Charles University in Prague

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DOCTORAL THESIS

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The mathematical theory of perturbations in cosmology

Ústav teoretické fyziky

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Na tomto místě bych chtěl poděkovat mnoha lidem, kteří mi pomáhali v psaní této práce. Zvláštní dík bych chtěl vyjádřit především svým blízkým kamarádům Maximu Eingornovi, Filipu Hořínkovi a Alexandře Kounitzké.

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Life is the most miraculous thing in this Universe
I dedicate this work to my girlfriend Ann I.Donskikh.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: v teto práci jsme studovali teorii kosmologických perturbací. Nejprve byla prezentována Obecná Teorie Relativity ve vyšší dimenzi. Potom jsme prezentovali Obecnou Teorii Relativity ve vyšší dimenzi. Potom jsme použili aparát GHP-formalizmu, což je zobecnění známého NP-formalizmu. Skalární perturbace v f(R) - kosmologiích je závěrečné téma, kde bylo ukázáno, že čtyřdimenzionální prostoročasy jsou speciální. Výsledkem bylo získání potenciálů Φ and Ψ for the case of box150 MpC. Použili jsme takzvaný mechanický přístup pro případ kosmologického pozadí. Náš výsledek je nový, je zajínavý také v kontextu simulací v tzv.nelineárních teorií.

Klíčová slova: teorie kosmologických perturbací, NP-formalizmus, f(R)-kosmologie, mechanický přístup, kvazistatická aproximace

Title: The mathematical theory of perturbations in cosmology

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Abstract: We have been studying Cosmological Perturbation Theory in this thesis. There was presented the Standard General Relativity in higher dimensions. Then we used the apparatus of so called GHP formalism and this is a generalization of the well-known NP-formalism. Scalar perturbations in f(R)-cosmology in the late Universe is the final topic, which was a logical step how to proceed further and to continue in work where was shown that four-dimensional spacetimes are special. We get the potentials Φ and Ψ for the case of a box 150 Mpc. We used the so called mechanical approach for the case of a cosmological background. Our approach of getting these potentials is in observable Universe new. It is interesting also in the context of simulations in these, so called nonlinear theories.

Keywords: cosmological perturbation theory, NP-formalism, f(R)-cosmologies, mechanical approach, quasi-static approximation

Contents

1. Introduction

1.1 General introduction

This work focuses mostly on cosmological perturbation theory. Let us start with a physical introduction. At this moment it is known that there are three fundamental interactions, which were described by Standard Model: electromagnetic, weak and strong nuclear forces; The only missing knowledge is the neutrino mass. Gravity is described by the Standard General Relativity (SGR), which is a theory of spacetime and matter. Until this moment there was no contradiction with empirical observations of this theory. One prediction of SGR which was not directly confirmed yet are the gravitational waves and some people are already very optimistic that they will be found soon. It looks like that we should be satisfied from purely empirical point of view. However, from a mathematical point of view, the situation is still not satisfactory. The Standard Model is based on Quantum Field Theory (QFT), theory of gravity is purely classical. We start with the action integral

$$S_{EH} = -\frac{1}{2\kappa^2}\int_{\Sigma}\sqrt{-g}(R - 2\kappa L_F)\ d^4x + \frac{1}{\kappa^2}\int_{\partial\Sigma}\sqrt{h}K, \tag{1.1}$$

where g is a determinant of the metric, R is the Ricci scalar and Λ is the cosmological constant, $\kappa^2 = 8\pi G$ and L_F is the lagrangian of matter fields, where h is a determinant of the metric on the boundary $\partial\Sigma$ and K is extrinsic curvature. When we apply normal quantization procedure in SGR, we don't get the same equations, which follow from the variational principle.

In addition to the two main terms, which consist of the integrals of the spacetime region Σ, there is a term that is defined on the boundary of this region $\partial\Sigma$.

One of the models for quantum gravity is the String Theory. According to this theory the elementary particles are small vibrating strings. Originally it was formulated in the dimension of spacetime 10 or 11 but from one of the previous articles (reference[45]in Chapter III) it is clear that we are not living in higher dimensional universe however String Theory can be formulated also in the four-dimensional spacetime. Physical idea behind the String Theory is different from other theories. It is not a direct quantization of SGR or any other classical theory of gravity. It is a prototype of unified theory of all interactions. Gravity, as well as other interactions, only emerges in an appropriate limit. Strings are one dimensional objects characterized by one parameter α or the string length $l_s = \sqrt{2\alpha\hbar}$. In spacetime it forms a two dimensional surface, the world sheet. Closer inspection of strings needs also other objects known as D-branes. String necessarily contains gravity, because the graviton - the hypothetical particle - appears as an excitation of closed strings. String Theory requires also the presence of supersymmetry. One simply recognizes that gravity can be incorporated into this theory.

Another approach is loop quantum gravity. The variables used in this theory are close to Yang-Mills type variables. The loop variables are defined as follows. The role of the momentum variable is played by the densitized triad

$$E_i^a(x) := \sqrt{h}(x) e_a^i(x),$$

while the configuration variable is the connection

$$GA_a^i(x) = \Gamma_a^i(x) + \beta K_a^i(x),$$

$K_a^i(x)$ is related to the second fundamental form. The parameter β is called Barbero-Immirzi psrameter, it can assume any non-vanishing real value and this is a free parameter in loop quantum gravity. It may be fixed by the requirement that the black hole entropy calculated from loop quantum cosmology coincides with the Beckenstein-Hawking expression. One can find more information in recent work of C.Kiefer (reference is in Chapter III,[26]).

There were also other approaches toward quantum gravity, we will mention the so called twistor theory later. We will mention now the problem of time, It was clear many years ago that the notion of time was absolute in the theory of Quantum Mechanics (QM), however was relative in standard general relativity (SGR). Spacetime corresponds to what is a particle trajectory in mechanics. When we apply the quantization rules to SGR we get that the classical trajectories disappear.

The major conceptual problem concerns the arrow of time. Although our fundamental laws were time reversal invariant, there was a problem with entropy. R. Penrose wrote that it is interesting that the universe began in a very low entropic state, he meant in a very special state. It was also him who pointed out that he didn't believe in cosmological inflation.

Quantum Gravity when applied in cosmology could shed light to interpretation of QM. There were various interpretations of QM in the past. Let us mention for example the Feynman's approach, which was a beautiful combination of classical mechanics with probabilistic approach. We mean that we integrate $\int \exp iS$ in this reformulation over all trajectories. The particle could possibly travel over all paths between the first and final point. But the biggest contribution to the wave function is only from the classical path. R. Feynman in his original paper showed that the standard Schrodinger equation naturally emerges. His reformulation was a convenient way how to look at computations in QM. It had further applications to QFT. He formulated in this language so called quantum electrodynamics, which shed more light on interaction of photons with matter, which was in his time revolutionary. Mathematicians studied in connection with his works so called Feynman integral, which is a big unsolved problem of theory of integral. (The usual procedure in building abstract integral was not working.)

There existed other interpretations of Quantum Mechanics, so called Everett's interpretation where all the components of the wave function are equally real. It was possible to apply the Everett's approach in Quantum Cosmology, when it was combined with the process of decoherence. Decoherence was formulated like irreversible emergence of classical properties from unavoidable interaction with the environment.

Quantum Gravity remained a big challenge for theoretical physicists for many years and it will be nice to formulate a consistent theory, which could be applied in Cosmology.

1.2 Cosmology-historical background

We want to devote this part to Cosmology. In recent decaded Cosmology became a real science and according to some authors there is now a golden age of Cosmology. One can half-jokingly say that this scientific discipline is like archeology. Something happened in the past and now we uncover the remnants of events by modern technologies. The disadvantage is that we have only one universe. However, we use, of course, accelerators for simmulation of very hot and dense state of the universe.

Our present understanding of the universe is based upon the successful hot Big Bang theory, which explains its evolution from the first fraction of a second to our present age, 13 billion years later. This theory rests upon Standard General Relativity (SGR) and was experimentally verified by three observational facts: the expansion of the universe (Edwin P. Hubble in 1930's), the relative abundance of light elements (George Gamow in 1940's) and finally cosmic microwave background (Arno A.Penzias and Robert W.Wilson in 1965).

1.3 Basics

Modern Cosmology is based on the, so called, cosmological principle: universe looks the same for observers at all points and all directions. It is something like the Copernican principle taken to the extreme. So, universe looks very homogeneous and isotropic [1] on big scales (100 Mpc and bigger), which leads to an essential simplification of our models in the form of the so called FLRW (Friedmann-Lemaitre-Robertson-Walker) metric. Let us now present FLRW metrics for three values of spatial curvature of the universe $K = -1, 0, 1$. Open, flat and close universe correspond to the 3-dimensional spatial slices being hyperbolic surfaces with negative curvature, flat Euclidean surfaces with zero

[1] We have two terms: homogeneity and isotropy in a point; Isotropy in every point implies homogeneity, but global homogeneity - it means also that we have local homogeneity in sufficiently small sphere around this point- does not imply isotropy.

curvature or 3 spheres with positive curvature

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-\mathcal{K}r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] = g_{\mu\nu}dx^\mu dx^\nu,$$

where $a(t)$ is so called scale factor, which determines the physical size of the universe. $\{r, \theta, \phi\}$ are comoving coordinates, a particle initially at rest in these coordinates remains at rest. The physical separation between freely moving particles at $t = 0$ and $t = r$ is

$$d(t,r) = \int ds = a(t)\int_0^r \frac{ds}{\sqrt{1-Ks^2}}.$$

In an expanding universe ($\dot{a} > 0$) the distance increases with time:

$$\dot{d} = \frac{\dot{a}}{a}d \equiv Hd,$$

with $H(t)$ the Hubble parameter or constant. The above is nothing but Hubble's law: galaxies recede from each other with a velocity which is proportional to the distance. Hubble's law is supported by observations: the present day value of the Hubble law parameter is $H_0 \approx 72 \pm 8$ km/sec/Mpc.

We can write the metric (2) also in other form where we will use notation and trick with complex numbers: $S(r) = \frac{\sin(\sqrt{\mathcal{K}}r)}{\sqrt{\mathcal{K}}}$ (where the case $\mathcal{K} = 0$ can be obtained by limiting procedure);

$$ds^2 = dt^2 - a^2(t)\left[dr^2 + S^2(r)(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

After some computation the form of the FLRW metric in the $K = 0$ case can be changed in such way that it will have the same structure as the Schwarzschild metric in standard coordinates with the difference that we will have a function of time and radial coordinate in front of the $\mathrm{d}T^2$ and $\mathrm{d}R^2$:

$$ds^2 = F\left(T,R\right)dT^2 - \frac{1}{F\left(T,R\right)}dR^2 - \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where $F(T,R)$ is a function of $T(t,r)$ and $R(t,r)$.

The spatial curvature of the universe is equal to the following expression:

$$R^{(3)} = \frac{6\mathcal{K}}{a^2(t)}$$

Spatially open, flat and closed universes have different geometries. Light geodesics in these universes behave differently, and thus can be in principle distinguished experimentally. We can also compute a four-dimensional spacetime curvature (for example in the Lectures on Cosmology, J. Garcia- Bellido, CERN JINR European School) :

$$R^{(4)} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}}{a^2}\right)$$

Finally, we could also change the time coordinate to conformal time: $\mathrm{dt} = \mathrm{a}d\eta$. We get from (5):

$$ds^2 = a^2(t)\{d\eta^2 - [dr^2 + S^2(r)(d\theta^2 + \sin^2\theta d\phi^2)]\}.$$

This metric is conformal to Minkowski in the case of $\mathcal{K} = 0$ and for us this will be the most interesting case.

The metric in SGR is a dynamical object. The time evolution of the scale factor is governed by Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu},$$

with R and $R_{\mu\nu}$ the scalar curvature and Ricci curvature tensor respectively (which are both functions of the metric with up to the second metric derivatives). We will use units in which $m_p^2 = (8\pi G)^{-1}$. Depending on the dynamics - and thus matter-energy content of the universe - we will have different possible outcomes of the evolution. The universe may expand forever, re-collapse in the future or approach an asymptotic state in between. So now we will consider the matter-energy content of the universe. The matter fluid which is consistent with the homogeneity and isotropy is a perfect fluid, one in which an observer, co-moving with the fluid, would see the universe around it as isotropic. The energy momentum tensor associated with such a fluid can be written as

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu - pg^{\mu\nu},$$

where $p(t)$ and $\rho(t)$ are pressure and energy density of the matter in given time of the expansion, and U^μ is the co-moving four-velocity satisfying $U^\mu U_\mu = 1$. Let us now write the equations of motion in an expanding universe. According to SGR, these equations can be deduced from Einstein equations (??), where we substitute the FLRW metric and the perfect fluid tensor (??). This leads to the famous Friedmann equation

$$\frac{\dot{a}^2}{a^2} = 8\pi G\frac{\rho}{3} - \frac{mathcalK}{a^2}. \tag{1.2}$$

The conservation of energy, a direct consequence of general covariance of the theory, can be written as

$$\frac{d}{dt}(\rho a^3) + p\frac{d}{dt}(a^3) = 0. \tag{1.3}$$

We will introduce the equation of state parameter $p = w\rho$. Then the continuity equation can be integrated to give

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \implies \rho \sim \ a^{-3(1+w)}. \tag{1.4}$$

From the two equations (1.2) and (1.3) we could by mathematical manipulations derive the third Raychaudhuri equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.5}$$

From (1.2), neglecting the curvature terms, it then follows

$$a \sim \begin{cases} t^2/(3(1+w)) & w \neq 1 \\ e^{Ht} & w = 1 \end{cases} \tag{1.6}$$

The matter in the universe consists of several fluids $T^{\nu}_{\mu} = \sum_i T^{(i)\nu}{}_{\mu}$, with i corresponding to radiation, non-realativistic matter or cosmological constant. If the energy exchange between these components is negligible, it follows that all fluids separately satisfy the continuity equation. We can define an equation of state for each fluid separately $p_i = w_i \rho_i$.

Radiation include, for example, photons. For radiation $w_{rad} = \frac{1}{3}$ and from (1.5) we have that $\rho_{rad} \sim \frac{1}{a^4}$. If the universe is dominated by radiation, it follows from (1.6) that $a \sim \sqrt{t}$.

Vacuum energy remains constant with time. If it dominates universe, then $a(t) \sim e^{Ht}$. Define Ω_i with ρ_c being the critical density. Then the Friedmann equation becomes open, close, or flat with depending on $\Omega = \Omega_i$. Thus Ω is larger, smaller, or equal to one for open, close, or flat universe, respectively. We find for the present values $\Omega_B \sim 0.04$ (baryons), $\Omega_{DM} \sim 0.31$ (dark matter), $\Omega_\gamma \sim 10^{-5}$ (radiation) $\Omega_\Lambda \sim 0.069$, (cosmological constant) - Planck collaboration.

Hubble's law and other observations indicate that the universe is expanding. The temperature of the radiation bath of the universe is $T^4 \sim \frac{1}{a^4}$. Where for the first expression we used Stephan-Boltzmann law. It follows that the temperature decreases with $T \sim \frac{1}{a}$ with the expansion. Initially the universe was hot and dense and it cooled as it expanded. The key ingredients of the Big Bang model are nucleosynthesis matter-antimatter relation, matter-radiation equality, recombination, formation of gravitationally-bounded systems and temperature of relic radiation.

We will not discuss now the basic cosmological models, these can be found for example in the book of J. Garcia-Bellido[]. But we will rather say more about cosmological constant puzzle. It is a mystery - because the cosmological constant could be associated with the vacuum energy of QFT - why it has such a small value (approximately 120 orders smaller than predicted by QFT).

In spite of theoretical prejudice towards $\Lambda = 0$, there are new observational arguments for a non-zero value. The most important ones are recent evidence that we live in a flat universe, together with indications of low mass density. That indicates that some kind of dark energy must make up the rest of the energy density. In addition, the disagreement between the ages of globular clusters and the expansion age of the universe may be resolved with $\Lambda \neq 0$. Finally, it was experimentally verified that we live in an accelerating universe!

The so called dark energy have to resist gravitational collapse, otherwise it would have been detected already as a part of the energy in the halos of the galaxies. However, if most of the energy of the universe resists gravitational collapse, it is impossible for structure in the universe to grow. This dilemma can be resolved if the hypothetical dark energy was negligible in the past and only recently became the dominant component.

The dark energy has negative pressure. This rules out all of the usual suspects like neutrinos, cold dark matter, radiation, etc. It is possible that the non-zero cosmological constant has something to do with limits of Standard General Relativity, so that we will need other classical theory of Gravity.

What are the shortcomings of the Big-Bang model?

Photons travel along null geodesics with $ds^2 = 0 \rightarrow dr = dt/a(t)$ for a radial path. The particle horizon (opposite to Hubble horizon) is the type of horizon that light can travel between 0 and t and which is equal to

$$R_p(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a(t) \int_0^a \frac{d(lna)}{aH} = \frac{t}{1-n}.$$

Note that the particle horizon is set by comoving Hubble radius $(aH)^{-1}$. Physical lengths are stretched by the expansion $\lambda \approx a$. Since λ grows with time, so thus the ratio $\frac{R_p}{\lambda}$. Scales that are inside the horizon at present were outside in earlier times. Concretely consider two CMB photons emitted, which were emitted at the time of last scattering. Nowadays we see on the sky two points separated by distance $\lambda(t_0) < R_p(t_0)$. Extrapolating back in time to the surface of the last scattering, it follows that $\lambda(t_{ls}) > R_p(t_{ls})$ was bigger than the horizon. No causal physics could have acted at such large scales. Yet, although these photons came from two disconnected regions, to a very good precision they have nearly the same temperature. People were asking, how can this be possible.

- Horizon problem: Although the universe was vanishingly small, the rapid expansion didn't allow causal contact from being established throughout. The CMB(cosmic microwave background) has a perfect black body spectrum. Two photons coming from the opposite directions of the universe have nearly equal temperatures. Yet the photons coming from the different parts of the sky, could not have a causal contact with each other.

- Flatness problem:

  Consider the Friedmann equation in the form $\Omega - 1 = \frac{\mathcal{K}}{(aH)^2}$. The comoving Hubble radius $(aH)^{-1}$ grows with time, and thus $\Omega = 1$ is an unstable fixed point, in the language of ODE's. Therefore the value of Ω had to be extremely fine-tuned.

- Monopole problem:

  If the universe can be extrapolated back in time to high temperatures (remember we only have direct evidence for the big bang picture for low temperatures) , the universe went through a series of phase transitions during its evolution. There were considered the electroweak and QCD phase transitions, and possibly other ones at the Grand Unified Theory scales. Depending on the symmetry broken in the phase transition topological defects - domain walls, cosmic strings, monopoles or textures may form. So called Polonyi fields also presented a problem. If a semi-simple GUT group is broken down to the Standard Model, either directly or via some intermediate steps, monopoles form. Monopoles are heavy pointlike objects, which behave as cold matter $\rho_{mp} \approx \frac{1}{a^3}$. If produced in the early universe, the energy density in monopoles decreases slowlier than the radiation background, and comes to dominate the energy density in the universe early on, in conflict with observations.

Inflation

The hot Big Bang theory could not explain the origin of structure in the universe, the origin of matter and radiation, and the initial singularity. Especially, the questions why is this universe so close to spatially flat one and why is the matter so homogeneously distributed on large scales, could be resolved by the so called Cosmological Inflation.[2] This theory was invented at the beginning of 1980's by A. Guth, A. Linde and A. A.Starobinsky like an epoch in the evolution of the universe before the radiation epoch - phase transition - which is characterized by $\ddot{a} > 0$ when it was approximately only $10^{-43} - 10^{-32}$ second old. It is an epoch when the universe was exponentially expanding for a tiny moment. People used like a trigger a homogeneously distributed scalar field, which then decayed. There is a similarity to the current situation in our universe because we have also an accelerating epoch, but the difference is, for example, in the duration how long it was accelerating. (The beginning of today's acceleration is approximately 5 billion years old.) It was announced that from the result of experiment BICEP2, which was published this year in March, that there were indirectly measured gravitational waves. However, this result must be confirmed at this moment. It look like that at this moment that there was a contribution from "magnetized gas". (Result from September 2014.)

The vacuum like period that drives inflation must be dynamic, it can't be true cosmological constant, because inflation must end. If we want to violate the strong energy condition and get a system with $\rho = -p$, we can use scalar fields. We will explain the basic concept of scalar fields minimally coupled to matter, which are one of the triggers of the Cosmological Inflation. We will consider for simplicity the single

[2] Details could be found in the Linde's book.

scalar field. Let's take the following action for the scalar field, which we will call the inflaton field (we could, for example consult the lectures of T.Prokopec, Lectures notes on Cosmology):

$$S = \int \sqrt{-g}[-\frac{1}{2}R + L_{\varphi}]\ d^4x., \tag{1.7}$$

with $L_{\varphi} = \frac{1}{2}g^{\mu\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi - V(\varphi)$ **where** $g = \det[g_{\mu\nu}] = -a^6$ **(FLRW).**
We can make a variation with respect to the scalar field and we get Euler-Lagrange equations:

$$\frac{\partial L_{\varphi}}{\partial \varphi} - \nabla_{\mu}[\frac{\partial L_{\varphi}}{\partial(\nabla_{\mu}\varphi)}] = 0 \tag{1.8}$$

But

$$\frac{\partial L_{\varphi}}{\partial \varphi} = -V(\varphi), \quad \frac{\partial L_{\varphi}}{\partial(\nabla_{\rho}\varphi}) = \nabla_{\rho}\varphi.$$

So the standard result is

$$\Box\varphi + \frac{\partial V}{\partial \varphi} = 0. \tag{1.9}$$

With the definition

$$T_{\mu\nu} \equiv -2\frac{1}{\sqrt{-g}}\frac{\partial S_{\varphi}}{\partial g^{\mu\nu}}, \tag{1.10}$$

we get also

$$T_{\mu\nu} \equiv -\partial_{\mu}\varphi\partial_{\nu}\varphi + g_{\mu\nu}(\frac{1}{2}\partial_{\rho}\varphi\partial^{\rho}\varphi - V(\varphi)). \tag{1.11}$$

1.4 Cosmological perturbation theory

Let us make an introduction to Cosmological Perturbation Theory in SGR. We mention that we use in this thesis a signature $(-,+,+,+)$ **except of the part two, where we use** $(+,-,-,-)$**. We consider a ST , a perturbed ST that is close to the background ST. We have an example of the background and a perturbed ST on the Figure 1. The metric on the perturbed ST will be the following metric:**

$$g_{\mu\nu}(t,\vec{x}) = \overline{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\vec{x}), \tag{1.12}$$

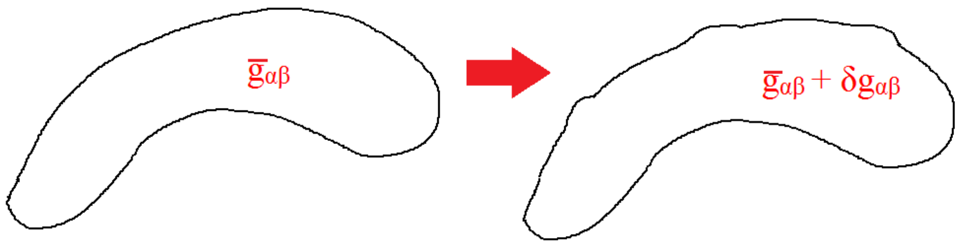
where bar means the background and δ **is a small change - perturbation - of the metric. We also assume that first and second partial derivatives are small, because we have second order PDE's. The field equations after subtraction:**

$$\delta G_{\mu\nu} = \kappa\delta T_{\mu\nu}, \tag{1.13}$$

where $\delta G_{\mu\nu}$ **is a perturbation of the Einstein tensor,** $\delta T_{\mu\nu}$ **is a perturbation of energy-momentum tensor and** $\kappa = 8\pi G$ **when the gravitational constant is equal to** 1**.**

The things above require a pointwise correspondence, so we can make comparisons and subtractions. Given a background coordinate

Figure 1.1: **Perturbation of background spacetime**



system, we have many coordinate system in the perturbed one, for which (??) holds. The choice among coordinates is called a gauge choice.

In first order prturbation theory we drop all terms from our equations which are products of small quantities of $\delta g_{\mu\nu}$, $\delta g_{\mu\nu,\sigma}$ and $\delta g_{\mu\nu,\sigma\tau}$. The field equations become then the linear differential equations for $\delta g_{\mu\nu}$.

So, as the background ST we will take the Friedmann-Lemaitre-Robertson-Walker ST (FLRW). And we will concentrate mainly on flat space (FLRW(0)). The metric is in co-moving coordinates

$$ds^2 = g_{\mu\nu}\mathrm{dx}^\mu \mathrm{dx}^\nu = \mathrm{a}^2(\eta)(\mathrm{d}\eta^2 + \mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2),$$

where $a(t)$ can be obtained with the cosmological constant equal to zero from Friedmann equations with cosmological constant equal to zero. We will denote again the backfground quantities by overbar. We could rewrite the Friedmann equations as

$$H_c^2 = \frac{8\pi\bar{\rho}\ a^2(\eta)}{3}, \qquad (1.14)$$

$$H_c^{'} = \frac{-4\pi}{3}(\bar{\rho} + 3\bar{p})\ a^2(\eta), \qquad (1.15)$$

where $H_c^{'} = \frac{dH_c(\eta)}{d\eta}$ is the derivative with respect to the conformal time. The energy- continuity equation becomes just

$$\bar{\rho}^{'} = -3H_c(\bar{\rho} + \bar{p}). \qquad (1.16)$$

We could derive further (with notation $w \equiv \frac{\bar{p}}{\bar{\rho}}$) also

$$H_c^{'} = \frac{(-1 - 3w)}{2} H_c^2. \qquad (1.17)$$

These equations show that $w = \frac{-1}{3}$ corresponds to constant comoving Hubble length, when the RHS of the previous equation is zero. But for $w < \frac{-1}{3}$ the comoving Hubble length shrinks with time (this is a typical situation for Cosmological Inflation), whereas for $w > \frac{-1}{3}$ it grows with time.

We can write the metric of the perturbed FLRW(0) universe as

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}), \tag{1.18}$$

where $h_{\mu\nu}$, as well as $h_{\mu\nu,\rho}$ and $h_{\mu\nu,\rho\sigma}$ are assumed small. We are doing the first order perturbation theory, so we shell drop from the equations all the terms which are of order $O(h^2)$ or higher. We define

$$h^{\mu}{}_{\nu} \equiv \eta^{\mu\rho}\eta^{\sigma}{}_{\nu}h_{\rho\sigma}, \;\; h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}. \tag{1.19}$$

The inverse metric of the perturbed spacetime is in first order

$$g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} - h^{\mu\nu}).$$

We shall now give different names for the time and space components of the perturbed metric, defining

$$h_{\mu\nu} = \begin{pmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{pmatrix}$$

.
where $D = -\frac{1}{6}h^i{}_i$ carries the trace of the spatial metric perturbation h_{ij}, and E_{ij} is traceless,

$$\delta^{ij}E_{ij} = 0.$$

Since indices on $h_{\mu\nu}$ are raised and lowered with $\eta_{\mu\nu}$, we immediately have

$$h^{\mu\nu} = \begin{pmatrix} -2A & B_i \\ B_i & -2D\delta_{ij} + 2E_{ij} \end{pmatrix}$$

.
The line element is thus

$$\mathrm{ds^2 = a^2(\eta)\{-(1+2A)d\eta^2 - 2\ B_i\ d\eta dx^i + [(1-2D)\delta_{ij} + 2E_{ij} + h_{ij}]dx^i dx^j\}}. \tag{1.20}$$

The association between the background and perturbed ST will be due to the coordinate system x^α. There are many possible coordinate systems in the perturbed STs for a given coordinate system in the background. (GR is diffeomorphism-invariant theory and we fixed the background.) Now we denote coordinates of the background by x^α and two different coordinates on the perturbed spacetime by $\hat{x}^\alpha$ and $\tilde{x}^\alpha$. These coordinates are related via the following relation

$$\tilde{x}^\alpha = \hat{x}^\alpha + \xi^\alpha, \tag{1.21}$$

where ξ^α and $\xi^\alpha{}_{,\beta}$ are small quantities (zero and first derivative is small) . And we shall think of ξ^α as living on the background ST.

