r)$. These functions fulfill the relation

$$g(r) \equiv 4(AC - B_1{}^2 - B_2{}^2) = \frac{1}{h^2(r)} + k(r). \tag{4.13}$$

The mass density can be computed by

$$8\pi G\rho = \frac{(2Me^{3\nu})_{,r}}{e^{2\beta}(e^{\beta})_{,r}}. \tag{4.14}$$

We can integrate equation (4.12) to obtain

$$\int_0^{\Phi} \frac{\mathrm{d}\tilde{\Phi}}{\sqrt{-k + \frac{2M}{\tilde{\Phi}} + \frac{1}{3}\Lambda\tilde{\Phi}^2}} = t - t_B(r), \tag{4.15}$$

where $t_B(r)$ is another arbitrary function which can be interpreted in the same way as in LTB model, i.e. describing non-simultaneity of big-bang. The function $g(r)$ determines the geometry of the surface $t = const.$, $r = const.$ The sign of $k(r)$ determines the type of evolution. With $k(r) > 0 = \Lambda$ model starts from an initial singularity and recollapses to final singularity. The case with $k(r) < 0 = \Lambda$ is ever-expanding or ever-collapsing depending on initial conditions and $k(r) = 0 = \Lambda$ is an intermediate case.

Depending on the sign of $g(r)$ we can divide Szekeres model into quasi-hyperbolic with $g < 0$, quasi-planar $g = 0$ and quasi-spherical model $g > 0$. The first two are not usually applied in astrophysical applications and are not so well investigated as the quasi-spherical case. Recent work related to quasi-hyperbolic and quasi-planar model can be found in [62] and [63]. In the Szekeres spacetime there can be regions with a different sign of g, e.g. quasi-spherical and quasi-hyperbolic regions separated by quasi-planar region. Quasi-spherical model can be considered as a generalization of the LTB spacetime. Spherical shells are not concentric but their centers are shifted. Its positions are determined by the functions $A(r), B_1(r), B_2(r)$.

It is possible to find different coordinates to represent Szekeres metric. We will show derivation for the case with $g \neq 0$ [9]. If we write

$$\begin{aligned} A &= \frac{\sqrt{|g|}}{2S}, \;\; B_1 = -\frac{\sqrt{|g|}P}{2S}, \;\; B_2 = -\frac{\sqrt{|g|}Q}{2S}, \\ \epsilon &= \frac{g}{|g|}, \;\; k = |g|\,\tilde{k}, \;\; \Phi = \sqrt{|g|}\tilde{\Phi}, \;\; e^{-\nu} = \sqrt{|g|}\mathcal{E}, \end{aligned} \tag{4.16}$$

where the function $\mathcal{E}$ is defined by

$$\mathcal{E} = \frac{S}{2}\left[\left(\frac{x-P}{S}\right)^2 + \left(\frac{y-Q}{S}\right)^2 + \epsilon\right]. \tag{4.17}$$

The metric tensor then reads

$$ds^2 = dt^2 - \frac{\left(\Phi_{,r} - \Phi\mathcal{E}_{,r}/\mathcal{E}\right)^2}{\epsilon - k(r)}dr^2 - \frac{\Phi^2}{\mathcal{E}^2}\left(dx^2 + dy^2\right). \tag{4.18}$$

In this form tilde was removed for better readability. The sign of ϵ determines type of the model. The advantage of this type of coordinates is that the constraint (4.13) is identically satisfied. It means that the functions presented are independent.

4.4 Lemaître model

Lemaître model generalizes LTB model for a fluid with nonzero pressure. The metric is spherically symmetric and the line element reads [9]

$$ds^2 = e^{A(t,r)}dt^2 - e^{B(t,r)}dr^2 - R^2(t,r)(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{4.19}$$

The Einstein equations are in the form

$$8\pi G R^2 R_{,r}\,\rho = 2M_{,r}\,, \tag{4.20}$$

$$8\pi G R^2 R_{,t}\,p = -2M_{,t}\,, \tag{4.21}$$

where the function $M(t,r)$ represents the mass inside the shell labeled by coordinate r and can be computed by the formula

$$\begin{aligned} 2M(t,r) &= R(t,r) + R(t,r)e^{-A(t,r)}{R_{,t}}^2 \\ &- e^{-B(t,r)}{R_{,r}}^2\,R(t,r) - \frac{1}{3}\Lambda R^3(t,r). \end{aligned} \tag{4.22}$$

The conservation equations ${T^{\alpha\beta}}_{;\beta}$ give us

$$B_{,t} + 4\frac{R_{,t}}{R} = -\frac{2\rho_{,t}}{\rho + p}, \tag{4.23}$$

$$A_{,r} = -\frac{2p_{,r}}{\rho + p}, \tag{4.24}$$

$$\frac{\partial p}{\partial \theta} = 0, \tag{4.25}$$

$$\frac{\partial p}{\partial \phi} = 0. \tag{4.26}$$

From the last two equations we can see that the perfect fluid has the same symmetries as the spacetime metric. The function $e^{B(t,r)}$ can be integrated to give

$$e^{B(t,r)} = \frac{R_{,r}{}^2(t,r)}{1+2E(r)} \exp\left(\int_{t_0}^{t} d\tilde{t} \frac{2R_{,t}(\tilde{t},r)}{\left[\rho(\tilde{t},r)+p(\tilde{t},r)\right] R_{,r}(\tilde{t},r)} p_{,r}(\tilde{t},r) \right), \tag{4.27}$$

where $E(r)$ is an arbitrary function. The LTB limit appears when the fluid is reduced to dust.

4.5 The Stephani – Barnes family

The Stephani – Barnes (S–B) family is invariantly characterized by the following relations [61].

- It has zero shear
- It has zero rotation
- It has non-zero expansion

There exist two subclasses of this family

4.5.1 The conformally flat solution

This class of spacetime represents the most general conformally flat solution with a perfect fluid with nonzero expansion. For the conformally flat S-B solution the metric tensor reads

$$ds^2 = D^2 dt^2 - V^{-2}(t,x,y,z)(dx^2+dy^2+dz^2), \tag{4.28}$$

where the functions D and V have a form

$$D = F(t)\frac{V_{,t}}{V}, \tag{4.29}$$

$$V = \frac{1}{R}\left\{1+\frac{1}{4}k(t)\left[(x-x_0(t))^2+(y-y_0(t))^2+(z-z_0(t))^2\right]\right\}. \tag{4.30}$$

Here $F(t), R(t), k(t), x_0(t), y_0(t)$ and $z_0(t)$ are arbitrary functions of time. The expansion scalar is related to the function F by the relation

$$\theta = \frac{3}{F}. \tag{4.31}$$

$k(r)$ generalizes curvature parameter in FRW model and can change the sign during evolution. The matter density and pressure can be computed by the formulas

$$8\pi G\rho = 3kR^2 + \frac{3}{F^2} := 3C^2(t), \tag{4.32}$$

$$8\pi G p = -3C^2(t) + 2CC_{,t}\frac{V}{V_{,t}}. \tag{4.33}$$

Here we can see that the matter density depends only on radial coordinate, but the pressure depends on all coordinates. The solution in general has no symmetries.

4.5.2 The Petrov type D solutions

In this class of spacetimes the equations for conformally flat solution (4.28) and (4.29) still hold but the function $V(t, x, y, z)$ is determined from the Einstein equation

$$\frac{w_{,uu}}{w^2} = f(u). \tag{4.34}$$

Here $f(u)$ is an arbitrary function. The variable u and the function w relates to the coordinates x, y, z and the function $V(t, x, y, z)$ depending on a given type of model. There are three possibilities

- $(u, w) = (r^2, V)$ for spherically symmetric models,
- $(u, w) = (z, V)$ for plane symmetric models,
- $(u, w) = (x/y, V/y)$ for hyperbolically symmetric models,

where $r^2 = x^2 + y^2 + z^2$.
These three classes were derived by Barnes [2]. An important solution inside this class is so called McVittie solution. The line element reads

$$ds^2 = \left[\frac{1-\mu(t,r)}{1+\mu(t,r)}\right]^2 - R^2(t)\frac{[1-\mu(t,r)]^4}{\left(1+\frac{1}{4}kr^2\right)^2}\left[dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right], \tag{4.35}$$

$$\mu(t,r) = \frac{M}{2rR}\sqrt{1+\frac{1}{4}kr^2}, \tag{4.36}$$

where m and k are arbitrary constrants and $R(t)$ is an arbitrary function. For the case $m = 0$ the metric can reproduce the whole FRW class. When we put $k = 0$ and $R = 1$ the solution reduces to Schwarzschild spacetime. McVittie spacetime can be interpreted as an exact superposition of the FRW and Schwarzschild metrics, with a perfect fluid source. The disadvantage of this solution is that it contains an arbitrary function of time so that the evolution law for the Universe is not well defined. This problem is shared with the whole S-B family. One way of overcoming this problem is to impose an equation of state. The problem is that for barotropic equation of state $f(p, \rho) = 0$ McVittie solution reduces to FRW model.

4.6 LRS spacetime

In this section we will review locally rotationally symmetric (LRS) spacetime. We include this family into this chapter about inhomogeneous cosmological models despite the fact that only some representatives belong to this kind of models - some of the LRS models are inhomogeneous, but there exist also homogeneous LRS models, e.g. LRS Bianchi cosmologies. In the later text LRS class II dust spacetime will be used for averaging and the averaged equation will generalize the Buchert equations in the sense that all Einstein equations will be averaged.

LRS dust spacetimes are defined by the following characterization [32]: In an open neighborhood of each point p, there exists a nondiscrete subgroup of the Lorentz group which leaves the Riemann tensor and its covariant derivatives up to third order invariant. Therefore, in LRS spacetimes there exists a preferred direction e^{μ} (the axis of symmetry) in every point. The subgroup can be one or three dimensional. In the second case, we can rotate the axis of symmetry and spacetimes are everywhere isotropic - these are FRW models.

We will use the covariant 3+1 splitting of spacetime with the timelike vector field u^{μ} normalized by the condition $u_{\rho}u^{\rho} = -1$ and the projection tensor $h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$. In this section we will follow the article of van Elst and Ellis [36].

Preferred spacelike vector field e^{μ} satisfies the following conditions:

$$e_{\rho}u^{\rho} = 0, \quad e_{\rho}e^{\rho} = 1. \tag{4.37}$$

Because of the property of the LRS spacetime, all covariantly defined spacelike vectors orthogonal to u^{μ} (acceleration $\dot{u}^{\mu}$, vorticity ω^{μ}, projected gradient of density $h^{\sigma}_{\ \mu}\nabla_{\sigma}\rho$, pressure $h^{\sigma}_{\ \mu}\nabla_{\sigma}p$ and expansion $h^{\sigma}_{\ \mu}\nabla_{\sigma}\theta$) must be proportional to e^{μ} - if this condition does not hold, spacelike vectors will not be invariant under the rotation about e^{μ}.

$$\dot{u}^{\mu} = \dot{u}e^{\mu}, \qquad \omega^{\mu} = \omega e^{\mu}, \tag{4.38}$$

$$h^{\sigma}_{\ \mu}\nabla_{\sigma}\rho = \rho' e_{\mu}, \quad h^{\sigma}_{\ \mu}\nabla_{\sigma}p = p' e_{\mu}, \quad h^{\sigma}_{\ \mu}\nabla_{\sigma}\theta = \theta' e_{\mu}. \tag{4.39}$$

Dot here denotes the covariant derivative along the flow vector u^{μ} and the prime denotes covariant derivative along the vector e^{μ}. We define the magnitude of the spatial rotation k and the magnitude of the spatial divergence a as

$$k := \left|\eta^{\alpha\beta\gamma\delta}\left(\nabla_{\beta}e_{\gamma}\right)u_{\delta}\right|, \tag{4.40}$$

$$a := h^{\alpha}_{\ \beta}\left(\nabla_{\alpha}e^{\beta}\right), \tag{4.41}$$

where $\eta^{\alpha\beta\gamma\delta}$ is totally antisymmetric object ($\eta_{1234} = -\sqrt{-g}$). Similar rule works also for the spacelike tracefree symmetric tensors orthogonal to u^{μ}. We will create a new tensor field $e_{\mu\nu}$ defined from e^{μ}

$$e_{\mu\nu} := \frac{1}{2}\left(3e_{\mu}e_{\nu} - h_{\mu\nu}\right). \tag{4.42}$$

Then we have the relations for the shear tensor and the electric and magnetic parts of the Weyl tensor

$$\sigma_{\mu\nu} = \frac{2}{\sqrt{3}}\sigma e_{\mu\nu}, \quad E_{\mu\nu} = \frac{2}{\sqrt{3}}E e_{\mu\nu}, \quad H_{\mu\nu} = \frac{2}{\sqrt{3}}H e_{\mu\nu}. \tag{4.43}$$

Here we can see that the LRS spacetimes are characterized only by the finite set of the scalar functions: ρ, p, θ, σ, ω, $\dot{u}$, a, k, E and H, where ρ is the mass density.

There exists three classes of LRS family:

- LRS Class I (Rotating solutions): $\omega \neq 0$. This class includes Gödel rotating model of the Universe.
- LRS Class II (Inhomogeneous orthogonal family): $k = \omega = 0$. This class includes LTB solution and its generalization to the case with zero or negative spatial curvature.
- LRS Class III (Homogeneous orthogonal models with twist): $k \neq 0$. This class includes LRS Bianchi cosmology of type II, III, VIII or IV.

In the later part of the section we will summarize some basic facts about LRS class II dust spacetime. The reason for choosing this class is because of its simple properties which makes it easy to average the Einstein equations. Another reason for choosing this class is because LTB model which lies within this family is often used for cosmological and astrophysical applications.

For LRS class II dust model, the magnetic part of the Weyl tensor is equal to zero, $H = 0$. The relevant evolution equations are

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi\rho, \tag{4.44}$$

$$\dot{\sigma} = -\frac{1}{\sqrt{3}}\sigma^2 - \frac{2}{3}\theta\sigma - E, \tag{4.45}$$

$$\dot{E} = -4\pi\rho\sigma + \sqrt{3}E\sigma - \theta E, \tag{4.46}$$

$$\dot{\rho} = -\rho\theta, \tag{4.47}$$

$$\dot{a} = -\frac{1}{3}a\theta + \frac{1}{\sqrt{3}}a\sigma. \tag{4.48}$$

and the constraints

$$\sigma' = \frac{1}{\sqrt{3}}\theta' - \frac{2}{3}a\sigma, \tag{4.49}$$

$$E' = -\frac{3}{2}aE + \frac{4\pi}{\sqrt{3}}\rho', \tag{4.50}$$

$$a' = \frac{2}{9}\theta^2 + \frac{2}{3\sqrt{3}}\theta\sigma - \frac{4}{3}\sigma^2 - \frac{2}{\sqrt{3}}E - \frac{1}{2}a^2 - \frac{16\pi}{3}\rho. \tag{4.51}$$

If we perform time derivative of the constraints, we can see that they do not change with time. In the later text evolution and constrained equations will be averaged.

4.7 Other cosmological models

In this section we will mention some models which were not classified in previous text [10]. For LTB model there exists a generalized solution where the source is in the form of a charged dust. Charged perfect fluid solution was also found for Barnes class. For both LTB and Barnes class the source can also have nonzero viscosity or heat conduction.

There exist models with null radiation. These are superpositions of FRW spacetime with vacuum models like Schwarzschild, Kerr or Kerr–Newman. The superposition does not have perfect fluid as a source but it has a mixture of perfect fluid with null radiation, sometimes also with electromagnetic field. In this family superposition of Schwarzschild metric with FRW model does not result in McVittie model.

Another models which are applied in the early universe are spacetimes with a perfect fluid obeying stiff equation of state - i.e. energy density = pressure. These solutions have two-dimensional Abelian group acting on spacelike orbits. For an extensive review of inhomogeneous cosmological models see Krasiński's book [61].

5. Averaging in cosmology based on Cartan scalars

- Kašpar, P., Svítek, O.: *Averaging in cosmology based on Cartan scalars*, Class. Quantum Grav. 31, 095012 (2014).

In this chapter we will show the method how to average not only the spacetime geometry, but also the Einstein equations. It utilizes the theory of Cartan scalars originally used for comparing spacetimes in the so called equivalence problem. This method is closely related to the work of Coley [28] presented in chapter 1, who used scalar invariants created from Riemann tensor and its covariant derivatives. For application of the theory of averaging by Cartan scalars we considered LTB model which was described in chapter 4. For special ansatz of the radial function we found that correlation term behaves like a positive cosmological constant.

IOP Publishing | Classical and Quantum Gravity

Class. Quantum Grav. **31** (2014) 095012 (11pp) doi:10.1088/0264-9381/31/9/095012

Averaging in cosmology based on Cartan scalars

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Received 23 January 2014, revised 21 March 2014
Accepted for publication 24 March 2014
Published 16 April 2014

Abstract

We present a new approach for averaging in general relativity and cosmology. After a short review of the theory originally taken from the equivalence problem, we consider two ways of dealing with averaging based on Cartan scalars. We apply the theory for two different Lemaître–Tolman–Bondi models. In the first one, the correlation term behaves as a positive cosmological constant, in the second example, the leading correlation term behaves like spatial curvature. We also show the non-triviality of averaging for linearized monochromatic gravitational wave.

