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May 10, 2015

Report on the dissertation of Karel Chvalovský: Undecidability of Some Substructural Logics.

The reviewed dissertation consists of the introductory chapter and two appendices, which are the same as two papers accepted for publication in *Journal of Symbolic Logic* (up to minor changes in numbers of sections and the style of bibliography). These papers are:

(**A**) K. Chvalovský, Undecidability of Consequence Relation in Full Nonassociative Lambek Calculus.

(**B**) K. Chvalovský and R. Horčik, Full Lambek Calculus with Contraction is Undecidable.

Each of these parts has its own bibliography. Additionally the dissertation contains Abstract, Acknowledgements, Foreword and Table of Contents.

The results presented in (**A**) and (**B**) show the undecidability of several substructural logics, i.e. nonclassical (propositional) logics whose Gentzen-style sequent systems omit some structural rules, characteristic of Gentzen systems for classical and intuitionistic logic (exchange, weakening, contraction). Nowadays these logics enjoy much attention in the community of algebraic logic. They are also popular in the logico-linguistic community as type change logics in type grammars (or: categorial grammars).

The thesis focuses on basic substructural logics, extending the Lambek calculus (both its nonassociative and associative version) by lattice connectives $\vee, \wedge$; these logics are called Full Nonassociative Lambek Calculus (FNL) and Full Lambek Calculus (FL), respectively. FNL is also known as Groupoid Logic (GL) (the term coined by K. Došen in 1980-ties).

The Lambek calculus (L), introduced by J. Lambek in 1958 under the name 'Syntactic Calculus', admits three connectives: product $\cdot$, right residual $\backslash$ and left residual $/$. The provable sequents $\varphi \Rightarrow \psi$ are precisely the sequents valid in residuated semigroups, i.e. partially ordered semigroups $(A, \cdot, \leq)$ augmented with operations $\backslash, /$, satisfying the residuation laws: $a \cdot b \leq c$ iff $a \leq c/b$ iff $b \leq a\backslash c$. Dropping the associativity of product, one obtains the nonassociative Lambek calculus (NL), complete with respect to residuated groupoids (introduced by J. Lambek in 1961). Both L and NL are basic type change logics. It is natural to consider stronger logics L1 and NL1, admitting the constant 1 (the unit of product); they are complete with respect to residuated monoids and residuated unital groupoids, respectively. FL and FNL are usually defined as extensions of L1 and NL1, respectively, by $\wedge, \vee$. FNL is complete with respect to lattice ordered residuated unital groupoids, and FL with respect to residuated lattices (i.e. lattice ordered residuated monoids). There are good reasons to consider weaker logics, corresponding to algebras without 1: lattice ordered residuated groupoids and semigroups. These logics are denoted here by FNL^- and FL^- (no standard notation is commonly accepted). FNL^- (resp. FL^-) is a conservative extension of NL (resp. L).

Already Lambek (1958, 1961) presented L and NL as sequent systems. In L sequents are of the form $\varphi_1, \cdots, \varphi_n \Rightarrow \psi$ (comma is interpreted as product). In NL sequences of formulas are replaced by bracketed sequences. Lambek proved the cut-elimination theorem for these systems. As a consequence, L and NL are decidable and possess the subformula property. Analogous results for FL and FNL were obtained by different authors. The cut-elimination theorem does not hold for theories based on these logics, i.e. logics augmented with nonlogical axioms (assumptions).

In 2005 it was shown that the consequence (or: deducibility) relation of NL and NL1 is decidable (in fact, polynomial). Much earlier it was known that the consequence relation of L (even its $/-$fragment) is undecidable. (By the consequence relation we mean the relation of provability from finitely many assumptions). In 2009 it was shown that the consequence relation remains decidable for DFNL, DFNL^-, i.e. FNL and FNL^- extended by the distributive laws for $\vee, \wedge$, and their extensions admitting boolean negation or intuitionistic implication.