$\tilde{x}^\alpha$ associates background point $\overline{P}$ with a point $\tilde{P}$ and analogically $\hat{x}^\alpha$ associates background point $\overline{P}$ with a point $\hat{P}$. We plug to the formula (1.21) for points $\hat{P}$ and $\tilde{P}$:

$$\tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\tilde{P}) + \xi^\alpha, \tag{1.22}$$
$$\tilde{x}^\alpha(\hat{P}) = \hat{x}^\alpha(\hat{P}) + \xi^\alpha. \tag{1.23}$$

Now the difference $\xi^\alpha(\tilde{P}) - \xi^\alpha(\hat{P})$ is second order small, so we just write ξ^α and associate it with the background point:

$$\xi^\alpha = \xi^\alpha(\overline{P})$$

Using previous knowledge, we get the relation between the coordinates of two different points in a given coordinate system,

$$\hat{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}) - \xi^\alpha, \tag{1.24}$$
$$\tilde{x}^\alpha(\tilde{P}) = \tilde{x}^\alpha(\hat{P}) - \xi^\alpha. \tag{1.25}$$

Let us now perturb various quantities now. We could have in the background ST 4-scalar fields $\overline{s}$, 4-vector fields $\overline{w}^\alpha$ and tensor fields $\overline{A}^\alpha{}_\beta$. In the background spacetime we have corresponding perturbed quantities in the perturbed ST.

$$s = \overline{s} + \delta s, \tag{1.26}$$
$$w^\alpha = \overline{w}^\alpha + \delta w^\alpha, \tag{1.27}$$
$$A^\alpha{}_\beta = \overline{A}^\alpha{}_\beta + \delta A^\alpha{}_\beta. \tag{1.28}$$

Now let us talk about 4-scalar. The full quantity $s = \overline{s} + \delta s$ lives on the perturbed ST. However, there is no unique background quantity s which could we assign to a point in the perturbed ST, because these points are assigned to different points $\overline{s}$ in the background. Therefore, we do not have unique perturbation δs, but the perturbation is gauge dependent. The perturbations in different gauges are defined as

$$\widehat{\delta s}(x^\alpha) \equiv s(\hat{P}) - \overline{s}(\overline{P}), \widetilde{\delta s}(x^\alpha) \equiv s(\tilde{P}) - \overline{s}(\overline{P}). \tag{1.29}$$

The perturbation δs is obtained from a subtraction between two STs, but we will consider it as living on background ST. It changes 4nder the gauge transformation. We will us now this knowledge and we will apply them $\widehat{\delta s}$ to the Weyl spinor and $\widetilde{\delta s}$:

$$s(\tilde{P}) = s(\hat{P}) + \frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P})[\hat{x}^\alpha(\tilde{P}) - \hat{x}^\alpha(\hat{P})] = s(\hat{P}) - \frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P})\xi^\alpha = s(\hat{P}) - \frac{\partial \overline{s}}{\partial x^\alpha}(\overline{P})\xi^\alpha, \tag{1.30}$$

used approxination $\frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P}) \approx \frac{\partial \overline{s}}{\partial x^\alpha}(\overline{P})$, because the difference is first order perturbation and ξ^α makes it second order.

The background is homogeneous: $\overline{s} = s(\eta, x^i) = \overline{s}(\eta)$, and

$$\frac{\partial \overline{s}}{\partial x^\alpha}(\overline{P})\xi^\alpha = \overline{s}'\xi^\alpha.$$

Thus, we get

$$s(\tilde{P}) = s(\hat{P}) - \overline{s}'\xi^0.$$

Our final result for a gauge transformation of δs is

$$\widetilde{\delta s}(x^\alpha) = \widehat{\delta s}(x^\alpha) - \overline{s}'\xi^0. \tag{1.31}$$

Similar results holds for vector and tensor PB's in the two gauges. So, for example:

$$\widetilde{\delta w^\alpha} = \delta w^\alpha + \xi^\alpha{}_{,\beta}\overline{w}^\beta - \overline{w}^\alpha{}_{,\beta}\xi^\beta, \tag{1.32}$$

where we dropped the hats from the first gauge, we will do the same in the following text. By applying the gauge transformation equation to the metric perturbation, we get

$$\widetilde{\delta g_{\mu\nu}} = \delta g_{\mu\nu} - \xi^\rho{}_{,\mu}\overline{g}_{\rho\nu} - \xi^\sigma{}_{,\nu}\overline{g}_{\mu\sigma} - \overline{g}_{\mu\nu,\eta}\xi^0, \tag{1.33}$$

where we have replaced the sum $\overline{g}_{\mu\nu,\alpha}\xi^\alpha$ with $\overline{g}_{\mu\nu,0}\xi^0$, since the background metric depends only on the time coordinate η. After some conputation we obtain after some computation the gauge transformation laws:

$$\begin{aligned}
\tilde{A} &= A - \xi^0_{,0} - H_c, \\
\tilde{B}_i &= B_i + \xi^i_{,0} - \xi^0_{,i}, \\
\tilde{D} &= D + \frac{1}{3}\xi^k_{,k} + H_c\xi^0, \\
\tilde{E}_{ij} &= E_{ij} - \frac{1}{2}(\xi^i_{,j} + \xi^j_{,i}) + \frac{1}{3}\delta_{ij}\xi^k_{,k}.
\end{aligned} \tag{1.34}$$

However we could look at the transformations differently. We fix the correspondence between the background and perturbed ST. Now we make coordinate transformations on the background and we induces - via the correspondence mapping - the coordinate transformations in the perturbed ST. We respect the homogeneity property on the background, which gives us unique slicing of the ST into homogeneous, $t = const.$, spacelike slices. This leaves us homogeneous transformations of the time coordinate, which we have as an example, when we switch from the cosmic time t to the conformal time η, (??). We can make transformations in the space coordinates

$$x^{i'} = X^{i'}{}_k x^k,$$

where $X^{i'}{}_k$ is independent of time. For the three metric in our background we had chosen Euclidean coordinates for the 3-metric in our background and this leaves us rotations.

$$g_{ij} = a^2\delta_{ij}.$$

We have transformation matrices

$$X^{\mu'}{}_\rho = \begin{pmatrix} 1 & 0 \\ 0 & X^{i'}{}_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R^{i'}{}_k \end{pmatrix},$$

and

$$X^{\mu}{}_{\rho'} = \begin{pmatrix} 1 & 0 \\ 0 & R^{i}{}_{k'} \end{pmatrix},$$

where $R^{i'}{}_{k}$ **is a rotation matrix, with the property** $R^T R = I$ **or** $R^{i'}{}_{k} R^{i'}{}_{l} = \delta_{kl}$**. Thus** $R^T = R^{-1}$**, so that** $R^{i'}{}_{k} = R^{k}{}_{i'}$**.**

This coordinate transformation in the background induces the corresponding transformation,

$$x^{\mu'} = X^{\mu'}{}_{\rho} x^{\rho},$$

into the perturbed ST. Here the metric is

$$g_{\mu\nu} = \begin{pmatrix} -1-2A & -B_i \\ -B_i & 1-2D\delta_{ij}+2E_{ij} \end{pmatrix}$$
$$= a^2\eta_{\mu\nu} + a^2 \begin{pmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij}+2E_{ij} \end{pmatrix}$$

.

Transforming the metric

$$g_{\rho'\sigma'} = X^{\mu}{}_{\rho'} X^{\nu}{}_{\sigma'} g_{\mu\nu},$$

after computation for the perturbations in the new coordinates we get,

$$A' = A, \tag{1.35}$$

$$D' = D, \tag{1.36}$$

$$B_{l'} = R^{j}{}_{l'} B_j, \tag{1.37}$$

$$E_{k'l'} = R^{i}{}_{k'} R^{j}{}_{l'} E_{ij}. \tag{1.38}$$

So we see that A **and** D **transform like scalars in the background spacetime coordinates,** B_i **like a 3-vector and** E_{ij} **like a tensor. But we could think of them as scalar, vector and tensor fields on the 3-dimensional background spacetime. However, we can extract two more scalar quantities from** B_i **and** E_{ij}**, and a vector quantity from** B_i **and** E_{ij}**.**

We know from the 3-dimensional calculus - Helmholtz theorem - that a vector field could be divided into 2 parts: the first one with zero curl and the second one with zero divergence

$$\vec{B} = \vec{B}_S + \vec{B}_V,$$

with $\nabla \times \vec{B}^S = 0$ and $\nabla.\vec{B}^V = 0$, so the first one could be expressed as a gradient of some scalar field $\vec{B}^S = -\nabla B$. In component notation, $B_i = -B_{,i} + B_i^V$, where $\delta^{ij} B^V_{i,j} = 0$. In like manner, the symmetric traceless tensor field E_{ij} can be divided into three parts,

$$E_{ij} = E^S_{ij} + E^V_{ij} + E^T_{ij},$$

where E^S_{ij} and E^V_{ij} can be expressed in terms of scalar field E and vector field E_i,

$$E^S_{ij} = \quad (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E = E_{,ij} - \frac{1}{3}\delta_{ij}\delta^{kl}E_{,kl}, E^V_{ij} = \quad -\frac{1}{2}(E_{i,j} + E_{j,i}), \quad (1.39)$$

where $\delta^{ij} E_{i,j} = \nabla \vec{E} = 0$, $\delta^{ik} E^T_{ij,k} = 0$ and $\delta^{ij} E^T_{ij} = 0$.

We see that E^S_{ij} is symmetric and traceless by construction. E^V_{ij} is symmetric by definition and the condition on E_i makes it traceless. The tensor E^T_{ij} is assumed to besymmetric. And the two conditions on it make it transverse and traceless. Under rotation in the background space,

$$A^{'} = A, \; B^{'} = B, \; D^{'} = D, \; E^{'} = E,$$
$$B^V_{l'} = R^j{}_{l'} B^V_j, \; E_{l'} = R^j{}_{l'} E_j,$$
$$E^T_{j'l'} = R^i{}_{j'} R^j{}_{l'} E^T_{ij}.$$

The metric perturbation can those be divided into scalar, vector and tensor part and these names refers to their transformation property in the background spacetime. In all textbooks is written that scalar, vector and tensor perturbations do not couple to each other but they evolve independently. We had a comment already in the previous chapter. We imposed one constraint on each of the 3-vectors B^V_i and E_i, and 4 constraints on the symmetric 3-dimensional tensor E^T_{ij} leaving each of them 2 independent components. Thus the 10 degrees of freedom corresponding to the 10 components of the metric perturbation $h_{\mu\nu}$ are divided into $1+1+1+1=4$ scalar, $2+2=4$ vector, and 2 tensor degrees of freedom.

The scalar perturbations are for us the most important. They couple to the density and pressure perturbations and exhibit gravitational instability: overdense regions grow more overdense; They are responsible for formation of structure in the universe from small initial perturbations. We have an Appendix in Chapter 2 is devoted to scalar perturbations.

The vector perturbations couple to rotational velocity perturbations in the cosmic fluid. They tend to decay in the expanding universe and are therefore not important in cosmology. Tensor perturbations have cosmological importance, since they have an observable effect on the anisotropy of the cosmic microwave background.

We will now consider only scalar perturbations. The metric is now

$$\mathrm{ds^2 = a^2(\eta)\{-(1+2A)d\eta^2 + (1-2\psi)\delta_{ij}dx^idx^j\}}, \tag{1.40}$$

where we use the curvature perturbation

$$\psi \equiv D + \frac{1}{3}\nabla^2 E. \tag{1.41}$$

If we start from a pure scalar perturbation and we make an arbitrary gauge transformation $\xi^\mu = (\xi^0, \xi^i)$, we may introduce also a vector perturbation. This is pure gauge transformation and thus of no interest. As we did in the previous part for B_i, we could divide ξ into part with zero divergence and part with zero curl, expressible as a gradient of some function ξ,

$$\xi^i = \xi^i_{vec} - \delta^{ij}\xi_{,j} \leftrightarrow \vec{\xi}_{vec} - \nabla\xi,$$

where $\xi^i_{vec,i} = 0$. The part ξ^i_{vec} is responsible for spurious vector perturbation, where ξ_0 and ξ_j change the scalar perturbation. For our discussion of scalar perturbations we thus lose nothing, if we decide that we only consider gauge transformations, where the ξ^i_{tr} is absent. These scalar gauge transformations are fully specified by two functions, ξ^0 and ξ,

$$\begin{aligned} \tilde{\eta} &= \eta + \xi^0(\eta, \vec{x}), \\ \tilde{x}^i &= x^i - \delta^{ij}\xi_{,j}(\eta, \vec{x}). \end{aligned} \tag{1.42}$$

and they preserve scalar nature of the perturbation. Applied to scalar perturbations and gauge transformations, our transformation equations become

$$\begin{aligned} \tilde{A} &= A - \xi^{0\prime} - \frac{a'}{a}\xi^0, \\ \tilde{B} &= B + \xi' + \xi^0, \\ \tilde{D} &= D - \frac{1}{3}\nabla^2\xi + \frac{a'}{a}\xi^0, \\ \tilde{E} &= E + \xi. \end{aligned} \tag{1.43}$$

where we have used the notation $' \equiv \frac{\partial}{\partial\eta}$ for quantities which depend on both η and $\vec{x}$. The quantity ψ defined in (1.41) is often used as the fourth scalar variable instead of D. For it, we get

$$\tilde{\psi} = \psi + \frac{a'}{a}\xi^0.$$

We now define the following two quantities called the Bardeen potentials:

$$\boxed{\begin{aligned} \Phi &\equiv A + H_c(B - E') + (B - E')', \\ \Psi &\equiv D + \tfrac{1}{3}\nabla^2 E - H_c(B - E') = \psi - H_c(B - E'). \end{aligned}}$$

The quantities are invariant under gauge transformations when we use that $\tilde{H}_c = H_c$, because H_c do not transform - the background quantity. These potentials were introduced by Bardeen and they are the simplest gauge-invariant linear combination of A, D, B and E, which span a two dimensional space of gauge invariant variables and which can be be constructed from metric-variables alone.

We can use the gauge freedom to set the scalar perturbations B and E equal to zero. From equation (1.43) we see that this is accomplished by choosing

$$\begin{aligned} \xi &= -E, \\ \xi^0 &= -B + E'. \end{aligned} \tag{1.44}$$

Doing this gauge transformation we arrive at a commonly used gauge, which has a name conformal-Newtonian gauge (people are using also other names for this gauge). We will denote quantities in this gauge with the superscript N. Thus $B^N = E^N = 0$, whereas we immediately see that

$$\begin{aligned} A^N &= \Phi, \\ D^N = \psi^N &= \Psi. \end{aligned} \tag{1.45}$$

Thus the Bardeen potentials are equal to the two nonzero metric perturbations in the conformal-Newtonian gauge. We could use also different gauges but we are interested mainly in the conformal-Newtonian gauge. But we will use now the computations of Riemann and energy momentum tensor in this gauge from the lectures, for example, of H.- K.Suonio. We will apply it to perfect fluid scalar perturbations, especially to scalar perturbations in the matter-dominated universe.

By matter we mean here the non-relativistic matter, whose pressure is so small to energy density that we could ignore it here. It is usually called dust. Until the $1990's$ it was believed that this matter-dominated universe persists until present time. However now we know that we are living in accelerated epoch, which means that there is an other component of energy density of the universe with negative pressure. This component is called dark energy (Chapter 2). So, the validity of the matter dominated approximation is not as extensive as was thought before, anyway there was a significant period inthe history of the universe when it was valid.

So, we now make the matter dominated approximation and we ignore pressure $p = 0$. We talked about this example already in the Chapter 2. The order of work is always the same. We solve the background problem and we use the background quantities as known functions of time to solve the perturbation problem.

The background equations - we will write an overbar - are

$$(A1 H_c^2 = \frac{8\pi G}{3} \bar{\rho} a^2, \tag{1.46}$$

$$A2H_c^{'} = -\frac{4\pi G}{3}\overline{\rho}a^2. \qquad (1.47)$$

from which we have $2H_c^{'} + H_c^2 = 0$. The background solution is the familiar $k = 0$ matter-dominated Friedmann model, $a \sim t^{2/3}$. But we will review the solution in terms of a conformal time. Since $\overline{\rho} \sim a^{-3}$, the solution of (??) gives

$$a(\eta) \sim \eta^2.$$

From $a \sim \eta^2$ we get

$$H_c = \frac{a^{'}}{a} = \frac{2}{\eta},$$

and

$$H_c^{'} = -\frac{2}{\eta^2}.$$

Thus from equations (??) ,

$$4\pi G a^2\overline{\rho} = \frac{3}{2}H_c^2 = \frac{6}{\eta^2}. \qquad (1.48)$$

According to [9], the perturbation equations are for $\overline{p} = \delta p = 0$

$$\nabla^2\Phi = 4\pi G a^2\overline{\rho}[\delta^N + 3H_c v^N], \qquad (1.49)$$

$$\Phi^{'} + H_c\Phi = 4\pi G a^2\overline{\rho}v^N, \qquad (1.50)$$

$$\Phi^{''} + 3H_c\Phi^{'} + (2H_c + H_c^2) = 0. \qquad (1.51)$$

Here we use the notation $v_i = -v_{,i}$ and $v_i = \frac{\delta u_i}{a}$, index N denotes again the conformal-Newtonian gauge.

Now, we will use $2H_c^{'} + H_c^2 = 0$ and the last equation (1.51). We wil get that

$$\Phi(\eta, \vec{x}) = C_1(\vec{x}) + C_2(\vec{x})\eta^{-5}. \qquad (1.52)$$

The second term is the decaying part. We get $C_1(\vec{x})$ from the initial values $\Phi_{in}(\vec{x})$ and $\Phi_{in}^{'}(\vec{x})$ at some intial time $\eta = \eta_{in}$,

$$\Phi_{in}(\vec{x}) = C_1(\vec{x}) + C_2(\vec{x})\frac{1}{\eta_{in}^5}, \Phi_{in}^{'}(\vec{x}) = -5C_2(\vec{x})\eta_{in}^{-6}, \qquad (1.53)$$

where

$$C_1(\vec{x}) = \Phi_{in}(\vec{x}) + \frac{1}{5}\eta_{in}\Phi_{in}^{'}(\vec{x}) \ , C_2(\vec{x}) = -\frac{1}{5}\eta_{in}^6\Phi_{in}^{'}(\vec{x}). \qquad (1.54)$$

Unless we have very special initial conditions, conspiring to make $C_1(\vec{x})$ vanishingly small, the decaying part soon becomes much smaller than $C_1(\vec{x})$ and can be ignored. Thus we have the important result that the Bardeen potential Φ is constant in time for perturbations in the flat matter dominated universe.

Ignoring the decaying part, we have $\Phi^{'} = 0$ and we get for the velocity perturbation from (1.50)

$$v^N = \frac{H_c\Phi}{4\pi G a^2\overline{\rho}} = \frac{2\Phi}{3H_c} = \frac{1}{3}\Phi\eta = t^{\frac{1}{3}}, \qquad (1.55)$$

and from (1.49) we have

$$\nabla^2\Phi = 4\pi G a^2\overline{\rho}[\delta^N + 2\Phi] = \frac{3}{2}H_c^2[\delta^N + 2\Phi], \tag{1.56}$$

or

$$\delta^N = -2\Phi + \frac{2}{2H_c^2}\nabla^2\Phi. \tag{1.57}$$

Because our background space is flat we can Fourier expand the perturbations. For an arbitrary perturbation $f = f(\eta, x^i) = f(\eta, \vec{x})$, we write

$$f(\eta, \vec{x}) = \sum_{\vec{k}} f_{\vec{k}}(\eta)e^{i\vec{k}\vec{x}}. \tag{1.58}$$

Using a Fourier sum implies using a fiducial box with volume V. Finally we can let $V \to \infty$, and replace remaining Fourier sums with integrals. In first order perturbation theory each Fourier component evolves independently, so we can just study the evolution of a single Fourier component, with some arbitrary wave vector $\vec{k}$, and we drop the subscript $\vec{k}$ from the Fourier amplitudes. Since $\vec{x} = (x^1, x^2, x^3)$ is a co-moving coordinate, $\vec{k}$ is a co-moving wave vector. The co-moving wave number k and wavelength $\lambda = \frac{2\pi}{k}$ are related to the physical wavelength and wave number of the Fourier mode by

$$k_{phys} = \frac{2\pi}{\lambda_{phys}} = \frac{2\pi}{a\lambda} = a^{-1}k. \tag{1.59}$$

Thus the wavelength λ_{phys} of the Fourier mode grows in time as the universe expands. Details are written in the chapter 6 of [9]. Now we will return to equation (1.57). In Fourier space this reads

$$\delta_{\vec{k}}^N(\eta) = -[2\Phi + \frac{2}{3}(\frac{k}{H_c^2})^2]\Phi_{\vec{k}}, \tag{1.60}$$

Thus we see that for the superhorizon scales, $k << H_c$, the density perturbation stays constant

$$\delta_{\vec{k}}^N = -2\Phi_{\vec{k}} = const. \tag{1.61}$$

whereas for subhorizon scales, $k >> H_c$, they grow proportional to the scaling factor

$$\delta_{\vec{k}}^N \sim a \sim t^{2/3}. \tag{1.62}$$

Since the comoving Hubble scale H_c grows with time, various scales k are superhorizon to begin with, but later become subhorizon. We say that the scale in question "enters the horizon". We see that the density perturbations begin to grow when they enter the horizon, and after that they grow proportionally to the scale factor.
But one has to remember that these results refer to the density and velocity perturbations in the conformal-Newtonian gauge only. In some other gauges these perturbations, and their growth laws would be different. However, for subhorizon scales general relativistic effects become unimportant and a Newtonian description becomes valid. In

this limit, the issue of gauge choice become irrelevant as all "sensible" gauges approach each other, and the conformal-Newtonian density and velocity perturbations become those of a Newtonian description. The Bardeen potential can then be understood as a Newtonian gravitational potential due to density perturbations.

More about perturbations can found in the lectures of M. Postma, T. Prokopec, H. - K. Suonio. General introduction to cosmology we could find, for example, in the text of J. G. - Bellido.

1.5 Gauge invariance

SGR is gauge invariant theory, what we already mentioned in the previous, where the gauge transformations are the generic coordinate transformations from local reference frame to another. The coordinates t, x carry an independent physical meaning. By performing a coordinate transformation, we can create fictious fluctuations in a homogeneous and isotropic universe, which are just gauge artefacts. For a FLRW universe there is a special gauge choice in which the metric is homogeneous and isotropic, which singles out a preferred coordinate choice. But he situation is more complicated in a perturbed universe and we have to be careful in that. Consider first a scalar perturbation in a fixed ST. It can be defined via $\delta\phi(p) = \phi_p - \phi_0(p)$ with ϕ_0 the unperturbed field and p is any point of the ST. Generalizing this to the standard General Relativity, where ST is not a fixed background, but is perturbed, if matter is perturbed, the above definition is ill defined. Indeed, ϕ lives in the perturbed real ST M where as ϕ_0 lives in another ST, the reference spacetime M_0. To define a perturbation requires an identification that maps points in M_0 to points in M. The perturbation can then be defined via $\delta\phi = \phi((p_0)) - \phi_0(p_0)$. However, the identification is not uniquely defined, and therefore the definition of the perturbation depends on the choice of map. This freedom of choising map is the freedom of choosing coordinates. The choice of map is a gauge choice, changing the map is a gauge transformation.

Thus fixing a gauge in SGR implies choosing a coordinate system, threading a ST into lines (corresponding to fix x) and slicing into hypersurface of fixed time. There are two ways to proceed, and remove the gauge artifacts. Perform the computation in terms of the gauge invariant quantities or in a fixed gauge.

2. Standard General Relativity in Higher Dimensions

2.1 Introduction

Studying SGR in higher dimensions - generally for dimension of spacetime d with non-compact dimensions - served us as a nice preparation to our other works in Chapter II and Chapter III. Therefore we are beginning with this topic in Chapter I and we briefly discuss at the beginning the GHP formalism in higher dimensions. Then a section about the classification of the Weyl tensor in higher dimensions follows. We also included a related section about classification in spinors. In the following we also study Kundt spacetimes (ST's), because of their usefulness in perturbations of black holes. We develop some basic concepts for dimension d.

Before we will begin our own work, let us mention also the following inspirative ideas in connection with cosmology. Details could be found, for example, in [1], however various authors also discuss this topic in other sources. The author of [1] was looking on the matter in a curved $4d$ ST which can be regarded as the result of the embedding in a x^4 - dependent $5d$ ST. The nature of the 4d matter depends on the signature of the $5d$ metric. And finally, what is most important for us, the $4d$ source depends on the extrinsic curvature of the embedded $4d$ ST and the scalar field associated with the extra dimension. Various cosmological models are also discussed in this book.

First of all, note that the field equations of SGR in higher dimensions are more complicated and the computations more involved. Because in this thesis we want to concentrate on perturbation theory, we should mention that perturbations of rotating objects are more complex. For example, the perturbation theory of Schwarzchild black hole was studied, even by analytical methods, already by Chandrasekhar [38], in 1983. When we consider the Kerr black hole, which was found in 1963, we have already much more difficult problem. And the difficulty increases as we go to higher and higher dimensions. People are using numerical simulations for studying the stability of such objects [41]. The features of event horizons are strongly dimension dependent as was pointed out already in [39]. Black hole thermodynamics is also used in this analysis, [41].

The generalization of the Kerr solution - the rotating black hole - into higher dimensions is so called Myers Perry solution. It is a hard problem to solve the stability issues for this solution. When people try to solve these questions, they usually begin with rotations in single plane. Natural parameters of this solution are angular momentum parameters and mass. From the formula for mass, it seems that the properties of these black holes do not differ too much from their counterparts in four dimensions, however this in not true, as we can see from the formula for gravitational attraction and centrifugal repulsion.

$$\frac{\Delta}{r^2} - 1 = \frac{-\mu}{r^{d-3}} + \frac{a^2}{r^2}, \tag{2.1}$$

details in [41].

The full solution can be analyzed using the method of the phase space. We know the term phase space from the theory of ODE's, however here it means that we fix the mass and to every angular momentum parameter we define a dimensionless quantity j_i for each angular momentum parameter, we have a system of $(j_1, ...j_N)$,where N goes to $d-3$. Only some values of these parameters - regions - are allowed. We can find the qualitative behaviour of solutions in dimension d, if we know it already in dimension $d-2$. If we are able to achieve to find the region where the regular black hole exist, we could express all physical magnitudes as functions of phase space variables j_i.

The global topology of these solutions is the same as for the Kerr solutions, however there are some differences in the causal nature of the singularities. The Myers-Perry solution is manifestly invariant under translations and also under rotations generated by N Killing vectors.

People faced for many years the problems with the stability issues and some questions are not answered yet. The interesting feature occured for the ultraspinning regime of rotating black holes. They are dynamically unstable for dimension $d \geq 6$ and they come apart into pieces.

General structure of these solutions is more complex than in 4 dimensions. Black rings and black saturns also exist. Therefore it is important to find which solutions are stable in the linearized sense.

We are mostly interested in black holes. These objects are for us the most mysterious objects in the universe. People found already many years ago that they, however, are very easy to describe thanks to no-hair theorems. We were able to describe such an object with only few quantities. Also, it was found by Roger Penrose and Stephen Hawking that there was an analogy between cosmological model and black hole, because both have singularities. So for us, this is a motivation to study these objects.

Let us mention here that perhaps the most fascinating objects in this universe are various galaxies and quasars. We suppose and all indirect measurements are in concordance with this, that super-massive black holes are in the centers of such objects.

The study of the linearized perturbations have connections with isometries of black hole spacetimes. But an alternative approach was made by Teukolsky. His approach was used for dealing with perturbations of Kerr solution, [38]. As we write later, the Newman-Penrose scalars Ψ_i encode the information about Weyl tensor. The perturbation of Ψ_i will be denoted by $\Psi_i^{(1)}$ and the unperturbed value is Ψ_0. We have a gauge freedom in infinitesimal coordinate transformations and infinitesimal changes of tetrad.

$\Psi_0^{(1)}$ is gauge invariant, if l is repeated principal null direction of background ST.

In general spacetime, the linearized equations will lead to coupled equations of motion for the quantities $\Psi_A^{(1)}$. In an algebraically special vacuum ST, Teukolsky showed that one can decouple these equations to obtain a single second order wave equation for Ψ_0^1 . If the ST is of the algebraical type D, both $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ are gauge invariant. They satisfy decoupled equations of motion. We will use the GHP formalism in Chapter 2 to show that $d\Psi_0$ and $d\Psi_4$ decouple for the case of FLRW ST's , when we make appropriate simplifications of the RHS.

For the case of Kundt ST's, we obtain decoupling these quantities also decouple in higher dimensions. We will present it on the example of electromagnetism, because this result is considerably simpler. Then we will mention gravitational perturbations. The highest boost weights components in electromagnetism are denoted by ϕ_j. In $4d$ these quantities satisfy a decoupled system of equations and it was investigated how this works in general dimension d:

$$
\begin{aligned}
&(2\text{þ}'\text{þ} + \eth_j\eth_j + \rho'\text{þ} - 4\tau_j\eth_j + \phi - \frac{2(d-3)}{(d-1)}\Lambda)\phi_i \\
&\quad + (-2\tau_i\eth_j + 2\tau_j\eth_i + 2\phi_{ij}^S + 4\phi_{ij}^A)\phi_j = 0
\end{aligned}
\tag{2.2}
$$

It is interesting to compare this with the equation for massive scalar field ϕ

$$
(\nabla_\mu\nabla^\mu - \mu^2)\phi = 0. \tag{2.3}
$$

When we write this equation in the GHP formalism, we get the following result:

$$
(2\text{þ}'\text{þ} + \eth_i\eth_i + \rho'\text{þ} + 2\tau_i\eth_i + \rho\text{þ}' - \mu^2)\phi = 0. \tag{2.4}
$$

The basic objects are the gravitational perturbations Ω_{ij}. $\Omega_{ij}^{(1)}$ is gauge invariant quantity, if l is a multiple WAND of the background spacetime. Thus we should study Einstein spacetimes for which l is a multiple WAND. In that case we have that Ω_{ij} and Ψ_{ijk} vanish in the background and that $\Omega_{ij} = \Omega_{ij}^{(1)}$ and $\Psi_{ijk} = \Psi_{ijk}^{(1)}$. And the final result is the same: we could achieve decoupling when the ST is Kundt.