Keywords: cosmology, Cartan scalars, averaging
PACS number: 98.80.Jk

1. Introduction

In general relativity and cosmology, we often deal with spacetimes that have many symmetries. We can justify this step by choosing some particular length scale and claim that our simple spacetime is the average of some more realistic model. The main motivation for the averaging comes from cosmology. Gravity is well tested within our solar system. On cosmological scales, we do not need to know the details about a fluctuating gravitational field. In order to obtain a 'macroscopic' theory of gravity, we should perform averaging of Einstein equations. These equations are strongly nonlinear, so if we want to use averaged metric, we have to add a correlation term which does not need to satisfy the usual energy conditions and can act as a dark energy. The problem is that averaging involves integration of the tensor field on the curved manifold and this operation is not well-defined.

The most popular approach to averaging is scalar averaging and investigation of the so-called Buchert equations [1, 2], where only the scalar part of the Einstein equations is averaged (see [3] for a recent review). All Einstein equations are averaged in the context of macroscopic

0264-9381/14/095012+11$33.00

gravity (MG) [4, 5], and at the same time the Cartan structure equations which describe the geometry of spacetime are averaged. A theorem about isometric embedding of a 2-sphere into Euclidean space is applied for averaging by Korzyński [6]. In [7] the Weitzenböck connection for parallel transport is used for the definition of the average value of tensor field.

The theory of Cartan scalars was developed in order to decide if two spacetimes are locally equivalent [8, 9]. We can use this theory for local characterization of a given spacetime. Then, inspired by the method given by Coley [10], who investigated averaged scalar invariants constructed from the Riemann tensor and a finite number of its covariant derivatives, we average the left-hand side of Einstein equations (which contain a finite number of the Cartan scalars if rewritten in tetrad form) and we give the prescription for the computation of the correlation term.

In the first section, we review the theory of Cartan scalars, then after a short introduction of Lemaître–Tolman–Bondi (LTB) spacetime, we give two different examples of averaging by Cartan scalars. The first one utilizes approximation for the areal function $R(t, r)$. In the second example we investigate backreaction for the LTB metric given by Biswas *et al* [11]. Then we consider the averaged linearized monochromatic gravitational wave and we end with the conclusion.

2. Cartan scalars

If we want to specify the geometry of spacetime, we are allowed to choose the $\frac{n(n+1)}{2}$ components of the metric tensor. There also exists another possibility. It can be shown that the tetrad projection of Riemann tensor and the finite number of its covariant derivatives (called Cartan scalars) completely (locally) specify the geometry of Riemannian manifold [8]. Cartan scalars are true scalars on the bundle of frames $F(\mathcal{M})$, but if we fix the tetrad, they behave as scalars on the manifold as well. Because it is still not clear how to unambiguously average a metric tensor, there exists a possibility of describing the geometry with Cartan scalars and average them (which is straightforward in the case of scalars).

There exists another advantage within this formalism. The left-hand side of the Einstein equations can be rewritten in the tetrad form, so it consists of the finite sum of Cartan scalars. Using Cartan scalars we can average not only the spacetime geometry but also the left-hand side of the Einstein equations. From the Cartan scalars we can easily read off a dimension of an isometry group and we can obtain an algebra of the Killing vectors [12].

We will review the construction of the Cartan scalars [9], [13], [14]. Let $\mathcal{M}$ be an n-dimensional differentiable manifold with a metric

$$\mathbf{g} = \eta_{ij}\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{1}$$

where η_{ij} is a constant symmetric matrix and $\boldsymbol{\omega}^i$, i = 1,2...,n form a basis of the cotangent space at the point x^μ. The tetrad (frame) $\boldsymbol{\omega}^i$ is for a given $\mathbf{g}$ and η_{ij} fixed up to the generalized rotations.

$$\boldsymbol{\omega}^i = \omega^i_\nu(x^\mu, \xi^\Upsilon)\,\mathbf{d}x^\nu, \tag{2}$$

where ξ^Υ, $\Upsilon = 1, \ldots, \frac{1}{2}n(n-1)$, denotes the coordinates of an orthogonal group. For simplicity, we will define all geometrical objects on the enlarged $\frac{1}{2}n(n+1)$-dimensional space—the bundle of frames $F(\mathcal{M})$. $F(\mathcal{M})$ is locally isomorphic to the Cartesian product of an open set on the manifold (spacetime) and the orthogonal (Lorentz) group G—it means that in every point x^μ there exists a fiber with coordinates ξ^Υ. In the following we will use an enlarged exterior derivative in the form $\mathbf{d} = \mathbf{d}_x + \mathbf{d}_\xi$. Cartan structure equations read

$$\mathbf{d}\boldsymbol{\omega}^i = \boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^i{}_j, \tag{3}$$

$$\mathbf{d}\boldsymbol{\omega}^i{}_j = -\boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j + \tfrac{1}{2} R^i{}_{jkl} \boldsymbol{\omega}^k \wedge \boldsymbol{\omega}^l. \tag{4}$$

with a condition

$$\eta_{ik} \boldsymbol{\omega}^k{}_j + \eta_{jk} \boldsymbol{\omega}^k{}_i = 0. \tag{5}$$

From the first equation we can compute the connection 1-form $\boldsymbol{\omega}^i{}_j$, next equation serves as a definition of the curvature tensor $R^i{}_{jkl}$. To generate covariant derivatives of the Riemann tensor, we repeatedly apply an exterior derivative:

$$\begin{aligned}
\mathbf{d}R_{ijkl} &= R_{mjkl}\boldsymbol{\omega}^m{}_i + R_{imkl}\boldsymbol{\omega}^m{}_j + R_{ijml}\boldsymbol{\omega}^m{}_k + R_{ijkm}\boldsymbol{\omega}^m{}_l + R_{ijkl;m}\boldsymbol{\omega}^m, \\
\mathbf{d}R_{ijkl;n} &= R_{mjkl;n}\boldsymbol{\omega}^m{}_i + R_{imkl;n}\boldsymbol{\omega}^m{}_i + \cdots + R_{ijkl;nm}\boldsymbol{\omega}^m, \\
&\;\;. \\
&\;\;. \\
&\;\;.
\end{aligned} \tag{6}$$

Let R^p denote the set $\{R_{ijkm}, R_{ijkm;n_1}, \ldots, R_{ijkm;n_1\ldots n_p}\}$ where p is such that R^{p+1} contains no element that is functionally independent of the elements in R^p. Two functions f and g are functionally independent if the one-form $\mathbf{d}f$ and $\mathbf{d}g$ are linearly independent. Then the set R^{p+1} characterizes the geometry completely and its elements are called Cartan scalars. There exists an algorithmic way to compute Cartan scalars [15]. It uses the standard form of the Riemann tensor that can be found by the Petrov and Segre algorithm (and its generalization for tensors with more indices). However, the tetrad does not need to be fixed completely. There exist some degrees of freedom which can nontrivially transform the components of other tensors, but the Cartan scalars remain fixed. This property allows us to integrate Cartan scalars over some domain $\mathcal{D} \subset \mathcal{M}$ as we will see later.

If we want to specify the geometry of spacetime, we are allowed to choose the $\frac{n(n+1)}{2}$ components of the metric tensor, which satisfy the Einstein equations. If we want to use the Cartan scalars instead, there must exist some algebraic and differential equations that they have to fulfil. In other words, from a given set R^{p+1} we have to find the conditions necessary to construct one-form $\boldsymbol{\omega}^i$, which satisfies the equations (3)–(6). These constraints should be respected also by the averaged Cartan scalars.

To see explicitly the form of the constraints it is easier to rewrite equations (3)–(6) in a more compact way. The connection one form is defined on the bundle of frames $F(\mathcal{M})$ as

$$\boldsymbol{\omega}^i{}_j = \gamma^i{}_{jk}\boldsymbol{\omega}^k + \boldsymbol{\tau}^i{}_j, \tag{7}$$

where $\boldsymbol{\tau}^i{}_j = \tau^i{}_{j\Upsilon}\mathbf{d}\xi^\Upsilon$ generates the orthogonal group and $\gamma^i{}_{jk}$ are the Ricci rotation coefficients. It means that $\boldsymbol{\omega}^i{}_j$ and $\boldsymbol{\omega}^k$ are independent objects on $F(\mathcal{M})$ and we can denote them collectively as $\{\boldsymbol{\omega}^I\} \equiv \{\boldsymbol{\omega}^i, \boldsymbol{\omega}^i_j\}$, $I = 1, 2, \ldots \frac{1}{2}n(n+1)$. Cartan structure equations can be rewritten in the simple form as

$$\mathbf{d}\boldsymbol{\omega}^I = \tfrac{1}{2} C^I{}_{JK}\boldsymbol{\omega}^J \wedge \boldsymbol{\omega}^K. \tag{8}$$

$C^I{}_{JK}$ essentially represents the Riemann tensor on $F(\mathcal{M})$. We will denote a maximal set of the functionally independent objects in R^p as I^α, $\alpha = 1,\ldots,k \leqslant \frac{1}{2}n(n+1)$, which can be thought of as the coordinates on the bundle of frames. It means that all objects in R^{p+1} are functions of I^α only. By applying an exterior derivative we will obtain an analogue of the equation (6)

$$\begin{aligned}
\mathbf{d}C^I{}_{JK} &= C^I{}_{JK,\alpha}\mathbf{d}I^\alpha \equiv C^I{}_{JK,\alpha}I^\alpha{}_{|L}\boldsymbol{\omega}^L \equiv C^I{}_{JK|L}\boldsymbol{\omega}^L,\\
\mathbf{d}C^I{}_{JK|L} &= C^I{}_{JK|LM}\boldsymbol{\omega}^M,\\
&\ \ \vdots
\end{aligned} \tag{9}$$

Symbol | here denotes the derivative with respect to the vector field dual to the 1-form $\boldsymbol{\omega}^L$ and similarly, symbol ‘,’ represents the derivative with respect to the vector field dual to $\mathbf{d}I^\alpha$. We can see from the above equations that R^{p+1} can be constructed from the set $\{C^I_{JK}, I^\alpha{}_{|L}\}$. The constraints that have to be satisfied then read

$$\begin{aligned}
&I^\alpha{}_{|K,\beta}I^\beta{}_{|J} - I^\alpha{}_{|J,\beta}I^\beta{}_{|K} + I^\alpha{}_{|L}C^L_{JK} = 0,\\
&C^P_{[JK|L]} + C^P_{M[K}C^M_{LJ]} = 0.
\end{aligned} \tag{10}$$

3. Averaging Cartan scalars

Let us suppose that we have a given manifold $\mathcal{M}$ characterized by the set of scalar functions R^{p+1} and a given domain $\mathcal{D}$. We would like to obtain a new manifold $\langle\mathcal{M}\rangle$—identical as a set but with a smooth metric structure, which would not recognize quickly fluctuating inhomogeneities of the gravitational field. The naive approach would consist of the integration of the scalar function $f \in R^{p+1}$ according to the rule

$$\langle f\rangle(x) = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} f(x+x')\,\mathrm{d}^N x', \tag{11}$$

where $\mathrm{d}^N x$ is an invariant metric volume element. Following this rule we would obtain a new set $\langle R^{p+1}\rangle$. The problem is that the elements of $\langle R^{p+1}\rangle$ would not satisfy the constraints (which can be written as (10)) because of the nonlinearity of the equations.

We will deal with the problem in a similar way as Coley did [10]. First we will restrict ourselves to the smallest possible set of independent functions $R'^{p+1} \subseteq R^{p+1}$ (with the help of the constraints it would be possible to generate the whole set R^{p+1}) and proceed with averaging of R'^{p+1}. We will obtain a new set $\langle R'^{p+1}\rangle$. In the next step, we have to suppose that the constraints will have the same form (they are not modified by correlation terms) and as a result we can generate the whole set $\langle R^{p+1}\rangle$ from $\langle R'^{p+1}\rangle$. The theory then guarantees that there exists the metric tensor $\langle g_{\mu\nu}\rangle$ (or equivalently the 1-forms $\langle\boldsymbol{\omega}^i\rangle$). With the help of the equations (3)–(6) it will give rise to the known functions $\langle R^{p+1}\rangle$.

If we apply averaging to R'^{p+1}, the number of independent functions will be usually decreasing as a consequence of an enlarged isotropy group of the new spacetime $\langle\mathcal{M}\rangle$. We can also obtain an algebra of the Killing vectors [12].

In practice there are two goals of averaging—the first is an averaging of the spacetime geometry and the second is an averaging of the Einstein equations. We can see that the left-hand side of the Einstein equations (rewritten in the tetrad form when the frame is fixed by the Cartan–Karlhede algorithm) contains the sum of the Cartan scalars and these can be integrated simply as scalar functions. Einstein equations are nonlinear in metric tensor, so we can expect that after averaging we will obtain equations in the form

$$R^\mu{}_\nu(\overline{g_{\alpha\beta}}) - \tfrac{1}{2}R(\overline{g_{\alpha\beta}})\delta^\mu{}_\nu + C^\mu{}_\nu = 8\pi T^\mu{}_\nu(\overline{g_{\alpha\beta}}). \tag{12}$$

Here we suppose that $R^\mu{}_\nu(\overline{g_{\alpha\beta}})$ is the macroscopic Ricci tensor, which is obtained from the averaged metric $\overline{g_{\alpha\beta}}$. The same holds for $T^\mu{}_\nu(\overline{g_{\alpha\beta}})$. In several cases we explicitly suppose the form of the metric structure on the averaged manifold $\overline{\mathcal{M}}$—for example in cosmology it is usual to suppose homogeneous and isotropic FRW models. It is questionable whether this kind of ansatz is adequate. It is straightforward to create perturbations from the symmetric spaces but the inverse procedure is not so clear. By averaging inhomogeneous metric we could also obtain a situation where the averaged spacetime has a nonzero Weyl tensor or where the correlation term is not in the form of a homogeneous and isotropic perfect fluid. It is also ambiguous how to interpret the correlation term.

First we could use the averaging of Cartan scalars described above and obtain a new macroscopic metric tensor $\langle g_{\alpha\beta}\rangle$ (in general not very simple). Einstein tensor is created from $\langle g_{\alpha\beta}\rangle$. The correct averaging procedure is guaranteed, but the macroscopic metric is gained by a rather difficult method (how to obtain the one-form $\boldsymbol{\omega}^i$ from the Cartan scalars R^{p+1} is shown e.g. in [14]). The correlation term is equal to zero—or more precisely, the geometrical correction is hidden into the macroscopic Ricci tensor $\langle R^{\mu}{}_{\nu}\rangle$ ($\langle R^{\mu}{}_{\nu}\rangle$ is constructed using Cartan scalars averaged according to the definition (11)). The advantage of this approach is the possibility to see how the symmetry is increasing after averaging.

More straightforward and, for its simplicity, more acceptable is the second approach: suppose the averaged (macroscopic) metric tensor $\overline{g_{\alpha\beta}}$ is given (e.g. spherical symmetric, homogeneous, Friedmann–Robertson–Walker (FRW), . . .). Then compute the averaged Cartan scalars and compare it with the Cartan scalars for the macroscopic metric—it is possible to see if the form is the same and under which conditions these two are comparable. Now we have two Ricci tensors—the first one is the macroscopic $R^{\mu}{}_{\nu}(\overline{g_{\alpha\beta}})$ (built from the known $\overline{g_{\alpha\beta}}$) and the second one is $\langle R^{\mu}{}_{\nu}\rangle$ (in the previous paragraph these two were the same). We can define the correlation term as

$$C^{\mu}{}_{\nu} = \langle R^{\mu}{}_{\nu}\rangle - \tfrac{1}{2}\langle R\rangle\delta^{\mu}{}_{\nu} - R^{\mu}{}_{\nu}(\overline{g_{\alpha\beta}}) - \tfrac{1}{2}R(\overline{g_{\alpha\beta}})\delta^{\mu}{}_{\nu}. \tag{13}$$

The Ricci tensor $R^{\mu}{}_{\nu}(\overline{g_{\alpha\beta}})$ satisfies the contracted Bianchi identities and as a consequence the locally conserved object is not the tensor $T^{\mu}{}_{\nu}(\overline{g_{\alpha\beta}})$ but the expression $T^{\mu}{}_{\nu}(\overline{g_{\alpha\beta}}) - C^{\mu}{}_{\nu}$. Correlation term can be interpreted as a part of the conserved stress–energy tensor

$${}^{(ef)}T^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}(\overline{g_{\alpha\beta}}) - C^{\mu}{}_{\nu}. \tag{14}$$

We can divide averaging into several steps: guess the right macroscopic metric, compute an averaged Cartan scalars and find the correlation term, which can modify the macroscopic metric.

The question is how to decide between these two approaches [10]. In the first one, the procedure is unambiguous and the averaged metric tensor can be constructed (despite technical difficulty). The second one is much easier—it remains to be clarified whether it is possible to use the simplified metric without losing important information about the inhomogeneous metric. In cosmology, the question is under which circumstances it is possible to characterize the spacetime by only one scale function $a(t)$ and how the form of $a(t)$ is changed by the correlation term. It would cause a problem, if the correlation term did not satisfy the form of stress–energy tensor of the 'guessed' metric (a homogeneous and isotropic perfect fluid in the case of FRW spacetime) and its magnitude would not be negligible. Then we have to use the first approach.