Paper (**A**). Since the distributive laws were quite essential in the decidability proofs, it was an open problem of whether the consequence relation for FNL is decidable. The negative answer is given in (**A**). The proof works for FNL and FNL^- as well as their extensions by structural rules of contraction (c) and exchange (e).

K. Chvalovský finds an encoding of 2-tag systems (whose halting problem is undecidable) in FNL with assumptions. This encoding is too involved to be presented here. I merely note some basic tools. Each word w, processed by the 2-tag system, is translated into a word $\delta(w)$, defined as follows. Pairs of symbols $a_i a_j$ are represented as special symbols c^i_j (this is needed, since 2-tag systems delete pairs of symbols in front). If $|w|$ is even, then $\delta(w)$ is the string of the special symbols corresponding to the consecutive pairs of symbols; for example, $w = a_1 a_2 a_2 a_1$ is translated into $\delta(w) = c^1_2 c^2_1$. If $|w|$ is odd, then special symbols c_i are employed; for example, $w = a_1 a_2 a_2 a_1 a_2$ is translated into $\delta(w) = c^1_2 c^2_1 c_2$. For any 2-tag system, one defines a finite set of assumptions Φ, satisfying the equivalence: the 2-tag system terminates on w if and only if $e\delta(w)X \Rightarrow eX \vee eA$ is provable in FNL from Φ, which yields the undecidability. Here e, X, A are certain auxiliary symbols, essential for the encoding. Symbols are identified with propositional variables. Words in antecedents of sequents are interpreted as bracketed strings of variables, while in succedents as products of variables (in both cases, parentheses are associated to the right). For example, $ec^1_2 X \Rightarrow eX$ means $e(c^1_2 X) \Rightarrow e \cdot X$.

The implication ($\Rightarrow$) of this equivalence is relatively easy, but the converse implication has a quite sophisticated proof. The author's proofs are correct and elegant; they use some normalization of formal derivations in FNL with regular assumptions. The methods are proof-theoretic.

Since the encoding uses sequents with product and join only, then it yields the undecidability of the consequence relation of the $(\cdot, \vee)-$fragments of FNL and FNL^-. In algebraic terms, the quasi-equational theory of lattice ordered residuated (unital) groupoids is undecidable, and similarly for join-semilattice ordered (unital) groupoids (product distributes over join).

Paper (**B**). This paper proves the undecidability of FL with (c). At the end, the authors show that a modification of this proof also yields the undecidability of FL^- with (c). In algebraic terms, this amounts to the undecidability of the equational theories of square-increasing residuated lattices and square-increasing lattice-ordered residuated semigroups. Recall that an ordered algebra with product is square-increasing (or: contractive), if $a \leq a \cdot a$ holds for all elements a.

These results are interesting, since they solve a long-standing open problem. The pure FL and its extensions by any collection of structural rules (exchange, left weakening, contraction) are decidable except this particular case: contraction only. Although the cut-elimination theorem holds for FL with (c), it does not yield the decidability; the premise of (c) $\Gamma, \Delta, \Delta, \Gamma' \Rightarrow \psi$ is longer than its conclusion $\Gamma, \Delta, \Gamma' \Rightarrow \psi$, hence the proof-search tree may contain infinite branches.

The methods of (**B**) are algebraic and combinatorial. In fact, the authors work in the equational theory of square-increasing residuated lattices and show its undecidability. This class of algebras is denoted by $\mathcal{RL}_c$. The undecidability of FL with (c) immediately follows from the completeness of this logic with respect to $\mathcal{RL}_c$.