We have already enough motivation for presenting the following concept of the algebraic classification. [41]

2.2 Algebraic classification of spacetimes in higher dimensions

We will mention the most important concepts from algebraic classification of Weyl tensor in higher d-dimensional Lorentzian manifolds, [2]. The present classification reduces to the classical Petrov classification in 4 dimensions. We shall consider null frame l, n, $m_{(i)}$ (l and n null with $l^\mu l_\mu = n^\mu n_\mu = 0$, $l^\mu n_\mu = 1$, $m^{(i)}$ real and spacelike with $m^a_{(i)} m^{(j)}_a = \delta^{(j)}_{(i)}$; all other products vanish) in a d-dimensional Lorentzian ST with signature $(-,+,+...,+)$ (we could choose also $l^\mu n_\mu = -1$, both alternatives are possible), so that

$$g_{\mu\nu} = 2l_{(\mu}n_{\nu)} + \delta_{(j)(k)} m^{(j)}_\mu m^{(k)}_\nu. \tag{2.5}$$

The frame is covariant relative to the group of real orthochronous (=preserves the direction of time) linear Lorentz transformations, generated by null rotations, boosts and spins:

Null rotations:
rotations of one of the null basis vectors about the other. A null rotation about l takes the form

$$l \mapsto l,\ \ n \mapsto n + z_i m_{(i)} - \frac{1}{2} z^2 l,\ \ m_{(i)} \mapsto m_{(i)} - z_i l, \tag{2.6}$$

where $z^2 = z_i z_i$.
Boosts:
these are rescalings of the null basis vectors that preserve the scalar product $l.n = 1$

$$l \mapsto \lambda l,\ \ n \mapsto \lambda^{-1} n,\ \ m_{(i)} \mapsto m_{(i)}, \tag{2.7}$$

where λ is an arbitrary non-zero function. We shall say that l, n and $m_{(i)}$ have boost weights -1, 0 and 1. Generally, we say that a tensor quantity $T_{i_s...i_s}$ has a boost weight b, if it transforms as

$$T_{i_1....i_s} \mapsto \lambda^b T_{i_1...i_s} \tag{2.8}$$

under boosts.
Spins:
these are $SO(d-2)$ rotations of the spatial basis vectors

$$m_{(i)} \mapsto X_{ij} m_{(j)} \tag{2.9}$$

Any tensor T can be expanded with respect to the basis $l, n, m_{(i)}$, where we use a collective notation for all three vectors $l^{(a)}$,

$$T_{\mu\nu...\sigma} = l^{(a)}_\mu l^{b)}_\nu ... l^{(d)}_\sigma T_{(a)(b)...(d)}, \tag{2.10}$$

so, for example, (lowered) indices 0 correspond to contractions with l. The objects $T_{(a)(b)...(d)}$ are ST scalars, but transform as tensor components under local Lorentz transformations, corresponding to changes in the choice of basis vectors. We write the covariant derivatives of the basis vectors as

$$L_{\mu\nu} = \nabla_\nu l_\mu,\ \ N_{\mu\nu} = \nabla_\nu n_\mu,\ \ M^{(i)}_{\mu\nu} = \nabla_\nu m_{(i)\mu}, \tag{2.11}$$

and then project into the basis to obtain the scalars $L_{(a)(b)}, N_{(a)(b)}, M^{(i)}_{(a)(b)}$. From the orthogonality properties of the basis vectors we have the identities

$$N_{0(a)} + L_{1(a)} = 0,\ \ M^{(i)}_{0(a)} + L_{(i)(a)} = 0,\ \ M^{(i)}_{1(a)} + N_{(i)(a)} = 0,\ \ M^{(i)}_{j(a)} + M^{(j)}_{(a)(b)} = 0 \tag{2.12}$$

We introduce the notation

$$T_{\{pqrs\}} \equiv \frac{1}{2}(T_{[pq][rs]} + T_{[rs][pq]}).$$

We can decompose the Weyl tensor and sort the components of the Weyl tensor by boost weight:

$$\begin{aligned}
C_{\mu\nu\dots\sigma} = {} & \overbrace{4C_{0(i)0(j)} n_{\{\mu} m^{(i)}_{\nu} n_{\rho} m^{(j)}_{\sigma\}}}^{\textbf{boost weight 2}} \\
& + \overbrace{8\,C_{010(i)} n_{\{\mu} l_{\nu} n_{\rho} m^{(i)}_{\sigma\}} + 4C_{0(i)(j)(k)} n_{\{\mu} m^{(i)}_{\nu} m^{(j)}_{\rho} m^{(k)}_{\sigma\}}}^{\textbf{boost weight 1}} \\
& \left.\begin{aligned} &+4C_{0101} n_{\{\mu} l_{\nu} n_{\rho} l_{\sigma\}} + 4C_{01(i)(j)} n_{\{\mu} l_{\nu} m^{(i)}_{\rho} m^{(j)}_{\sigma\}} \\ &+8C_{0(i)1(j)} n_{\{a} m^{(i)}_{b} l_{c} m^{(j)}_{d\}} + C_{(i)(j)(k)(l)} m^{(i)}_{\{a} m^{(j)}_{b} m^{(k)}_{c} m^{(l)}_{d\}} \end{aligned}\right\}\textbf{boost weight 0} \\
& + \overbrace{8C_{101(i)} l_{\{a} n_{b} l_{c} m^{(i)}_{d\}} + 4C_{1(i)(j)(k)} l_{\{a} m^{(i)}_{b} m^{(j)}_{c} m^{(k)}_{d\}}}^{\textbf{boost weight -1}} \\
& + \overbrace{4C_{1(i)1(j)} l_{\{a} m^{(i)}_{b} l_{c} m^{(j)}_{d\}}}^{\textbf{boost weight -2}}
\end{aligned} \tag{2.13}$$

The Weyl tensor is generically of boost order 2. If all $C_{0(i)0(j)}$ vanish, but some $C_{010(i)}$ or $C_{0(i)(j)(k)}$ do not, then the boost order is 1, etc. The Weyl scalars also satisfy a number of additional relations, which follow from the curvature tensor symmetries and from the trace-free condition:

$$C^{(i)}{}_{0(i)0} = 0,\ C_{010(j)} = C^{(i)}{}_{0(i)(j)},\ C_{0\langle(i)(j)(k)\rangle} = 0, \tag{2.14}$$

$$C_{0101} = C^{(i)}{}_{0(i)1},\ C_{(i)\langle(j)(k)(l)\rangle} = 0\ C_{0(i)1(j)} = -\frac{1}{2}C^{(k)}{}_{(i)(k)(j)} + \frac{1}{2}C_{01(i)(j)}, \tag{2.15}$$

$$C_{011(j)} = -C^{(i)}{}_{1(i)(j)},\ C_{1\langle(i)(j)(k)\rangle} = 0,\ C^{(i)}{}_{1(i)1} = 0, \tag{2.16}$$

where we use for the symmetrization notation $\langle ... \rangle$. A real null rotation about l fixes the leading terms of a tensor, while boosts and spins subject the leading terms to an invertible transformation. It follows that the boost order (along l) of a tensor is a function of the null direction l(only). We shall therefore denote boost order by $B(l)$. We define a null vector l to be aligned with the Weyl tensor whenever $B(l) \leq 1$ (and we shall therefore refer to l as a Weyl aligned null direction (WAND)). We call an integer $1 - B(l) \in \{0, 1, 2, 3\}$ the order of alignment. The alignment equations are $d(d-3)/2$ degree-4 polynomial equations in $(d-2)$ variables, which are in general overdetermined and hence have no solutions for $d > 4$.
We say that the principal type of the Weyl tensor in a Lorentzian manifold is I, II, III, N according to whether there exists an aligned l of alignment order $0, 1, 2, 3$, respectively. If no aligned l exists we will say that the manifold is of general type G. If the Weyl tensor vanishes, we will say that the manifold is of type O. The algebraically special

types are summarized as follows:

$$
\begin{aligned}
I &: C_{0(i)0(j)} = 0 \\
II &: C_{0(i)0(j)} = C_{0(i)(j)(k)} = 0 \\
III &: C_{0(i)0(j)} = C_{0(i)(j)(k)} = C_{(i)(j)(k)(l)} = C_{01(i)(j)} = 0 \\
N &: C_{0(i)0(j)} = C_{0(i)(j)(k)} = C_{(i)(j)(k)(l)} = C_{01(i)(j)} = C_{1(i)(j)(k)} = 0
\end{aligned} \tag{2.17}
$$

A 4-dimensional Weyl tensor always possesses at least one aligned direction. For higher dimension, in general, a Weyl tensor does not possess any aligned directions. Like a remark, it was shown that if $d \geq 5$, then the set of Weyl tensors with alignment type G is a dense, open subset of the set of all d-dimensional Weyl tensors.

2.2.1 Shear, twist and expansion

We can write

$$l_{\alpha;\beta} l^{\beta} = L_{10} l_{\alpha} + L_{(i)0} m_{\alpha}^{(i)}$$

according to [7].
So

$$l \text{ is geodetic} \Leftrightarrow L_{(i)0} = 0 \tag{2.18}$$

In this case the matrix $L_{(i)(j)}$ acquires a special meaning since it is then invariant under the null rotations preserving l. It is then convenient to decompose $L_{(i)(j)}$ into its tracefree symmetric part $\sigma_{(i)(j)}$ (shear), its trace θ (expansion) and its antisymmeric part $A_{(i)(j)}$ (twist) as

$$L_{(i)(j)} = \sigma_{(i)(j)} + \theta \delta_{(i)(j)} + A_{(i)(j)} \tag{2.19}$$

$$\sigma_{(i)(j)} \equiv L_{\langle(i)(j)\rangle} - \frac{1}{n-2} L_{(k)(k)} \delta_{(i)(j)}, \ \theta \equiv \frac{1}{n-2} L_{(k)(k)}, \ A_{(i)(j)} \equiv L_{[(i)(j)]} \tag{2.20}$$

If l is affinely parametrized, i.e. $L_{10} = 0$, the optical scalars take the form

$$\sigma^2 = l_{(\alpha;\beta)} l^{(\alpha;\beta)} - \frac{1}{d-2} (l^{\alpha}{}_{;\alpha})^2 \tag{2.21}$$

$$\theta = \frac{1}{d-2} l^{\alpha}{}_{;\alpha} \tag{2.22}$$

$$w^2 = l_{[\alpha;\beta]} l^{\alpha;\beta}. \tag{2.23}$$

2.3 Spinor approach

Spinors are more simpler objects than tensors (we denote them usually with capital index), intuitive comparison is that it is a square root from a tensor. Why were spinors so interesting for us in the approach for SGR ? Authors [30] and [31] showed in their book that every vector could be written in the language of spinors. We could translate every vectorial equation into this formalism. Some equations looks simpler in this formalism. But very interesting is their remark that spinor formalism is possible to apply in dimension four, although they had an appendix about spin spaces in all dimensions.

There are more consistent definitions. The basic spin-space for us is 2-dimensional complex vector space equipped with skew-symmetric bilinear form a and the objects from this space are 2-components spin-vectors, which are the simplest spinors. [1] We denote the symplectic form by ϵ_{AB}, which plays the role of the metric. We could raise and lower indices with this object and if we have two contravariant spinors ψ^A and ϕ^B, we could make an action of the bilinear form on these spinors and we get a complex number. The forms ϵ^{AB} and ϵ_{AB} provides a natural isomorphism between spin-space and its dual.

From a group theoretical methods are the spinors are elements of the representation of the group $SL(2, \mathbf{C})$. And for us are interesting the mix spinor-tensor objects which make the bridge between spinors and tensors. So we have the correspondence

$$\sigma^a_{AA'}\sigma^b_{BB'} \leftrightarrow \epsilon_{AB}\epsilon_{A'B'}. \tag{2.24}$$

But all this is written and is well known from the literature, for example [30], [31], [36], in various notations. But as authors from [30] wrote in their book the complications with formulating the physical laws were due to the tensorial approach. When we take the spinors as basic buliding blocks, the difficulties disappears. As we already wrote, it is possible to build spinors in all dimensions, however the dimension of the spin-space goes like $2^{\frac{n}{2}}$, so the efficiency of the spinor formalism is very low in higher dimensions. Later we will show how to make the spinor classification of the Weyl tensor in higher dimensions and it was shown in the work of [37] that it is not so useful with the standard classification (and not equivalent).

We will begin with classification of Maxwell tensor which will be a toy-model for us. Then we will classify the gravitational field. The discussion of the electromagnetic field will be made in such a way as to emphasize the analogy to classification of the Weyl tensor for the gravitational field case.

2.3.1 Complex three space

We will follow the notation of [23]. Let $F_{\mu\nu}$ be the Maxwell tensor and let $*F_{\mu\nu}$ be its dual. The $F_{\mu\nu}$ carries the information about the electromagnetic field. Let us also define the tensor $F^+_{\mu\nu}$ by

$$F^+_{\mu\nu} = F_{\mu\nu} + i * F_{\mu\nu}, \tag{2.25}$$

[1] It is an interesting observation that complex numbers is not only an artificial tool, but they are built in formulations of physical laws, as we could see from the examples of QFT, and they played an imporant role in physics. But we could look at the complex analysis as a real analysis, with two real variables. Why I am making this aside will be clearer from the end of this paragraph.

so that $*F^+_{\mu\nu} = -iF^+_{\mu\nu}$. In section 8.2 of [23] the spinor equivalent of the tensor $F_{\mu\nu}$ was found, whereas that of the tensor $F^+_{\mu\nu}$ is given by

$$F^+_{AB'CD'} = 2\phi_{AC}\ \epsilon_{B'D'}, \tag{2.26}$$

where ϕ_{AC} is the Maxwell spinor. Classification of the electromagnetic field can be made by classifying ϕ_{AB}. Therefore one studies eigenspinors and eigenvalues of the spinorial equation

$$\phi^A{}_B \alpha^B = \lambda \alpha^A. \tag{2.27}$$

To study this equation one introduces the basis in the spin space. Let the two spinors of the basis be denoted by l_A and n_A, satisfying the normalization condition $l_A n^A = 1$. This basis induces another basis, given by

$$\xi_{0AB} = n_A n_B, \quad \xi_{1AB} = -2l_{(A}n_{B)}, \quad \xi_{2AB} = l_A l_B, \tag{2.28}$$

in the three dimensional space, E_3 of bispinors. This means a bispinor ϕ_{AB} can be written in terms of the basis (2.28) as

$$\phi_{AB} = \sum_{m=0}^{2} \phi_m\ \xi_{mAB}, \tag{2.29}$$

where ϕ_0, ϕ_1 and ϕ_2 are called dyad components of the bispinor and corresponds to six real components of the tensor $F_{\mu\nu}$. The spin frame l_A and n_A induces other basis in E_3

$$\eta_{0AB} = \frac{1}{\sqrt{2}} i\ (l_A n_B + l_B n_A), \tag{2.30}$$

$$\eta_{1AB} = \frac{1}{\sqrt{2}} (l_A l_B + n_A n_B), \tag{2.31}$$

$$\eta_{2AB} = \frac{1}{\sqrt{2}} i\ (l_A l_B - n_A n_B). \tag{2.32}$$

This basis satisfies the orthogonality relation

$$\eta_{mAB}\eta_n^{AB} = \delta_{mn}. \tag{2.33}$$

In terms of this last basis ϕ_{AB} can now be written as

$$\phi_{AB} = \sum_{m=0}^{2} \chi_m \eta_{mAB}.$$

The two sets of three components χ and ϕ are then related by

$$\chi_0 = \sqrt{2} i\ \phi_1, \tag{2.34}$$

$$\chi_1 = \frac{1}{\sqrt{2}} (\phi_0 + \phi_2), \tag{2.35}$$

$$\chi_2 = \frac{1}{\sqrt{2}} i\ (\phi_0 - \phi_2). \tag{2.36}$$

This we do from the group theoretical reasons, because of symmetry.

2.3.2 Classification of the Maxwell spinor

In terms of the dyad components ϕ_m, the eigenvalue equation (2.27) becomes

$$\Phi\alpha = \lambda\alpha, \tag{2.37}$$

where Φ is a matrix of rank 2, given by

$$\begin{pmatrix} \phi_1 & \phi_2 \\ -\phi_0 & -\phi_1 \end{pmatrix}$$

and α is a column matrix given by $\begin{pmatrix} \alpha^0 \\ \alpha^1, \end{pmatrix}$,

where α^a are the dyad components of α^A, i.e., $\alpha^a = \zeta^a{}_A\alpha^A$, and we have denoted ${\zeta_0}^A = l^A$ and ${\zeta_1}^A = n^A$. The two eigenvalues of equation (2.27) are $\lambda = \pm\sqrt{({\phi_1}^2 - \phi_0\phi_2)}$. One, therefore, has two cases: $\sqrt{({\phi_1}^2 - \phi_0\phi_2)} \neq 0$, in which case there are two different eigenspinors; and $\sqrt{({\phi_1}^2 - \phi_0\phi_2)} = 0$ in which case there is only one eigenspinor.

2.3.3 Note on the classification of the Weyl spinor

In the previous section bivectors were discussed. We will use now this knowledge and we will apply them to the Weyl spinor.
The Weyl tensor $C_{\alpha\beta\gamma\delta}$ has the same symmetry properties as the Riemann tensor. In addition, it satisfies

$$C^{\rho}{}_{\alpha\rho\beta} = 0. \tag{2.38}$$

These identities reduce the number of its independent components to ten. We could find that the spinor equivalent of $C_{\alpha\beta\gamma\delta}$ is completely symmetric spinor of four indices, ψ_{ABCD},

$$-C_{AB'CD'EF'GH'} = \epsilon_{AC}\epsilon_{EG}\ \overline{\psi}_{B'D'F'H'} + \psi_{ACEG}\ \epsilon_{B'D'}\epsilon_{F'H'}. \tag{2.39}$$

The classification of the Weyl spinor is analogical to the classification of the classification of the Weyl tensor.

2.3.4 Complex 5-space

In order to classify the Weyl tensor we classify the Weyl spinor ψ_{ABCD} in terms of its eigenvalues and eigenspinors. The characteristic equation is now:

$$\psi_{ABCD}\phi^{CD} = \lambda\phi_{AB} \tag{2.40}$$

The basis l_A, n_A in spinorial space induces the basis

$$\zeta_{0ABCD} = n_A n_B n_C n_D, \tag{2.41}$$

$$\zeta_{1ABCD} = -4l_{(A}n_B n_C n_{D)}, \tag{2.42}$$

$$\zeta_{2ABCD} = 6l_{(A}l_B n_C n_{D)}, \tag{2.43}$$

$$\zeta_{3ABCD} = -4l_{(A}l_B l_C l_{D)}, \tag{2.44}$$

$$\zeta_{4ABCD} = l_A l_B l_C l_D. \tag{2.45}$$

2.3.5 Change of frame

The key ingredient of these formulas is the following: we have 5 complex vectors ζ which correspond to 5 complex Newman-Penrose scalars; These vectors create 5 dimensional space of completely symmetric 4-spinors. These scalars transform as vectors in this space. It is not possible to do this in dimension 5 [35]. But why we are saying all this, even though it was well described in the literature [23]? For example, when we try to generalize this concept of classification, (see e.g. [35]) to dimension 5, where it is not possible to achieve this, because there is not such a nice relation like inclusion of E_5 to $E_3 \times E_3$ in contrast with four dimensions. And now comes the key thing, that the case of four dimensions is exceptional for spinors, as was already mentioned in the work [30], and as we already discussed in the Introduction, [45]. Also in pure geometry the dimension four has nice properties. Why are four dimensions so special? Let me end this section with this question mark.

2.4 Kundt class

The motivation for studying the Kundt class is from the point of view of perturbation theory, for example, the following: Kundt class does not contain any black holes, however the studying of the so-called near horizon geometry played a role in the past and lead toward the studies of this type of space-time. The motivation came also from the theory of supergravity, which we mentioned in the introduction. We could find informations about algebraic types of Kundt solutions in [25]. The Kundt class was generalized to arbitrary number of dimensions in [3]. As in four dimensions it is characterized by having a shear-free, non-expanding, non-twisting geodesic null congruence $l = \partial_v$. Higher-dimensional Kundt class can be written in canonical form as follows:

$$\mathrm{d}s^2 = 2\mathrm{d}u[\mathrm{d}v + H(u,v,x^k)\mathrm{d}u + W_i(u,v,x^k)\mathrm{d}x^i] + g_{ij}(u,x^k)\mathrm{d}x^i\mathrm{d}x^j \tag{2.46}$$

The spatial coordinates are $(x^1, ..., x^{d-2})$; g_{ij} is the Riemanian metric. It follows from [8] that

$$C_{vijk} = \frac{1}{2(d-2)}[g_{ik}W_{j,vv} - g_{ij}W_{k,vv}]. \tag{2.47}$$

and

$$C_{vivj} = 0. \tag{2.48}$$

Therefore the frame component $C_{0(i)0(j)} = 0$ for the natural tetrad:

$$l_\mu = (1, 0, 0, ...),\ n_\mu = (H, 1, W_1, W_2, ..., W_{d-2}), \tag{2.49}$$

(where the general prescription for m_μ depends on the form of g_{ij}.) Therefore

Lemma 2.1. *The Weyl type of Kundt metrics is I or more special,*

as follows from algebraical criteria, when we use the equation (2.48). All the coordinate transformations preserving the form of metric ([3], [8]) are:

- $(v', u', x'^i) = (v, u, f^i(x^k))$ **and** $J^i{}_j \equiv \frac{\partial f^i}{\partial x^j}$

$$H' = H, \quad W'_i = W_j (J^{-1})^j{}_i, \quad g'_{ij} = g_{kl}(J^{-1})^k{}_i (J^{-1})^l{}_j \tag{2.50}$$

- $(v', u', x'^i) = (v + h(u, x^k), u, x^i)$

$$H' = H - h_{,u}, \quad W'_i = W_i - h_{,i}, \quad g'_{ij} = g_{ij} \tag{2.51}$$

- $(v', u', x'^i) = (v/k_{,u}(u), k(u), x^i)$

$$H' = \frac{1}{k_{,u}^2}(H + v\frac{k_{,uu}}{k_{,u}}), \quad W'_i = \frac{1}{k_{,u}} W_i, \quad g'_{ij} = g_{ij} \tag{2.52}$$

- $(v', u', x'^i) = (v, u, f^i(u; x^k))$ **and** $J^i{}_j \equiv \frac{\partial f^i}{\partial x^j}$

$$\begin{aligned} H' = H + g_{ij} f^i_{,u} f^j_{,u} - W_j (J^{-1})^j{}_i f^i_{,u}, \quad W'_i = W_j (J^{-1})^j{}_i - g_{ij} f^j_{,u}, \\ g'_{ij} = g_{kl}(J^{-1})^k{}_i (J^{-1})^l{}_j \end{aligned} \tag{2.53}$$

The higher - dimensional Kundt class contains a number of interesting subclasses, which we describe in what follows.
pp-wave ST:
Higher-dimensional pp-wave spacetimes are defined as in four dimensions, as spacetimes which admit a covariantly constant null vector. The most general d-dimensional pp-wave ST is given by the Brinkmann metric (we could find reference in the section 8.3.3. of paper ([42] and [43]) :

$$ds^2 = 2du[dv + H(u, x^k)du + W^i(u, x^k)dx^i] + g_{ij}(u, x^k)dx^i dx^j. \tag{2.54}$$

From (2.54) it is clear that $k = k^\mu \partial_\mu = \partial_v$ **is a Killing vector.**

From the Bel-Debever criteria that the Weyl type is N **in four dimensions.**
Now we would like to solve the question, if there exist pp-waves of Weyl type I in higher dimensions. I assert that no: according to [8], $\omega = R_{(0)(0)} = R_{vv} = 0$ **for natural frame vector** $l^\mu = (0, 1, 0, ..., 0)$ **and** $\psi_i = R_{(0)(i)} = 0$ **for natural frame vectors** l^μ **and** $m^\mu_{(i)}$ **; Now we use Proposition 2 from [7]. It follows that pp-waves couldn't be of Weyl type I. In fact, pp-waves are of Weyl type II, III or N in higher dimensions.**

Higher-dimensional Lorentzian ST's with vanishing scalar curvature invariants of all orders are so called VSI ST's. It was found that all

such ST's belong to the Kundt class and in fact, can be written in the canonical form, [8]

$$ds^2 = 2(e^i(x^k,u) + f^i(x^k,u)v)dx^i du + 2dudv + (av^2 + bv + c)du^2 + \delta_{ij}dx^i dx^j.$$

Such ST's have a Weyl tensor of algebraic type III, or more special. Two subclasses can be distinguished, namely the case $f^i = 0$ for all $i = 1, 2, ..., D-2$ and the case $f_1 \neq 0$ with $f_i = 0, i = 2, 3, ..., D-2$. A generalization of the VSI ST's belonging to the Kundt class, such that all polynomial scalar invariants constructed from the Riemann tensor and its derivatives are constant, are called CSI ST's.

In fact both alternative assumptions 1) and 2) in the lemma below uniquely identify the Kundt class of non-expanding, twist-free and shear-free ST, i.e. $L_{(i)(j)} = 0$ ($L_{(i)0} = 0$), [7]

Lemma 2.2. *Given a geodetic null congruence with tangent vector l in an arbitrary d-dimensional ST (d $\geq$ 4), the following implications hold:*
1)$R_{00} = 0,\ \theta = 0 = \sigma_{(i)(j)} \Longrightarrow A_{(i)(j)} = 0,\ C_{0(i)0(j)} = 0$
2)$R_{00} = 0,\ \theta = 0 = A_{(i)(j)} \Longrightarrow \sigma_{(i)(j)} = 0,\ C_{0(i)0(j)} = 0$

In view of 2), we can conclude that one can not generalize the Kundt solutions by allowing for non-zero shear (as long as $R_{00} = 0$ and one insists on the non-expanding and twistfree conditions). Note that the assumed condition $R_{00} = 0$ from previous lemma 2.2 on the matter content is satisfied in a large class of STs such as vacuum with a possible cosmological constant, aligned pure radiation and aligned Maxwell fields.

Further, in both above cases 1) and 2), the fact that the tangent vector is necessary a WAND (because of $C_{0(i)0(j)} = 0$) implies for $d > 4$ that the considered ST is algebraically special, i.e. it can not be of type G. In addition, if we now substitute $L_{(i)0} = 0 = L_{(i)(j)}$ in one of the Ricci identities (11k) in [7] and we further assume $R_{0(i)} = 0$, we obtain $C_{0(i)(j)(k)} = 0$. Recalling the identity $C_{0101} = C_{0(j)(i)(j)}$, we find also $C_{010(i)} = 0$, so that with previous lemma we conclude [7]:

Lemma 2.3. *Under the assumption $R_{00} = 0 = R_{0(i)}$ on the matter fields, d $\geq$ 4 Kundt STs ($L_{(i)0} = 0 = L_{(i)(j)}$) are of type II (or more special).*

2.4.1 Recurrent spacetimes

We define a special class of Kundt spacetimes , which we will call recurrent (RNV): there must exist a null vector l_μ in a neighborhood of every point such that

$$l_{\mu;\nu} = \alpha\ l_\mu l_\nu, \tag{2.55}$$

where α is a scalar function;
As we could see from standard literature, there exist coordinates u,v,x^i (where $i = 1, ..., d-2$), such that the metric has the form

$$\mathrm{d}s^2 = 2\mathrm{d}u[\mathrm{d}v + H(u,v,x^k)\mathrm{d}u + W_i(u,x^k)\mathrm{d}x^i] + g_{ij}(u,x^k)\mathrm{d}x^i\mathrm{d}x^j \tag{2.56}$$

where $g = g_{ij}(x^k, u)\mathrm{d}x^i\mathrm{d}x^j$ is a u-dependent family of Riemanian metrics. The vector field $\partial_v = \frac{\partial}{\partial v}$ is null and recurrent, [5].

We would like to stress that W_i are not functions of v. (If W_i was also a function of v, then the metric (2.56) would be a general Kundt metric.)

We would like to understand more geometrically what does it mean that a spacetime is RNV. We can contract the equation (2.55) with vector l, n and m. After doing this we get:

$$Dl^\mu = 0 \tag{2.57}$$
$$\Delta l^\mu = \alpha l^\mu \tag{2.58}$$
$$\delta l^\mu = 0, \tag{2.59}$$

where α is a function.

From the second equation we see the geometrical picture. Now we transport the vector l in direction n and the change of the vector l is again in the direction l.

Further, from the well known propagation equations

$$Dl^\mu = (\epsilon + \overline{\epsilon})l^\mu - \overline{\kappa}m^\mu - \kappa\overline{m}^\mu \tag{2.60}$$
$$\Delta l^\mu = (\gamma + \overline{\gamma})l^\mu - \overline{\tau}m^\mu - \tau\overline{m}^\mu \tag{2.61}$$
$$\delta l^\mu = (\overline{\alpha} + \beta)l^\mu - \overline{\rho}m^\mu - \sigma\overline{m}^\mu \tag{2.62}$$

we get for the natural l vector the conditions for the spin coefficients:

$$\kappa = 0, \quad \sigma = 0, \quad \tau = 0, \quad \rho = 0. \tag{2.63}$$

As was already written (and this is clear from definition), recurrent spacetimes are a subclass of Kundt spacetimes. But we have a natural division of the Kundt class to subclasses according to the form of the function W_i for vacuum, with possibly cosmological constant Λ or those that include a Maxwell field aligned with the geometrically privileged null vector l such that

$$F_{\alpha\beta}l^\beta = Ql_\alpha,$$

where Q is a function, [8].