A similar situation presents itself in the theory of MG [4, 5]—it is necessary to choose which averaged object will be considered as fundamental. In MG the main geometrical objects used in the averaging procedure are Christoffel symbols. In our case, the first possibility is to choose the Riemann tensor (and it's covariant derivatives) because we average Cartan scalars, the second one is the macroscopic metric.

So far, we were dealing with scalars averaged at a single point. If we want to obtain a unique prescription for the averaged scalar field, we should have a rule how to choose a domain at the point x' from a given domain at x. This problem was discussed by Zalaletdinov in the context of MG [16], where the definition of the averaged geometrical objects depends on the choice of the bilocal operators. We will leave this rule unspecified but we will be guided by the symmetries of spacetime. In the next chapter we will assume thick spherical shells for averaging Cartan scalars in a LTB spacetime.

The problem that remains is how to practically use the constraints (10). For making some explicit calculations, we usually use the fixed frame formalism [14], where $I^{\alpha}{}_{|K}$ correspond to the gradients of coordinates and Ricci rotation coefficients $\{x^{\mu}{}_{|k}, \gamma^{m}{}_{kn}\}$ and we have to deal with the difficulty of how to average tetrad. In the next chapters we will use the minimal set of Cartan scalars introduced by MacCallum and Åman [17] and implemented in the algebraic program SHEEP [18].

Next, remark should be added. The whole averaging procedure strongly depends on the choice of the frame. In some spacetimes the tetrad can be chosen in a well-defined way. This usually works well for spacetimes with an additional symmetry (as will be the case for the spherically symmetric LTB metric discussed in the next section), but the method is not suited e.g. for the general perturbations of FRW, where the frame is restricted only by the algebraic property of spacetime. Another possibility would be to choose the frame by minimizing a certain kind of functional as done by Behrend [19] in the context of averaging.

Correct averaging should not change the metric structure of the space with a constant curvature. In this case there is only one nonzero Cartan scalar (Ricci scalar or lambda term in NP formalism), which is constant and the averaging does not change its value. If we have a constant curvature space and perform averaging by Cartan scalars, we obtain the same space.

4. Cartan scalars of FRW spacetime

It is most common to use, for its simplicity, the FRW model as a template for interpreting the cosmological data. It is believed that it is a good approximation of the universe over the large scales. We will consider a flat FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{15}$$

The following computations are performed using the algebraic program SHEEP [18]. Nonzero Cartan scalars are

$$\phi_{00'} = \phi_{22'} = 2\phi_{11'} = -\frac{1}{2}a^{-1}a_{,tt} + \frac{1}{2}a^{-2}(a_{,t})^2, \tag{16}$$

$$\Lambda = \frac{1}{4}a^{-1}a_{,tt} + \frac{1}{4}a^{-2}(a_{,t})^2, \tag{17}$$

$$D\phi_{00'} = D\phi_{33'} = 3D\phi_{11'} = 3D\phi_{22'} = -\frac{1}{2\sqrt{2}}a^{-1}a_{,ttt} + \frac{5}{2\sqrt{2}}a^{-2}a_{,t}a_{,tt} - \sqrt{2}a^{-3}(a_{,t})^3, \tag{18}$$

$$D\Lambda_{00'} = D\Lambda_{11'} = \frac{1}{4\sqrt{2}}a^{-1}a_{,ttt} + \frac{1}{4\sqrt{2}}a^{-2}a_{,t}a_{,tt} - \frac{1}{2\sqrt{2}}a^{-3}(a_{,t})^3. \tag{19}$$

Now, if we have an inhomogeneous model, we can compare the averaged Cartan scalars with the FRW case. By comparing two different sets of scalars, we can see under which conditions we can obtain an effective FRW metric by averaging.

5. LTB metric

The LTB metric [20–22] is a spherically symmetric exact solution of the Einstein equations. It corresponds to an inhomogeneous dust with the stress–energy tensor

$$T_{\mu\nu} = \rho u_{\mu}u_{\nu}, \tag{20}$$

where u_μ is 4-velocity of a dust with a density ρ. For a recent review of LTB metric see e.g. [23, 24]. The line element reads

$$ds^2 = -dt^2 + \frac{(R')^2}{1+2E(r)}\,dr^2 + R^2(t,r)(d\theta^2 + \sin^2(\theta)\,d\phi^2), \tag{21}$$

where $E(r)$ is an arbitrary function and the prime denotes partial derivative with respect to r. Function $R(t,r)$ obeys the Einstein equations if

$$R_{,t}^2 = 2E + \frac{2M}{R} + \frac{\Lambda}{3}R^2, \tag{22}$$

where $M = M(r)$ is another arbitrary function of integration. The energy density ρ is determined by the equation

$$4\pi\rho = \frac{M'}{R'R^2}. \tag{23}$$

The function $E(r)$ determines a curvature of the space $t = const.$ (which is flat for $E(r) = 0$) and the function $M(r)$ is the gravitational mass contained within the comoving spherical shell at any given r. Equation (22) can be integrated to give the result

$$\int_0^R \frac{d\tilde{R}}{\sqrt{2E + \frac{2M}{\tilde{R}} + \frac{1}{3}\Lambda\tilde{R}^2}} = t - t_B(r), \tag{24}$$

$t_B(r)$ is the third free function of r (called the bang-time function). In the LTB model, in general, the big bang is not simultaneous as in the FRW case, but it depends on the radial coordinate r. The given formulas are invariant under transformation $\tilde{r} = g(r)$. We can use this freedom to choose one of the functions $E(r)$, $M(r)$ and $t_B(r)$. For $\Lambda = 0$ the above equation can be solved explicitly—when $E < 0$ (elliptic evolution)

$$R(t,r) = \frac{M}{(-2E)}(1-\cos\eta),$$
$$\eta - \sin\eta = \frac{(-2E)^{3/2}}{M}(t - t_B). \tag{25}$$

If E = 0 (parabolic evolution)

$$R(t,r) = \left[\tfrac{9}{2}M(t-t_B)^2\right]^{1/3}, \tag{26}$$

when $E > 0$ (hyperbolic evolution)

$$R(t,r) = \frac{M}{2E}(\cosh\eta - 1),$$
$$\sinh\eta - \eta = \frac{(2E)^{3/2}}{M}(t - t_B(r)). \tag{27}$$

6. Averaging LTB spacetime

For simplicity we will consider the situation when $E = 0$. Unfortunately Cartan scalars for the exact solution listed above are too complicated. We will deal only with an areal function $R(t,r)$. The first guess would be to investigate the separated form $R(t,r) = A(t)B(r)$. However, by the simple radial transformation $dr' = B'(r)\,dr$ we obtain flat FRW spacetime (the result is easily checked by computing the Cartan scalars, which depend only on the t coordinate).

Next, we will assume the ansatz

$$R(t,r) = A(t,r)\exp\psi(t,r), \tag{28}$$

where $\psi(t, r)$ is a quickly varying function, $\psi \ll \psi_{,x} \sim \psi_{,xy} \sim \psi_{,xyz}$, where x, y and z denote time or radial coordinate. $\psi_{,x}$ is also much bigger than $A(t, r)$ and its derivatives. In order to compute the Cartan scalars we will use the null tetrad

$$\begin{aligned}
\boldsymbol{\omega}^0 &= \frac{1}{\sqrt{2}}(\mathrm{d}t + R_{,r}\,\mathrm{d}r),\\
\boldsymbol{\omega}^1 &= \frac{1}{\sqrt{2}}(\mathrm{d}t - R_{,r}\,\mathrm{d}r),\\
\boldsymbol{\omega}^2 &= \frac{1}{\sqrt{2}}(R\,\mathrm{d}\theta + \mathrm{i}R\sin\theta\,\mathrm{d}\phi),\\
\boldsymbol{\omega}^3 &= \frac{1}{\sqrt{2}}(R\,\mathrm{d}\theta - \mathrm{i}R\sin\theta\,\mathrm{d}\phi).
\end{aligned} \tag{29}$$

Nontrivial zero-order Cartan scalars are

$$\psi_2 = -\tfrac{1}{6}(R_{,r})^{-1}R_{,ttr} + \tfrac{1}{6}R^{-1}R_{,t}(R_{,r})^{-1}R_{,tr} + \tfrac{1}{6}R^{-1}R_{,tt} - \tfrac{1}{6}R^{-2}(R_{,t})^2, \tag{30}$$

$$\phi_{00'} = \phi_{22'} = \tfrac{1}{2}R^{-1}R_{,t}(R_{,r})^{-1}R_{,tr} - \tfrac{1}{2}R^{-1}R_{,tt}, \tag{31}$$

$$\phi_{11'} = -\tfrac{1}{4}(R_{,r})^{-1}R_{,ttr} + \tfrac{1}{4}R^{-2}(R_{,t})^2, \tag{32}$$

$$\Lambda = \tfrac{1}{12}(R_{,r})^{-1}R_{,ttr} + \tfrac{1}{6}R^{-1}R_{,t}(R_{,r})^{-1}R_{,tr} + \tfrac{1}{6}R^{-1}R_{,tt} + \tfrac{1}{12}R^{-2}(R_{,t})^2. \tag{33}$$

We plug the form (28) into the spinors. The most important terms are the ones with higher powers of various derivatives of the function ψ. Function $A(t, r)$ appears in the same power in the numerator and in the denominator and is canceled. If we assume the condition $\psi \ll \psi_{,x} \sim \psi_{,xy} \sim \psi_{,xyz}$, in the leading order all quantities are equal to zero except

$$\Lambda = \tfrac{1}{2}\psi_{,t}^2. \tag{34}$$

Averaging Λ over the domain $\mathcal{D}$ of the shape of the thick shell (times a certain time interval) gives a nonzero contribution which can be constant by a suitable choice of ψ and $\mathcal{D}$. The first order Cartan scalars contain more terms (higher order Cartan scalars are equal to zero). A lengthy but straightforward calculation shows, that in this approximation they are (in the leading order) all equal to zero. For example the simplest one is

$$\begin{aligned}
D\phi_{00'} = {}& \frac{1}{2\sqrt{2}}R^{-1}R_{,t}(R_{,r})^{-1}R_{,ttr} - \frac{3}{2\sqrt{2}}R^{-1}R_{,t}(R_{,r})^{-2}(R_{,tr})^2\\
&+\frac{1}{2\sqrt{2}}R^{-1}R_{,t}(R_{,r})^{-2}R_{,ttr} - \frac{1}{2\sqrt{2}}R^{-1}R_{,t}(R_{,r})^{-3}R_{,tr}R_{,rr}\\
&-\frac{1}{2\sqrt{2}}R^{-1}R_{,ttt} + \frac{3}{2\sqrt{2}}R^{-1}(R_{,r})^{-1}R_{,tt}R_{,tr} - \frac{1}{2\sqrt{2}}R^{-1}(R_{,r})^{-1}R_{,ttr}\\
&+\frac{1}{2\sqrt{2}}R^{-1}(R_{,r})^{-2}(R_{,tr})^2 - \frac{1}{2\sqrt{2}}R^{-2}(R_{,t})^2(R_{,r})^{-1}R_{,tr}\\
&+\frac{1}{2\sqrt{2}}R^{-2}R_{,t}R_{,tt} - \frac{1}{2\sqrt{2}}R^{-2}R_{,t}(R_{,r})^{-1}R_{,tr} + \frac{1}{2\sqrt{2}}R^{-2}R_{,tt}.
\end{aligned} \tag{35}$$

Now we suppose that the macroscopic metric is a flat FRW spacetime. To have a correct averaging procedure we should have the spinors (16), (18) and (19) equal to zero. If we also assume the class of LTB spacetimes (in our approximation given by $\psi(t, r)$) and the domain $\mathcal{D}$, where the average of Λ is constant, these conditions are fulfilled by (anti-)de Sitter space. Correlation term is in the form of a positive cosmological constant, so the averaged

LTB spacetime behaves (in the leading order) as an FRW model with a positive cosmological constant—de Sitter spacetime.

In the flat solution without cosmological constant we know the explicit form of $R(t, r)$ (26). If we choose the coordinates where the mass function reads $M(r) = \frac{4}{3}\pi M_0^4 r^3$ we can relate the bang-time function $t_B(r)$ to our ansatz

$$t_b(r) = t - \frac{[A(t, r)\exp\psi(t, r)]^{3/2}}{(9/2)^{3/2}\sqrt{4/3\pi}M_0^2 r^{3/2}}, \tag{36}$$

that gives (together with our conditions) big restrictions on the form of $A(t, r)$ and $\psi(t, r)$. This requirement could be relaxed if we allow LTB solution with cosmological constant, where the solutions for $R(t, r)$ involve elliptic functions. This does not give us a very strict formula for areal function $R(t, r)$ as in the flat LTB spacetime without cosmological constant. Regularity conditions in the origin $r = r_c$, where time derivatives of $R(t, r_c)$ have to be equal to zero, and no shell crossing condition $R'(t, r) \neq 0$ has to be also fulfilled. Other constraints which would be difficult to satisfy are Bianchi identities.

We can compare our result with a different approach to averaging in LTB spacetime. Paranjape and Singh [25] showed that in the Buchert equations the backreaction term is equal to zero for a general flat LTB metric (which they call marginally bound LTB)—see [3] for a generalization of this result. We obtained a different result. The first reason is that they used only spatial averaging while we have used a spacetime one. But most importantly, different objects were averaged. Paranjape and Singh [25] averaged a subset of Ricci rotation coefficients for orthogonal frame, namely optical scalars. On the other hand, we average all scalars made from the Riemann tensor and its covariant derivatives. As already mentioned in [25] one can expect an additional influence coming from objects not considered in the averaging procedure. Moreover, the problem of directly comparing these results is rather difficult due to the nonlinear relation between curvature scalars and Ricci rotation coefficients (as can be seen from Newmann–Penrose equations) which would again introduce correlation terms during averaging. Both approaches have their value, the one used in [25] is better suited for direct cosmological application, but the method presented here takes more effects into account.

7. Onion LTB model

As the next example we investigate the onion model used in [11] by Biswas *et al* who computed the corrections to luminosity distance–redshift relation. It represents spacetime with radial shells of overdense and underdense regions. The curvature of three-dimensional spaces is nonzero ($E(r) > 0$), so the evolution of LTB model is hyperbolic. For convenience we will use the rescaled function $a(t, r) := \frac{R(t,r)}{r}$ which is suitable for comparison with the FRW model. It reads

$$a(t, r) := \left(\frac{6}{\pi}\right)^{1/3} t^{2/3}(1 + Lt^{2/3}\frac{1}{r}\sin\pi r\sin\pi r). \tag{37}$$

If we take the trace of the Einstein equations we will find that Ricci scalar (which is proportional to Λ term in the NP formalism) behaves in the same way as the matter density (assuming zero cosmological constant and the equation of state $p = (\gamma - 1)\rho$). The metric function $a(t, r)$ looks like perturbation of the flat dust FRW spacetime, where density scales like $\rho \propto \frac{1}{a(t)^{3\gamma}}$ and we assume that L is a small parameter. From the form of the metric we demand that the averaged spacetime is Einstein-de Sitter (EdS). The Ricci spinor of LTB spacetime is in the form of the perfect fluid. If we perform averaging, the condition for the Ricci spinor to describe perfect fluid does not change. Weyl spinor and higher order Cartan scalars will be discussed later. The most important Cartan scalar is Λ term which reads

$$\Lambda(r,t,L) = \frac{1}{12}\frac{1}{a^2(a+a_{,r})r}[3a^2a_{,tt} + a^2ra_{,ttr} + 3a(a_{,t})^2 + 2a_{,tt}aa_{,r}r + 2a_{,t}raa_{,tr} + (a_{,t})^2ra_{,r} + a_{,r}rK + 3Ka + aK_{,r}]. \quad (38)$$

Function $K(r, L)$ is related to curvature function $E(r)$ by

$$K(r,L) = -\frac{2E(r)}{r^2} = \frac{-L}{\pi r}\sin\pi r \sin\pi r. \quad (39)$$

Here we perform averaging on the constant time surface. We choose one point and the domain Ω and denote a new averaged function $\langle\Lambda\rangle$ which is only time dependent. Next, we expand an averaged Λ term in powers of L and we obtain a series that looks like

$$\langle\Lambda\rangle \approx \frac{A}{t^2} + \frac{B}{t^{4/3}}L + \frac{C}{t^{2/3}}L^2 + \frac{D}{t^0}L^3. \quad (40)$$

The coefficients in front of different powers of L depend on the chosen point and the domain Ω and can be calculated as follows. In the definition of the average value of Λ (11) (but here $\mathcal{D}$ denotes three-dimensional surface), we expand in powers of L the integrated expressions in the numerator and the denominator separately and compute coefficients in front of the time-dependent terms. Then we expand the whole expression and obtain equation (40). Now, we have an averaged EdS background, so the scale factor $a(t)$ is proportional to $t^{2/3}$ and the density scales like $\rho \propto \frac{1}{t^{2\gamma}}$. Now let us assume that we can use this expression for the additional terms in $\langle\Lambda\rangle$ that deviate from EdS. The dominant term describes the dust as expected. The expression proportional to L has an equation of state $p = -\frac{1}{3}\rho$ and behaves like curvature (as interpreted in [26]). Next term can already cause acceleration with the dependence of density on pressure $p = -\frac{2}{3}\rho$ and the term proportional to L^3 behaves like cosmological constant.

