Charles University in Prague

Faculty of Social Sciences
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MASTER THESIS

Fractal Dimension and Efficient Markets

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Declaration of Authorship

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Signature

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Abstract

The efficient market hypothesis is one of the most important propositions in finance theory and has been subjected to years of rigorous empirical testing. We examine power of a new tool for evaluating market efficiency, fractal dimension. Characteristics and abilities of fractal dimension measure are explored through extensive Monte Carlo simulations. We prove that it provides an accurate evaluation of market's efficiency and its changes. This approach is highly innovative and creates new possibilities for examination of markets. The uniqueness of fractal dimension is in its ability to assign a numerical ranking to examined series describing the level of (in)efficiency; it is accurate for small samples of observations and quickly reflects changes in market efficiency structure.

JEL Classification	G14
Keywords	Market efficiency, fractal dimension, Monte Carlo simulation

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Abstrakt

Hypotéza efektivních trhů je jedním z nejdůležitějších tezí ve finančních teoriích a byla zkoumána v mnoha empirických studiích. V naší práci se věnujeme zkoumání schopností nového nástroje pro ohodnocení tržní efektivity, fraktální dimenzi. Charakteristiky a schopnosti fraktální dimenze jsou zkoumány pomocí rozsáhlé Monte Carlo simulace. Simulace ukazuje, že fraktální dimenze poskytuje přesné ohodnocení efektivity trhů a jejích změn. Tento přístup je vysoce inovativní a vytváří nové možnosti pro zkoumání trhů. Jedinečnost fraktální dimenze jako měřítka tržní efektivity spočívá v její schopnosti přiřadit zkoumanému trhu číselné ohodnocení vypovídající o stupni (ne)efektivnosti, fraktální dimenze má přesné výsledky i pro malé počty pozorování a dokáže rychle zachytit změny ve struktuře tržní efektivity.

JEL klasifikace	G14
Klíčová slova	Efektivní trhy, fraktální dimenze, Monte Carlo simulace

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Acronyms

ARFIMA Autoregressive fractionally integrated moving average

BM Brownian motion

M Martingale

RW Random walk

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Proposed Topic:

Fractal Dimension and Efficient Markets.

Topic Characteristics:

The thesis will examine explanatory power of time series' fractal dimension in context of efficient markets. Our hypothesis is that fractal dimension can capture both short and long term deviations of time series from market efficiency conditions and evaluate size of the deviation. This hypothesis will be tested through simulation of efficient financial time series and time series with various deviations from efficiency and measurement of their fractal dimensions. If the power of fractal dimension is proved, we provide a market efficiency ranking of real stock indexes.

Many academics believe that the weak form of the efficient market hypothesis holds meaning that stock price follows martingales and given all pieces of information about past prices and returns, the best predictor of a future price is the current price, the expected return is zero and investors cannot beat the market using technical analysis. Ranking of real stock indexes according to level of market efficiency may be useful for investors, because it is possible to make abnormal returns on inefficient markets in contrary to efficient markets.

The fractal dimension measures roughness of n-dimensional sphere and takes values [n, n+1) for hyperplane R^{n+1}. A straight line dimension is 1 and a dimension of a flat surface is 2. What is a dimension of an irregular rough index? It should be somewhere between 1 and 2. The fractal dimension of a random series is equal to 1.5 (Kristoufek & Vosvrda 2013). Market mood is reflected in short-term trends and makes the time series locally smoother. Hence, fractal dimension is lower than 1.5. Analogically, a short term increase in volatility increases fractal dimension.

The hypothesis of market efficiency has been widely tested with miscellaneous results (e.g. Kima & Shamsuddinb 2008, Dickinson & Muragu 1994). The most widely used test of market efficiency is autocorrelation test which identifies possible long time dependencies in the series and therefore a violation of the market efficiency. However, short term deviation from martingale is not captured by the autocorrelation test. Further, the autocorrelation test either rejects or does not reject the null hypothesis of market efficiency, but ranking of stock indexes according to their level of efficiency is hardly seen.

This approach is innovative and as far as we know, there are no other studies examining the power of fractal dimension to capture efficiency of markets. Finally, fractal dimension as a measurement of market efficiency is used only in a few studies (e.g. Kristoufek & Vosvrda 2013).