- We will consider the following metric in dimension 5:

$$g = 2\mathrm{d}u(\mathrm{d}v + H(u, v, x^k)\mathrm{d}u + W_i(u, x^k)\mathrm{d}x^i) + g_{ij}(u, x^k)\mathrm{d}x^i\mathrm{d}x^j,$$

where g_{ij} is again Riemannian metric. Then

$$l_\mu = (1, 0, 0, 0, 0), \ \ n_\mu = (H, 1, W_1, W_2, W_3)$$

is a choice of l and n. Non-zero Christoffel symbols of second kind with zero up are:

$$\Gamma^u_{uu} = -\frac{\partial H(u, v, x^k)}{\partial v}; \quad \Gamma^u_{u2} = -\frac{\partial W_1(u, x^k)}{\partial v}; ...; \quad \Gamma^u_{u4} = -\frac{\partial W_3(u, x^k)}{\partial v} \tag{2.64}$$

We have the following lemma:

Lemma 2.4. *For recurrent spacetimes ($W_i = W_i(u, x^k)$) the vector* $\frac{\partial}{\partial v} = v^\mu = (0, 1, 0, ..., 0)$ *is the recurrent vector and* $v_{0;0} = \alpha = \frac{\partial H}{\partial v}$.

pp-waves (ST's admitting covariantly constant null vector, $l_{\mu;\nu} = 0$) are a subclass of recurrent spacetimes ($\alpha = 0$). Einstein pp-waves spacetimes do not admit a non-zero cosmological constant, but recurrent STs do, as was observed in [5].

It follows from [8] that $C_{0(i)(j)(k)} = 0$. Using Bel-Debever criteria in [6] we arirved to:

Lemma 2.5. *The recurrent STs are of algebraical type II or more special. (Ricci flat recurrent STs are of algebraic type IIa or more special.)*

Note that in contrast with the result of [7], we do not make additional assumptions about Ricci tensor.

This lemma 2.5 is in correspondence with the results of T. Málek. ([44]). We would like to find the conditions on metric for Weyl types II, III, N and 0. We will solve this problem for recurrent Einstein spacetimes or recurrent spacetimes with null radiation again with the help of Bel-Debever criteria. The components of Weyl tensor in coordinate basis could be found in [8]. After a short computation we get the following formula

$$\frac{R}{d-1} g_{ij} = R^{(S)}_{ij} \tag{2.65}$$

It immediately follows that, $g_{ij} = 0$ for Einstein and null radiation spacetimes if we want a recurrent ST of type III or N for non-zero cosmological constant. Therefore Einstein recurrent spacetimes of Weyl type III and N with non-zero cosmological constant do not exist. We have type N recurrent ST's, if following conditions (and (2.65)) are satisfied for the metric (and type III if (2.65) and (2.66) are satisfied) :

$$R^S_{ijkl} - \frac{1}{D-2}(g_{ik}R_{lj} - g_{il}R_{kj} - g_{jk}R_{li} + g_{jl}R_{ki}) + \frac{R}{(D-1)(D-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) = 0 \tag{2.66}$$

$$R_{uijk} - \frac{1}{D-2}(-g_{ij}R_{uk} + g_{ik}R_{ju} - g_{uk}R_{ji} + g_{uj}R_{ki}) + \frac{R}{(D-1)(D-2)}(g_{uj}g_{ik} - g_{uk}g_{ij}) = 0, \tag{2.67}$$

where

$$R_{ui} = \frac{1}{2}[g^{jk}(g_{ij,u} + g_{uj,i} - g_{ui,j})]_{,k} + \frac{1}{2}[g^{jk}(g_{ij,u} + g_{uj,i} - g_{ui,j})](\ln\sqrt{g})_{,k} - \frac{1}{2}H_{,ri} + \frac{1}{2}g^{jk}g^{lm}g_{im,k}[g_{ul,j} - g_{uj,l}] - (\ln\sqrt{g})_{,ui} - \frac{1}{4}g^{jk}g^{lm}g_{km,i}g_{jl,u} \tag{2.68}$$

Conditions for type 0 metric:

$$0 = -\frac{1}{2}H_{,rr} + (\frac{H_{,rr}}{d-2}) - \frac{R}{(d-1)(d-2)} \quad (2.69)$$

$$0 = \frac{R_{ui}}{d-2} + \frac{W_i H_{,rr}}{2(d-2)} - \frac{W_i R}{(d-1)(d-2)} \quad (2.70)$$

$$0 = R_{uiuj} - \frac{1}{d-2}(HR_{ij} - W_j R_{ui} - W_i R_{ju} + g_{ij}R_{uu}) \quad (2.71)$$

$$+ \frac{R}{(d-1)(d-2)}(Hg_{ij} - W_i W_j), \quad (2.72)$$

where

$$R_{uiuj} = \frac{1}{2}(g_{ui,uj} + g_{uj,ui} - H_{,ij} - g_{ij,uu}) - \Gamma^v_{ij}\Gamma^u_{uu} + g_{kl}(\Gamma^k_{ui}\Gamma^l_{uj} - \Gamma^k_{ij}\Gamma^l_{uu}) - g_{uk}\Gamma^u_{uu}\Gamma^k_{ij},$$

$$\Gamma^v_{ij} = \frac{1}{2}(g_{ij,u} - W_{i,j} - W_{j,i}) + \frac{1}{2}g^{vk}(g_{ik,j} + g_{jk,i} - g_{ij,k}),$$

$$\Gamma^l_{uu} = g^{lj}g_{uj,u} - \frac{1}{2}g^{lj}H_{,j} - \frac{1}{2}g^{lv}g_{uu,v},$$

$$\Gamma^k_{ui} = \frac{1}{2}g^{kl}(g_{il,u} + g_{ul,i} - g_{ui,l}),$$

$$\Gamma^u_{uu} = \frac{1}{2}H_{,r}. \quad (2.73)$$

Examples of recurrent ST's of Ricci type N of Weyl type III, N and 0 are the same as for VSI ST's and examples of recurrent ST's of type II could be found in [5].

Will be the generalized warped-product of a recurrent Einstein ST again a recurrent Einstein ST? We will use metric (2.55) as a seed-metric:

$$\mathrm{d}s^2 = \frac{1}{f}\mathrm{d}\tilde{z}^2 + f[2\mathrm{d}u(\mathrm{d}v + H(u,v,x^k)\mathrm{d}u + W_i(u,x^k)\mathrm{d}x^i) + g_{ij}(u,x^k)\mathrm{d}x^i\mathrm{d}x^j], \quad (2.74)$$

where $f = -\lambda\tilde{z}^2 + 2d\tilde{z} + b$ **and** $i,j = 1,...,d-2$**. We would like to rewrite this metric in the Kundt-form (2.46). We will use the following transformation**

$$v = \frac{\tilde{v}}{-\lambda\tilde{z}^2 + 2d\tilde{z} + b}. \quad (2.75)$$

We will also denote $x^{d-1} = \tilde{z}$ **and** $\tilde{W}_{d-1} = -\frac{\tilde{v}(-2\lambda\tilde{z}+2d)}{-\lambda\tilde{z}^2+2d\tilde{z}+b}$**.** $\tilde{W}_s = fW_s$ **for** $s = 1,...,d-2$**.** $\tilde{H} = fH$**,** $\tilde{g}_{ij} = fg_{ij}$**,** $\tilde{g}_{d-1d-1} = \frac{1}{f}$**. And indices** $\tilde{i},\tilde{j},\tilde{k} = 1,...,d-1$**:**

$$\mathrm{d}s^2 = 2\mathrm{d}u[\mathrm{d}\tilde{v} + \tilde{H}(u,\tilde{v},x^{\tilde{k}})\mathrm{d}u + \tilde{W}_{\tilde{i}}(u,\tilde{v},x^{\tilde{k}})\mathrm{d}x^{\tilde{i}}] + \tilde{g}_{\tilde{i}\tilde{j}}(u,x^{\tilde{k}})\mathrm{d}x^{\tilde{i}}\mathrm{d}x^{\tilde{j}} \quad (2.76)$$

So, we see that we also get a dependence on $\tilde{v}$ **for** $\tilde{W}_{\tilde{i}}$**!**

Lemma 2.6. *Direct product of Ricci flat Einstein ST is again a recurrent Ricci flat Einstein ST, but Brinkmann warped product of a recurrent Einstein ST is not a recurrent Einstein ST.*

We used software Maple for some computations.

2.4.2 Spacetimes with null Killing vector fields

First of all I would like to mention the concept of Killing vectors (Wilhelm Killing (1847-1923)). Existence of Killing vectors in ST's are connected with symmetries. A vector l_μ is a Killing vector, if it satisfies an equation

$$l_{\mu;\nu} + l_{\nu;\mu} = 0. \tag{2.77}$$

We say that a spacetime is called a null Killing vector field ST (NKVF) if it admits a null Killing vector field. Some basic information we could find in [46].
We have a Theorem 8.21 in [21]. A null Killing vector field k is necessary geodesic, shearfree and non-expanding, with twist given by $w^2 = R_{ab}k^a k^b$. If k is twistfree then it must be a WAND (and vice versa if n is odd), so that the ST is of Weyl type I and belongs to the Kundt class. If, additionally, $R_{ab}k^a \; k_b$ then the Weyl type is II.

2.5 Conclusion

In this first Chapter 1 we studied SGR in higher dimensions. We reviewed the algebraic classification of Weyl tensor in higher dimensions, we mentioned the classification in spinors and then we studied the so called Kundt class, which admits twistfree, nonexpanding and shearfree null direction. This class played a role in the perturbation theory in studying the so called near-horizon geometries. We provided an example for the case of recurrent spacetimes, which contains the pp-waves as a special case. This exact solution has a connection in the search for the gravitational waves, which is as we mentioned in the Introduction missing experimental consequence of SGR. People expect to find the gravitational waves with future generation of detectors.

The algebraic classification in dimension four - so called Petrov classification - was important from other physical point of view. For example, type D spacetimes contains black holes. [2]

And finally this interesting discipline served us as motivation for other works. Here we present literature of our team here in the Mathematical Institute ([2], [4], [7], [21], [33]) and other sources. Gravitational instabilty in higher dimensions was studied, for example, [15], [18],[19] and [20].

Other sources for higher dimensions are, for example, [13], [16].

Some information about recurrent spacetimes we could find in [5], [32] and [8] .

General books about SGR are [23], [38] and [25],.

[2]In SGR holds also the so called peeling theorem that far from the source are the waves of type N, so we see the advantage of this formalism.

Bibliography

[1] Wesson, P.S. Five-dimensional physics: classical and quantum consequences of Kaluza-Klein cosmology, Hackensack, N.J.: World Scientific c2006, 222

[2] Coley A., Milson R., Pravda V., Pravdová A.: Classification of Weyl tensor in higher dimensions, Class.Quantum Grav. 21, 2004, arXiv: gr-qc/0401008

[3] Fuster-Perez A.: Kundt spacetimes in general relativity and supergravity, PhD-thesis , 2007, Amsterdam

[4] Durkee M., Pravda V., Pravdová A., Reall H.S.: Generalization of Geroch-Held-Penrose formalism to higher dimensions, Class.Quantum Grav.27, 2010, arXiv:1002.4826v1

[5] Galaev A.S.: Examples of Einstein spacetimes with recurrent null vector fields Class.Quantum Grav. 26, 2009, arXiv: 0906.3818

[6] Ortaggio M.,Bel-Debever criteria for the classification of the Weyl tensors in higher dimensions, Class.quantum Grav. 26:195015, 2009, vol.26, no 195015, 2009, arXiv: 0906.3818

[7] Ortaggio M., Pravda V., Pravdová A.: Ricci identities in higher dimensions, Class.Quantum.Grav. 24, 2007, arXiv: gr-qc/0701150

[8] Podolský J., Zofka M.: General Kundt spacetimes in higher dimensions, gr-qc 105008, 2009, Class.Quantum Grav. 26, no 10, 1005008,2009 ,arXiv: 0812.4928

[9] Walker A.G.: On parallel fields of partially null vector spaces, Quarterly J.Math. 20, no 1, 135-145, 1949

[10] Kleihaus B., Kunz J., Radu E.: Black rings in more than five dimensions, 2012 , arXiv: 1205.5437

[11] Andrade, Garcia de, Ricci recurrent spacetimes with torsion, International journal of theoretical physics, vol 30, no.2., p.235-238,1991

[12] Hall, G.S., Reccurence conditions in spacetime, J.Phys.A: Math.Gen.,Vol.10, No.1,1977

[13] Emparan, R., Reall, H.S. A rotating black ring in five dimensions, Phys.Rev.Lett.88, 101101, 2002, arXiv: hep-th/0110260v2

[14] Hawking S.W., Ellis G.F.R., The large scale structure of space-time, Cambridge Univerisity Press, Cambridge, 1973

[15] Myers R.C., Emparan R., Instability of ultra-spinning black holes, JHEP 0309, 025, 2003, arXiv:1006.1904v1

[16] Emparan R. and Reall H.S., Black rings, Class. Quant. Grav. 23, 2006, R169, arXiv:hep-th/0608012.

[17] Gibbons G., Hartnoll, H.S., A gravitational instability in higher dimensions, Phys.Rev.D66, 2002, 064024, arXiv: hep-th/0206202

[18] Kodama H., Ishibashi A., A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions., Prog.Theor.Phys.110, 701 2003, hep-th/0305147

[19] Kodama H., Ishibashi A., Stability of higher-dimensional Schwarzschild black holes, Prog.Theor.Phys.110, 901, 2003

[20] Gregory R., Laflamme R., The instability of charged black strings and p-branes, Nuclear.Phys.B 428, 399, 1994,

[21] Ortaggio M., Pravda V., Pravdová A., Algebraic classification of higher dimensional spacetimes based on null alignment, Class.Quantum Grav.30: 013001, 2013

[22] Jakimowicz M., Tafel J., Generalization of the Gross-Perry metrics, Int J Theor Phys, 48, 2876-2833, 2009

[23] Carmeli M., Group theory and general relativity, McGraw-Hill Inc., 1977, 391, New York

[24] Felice A. de, Tsujikawa S., f(R) theories, Living Rev.Relativity 13 (2010), ISBN 9780521889278

[25] Griffiths J.B., Podolský J., Exact space-times, Cambridge University Press (2009), 578

[26] Pirani, F.A.E., Invariant Formulation of Gravitational Radiation Theory, Physical Review, 105, 3, 1957

[27] Lorente M., Kramer P., Tensor and spin representation of $SO(4)$ and discrete quantum gravity, Symmetries in Science, Springer, 2005, 377-394, ISBN 971-1-4020-2633, arXiv: gr-qc/04020504v4

[28] Ghanam R., Thompson G., Oktem F., Two special metrics with R_{14}-type holonomy, Classical and Quantum Gravity 18 (2001)

[29] Oktem F., On Parallel Null 1-Planes in Space-Time, Il nuovo cimento, Vol.34, B 11, N.1, 1976

[30] Penrose R., Rindler W., Spinors and spacetime, Two spinors calculus and relativistic fields Vol.1, Cambridge, Cambridge University Press, 1987

[31] Penrose R., Rindler W., Spinors and spacetime, Spinor and twistor methods in space-time geometry, Cambridge, Cambridge University Press, Vol.2, 1988, 5030, ISBN- 0521-34786-6

[32] Garcia de Andrade L.C., Ricci recurrent space-times with torsion, International Journal of Theoretical Physics, Vol.30, No.2 (1991)

[33] Orttagio M., Podolsky J., Explicit Kundt type II and N solutions as gravitational waves in various type D and O universes, Class. Quantum Grav. 20, 2003

[34] Wald R.M., Note on the stability of Schwarzschild metric, Journal of mathematical physics, 20, 1056, 1979

[35] Garcia-Martin, J.M., A.Garcia-Lobo , Spinor calculus on 5-dimensional spacetimes, gr-qc, arXiv 0905.2846v2, 2009, arXiv 0905.2846v2

[36] Ward R.S., Twistor geometry and field theory, Cambridge Monograph on Mathematical Physics, Cambridge University Press, 520, 1999, ISBN 97-80521268905

[37] Godazgar M., Spinor classification of the Weyl tensor in dimension 5, 0905.2846v2, 2010, arXiv: 10082955v1

[38] Chandrasekhar S., Mathematical theory of black holes, Oxford University Press, 668, 1983, ISBN:0198512910

[39] Reall H., Emparan R., Black holes in higher dimensions, 2008, arXiv:0801.3471v2

[40] Kubiznak D., Hidden symmetries of higher dimensional rotating black holes, PhD-thesis, Alberta

[41] Durkee M., Reall, H.S., Perturbations of higher dimensional spacetimes, arXiv:1009.0015v2, 2010

[42] Pravda V., Pravdova A., Ortaggio M., Algebraic classification of spacetimes based on null alignment, Class.Quantum Gravity 30, 013001, 2013

[43] Brinkmann H.W., Einstein spaces which are mapped conformally on each other. Math. Ann., 94: 119-145, 1925

[44] Málek T., Exact solutions of General Relativity and Quadratic Gravity in Arbitrary Dimension, 2012 ,arXiv:1204.0291

[45] Eingorn, M. and Zhuk, A., Kaluza-Klein models: can we construct a viable example?, Physical Review D 83, 044005, 2011, arXiv:gr-qc/1010.5740

[46] Lukacs, B., Perjes, Z., Sebestyen, A., Null Killing Vector, J.Math.Physics, 22, 1248, 1981

3. Perturbation of FLRW spacetimes in GHP formalism

3.1 Introduction

Our goal is to reformulate the theory of Cosmological Perturbation Theory in the compact language of GHP formalism. This is a convenient formalism, because equations are of the first order. We would like to show the correspondence with metric perturbations in NP and if possible also in GHP. For us, the most interesting are scalar and tensor perturbations. We will show how to reformulate the central equation of [2] for the case of tensor perturbations in GHP. We will perform calculations similar to those already done in previous literature for the case of tensor perturbations. Later we will apply this to the case with sources. We wish to apply this machinery to the phase transition, which means first on the Cosmological Inflation, at the beginning of evolution of our universe.

3.2 GHP-formalism

In NP notation the components of traceless Ricci read:

$$\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{11}, & \Phi_{01} &= -\frac{1}{2}R_{13}, & \Phi_{10} &= -\frac{1}{2}R_{14},\\
\Phi_{11} &= -\frac{1}{4}(R_{12}+R_{34}), & \Phi_{21} &= -\frac{1}{2}R_{24}, & \Phi_{22} &= -\frac{1}{2}R_{22},\\
\Phi_{12} &= -\frac{1}{2}R_{23}, & \Phi_{02} &= -\frac{1}{2}R_{33}, & \Phi_{20} &= -\frac{1}{2}R_{44},\\
\Lambda &= \frac{1}{12}(R_{12}-R_{34}). & & & &
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\Phi'_{00} = \Phi_{22}, && \Phi'_{11} = \Phi_{11}, && \Phi'_{10} = \Phi_{12}, && \Phi'_{02} = \Phi_{20}, && \Lambda' = \Lambda,\\
&\Phi^*_{00} = \Phi_{02}, && \Phi^*_{01} = -\Phi_{01}, && \Phi^*_{10} = \Phi_{12}, && \Phi^*_{11} = -\Phi_{11}, && \Phi^*_{21} = -\Phi_{21},\\
&\Phi^*_{22} = \Phi_{20}, && \Phi^*_{12} = \Phi_{10}, && \Phi^*_{02} = \Phi_{00}, && \Phi^*_{20} = \Phi_{22}, && \Lambda^* = -\Lambda.
\end{aligned} \tag{3.2}$$

Let's make the following change of tetrad:

$$l^\mu \to a l^\mu, \tag{3.3}$$

$$l^\nu \to a^{-1} n^\nu, \tag{3.4}$$

$$m^\rho \mapsto e^{i\theta} m^\rho. \tag{3.5}$$

The quantity which transforms under these changes of tetrad like

$$\eta \to a^{(p+q)/2}\overline{a}^{(p-q)/2}\eta \tag{3.6}$$

is said to be a GHP-quantity of type (p, q). Now we make the standard definitions, [1],

$$\begin{aligned}
&\Phi_{00} : [2,2], \quad \Phi_{01} : [2,0], \quad \Phi_{10} : [0,2], \quad \Phi_{11} : [0,0], \quad \Phi_{20} : [-2,2],\\
&\Phi_{22} : [-2,-2], \quad \Phi_{12} : [0,-2], \quad \Phi_{21} : [-2,0], \quad \Phi_{02} : [2,-2], \quad \Lambda : [0,0].
\end{aligned} \tag{3.7}$$

3.3 Computations

Reference, where the following fact can be found, [2] will be of great importance for us : the only non-vanishing spin coefficients for the case of FLRW are α, β, γ, μ and ρ. These are the same non-zero spin coefficients as for the case of the Schwarzschild solution. This fact can be employed in the analysis of unperturbed equations. This means that we can get rid of many terms in resulting equations. We get rid of α, β, γ and ϵ because they are absorbed into þ and ð (þ$'$ and ð$'$). We have together 12 spin coefficients, which means that there remain yet 8 more: τ, σ, κ, μ, ρ, λ, π and ν .

For the case of FLRW ST we have the following 2 equations in GHP formalism. In standard NP we have 8 equations, but this formalism is even more efficient. But contrary to the Schwarzschild ST, we have sources on the right hand side of the equations. The equations read

$$\begin{aligned}
þ\Psi_1 - ð'\Psi_0 + \tau'\Psi_0 - 4\rho\Psi_1 + 3\kappa\Psi_2 = þ\Phi_{01} - ð\Phi_{00} - \overline{\pi}\Phi_{00} - 2\overline{\rho}\Phi_{01} + \\
\overline{\kappa}\Phi_{02} + 2\kappa\Phi_{11} - 2\sigma\Phi_{10},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
þ\Psi_2 - ð'\Psi_1 - \sigma'\Psi_0 + 2\tau'\Psi_1 - 3\rho\Psi_2 + 2\kappa\Psi_3 = -ð'\Phi_{01} + þ'\Phi_{00} - \overline{\rho}'\Phi_{00} + \\
2\overline{\tau}\Phi_{01} - 2\rho\Phi_{11} - \overline{\sigma}\Phi_{00}^{*} + 2\tau\Phi_{10} + 2þ\Lambda,
\end{aligned} \tag{3.9}$$

where the NP components of the Weyl tensor are defined in the standard way by formulas

$$\begin{aligned}
\Psi_0 &= l^\mu m^\nu l^\rho m^\sigma C_{\mu\nu\rho\sigma},\\
\Psi_1 &= l^\mu n^\nu l^\rho m^\sigma C_{\mu\nu\rho\sigma},\\
\Psi_2 &= l^\mu m^\nu \overline{m}^\rho n^\sigma C_{\mu\nu\rho\sigma},\\
\Psi_3 &= n^\mu l^\nu n^\rho \overline{m}^\sigma C_{\mu\nu\rho\sigma},\\
\Psi_4 &= n^\mu \overline{m}^\nu n^\rho \overline{m}^\sigma C_{\mu\nu\rho\sigma}.
\end{aligned}$$

According to the [2] - and this should be clear because of their boost weights - the Ψ_0 and Ψ_4 are connected with the tensor perturbations, Ψ_1 and Ψ_3 are connected with the vector perturbations and ψ_2 is connected with the scalar perturbations.

(In the case of non-zero sources we have also other two equations:

$$\begin{aligned}
&-[\text{þ}' - 2\overline{\tau}^* + \pi^*]\Phi_{01} + [-\text{þ} - 2\tau^* + \overline{\pi}^*]\Phi_{12} + \\
&\quad [\eth - 2(\rho^* + \overline{\rho}^*]\Phi_{11} - [-\eth' + \mu^* + \overline{\mu}^*]\Phi_{02} + \\
&\quad \overline{\sigma}^*\Phi^*_{02} + \sigma^*\Phi^*_{20} - \overline{\kappa}^*\Phi^*_{12} - \kappa^*\Phi^*_{21} + 3\eth\Lambda = 0,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&[\eth - 2\tau + 2\pi^*]\Phi_{11} - 3\eth\Lambda + [-\text{þ} + 2\rho + \overline{\rho}]\Phi_{12} + \\
&\quad [-\text{þ}' - 2\overline{\mu} - \mu]\Phi_{01} + [\eth' - \tau^* + \pi]\Phi_{02} \\
&\quad - \kappa\Phi_{22} + \overline{\nu}\Phi_{00} + \sigma\Phi_{21} - \overline{\lambda}\Phi_{10} = 0.)
\end{aligned} \tag{3.11}$$

Now we will follow the approach of [5]. The difference, as we already mentioned, is that we have sources on the RHS. However, we could make the same steps: we will take the first equation, we make the star duality and we add these two equations. Then we plug from the Ricci itentities, we eliminate some of these combinations of spin coefficients (we make also prime and star dualities of these Ricci identities) and we arrive at the following result (the second equation could be obtained in similar way, this equation contain information from both (3.8) and (3.9) [1], it should be obtained by duality) :

$$\begin{aligned}
&[\text{þ}'\text{þ} - \eth'\eth - (4\rho' + \overline{\rho}')\,\text{þ} - \rho\text{þ}' + (4\tau' + \overline{\tau})\,\eth + \tau\eth' + 4\rho\rho' - 4\tau\tau' - 2\Psi_2 + 2\Lambda]\,\Psi_4 + \\
&\quad + [4\text{þ}\kappa' - 4\eth\sigma' - 4\,(\overline{\rho} - 2\rho)\,\kappa' + 4\,(\overline{\tau} - 2\tau)\,\sigma' + 10\Psi_3]\,\Psi_3 \\
&\quad + [-4\sigma'\text{þ}' + 4\kappa'\eth' - 12\kappa'\tau' + 12\rho'\sigma' - 3\Psi_0]\,\Psi_2 = 0.
\end{aligned} \tag{3.12}$$

It is interesting that for this case of FLRW spacetimes, we have cancellations of all extra terms in front of Ψ_2 and Ψ_3. So these terms in the brackets are exactly the same as for the case of the Schwarzschild spacetime. This means that when we will make perturbations of these equations, the second and third term disappear. And this is a reformulation according to [2], when we are interested in equations without sources, i.e. when we put just delta-function on the RHS. But in later work we will be interested in the same problem but with sources, as was already suggested in ([2]). The advantage of our approach is that all source terms could be written in one compact form. When we write also the sources on the RHS of equation (3.8) and (3.9) we get the following expressions on the RHS of these two equations (however we are interested only in the first equation):

[1] In [5] is the second equation little different, however we will deal only with this first equation

$$\eth\text{þ}\Phi_{01}-\eth\eth\Phi_{00}-\eth(\overline{\pi}\Phi_{00})-2\eth(\overline{\rho}\Phi_{01})+\eth(\overline{\kappa}\Phi^*_{00})+2\eth(\kappa\Phi_{11})-2\eth(\sigma\Phi_{10})$$
$$\text{þ}\eth\Phi_{01}-\text{þ}\text{þ}\Phi_{02}+\text{þ}(\overline{\mu}\Phi_{02})-2\text{þ}(\overline{\tau}\Phi_{01})-\text{þ}(\overline{\sigma}\Phi_{00})+2\text{þ}(\sigma\Phi_{11})-2\text{þ}(\kappa\Phi_{12})$$
$$-\overline{\tau}'(\text{þ}\Phi_{01}-\eth\Phi_{00}-\overline{\pi}\Phi_{00}-2\overline{\rho}\Phi_{01}+\overline{\kappa}\Phi_{02}+2\kappa\Phi_{11}-2\sigma\Phi_{10})+$$
$$\kappa(-\eth'\Phi_{02}+\text{þ}'\Phi_{01}+\overline{\kappa}\Phi_{00}+2\overline{\rho}\Phi_{01}+\overline{\tau}'\Phi_{02}+2\tau\Phi_{11}-2\rho\Phi_{12}-2\eth\Lambda)+$$
$$\overline{\rho}(-\eth\Phi_{01}+\text{þ}\Phi_{02}-\overline{\mu}\Phi_{02}+2\overline{\tau}\Phi_{01}+\overline{\sigma}\Phi_{00}-2\sigma\Phi_{11}+2\kappa\Phi_{12})$$
$$-\sigma(\text{þ}'\Phi_{00}-\eth'\Phi_{01}-\overline{\sigma}\Phi_{02}+2\overline{\tau}\Phi_{01}-\overline{\rho}'\Phi_{00}+2\tau\Phi_{10}+2\text{þ}\Lambda-2\rho\Phi_{11})$$
$$-4\tau(\text{þ}\Phi_{01}-\eth\Phi_{00}+\overline{\pi}\Phi_{00}-2\overline{\rho}\Phi_{01}+\overline{\kappa}\Phi_{02}+2\kappa\Phi_{11}-2\sigma\Phi_{10})+$$
$$4\rho(-\eth\Phi_{01}+\text{þ}\Phi_{02}-\overline{\mu}\Phi_{02}+2\overline{\tau}\Phi_{01}+\overline{\sigma}\Phi_{00}-2\sigma\Phi_{11}+2\kappa\Phi_{12}). \tag{3.13}$$

We can collect terms now, so we will get the following equation:

$$[\text{þ}'\text{þ}-\eth'\eth-(4\rho'+\overline{\rho}')\,\text{þ}-\rho\text{þ}'+(4\tau'+\overline{\tau})\,\eth+\tau\eth'+4\rho\rho'-4\tau\tau'-2\psi_2+2\Lambda]\Psi_4+$$
$$[4\text{þ}\kappa'-4\eth\sigma'-4\,(\overline{\rho}-2\rho)\,\kappa'+4\,(\overline{\tau}-2\tau)\,\sigma'+10\psi_3]\,\Psi_3++[-4\sigma'\text{þ}'+4\kappa'\eth'-$$
$$-12\kappa'\tau'+12\rho'\sigma'-3\psi_0]\Psi_2=[\eth\text{þ}-2\eth\overline{\rho}-2\overline{\rho}\eth+\text{þ}\eth-2\text{þ}\overline{\tau}-2\overline{\tau}\text{þ}-\overline{\tau}'\text{þ}$$
$$+2\overline{\rho\tau}'+\kappa\text{þ}'+2\overline{\rho}\kappa-\overline{\rho}\eth+2\overline{\tau\rho}+\sigma\eth'-2\overline{\tau}\sigma-4\tau\text{þ}+8\tau\overline{\rho}-4\rho\eth$$
$$+8\overline{\tau}\rho]\Phi_{01}+[-\eth\eth-\eth\overline{\pi}-\overline{\pi}\eth-\text{þ}\overline{\sigma}-\overline{\sigma}\text{þ}+\overline{\pi\tau}'+\overline{\tau}'\eth+\kappa\overline{\kappa}+\overline{\rho\sigma}$$
$$-\sigma\text{þ}'+\sigma\text{þ}'+\sigma\overline{\rho}'+4\tau\eth-4\tau\overline{\pi}+4\rho\overline{\sigma}]\Phi_{00}+[-2\eth\sigma-2\sigma\eth+2\overline{\tau}'\sigma+6\sigma\tau]\,\Phi_{10}+$$
$$+[2\eth\kappa+2\kappa\eth+2\text{þ}\sigma+2\sigma\text{þ}-2\kappa\overline{\tau}'-2\sigma\overline{\rho}-6\kappa\tau+10\rho\sigma]\,\Phi_{11}-[-2\text{þ}\kappa-2\kappa\text{þ}+$$
$$6\rho\kappa+2\kappa\overline{\rho}]\Phi_{12}+[-2\kappa\eth-2\sigma\text{þ}]\,\Lambda+[\eth\overline{\kappa}+\overline{\kappa}\eth-\text{þ}\text{þ}+\text{þ}\overline{\mu}+\overline{\mu}\text{þ}-\overline{\kappa\tau}'$$
$$-\kappa\eth'+\kappa\overline{\tau}'+\overline{\rho}\text{þ}-\overline{\mu}\rho+\sigma\overline{\sigma}-4\overline{\kappa}\tau+4\rho\text{þ}-4\rho\overline{\mu}]\Phi_{02}. \tag{3.14}$$

But let's comment more the approach from this article. It is a nice exercise with Green's functions, raising and lowering operators. Their approach for the case of scalar perturbations leads also to usage of residue theorem. Our approach should be applicable for the case of scalar, vector and tensor perturbations together.