We can play the same game with nonzero Weyl scalar ψ_2 and we can see that the first nonzero contribution to $\langle\psi_2\rangle$ is proportional to L. In order to obtain an EdS background, we need to have $\langle\psi_2\rangle = 0$. Also, all higher order Cartan scalars should be comparable with the averaged spacetime up to the corrections in powers of L.

8. Linearized gravitational wave

In the last simple example, we will show the non-triviality of averaging. We assume monochromatic linearized gravitational wave with selected polarization propagating in the direction z on the Minkowski background

$$ds^2 = -dt^2 + (1 + A\sin(t-z))\,dx^2 + (1 - A\sin(t-z))\,dy^2 + dz^2. \quad (41)$$

A is a small parameter describing the amplitude of the gravitational wave. Non-zero lowest-order Cartan scalars read

$$\Psi_4 = \left(\tfrac{1}{2}\sin(t)\cos(z) - \tfrac{1}{2}\sin(z)\cos(t)\right)A + O(A^3),$$
$$\Phi_{22} = \left(\tfrac{1}{4} - \tfrac{3}{4}\cos(z)^2 - \tfrac{3}{4}\cos(t)^2 + \tfrac{3}{2}\cos(t)^2\cos(z)^2\right)A^2 + \left(\tfrac{3}{2}\sin(t)\cos(z)\sin(z)\cos(t)\right)A^2 + O(A^4). \quad (42)$$

If we integrate over several wave lengths, the Weyl scalar vanishes and nonzero Ricci scalar Φ_{22} is constant. Now we assume that averaged spacetime is Minkowski background. Φ_{22} can be put into the right-hand side of the Einstein equation and interpreted as the correlation term which behaves like a null fluid and serves as an effective stress–energy tensor of the gravitational wave. If we would like to determine its influence on the background we would need to consider the next-order Cartan scalars.

9. Conclusion

Theory of Cartan scalars is commonly used for equivalence problem. We have applied this theory in the context of averaging in GR and cosmology. There are two different ways to perform averaging. In the first one, the correlation term is equal to zero, but the averaged geometry is explicitly constructed. In the second approach we assume the form of the smooth metric tensor and compute the correlation term. We used the second approach for computation of backreaction in two different LTB models. Correlation term behaves as a cosmological constant in the first example and the curvature term plus small terms causing acceleration in the second example. Thus the inhomogeneity of spacetime may serve as a reason for accelerated expansion when viewed in averaged picture of standard cosmological models. This is in contrast with the solutions of [26] and [10] where correlation term behaves as a curvature term and does not lead to acceleration. We have also shown the non-triviality of averaging in the case of monochromatic linearized gravitational wave.

Acknowledgments

We would like to thank M Bradley, A A Coley and J Hruška for useful discussions. We would also like to thank J E Åman and M Bradley for providing us with the algebraic program SHEEP. PK was supported by grants GAUK 398911 and SVV-267301. OS was supported by grant GAČR 14-37086G.

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6. Averaging in LRS class II spacetimes

- Kašpar, P., Svítek, O.: *Averaging in LRS class II spacetimes*, submitted to Gen. Rel. Grav.

In this chapter we will introduce method of averaging which holds for LRS class II dust family of spacetimes. This method generalizes Buchert's approach which was introduced in chapter 1 in the sense that all the Einstein equations are averaged. This family of spacetimes was reviewed in chapter 4 about inhomogeneous cosmology. LTB model is the most important representative which belongs to this class. By averaging inside this class we can investigate not only the backreaction on an expansion, but also on a shear scalar. We considered two different ways how to close the system of equations. In a concrete example we obtained deceleration parameter which changes its sign from positive to negative.

Noname manuscript No.
(will be inserted by the editor)

Averaging in LRS class II spacetimes

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Received: date / Accepted: date

Abstract We generalize Buchert's averaged equations [Gen. Rel. Grav. 32, 105 (2000); Gen. Rel. Grav. 33, 1381 (2001)] to LRS class II dust model in the sense that all Einstein equations are averaged, not only the trace part. We derive the relevant averaged equations and in order to close this system we first consider an exact LTB model, where we investigate backreaction on an expansion and a shear scalar. Then we propose an ansatz for closing the system and construct two models with different correlation terms. We numerically compute deceleration parameter, which is initially positive, then changes the sign and goes asymptotically to zero (or a negative constant).

Keywords LRS family; Cosmology; Averaging

1 Introduction

Our universe is considered to be homogeneous and isotropic on the large scale leading to FLRW model. However, if we move to smaller scales, we can observe strongly inhomogeneous distribution of the structures. If we want to deal with inhomogeneity rigorously and at the same time use FLRW model, we should apply some averaging procedure to smooth out the metric tensor and also average Einstein equations. The problem is that Einstein equations are nonlinear and if we average Einstein equations generically we do not obtain averaged metric tensor as a solution of an averaged equations. Instead, we should consider an additional term - so called correlation term, which can change the evolution of the smooth metric tensor and lead to the so called backreaction. This term is created due to the nonlinearity of the Einstein equations. It does

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not need to satisfy usual energy conditions so it can possibly act as a dark energy.

While building rigorous averaging scheme we face a problem that the average value of a tensor field is not well defined. There are several different approaches how to define averaging of tensors. One of the most promising one is the scheme by Zalaletdinov [1], [2] where not only Einstein equations but also Cartan structure equations (and its integrability conditions) are averaged. Theorem about isometric embedding of 2-sphere into Euclidian space is applied for averaging by Korzyński [3]. In [4] Weitzenbck connection for parallel transport is used for the definition of average value of a tensor field.

One of the most popular approaches for averaging is the one investigated by Buchert [5], [6], where only scalar part of the Einstein equations is averaged. Wiltshire used this approach for alternative explanation of the cosmic acceleration [7]. The theory was also applied in a cosmological perturbation theory [8], [9], [10], [11]. For observational issues see e.g. [12]. In this article, we will generalize Buchert's equations for the locally rotationally symmetric (LRS) class II dust family of spacetime. LRS family was classified in [13], [14] and recently in [15]. LRS family contains e.g. LRS Bianchi cosmologies, Kantowski-Sachs model or LTB model and its generalizations. We will use the fact that this family is described by scalars to average the complete set of Einstein equations, including constraints. Although the averaged constraints are shown to be preserved during evolution the averaged system of equations is not closed and additional information has to be supplemented.

Naturally, one can study inhomogeneities perturbatively on a homogeneous background and many important results are based on this approach. However, we should be cautious about relying solely on a linear perturbative analysis when dealing with nonlinear theory. The effects of the correlation term indicate what kind of effects one might be missing when using simple approach. In this sense, rigorous averaging of exact inhomogeneous spacetimes leading to standard cosmological models provides a possibility to qualitatively estimate these effects.

The paper is organized as follows. In the section 2 we review LRS family and its characterizations, then we briefly mention Buchert equations. Next, we average equations describing dust LRS class II family. After a short review of LTB metric in section 5 we investigate backreaction in the so called onion model. We proceed by attempting to close the averaged equation and we investigate two simple examples. We finish with conclusion.

2 LRS family

Locally rotationally symmetric (LRS) dust spacetimes are defined by the following characterization [13]: In an open neighborhood of each point p, there exists a nondiscrete subgroup of the Lorentz group which leaves the Riemann tensor and its covariant derivatives up to third order invariant. Therefore, in

LRS spacetimes there exists a preferred direction e^μ (the axis of symmetry) in every point. The subgroup can be one or three dimensional. In the second case, we can rotate the axis of symmetry and spacetimes are everywhere isotropic - these are FRW models.

We will use the covariant 3+1 splitting of spacetime with the timelike vector field u^μ normalized by the condition $u_\rho u^\rho = -1$ and the projection tensor $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. In this section we will follow the article of van Elst and Ellis [15].

Preferred spacelike vector field e^μ satisfies the following conditions:

$$e_\rho u^\rho = 0, \quad e_\rho e^\rho = 1. \tag{1}$$

Because of the property of the LRS spacetime, all covariantly defined spacelike vectors orthogonal to u^μ (acceleration $\dot{u}^\mu$, vorticity ω^μ, projected gradient of density $h^\sigma_{\ \mu}\nabla_\sigma\rho$, pressure $h^\sigma_{\ \mu}\nabla_\sigma p$ and expansion $h^\sigma_{\ \mu}\nabla_\sigma\theta$) must be proportional to e^μ - if this condition does not hold, spacelike vectors will not be invariant under the rotation about e^μ.

$$\dot{u}^\mu = \dot{u}e^\mu, \qquad \omega^\mu = \omega e^\mu, \tag{2}$$

$$h^\sigma_{\ \mu}\nabla_\sigma\rho = \rho' e_\mu, \quad h^\sigma_{\ \mu}\nabla_\sigma p = p' e_\mu, \quad h^\sigma_{\ \mu}\nabla_\sigma\theta = \theta' e_\mu. \tag{3}$$

Dot here denotes the covariant derivative along the flow vector u^μ and the prime denotes covariant derivative along the vector e^μ. We define the magnitude of the spatial rotation k and the magnitude of the spatial divergence a as

$$k := \left|\eta^{\alpha\beta\gamma\delta}\left(\nabla_\beta e_\gamma\right)u_\delta\right|, \tag{4}$$

$$a := h^\alpha_{\ \beta}\left(\nabla_\alpha e^\beta\right), \tag{5}$$

where $\eta^{\alpha\beta\gamma\delta}$ is totally antisymmetric object ($\eta_{1234} = -\sqrt{-g}$) Similar rule works also for the spacelike tracefree symmetric tensors orthogonal to u^μ. We will create a new tensor field $e_{\mu\nu}$ defined from e^μ

$$e_{\mu\nu} := \frac{1}{2}\left(3e_\mu e_\nu - h_{\mu\nu}\right). \tag{6}$$

Then we have the relations for the shear tensor and the electric and magnetic parts of the Weyl tensor

$$\sigma_{\mu\nu} = \frac{2}{\sqrt{3}}\sigma e_{\mu\nu}, \quad E_{\mu\nu} = \frac{2}{\sqrt{3}}E e_{\mu\nu}, \quad H_{\mu\nu} = \frac{2}{\sqrt{3}}H e_{\mu\nu}. \tag{7}$$

Here we can see that the LRS spacetimes are characterized only by the finite set of the scalar functions.

For simplicity we will restrict to the LRS class II dust models defined by the relation $k = \omega = 0$.). It can also be shown that the magnetic part of the Weyl tensor is equal to zero, $H = 0$. This family of spacetimes includes

LTB metric and its generalizations to spacelike 2-surfaces with negative or zero curvature scalar. The relevant evolution equations are

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi\rho, \tag{8}$$

$$\dot{\sigma} = -\frac{1}{\sqrt{3}}\sigma^2 - \frac{2}{3}\theta\sigma - E, \tag{9}$$

$$\dot{E} = -4\pi\rho\sigma + \sqrt{3}E\sigma - \theta E, \tag{10}$$

$$\dot{\rho} = -\rho\theta, \tag{11}$$

$$\dot{a} = -\frac{1}{3}a\theta + \frac{1}{\sqrt{3}}a\sigma. \tag{12}$$

and the constraints

$$\sigma' = \frac{1}{\sqrt{3}}\theta' - \frac{2}{3}a\sigma, \tag{13}$$

$$E' = -\frac{3}{2}aE + \frac{4\pi}{\sqrt{3}}\rho', \tag{14}$$

$$a' = \frac{2}{9}\theta^2 + \frac{2}{3\sqrt{3}}\theta\sigma - \frac{4}{3}\sigma^2 - \frac{2}{\sqrt{3}}E - \frac{1}{2}a^2 - \frac{16\pi}{3}\rho. \tag{15}$$

If we perform time derivative of the constraints, we can see that they do not change with time.

3 Buchert equations

In this section we will review averaging method developed by Buchert [5]. We will consider only dust case - for generalization to perfect fluid see [6]. This approach uses 3+1 splitting of spacetime, which is well defined by irrotational dust 4-velocity. However, averaging is defined only for scalars, so only scalar part of the Einstein equations is averaged. Given a scalar field A, average value over the three dimensional spacelike domain $\mathcal{D}$ is defined by

$$\langle A\rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} d^3X JA = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} d^3X \sqrt{\det g_{ij}}A, \tag{16}$$

where $J := \sqrt{detg_{ij}}$, g_{ij} is the metric of the spacelike hypersurface, X^i are the comoving coordinates and $V_{\mathcal{D}}$ is the volume of the three dimensional domain $\mathcal{D}$. From this definition we can see that time derivative and averaging do not commute. We have a commutation relation

$$\begin{aligned}\dot{\langle A\rangle}_{\mathcal{D}} &= \frac{d}{dt}\left(\frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} d^3X JA\right) = -\frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}}\langle A\rangle_{\mathcal{D}} + \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} d^3X\left(\dot{J}A + J\dot{A}\right)\\ &= -\langle\theta\rangle_{\mathcal{D}}\langle A\rangle_{\mathcal{D}} + \langle A\theta\rangle_{\mathcal{D}} + \left\langle\dot{A}\right\rangle_{\mathcal{D}}.\end{aligned} \tag{17}$$

where the expansion rate Θ is related to the velocity of the fluid u^μ according to the definition by $\Theta = u^\mu{}_{;\mu}$. Next, we introduce in analogy with FRW spacetime the dimensionless scale factor $a_{\mathcal{D}}$ and the effective Hubble parameter $H_{\mathcal{D}}$

$$a_{\mathcal{D}} = \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}_i}}\right)^{\frac{1}{3}}, \tag{18}$$

$$\langle\Theta\rangle_{\mathcal{D}} = \frac{\dot{V_{\mathcal{D}}}}{V_{\mathcal{D}}} = 3\frac{\dot{a_{\mathcal{D}}}}{a_{\mathcal{D}}} =: 3H_{\mathcal{D}}. \tag{19}$$

$V_{\mathcal{D}_i}$ is the volume of the initial domain which geodetically evolved to $V_{\mathcal{D}}$. Now we have a formalism how to average scalars. To obtain scalar equation from the Einstein equation, we have to contract it with available tensors - i.e. $g^{\mu\nu}, u^\mu$ and ∇^μ. After contraction we obtain the Raychaudhuri equation, the Hamiltonian constraint and the continuity equation. Now we perform averaging and use the commutation rule (17)

$$3\frac{\ddot{a_{\mathcal{D}}}}{a_{\mathcal{D}}} + 4\pi G \langle\rho\rangle_{\mathcal{D}} - \Lambda = \mathcal{Q}_{\mathcal{D}}, \tag{20}$$

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - \frac{8\pi G}{3}\langle\rho\rangle_{\mathcal{D}} + \frac{\langle\mathcal{R}\rangle_{\mathcal{D}}}{6} - \frac{\Lambda}{3} = -\frac{\mathcal{Q}_{\mathcal{D}}}{6}, \tag{21}$$

$$\partial_t \langle\rho\rangle_{\mathcal{D}} + 3\frac{\dot{a_{\mathcal{D}}}}{a_{\mathcal{D}}}\langle\rho\rangle_{\mathcal{D}} = 0. \tag{22}$$

$\langle\mathcal{R}\rangle_{\mathcal{D}}$ denotes average value of the spatial Ricci scalar, $\langle\rho\rangle_{\mathcal{D}}$ means average density of the fluid and $\mathcal{Q}_{\mathcal{D}}$ that shows possible backreaction (due to inhomogeneity and anisotropy) is defined by

$$\mathcal{Q}_{\mathcal{D}} := \frac{2}{3}\left\langle(\Theta - \langle\Theta\rangle_{\mathcal{D}})^2\right\rangle_{\mathcal{D}} - 2\left\langle\sigma^2\right\rangle_{\mathcal{D}}. \tag{23}$$

The scalar $\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}$ is constructed from the shear tensor.

4 Averaging LRS class II dust spacetime

Now, we will generalize the above approach to LRS class II dust solutions. Originally Buchert considered spacetimes with a dust [5] or a perfect fluid [6] source. He did not suppose any symmetries or simplifications and his equations can be applied to a large class of metrics. Here we will restrict to spacetimes with the special LRS symmetry. For this family we will generalize Buchert equations in the sense that all Einstein equations are averaged consistently.