The proof is based on three main constructions. First, the configurations (q_i, m, n) of a 2-counter machine (i.e. the machine in state q_i has the integers m, n in counters) are represented by words $Ah^m(a)Bq_iCh^n(a)D$, where h is a morphism on $\{a, b, c\}^*$, defined by: $h(a) = abc$, $h(b) = ac$, $h(c) = b$. It is known that all words $h^n(a)$ are square-free, i.e. they do not contain a factor xx, where x is nonempty. A language is square-free, if it consists of square-free words. Since the halting problem for 2-counter machines is undecidable, then there exists a non-recursive language U consisting of such words. This language can be generated by a string rewriting system (SRS); precisely, the SRS contains some auxiliary symbols and generates a non-recursive language U' on the larger alphabet, but U is the restriction of U' to words on $\{A, B, C, D, a, b, c\}$. Furthermore, U' is square-free. (Here I use U, U' instead of more standard L, L' to avoid confusion with symbols for logics.) This part of the paper is based on an earlier paper of R. Horčik, proving the undecidability of the consequence relation of FL with (c); many details are omitted in (**B**).

The second construction is a simulation of an arbitrary SRS by an atomic conditional SRS (CSRS). In a CSRS, rewriting rules are of the form $(x \rightarrow y, L_l, L_r)$, where L_l, L_r are some regular languages; this means that the rule can transform uxv into uyv provided that $u \in L_l$, $v \in L_r$. A CSRS is atomic, if in each rewriting rule y is a symbol. The atomic CSRS simulating the SRS from the preceding paragraph generates a non-recursive, square-free language U_0, which consists of all words reducible to a fixed word w_0 in the sense of the system. Additionally, all regular languages appearing in rules are closed under contraction. (A language L is closed under contraction, if $uxxv \in L$ entails $uxv \in L$; notice that square-free languages fulfill this condition.) U_0 does not contain the empty word ϵ.

In the third step, the alphabet Σ of the atomic CSRS is enriched by new symbols $r_1, \ldots, r_k$, corresponding to all rewriting rules $R_1, \ldots, R_k$; the extended alphabet is denoted by Σ_e. The auxiliary language AUX is the union of all languages $L_l^{(i)} r_i L_r^{(i)}$. AUX is regular and closed under contraction. Let G be a right-linear context-free grammar with m production rules, generating AUX, whose start symbol is S. One defines the term $\delta_G = 1 \wedge \delta_1 \wedge \cdots \wedge \delta_m$, where $\delta_i = (xB)\backslash A$ whenever the $i-$th production rule of G is $A \rightarrow xB$, and $\delta_i = A$ whenever the rule is $A \rightarrow \epsilon$. (δ_G is a term of the first-order language of $\mathcal{RL}_c$, but it is a formula of FL.) One defines $L = U_0 \cup$ AUX. The language L is closed under contraction.

For any rewriting rule $R_i = (x \rightarrow a, L_l, L_r)$ of the atomic CSRS, one defines a term $\theta_i = x\backslash(a \vee r_i)$. One defines $\theta = 1 \wedge \theta_1 \wedge \cdots \wedge \theta_k$. A nonempty word w on Σ_e, $w = a_1 \ldots a_n$, is represented as $w^\theta = a_1\theta a_2\theta \cdots a_n\theta$. The crucial equivalence is: $w \in U_0$ if and only if $w^\theta \delta_G \leq w_0 \vee S$ is valid in $\mathcal{RL}_c$, for any nonempty word w on Σ_e. This yields the undecidability of the equational theory of $\mathcal{RL}_c$. The proof of ($\Rightarrow$) is easy; it employs basic lattice laws, $\theta \leq 1$, $\delta_G \leq 1$, the distributivity of product over join, and $x \cdot (x\backslash y) \leq y$.

The proof of ($\Leftarrow$) is more involved. One defines an algebra W_L^+, whose elements are closed languages on Σ_e with respect to the closure operation: $\gamma(X) = X^{\rhd\lhd}$, where $X^\rhd = \{(u,v) \in (\Sigma_e^*)^2 : (\forall x \in X) uxv \in L\}$, $Y^\lhd = \{x \in \Sigma_e^* : (\forall (u,v) \in Y) uxv \in L\}$. W_L^+ is a square-increasing residuated lattice with operations $X \cdot_\gamma Y = \gamma(X \cdot Y)$, $X \vee_\gamma Y = \gamma(X \cup Y)$, $1_\gamma = \gamma(\{\epsilon\})$, and the remaining operations inherited from the algebra of all languages on Σ_e. One defines a valuation e in W_L^+ by setting: $e(a) = \gamma(\{a\})$ for $a \in \Sigma_e$, $e(A) = \gamma(L_G(A))$, for any nonterminal symbol A of G. (Here $L_G(A)$ is the set of all terminal words derivable from A in G.) Now, if $w^\theta \delta_G \leq w_0 \vee S$ is valid in $\mathcal{RL}_c$, then $e(w^\theta \delta_G) \leq e(w_0 \vee S)$ in W_L^+, and this entails $w \in U_0$.