Let's follow now the main road of our approach to tensor perturbations. In notation of [2] (in our notation (3.12) it is the first bracket before $d\Psi_4$) we want to solve the following equation which is a standard method for solving linear equations with delta function source on the right hand side:

$$(SD_1^+SD_2^-+L_3^+L_{-2}^-)\Phi_2=\delta(r-r')\delta(\eta-\eta')\delta(\theta-\theta')\delta(\phi-\phi').$$

Using the following formula for the Green functions

$$G(x,x')=\sum_n\frac{\phi_n^*(x')\phi_n(x)}{\lambda_n},$$

we get

$$G=\int\sum_{k,l,m}\frac{Y^m_{(2)l}Y^{m*}_{(2)l}R^w_{(2)k}R^{w*}_{(2)k}}{(k-2)(k+3)-(l-2)(l+3)}\;dw$$
$$=\sum_{k,l}\frac{(\sum_m Y^m_{(2)l}Y^{m*}_{(2)l})(\int R^w_{(2)k}R^{w*}_{(2)k}\;dw)}{(k+l+1)(k-l)}, \tag{3.15}$$

where Y are spherical harmonics and R are corresponding functions for radial-conformal part.

3.4 Remark

For the case of scalar perturbations we need to compute

$$G = \int \sum_{k,l,m} \frac{Y^m_{(0)l} Y^{m*}_{(0)l} R^w_{(0)k} R^{w*}_{(0)k}}{(k-2)(k+3)-(l-2)(l+3)}\ dw. \tag{3.16}$$

and for the case of vector perturbations

$$G = \int \sum_{k,l,m} \frac{Y^m_{(1)l} Y^{m*}_{(1)l} R^w_{(0)k} R^{w*}_{(0)k}}{(k-2)(k+3)-(l-2)(l+3)}\ dw.$$

Let's now do the steps of the computations for the scalar case (3.16):

The nominator is equivalent to $\frac{(2l+1)}{4\pi} C^{(1/2)}_l(x)$, where C are Gegenbauer polynomials. But we know that

$$\sum_{l=0}^{\infty} C^{(1/2)}_l(x) t^l = f_{(1/2)}(x,t) = \frac{1}{\sqrt{(1-2xt+t^2)}}.$$

Now we use methods from mathematical analysis known as generating function methods, see, e.g. [7], for gaining the following expression

$$\sum_{l=0}^{\infty} \frac{C^{(1/2)}_l(x)(2l+1)}{(k+l+1)(k-l)}.$$

We proceed in three steps. Two times integration and one derivative

$$\sum_{l=0}^{\infty} C^{(1/2)}_l(x) \frac{t^{l-k-1}}{l+k+1} = \frac{1}{t^{2k+2}} \int f_{(1/2)}(x,t)\ t^k = \frac{1}{t^{2k+2}} F_{(1/2)}(x,t),$$

$$\begin{aligned} F_{(1/2)}(x,t) &= \int \frac{t^k}{\sqrt{1-2xt+t^2}} \\ &= (\sum_{i=0}^{\infty} A_i t^i)\sqrt{(1-2xt+t^2)} + A_k \int \frac{1}{\sqrt{1-2xt+t^2}}, \end{aligned} \tag{3.17}$$

$$\sum_{l=0}^{\infty} C^{(1/2)}_l(x) \frac{t^{l-k}}{(l+k+1)(k-l)} = \int \frac{1}{t^{2k+2}} F_{(1/2)}(x,t) = G_{(1/2)}(x,t).$$

All these calculations, although lenghty, are feasible. For some relevant formulae, see [8]. The result of all these computations is similar to the scalar case.

We will relate this to metric perturbations. We can write the unperturbed tetrad as $e^1_{(a)} = A^k_{(a)} e_{(k)}$, where we write as usual by (1) the first order in perturbation, where the matrix A expresses the perturbations by the tetrad. Hence we need 16 functions, 10 degrees of freedom, 4 of general covariance and 6 of tetrad rotations.

The Ricci identities, which we have in Appendix A, provide the equations satisfied by the spin coefficients. For example the 3.48 relates the shear σ to the gravitational radiation. For relating the spin coefficients to A^k_a we need to linearize the commutation relation $c^{(k)}_{(a)(b)} e^{(k)} = [e_{(a)}, e_{(b)}]$ to first order in perturbation. With these solutions, we can determine some of the components of the metric tensor with the help of the formula

$$h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}, \tag{3.18}$$

[2].

Let's make the final comment: we will show that the central equation (6) in [2] can be formulated in GHP formalism. So we have the following system of 8 equations. They are already linearized:

$$SD^{\pm}_{\mp p}\Phi_p \pm L^{\mp}_{1\mp p}\Phi_{p\mp 1} = ... \tag{3.19}$$

p takes values from the set $\{-2, -1, 0, 1, 2\}$, but two equations are excluded. So together 8 equations. (The dots means RHS-sources, which are not important for us now.)

$$SD^{+}_{0}\Phi_0 + L^{-}_{1}\Phi_{-1} = ..., \tag{3.20}$$

$$SD^{-}_{0}\Phi_0 - L^{+}_{1}\Phi_1 = ..., \tag{3.21}$$

$$SD^{+}_{1}\Phi_{-1} + L^{-}_{2}\Phi_{-2} = ..., \tag{3.22}$$

$$SD^{-}_{-1}\Phi_{-1} - L^{+}_{0}\Phi_0 = ..., \tag{3.23}$$

$$SD^{+}_{-1}\Phi_1 + L^{-}_{0}\Phi_0 = ..., \tag{3.24}$$

$$SD^{-}_{1}\Phi_1 - L^{+}_{2}\Phi_2 = ..., \tag{3.25}$$

$$SD^{+}_{-2}\Phi_2 + L^{-}_{-1}\Phi_1 = ..., \tag{3.26}$$

$$SD^{-}_{-2}\Phi_{-2} - L^{+}_{-1}\Phi_{-1} = ... \tag{3.27}$$

We will show in the first step that they are equivalent to

$$þ\Psi_1 - ð'\Psi_0 + \tau'\Psi_0 - 4\rho\Psi_1 + 3\kappa\Psi_2 = ..., \tag{3.28}$$

$$þ\Psi_2 - ð'\Psi_1 - \sigma'\Psi_0 + 2\tau'\Psi_1 - 3\rho\Psi_2 + 2\kappa\Psi_3 = ..., \quad (3.29)$$

when we use prime and star duality. (Terms with κ, σ and τ are zero, when we make the linearization.) First we make the prime duality, then star duality of both equations which gives six equations:

$$ð\Psi_1 - þ'\Psi_0 - \mu\Psi_0 - 4\tau\Psi_1 + 3\sigma\Psi_2 = ..., \quad (3.30)$$

$$þ'\Psi_3 - ð\Psi_4 + \tau\Psi_4 + 4\mu\Psi_3 - 3\nu\Psi_2 = ..., \quad (3.31)$$

$$ð\Psi_2 - þ'\Psi_1 + \nu\Psi_0 - 2\mu\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 = ..., \quad (3.32)$$

$$þ'\Psi_2 - ð\Psi_3 - \sigma\Psi_4 + 2\tau\Psi_3 + 3\mu\Psi_2 - 2\nu\Psi_1 = ..., \quad (3.33)$$

Remaining two equations are obtained by prime duality of (3.30) and prime duality of (3.32).

$$ð'\Psi_3 - þ\Psi_4 + \rho\Psi_4 + 4\pi\Psi_3 - 3\lambda\Psi_2 = ..., \quad (3.34)$$

$$ð'\Psi_2 - þ\Psi_3 - \kappa\Psi_4 + 2\rho\Psi_3 + 3\pi\Psi_2 - 2\lambda\Psi_1 = ... \quad (3.35)$$

Let's examine the first equation (3.28). þ is acting on Ψ_1, quantity of type (2,0) (according to notation in [5]), therefore $þ = D$. And $ð' = \overline{\delta} - p\alpha - q\overline{\beta}$. (Quantity Ψ_0 is (4,0).) So

$$(D - 4\rho)\Psi_1 - (\overline{\delta} - 4\alpha)\Psi_0 = ...$$

We will show equivalence with

$$SD_1^-\Phi_1 - L_2^+\Phi_2 = ..., \quad (3.36)$$

$$S\left(D^- + \frac{S'}{S}\right)\left(\frac{a^4S^3}{\sqrt{2}}\Psi_1\right) - \left(L^+ + 2\cot(\theta)\right)\left(\frac{a^5S^3\Psi_0}{2}\right) = ... \quad (3.37)$$

But we can multiply the second equation by $\frac{a^5S^3}{2}$, use the knowledge of spin coefficients $\rho = -\frac{\dot{a}}{a^3} - \frac{S'}{a^2S}$ and $\alpha = -\frac{\cot\theta}{2\sqrt{2}aS}$. We see after usage of $D^- = \frac{\partial}{\partial r} + \frac{\partial}{\partial\eta}$ that they are completely equivalent.

Second equation (3.29)

$$(D - 3\rho)\Psi_2 - (\overline{\delta} - 2\alpha)\Psi_1 = ...$$

is equivalent to (3.21)

$$SD^-(a^3S^3\Psi_2) - \left(L_1^+ + \cot(\theta)\right)\left(\frac{a^4S^3\Psi_1}{\sqrt{2}}\right) = ... \quad (3.38)$$

Third equation (3.30)

$$ð\Psi_1 - þ'\Psi_0 - \mu\Psi_0 - 4\tau\Psi_1 + 3\sigma\Psi_2 = ... \quad (3.39)$$

is equivalent to

$$SD^{+}_{-2}\Phi_2 + L^{-}_{-1}\Phi_1 = ... \tag{3.40}$$

Then equation (3.20) is equivalent to (3.33), (3.22), is equivalent to (3.31), (3.23) is equivalent to (3.35), (3.24), is equivalent to (3.32) and (3.25), is equivalent to (3.34).

We are now ready to show the equivalence of equations

$$ð'ð \ d\Psi_4 = S,$$

and

$$L^{+}_{-1}L^{-}_{+2} \ d\Psi_4 = S,$$

which is the angular part of central equation from [2] for $p = -2$ **, and** S **means for this moment the RHS. Following expressions are the identities:**

$$(\delta^* - p\alpha - q\overline{\beta})(\delta - p\beta - q\overline{\alpha}) \ d\Psi_4, \tag{3.41}$$

$$(\delta^* + 3\alpha + \beta)\left[(\delta + 4\beta) \ \ d\Psi_4\right], \tag{3.42}$$

$$(\delta^* + 2\alpha)\left[(\delta - 4\alpha) \ \ d\Psi_4\right], \tag{3.43}$$

which is equivalent to

$$(\frac{1}{2a^2S^2})L^{+}_{-1}L^{-}_{+2} \ d\Psi_4.$$

We need further to show that $þ'\rho + \mu\rho + 2\Lambda = 0$**. This follows from**

$$\frac{1}{a^2}(\frac{-S'S - S'^2}{S^2}) - \left[\frac{\ddot{a}a^3 - 3a^2\dot{a}^2}{a^6}\right] + 2\frac{S'\dot{a}}{a^3} + 2\frac{\dot{a}}{a}\left(-\frac{\dot{a}}{a^3} - \frac{S'}{a^2S}\right) + \frac{1}{a^2}\left(\left(\frac{S'}{S}\right)^2 - \left(\frac{\dot{a}}{a}\right)^2\right) + \frac{4\left(a + \ddot{a}\right)}{4a^3}. \tag{3.44}$$

And now it remains to prove that

$$D^{+}_{-3}\left[aSD^{-}_{-2}(aSd\Psi_4)\right] \tag{3.45}$$

is equal to

$$\left(\Delta + 2\gamma + 5\mu\right)\left(D - \rho\right)d\Psi_4. \tag{3.46}$$

Later it will be interesting to show the correspondence with central equation in [2].

3.5 Appendix A: Ricci identities

We will for the completness write here the 18 Ricci identities. 6 of them we will comment more in the next section of the appendix. In the remaining terms are also the perturbations of derivatives. As we know these equations are linear.

$$d(D\rho) - \delta^* d\kappa - 2\rho d\rho - \rho d\epsilon - \rho d\epsilon^* + 2\alpha d\kappa = d\Phi_{00}, \tag{3.47}$$

$$Dd\sigma - \delta d\kappa - 2\rho d\sigma - 2\alpha d\kappa - d\Psi_0 = 0, \tag{3.48}$$

$$-\Delta d\kappa + Dd\tau - \rho d\tau - \rho d\pi^* + 4\gamma d\kappa - d\Psi_1 = d\Phi_{01}, \tag{3.49}$$

$$d(D\alpha) - \delta^* d\epsilon - \rho d\alpha - \alpha d\rho + \alpha d\sigma^* - \alpha d\epsilon^* - \alpha d\epsilon + 2\alpha d\epsilon + \gamma d\kappa^* - \rho d\pi = d\Phi_{10}, \tag{3.50}$$

$$d(D\beta) - \delta d\epsilon - \alpha d\sigma - \rho^* d\beta - \alpha d\epsilon^* + \mu d\kappa + \gamma d\kappa + \alpha d\epsilon + \alpha d\rho^* - d\Psi_1 = 0, \tag{3.51}$$

$$d(D\gamma) - \Delta d\epsilon - \alpha(d\tau - d\tau^*) - \alpha(d\pi^* - d\pi) + \gamma(d\epsilon + d\epsilon^*) + 2\gamma d\epsilon - d\Psi_2 = d\Phi_{11} - d\Lambda, \tag{3.52}$$

$$Dd\lambda - \delta^* d\pi - \rho d\lambda - \mu d\sigma^* - 2\alpha d\pi = d\Phi_{20}, \tag{3.53}$$

$$d(D\mu) - \delta d\pi - \mu d\rho^* - \rho d\mu + 2\alpha d\pi + \mu d\epsilon + \mu d\epsilon^* - d\Psi_2 = d(2\Lambda), \tag{3.54}$$

$$Dd\nu - \Delta d\pi - \mu d\pi - \mu d\tau^* - d\Psi_3 = d\Phi_{21}, \tag{3.55}$$

$$\Delta d\lambda - \delta^* d\nu + 2\mu d\lambda + 2\gamma d\lambda - 2\alpha d\nu + d\Psi_4 = 0, \tag{3.56}$$

$$d(\delta\rho) - \delta^* d\sigma + 4\alpha d\sigma - \rho d\alpha^* - \rho d\beta + d\Psi_1 = d\Phi_{01}, \tag{3.57}$$

$$\begin{aligned} d(\delta\alpha) - d(\overline{\delta}\beta) - \mu d\rho - \rho d\mu - \alpha d\alpha^* - \alpha d\alpha - \beta d\beta^* - \\ \beta d\beta - 2\alpha d\beta - 2d\alpha\beta - \gamma(d\rho - d\rho^*) + d\Psi_2 = d\Phi_{11} + d\Lambda, \end{aligned} \tag{3.58}$$

$$\delta d\lambda - d(\delta^* \mu) - 4\alpha d\lambda + \mu d\alpha - \mu d\beta^* + d\Psi_3 = d\Phi_{21}, \tag{3.59}$$

$$\delta d\nu - d(\Delta\mu) - 2\mu d\mu - 2\gamma d\mu - 2\mu d\gamma - 2\alpha d\nu = d\Phi_{22}, \tag{3.60}$$

$$d(\Delta\gamma) - d(\Delta\beta) - \gamma d\tau + d\alpha^* \gamma + d\beta\gamma - d\tau\mu + \beta d\gamma - \beta d\gamma^* - \mu d\beta - d\mu\beta - \alpha d\lambda^* = d\Phi_{12}, \tag{3.61}$$

$$d(\Delta\alpha) - d(\delta^*\gamma) - \rho d\nu - \alpha d\lambda - \gamma^* d\alpha - \alpha d\gamma^* - \gamma d\beta^* - d\gamma\beta^* + d\tau^*\gamma + \alpha d\mu^* + d\alpha\mu^* + d\Psi_3 = 0, \tag{3.62}$$

$$\delta d\tau - \Delta d\sigma - \mu d\sigma - \rho d\lambda^* + 2\gamma d\sigma + 2\alpha d\tau = d\Phi_{02}, \tag{3.63}$$

$$d(\Delta\rho) - \delta^* d\tau + \mu d\rho + \rho d\mu^* - 2\gamma d\rho + 2\alpha d\tau - \rho d\gamma - \rho d\gamma^* + d\Psi_2 = -2d\Lambda. \tag{3.64}$$

3.6 Appendix B: Gauge invariant variables

If we make the following change of tetrad

$$\begin{aligned} &l^\mu \to l^\mu, \\ &m^\mu \to m^\mu + \epsilon a l^\mu, \\ &n^\mu \to n^\mu + \epsilon a \overline{m}^\mu + \epsilon \overline{a} m^\mu. \end{aligned} \tag{3.65}$$

we would like to know how will transform following entities Φ. We have several notions of gauge invariance, so called infinitesimal gauge invariance (i.g.i.) (entity is i.g.i., if it is, for example, zero in the baground and we can find more about it in [5] and this is an example of so called tetrad gauge invariance (t.g.i.[2]). Here are the explicit transformation properties of the entities Φ:

$$\Phi_{00} \to \Phi_{00}, \tag{3.66}$$

$$\Phi_{02} \to \Phi_{02} + 2\epsilon a \Phi_{01}, \tag{3.67}$$

$$\Phi_{20} \to \Phi_{20} + 2\epsilon \overline{a} \Phi_{10}, \tag{3.68}$$

$$\Phi_{22} \to \Phi_{22} + 2\epsilon a \Phi_{21} + 2\epsilon \overline{a} \Phi_{12}, \tag{3.69}$$

$$\Phi_{11} \to \Phi_{11} + \epsilon a \Phi_{10} + \epsilon \overline{a} \Phi_{01}, \tag{3.70}$$

$$\Phi_{01} \to \Phi_{01} + \epsilon a \Phi_{00}, \tag{3.71}$$

$$\Phi_{10} \to \Phi_{10} + \epsilon \overline{a} \Phi_{00}, \tag{3.72}$$

$$\Phi_{12} \to \Phi_{12} + 2\epsilon a \Phi_{11} + \epsilon \overline{a} \Phi_{02}, \tag{3.73}$$

$$\Phi_{21} \to \Phi_{21} + 2\epsilon \overline{a} \Phi_{11} + \epsilon a \Phi_{20}, \tag{3.74}$$

$$\Lambda \to \Lambda. \tag{3.75}$$

So, the non t.g.i are only Φ_{01}, Φ_{10}, Φ_{12} and Φ_{21}, because the non-diagonal entities Φ are zero.

[2]There are also other notions of gauge invariance.

3.7 Appendix C: General gauge invariance

3.7.1 Conventions

We employ abstract index formalism [4] and abstract indices will be denoted by $a, b, c....$ The components of tensors with respect to null tetrad will be labelled by concrete indices $(a), (b), \ldots$. Finally, components of tensors with respect to coordinate system will be labelled by Greek indices $\mu, \nu, \ldots$, while their spatial parts are denoted by Latin indices $i, j, \ldots$. We stress once again that we use the metric tensor with signature $(1, -1, -1, -1)$.

3.7.2 Basic setup

Let us denote the background FLRW spacetime endowed with the background metric by (M, g), while the perturbed spacetime is $(\tilde{M}, \tilde{g})$. We assume that there is a diffeomorphism $\tilde{\phi} : M \mapsto \tilde{M}$ which allows us to identify the points in the background spacetime with the points of perturbed spacetime. This identification is not unique and gives rise to the gauge transformations. We can construct another diffeomorphism $\hat{\phi}$ which is "infinitesimally close" to $\tilde{\phi}$ as follows.

Consider a point $P \in M$ and define arbitrary vector field ξ on M. Let $\phi_\epsilon : M \mapsto M$ be the flow for this vector field, i.e.

$$\phi_0 = \qquad \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \phi_\epsilon = \xi.$$

Next we lift the mapping ϕ_ϵ on $\tilde{M}$ by

$$\tilde{\phi}_\epsilon = \tilde{\phi} \circ \phi_\epsilon \circ \tilde{\phi}^{-1}$$

which is the flow for the push-forward of vector field ξ,

$$\tilde{\xi} \equiv \tilde{\phi}_* \xi = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \tilde{\phi}_\epsilon.$$

Desired new diffeomorphism of M and $\tilde{M}$ is now defined by

$$\hat{\phi} = \tilde{\phi}_\epsilon \circ \tilde{\phi} = \tilde{\phi} \circ \phi_\epsilon : M \mapsto \tilde{M}.$$

Let X be any (tensorial) quantity defined on M and let $\tilde{X}$ be corresponding perturbed quantity defined on $\tilde{M}$. We wish to define the perturbation of quantity X as the difference of $\tilde{X}$ and X, but these quantities live in different spacetimes. Hence, in order to compare them, we have to define the perturbation as

$$\widetilde{X} = \tilde{\phi}^* \tilde{X} - X.$$

However, this depends on the diffeomorphism chosen to identify the spacetimes. If we take infinitesimally close diffeomorphism $\hat{\phi}$, corresponding perturbation will be

$$\widehat{X} = \hat{\phi}^* \tilde{X} - X.$$

These perturbations both live in the background spacetime and can be compared:

$$\widehat{X} - \widetilde{X} = \hat{\phi}^* \tilde{X} - \tilde{\phi}^* \tilde{X} = (\phi_\epsilon^* - \mathrm{id})\, \tilde{\phi}^* \tilde{X} = \epsilon\, \pounds_\xi\, \tilde{\phi}^* \tilde{X} + c.$$

We adopt the convention that all perturbed quantities are automatically pulled-back to the background spacetime, so that we do not distinguish quantity $\tilde{X}$ living in the perturbed spacetime and its pull-back $\tilde{\phi}^* \tilde{X}$ living in the background spacetime. Then we simply write

$$\widehat{X} = \widetilde{X} + \epsilon\, \pounds_\xi \tilde{X}. \tag{3.76}$$

In particular, the perturbation of the metric tensor will be denoted by $\widetilde{g_{\mu\nu}}$ and under the gauge transformation it transforms via the rule

$$\widehat{g_{\mu\nu}} = \widetilde{g_{\mu\nu}} + \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu. \tag{3.77}$$

In what follows we omit the tildas over the perturbed quantities, but we occasionally use the hat to indicate the behaviour under the gauge transformation.

3.7.3 FLRW spacetimes

We take the line element of the background spacetime in the form

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = a^2(\eta) \left[\mathrm{d}\eta^2 - g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j \right], \tag{3.78}$$

where η is conformal time, $a(\eta)$ is the usual scale factor and g_{ij} is the spatial metric. Spatial indices $i, j, \dots$ will be lowered/raised using this spatial metric. General perturbation is written in the form [9]

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} 2\,\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 \\ \mathcal{B}_1 & & & \\ \mathcal{B}_2 & & -g_{ij} & \\ \mathcal{B}_3 & & & \end{pmatrix}, \tag{3.79}$$

where $\mathcal{A}$ is the lapse function and $\mathcal{B}_i$ is the shift vector. Perturbation of the spatial part g_{ij} can be further decomposed into $g^{ij}-$trace free part and $g^{ij}-$trace:

$$g_{ij} = 2\, \mathcal{E}_{ij} + 2\, \mathcal{D}\, g_{ij}, \tag{3.80}$$

so that

$$g^{ij}\, \mathcal{E}_{ij} = 0, \qquad \mathcal{D} = \frac{1}{6}\, g^{ij}\, g_{ij}.$$

3.7.4 Newman-Penrose formalism

In the background spacetime we introduce the Newman-Penrose null tetrad

$$e^\mu_{(a)} = (l^\mu, n^\mu, m^\mu, \overline{m}^\mu), \tag{3.81}$$

where a is an abstract index [4] and $(a) = 0, 1, 2, 3$ labels the elements of the tetrad. The Newman-Penrose null tetrad is chosen in such a way that

$$g_{(a)(b)} = e^{\mu}_{(a)}\, g_{\mu\nu}\, e^{\nu}_{(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.82}$$

Corresponding dual tetrad is

$$e^{(a)}_{\mu} = (n_\mu, l_\mu, -\overline{m}_\mu, -m_\mu)$$

and satisfies

$$e^{\mu}_{(a)}\, e^{(a)}_{\nu} = \delta^{\mu}_{\nu}, \qquad e^{\mu}_{(a)}\, e^{(b)}_{\nu} = \delta^{(b)}_{(a)},$$

where δ^{μ}_{ν} is the unit tensor and $\delta^{(a)}_{(b)}$ is the Kronecker symbol.

In the perturbed spacetime we introduce the perturbed null tetrad

$$\tilde{e}^{\mu}_{(a)} = (\tilde{l}^{\mu}, \tilde{n}^{\mu}, \tilde{m}^{\mu}, \tilde{\overline{m}}^{\nu}).$$

We also employ the notation

$$\tilde{e}^{\mu}_{(a)} = e^{\mu}_{(a)} + \mathbf{d}e^{\mu}_{(a)}.$$

Since both set of vectors $e^{\mu}_{(a)}$ and $\tilde{e}^{\mu}_{(a)}$ form a basis of the tangent space, we can expand

$$e^{\mu}_{(a)} = J^{(b)}_{(a)}\, e^{a}_{(b)}, \tag{3.83}$$

where $J^{(a)}_{(b)}$ are small quantities. Requiring that $\tilde{e}^{\mu}_{(a)}$ be the null tetrad with respect to $\tilde{g}_{\mu\nu}$, i.e.

$$\tilde{e}^{\mu}_{(a)}\, \tilde{g}_{\mu\nu}\, \tilde{e}^{\nu}_{b} = g_{(a)(b)},$$

we arrive at the condition

$$J_{(a)(b)} = -\frac{1}{2}\, g_{(a)(b)}, \tag{3.84}$$

where indices on J^{μ}_{ν} are lowered/raised with the tetrad metric (3.82) and

$$g_{\mu\nu} = e^{\mu}_{(a)}\, g_{\mu\nu}\, e^{\nu}_{(b)}.$$

Clearly, equation (3.84) fixes only the symmetric part of matrix $J_{\mu\nu}$ and thus we can perform a gauge transformation

$$J_{\mu\nu} \mapsto \hat{J}_{\mu\nu} = J_{\mu\nu} + F_{\mu\nu}, \tag{3.85}$$

where $F_{\mu\nu}$ is an arbitrary antisymmetric matrix, without changing the normalization of perturbed null tetrad. Let us denote the elements of $F_{\mu\nu}$ by

$$F_{\mu\nu} = \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ -A_1 & 0 & B_3 & -B_2 \\ -A_2 & -B_3 & 0 & B_1 \\ -A_3 & B_2 & -B_1 & 0. \end{pmatrix}$$

Gauge-transformed null tetrad is then

$$\widehat{e}^{\mu}_{(a)} = e^{\mu}_{(a)} + F^{(b)}_{(a)}\, e^{\mu}_{(b)},$$

or explicitly

$$\widehat{l}^{\mu} = l^{\mu} - A_1\, l^{\mu} + A_3\, m^{\mu} + A_2\, \overline{m}^{\mu}, \tag{3.86a}$$

$$\widehat{n}^{\mu} = n^{\mu} + A_1\, n^{\mu} - B_2\, m^{\mu} + B_3\, \overline{m}^{\mu}, \tag{3.86b}$$

$$\widehat{m}^{\mu} = m^{\mu} + B_3\, l^{\mu} + A_2\, n^{\mu} + B_1\, m^{\mu}, \tag{3.86c}$$

$$\widehat{\overline{m}}^{\mu} = \overline{m}^{\mu} - B_2\, l^{\mu} + A_3\, n^{\mu} - B_1\, \overline{m}^{\mu}. \tag{3.86d}$$

The reality of l^{μ} and n^{ν} (and their perturbations) and the relation $\overline{m^{\mu}} = \overline{m}^{\mu}$ imply

$$A_3 = \overline{A}_2, \qquad B_2 = -\overline{B}_3, \qquad \overline{B}_1 = -B_1, \qquad \overline{A}_1 = A_1. \tag{3.87}$$

Thus, the transformation (3.85) has 6 real parameters corresponding to 6 generators of the Lorentz group.