Given preferred spacelike direction e_μ, all the equations describing LRS metric are scalar. It means we can perform averaging (which is covariantly defined for scalars). We will use averaging over the spacelike domain $\mathcal{D}$ defined by (16). In order to obtain an averaged equations we need to derive the commutation relations for the time and the prime derivative (with respect to the preferred direction). In the similar way as in relation (17) we will derive the second commutation rule.

$$\langle A\rangle'_{\mathcal{D}} = \mathbf{e}\left(\frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} d^3X JA\right) = -\frac{V'_{\mathcal{D}}}{V_{\mathcal{D}}}\langle A\rangle_{\mathcal{D}} + \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} d^3X\,(J'A + JA') \quad (24)$$

$$= -\langle\xi\rangle_{\mathcal{D}}\langle A\rangle_{\mathcal{D}} + \langle A\xi\rangle_{\mathcal{D}} + \langle A'\rangle_{\mathcal{D}}. \quad (25)$$

where we have defined ξ by the equation

$$J' = J\xi \quad (26)$$

In more detail

$$J' = \left(\sqrt{\det g_{ij}}\right)' = \sqrt{\det g_{ij}}\frac{(\det g_{ij})'}{2(\det g_{ij})} = \sqrt{\det g_{ij}}\xi = J\xi. \quad (27)$$

For simplicity we will restrict to the class II LRS spacetime with the condition $p = 0 \Leftrightarrow \dot{\rho} = 0$ (dust models) which includes LTB spacetimes and their generalizations. If we average the equations (8) - (15) we will obtain

$$\langle\theta\rangle^{\cdot} = -\frac{1}{3}\langle\theta\rangle^2 - 4\pi\langle\rho\rangle + \underline{\frac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - 2\langle\sigma^2\rangle} \quad (28)$$

$$\langle\sigma\rangle^{\cdot} = -\frac{1}{\sqrt{3}}\langle\sigma\rangle^2 - \frac{2}{3}\langle\theta\rangle\langle\sigma\rangle - \langle E\rangle + \underline{\frac{1}{\sqrt{3}}\left(\langle\sigma\rangle^2 - \langle\sigma^2\rangle\right)}$$
$$\underline{+\frac{1}{3}(\langle\theta\sigma\rangle - \langle\theta\rangle\langle\sigma\rangle)} \quad (29)$$

$$\langle E\rangle^{\cdot} = -4\pi\langle\rho\rangle\langle\sigma\rangle + \sqrt{3}\langle E\rangle\langle\sigma\rangle - \langle\theta\rangle\langle E\rangle$$
$$\underline{-4\pi(\langle\rho\sigma\rangle - \langle\rho\rangle\langle\sigma\rangle) + \sqrt{3}(\langle E\sigma\rangle - \langle E\rangle\langle\sigma\rangle)} \quad (30)$$

$$\langle\rho\rangle^{\cdot} = -\langle\rho\rangle\langle\theta\rangle \quad (31)$$

$$\langle a\rangle^{\cdot} = -\frac{1}{3}\langle a\rangle\langle\theta\rangle + \frac{1}{\sqrt{3}}\langle a\rangle\langle\sigma\rangle + \underline{\frac{2}{3}(\langle a\theta\rangle - \langle a\rangle\langle\theta\rangle)}$$
$$\underline{+\frac{1}{\sqrt{3}}(\langle a\sigma\rangle - \langle a\rangle\langle\sigma\rangle)} \quad (32)$$

$$\langle\sigma\rangle' = \frac{1}{\sqrt{3}}\langle\theta\rangle' - \frac{2}{3}\langle a\rangle\langle\sigma\rangle + \underline{\langle\sigma\xi\rangle - \langle\xi\rangle\langle\sigma\rangle - \frac{1}{\sqrt{3}}(\langle\xi\theta\rangle - \langle\xi\rangle\langle\theta\rangle)}$$
$$\underline{-\frac{3}{2}(\langle a\sigma\rangle - \langle a\rangle\langle\sigma\rangle)} \quad (33)$$

$$\langle E\rangle' = -\frac{2}{3}\langle a\rangle\langle E\rangle + \frac{4\pi}{\sqrt{3}}\langle\rho\rangle' - \underline{\frac{2}{3}(\langle aE\rangle - \langle a\rangle\langle E\rangle)}$$
$$\underline{+\langle\xi E\rangle - \langle\xi\rangle\langle E\rangle - \frac{4\pi}{\sqrt{3}}(\langle\xi\rho\rangle - \langle\xi\rangle\langle\rho\rangle)} \quad (34)$$

$$\langle a\rangle' = \frac{2}{9}\langle\theta\rangle^2 + \frac{2}{3\sqrt{3}}\langle\theta\rangle\langle\sigma\rangle - \frac{4}{3}\langle\sigma\rangle^2 - \frac{2}{\sqrt{3}}\langle E\rangle - \frac{1}{2}\langle a\rangle^2 - \frac{16\pi}{3}\langle\rho\rangle$$

$$\underline{+\langle a\xi\rangle - \langle a\rangle\langle\xi\rangle + \frac{2}{9}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) + \frac{2}{3\sqrt{3}}\left(\langle\theta\sigma\rangle - \langle\theta\rangle\langle\sigma\rangle\right)}$$
$$\underline{-\frac{4}{3}\left(\langle\sigma^2\rangle - \langle\sigma\rangle^2\right) - \frac{1}{2}\left(\langle a^2\rangle - \langle a\rangle^2\right)} \tag{35}$$

Underlined part of the equations denotes the additional terms created by averaging. We can recognize known Buchert equation (28) with the kinematical backreaction term and the mass conservation equation (31). It can be shown, that the averaged constraint equations (33) - (35) are preserved in time. The key role in the calculation is played by the equation [15]

$$(f')^{\cdot} = \left(\dot{f}\right)' - \frac{2}{\sqrt{3}}\sigma f' - \frac{1}{3}\theta f', \tag{36}$$

and its averaged version. Now we can perform the time derivative of the constrains (33) - (35). Using commutation rules and valid equations (8) - (15) slow but straightforward computation will show that the constraints do not evolve in time.

All of the Einstein equations are averaged. It means that we can investigate not only backreaction on the expansion rate but also on shear scalar or electric part of the Weyl scalar. The problem is that the equations are not closed. We need additional relations to close the system because for example $\langle\theta\rangle$ is independent of $\langle\theta^2\rangle$. In the next chapters we will see the proposals how to close the system of the equations.

So far we have not seen the analog of the Hamiltonian constraint. Our unaveraged model is characterized by the set of five scalar function. However, we could use another set. For example Sussman in [16] used three dimensional curvature ${}^3R$ instead of the function a. We could enlarge our set of functions to include ${}^3R$ and the definition of ${}^3R$ would play the role of the Hamiltonian constraint.

5 LTB metric

The most important representative of the dust LRS class II family is LTB spacetime. In this section we will briefly review its properties.

The Lematre-Tolman-Bondi (LTB) metric [17], [18], [19] is a spherically symmetric exact solution of the Einstein equations. It corresponds to an inhomogeneous dust with the stress energy tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu, \tag{37}$$

where u_μ is 4-velocity of a dust with density ρ. For the recent review of LTB metric see e.g. [20], [21]. The line element reads

$$ds^2 = -dt^2 + \frac{(R')^2}{1+2E(r)}dr^2 + R^2(t,r)[d\theta^2 + \sin^2(\theta)d\phi^2], \tag{38}$$

where $E(r)$ is an arbitrary function and the prime denotes partial derivative with respect to r. Function $R(t,r)$ obeys Einstein equations if

$$R_{,t}^2 = 2E + \frac{2M}{R} + \frac{\Lambda}{3}R^2. \tag{39}$$

$M = M(r)$ is another arbitrary function of integration. The energy density ρ is determined by the equation

$$4\pi\rho = \frac{M'}{R'R^2}. \tag{40}$$

The function $E(r)$ determines a curvature of the space $t = const.$ (which is flat for $E(r) = 0$) and the function $M(r)$ is the gravitational mass contained within the comoving spherical shell at any given r. Equation (39) can be integrated to give the result

$$\int_0^R \frac{d\tilde{R}}{\sqrt{2E + \frac{2M}{\tilde{R}} + \frac{1}{3}\Lambda\tilde{R}^2}} = t - t_B(r). \tag{41}$$

$t_B(r)$ is the third free function of r (called the bang time function). In the LTB model, in general, the Big Bang is not simultaneous as in the FRW case, but it depends on the radial coordinate r. The given formulas are invariant under transformation $\tilde{r} = g(r)$. We can use this freedom to choose one of the functions $E(r), M(r)$ and $t_B(r)$. For $\Lambda = 0$ the above equation can be solved explicitly. The evolution can be elliptic ($E < 0$), parabolic ($E = 0$) or hyperbolic ($E > 0$).

6 Backreaction inside the LTB onion model

We close the system of equations by using an exact LTB model. We will consider an onion model investigated in [22] by Biswas, Mansouri and Notari, who computed the corrections to luminosity distance–redshift relation. It represents spacetime with radial shells of overdense and underdense regions. Density profile at the time $t = 20$ can be seen at the figure 1. The curvature of three dimensional spaces is nonzero ($E(r) > 0$), so the evolution of the LTB model is hyperbolic. The metric function $R(t,r)$ reads

$$R(t,r) := \left(\frac{6}{\pi}\right)^{1/3} t^{2/3} r \left(1 + \left(\frac{81}{4000\pi^2}\right)^{1/3} \left(\frac{1}{2\pi}\right) L t^{2/3} \frac{1}{r} \sin \pi r \sin \pi r\right). \tag{42}$$

The curvature function $E(r)$ reads

$$E(r) = \frac{r}{2\pi} \sin \pi r \sin \pi r. \tag{43}$$

First we investigate backreaction term in Buchert equation (28). We numerically integrate underlined part of the equation (28) depending on the averaging

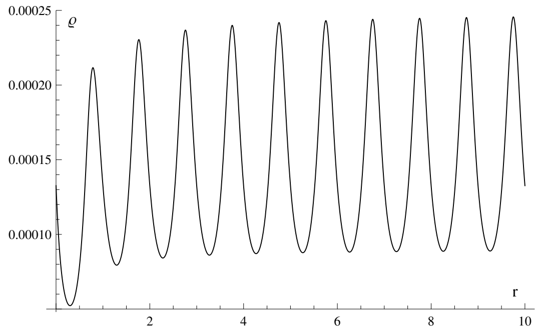

Fig. 1 Density profile at the time $t = 20$

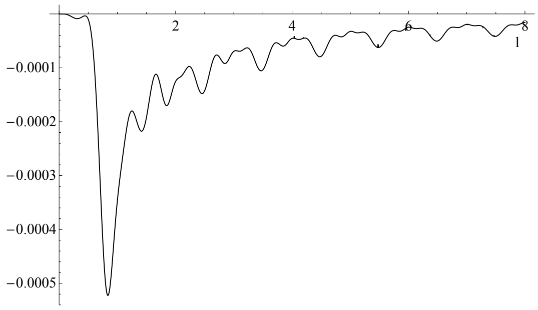

Fig. 2 Backreaction term in the evolution equation for expansion depending on an averaging scale l normalized by $\langle\theta\rangle^{\cdot}$

scale l. As we can see from figure 2, backreaction is positive and it leads to increase of an expansion. It has a peak for the value of averaging scale $l \approx 0,8$. The value of backreaction normalized by $\langle\theta\rangle^{\cdot}$ is of order 10^{-4}. We can investigate also backreaction terms in other equations which do not appear in the Buchert framework and which can supplement his equations. For example here we will show the result for backreaction in the averaged evolution equation for shear (29). All results depend on an averaging scale l. As we can see from figure 3 - for small scales, contribution in the equation for shear is positive with a peak around $l \approx 0,9$. For larger scales the contribution is smaller and negative. The turning point is for $l \approx 1,2$. To be more precise there exist regions where the backreaction change the sign twice for very small neighbourhood of l. If we compare backreaction with the time derivative of a shear scalar, we can see that its ratio is of order $10^{-4} - 10^{-3}$. It means that the backreaction plays more important role in the averaged equation for shear than in the averaged equation for expansion.

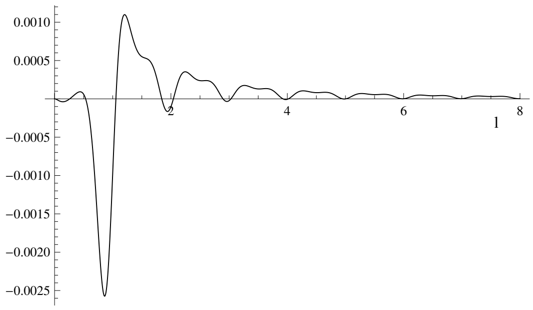

Fig. 3 Backreaction term in the evolution equation for shear depending on an averaging scale l normalized by $\langle\sigma\rangle^{\cdot}$

7 Averaged FRW-like LRS dust class II equations

One of the most important equations in cosmology is the evolution equation for an expansion scalar. In the averaged equation (28) we have an independent variables $\left\langle\theta^2\right\rangle$ and $\left\langle\sigma^2\right\rangle$. All averaged quantities are functions of only time coordinate t. To obtain evolution equation for $\left\langle\theta^2\right\rangle$ we multiply (8) by 2θ. Then we perform averaging and we have the equation

$$\begin{aligned}\left\langle\theta^2\right\rangle^{\cdot} = & -\frac{2}{3}\left\langle\theta\right\rangle^3 - 4\left\langle\theta\right\rangle\left\langle\sigma\right\rangle^2 - 8\pi\left\langle\rho\right\rangle\left\langle\theta\right\rangle + \underline{\frac{1}{3}\left(\left\langle\theta^3\right\rangle - \left\langle\theta\right\rangle^3\right)} \\ & \underline{+\left(\left\langle\theta\right\rangle^3 - \left\langle\theta\right\rangle\left\langle\theta^2\right\rangle\right) - 4\left(\left\langle\theta\sigma^2\right\rangle - \left\langle\theta\right\rangle\left\langle\sigma\right\rangle^2\right) - 4\pi\left(\left\langle\rho\theta\right\rangle - \left\langle\rho\right\rangle\left\langle\theta\right\rangle\right)}\end{aligned} \tag{44}$$

In the similar way we obtain evolution equation for $\left\langle\sigma^2\right\rangle$

$$\begin{aligned}\left\langle\sigma^2\right\rangle^{\cdot} = & -\frac{2}{\sqrt{3}}\left\langle\sigma\right\rangle^3 - \frac{4}{3}\left\langle\theta\right\rangle\left\langle\sigma\right\rangle^2 - 2\left\langle E\right\rangle\left\langle\sigma\right\rangle + \underline{\frac{1}{3}\left(\left\langle\theta\right\rangle\left\langle\sigma\right\rangle^2 - \left\langle\theta\sigma^2\right\rangle\right)} \\ & \underline{-\frac{4}{3}\left(\left\langle\theta\right\rangle\left\langle\sigma^2\right\rangle - \left\langle\theta\right\rangle\left\langle\sigma\right\rangle^2\right) - \frac{2}{\sqrt{3}}\left(\left\langle\sigma^3\right\rangle - \left\langle\sigma\right\rangle^3\right)} \\ & -\underline{2\left(\left\langle E\sigma\right\rangle - \left\langle E\right\rangle\left\langle\sigma\right\rangle\right)}\end{aligned} \tag{45}$$

Now we also need the equations for $\left\langle\theta^3\right\rangle$, $\left\langle\theta\sigma^2\right\rangle$ or $\left\langle\sigma^3\right\rangle$. We could obtain these evolution equations by the same procedure. Thus we have an infinite number of equations for the correlation terms. Here we need to consider some ansatz. For example we can consider reasonable assumption that for a given order the correlation terms are negligibly small and we can truncate the hierarchy to obtain the finite set of equations. We can also assume that some terms are proportional to each other. Thus we have the quantities which depend only on time and the inhomogeneities which are modeled by different type of the correlation functions.

7.1 Model 1

In the first example we numerically integrate six coupled ordinary differential equations (28), (29), (30), (31), (44) and (45) for $\langle\theta\rangle$, $\langle\sigma\rangle$, $\langle\rho\rangle$, $\langle E\rangle$, $\left\langle\theta^2\right\rangle$ and $\left\langle\sigma^2\right\rangle$. Here we consider simple example where most of the correlation functions are set to zero. We consider following ansatz:

$$\langle E\sigma\rangle = \langle E\rangle\langle\sigma\rangle \tag{46}$$
$$\left\langle\theta^3\right\rangle = \langle\theta\rangle\left\langle\theta^2\right\rangle \tag{47}$$
$$\left\langle\theta\sigma^2\right\rangle = 0,3\langle\theta\rangle\left\langle\sigma^2\right\rangle \tag{48}$$
$$\langle\rho\theta\rangle = \langle\rho\rangle\langle\theta\rangle \tag{49}$$
$$\left\langle\sigma^3\right\rangle = \langle\sigma\rangle^3 \tag{50}$$
$$\langle\theta\sigma\rangle = 0,3\langle\theta\rangle\langle\sigma\rangle \tag{51}$$
$$\langle\rho\sigma\rangle = \langle\rho\rangle\langle\sigma\rangle \tag{52}$$

The factor 0,3 in the ansatz was used in order to have nontrivial correlation term modeling inhomogeneity and also well behaved averaged functions - e.g. $\left\langle\theta^2\right\rangle$ and $\left\langle\sigma^2\right\rangle$ should be nonnegative. We consider the following initial conditions at $t = 0$: $\langle\theta\rangle_0 = 0,9$, $\langle\sigma\rangle_0 = 0,3$, $\langle\rho\rangle_0 = 0,000001$, $\langle E\rangle_0 = 0,01$, $\left\langle\theta^2\right\rangle_0 = 0,8$ and $\left\langle\sigma^2\right\rangle_0 = 0,08$. The important observable in cosmology is decceleration parameter which is defined by

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \tag{53}$$

As we can see from figure 4 expansion scalar is monotonically decreasing function. Despite this fact, deceleration parameter isn't strictly positive function. The evolution begins with a positive deceleration parameter as can be seen from figure 5. At the time $t \approx 4,3$ the fluid starts to accelerate. From the time $t \approx 7$ acceleration is slowing down and goes asymptotically to zero. In [23] a correspondence was given in the Buchert formalism between the backreaction terms and the scalar field (so called morphon field). For this fluid we can define equation of state $p = w\rho$. This parameter is shown in figure 6, where we can see that w goes asymptotically to $w = -\frac{1}{3}$, which is a transition between an accelerating and decelerating solution and can be interpreted as a curvature term [24].