The constant 1 can be eliminated from this argument by replacing 1 in θ and δ_G with the meet of all $a\backslash a$ such that a is a variable appearing in the encoding. The authors also note that product can be eliminated; so it suffices to use the operation symbols $\backslash, \wedge, \vee$. Furthermore, contraction $a \leq a \cdot a$ can be weakened to $a^k \leq a^l$, for some fixed positive integers $k < l$.

Let me add that this proof also yields the undecidability of a weaker logic, arising from FL$^-$ with (c), after one has dropped the right-introduction rules for $\backslash, /$, since the proof of ($\Rightarrow$) only uses the laws $x(x\backslash y) \leq y$ for $\backslash$. In fact, all intermediate logics between this weak logic and FL with (c) are undecidable.

I have outlined the methods of (**B**) in some detail just to show the rich methodology of this paper: formal languages and grammars, combinatorics of words, and algebraic tools (in particular, W_L^+ is constructed by nuclear completion).

Critical remarks. Although both papers are written clearly and elegantly, some points are handled too shortly, which may cause misunderstanding.

In (**B**), the authors do not write precisely that the proof also yields the undecidability of the equational theory of square-increasing lattice ordered residuated semigroups, and consequently, the logic FL^- with (c), which is a nonconservative fragment of FL with (c). Their short remarks might be understood as the possibility of eliminating 1 from the language, without changing the class of algebras. Also, in the proof of ($\Rightarrow$) of the crucial equivalence, the case $x = \epsilon$ needs a slightly different treatment than the case $x \neq \epsilon$, worked out in the paper.

In (**A**), the final parts, discussing contraction and the one-variable fragment, seem too sketchy.

The introductory chapter of the dissertation aims to explain the matters to a general logician. Although the author formulates many interesting remarks, I'm not fully satisfied. These remarks are often too general, and the introduction leaves many important things untouched. For instance, it provides no concrete examples of algebras under consideration, nor explains the difference between FL and FL^- (the latter is restricted to sequents with nonempty antecedents), nor shows any laws provable in these logics. It does not discuss the relation of Lambek calculi to other substructural logics (e.g. linear, relevant, many-valued). Furthermore, since (**B**) omitted certain essential details on square-free words and the SRS generating U' (with a reference to an unpublished paper of R. Horčik), it would be reasonable to include these matters in the introduction (just to help the reader). The only complete proof in the introduction shows the undecidability of the halting problem for 2-tag systems. The author announces a new result, namely the undecidability of the consequence relation of FNL restricted to $\backslash, \vee$, but the idea of proof is not clear. How to express - without associativity - the context-dependence of rewriting rules of the atomic CSRS?

Conclusion. The above critical remarks do not essentially influence my very positive opinion about this PhD thesis. The main results are important: they solve basic open problems. The proofs are highly nontrivial and well-written. Especially (**B**) employs a bundle of different mathematical tools, but also (**A**), though purely proof-theoretic, invents certain new, subtle methods of handling nonassociative logics. The undecidability proofs resemble in some points the earlier proofs of undecidability of linear logics (P. Lincoln et al., M. Kanovich, Y. Lafont), but the presence of (c) in (**B**) and the nonassociativity in (**A**) require substantial innovations.

Therefore I'm fully convinced that this thesis is an outstanding Ph.D. thesis in logic, which strongly justifies awarding Karel Chvalovský the Ph.D. degree.

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