Basis dual to $e^{\mu}_{(a)}$ will be denoted by $e^{(a)}_{\mu}$ and its elements are

$$e_{\mu} = (n_{\mu}, l_{\mu}, -\overline{m}_{\mu}, -m_{\mu}).$$

Corresponding perturbed dual tetrad is

$$\tilde{e}^{a}_{\mu} = e^{a}_{\mu} + \mathbf{d}e^{a}_{\mu},$$

where

$$e^{(a)}_{\mu} = -J^{(a)}_{(b)}\, e^{(b)}_{\mu}.$$

The connection ∇_{μ} associated with the background spacetime is encoded in the Ricci rotation coefficients defined by

$$\omega_{(a)}{}^{(b)}{}_{(c)} = e^{(b)}_{\mu} \nabla_{(c)} e^{\mu}_{(a)} \equiv e^{(b)}_{\mu} e^{(d)}_{(c)} \nabla_{(d)} e^{\mu}_{(a)}. \tag{3.88}$$

The Newman-Penrose spin coefficients are related to the Ricci rotation coefficients by relations

$$\kappa = -\omega_0{}^3{}_0, \qquad \varepsilon = \frac{1}{2}\left(\omega_0{}^0{}_0 - \omega_3{}^3{}_0\right), \qquad \pi = \omega_1{}^2{}_0, \tag{3.89a}$$

$$\tau = -\omega_0{}^3{}_1, \qquad \gamma = \frac{1}{2}\left(\omega_0{}^0{}_1 - \omega_3{}^3{}_1\right), \qquad \nu = \omega_1{}^2{}_1, \tag{3.89b}$$

$$\sigma = -\omega_0{}^3{}_2, \qquad \beta = \frac{1}{2}\left(\omega_0{}^0{}_2 - \omega_3{}^3{}_2\right), \qquad \mu = \omega_1{}^2{}_2, \tag{3.89c}$$

$$\rho = -\omega_0{}^3{}_3, \qquad \alpha = \frac{1}{2}\left(\omega_0{}^0{}_3 - \omega_3{}^3{}_3\right), \qquad \lambda = \omega_1{}^2{}_3. \tag{3.89d}$$

Let $\widetilde{\nabla}_{\mu}$ be connection associated with the perturbed spacetime. Then [6]

$$\widetilde{\nabla}_{\mu} X_{\nu} = \nabla_{\mu} X_{\nu} - \Gamma^{\gamma}_{\mu\nu} X_{\gamma}, \tag{3.90}$$

which implies

$$\Gamma^{\gamma}_{\mu\nu} = \frac{1}{2}\, g^{\gamma\delta}\left(\nabla_\mu g_{\nu\delta} + \nabla_\nu g_{\mu\delta} - \nabla_\delta g_{\mu\nu}\right). \tag{3.91}$$

Under the gauge transformation (3.77) we have

$$\widehat{\Gamma}^{\gamma}_{\mu\nu} = \Gamma^{\gamma}_{\mu\nu} + \nabla_\nu\nabla_\mu\xi^\gamma + R^{\gamma}{}_{\mu\nu\delta}\,\xi^\delta, \tag{3.92}$$

where $R^{\mu}{}_{\nu\delta\gamma}$ is the Riemann tensor of the background spacetime.

The Ricci rotation coefficients associated with perturbed null tetrad are

$$\widetilde{\omega}_{(a)}{}^{(b)}{}_{(c)} = \omega_{(a)}{}^{(b)}{}_{(c)} + \nabla_{(c)}J^{(b)}_{(a)} + \Gamma^{(b)}_{(a)(c)} + J^{(d)}_{(a)}\,\omega_{(d)}{}^{(b)}{}_{(c)} - J^{(b)}_{(d)}\,\omega_{(a)}{}^{(d)}{}_{(c)} + J^{(d)}_{(c)}\,\omega_{(a)}{}^{(b)}{}_{(d)}. \tag{3.93}$$

Under the gauge transformation (3.85) (which does not affect the Christoffel symbols) we have

$$\widehat{\omega}_{(a)}{}^{(b)}{}_{(c)} = \widetilde{\omega}_{(a)}{}^{(b)}{}_{(c)} + \nabla_{(c)}F^{(b)}_{(a)} + F^{(d)}_{(a)}\,\omega_{(d)}{}^{(b)}{}_{(c)} - F^{(b)}_{(d)}\,\omega_{(a)}{}^{(d)}{}_{(c)} + F^{(d)}_{(c)}\,\omega_{(a)}{}^{(b)}{}_{(d)}. \tag{3.94}$$

We introduce the notation

$$\widehat{\omega}_{(a)}{}^{(b)}{}_{(c)} = \widehat{\omega}_{(a)}{}^{(b)}{}_{(c)} - \omega_{(a)}{}^{(b)}{}_{(c)}, \qquad \omega_{(a)}{}^{(b)}{}_{(c)} = \widetilde{\omega}_{(a)}{}^{(b)}{}_{(c)} - \omega_{(a)}{}^{(b)}{}_{(c)}.$$

The transformation of the spin coefficients is found from (3.89):

$$\hat{\kappa} = \kappa - DA_2 + (\rho + 2\varepsilon)A_2 + \kappa\,(B_1 - 2A_1) + \sigma\,\overline{A}_2, \tag{3.95a}$$
$$\hat{\tau} = \tau - \Delta A_2 + +2\,\gamma\,A_2 + \tau\,B_1 - \sigma\,B_2 - \rho\,\overline{B}_2, \tag{3.95b}$$
$$\hat{\sigma} = \sigma - \delta A_2 + (2\beta + \tau)A_2 + \sigma(2\,B_1 - A_1) - \kappa\,\overline{B}_2, \tag{3.95c}$$
$$\hat{\rho} = \rho - \overline{\delta}A_2 + 2\,\alpha\,A_2 - \rho\,A_1 - \kappa\,B_2 + \tau\,\overline{A}_2, \tag{3.95d}$$
$$\hat{\varepsilon} = \varepsilon + \frac{1}{2}\,(DB_1 - DA_1) - \varepsilon\,A_1 + (\alpha + \pi)A_2 + \beta\,\overline{A}_2 - \kappa\,B_2, \tag{3.95e}$$
$$\hat{\gamma} = \gamma + \frac{1}{2}\,(\Delta B_1 - \Delta A_1) - (\beta + \tau)B_2 - \alpha\,\overline{B}_2 + \gamma\,A_1 + \nu\,A_2, \tag{3.95f}$$
$$\hat{\beta} = \beta + \frac{1}{2}\,(\delta B_1 - \delta A_1) + (\mu + \gamma)A_2 + \beta\,B_1 - \sigma\,B_2 - \varepsilon\,\overline{B}_2, \tag{3.95g}$$
$$\hat{\alpha} = \alpha + \frac{1}{2}\left(\overline{\delta}B_1 - \overline{\delta}A_1\right) + \lambda\,A_2 + \gamma\,\overline{A}_2 - \alpha\,B_1 - (\varepsilon + \rho)B_2, \tag{3.95h}$$
$$\hat{\pi} = \pi - DB_2 + \lambda\,A_2 + \mu\,\overline{A}_2 - \pi\,B_1 - 2\,\varepsilon\,B_2, \tag{3.95i}$$
$$\hat{\nu} = \nu - \Delta B_2 + \nu(2\,A_1 - B_1) - (\mu + 2\,\gamma)B_2 - \lambda\,\overline{B}_2, \tag{3.95j}$$
$$\hat{\mu} = \mu - \delta B_2 + \mu\,A_1 + \nu\,A_2 - 2\,\beta\,B_2 - \pi\,\overline{B}_2, \tag{3.95k}$$
$$\hat{\lambda} = \lambda - \overline{\delta}B_2 + \lambda\,A_1 + \nu\,\overline{A}_2 - 2\,\lambda\,B_1 - (\pi + 2\,\alpha)B_2. \tag{3.95l}$$

The gauge freedom (3.85) can be exploited in order to eliminate some of the spin coefficients. Typically we want to eliminate κ and $(\varepsilon + \overline{\varepsilon})$, for then the vector l^a is tangent to affinely parametrized geodesic and $Dl^a = 0$.

Let us now return to the gauge transformation (3.76)**. If**

$$g_{(a)(b)} = e^\mu_{(a)}\, e^\nu_{(b)}\, g_{\mu\nu}$$

are the tetrad components of the perturbation, then after the gauge transformation (3.76) **the new components will be**

$$\widehat{g}_{(a)(b)} = g_{(a)(b)} + e^\mu_{(a)}\, e^\nu_{(b)}\, (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu)\,.$$

Explicitly in the Newman-Penrose formalism:

$$\widehat{g}_{00} = g_{00} + 2\, D\xi_0 - 2(\varepsilon + \overline{\varepsilon})\xi_0 + 2\,\overline{\kappa}\,\xi_2 + 2\,\kappa\,\overline{\xi}_2, \tag{3.96a}$$

$$\widehat{g}_{01} = g_{01} + D\xi_1 + \Delta\xi_0 - (\gamma + \overline{\gamma})\xi_0 + (\varepsilon + \overline{\varepsilon})\xi_1 + (\overline{\tau} - \pi)\xi_2 + (\tau - \overline{\pi})\overline{\xi}_2, \tag{3.96b}$$

$$\widehat{g}_{02} = g_{02} + D\xi_2 + \delta\xi_0 - (\overline{\alpha} + \beta + \overline{\pi})\xi_0 + \kappa\,\xi_1 + (\overline{\rho} - \varepsilon + \overline{\varepsilon})\xi_2 + \sigma\,\overline{\xi}_2, \tag{3.96c}$$

$$\widehat{g}_{11} = g_{11} + 2\Delta\xi_1 + 2(\gamma + \overline{\gamma})\xi_1 - 2\,\nu\,\xi_2 - 2\,\overline{\nu}\,\overline{\xi}_2, \tag{3.96d}$$

$$\widehat{g}_{12} = g_{12} + \Delta\xi_2 + \delta\xi_1 - \overline{\nu}\,\xi_0 + (\overline{\alpha} + \beta + \tau)\xi_1 + (\overline{\gamma} - \gamma - \mu)\xi_2 - \overline{\lambda}\,\overline{\xi}_2, \tag{3.96e}$$

$$\widehat{g}_{22} = g_{22} + 2\delta\xi_2 - 2\,\overline{\lambda}\,\xi_0 + 2\,\sigma\,\xi_1 + 2(\overline{\alpha} - \beta)\xi_2, \tag{3.96f}$$

$$\widehat{g}_{23} = g_{23} + \delta\overline{\xi}_2 + \overline{\delta}\xi_2 - (\mu + \overline{\mu})\xi_0 + (\rho + \overline{\rho})\xi_1 + (\overline{\beta} - \alpha)\xi_2 + (\beta - \overline{\alpha})\overline{\xi}_2. \tag{3.96g}$$

Following [2] we introduce operators

$$L^\pm = \partial_\theta \mp \frac{i}{\sin\theta}\,\partial_\phi, \qquad \mathcal{L}^\pm_n = L^\pm + n\,\cot\theta, \tag{3.97}$$

$$\mathcal{D}^\pm = \partial_r \mp \partial_\eta \qquad \mathcal{D}^\pm_n = \mathcal{D}^\pm + n\,\cot r. \tag{3.98}$$

With this notation, the Newman-Penrose operators acquire the form

$$D = \frac{1}{a^2}\,\mathcal{D}^-, \qquad \Delta = -\frac{1}{2}\,\mathcal{D}^+, \tag{3.99}$$

$$\delta = \frac{1}{\sqrt{2}\,S\,a}\,L^-, \qquad \overline{\delta} = \frac{1}{\sqrt{2}\,S\,a}\,L^+. \tag{3.100}$$

We will need the following identities. Let X **and** f **be real functions. Then the equation**

$$\mathcal{D}^-_p X = f(\eta, r, \theta, \phi) \tag{3.101}$$

has explicit solution

$$X = \frac{1}{\sin^p r}\int_0^\eta \sin^p(z - \eta + r)\; f(z, z - \eta + r, \theta, \phi)\,\mathrm{d}z + \sin^{-p} r\, C(\eta - r, \theta, \phi), \tag{3.102}$$

where C **is an arbitrary function of three variables. In order to simplify notation we introduce the integral operator**

$$(\mathcal{D}^-_p)^{-1} f = \frac{1}{\sin^p r}\int_0^\eta \sin^p(z - \eta + r)\; f(z, z - \eta + r, \theta, \phi)\,\mathrm{d}z,$$

so that the solution (3.102) **acquires the form**

$$X = (\mathcal{D}^-_p)^{-1} f + \frac{1}{\sin^p r} C(\eta - r, \theta, \phi).$$

3.7.5 Spatially flat case

Let us assume that the backround universe is spatially flat, so that the spatial part of the metric (3.78) **in the spherical coordinates reads**

$$g_{ij} = \mathrm{diag}\ (1, r^2, r^2 \sin^2\theta).$$

The perturbation of the metric (3.79) **then reads**

$$g_{\mu\nu} = \begin{pmatrix} 2\,\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 \\ \mathcal{B}_1 & -2\,\mathcal{E}_{rr} - 2\,\mathcal{D} & -2\,\mathcal{E}_{r\theta} & -2\,\mathcal{E}_{r\phi} \\ \mathcal{B}_2 & -2\,\mathcal{E}_{r\theta} & -2\,\mathcal{E}_{\theta\theta} - 2\,r^2\,\mathcal{D} & -2\,\mathcal{E}_{\theta\phi} \\ \mathcal{B}_3 & -2\,\mathcal{E}_{r\phi} & -2\,\mathcal{E}_{\theta\phi} & -2\,\mathcal{E}_{\phi\phi} - 2\,r^2\,\mathcal{D}\,\sin^2\theta \end{pmatrix}.$$

In the conformal Newtonian gauge, the perturbation acquires the form

$$g_{\mu\nu} = 2\,a^2(\eta)\,\mathrm{diag}\ (\Phi, \Psi, \Psi, \Psi).$$

Then the tetrad components of the perturbation are

$$g_{(a)(b)} = \begin{pmatrix} 2\,a^{-2}(\Psi+\Phi) & \Phi-\Psi & 0 & 0 \\ \Phi-\Psi & (1/2)a^2(\Psi+\Phi) & 0 & 0 \\ 0 & 0 & -r^{-2}\,\cot^2\theta\,\Psi & r^{-2}\Psi(1+\csc^2\theta) \\ 0 & 0 & r^{-2}\,\Psi(1+\csc^2\theta) & -r^{-2}\,\cot^2\theta\,\Psi \end{pmatrix}. \tag{3.103}$$

Using (3.83) **and** (3.84) **we can immediately write down the perturbations to the elements of the null tetrad:**

$$l^\mu = \frac{1}{a^2}\,(-\Phi, \Psi, 0, 0)\,, \tag{3.104a}$$

$$n^\mu = -\frac{1}{2}\,(\Phi, \Psi, 0, 0)\,, \tag{3.104b}$$

$$m^\mu = \frac{1}{\sqrt{2}\,a\,r^3}\left(0, 0, \Psi, \frac{i\,\Psi}{\sin^3\theta}\right). \tag{3.104c}$$

Perturbation of the spin coefficients are

$$\kappa = -\frac{1}{a^2}\,(\delta\Phi + \delta\Psi)\,, \tag{3.105a}$$

$$\varepsilon = \frac{1}{a^2}\,(\Phi_{,r} - \Psi_{,\eta})\,. \tag{3.105b}$$

Thus, the perturbed vector $\tilde{l}^a$ **is not tangent to a geodesic** ($\kappa \neq 0$)**. Now we can exploit the gauge freedom** (3.85) **under which** κ **transforms according to the equation** (3.95a)**,**

$$\widehat{\kappa} = -\frac{1}{a^2}\,(\delta\Phi + \delta\Psi) - (D - \rho)A_2.$$

In order to set $\hat{\kappa} = 0$ **we have to solve this equation, which in terms of the operator** $\mathcal{D}$ **reads**

$$\mathcal{D}_p^-\left(a\,r^{1-p}A_2\right) = -a\,r^{1-p}\,\delta(\Phi+\Psi).$$

This is of the form (3.101) **and hence the solution is**

$$A_2 = -\frac{r^{p-1}}{a}(\mathcal{D}_p^-)^{-1}\left(a\, r^{1-p}\,\delta(\Phi+\Psi)\right) + \frac{r^{p-1}}{a\,\sin^p r}\,C(\eta - r, \theta, \phi).$$

Notice that there is a residual gauge freedom in the choice of function C**. Performing this gauge transformation we achieve**

$$\widehat{\kappa} = 0.$$

With this choice, $\tilde{l}^a$ **is a geodesic, but since** $(\varepsilon + \overline{\varepsilon}) \neq 0$**, it is not affinely parametrized. This can be cured by additional gauge transformation, for the relation** (3.95e) **implies**

$$\widehat{\varepsilon} + \widehat{\overline{\varepsilon}} = \varepsilon + \overline{\varepsilon} - DA_1.$$

Solving equation

$$DA_1 = \frac{1}{a^2}\left(\Phi_{,r} - \Psi_{,\eta}\right)$$

we set $\hat{\varepsilon} + \hat{\overline{\varepsilon}} = 0$ **and from now** $\tilde{l}^a$ **is an affinely parametrized geodesic.**

3.8 Appendix D: Remark about Ricci identites

We could rewrite the 6 Ricci identities followingly ([1], equation (310), b, c, g, i, j, p), where we make the similar notation from [2] :

$$\frac{1}{2}D^-_{-2}(S_1) - \mu S_1 - L^-_{-1}(\frac{a^4 S^2 d\kappa}{2\sqrt{2}}) = \Phi_2, \tag{3.106}$$

$$D^+_{-2}(S_2) - \rho a^2 S_2 + L^+_{-1}(\sqrt{2} d\nu S^2) = \Phi_{-2}, \tag{3.107}$$

$$\frac{1}{2}D^+_{-2}(S_1) - a^2\rho\overline{S_2} + L^-_{-1}(\frac{d\tau a^2 S^2}{\sqrt{2}}) = a^3 S^3 d\Phi_{02}, \tag{3.108}$$

$$D^-_{-2}(S_2) - \mu\overline{S_1} - L^+_{-1}(\frac{d\pi a^2 S^2}{\sqrt{2}}) = a^3 S^3 d\Phi_{20}, \tag{3.109}$$

$$\frac{1}{a^2}D^-(d\nu) + \frac{1}{2aS}D^+(aSd\pi) - \mu d\overline{\tau} = d\Psi_3 + d\Phi_{21}, \tag{3.110}$$

$$\frac{1}{a^3 S}D^-(aSd\tau) + \frac{1}{2a^4}D^+(a^4 d\kappa) - \rho d\overline{\pi} = d\Psi_1 + d\Phi_{01}, \tag{3.111}$$

$S_1 = S^3 a^3 d\sigma$ **and** $S_2 = S^3 a d\lambda$**. And now equations a,h,k:**

$$\frac{1}{a^4 S^2}D^-(a^2 S^2 d\rho) - \frac{1}{\sqrt{2}S^3 a^5}L^+_1(a^4 S^2 d\kappa) = \rho d\epsilon + \rho d\overline{\epsilon} + AD\rho + E\delta\rho + \overline{E\delta}\rho + d\Phi_{00}, \tag{3.112}$$

$$\frac{1}{a^3 S}D^-(aSd\mu) - \frac{1}{\sqrt{2}a^5 S^3}L^-_1(a^4 S^2 d\pi) = \mu d\epsilon + \mu d\overline{\epsilon} - \mu d\overline{\rho} + d\Psi_2 + 2d\Lambda, \tag{3.113}$$

$$\frac{1}{\sqrt{2}aS}L^{-}d\rho-\frac{1}{\sqrt{2}a^2S^2}L^{-}_{2}(aSd\sigma)=-CD\rho-\overline{E}\Delta\rho+\rho d\overline{\alpha}+\rho d\beta-d\Phi_1+d\Phi_{01}, \tag{3.114}$$

Now equations d,e,f:

$$\frac{1}{a^3S}D^{-}(aSd\alpha)-\frac{1}{\sqrt{2}aS}L^{+}_{\frac{1}{2}}(d\epsilon)=-AD\alpha-E\delta\alpha-\overline{E\delta}\alpha+\alpha d\rho$$
$$-\alpha d\overline{\sigma}+\alpha d\overline{\epsilon}-\gamma d\overline{\kappa}+\rho d\pi+d\Phi_{10}, \tag{3.115}$$

$$\frac{1}{a^3S}D^{-}(aSd\beta)-\frac{1}{\sqrt{2}aS}L^{-}_{\frac{1}{2}}(d\epsilon)=\alpha d\sigma+\alpha d\overline{\epsilon}-\mu d\kappa-\gamma d\kappa$$
$$-\alpha d\overline{\rho}+d\Psi_1-AD\beta-E\delta\beta-\overline{E\delta}\beta, \tag{3.116}$$

$$Dd\gamma+\frac{1}{2a^3}D^{+}(a^3d\epsilon)=-AD\gamma+\alpha(d\tau-d\overline{\tau})+\alpha(d\overline{\pi}-d\pi)-\gamma d\overline{\epsilon}+d\Psi_2+d\Phi_{11}-d\Lambda, \tag{3.117}$$

Now equations o,q,r:

$$L^{-}_{-\frac{1}{2}}(d\gamma)+\frac{1}{2a^2S}D^{+}(a^2Sd\beta)=A\Delta\gamma-A\Delta\beta+\overline{C}\delta\beta+C\overline{\delta}\beta+\gamma d\tau$$
$$-\gamma d\overline{\alpha}+\mu d\tau+\beta d\overline{\gamma}+\beta d\mu+\alpha d\overline{\lambda}+d\Phi_{12}, \tag{3.118}$$

$$-\frac{1}{2a^3S}D^{+}(a^3Sd\rho)-\frac{1}{\sqrt{2}aS}L^{+}_{-1}d\tau=A\Delta\rho-\rho d\overline{\mu}+\rho d\gamma+\rho d\overline{\gamma}-d\Psi_2-2d\Lambda, \tag{3.119}$$

$$\frac{1}{a^2S}D^{+}(a^2Sd\alpha)-L^{+}_{\frac{1}{2}}(d\gamma)=A\Delta\alpha-\overline{C}\delta\alpha-C\overline{\delta}\alpha+\overline{C}D\gamma+E\Delta\gamma$$
$$+\rho d\nu+\alpha d\lambda+\alpha d\overline{\gamma}+\gamma d\overline{\beta}-\gamma d\overline{\tau}-\alpha d\overline{\mu}-d\Psi_3=0, \tag{3.120}$$

And finally l,m,n:

$$\frac{1}{\sqrt{2}aS}L^{-}_{-\frac{1}{2}}(d\alpha)-\frac{1}{\sqrt{2}aS}L^{+}_{\frac{-1}{2}}d\beta=-CD\alpha-\overline{E}\Delta\alpha-B\delta\alpha+\overline{C}D\beta+E\Delta\beta$$
$$+\overline{B\delta}\beta+\mu d\rho+\rho d\mu+\alpha d\overline{\alpha}+\beta d\overline{\beta}+\gamma(d\rho-d\overline{\rho})+d\Phi_{11}+d\Lambda+d\Psi_2, \tag{3.121}$$

$$\frac{1}{\sqrt{2}a^2S^2}L^{-}_{2}(aSd\lambda)-\frac{1}{\sqrt{2}aS}L^{+}d\mu=\overline{C}D\mu+E\Delta\mu+\overline{B\delta}\mu-d\Psi_3+d\Phi_{21}, \tag{3.122}$$

$$\frac{1}{\sqrt{2}a^3L^3}L^{-}_{1}(a^2S^2d\nu)+\frac{1}{2S^2}D^{+}(S^2d\mu)=-A\Delta\mu+\overline{C}\delta\mu+C\overline{\delta}\mu+2\mu d\gamma+d\Phi_{22}, \tag{3.123}$$

Now we will deal with the first 6 equations. We have

$$D^{+}_{-q}[\frac{1}{a^4S^{2-p_1}}D^{-}_{p_1}(a^2S^{2-p_1}d\sigma)] - L^{-}_{-1}[D^{+}_{-q}(\frac{1}{\sqrt{2}aS}d\kappa)] = D^{+}_{-q}d\Psi_0, \tag{3.124}$$

$$-D^{-}_{p}[\frac{1}{2a^3S^{1-p_2}}D^{+}_{p_2}(a^3S^{1-p_2}d\sigma)] - L^{-}_{-1}[D^{-}_{p}(\frac{1}{\sqrt{2}aS}d\tau)] = -D^{-}_{p}(\rho d\overline{\lambda}) - D^{-}_{p}(d\Phi_{02}), \tag{3.125}$$

and

$$\frac{1}{a^3S^{1-p}}D^{-}_{p}(aS^{1-p}d\tau) + \frac{1}{2a^4S^q}D^{+}_{-q}(a^4S^q d\kappa) = \rho d\overline{\pi} + d\Psi_1 + d\Phi_{01}, \tag{3.126}$$

We will continue to work with the 6 Ricci equations (31a)-(31f), because we would like to rewrite them in the manner of [2].($\Psi = 0$ **and** $\Phi = 0$ **for simplicity but we could add these terms at the end.)**

We apply operator $-D$ **on the last equation and operator** Δ **on the first. We add these two equations. Now we will use the second equation and the commutation relation** $\Delta D - D\Delta = 2\gamma D$ **:**

$$\begin{aligned} 2\gamma Dd\sigma - \Delta\delta d\kappa + D\delta d\tau = 2\gamma Dd\sigma + \delta(d\tau + d\overline{\pi}) + \\ \rho(\delta d\tau + \delta d\overline{\pi}) - 4(\delta\gamma)d\kappa - 4\gamma(\delta d\kappa) = \\ = \Delta(2\rho)d\sigma + 2\rho(\Delta d\sigma) + \Delta(2\alpha)d\kappa + 2\alpha\Delta d\kappa + D(\mu - 2\gamma)d\sigma \\ +(\mu - 2\gamma)Dd\sigma + D\rho d\overline{\lambda} + \rho Dd\overline{\lambda} + D(2\beta)d\tau + 2\beta D(d\tau) \end{aligned} \tag{3.127}$$

At this moment we eliminated all second order operators and after usage of other equations (31a)-(31f) and simplifications we get a final result (where we omitted terms which are zero for our spin coefficients):

$$\begin{aligned} \rho\Delta d\sigma + \mu Dd\sigma + \rho\mu(\overline{d\sigma} - d\sigma) - 6\gamma\rho d\sigma + \Delta(2\rho)d\sigma + [D(\mu - 2\gamma)]d\sigma \\ +[D\rho]\overline{d\lambda} + [\Delta(2\alpha)]d\kappa + [D(2\beta)]d\tau = 0. \end{aligned} \tag{3.128}$$

What is interesting that the same method will lead to the following equation (in case of equations (31c)-(31e)):

$$\rho\Delta d\lambda + \mu Dd\lambda + [\Delta\rho]d\lambda + 2[D\mu + D\gamma]d\lambda + (2\mu\gamma + \Delta\mu)\overline{d\sigma} - 2[D\alpha]d\nu + 2[\Delta\alpha]d\pi = 0. \tag{3.129}$$

Remark. If we want to include the RHS, there occures expressions like that:

$$\delta d\Psi_1 - Dd\Phi_{02} + \Delta d\Psi_0 - \delta d\Phi_{01}. \tag{3.130}$$

3.9 Appendix E: One remark about scalar, vector, tensor decouplings in linear perturbation theory

The following argument could be found in every good lecture book on cosmological perturbation theory. We Taylor expand the action, the first term is satisfied from the dynamical equations and then we

analyze the second term. However we present here for completness the following, because we didn't see that it was presented somewhere in this manner. We use the conformal Newtonian gauge and we use coordinate approach.

Now we want to see, how will slightly change the spin coefficients. We will make a comparison with unperturbed values and so we get also the perturbations. This road is promising, because we could observe, what kind of NP equations are influenced by such scalar perturbations. (We could do that also for vector and tensor perturbations.)