7.2 Model 2

In this toy model the set of equations is closed by the following ansatz:

$$\langle E\sigma\rangle = 1,7\langle E\rangle\langle\sigma\rangle \tag{54}$$
$$\left\langle\theta^3\right\rangle = 1,2\langle\theta\rangle\left\langle\theta^2\right\rangle \tag{55}$$
$$\left\langle\theta\sigma^2\right\rangle = 0,4\langle\theta\rangle\left\langle\sigma^2\right\rangle \tag{56}$$
$$\langle\rho\theta\rangle = 0,7\langle\rho\rangle\langle\theta\rangle \tag{57}$$

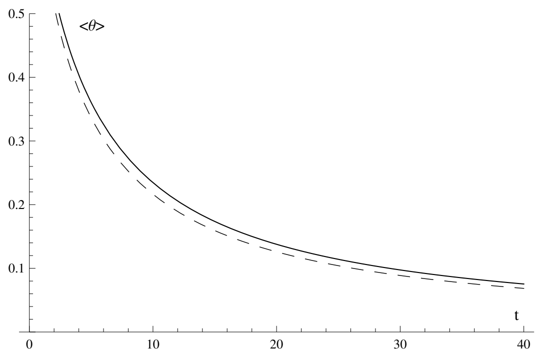

Fig. 4 Average expansion scalar $\langle\theta\rangle$ in the model 1 (dashed line) and the model 2

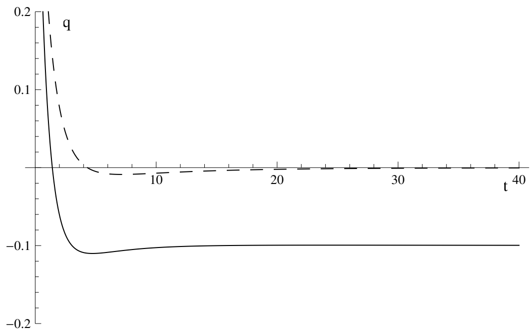

Fig. 5 Decceleration parameter q in the model 1 (dashed line) and the model 2

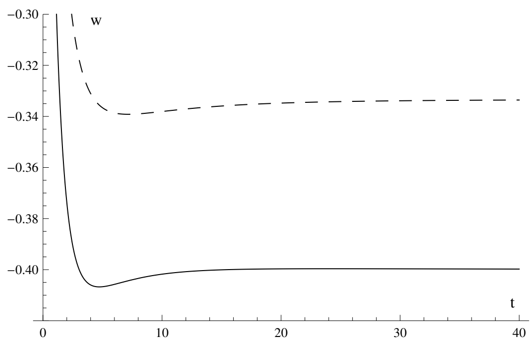

Fig. 6 Parameter of the equation of state w of the effective fluid describing backreaction in model 1 (dashed line) and the model 2

$$\left\langle \sigma^3 \right\rangle = 1,3 \left\langle \sigma \right\rangle^3 \tag{58}$$

$$\left\langle \theta\sigma \right\rangle = 0,4 \left\langle \theta \right\rangle \left\langle \sigma \right\rangle \tag{59}$$

$$\left\langle \rho\sigma \right\rangle = 1,5 \left\langle \rho \right\rangle \left\langle \sigma \right\rangle \tag{60}$$

This ansatz generates richer structure of inhomogeneities. We consider the same initial conditions. Here we can see that the expansion and the deceleration parameter behaves in a similar way as in the previous example. In this model, deceleration parameter does not go asymptotically to zero but instead it has constant nonzero value $q \approx -0,1$. This solution describes the fluid which starts from the deceleration phase and at late time accelerates with a constant deceleration parameter.

8 Conclusion

We have generalized the Buchert equations for the LRS class II dust model. We used the property that this family is characterized only by scalars and we employed similar technique for averaging. However, averaged equations do not close. Buchert considered so called scaling solutions [23] to close the system of equations. In our work we have first closed the system for the exact LTB model which describes fluctuating radial inhomogeneities and we have investigated influence of backreaction on the expansion and the shear scalar. Then we have proposed an infinite system of equations which supplement averaged equations for the expansion. All averaged variables depend only on time variable as in homogeneous model and inhomogeneities are modeled by the form of the correlation terms. We have given two different forms of ansatz for its behaviour which allow to cut off the infinite hierarchy of equations. In both examples the deceleration parameter changes its sign from positive to negative values. In the first model it goes asymptotically to zero, in the second model it goes to nonzero negative value. In the classical cosmological models we usually introduce cosmological constant or modify gravity (stress energy tensor) to explain observed acceleration of the universe. In our simple example we obtain acceleration from the equations describing averaged LRS class II dust spacetime with a certain prescription for the correlation terms.

Acknowledgments

We would like to thank R. Sussman for useful discussion. P.K. was supported by grants GAUK 398911 and SVV-267301. O.S. acknowledges the support of grant GAČR 14-37086G.

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7. Averaging Problem in General relativity and Cosmology

- Kašpar, P. *Averaging Problem in General relativity and Cosmology*, Acta Universitatis Carolinae. Mathematica et Physica, Vol. 53 (2012).

This chapter can be thought of as a brief introduction to the chapter 1 and the chapter 2. Its aim is to review different averaging methods. It starts with the Isaacson approach followed by Macroscopic Gravity, Buchert's equations, Ricci flow, averaging by scalar invariants and finally gives a motivation for averaging by Cartan scalars which was described in chapter 5. This chapter tries to briefly describe some averaging methods with an emphasis on cosmological application.

Averaging problem in general relativity and cosmology

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Abstract. It is tradition in cosmology to use the homogeneous and isotropic FRW (Friedmann-Robertson-Walker) spacetime. However, the real universe is inhomogeneous and anisotropic on small scales so if we want to retain the FRW approach, we should at least perform some averaging procedure. Because of the nonlinearity of the Einstein field equations, we will in general obtain a nonzero correlation term, which does not necessarily obey the energy condition and so it can mimic the dark energy term. In this article I will try to review different approaches to the averaging problem with the emphasis on cosmology.

Introduction

In General relativity (GR) the evolution of the metric tensor is driven by the Einstein field equations. As emphasized in 80's by [*Ellis*, 1984], averaging and evolution do not commute, i.e. $\langle E_{\mu\nu}(g_{\mu\nu})\rangle \neq E_{\mu\nu}(\langle g_{\mu\nu}\rangle)$. $E_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor and $\langle\rangle$ is some unspecified averaging procedure. On the other hand, in cosmology one usually uses the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric and the smooth stress energy tensor of the perfect fluid. If we want to use a simple model and represent the dynamics of the universe by one single scale function $a(t)$ (not to use more general inhomogeneous cosmological model), we should put a new correlation term $C_{\mu\nu}$ into the equations .

$$E_{\mu\nu}(\langle g_{\mu\nu}\rangle) = 8\pi \langle T_{\mu\nu}\rangle + C_{\mu\nu}, \tag{1}$$

which is defined by the construction

$$C_{\mu\nu} = E_{\mu\nu}(\langle g_{\mu\nu}\rangle) - \langle E_{\mu\nu}(g_{\mu\nu})\rangle . \tag{2}$$

It does not necessarily obey the usual energy condition and it can act as dark energy [*Buchert*, 2008]. Averaging can be considered over some spacelike hypersurface, which depends on the selected slicing or over some spacetime interval, which can be covariantely defined. There are two main goals concerning averaging - the first is to construct averaged metric and the second is to obtain correlation term modifying Einstein equations.

There is a technical problem in a definition of an averaged tensor: Integrating a tensor field in curved spacetime does not result in a new tensor field (this is because of the addition of the tensors living in different spaces). In the next sections we will show some attempts how to solve this problem.

Isaacson's approach

Following the work of [*Brill, Hartle*, 1964], Isaacson used an averaging method for computing the effective gravitational stress energy tensor [*Isaacson*, 1968]. In order to compute an average value of the general tensor over the domain $\mathcal{D}$ at the base point x, he parallel transports tensors from points in $\mathcal{D}$ to x and then integrates.

$$\langle A_{\mu\nu}(x)\rangle_{BH} = \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} g_{\mu}^{\alpha'}(x,x') g_{\nu}^{\beta'}(x,x')\, A_{\alpha'\beta'}(x')\sqrt{-g(x')} d^4x'. \tag{3}$$

$g(x')$ denotes the determinant of the metric. $g_{\mu}^{\alpha'}(x,x')$ is the bivector of geodesic parallel displacement that serves to parallel transport of $A_{\alpha'\beta'}(x')$ and $V_{\mathcal{D}}$ is the volume of $\mathcal{D}$. Integration over x' is justified because of the contraction over the prime indices. It can be shown that the following properties hold:

- One can ignore the terms $\left\langle A_{\mu\nu}{}^{\rho}{}_{;\rho} \right\rangle_{BH}$.
- One can integrate by parts.
- Covariant derivatives commute.

Macroscopic Gravity

Another promising approach to the averaging problem is the method (valid for n-dimensional manifolds) developed by Zalaletdinov who also gives several conditions for the correlation term to be fulfilled [*Zalaletdinov*, 1992, 1993, 2004]. One of the big problem of BH averaging scheme is that it leaves the metric tensor unchanged. To overcome this trouble, Zalaletdinov introduced a bilocal averaging operator $\mathcal{W}^{\alpha'}_{\beta}(x', x)$ which transforms as a vector at the point x' and as a covector at the point x. Its construction follows from the demanded properties:

$$\lim_{x' \to x} \mathcal{W}^{\alpha'}_{\beta}(x', x) = \delta^{\alpha}_{\beta}, \tag{4}$$

$$\mathcal{W}^{\alpha'}_{\gamma''}(x', x'')\mathcal{W}^{\gamma''}_{\beta}(x'', x) = \mathcal{W}^{\alpha'}_{\beta}(x', x). \tag{5}$$

It can be shown that these two properties are equivalent to the following form of the bilocal operator:

$$\mathcal{W}^{\alpha'}_{\beta}(x', x) = F^{\alpha'}_{\gamma}(x')F^{-1\gamma}_{\beta}(x). \tag{6}$$

Now it is possible for a given compact region $\mathcal{D}$ of a differentiable space-time manifold $(\mathcal{M}, g_{\alpha\beta})$ with a volume n-form to define the average value of the tensor field $t^{\alpha\cdots}_{\beta\ldots}(x)$, $x \in \mathcal{M}$ as

$$\bar{t}^{\alpha\cdots}_{\beta\ldots}(x) = \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}} \tilde{t}^{\alpha\cdots}_{\beta\ldots}(x, x')\sqrt{-g'}d^n x', \tag{7}$$

g'=det$(g_{\alpha\beta}(x'))$, $V_{\mathcal{D}}$ is the volume of $\mathcal{D}$ and the object $\tilde{t}^{\alpha\cdots}_{\beta\ldots}(x, x')$ is the bilocal extension of the tensor $t^{\alpha\cdots}_{\beta\ldots}(x)$ using the bivector $\mathcal{W}^{\alpha'}_{\beta}(x', x)$

$$\tilde{t}^{\alpha\cdots}_{\beta\ldots}(x, x') = \mathcal{W}^{\alpha}_{\alpha'}(x', x)\ldots\mathcal{W}^{\beta'}_{\beta}(x', x)\ldots t^{\alpha'\cdots}_{\beta'\ldots}(x'). \tag{8}$$

Now it is possible to bilocally extend Eistein equations and then perform averaging. The theory of Macroscopic Gravity not only averages Einstein equations but also geometry itself. From the consistent procedure how to average Cartan structure equations and their integrability equations it is possible to find a system of algebraic and differential equations that must be fulfilled by the correlation term term.

The first exact solution of Macroscopic Gravity was published by [*Coley, Pelavas, Zalaletdinov*, 2005]. Resulting correlation term can be interpreted as an additional space curvature.

Buchert equations

In the last two sections we have seen that it isn't very obvious how to average tensors. However, averaging scalars has a clear rule. In most of the cosmological models there is a preferred timelike vector (cosmic time) so it is useful to perform 3+1 splitting of the variables. Here we will restrict ourselves only to the dust source [*Buchert*, 2000] (it can be generalized for the perfect fluid [*Buchert*, 2001]).

For the metric $ds^2 = -dt^2 + g_{ij}dX^i dX^j$ spatial averaging of the scalar field Ψ over the domain $\mathcal{D}$ is defined by

$$\left\langle \Psi(t, X^i) \right\rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}} \int\limits_{\mathcal{D}} J d^3 X \Psi(t, X^i), \tag{9}$$

$$V_{\mathcal{D}} = \int\limits_{\mathcal{D}} J d^3 X, \tag{10}$$

where $J := \sqrt{det g_{ij}}$, g_{ij} is the metric of the spacelike hypersurface and X^i are the comoving coordinates. Taking time derivative of this definition we can obtain the following important commutation rule:

$$\partial_t \left\langle \Psi(t, X^i) \right\rangle_{\mathcal{D}} - \left\langle \partial_t \Psi(t, X^i) \right\rangle_{\mathcal{D}} = \left\langle \Psi(t, X^i) \right\rangle_{\mathcal{D}} \langle \Theta \rangle_{\mathcal{D}} - \left\langle \Psi(t, X^i) \Theta \right\rangle_{\mathcal{D}}, \tag{11}$$

where the expansion rate Θ is related to the velocity of the fluid u^μ according to the definition by $\Theta = u^\mu{}_{;\mu}$. Next we introduce in analogy with FRW spacetime a dimensionless scale factor $a_{\mathcal{D}}$ and the effective Hubble parameter $H_{\mathcal{D}}$

$$a_{\mathcal{D}} = \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D}i}} \right)^{\frac{1}{3}}, \tag{12}$$

$$\langle \Theta \rangle_{\mathcal{D}} = \frac{\dot{V_{\mathcal{D}}}}{V_{\mathcal{D}}} = 3 \frac{\dot{a_{\mathcal{D}}}}{a_{\mathcal{D}}} =: 3 H_{\mathcal{D}}. \tag{13}$$

A dot denotes partial derivative with respect to time, $V_{\mathcal{D}i}$ is the volume of the initial domain which geodetically evolved to $V_{\mathcal{D}}$. Now we have a formalism how to average scalars. To obtain scalar equation from the Einstein equation, we have to contract it with available tensors - i.e. $g^{\mu\nu}, u^\mu$ and ∇^μ. After contraction we obtain the Raychaudhuri equation, the Hamiltonian constraint and the continuity equation. Now we perform averaging and use the commutation rule (11).

$$3 \frac{\ddot{a_{\mathcal{D}}}}{a_{\mathcal{D}}} + 4\pi G \langle \rho \rangle_{\mathcal{D}} - \Lambda = \mathcal{Q}_{\mathcal{D}}, \tag{14}$$

$$\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 - \frac{8\pi G}{3} \langle \rho \rangle_{\mathcal{D}} + \frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{6} - \frac{\Lambda}{3} = -\frac{\mathcal{Q}_{\mathcal{D}}}{6}, \tag{15}$$

$$\partial_t \langle \rho \rangle_{\mathcal{D}} + 3 \frac{\dot{a_{\mathcal{D}}}}{a_{\mathcal{D}}} \langle \rho \rangle_{\mathcal{D}} = 0. \tag{16}$$

$\langle \mathcal{R} \rangle_{\mathcal{D}}$ denotes average value of the spatial Ricci scalar, $\langle \rho \rangle_{\mathcal{D}}$ means average density of the averaged fluid and $\mathcal{Q}_{\mathcal{D}}$ that shows possible backreaction (by present inhomogeneity and anisotropy) is defined by

$$\mathcal{Q}_{\mathcal{D}} := \frac{2}{3} \left\langle (\Theta - \langle \Theta \rangle_{\mathcal{D}})^2 \right\rangle_{\mathcal{D}} - 2 \left\langle \sigma^2 \right\rangle_{\mathcal{D}}. \tag{17}$$

The scalar $\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}$ is constructed from the shear tensor. The time derivative of the averaged Hamiltonian constrain agrees with the Raychaudhuri equation when the integrability equation is fulfilled

$$\partial_t \mathcal{Q}_{\mathcal{D}} + 6 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \mathcal{Q}_{\mathcal{D}} + \partial_t \langle \mathcal{R} \rangle_{\mathcal{D}} + 2 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \langle \mathcal{R} \rangle_{\mathcal{D}} = 0. \tag{18}$$

In a similar way as in the FRW approach we can define dimensionless variables (omega factors)

$$\Omega_m^{\mathcal{D}} := \frac{8\pi G}{3 H_{\mathcal{D}}^2} \langle \rho \rangle_{\mathcal{D}}; \quad \Omega_\Lambda^{\mathcal{D}} := \frac{\Lambda}{3 H_{\mathcal{D}}^2}; \quad \Omega_{\mathcal{R}}^{\mathcal{D}} := -\frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{6 H_{\mathcal{D}}^2}; \quad \Omega_{\mathcal{Q}}^{\mathcal{D}} := -\frac{\mathcal{Q}_{\mathcal{D}}}{6 H_{\mathcal{D}}^2} \tag{19}$$

and Hamiltonian constraint will be written in the standard form

$$\Omega_m^{\mathcal{D}} + \Omega_\Lambda^{\mathcal{D}} + \Omega_{\mathcal{R}}^{\mathcal{D}} + \Omega_{\mathcal{Q}}^{\mathcal{D}} = 1. \tag{20}$$

The formalism can be extended [*Larena*, 2009] to arbitrary coordinate system. In addition to the fluid 4-velocity u^μ, there is another velocity n^μ of the observer. In the Buchert equations there are together with the kinematic term $\mathcal{Q}_\mathcal{D}$ (and the dynamic term if the fluid has nonzero pressure) other corrections which complicate the resulting equations.