Hypotheses:

1. Fractal dimension **captures** short and long term deviations of time series from market efficiency conditions.
2. Fractal dimension **evaluates** the deviation of time series from market efficiency conditions.
3. We are able to create a market efficiency ranking of real stock indexes based on fractal dimension.

Methodology:

At first, we will simulate time series corresponding to the efficient market hypothesis and insert some phases of inefficiency. The simulation of various time series will be done in R project. Then the fractal dimension value for the series will be determined. There are many ways how to estimate fractal dimension (Box-Count, Periodogram, DCT-II, Hall-Wood, Variogram, etc.). Gneiting et al. (2010) make an extensive research on quality of these estimators and recommends using madogram for fractal dimension estimation. While estimating the fractal dimension we will utilize the findings of Gneiting.

As the size of inserted inefficiencies differ across the simulated time series, their fractal dimensions capturing the level of inefficiency should vary. If the fractal dimension's power of capturing the inefficiency will be proved, we measure the fractal dimension of some real stock indexes. The values will enable us to rank individual markets according to their efficiency.

Outline:

1. Introduction
2. Literature Overview
3. Simulation of time series
4. Estimation of fractal dimension
5. Discussion of results
6. Real data description
7. Market efficiency ranking of stock indexes
8. Conclusion

Core Bibliography:

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Chapter 1

Introduction

The efficient market hypothesis is one of the most important financial theories. Its weak form claims that all past information is reflected in the current price; it means that investors using technical analysis cannot systematically achieve abnormal returns in an efficient market. The market efficiency research is thus in interest of economists, as well as investment communities concerned about prices predictability.

The efficient market hypothesis is discussed and tested in many studies. Most of them focus on testing whether a market is efficient or inefficient for a selected sample period and employ various statistical tests, such as the serial correlation test. The problem with this type of tests is inability to capture short-term deviations from market efficiency. In our thesis, we examine power of a new tool for evaluation of market efficiency: fractal dimension, which as discussed below captures both global and local inefficiencies.

Fractal dimension measures roughness of time series data and takes value on the interval [1, 2); random walk has fractal dimension equal to 1.5. Any persistence makes time series smoother and fractal dimension is thus lower than 1.5. On the contrary, higher volatility increases fractal dimension above 1.5. We argue that fractal dimension accurately captures all changes in efficiency structure of examined market by taking values lower or higher than 1.5, and the size of inefficiency is reflected in the size of deviation from 1.5.

Before using fractal dimension in practice, it is necessary to perform a comprehensive study of fractal dimension's power in the context of efficient markets using simulations; this is the first analysis of fractal dimension as a tool for market efficiency evaluation. If the explanatory power of fractal dimension in the context of efficient markets is proved, researches can use fractal dimension for a detailed analysis and comparison of individual markets, whereas investors can utilize it in strategy planning to identify less efficient markets with possible abnormal returns and to track changes in market efficiency.

Our analysis utilizes an extensive Monte Carlo simulation of time series with different types of inefficiencies and estimates their fractal dimensions. We use simulations of random walk, Brownian motion, and martingales for the efficient part of time series and inter-space it with different settings of ARFIMA(1, d, 1) representing the inefficient part. The madogram, Hall-Wood, and box-count estimators are employed. Essentially, we examine how estimated fractal dimension reacts to changes in ARFIMA coefficients and in the length of the inefficient parts. In addition to time series analysis with one inefficient part, we examine the impact of more inefficient parts with reverse effects and the reaction of fractal dimension to gradual adding of observations with given settings to the examined sample. The gradual estimation enables us to evaluate the reaction of fractal dimension to small changes in the time series.

Except for extreme conditions, fractal dimension indeed reflects all changes in ARFIMA setting as assumed, so we conclude that it is a powerful measure of market efficiency enabling us to not only decide on efficiency/inefficiency of a market, but to compare efficiency levels of individual markets as well.

The work is completed by an illustration of usage of fractal dimension in practice; we evaluate efficiency of 28 stock indexes representing individual markets and compile their ranking of the markets based on efficiency. The evaluation reveals that indices such as FTSE (British), SPX (American), CAC (French) and ESTX (European) have higher volatility than is characteristic for efficient markets. Moreover, the least efficient markets are Chilean (IPSA) and Malaysian (KLSE) ones, while Dutch and Canadian markets are the most efficient ones. Lastly, we divide the indexes into four groups - high volatility markets, inefficient markets, constantly improving markets, and markets hit by the recent crisis.