We will make the following simplification first. $\psi = \psi(\eta, r, \theta)$, so ψ is not a function of ϕ. This is according to the known result that perturbations of spherically symmetric spacetime is axially symmetric. (However, this is only for the simplification, we could switch on the dependence on ϕ.)

The tetrad is for the case, when we switch on only the scalar perturbations the following. (We can check the consistency by $l^\mu n_\mu = 1$ and $m^\mu \overline{m}_\mu = -1$.)

$$l_\mu = (1, -(1-2\psi), 0, 0) \quad , \tag{3.131}$$

$$n_\mu = \left(\frac{a^2}{2}(1+2\psi), \frac{a^2}{2}, 0, 0 \right) \quad , \tag{3.132}$$

$$m_\mu = \left(0, 0, -\frac{aS}{\sqrt{2}}(1-\psi), -\frac{iaS\sin\theta}{\sqrt{2}}(1-\psi) \right) . \tag{3.133}$$

Now we can compute NP-scalars and $\Phi_{(a)(b)}$:

$$\rho = \overline{\rho} + \delta\rho \text{ and } u^\mu = \overline{u}^\mu + \delta u^\mu,$$

$\overline{u}^\mu = \frac{1}{a}(1, \vec{0})$ and because $\overline{u}_\mu \overline{u}^\mu = 1$ we have also $\overline{u}_\mu = a(1, \vec{0})$.

$$g_{\mu\nu} = \begin{pmatrix} a^2(1+2\psi) & 0 \\ 0 & a^2(1-2\psi)\delta_{ij} \end{pmatrix}$$

Now we have for the perturbations:

$$\delta u^i \equiv \frac{1}{a} v_i,$$
$$\delta u_i = a v_i,$$
$$\delta u^0 = -\frac{1}{a}\psi,$$
$$\delta u_0 = a^2 \delta u^0 + 2a\psi.$$

We can now compute, for example, Φ_{00}:

$$\Phi_{00} = -4\pi(\overline{\rho} + \delta\rho)(l^\mu u_\mu)^2 = \frac{-4\pi}{a^2}(\overline{\rho} + \delta\rho)[(1-2\psi)(1+\psi) + v_1]^2$$

We will start with equations (3.48) and (3.56) now. The LHS for (3.56) is equal to

$$\frac{1}{4S^2}(\frac{\partial^2\psi}{\partial\theta^2} - \frac{i}{\sin(\theta)}\frac{\partial^2\psi}{\partial\theta\partial\phi} + \cot(\theta)\frac{\partial\psi}{\partial\theta})$$

We could see the presence of imaginary part, which is equal to 0 because ψ does not depend on ϕ. The LHS of (3.48) is very similar. We compute Weyl tensor in Mathematica software and we could find that LHS = RHS for both equations (3.48) and (3.56) as it should be.
Now we will examine the next two equations (3.53) and (3.63). Because variations of spin coefficients $d\lambda$, $d\sigma$, $d\pi$ and $d\tau$ are zero in this case, we have only terms with $d\Phi_{02}$ and $d\Phi_{20}$, but these two pertrubations are zero in the first order approximation.
So first nontrivial equation is (3.55). We need to compute

$$d\Phi_{21} = -4\pi n^\mu u_\mu \overline{m}^\nu u_\nu(\overline{\rho} + \delta\rho)$$

and also $Dd\nu - d\Psi_3$. After some computation - during which we get $v_3 = 0$ because of the imaginary part - we get the equation

$$H_c\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\psi}{\partial\theta\partial\eta} = a^2 v_2(-4\pi\overline{\rho}),$$

which is one of the desired equations. We could could work with the second nontrivial equation (3.49) in similar way. Here are the components of Φ

$$d\Phi_{21} = \frac{-4\pi\overline{\rho}av_2}{2\sqrt{2}S}, \tag{3.134}$$

$$d\Phi_{01} = \frac{-4\pi\overline{\rho}v_2}{\sqrt{2}aS}. \tag{3.135}$$

We need the expressions for the work with the next twelve equations the perturbations of NP derivatives. So:

$$d(D\rho) = \frac{1}{a^2}[2\frac{\partial\psi}{\partial r}\frac{\dot{a}}{a^3} + \frac{1}{a^2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{a^2}\frac{\partial^2\psi}{\partial r\partial\eta} + 2(\frac{\partial\psi}{\partial\eta}\dot{a}a^3 + \psi\frac{\ddot{a}a^3 - 3\dot{a}^2a^2}{a^6}) + \frac{1}{a^2}\frac{\partial^2\psi}{\partial r\partial\eta} - \frac{\partial\psi}{\partial r}\frac{\dot{a}}{a^3} + \frac{1}{a^2}\frac{\partial^2\psi}{\partial\eta^2} - \frac{\partial\psi}{\partial\eta}\frac{\dot{a}}{a^3} - \frac{2\psi}{a^2}[\frac{\dot{a}}{ra^3} - \frac{\ddot{a}a^3 - 3\dot{a}^2a^2}{a^6}],$$

$$d(D\alpha) = \frac{-\cot\theta}{2\sqrt{2}a^2}[\frac{1}{a}(\frac{\frac{\partial\psi}{\partial r}r - \psi}{r^2}) + \frac{1}{ar}\frac{\partial\psi}{\partial\eta} + \frac{\psi\dot{a}}{a^2r}], \tag{3.136}$$

$$d(D\beta) = \frac{\cot\theta}{2\sqrt{2}a^2}\left[\frac{1}{a}(\frac{\frac{\partial\psi}{\partial r}r - \psi}{r^2}) + \frac{1}{ar}\frac{\partial\psi}{\partial\eta} + \frac{\psi\dot{a}}{a^2r}\right] - \frac{1}{\sqrt{2}a^2}\left[\frac{1}{a}(\frac{\frac{\partial^2\psi}{\partial r\partial\theta}r - \frac{\partial\psi}{\partial\theta}}{r^2}) + \frac{1}{r}(\frac{\frac{\partial^2\psi}{\partial\eta\partial\theta}a - \frac{\partial\psi\dot{a}}{\partial\theta}}{a^2})\right], \tag{3.137}$$

$$d(D\gamma) = \frac{1}{a^2}\left[\frac{1}{2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{2}\frac{\partial^2\psi}{\partial r\partial\eta} - \psi\frac{\partial}{\partial\eta}(\frac{\dot{a}}{a})\right], \tag{3.138}$$

$$d(D\mu)=\frac{1}{a^2}\left[-\frac{\frac{\partial\psi}{\partial r}r-\psi}{r^2}-\frac{1}{r}\frac{\partial\psi}{\partial\eta}+\frac{1}{2}\frac{\partial^2\psi}{\partial r^2}-\frac{1}{2}\frac{\partial^2\psi}{\partial\eta^2}-\psi\frac{\partial}{\partial\eta}(\frac{\dot{a}}{a})\right], \quad (3.139)$$

$$d(\delta\rho)=\frac{1+\psi}{\sqrt{2}a^4r}(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi})\left[2\psi\dot{a}+a\frac{\partial\psi}{\partial r}+a\frac{\partial\psi}{\partial\eta}\right], \quad (3.140)$$

$$d(\delta\alpha)=\frac{1+\psi}{\sqrt{2}a^4r}(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi})\left[\frac{-\cot\theta}{\sqrt{2}2ra}-\frac{\cot\theta\psi}{2\sqrt{2}ra}\right], \quad (3.141)$$

$$d(\overline{\delta}\mu)=\frac{1+\psi}{\sqrt{2}a^4r}(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi})\left[-\frac{\psi}{r}+\frac{1}{2}\frac{\partial\psi}{\partial r}-\frac{1}{2}\frac{\partial\psi}{\partial\eta}\right]. \quad (3.142)$$

Remark. This remark is about scalar, vector and tensor decoupling for the case of linear perturbation theory, which is a basic fact, which people used and uses in cosmology. We present here computations in coordinate-depednent approach, which should be useful for gaining some inside, how perturbations work. It is only a sketch of full computations, however should be useful.

We can take equations from [9] and we can compare them with corresponding equations in Ricci identities in [1]. We haven't to go through all the terms. It is enough just to look at the RHS and we could decide which equation from I,II and III fit to the right place. This table could be, of course, made for the case of vector and as well tensor perturbations.

Table 3.1: Correspondence between the Ricci identities in [1], equation 310 and in (13.8) - (13.10) in [9]

Equation	**Equation from Suonio**
2.	I.
3.	Identity
4.	II.
5.	II.
6.	Identity
7.	III.
8.	Identity(first order)
9.	Identity
10.	II.
11.	Identity
12.	II.
13.	III.
14.	II.
15.	I.
16.	II.
17.	Identity
18.	Identity (first order)
19.	Identity

Bibliography

[1] Chandrasekhar S., The mathematical theory of black holes (1983)

[2] Sharma S., Khanal U., Perturbations of FRW ST's in NP formalism, International Journal of Theoretical Physics, arXiv: 1109.6411v2 [astro-ph.co] 2013

[3] personal collaboration with M.Scholtz

[4] Penrose R., Rindler W., Spinors and spacetime, Volume 1

[5] Stewart J.M., Walker M., Perturbation of Spacetimes in General Relativity, 2013, Royal Society London

[6] General Relativity, Wald R.M., The University of Chicago Press, Chicago and London, 1984

[7] Matousek J., Nesetril J., Kapitoly s diskretni matematiky,

[8] Rektorys K., Prehled uzite matematiky I, II, Prometheus

[9] Suonio H.-K., Introduction to cosmological perturbation theory I

[10] Suonio H.-K., Introduction to cosmological perturbation theory II

[11] Prokopec T. Lectures in cosmology

[12] Cosmological inflation, Linde A., Inflation and Quantum Cosmology, Academic Press,1990

4. Scalar perturbations in $f(R)$-cosmology in the late universe

Now we will use the knowledge from the Appendix A and we will study scalar perturbations of the metrics for non-linear $f(R)$-models, [17], which are examples of the so called scalar-tensor theories. We will consider the universe at the late stage of its evolution and deep inside the cell of uniformity. We investigate the astrophysical approach in the case of Minkowski spacetime background and two case in the cosmological approach: the large scalaron mass approximation and the quasistatic approximation, and get explicit expressions for scalar perturbations for both these cases. The previous section will be used as a preliminary, because we can use the knowledge which is independent on the field equations.

We will consider a special class of $f(R)$ - models which have solutions R_{dS} of the equation (Appendix A)

$$F(R)R - 2f(R) = 0. \tag{4.1}$$

This equation follows from (4.51) in Appendix A for the case of the vacuum solutions for which Ricci scalar is constant. Such solutions are called de Sitter points. We can expand the function $f(R)$ in the vicinity of one of these points:

$$f(R) = f(R_{dS})+F(R_{dS})(R-R_{dS})+o(R-R_{dS}) = -f(R_{dS})+\frac{2f(R_{dS}}{R_{dS}}R+o(R-R_{dS}), \tag{4.2}$$

where we used equation (4.1). Now we suppose that parameters of the model can be chosen in such a way that

$$\frac{2f(R_{dS})}{R_{dS}} = 1 \Rightarrow f(R_{dS}) = \frac{R_{dS}}{2}. \tag{4.3}$$

Therefore we get

$$f(R) = -2\Lambda + R + o(R - R_{dS}),$$

where $\Lambda = \frac{R_{dS}}{4}$. The stability of these points was discussed in [1] and [17]. Obviously, these models go asympotically to the de Sitter space when $R \to R_{dS} \neq 0$ with a cosmological constant $\Lambda = \frac{R_{dS}}{4}$. This happens when the matter content becomes negligible with respect to Λ as it is the case with late Friedmann-Lemaitre-Robertson-Walker cosmology. We can also consider a zero solution $R_{dS} = 0$ of equation (4.1) It is correct to call these points a Minkowski one. Here, $\Lambda = 0$ and such models go asymptotically to the Minkowski space. In particular, three popular models, Starobinsky, Hu-Sawicky and MJWQ ([20], [21], [22]), have stable de Sitter points in the future (approximately at the redshift $z = -1$).

We have basic Friedmann equations in case of $f(R)$-theories in the Appendix A, (4.55). They describe homogeneous background. We consider the universe at late stages of its evolution, when galaxies and cluster of galaxies have already formed and when the universe is highly inhomogeneous inside the cell of uniformity, which is approximately 150 Mpc in size. These inhomogeneities perturb the homogeneous background. At scales larger than the cell of the uniformity, the matter fields are well described by the hydrodynamical approach. On the smaller scales is the mechanical approach more adequate. In the mechanical approach, galaxies, dwarf galaxies, and clusters of galaxies (composed of baryonic and dark matter) can be considered as separate compact objects. Moreover, at distances much greater than their characteristic sizes they can be well described as point-like matter sources with the rest mass density

$$\rho = \frac{1}{a^3}\sum_i m_i \delta(\vec{r} - \vec{r_i}) \equiv \frac{\rho_c}{a^3} \tag{4.4}$$

where $\vec{r_i}$ is the radius-vector of the i-th gravitating mass in the comoving coordinates. This is the generalization of the well known astrophysical approach to the case of dynamical cosmological background, [14]. Usually, the gravitational fields of these inhomogeneities are weak and their peculiar velocities are much less than the speed of light. All these inhomogeneities result in scalar perturbations of the FLRW metrics. In the conformal Newtonian gauge such perturbed metrics are

$$ds^2 = -(1+2\Phi)dt^2 + a^2(1-2\Psi)(dx^2+dy^2+dz^2), \tag{4.5}$$

where scalar perturbations $\Phi, \Psi << 1$. The smallness of non-relativistic gravitational potentials Φ and Ψ and the smallness of peculiar velocities are two independent conditions. We will split the investigation of galaxy dynamics into two steps. First we neglect the peculiar velocities and we define gravitational potential Φ. Then we use this potential to determine dynamical behavior of galaxies. This enables us to take into account both the gravitational attraction between inhomogeneities and the global cosmological expansion of the universe. The case $f(R) = R$ was already investigated in [3]. This result is devoted to the first step in the program. We are going to define scalar perturbations Φ and Ψ for the $f(R)$ gravitational models.

Under our assumptions and according to [17], these perturbations satisfy the following system of equations:

$$-\frac{\Delta\Psi}{a^2} + 3H\left(H\Phi + \dot{\Psi}\right) = -\frac{1}{2F}[\left(3H^2 + 3\dot{H} + \frac{\Delta}{a^2}\right)\delta F - 3H\dot{\delta F} + \\ +3H\dot{F}\Phi + 3\dot{F}\left(H\Phi + \dot{\Psi}\right) + \kappa^2\delta\rho], \tag{4.6}$$

$$H\Phi + \dot{\Psi} = \frac{1}{2F}\left(\dot{\delta F} - H\delta F - \dot{F}\Phi\right), \tag{4.7}$$

$$-F(\Phi - \Psi) = \delta F\,, \tag{4.8}$$

$$3(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Phi}) + 6H(H\Phi + \dot{\Psi}) + 3\dot{H}\Phi + \frac{\nabla\Phi}{a^2} = \frac{1}{2F}[3\delta\ddot{F} + 3H\delta\dot{F} - $$
$$-6H^2\delta F - \frac{\Delta\delta F}{a^2} - 3\dot{F}\dot{\Phi} - 3\dot{F}(H\Phi + \dot{\Psi}) - (3H\dot{F} + 6\ddot{F})\Phi + \kappa^2\delta\rho], \tag{4.9}$$

$$\delta\ddot{F} + 3H\delta\dot{F} - \frac{\Delta\delta F}{a^2} - \frac{1}{3}R\delta F = \frac{1}{3}\kappa^2(\delta\rho - 3\delta P) + \dot{F}(3H\Phi + 3\dot{\Psi} + \dot{\Phi}) +$$
$$+2\ddot{F}\Phi + 3H\dot{F}\Phi - \frac{1}{3}F\delta R, \tag{4.10}$$

$$\delta R = -2\left[3\left(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}\right) + 12H\left(H\Phi + \dot{\Psi}\right) + \frac{\Delta\Phi}{a^2} + 3\dot{H}\Phi - 2\frac{\Delta\Psi}{a^2}\right],$$

$$\delta F = F'\delta R. \tag{4.11}$$

In these equations, the function F, its derivative $F^{'}$ and the scalar curvature R are unperturbed background quantities. Here, Δ is a Laplacian in the comoving coordinates. As a matter source, we consider dust like matter. Therefore $\delta P = 0$ and

$$\delta\rho = \rho - \overline{\rho} = \frac{(\rho_c - \overline{\rho}_c)}{a^3}, \tag{4.12}$$

where $\overline{\rho}$ and ρ are defined in previous text.

It can be easily verified that in the linear case $f(R) = R \Rightarrow F(R) = 1$, this system of equations is reduced to equations (2.18) - (2.20) in [3]. Now we will consider previous equations (4.6) - (4.11) in the astrophysical approach. This means that we neglect the time dependence of functions in these equations by setting all time derivatives equal to zero. It is supposed also that the background model is matter-free, i.e. $\overline{\rho} = 0$. There are two types of vacuum background solutions of the equation (4.51): de Sitter spacetime with $R_{dS} = 12H^2 =$ const. $\neq 0$ and Minkowski spacetime with $R = 0$ and $H = 0$. However the system of equations was obtained for FLRW metrics, where we explicitly took into account the dependence of the scale factor a on time. Therefore if we want to get the time independent astrophysical equations directly from (4.6-4.11), we should also neglect the time dependence of a, the background parameter $H = 0$. This means that the background solution is the Minkowski spacetime. This background is perturbed by dust-like matter with the rest mass density, (4.4). Keeping in mind that $\overline{\rho} = 0$ we have $\delta\rho = \rho$.

In the case of Minkowski background and dropping the time derivatives, equations (4.6-4.11) in the astrophysical approach are reduced

to the following system:

$$-\frac{\Delta}{a^2}\Psi = -\frac{1}{2F}\left(\frac{\Delta}{a^2}\delta F + \kappa^2\delta\rho\right)\,, \tag{4.13}$$

$$-F(\Phi - \Psi) = \delta F\,, \tag{4.14}$$

$$\frac{\Delta}{a^2}\Phi = \frac{1}{2F}\left(-\frac{\Delta}{a^2}\delta F + \kappa^2\delta\rho\right)\,, \tag{4.15}$$

$$-\frac{\Delta}{a^2}\delta F = \frac{1}{3}\kappa^2\delta\rho - \frac{1}{3}F\delta R\,, \tag{4.16}$$

$$\delta F = F'\delta R, \quad \delta R = -2\left(\frac{\Delta}{a^2}\Phi - 2\frac{\Delta}{a^2}\Psi\right), \tag{4.17}$$

From (4.13) and (4.15) we obtain respectively

$$\Psi = \frac{1}{2F}\delta F + \frac{\varphi}{a} = \frac{F'}{2F}\delta R + \frac{\varphi}{a}, \quad \Phi = -\frac{1}{2F}\delta F + \frac{\varphi}{a} = -\frac{F'}{2F}\delta R + \frac{\varphi}{a}\,, \tag{4.18}$$

where the function φ satisfies the equation

$$\Delta\varphi = \frac{1}{2F}\kappa^2 a^3\delta\rho = \frac{1}{2F}\kappa^2\delta\rho_c = 4\pi G_N\delta\rho_c, \quad G_N = \frac{\kappa^2}{8\pi F}. \tag{4.19}$$

Here we took into consideration that in the astrophysical approach $\delta\rho_c = \rho_c$ where ρ_c is defined by (4.4). It is worth noting that in the Poisson equation the Newtonian gravitational constant G_N is replaced by an effective one $G_{eff} = G_N/F$.

Equation (4.14) follows directly from (4.18) and consequently, may be dropped, while from (4.16) we get the following Helmholtz equation with respect to the scalaron function δR :

$$\Delta\delta R + \frac{a^2}{3}\left(R - \frac{F}{F'}\right)\delta R = -\frac{a^2}{3F'}\kappa^2\delta\rho \tag{4.20}$$

On the other hand, it can be easily seen that the substitution of equations (4.18) and (4.19) into (4.17) results in the same equation (4.20). Therefore, in the case of Minkowski background, the mass squared of the scalaron is

$$M^2 = \frac{a^2}{3}\frac{F}{F'}. \tag{4.21}$$

Now we want to take into consideration cosmological evolution. This means that the background functions may depend on time. In this case, it is hardly possible to solve the system directly. Therefore, first we study the case of very large mass of the scalaron. It should be noted also that we investigate the universe filled with nonrelativistic matter with the rest mass density $\rho \sim \frac{1}{a^3}$. Hence we will drop all terms which decrease (with increasing a) faster than $\frac{1}{a^3}$. This is the accuracy of our approach. Within this approach, $\delta\rho \sim \frac{1}{a^3}$, [3].

4.1 Large scalaron mass

As we can see from equation (4.4), the limit of large scalaron mass corresponds to $F^{'} \to 0$. Then δF is also negligible. Therefore, equations (4.6)-(4.11) read

$$-\frac{\Delta\Psi}{a^2} + 3H\left(H\Phi + \dot{\Psi}\right) = -\frac{1}{2F}\left[3H\dot{F}\Phi + 3\dot{F}\left(H\Phi + \dot{\Psi}\right)\right], \tag{4.22}$$

$$H\Phi + \dot{\Psi} = \frac{1}{2F}\left(-\dot{F}\Phi\right), \tag{4.23}$$

$$\Phi - \Psi = 0\,, \tag{4.24}$$

$$3\left(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}\right) + 6H\left(H\Phi + \dot{\Psi}\right) + 3\dot{H}\Phi + \frac{\Delta\Phi}{a^2} = \frac{1}{2F}\left[-3\dot{F}\dot{\Phi} - 3\dot{F}\left(H\Phi + \dot{\Psi}\right) - \left(3H\dot{F} + 6\ddot{F}\right)\Phi\right], \tag{4.25}$$

$$0 = \dot{F}(3H\Phi + 3\dot{\Psi} + \dot{\Phi}) + 2\ddot{F}\Phi + 3H\dot{F}\Phi\,, \tag{4.26}$$

$$0 = 3\left(\dot{H}\Phi + H\dot{\Phi} + \ddot{\Psi}\right) + 12H\left(H\Phi + \dot{\Psi}\right) + \frac{\Delta\Phi}{a^2} + 3\dot{H}\Phi - 2\frac{\Delta\Psi}{a^2} \tag{4.27}$$

From (4.23) and (4.24) we get

$$\Psi = \Phi = \frac{\varphi}{a\sqrt{F}}, \tag{4.28}$$

where the introduced function φ depends only on spatial coordinates. Substituting (4.28) into (4.22), we obtain

$$\frac{1}{a^3\sqrt{F}}\Delta\varphi + \frac{3\dot{F}^2\varphi}{4aF^2\sqrt{F}} = \frac{1}{2F}\kappa^2\delta\rho \tag{4.29}$$

As we mentioned above, neglecting relativistic matter in the late universe we have $\delta\rho \sim \frac{1}{a^3}$([3]). This approximation is getting better and better performed in the limit $a \to \infty$. We assume that this limit corresponds to the final stage of universe evolution. The similar limit with respect to the scalar curvature is $R \to R_\infty$, where the value R_∞ is just finite. Then from (4.29) we immediately come to the condition

$$F = const. + o(1), \tag{4.30}$$

where $o(1)$ is decreasing function of a. This condition holds at the considered late stage. One can naively suppose that in the late universe $\dot{F} \approx \frac{1}{a} + o(\frac{1}{a})$. However this is wrong. Obviously, without loss of generality, we can suppose that $const. = 1$. From the condition (4.30) we get

$$F = 1 + o(1) \Rightarrow f = -2\Lambda + R + o(R - R_\infty), \tag{4.31}$$

where Λ is the cosmological constant. Therefore the limit of the large scalaron mass takes place for models which possess the asymptotic form of (4.31). For example, R_{∞} may correspond to the de Sitter point R_{dS} in future. All three popular models, Starobinsky, Hu-Sawicky and MJWQ [20],[21], [22] have such stable de-Sitter points in the future (approximately at the redshift $z = -1$) ([18],[19]). The condition of stability is $0 < \frac{RF'}{F} < 1$. Since $F \approx 1$, this condition reads $0 < R < \frac{1}{F'}$, which is fulfilled for the de Sitter points in the above-mentioned models. The reason of it consists in the smallness of F'.
We now return to the remaining equations (4.25) - (4.27) to show that they are satisfied within the considered accuracy. First, we study (4.25) which after the substitution of (4.28) and (4.29) and some simple algebra takes the form

$$\frac{\varphi \dot{H}}{a} - \frac{\varphi}{2aF}(H\dot{F} - \ddot{F}) = 0 \tag{4.32}$$

To estimate $\dot{F}$ and $\ddot{F}$, we take into account that in the limit $R \to R_{\infty}$, $F \approx 1$, $H \approx const. \Rightarrow \dot{H} \approx 0$, and $F'(R_{\infty})$ is some finite positive value. Then,

$$\dot{F} = F'\dot{R} \approx F'(R_{\infty})\dot{R} \approx \dot{T} \approx d(1/a^3)/dt \approx H(1/a^3) \approx 1/a^3$$

and $\ddot{F} \approx \dot{a}/a^4 \approx \frac{1}{a^3}$. Therefore, the LHS of equation (4.32) is of order $o(1/a^3)$ and we can put it zero within the accuracy of our approach. Similarly, equations (4.26) and (4.27) are satisfied within the considered accuracy. It can be also seen that the second term on the left hand side of equation (4.29) is of order $O(1/a^7)$ and should be eliminated. Thus, in the case of the large enough scalaron mass we reproduce the linear cosmology from the nonlinear one, as it should be.