It is still not clear how big the correction to the Friedmann eqautions are. They are some claims that they are negligible [*Ishibashi, Wald,*, 2006], however there some models which are able to explain observed acceleration of the universe [*Wiltshire*, 2007]. More references can be found e.g. in [*Ellis*, 2011]. For the scale issue see for example [*Li, Schwarz*, 2009].

Ricci flow

In the last section it was shown how to average scalars on an inhomogeneous manifold. However, cosmological data are most often interpreted in the FRW spacetime. In addition to the averaging (9), there should also be some procedure how to smooth geometry itself. The theory of Macroscopic Gravity uses averaging of the Cartan structure equations. There exists mathematically interesting alternative how to reach 3-spaces of constant curvature. Let g_{ab} be a given metric on the closed 3-manifold without boundary, which depends on the parametr β (typically cosmic time) and let it evolve in the direction of the Ricci tensor

$$\frac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta), \quad 0 \le \beta \le T_0$$

$$g_{ab}(\beta = 0) = g_{ab}. \tag{21}$$

It can be shown that on the compact manifold for the sufficiently small β local solution exist and if the initial metric has a positive Ricci curvature, solution exists for all β converging exponentially to the space of the constant curvature (technical details and other references can be found in [*Buchert, Carfora*, 2002]).

By this procedure also the other parameters will change - the average value of the density will change after smoothing $\mathcal{D}_0$ to $\overline{\mathcal{D}}$ as $\langle\rho\rangle_{\overline{\mathcal{D}}} = M_{\overline{\mathcal{D}}}/V_{\overline{\mathcal{D}}}$. Similarly we would obtain a new set of the normalized omega factors, which can be very different from the original ones.

Averaging using scalar curvature invariants

If we can average scalars it is natural to ask how we can represent spacetime by scalar quantities. In [*Coley et al.*, 2009] it was proven that the class of four-dimensional Lorentzian manifolds that cannot be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives must be of Kundt form (e.g. admitting geodetic null vector with a null expansion, rotation and shear).

For a given spacetime $(\mathcal{M}, g_{\alpha\beta})$ we define the set of scalar invariants [*Coley*, 2010]

$$\mathcal{I} \equiv \left\{R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}, R_{\mu\nu\alpha\beta;\gamma}R^{\mu\nu\alpha\beta;\gamma}, R_{\mu\nu\alpha\beta;\gamma\delta}R^{\mu\nu\alpha\beta;\gamma\delta}, ...\right\}. \tag{22}$$

Integrating over the domain $\mathcal{D}$ we obtain another set $\bar{\mathcal{I}}$ characterizing a smoother geometry. As we can see from relations like $\overline{R_{\mu\nu}R^{\mu\nu}} \neq \bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$ it is possible that there does not exist any metric tensor $\bar{g}_{\mu\nu}$ which would be constructed from the set $\bar{\mathcal{I}}$. To overcome this difficulty we will first remove the scalars which are not algebraically independent. It means that we will restrict our discussion to the subset $\mathcal{I}_A \subseteq \mathcal{I}$. Then we will omit any scalars that can be computed from the equations ("syzygies") characterizing particular spacetimes (e.g. defining the algebraic type of the spacetime, like the Segre type or the Petrov type). We will obtain the new set $\mathcal{I}_{SA} \subseteq \mathcal{I}_A$ and by averaging we will get $\bar{\mathcal{I}}_{SA}$. By the inverse procedure we will acquire a complete set $\bar{\mathcal{I}}$ (here we suppose that averaging will not change the form of the equations which allowed the construction $\mathcal{I}_{SA} \subseteq \mathcal{I}_A$).

Averaging Cartan scalars

In the last section geometry was characterized by the curvature scalars. This procedure works well only in four dimensions and it is rather difficult to obtain the metric or the Ricci tensor from the averaged scalars. It can be shown [*Cartan*, 1946] that the geometry may be completely characterized by the Riemann tensor and the finite number of its covariant derivatives (Cartan scalars). Because the Einstein tensor consists of the sum of the Riemann tensor, it is possible to average geometry and the Einstein equations together.

We will start with the construction of the Cartan scalars (for the texts concerning equivalence problem see e.g. [*Karlhede*, 1980, 2006]). Let $(\mathcal{M}, g)$ be n-dimensional differentiable manifold with a metric

$$\mathbf{g} = \eta_{ij}\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{23}$$

η_{ij} is constant symmetric matrix and $\boldsymbol{\omega}^i$, i=1,2...,n form the base of the cotangent space at the point x^μ. The tetrad (frame) is defined up to generalized rotations

$$\boldsymbol{\omega}^i = \omega^i_\nu(x^\mu, \xi^\Upsilon)\mathbf{d}x^\nu, \tag{24}$$

ξ^Υ, Υ=1,...,$\frac{1}{2}n(n-1)$ denotes the coordinates of the orthogonal group. In Macroscopic Gravity, theory uses (bilocally extended) Cartan equation. Now all the geometrical objects will be defined on the enlarged $\frac{1}{2}n(n+1)$ dimensional space $F(\mathcal{M})$ - the frame bundle of $\mathcal{M}$. The exterior derivative will be extended to $d = d_x + d_\xi$ and the Cartan equations have the form

$$\boldsymbol{d}\boldsymbol{\omega}^i = \boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^i{}_j, \tag{25}$$

$$\boldsymbol{d}\boldsymbol{\omega}^i{}_j = -\boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j + \frac{1}{2}R^i_{jkl}\boldsymbol{\omega}^k \wedge \boldsymbol{\omega}^l. \tag{26}$$

with the condition

$$\eta_{ik}\boldsymbol{\omega}^k{}_j + \eta_{jk}\boldsymbol{\omega}^k{}_i = 0. \tag{27}$$

Applying next the exterior derivative we will obtain covariant derivatives of the curvature tensor.

$$\mathbf{d}R_{ijkl} = R_{mjkl}\boldsymbol{\omega}^m_i + R_{imkl}\boldsymbol{\omega}^m_j + R_{ijml}\boldsymbol{\omega}^m_k + R_{ijkm}\boldsymbol{\omega}^m_l + R_{ijkl;m}\boldsymbol{\omega}^m,$$

$$\mathbf{d}R_{ijkl;n} = R_{mjkl;n}\boldsymbol{\omega}^m_i + R_{imkl;n}\boldsymbol{\omega}^m_j + ... + R_{ijkl;nm}\boldsymbol{\omega}^m,$$

$$\vdots \tag{28}$$

Let R^p denote the set $\left\{R_{ijkm}, R_{ijkm;n_1}, ..., R_{ijkm;n_1...n_p}\right\}$, p is the lowest number such that R^{p+1} contains no element that is functionally independent (over $F(\mathcal{M})$) of the elements in R^p (two functions f, g are functionally independent iff the 1-forms $\mathbf{d}f$ and $\mathbf{d}g$ are linearly independent).

There exist a quite elaborate algorithm [*Karlhede*, 2006] how to compute Cartan scalars. It uses the structure of isotropy group of R^q and in every step it restrict the frame requiring that R^q takes a standard form.

Now we can use the same algorithm as in the previous section. It can be shown that the Cartan scalars satisfy some algebraic and differential relations which are in general nonlinear. It means that we have to restrict to the smaller set of the scalars, perform averaging and then construct a new set $\overline{R}^{p+1}$, from which it is possible to construct a new metric $\overline{g}_{\mu\nu}$.

Conclusion

The averaging problem in GR and especially in cosmology is of the fundamental importance. Backreaction term in the averaged Einstein equations will change the dynamic of the metric and affect the cosmological evolution. The question is how important these corrections are and when it is possible to neglect them [*Buchert*, 2008], [*Ellis*, 2011].

There is also a problem how to define average of tensors. In this review we have introduced several different candidates how to average Einstein field equations and also spacetime geometry. Macroscopic Gravity is the promising model how to average inhomogeneities, but only a few simplified solutions are known because of the complexity of the equations. The most popular approach to the averaging problem are the Buchert equations. However, only scalar part of the equations are averaged so we have less equation then variables and we have to put some relation by hand.

Acknowledgments. I would like to thank to Otakar Svítek and David Vrba for useful discussions. The present work was supported by GAUK 398911, GACR-205/09/H033 and SVV-263301.

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Conclusion

In the first part of this thesis we have reviewed different averaging methods with emphasis on Zalaletdinov's Macroscopic Gravity. Then we briefly introduced anisotropic cosmological models and exact inhomogeneous cosmological models.

We examined new method for averaging based on Cartan scalars. These consist of Riemann tensor and the final number of its covariant derivatives projected on a tetrad basis. We showed two ways how to proceed with averaging. In the first one an averaged metric tensor is explicitly constructed, in the second one the form of an averaged metric is assumed and the backreaction term is computed. We applied the theory for a flat LTB metric with a special ansatz for the radial function. We showed that the correlation term behaves as a positive cosmological constant. In the next example we considered LTB model with nonzero curvature, so called onion model. Correlation term here consists of a sum of several terms with a leading term behaving like spatial curvature. Finally, we gave an illustration of non-triviality of averaging for monochromatic linearized gravitational wave.

We also investigated averaging within LRS class II dust spacetime. We averaged all the Einstein equations due to the fact that these are represented by scalars. The resulting equations generalize the Buchert's result for this family of spacetimes. We closed the system of equations by considering exact LTB model (onion model) and computed backreaction for expansion and shear scalar. We examined another possibility how to close the system of the averaged equations. We obtained evolution equations for different backreaction terms. The problem is that the system of equations is infinite and needs to be truncated. Inhomogeneities are represented by correlation terms in this approach. Given an ansatz for behaviour of the correlation terms we numerically integrated resulting truncated system of equations. We considered two different ansätze for the correlation term resulting in two different models. For both of them the computed deceleration parameter changes its sign from positive to negative.

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Appendix 1: Fixed frame formalism

In this appendix we review the formalism in a fixed frame. This approach is often used for the practical calculations. In the chapter 5 we considered averaging by Cartan scalars. Here we will describe the way how to compute these scalars in a fixed frame.

An exterior derivative of the Riemann tensor can be computed in terms of the gradients of the coordinates $x^{\mu}{}_{|k}$ and the Ricci rotation coefficients γ^{m}_{kn} [12].

$$dR_{ijkl} = R_{ijkl,\mu}x^{\mu}{}_{|m}\boldsymbol{\omega}^{m} + R_{mjkl}\left(\boldsymbol{\omega}^{m}_{i} - \gamma^{m}_{in}\boldsymbol{\omega}^{n}\right) + R_{imkl}\left(\boldsymbol{\omega}^{m}_{j} - \gamma^{m}_{jn}\boldsymbol{\omega}^{n}\right)$$
$$+R_{ijml}\left(\boldsymbol{\omega}^{m}_{k} - \gamma^{m}_{kn}\boldsymbol{\omega}^{n}\right) + R_{ijkm}\left(\boldsymbol{\omega}^{m}_{l} - \gamma^{m}_{mn}\boldsymbol{\omega}^{n}\right). \tag{7.1}$$

This can be found using the definition of the exterior derivative on the bundle of frames. If we compare this expression with the exterior derivative of the C^{I}_{JK} which contains the information about Riemann tensor, we can see that $I^{\alpha}{}_{|K}$ corresponds to $\left\{x^{\mu}{}_{|k}, \gamma^{m}_{kn}\right\}$. Now we will show how the constraints are constructed. If we suppose that we have some symmetries (this is often true when we compute averaged Cartan scalars) such that R^{p+1} depend on x^{α}, $\alpha = 1, ..., l < n$ and rotations in the ab-planes, $\{^{a}{}_{b}\} = 1, ..., m < \frac{1}{2}n(n-1)$ (for the case without symmetries inequalities will be sharp). The first set of constraints which have to be satisfied can be written as

$$x^{\alpha}{}_{|[k,\beta|}x^{\beta}{}_{|l]} + x^{\alpha}{}_{|m}\gamma^{m}{}_{[lk]} = 0, \tag{7.2}$$

and

$$R^{a}{}_{bij} = 2\gamma^{a}{}_{b[j,|\alpha|}x^{\alpha}{}_{|i]} + 2\gamma^{ak}{}_{[j}\gamma_{|bk|i]} + 2\gamma^{a}{}_{bk}\gamma^{k}{}_{[ij]}. \tag{7.3}$$

The second set of constraints has a form

$$R^{t}{}_{[ijk]} = 0, \tag{7.4}$$

and

$$R^{p}{}_{q[ij;k]} = 0, \tag{7.5}$$

where $t = l+1, ..., n$ and $\left\{^{p}{}_{q}\right\} = m+1, ...\frac{1}{2}n(n-1)$. When the frame is fixed, the set of the Cartan scalars R^{p+1} can be replaced by the set $\left\{R^{p}{}_{qkl}, \gamma^{a}{}_{bi}, x^{\alpha}{}_{|i}, \eta_{ij}\right\}$.

Appendix 2: Minimal set of the Cartan scalars computed for flat LTB metric

In this appendix we list Cartan scalars computed by the algebraic program SHEEP for the flat LTB metric. These were used in chapter 5 for the computation of the correlation term. The computation was performed within the spinor formalism [81]. The Riemann tensor is replaced by the Newmann-Penrose quantities $\psi_{ABCD}, \phi_{AB\dot{C}\dot{D}}$ and Λ. They satisfy Bianchi identities

$$\nabla^{C}{}_{\dot{Y}}\psi_{CDEF} = \nabla_{(D}{}^{\dot{W}}\phi_{EF)\dot{W}\dot{Y}}, \tag{7.6}$$

$$\nabla^{C\dot{Y}}\phi_{CD\dot{U}\dot{Y}} = -3\nabla_{D\dot{U}}\Lambda. \tag{7.7}$$

In the Cartan-Karlehede algorithm we compute different derivatives of the Riemann tensor, but some of them are redundant and can be expressed by other Cartan scalars of the same order. The minimal set of derivatives of the Riemann tensor which can give rise to the Cartan scalars was found by MacCallum and Åman [69], [86]. For the qth step of the Cartan-Karlhede algorithm, this minimal set can be found by the following rule:

1. The totally symmetrized spinor qth derivatives of Λ.
2. The totally symmetrized spinor qth derivatives of $\phi_{AB\dot{A}\dot{B}}$.
3. The totally symmetrized spinor qth derivatives of ψ_{ABCD}.
4. For $q \geq 1$, the totally symmetrized (q-1)th derivatives of $\Xi_{ABC\dot{D}} = \nabla^{D}{}_{\dot{D}}\psi_{DABC}$.
5. For $q \geq 2$, the d'Alembertian $\nabla^{A\dot{A}}\nabla_{A\dot{A}}$ applied to all the quantities calculated for the derivatives of order q-2.

These variables are computed by the algebraic program SHEEP, which we extensively used for the calculations.

In the next part of this appendix we show Cartan scalars computation by the algebraic program SHEEP for the flat LTB metric.

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PSL version 3.40  on  IBM 486
SHEEP 2 version 59 ( Thu 18-2-1993 )
Classi made   Thu 18-2-1993
Started   Tue 3-1-2012   at  JANILO.cbpf.br

(TOTAL (TIME -2533259) (GCTIME 0) (NETTIME -2533259))

The following switches are on:  SUBPOT NOZERO SEQSUBS

Variables from 0 to 3 : T R H F

A depends on T and R

FRAME = LORENTZ

IFRAME = LORENTZ

  0
IZ   = 1
   0

  1
IZ   = A
   1    ,R

  2
IZ   = A
   2

  3
IZ   = Asin(H)
   3

((TIME 0) (GCTIME 0) (NETTIME 0))

 Changing to null frame

SHP> (WMAKE DS2)

  2     2       2  2   2  2   2   2     2
ds  = dT  -(A   ) dR  -A dH  -A sin (H)dF
             ,R

SHP> (WMAKE FORMSU)

 0         1/2          1/2
W  = 1/2(2)   dT +1/2(2)   A  dR
                            ,R

 1         1/2          1/2
W  = 1/2(2)   dT -1/2(2)   A  dR
                            ,R

 2         1/2           1/2
W  = 1/2(2)   AdH +1/2(2)   IAsin(H)dF

 3         1/2           1/2
W  = 1/2(2)   AdH -1/2(2)   IAsin(H)dF

SHP> (WMAKE DS2F)

  2           1/2          1/2               1/2          1/2
ds  = 2(1/2(2)   dT+1/2(2)   A  dR)(1/2(2)   dT-1/2(2)   A  dR)
                              ,R                            ,R
```

```
         1/2          1/2                   1/2          1/2
-2(1/2(2)   AdH+1/2(2)   IAsin(H)dF)(1/2(2)   AdH-1/2(2)   IAsin(H)dF)


SHP> (WMAKE UNSGAM)

   u                      1/2 -1            1/2 -1
GAM       = rho =  -1/2(2)   A  A   -1/2(2)   A
    0010'                        ,T

   u                        1/2     -1
GAM       = epsilon = 1/4(2)   (A  )  A
    0100'                        ,R    ,TR

   u                     1/2 -1         -1
GAM       = beta = 1/4(2)   A  cos(H)sin  (H)
    0101'

   u                        1/2 -1         -1
GAM       = alpha =  -1/4(2)   A  cos(H)sin  (H)
    0110'

   u                        1/2     -1
GAM       = gamma =  -1/4(2)   (A  )  A
    0111'                        ,R    ,TR

   u                   1/2 -1           1/2 -1
GAM       = mu = 1/2(2)   A  A   -1/2(2)   A
    1101'                     ,T

((TIME 15) (GCTIME 0) (NETTIME 15))

SHP> (PETROV)

Please check that PSI2 is really non-zero !
If so, Petrov type is D


                -1           -1         -1          -1          -2     2
PSI  =  -1/6(A  )  A     +1/6A  A  (A  )  A    +1/6A  A    -1/6A  (A  )
   2          ,R    ,TTR         ,T  ,R    ,TR          ,TT           ,T

SHP> (ISOTST PSI)

Remaining isotropy group is:
boosts and rotations (2-dim), (shorthand notation: e)

This is so far a standard frame.

SHP> (FUNTST PSI)

New function, probably independent:

f1 = Re(PSI )
           2


        -1       -1         -1      -1      -2     2
= ((A  )  A    -A  A  (A  )  A   -A  A   +A  (A  ) )
     ,R    ,TTR     ,T  ,R    ,TR     ,TT      ,T
```

```
SHP> (WMAKE LAMBD)

                 -1           -1        -1          -1           -2     2
LAMBD = 1/12(A  )  A     +1/6A  A  (A  )  A    +1/6A  A    +1/12A  (A  )
              ,R    ,TTR         ,T  ,R    ,TR         ,TT           ,T

SHP> (WMAKE PHISTD)

   std          -1        -1          -1
PHI       = 1/2A  A  (A  )  A    -1/2A  A
      00'          ,T  ,R    ,TR         ,TT

   std                -1           -2     2
PHI       =  -1/4(A  )  A     +1/4A  (A  )
      11'          ,R    ,TTR          ,T

   std          -1        -1          -1
PHI       = 1/2A  A  (A  )  A    -1/2A  A
      22'          ,T  ,R    ,TR         ,TT

SHP> (ISOTST CHI)

Remaining isotropy group is:
boosts and rotations (2-dim), (shorthand notation: e)

This is so far a standard frame.

SHP> (ISOTST PHISTD)

Remaining isotropy group is:
space (spin) rotations (1-dim), (shorthand notation: s)
and swap of null directions

This is so far a standard frame.

SHP> (SEGRE)

Plebanski-Petrov Type is D

Segre type is A1 [(11)1,1]

SHP> (ISOTST PHI)

Remaining isotropy group is:
space (spin) rotations (1-dim), (shorthand notation: s)
and swap of null directions

This is so far a standard frame.


SHP> (WMAKE DPSI)

                   1/2     -1               1/2     -2
DPSI    =  -1/20(2)   (A  )  A      +1/20(2)   (A  )  A   A
    20'                 ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2               1/2     -3
-1/20(2)   (A  )  A      +1/20(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

        1/2 -1        -1              1/2 -1        -2      2
+3/20(2)   A  A  (A  )  A     -1/20(2)   A  A  (A  )   (A  )
               ,T  ,R    ,TTR                ,T  ,R     ,TR
```

```
        1/2 -1        -2              1/2 -1        -3
+1/20(2)   A  A  (A  )  A     -1/20(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
+1/20(2)   A  A     +1/20(2)   A  (A  )  A   A
               ,TTT                 ,R    ,TT ,TR

        1/2 -1     -1              1/2 -1     -2      2
+3/20(2)   A  (A  )  A     +1/20(2)   A  (A  )  (A   )
                ,R    ,TTR                 ,R     ,TR

        1/2 -2     2     -1            1/2 -2
-3/20(2)   A  (A  ) (A  )  A    -1/4(2)   A  A  A
                ,T    ,R    ,TR               ,T ,TT

       1/2 -2        -1             1/2 -2            1/2 -3     3
-1/4(2)   A  A  (A  )  A    -3/20(2)   A  A    +1/5(2)   A  (A  )
              ,T  ,R    ,TR                ,TT                ,T

       1/2 -3     2
+1/5(2)   A  (A  )
               ,T

                   1/2     -1               1/2     -2
DPSI    =  -1/20(2)   (A  )  A      +1/20(2)   (A  )  A   A
    31'                 ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2               1/2     -3
+1/20(2)   (A  )  A      -1/20(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

        1/2 -1        -1              1/2 -1        -2      2
+3/20(2)   A  A  (A  )  A     -1/20(2)   A  A  (A  )  (A   )
               ,T  ,R    ,TTR                ,T  ,R     ,TR

        1/2 -1        -2              1/2 -1        -3
-1/20(2)   A  A  (A  )  A     +1/20(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
+1/20(2)   A  A     +1/20(2)   A  (A  )  A   A
               ,TTT                 ,R    ,TT ,TR

        1/2 -1     -1              1/2 -1     -2      2
-3/20(2)   A  (A  )  A     -1/20(2)   A  (A  )  (A   )
                ,R    ,TTR                 ,R     ,TR

        1/2 -2     2     -1            1/2 -2
-3/20(2)   A  (A  ) (A  )  A    -1/4(2)   A  A  A
                ,T    ,R    ,TR               ,T ,TT

       1/2 -2        -1             1/2 -2            1/2 -3     3
+1/4(2)   A  A  (A  )  A    +3/20(2)   A  A    +1/5(2)   A  (A  )
              ,T  ,R    ,TR                ,TT                ,T

       1/2 -3     2
-1/5(2)   A  (A  )
               ,T

SHP> (FUNISOTST DPSI)

2 independent functions found so far
```

Remaining isotropy group is:
space (spin) rotations (1-dim), (shorthand notation: s)

This is so far a standard frame.

```
SHP> (WMAKE XI)

               1/2     -1               1/2     -2
XI    = 1/12(2)   (A  )  A      -1/12(2)   (A  )  A   A
  10'               ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2               1/2     -3
+1/12(2)   (A  )  A      -1/12(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

       1/2 -1        -1               1/2 -1        -2      2
+1/6(2)   A  A  (A  )  A     +1/12(2)   A  A  (A  )  (A   )
              ,T  ,R    ,TTR                ,T  ,R     ,TR

        1/2 -1        -2               1/2 -1        -3
-1/12(2)   A  A  (A  )  A     +1/12(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
-1/12(2)   A  A     -1/12(2)   A  (A  )  A   A
               ,TTT                ,R    ,TT ,TR

       1/2 -1     -1              1/2 -1     -2      2
+1/6(2)   A  (A  )  A     -1/12(2)   A  (A  )  (A   )
               ,R    ,TTR                ,R     ,TR

       1/2 -2     2     -1            1/2 -2             1/2 -3     3
-1/6(2)   A  (A  ) (A  )  A    -1/6(2)   A  A    +1/12(2)   A  (A  )
               ,T    ,R    ,TR               ,TT                 ,T

        1/2 -3     2
+1/12(2)   A  (A  )
                ,T

                 1/2     -1               1/2     -2
XI    =  -1/12(2)   (A  )  A      +1/12(2)   (A  )  A   A
  21'                ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2               1/2     -3
+1/12(2)   (A  )  A      -1/12(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

       1/2 -1        -1               1/2 -1        -2      2
-1/6(2)   A  A  (A  )  A     -1/12(2)   A  A  (A  )  (A   )
              ,T  ,R    ,TTR                ,T  ,R     ,TR

        1/2 -1        -2               1/2 -1        -3
-1/12(2)   A  A  (A  )  A     +1/12(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
+1/12(2)   A  A     +1/12(2)   A  (A  )  A   A
               ,TTT                ,R    ,TT ,TR

       1/2 -1     -1              1/2 -1     -2      2
+1/6(2)   A  (A  )  A     -1/12(2)   A  (A  )  (A   )
               ,R    ,TTR                ,R     ,TR

       1/2 -2     2     -1            1/2 -2             1/2 -3     3
+1/6(2)   A  (A  ) (A  )  A    -1/6(2)   A  A    -1/12(2)   A  (A  )
```

```
               ,T    ,R    ,TR                 ,TT                  ,T

        1/2 -3     2
+1/12(2)   A  (A  )
                ,T

SHP> (FUNISOTST XI)

2 independent functions found so far

Remaining isotropy group is:
space (spin) rotations (1-dim), (shorthand notation: s)

This is so far a standard frame.

SHP> (WMAKE DPHI)

                1/2 -1        -1             1/2 -1        -2      2
DPHI    = 1/4(2)   A  A  (A  )  A     -3/4(2)   A  A  (A  )  (A   )
    00'                ,T  ,R    ,TTR               ,T  ,R     ,TR

       1/2 -1        -2             1/2 -1        -3
+1/4(2)   A  A  (A  )  A     -1/4(2)   A  A  (A  )  A   A
              ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

       1/2 -1             1/2 -1     -1               1/2 -1     -1
-1/4(2)   A  A     +3/4(2)   A  (A  )  A   A    -1/4(2)   A  (A  )  A
              ,TTT                ,R    ,TT ,TR                ,R    ,TTR

       1/2 -1     -2      2        1/2 -2     2     -1
+1/4(2)   A  (A  )  (A   )  -1/4(2)   A  (A  ) (A  )  A
               ,R     ,TR                ,T    ,R    ,TR

       1/2 -2               1/2 -2        -1             1/2 -2
+1/4(2)   A  A  A    -1/4(2)   A  A  (A  )  A    +1/4(2)   A  A
              ,T ,TT                ,T  ,R    ,TR               ,TT

                   1/2     -1              1/2     -2
DPHI    =  -1/18(2)   (A  )  A      +1/18(2)   (A  )  A   A
    11'                 ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2              1/2     -3
-1/18(2)   (A  )  A      +1/18(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

        1/2 -1        -1             1/2 -1        -2      2
+5/36(2)   A  A  (A  )  A     +1/36(2)   A  A  (A  )  (A   )
               ,T  ,R    ,TTR               ,T  ,R     ,TR

        1/2 -1        -2             1/2 -1        -3
-1/36(2)   A  A  (A  )  A     +1/36(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR               ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
-1/36(2)   A  A     -1/36(2)   A  (A  )  A   A
               ,TTT                ,R    ,TT ,TR

        1/2 -1     -1              1/2 -1     -2      2
+5/36(2)   A  (A  )  A     -1/36(2)   A  (A  )  (A   )
                ,R    ,TTR                ,R     ,TR

        1/2 -2     2     -1             1/2 -2
-5/36(2)   A  (A  ) (A  )  A    +1/4(2)   A  A  A
                ,T    ,R    ,TR                ,T ,TT
```

```
       1/2 -2       -1             1/2 -2            1/2 -3     3
+1/4(2)   A  A  (A  )  A    -5/36(2)   A  A    -2/9(2)   A  (A  )
              ,T  ,R    ,TR                ,TT                ,T

       1/2 -3     2
-2/9(2)   A  (A  )
               ,T

                   1/2     -1               1/2     -2
DPHI    =  -1/18(2)   (A  )  A      +1/18(2)   (A  )  A   A
    22'                 ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2               1/2     -3
+1/18(2)   (A  )  A      -1/18(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

        1/2 -1        -1              1/2 -1        -2      2
+5/36(2)   A  A  (A  )  A     +1/36(2)   A  A  (A  )  (A   )
               ,T  ,R    ,TTR                ,T  ,R     ,TR

        1/2 -1        -2              1/2 -1        -3
+1/36(2)   A  A  (A  )  A     -1/36(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
-1/36(2)   A  A     -1/36(2)   A  (A  )  A   A
               ,TTT                 ,R    ,TT ,TR

        1/2 -1     -1              1/2 -1     -2      2
-5/36(2)   A  (A  )  A     +1/36(2)   A  (A  )  (A   )
                ,R    ,TTR                 ,R     ,TR

        1/2 -2     2     -1            1/2 -2
-5/36(2)   A  (A  ) (A  )  A    +1/4(2)   A  A  A
                ,T    ,R    ,TR               ,T ,TT

       1/2 -2       -1             1/2 -2            1/2 -3     3
-1/4(2)   A  A  (A  )  A    +5/36(2)   A  A    -2/9(2)   A  (A  )
              ,T  ,R    ,TR                ,TT                ,T

       1/2 -3     2
+2/9(2)   A  (A  )
               ,T

                1/2 -1        -1             1/2 -1        -2      2
DPHI    = 1/4(2)   A  A  (A  )  A     -3/4(2)   A  A  (A  )  (A   )
    33'                ,T  ,R    ,TTR               ,T  ,R     ,TR

       1/2 -1        -2             1/2 -1        -3
-1/4(2)   A  A  (A  )  A     +1/4(2)   A  A  (A  )  A   A
              ,T  ,R    ,TRR               ,T  ,R    ,TR ,RR

       1/2 -1             1/2 -1     -1                1/2 -1     -1
-1/4(2)   A  A     +3/4(2)   A  (A  )  A   A    +1/4(2)   A  (A  )  A
              ,TTT                ,R    ,TT ,TR                ,R    ,TTR

       1/2 -1     -2      2        1/2 -2     2     -1
-1/4(2)   A  (A  )  (A   )  -1/4(2)   A  (A  ) (A  )  A
               ,R     ,TR                  ,T    ,R    ,TR

       1/2 -2               1/2 -2        -1            1/2 -2
+1/4(2)   A  A  A    +1/4(2)   A  A  (A  )  A    -1/4(2)   A  A
              ,T ,TT               ,T  ,R    ,TR               ,TT
```

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SHP> (FUNISOTST DPHI)

2 independent functions found so far

Remaining isotropy group is:
space (spin) rotations (1-dim), (shorthand notation: s)

This is so far a standard frame.

SHP> (WMAKE DLAMBDA)

                    1/2     -1               1/2     -2
DLAMBDA    = 1/24(2)   (A  )  A      -1/24(2)   (A  )  A   A
       00'               ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2               1/2     -3
+1/24(2)   (A  )  A      -1/24(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

        1/2 -1        -1              1/2 -1        -2      2
+1/12(2)   A  A  (A  )  A     -1/12(2)   A  A  (A  )  (A   )
               ,T  ,R    ,TTR                ,T  ,R     ,TR

        1/2 -1        -2              1/2 -1        -3
+1/12(2)   A  A  (A  )  A     -1/12(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
+1/12(2)   A  A     +1/12(2)   A  (A  )  A   A
               ,TTT                 ,R    ,TT ,TR

        1/2 -1     -1              1/2 -1     -2      2
+1/12(2)   A  (A  )  A     +1/12(2)   A  (A  )  (A   )
                ,R    ,TTR                ,R     ,TR

        1/2 -2     2     -1              1/2 -2              1/2 -3     3
-1/12(2)   A  (A  ) (A  )  A    -1/12(2)   A  A    -1/12(2)   A  (A  )
                ,T    ,R    ,TR                ,TT                 ,T

        1/2 -3     2
-1/12(2)   A  (A  )
                ,T

                    1/2     -1               1/2     -2
DLAMBDA    = 1/24(2)   (A  )  A      -1/24(2)   (A  )  A   A
       11'               ,R    ,TTTR              ,R    ,TR ,TTR

        1/2     -2               1/2     -3
-1/24(2)   (A  )  A      +1/24(2)   (A  )  A   A
             ,R    ,TTRR              ,R    ,RR ,TTR

        1/2 -1        -1              1/2 -1        -2      2
+1/12(2)   A  A  (A  )  A     -1/12(2)   A  A  (A  )  (A   )
               ,T  ,R    ,TTR                ,T  ,R     ,TR

        1/2 -1        -2              1/2 -1        -3
-1/12(2)   A  A  (A  )  A     +1/12(2)   A  A  (A  )  A   A
               ,T  ,R    ,TRR                ,T  ,R    ,TR ,RR

        1/2 -1              1/2 -1     -1
+1/12(2)   A  A     +1/12(2)   A  (A  )  A   A
               ,TTT                 ,R    ,TT ,TR

        1/2 -1     -1              1/2 -1     -2      2
-1/12(2)   A  (A  )  A     -1/12(2)   A  (A  )  (A   )
```

```
                ,R     ,TTR                 ,R      ,TR

        1/2 -2      2      -1              1/2 -2              1/2 -3      3
-1/12(2)    A   (A   ) (A   )   A     +1/12(2)    A   A     -1/12(2)    A   (A   )
                ,T     ,R     ,TR                ,TT                    ,T

        1/2 -3      2
+1/12(2)    A   (A   )
                ,T

SHP> (FUNISOTST DLAMBDA)

2 independent functions found so far

Remaining isotropy group is:
space (spin) rotations (1-dim), (shorthand notation: s)

This is so far a standard frame.

SHP> (ISOTROPY)

Isotropy group is:
space (spin) rotations (1-dim), (shorthand notation: s)

This is a standard frame.

Isometry group is of dimension 3.

(RUN (TIME 156) (GCTIME 0) (NETTIME 156))

(TOTAL (TIME -2533103) (GCTIME 0) (NETTIME -2533103))
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