4.2 Quasi-static approximation

Now we do not want to assume a priori that the scalaron mass is large, i.e. F' can have arbitrary values. Hence, we will preserve the δF terms in equations (4.6) - (4.11). Moreover, we should keep the time derivatives in these equations. Such a system is very complicated for direct integration. However, we can investigate it in the quasistatic approximation. According to this approximation, the spatial derivatives give the main contribution to equations (4.6)-(4.11), ([24], [23]). Therefore, first, we should solve "astrophysical" equations (4.13)-(4.17), and then check whether the found solutions satisfy (up to the adopted accuracy) the full system of equations. In the other words, in the quasi-static approximation it is naturally supposed that the gravitational potentials (the functions Φ, Ψ) are produced mainly by the spatial distribution of astrophysical/cosmological bodies. As we have seen, equations (4.13) - (4.17) result in (4.18) - (4.20). Now, we should keep in mind that we have the cosmological background. Moreover, we consider the late

universe which is not far from the de Sitter point R_{dS} in future. This means that $\delta\rho = \rho - \overline{\rho}$ in (4.19), all background quantities are calculated roughly speaking at R_{dS} and the scalaron mass squared (4.4) reads now

$$M^2 = \frac{a^2}{3}(\frac{F}{F'} - R_{dS}) \qquad (4.33)$$

Let us consider now equation (4.20) with the mass squared (4.33). Taking into account that now $\delta\rho_c = \rho_c - \overline{\rho_c}$, we can rewrite this equation as follows:

$$\Delta\widetilde{\delta R} - M^2\widetilde{\delta R} + \frac{a^2}{3F'}\frac{\kappa^2}{a^3}\sum_i m_i\delta(\vec{r} - \vec{r}_i) = 0, \qquad (4.34)$$

where

$$\widetilde{\delta R} = \delta R + \frac{\kappa^2}{(F - F'R_{dS})a^3}\kappa^2\rho_c.$$

Then, the general solution for a full system is the sum over all gravitating masses. As a boundary conditions, we require for each gravitating mass the behavior $\delta R \sim \frac{1}{r}$ at small distances r and $\widetilde{R} \to 0$ for $r \to \infty$. Taking all these remarks into consideration, we obtain for the full system

$$\delta R = \frac{\kappa^2}{12\pi aF'}\sum_i \frac{m_i \exp(-M_i|\vec{r} - \vec{r}_i|)}{|\vec{r} - \vec{r}_i|} - \frac{\kappa^2\overline{\rho_c}}{(F - F'R_{dS})a^3}. \qquad (4.35)$$

It is worth noting that averaging over the whole co-moving spatial volume V gives the zero value $\overline{\delta R}$ .This result is reasonable because the rest mass density fluctuation $\delta\rho$, representing the source of the metric and the scalar curvature fluctuations Φ, Ψ and $\delta\rho$, has a zero average value $\overline{\delta\rho} = 0$. Consequently, all enumerated quantities should also have zero average values : $\overline{\Phi} = \overline{\Psi} = 0$ and $\overline{\delta R} = 0$, in agreement with (4.35). From equation (4.18) we get the scalar perturbation functions Φ and Ψ in the following form:

$$\Psi = \frac{F'}{2F}[\frac{\kappa^2}{12\pi F'}\sum_i \frac{m_i \exp(-M|\vec{r} - \vec{r}_i|)}{|\vec{r} - \vec{r}_i|} - \frac{\kappa^2}{(F - F'R_{dS})a^3}\overline{\rho_c}] + \frac{\varphi}{a} \qquad (4.36)$$

$$\Phi = \frac{-F'}{2F}[\frac{\kappa^2}{12\pi F'}\sum_i \frac{m_i \exp(-M|\vec{r} - \vec{r}_i|)}{|\vec{r} - \vec{r}_i|} - \frac{\kappa^2}{(F - F'R_{dS})a^3}\overline{\rho_c}] + \frac{\varphi}{a} \qquad (4.37)$$

where φ satisfies equation (4.19) with $\delta\rho$ in the form (4.12) (i.e., $\overline{\rho}_c \neq 0$). Obviously when $F' \to 0$, $M \to \infty$, and we have $\exp(-M|\vec{r} - \vec{r}_i|)/|\vec{r} - \vec{r}_i| \to 4\pi\delta(\vec{r} - \vec{r}_i)/M^2$, so the expression in the square brackets in (4.36) and (4.37) is equal to $\kappa^2\delta\rho_c/[(F - F'R_{dS})a^3]$. Therefore, in the considered limit $F' \to 0$ we reproduce the scalar perturbations Φ, Ψ from the previous large scalaron mass case, as it certainly should be.
Thus neglecting for a moment the influence of the cosmological background, but not neglecting the scalaron's constribution, we have found

the scalar perturbations. They represent the mix of the standard potential $\frac{\varphi}{a}$ (see the linear case [3]) and the additional Yukawa term which follows from the nonlinearity.
Now we should check that these solutions satisfy the full system (4.6)-(4.11). To do it, we substitute (4.35), (4.36) and (4.37) into this system of equations. Obviously the spatial derivatives disappear. Keeping in mind this fact the system (4.6)-(4.11) is reduced to the following equations:

$$3H\left(H\Phi+\dot{\Psi}\right)=-\frac{1}{2F}\left[\left(3H^2+3\dot{H}+\frac{\Delta}{a^2}\right)\delta F-3H\dot{\delta F}+3H\dot{F}\Phi +3\dot{F}\left(H\Phi+\dot{\Psi}\right)\right], \quad (4.38)$$

$$H\Phi+\dot{\Psi}=\frac{1}{2F}\left(\dot{\delta F}-H\delta F-\dot{F}\Phi\right), \quad (4.39)$$

$$3\left(\dot{H}\Phi+H\dot{\Phi}+\ddot{\Psi}\right)+6H\left(H\Phi+\dot{\Psi}\right)+3\dot{H}\Phi+\frac{\Delta\Phi}{a^2}=\frac{1}{2F}[3\ddot{\delta F} +3H\dot{\delta F}-6H^2\delta F-\frac{\Delta\delta F}{a^2}-3\dot{F}\dot{\Phi}-3\dot{F}\left(H\Phi+\dot{\Psi}\right)-\left(3H\dot{F}+6\ddot{F}\right)\Phi], \quad (4.40)$$

$$\ddot{\delta F}+3H\dot{\delta F}-\frac{\Delta\delta F}{a^2}=\dot{F}(3H\Phi+3\dot{\Psi}+\dot{\Phi})+2\ddot{F}\Phi+3H\dot{F}\Phi\,, \quad (4.41)$$

$$\delta F=F'\delta R, \quad (4.42)$$

$$\frac{F'}{F}R_{dS}\delta R=-2\left[3\left(\dot{H}\Phi+H\dot{\Phi}+\ddot{\Psi}\right)+12H\left(H\Phi+\dot{\Psi}\right)+\frac{\Delta\Phi}{a^2} +3\dot{H}\Phi-2\frac{\Delta\Psi}{a^2}\right]. \quad (4.43)$$

Here the term $\frac{F'}{F}R_{dS}\delta R$ in the left hand side of (4.43) disappear due to the redefinition of the scalaron mass squared (4.33).
It can be easily seen that all terms in (4.35), (4.36) and (4.37) depend on time, and therefore may contribute to equations (4.38)-(4.43). As we wrote above, according to our nonrelativistic approach, we neglect the terms of the order $o(1/a^3)$. On the other hand, exponential functions decrease faster than any power function. Moreover, we can write the exponential term in (4.35) as follows:

$$\frac{\kappa^2}{12\pi F'}\sum_i\frac{m_i\exp(-\sqrt{\frac{1}{3}(\frac{F}{F'}-R_{dS})}|r_{ph}-r_{ph,i}|)}{|r_{ph}-r_{ph,i}|} \quad (4.44)$$

where we introduced the physical distance $r_{ph} = ar$. It is well known that astrophysical tests impose strong restrictions on the non-linearity [15, 16] (the local gravity tests impose even stronger constraints ma[10, 15, 16]). According to these constraints, (4.44) should be small at the astrophysical scales. Consequently, on the cosmological scales it will be even much smaller. So we will not take into account the exponential terms in the above equations. However, in (4.35), (4.36) and (4.37), we have also $\frac{1}{a^3}$ and $\frac{1}{a}$ terms which we should examine. Before performing this, it should be recalled that we consider the late universe which is rather close to the de Sittter point. Therefore, as we already noted in the previous subsection, $F \approx 1$, $H \approx$ const. $\to \dot{H} \approx 0$, $R_{dS} = 12H^2$ and $F^{'}(R_{dS})$ is some finite positive value. Additionaly, $\dot{F}, \ddot{F}, \dot{F}^{'} \approx \frac{1}{a^3}$. Hence, all terms of the form of $\dot{F}, \ddot{F}, \dot{F}^{'} \times \Phi, \Psi, \dot{\Phi}, \dot{\Psi}$ are of the order $o(1/a^3)$ and should be dropped. In other words, the functions F and $F^{'}$ can be considered as time independent.
First, let us consider the terms $\Psi = \Phi = \varphi/a$ in equations (4.36) and (4.37) and substitute them into equations (4.38) - (4.43). Such $1/a$ term is absent in δR. So we should put $\delta R = 0$, $\delta F = 0$. Obviously, this is the linear theory case. It can be easily seen that all equations are satisfied.
Now, we study the terms $\sim 1/a^3$, i.e.,

$$\delta R = -\frac{\kappa^2}{(F - F' R_{dS})}\frac{\overline{\rho}_c}{a^3}\ \Psi = -\frac{\kappa^2 F^{'}}{2F(F - F' R_{dS})}\frac{\overline{\rho}_c}{a^3}\ \Phi$$
$$= \frac{\kappa^2 F^{'}}{2F(F - F' R_{dS})}\frac{\overline{\rho}_c}{a^3} \qquad (4.45)$$

Let us examine, for example equation (4.38). Keeping in mind that $\delta F = F^{'}\delta R$, one can easily get

$$12H_c^2\frac{\kappa^2 F^{'}}{2F(F - F' R_{dS})}\frac{\overline{\rho}_c}{a^3} = 12H_c^2\frac{\kappa^2 F^{'}}{2F(F - F' R_{dS})}\frac{\overline{\rho}_c}{a^3} + o(1/a^3) \qquad (4.46)$$

Therefore, the terms $\sim \frac{1}{a^3}$ exactly cancel each other, and this equation is satisfied up to the adopted accuracy $o(1/a^3)$. One can easily show that the remaining equations are fulfilled with the same accuracy.

Thus we have proved that the scalar perturbation functions Φ and Ψ in the form (4.36) and (4.37) satisfy the system of equations (4.38)-(4.43) with the required accuracy. Both of these functions contain the nonlinearity function F and the scale factor a. Therefore both the effects of nonlinearity and the dynamics of the cosmological background are taken into account. The function Φ corresponds to the gravitational potential of the system of inhomogeneities. Hence we can study the dynamical behavior of the inhomogeneities including into consideration their gravitational attraction and cosmological expansion, and also taking into account the effects of nonlinearity. For example, the non-relativistic Lagrange function for a test body of the mass m in the gravitational field described by the metric (4.5) has the form ([3]):

$$L \approx -m\Phi + \frac{ma^2\vec{v}^2}{2},\ \vec{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

We can use this Lagrange function for analytical and numerical study of mutual motion of galaxies. In the case of the linear theory, such investigation was performed, e.g., in [6].

We will make a small conclusion: we have studied scalar perturbations of the metrics in nonlinear $f(R)$ gravity. The universe has been considered at the late stage of its evolution and at scales much less than the cell of uniformity size which is approximately 150 Mpc. At such distances, our universe is highly inhomogeneous, and the averaged hydrodynamic approach does not work here. We need to take into account the inhomogeneities in the form of galaxies, groups and clusters of galaxies. The peculiar velocities of these inhomogeneities are much less than the speed of light, and we can use the nonrelativistic approximation. This means that in equations for scalar perturbations, we first neglect peculiar velocities and solve these equations with respect to scalar perturbation functions Φ and Ψ. The function Φ represents the gravitational potential of inhomogeneities. Then we use the explicit expression for Φ to describe the motion of inhomogeneities. Such mechanical approach is well known in astrophysics ([14]). We generalized it to the case of dynamical cosmological background ([3],[4]). The main objective of this work was to find explicit expressions for Φ and Ψ in the framework of nonlinear $f(R)$ models. Unfortunately, in the case of nonlinearity, the system of equations for scalar perturbations is very complicated. It is hardly possible to solve it directly. Therefore, we have considered the following approximations: the astrophysical approach; the large scalaron mass case and quasistatic approximation. in all three cases, we found the explicit expressions for the scalar perturbation functions Φ and Ψ up to the required accuracy. The latter means that, because we considered nonrelativistic matter with the averaged rest mass density $\rho \sim \frac{1}{a^3}$, all quantities in the cosmological approximation are also calculated up to corresponding orders of $\frac{1}{a}$. It should be noted that in the cosmological approach our consideration is valid for nonlinear models where functions of $f(R)$ have the stable de Sitter points in the future with respect to the present time, and the closer to R_{dS} we are, the most correct our approximation is. All three popular models, Starobinsky, Hu-Sawicky, and Miranda, have such stable de Sitter points in the future.
The quasi-static approximation is of most interest from the point of view of the large scale structure investigations. Here, the gravitational potential Φ contains both the nonlinearity function F and the scale factor a. Hence we can study the dynamical behavior of the inhomogeneities including into consideration their gravitational attraction and the cosmological expansion, and also taking into account the effect of nonlinearity. All this make it possible to carry out the numerical and analytical analysis of the large scale structure dynamics in the late universe for $f(R)$ models as was done in [6] for the case of standard general relativity.

What we suggest is the following numerical scheme. Because the

generalized Friedmann equation is of the third order (it is formally possible to write it followingly: $x^{'''} = F(x, x^{'}, x^{''})$, we could rewrite it as a system of three ordinary differential equations of the first order and we make the following substitution

$$\dot{x_n} = y_n, \tag{4.47}$$

$$\dot{y_n} = z_n, \tag{4.48}$$

$$\dot{z_n} = F(x_n, y_n, z_n), \tag{4.49}$$

where we plug for the function F fom the RHS of equation (51) in article [25] and R_{HS} is the Hu-Sawicky function. We could use, for example, the explicit Euler method and we could be inspired by previous works in simulations in f(R)-gravities.

4.3 Appendix A: Basic facts from $f(R)$ - cosmology

We have the field equations in $f(R)$-theories

$$\Sigma_{\mu\nu} \equiv F(R)g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Box F g_{\mu\nu} - \nabla_\mu \nabla_\nu F = \kappa^2 T^M_{\mu\nu} \tag{4.50}$$

$T^M_{\mu\nu}$ is again the energy momentum tensor defined by the variational derivative of L_M in terms of $g^{\mu\nu}$:

$$T^M_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta L_M}{\delta g^{\mu\nu}}$$

This tensor satisfies the continuity equation, as well as $\Sigma_{\mu\nu}$, $\nabla^\mu \Sigma_{\mu\nu} = 0$. Now, Einstein gravity without the cosmological constant corresponds to $f(R) = R$ and $F(R) = 1$, so that the term $\Box F$ in

$$3\Box F - FR - 2f = \kappa^2 T \tag{4.51}$$

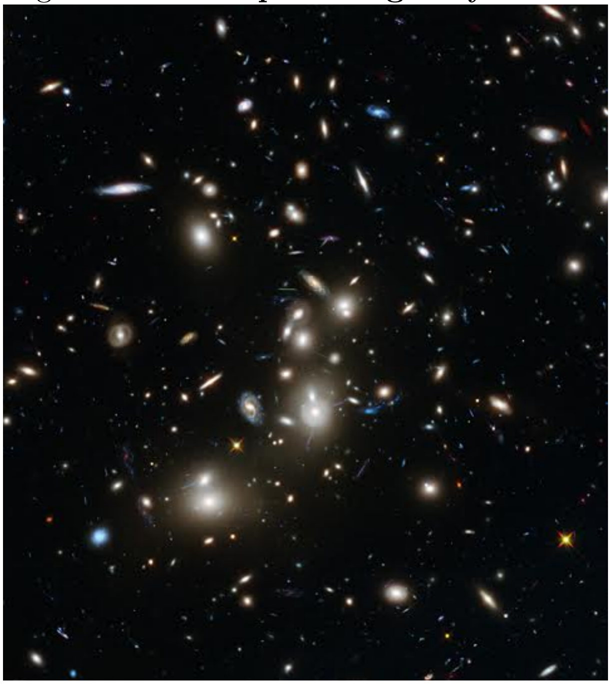
vanishes. In this case we have $R = -\kappa^2 T$ and hence the Ricci scalar is directly determined by the matter. In modified gravity the term $\Box F$ does not vanish which means there is a propagating scalar degree of freedom, $\phi = F(R)$. The trace equation determines the dynamics of the scalar field. Again,

$$G_{\mu\nu} = \kappa^2 (T^M_{\mu\nu} + T^{eff}_{\mu\nu}), \tag{4.52}$$

$$\kappa^2 T^{eff}_{\mu\nu} \equiv g_{\mu\nu} \frac{(f - R)}{2} + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F + (1 - F) R_{\mu\nu}. \tag{4.53}$$

Since $\nabla^\mu G_{\mu\nu} = 0$ and $T^M_{\mu\nu} = 0$, then $\nabla^\mu T^{eff}_{\mu\nu} = 0$. Hence the continuity equation holds not only for $\Sigma_{\mu\nu}$, but also for the effective $T^{eff}_{\mu\nu}$! We

Figure 4.1: **Example of a galaxy cluster**



consider only $f(R)$ -theories which admit de Sitter points: these points corresponds to vacuum solutions at which is the Ricci scalar constant. So,

$$RF - 2f(R) = 0. \tag{4.54}$$

We need the Sitter points, because we want to model inflation and accelerated expansion of our universe.
In the case of $f(R)$-cosmologies we get again two - but more complicated - Friedmann equations:

$$3FH^2 = \frac{FR - f}{2} - 3H\dot{F} + \kappa^2 \rho_M \tag{4.55}$$

$$-2F\dot{H} = \ddot{F} - H\dot{F} + \kappa^2(\rho_M + p_M), \tag{4.56}$$

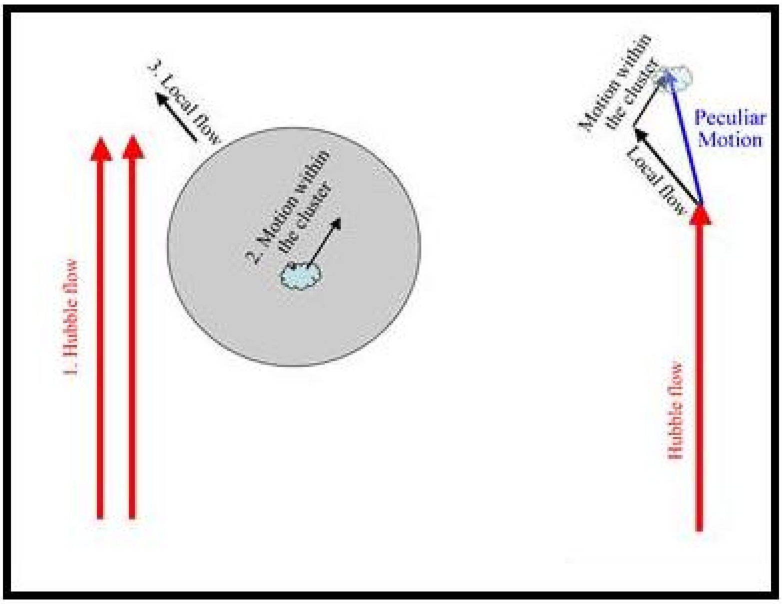
plus the continuity equation $\dot{\rho_M} + 3H(\rho_M + p_M) = 0$. We have again that the first equation with the continuity equation imply the second equation. But the steps are different than in the case of standard general realativity.

4.4 Appendix B: Hubble flows in observable universe

We could see at the previous picture one example of a galaxy cluster: Abell[1] 2744 galaxy cluster; As we all know galaxies are grouped into larger units called clusters and superclusters. But there are not any

[1]George Ogden Abell (1927-1983) was an american astronom. Abell's catalog is a list of approximately 4000 groups of galaxies, which have at least 30 members.

Figure 4.2: **Local Hubble flow**



bigger structures. This means that universe starts to be homogeneous and isotropic on scales bigger than 150 Mpc and is well described by the modern realization of Friedmann models, so called ΛCDM model. One of the characteristics of this model is linear velocity-distance relation between receding motion of galaxies due to the expansion of the universe, so called Hubble flow.
We can make a rough estimate, where the gravitational attraction prevails the cosmological expansion. If we plug v 300 km/s (peculiar velocities) and $H \approx 70km/s.Mpc$, we get a rough estimate 3-6 Mpc for our local group of galaxies. From this point of view it seems reasonable that Edwin Hubble observed the flow on distances 10 -30 Mpc.

But recent observations indicate the presence of Hubble flows on distances of few Mpc from the center of our group of galaxies. And we needed any theoretical substantiation for this result. There was a suggestion that the cosmological constant is responsible for this local Hubble flow, but the answer is no! The global cosmological exapansion is responsible for local cold flow, but there is a less diffusion in the

presence of cosmological constant. ([3])

4.5 Appendix C: Boundary terms in General relativity and f(R)-theories

We start with the action integral

$$S_{EH} = \int \sqrt{-g}(R - 2\kappa L_F)\ d^4x, \tag{4.57}$$

where L_F are an other fields, and demand its variation to be zero. So

$$\delta \int \sqrt{-g}R\ d^4x = \int \delta(\sqrt{-g}g^{\mu\nu})R_{\mu\nu}\ d^4x + \int \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\ d^4x \tag{4.58}$$

Now we need an equality which holds for all regular matrices A

$$\partial_\alpha A = -AA_{\mu\nu}\partial_\alpha A^{\mu\nu} \tag{4.59}$$

and which could be proven either by a help of the equation

$$\det A = \exp(\operatorname{tr}\log A)$$

or directly from definition of determinant. So we get for a metric $g_{\mu\nu}$ that

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{4.60}$$

The standard result is

$$\delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha{}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\alpha{}_{\mu\alpha}). \tag{4.61}$$

But this is a tensorial equation so it is valid for all coordinate systems. Consequently, we have for the second integral of (4.58) that

$$\int \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\ d^4x = \int \sqrt{-g}\ \nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\beta{}_{\mu\beta})\ d^4x \tag{4.62}$$

But this is zero according to Gauss's theorem, when we impose the suplementary condition $\delta(\partial_\mu g_{\rho\nu})|_{\partial\Sigma} = 0$
The second integral gives

$$\int \delta(\sqrt{-g}g^{\mu\nu})R_{\mu\nu}\ d^4x = \int \sqrt{-g}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu}\ d^4x \tag{4.63}$$

But we have also matter-term in (4.57), which gives us

$$\delta \int \sqrt{-g}L_F\ d^4x = -\frac{1}{2}\int T_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}\ d^4x, \tag{4.64}$$

where $T_{\mu\nu}$ is the energy-momentum tensor which is given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}[(\frac{\partial(\sqrt{-g}L_F)}{\partial g^{\mu\nu}{}_{,\alpha}})_{,\alpha} - \frac{\partial(\sqrt{-g}L_F)}{\partial g^{\mu\nu}}] \tag{4.65}$$

So, we reproduce finally the familiar Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{4.66}$$

We will get much more complex equation in case of the so called $f(R)$-gravities which we will explore latter. We saw that the boundary term disappeared, when we impose $\delta(\partial_\rho g_{\mu\nu})|_{\partial\Sigma} = 0$. What will happen when we drop this condition?
Then the variation of the metric derivatives no longer vanish on the boundary - we have a non-vanishing boundary term. We obtain no longer the Einstein equations in this case. Therefore we amend the action with the new Gibbons-York-Hawking term in order to fix this problem:

$$S = S_{EH} + S_B,$$

where S_{EH} is the Einstein-Hilbert action and S_B is the boundary term. A variation of this boundary term is equal to:

$$\delta S_B = \int \sqrt{-g}\nabla_\nu(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\sigma} - g^{\mu\sigma}\delta\Gamma^\nu_{\mu\sigma})\ d^4x \tag{4.67}$$

We will make a notation:

$$V^\nu = g^{\mu\nu}\delta\Gamma^\sigma_{\mu\sigma} - g^{\mu\sigma}\delta\Gamma^\nu_{\mu\sigma} \tag{4.68}$$

And so we can rewrite

$$\delta S_B = \int \sqrt{-g}\nabla_\nu V^\nu\ d^4x \tag{4.69}$$

We use the following Christoffel symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{4.70}$$

We have immediately

$$\begin{aligned}\delta\Gamma^\sigma_{\mu\nu} &= \delta\{\frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})\}\\ &= \frac{1}{2}\delta g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) + \frac{1}{2}g^{\sigma\lambda}\{\partial_\mu(\delta g_{\lambda\nu}) + \partial_\nu(\delta g_{\mu\lambda}) - \partial_\lambda(\delta g_{\mu\nu})\}\end{aligned} \tag{4.71}$$

Because of the boundary conditions $\delta g_{\mu\nu} = \delta g^{\mu\nu} = 0$ we see from (4.71)

$$\delta\Gamma^\sigma_{\mu\nu}|_{\partial\Sigma} = \frac{1}{2}g^{\sigma\lambda}\{\partial_\mu(\delta g_{\lambda\nu}) + \partial_\nu(\delta g_{\mu\lambda}) - \partial_\lambda(\delta g_{\mu\nu})\} \tag{4.72}$$

We can use this for computation of $V^\nu|_{\partial\Sigma}$, (4.68):

$$\begin{aligned}V^\nu|_{\partial\Sigma} &= \frac{1}{2}g^{\mu\nu}g^{\sigma\lambda}\{\partial_\mu(\delta g_{\lambda\sigma}) + \partial_\sigma(\delta g_{\mu\lambda}) - \partial_\lambda(\delta g_{\mu\sigma})\} - \frac{1}{2}g^{\mu\sigma}g^{\nu\lambda}\{\partial_\mu(\delta g_{\lambda\sigma})\\ &+\partial_\sigma(\delta g_{\mu\lambda}) - \partial_\lambda(\delta g_{\mu\sigma})\} = \frac{1}{2}g^{\mu\nu}g^{\sigma\lambda}\partial_\mu(\delta g_{\lambda\sigma}) - \frac{1}{2}g^{\mu\sigma}g^{\nu\lambda}(\ \partial_\mu(\delta g_{\lambda\sigma})\\ &+\partial_\sigma(\delta g_{\mu\lambda}) - \partial_\lambda(\delta g_{\mu\sigma})\}\end{aligned} \tag{4.73}$$

Bibliography

[1] L.Amendola, R.Gannouji, D.Polarski, S.Tsujikawa, Conditions for the cosmological viability of dark energy models. Phys. Rev. D75, 0803504 (2007). arXiv:gr-qc/0612180

[2] S.S.Carloni, P.K.S.Dunsby, On the existence of the Schwarzschild solution and Birkhoff's theorem in scalar tensor gravity, 2013, arXiv 1306.2473

[3] M.Eingorn, A.Zhuk, Hubble flows and gravitational potentials in observable Universe, Phys.Rev D 75, 2007, 083504

[4] M.Eingorn, A.Kudinova and A.Zhuk, Dynamics of astrophysical objects against the cosmological background, Journal of Cosmology and Astroparticle Physics 04, 010, 2013; arXiv:astro-ph/1211.4405

[5] M.Eingorn, A.Zhuk, Remarks on mechanical approach to observable Universe, Journal of Cosmology and Astroparticle Physics 05, 024 (2014); arXiv:astro-ph/1309.4924

[6] A.Burgazli, M.Eingorn, A.Zhuk, Rigorous theoretical constraint on constant negative EoS parameter w and its effect for late Universe, submitted to European Physical Journal C (2014); arXiv:astro-ph/1301.0418

[7] M.Eingorn, M.Brilenkov, B.Vlahovic, Zero average values of cosmological perturbations as an indispensable condition for the theory and simulations, submitted to Physical Review D (2014); arXiv:astro-ph/1407.3244

[8] M.Eingorn, J.Novák, A.Zhuk , f(R) gravity:scalar peturbations in the late Universe, European Physical Journal C 74, 3005(2014); arXiv:astro-ph/1401.5410

[9] A.Chopovsky, M.Eingorn and A.Zhuk, Many-body problem in Kaluza models with toroidal compactification, European Physical Journal C 74, 2700 (2014); arXiv:gr-qc/1302.0501

[10] Weak-field limit of f(R)-gravity in three and more spatial dimensions. Phys.Rev. 84, 024023(2011). arXiv:1104.1456[gr-qc]

[11] Testing gravity using cosmic voids, Y.-C.Cai, N.Padilla, B.Li, arXiv:1410.1510v1

[12] M.Brilenkov, M.Eingorn, L.Jenkovszky and A.Zhuk, Dark matter and dark energy from quark bag model, Journal of cosmology and Astroparticle Physics 08, 002, 2013; arXiv:astro-ph/1304.7521

[13] M.Brilenkov, M.Eingorn, L.Jenkovszky and A.Zhuk, Scalar perturbations in cosmological models with quark nuggets, European Physical Journal C 74, 3011 (2014); arXiv:astro-ph/1310.4540

[14] L.D.Landau and E.M.Lifshitz, The classical Theory of Fields, Fourth Edition: Volume 2 (Course of Theoretical Physics Series), Oxford Pergamon Press, Oxford, 2000

[15] C.P.L. Berry, J.R.Gair, Linearized f(R) gravity:gravitational radiation and solar system tests. Phys. Rev.D 83, 1004022(2011).arXiv:1104.0819

[16] J.Naf, P.Jetzer, On the 1/c expansion of f(R) gravity. Phys.Rev. D 81, 104003(2010). arXiv:1004.2014[gr-qc]

[17] Tsujikawa S., Felice A. de, f(R)-theories, Living Rev. Relativity 13, 3, 2010

[18] L.G.Jaime, L.Patino, M.Salgado, f(R) cosmology revisted. arXiv:1206.1642[gr-qc]

[19] L.G.Jaime, L.Patino, M.Salgado, Note on the equation of state of geometric dark-energy in f(R) gravity. Phys. Rev.D 89, 084010 (2014). arXiv:gr-qc/1312.5428

[20] A.A.Starobinsky, Disappearing cosmological constant in f(R) gravity. JETP Lett. 86, 157 (2007). arXiv:0706.2041

[21] W.Hu, I.Sawicky, Models of f(R) cosmic acceleration that evade solar system tests. Phys. Rev. D76, 064004 (2007). arXiv:0705.1158[astro-ph]

[22] V.Miranda. S.Joras, I.Waga, M.Quartin, Viable singularity-free f(R) gravity without cosmological constant. Phys. Rev.Lett. 102, 221101(2009).arXiv:1941[astro-ph]

[23] Matter density perturbations and effective gravitational constant in modified gravity models of dark energy. Phys. Rev.D 76, 023514(2007). arXiv:0705.1032[astro-ph]

[24] Density perturbations in f(R) gravity theories in metric and Palatini formalisms, Phys.Rev. D 77-04 3007. arXiv 0712.0082v2

[25] S.Domazet, V.Radovanovic, M.Simonovic, H.Stefancic, On analytical solutions of $f(R)$ modified gravity theories in FLRW cosmologies, gr-qc 2013, arXiv:1203.5220v5

[26] Kiefer C., Conceptual problems in Quantum Gravity and Quantum Cosmology, ISRN Math.Phys, 509316 , 2013, arXiv: 1401.3578v1

5. Conclusion

We have been studying cosmological perturbation theory in this work. In first part - Chapter I - we study general theory of relativity in higher dimensions with extended extra dimensions. We mentioned the usage of GHP formalism for perturbations in higher dimensions. We introduce what is the algebraic classification of spacetimes in higher-dimensions and we introduce the classification in spinors. This formalism we apply in the next Chapter II, where we use the GHP formalism for the perturbations of FLRW ST's. We want to use this result for the phase transition at the beginning of the Universe in next works.

In Chapter III we study so called f(R)-cosmologies, which are a promising road for modelling the accelerated expansion of the Universe. f(R)-gravities are a different theory than Standard General Relativity. We obtain by variational procedure more complicated equations. We study scalar perturbations, which are for us important because of coupling to matter. We want to model the origin of structures in the Universe. We used so called quasi-static approximation for obtaining the scalar potentials Φ and Ψ, because these equations were complicated for direct integration. We used an astrophysical approach first, where we neglect the time derivatives, and then the large scalaron mass approximation. This gave us the standard general relativity. There are written in our paper explicit expressions for scalar potentials Φ and Ψ for all three cases. One term is the Yukawa term and other part is the contribution from standard potential. There is used a generalization of the mechanical approach for the case of cosmological background. The hydrodynamical approach is not applicable for the cell of 150 Mpc, where the homogeneous Friedmann background is perturbed by inhomogeneities. Our approach is new. Next continuation of our work would be models with torsion. It would be interesting to concentrate first on the Hu-Sawicky function, from the recently published paper. This part should give us the model of evolution (dynamics) of Universe to the future. We all believe that it will help us to formulate one day the theory of Quantum Gravity, which is the challenge for community of theoretical physicists.

pwithtocSGR (Standard General Relativity),

QCD (Quantum Chromodynamics)