The thesis is structured as follows: the next chapter contains an introduction to the market efficiency theory and a literature overview. Chapter 3 describes the approaches to fractal dimension estimation, while Chapter 4 deals with the methodology and simulations settings. Consequently, Chapter 5 constitutes the principal part of the work and contains simulation results with discussion; Chapter 6 then illustrates usage of the fractal dimension measure in practise and presents efficiency ranking of the chosen markets. Finally, Chapter 7 concludes the thesis.

Chapter 2

Efficient market theory

The efficient market theory is the fundamental theory of capital markets developed by Fama (1965) and Samuelson (1965), and further expanded by Fama (1970). The efficient market concept was widely accepted for approximately 30 years, until the behavioural economics started to become mainstream; nevertheless, the efficient market theory still remains the central postulate of financial markets. A market is called efficient if all available information is immediately and accurately incorporated into prices, so investors cannot systematically make extensive profits.

2.1 Development of the theory

The idea of random walk model of security prices was firstly examined by Bachelier (1900) in his Ph.D. dissertation "*The Theory of Speculation*". He argues that a speculation on security prices should be a "fair game" and expected profits of a speculator should be zero. This principle was later named martingale. More than a half of the century later researchers started to empirically examine security price formation without any given theory; the theory was developed in 1960's after accumulation of some empirical evidence of security prices nearly following random walk. Kendall and Hill (1953) focus in their research on industrial share prices indices and spot prices of cotton and wheat. They state that it seems that there is a randomly chosen number from a symmetrical population of a fixed dispersion added to the series of prices. This result is supported by investigations of Roberts (1959), and Granger and Morgenstern (1963), who use spectral analysis. Fama firstly addressed the efficient market hypothesis in his Ph.D. dissertation, outlining the basic idea, in 1965. Samuelson (1965) and Mandelbrot (1966) extensively examine the expected return model in efficient market theory and study the relationship between this model and the model of random walk empirically examined in the previous studies. Samuelson (1965) provides the first rigorous formulation of the efficient market hypothesis using martingale principle.

Fama (1970) introduces three forms of efficient markets – strong-form efficiency, semi-strong-form efficiency, and weak-form efficiency and brings the term "efficient market" into general use. The difference among the three forms is in the information set which is fully reflected in security prices. The weak-form's information set is created by historical prices, meaning that investment strategies based on the historical data analysis cannot produce excessive returns, but one can earn excessive returns using current or private information. The semi-strong-form's information set consists of all publicly available information, so an excessive return may be earned only by using private information. Under the strong-form of efficient market hypothesis, no one can earn excessive returns because all information - public and private - is fully reflected in security prices.

The term "fully reflected" is too general and untestable, so the price formation process had to be defined more exactly and Fama uses expected return or "fair game" model for it.

The expected return model can be described by the following equation

$$E(p_{j,t+1}|\Phi_t) = [1 + E(r_{j,t+1}|\Phi_t)]p_{j,t}, \tag{2.1}$$

where E is expected value, $p_{j,t}, p_{j,t+1}$ are prices of security j at times t and $t+1$, respectively, $r_{j,t+1}$ is one period expected return, which can be expressed as $(p_{j,t+1}, p_{j,t})/p_{j,t}$, and Φ_t stays for the information set that is fully reflected in the price of security. This expression can be summarized into two assumptions - market equilibrium conditions can be stated in terms of expected returns and expected returns depend on the information set. These two assumptions ensure that expected returns of a trading system based on information set Φ_t cannot exceed the equilibrium expected returns.

Let

$$x_{j,t+1} = p_{j,t+1} - E(p_{j,t+1}|\Phi_t) \tag{2.2}$$

be the excess market value of security j at time $t+1$; in other words, it is the difference between observed and expected price, then

$$E(x_{j,t+1}|\Phi_t) = 0, \tag{2.3}$$

so x_j is a "fair game" with respect to Φ_t. It can be equivalently described in terms of returns. Let

$$z_{j,t+1} = r_{j,t+1} - E(r_{j,t+1}|\Phi_t), \tag{2.4}$$

$z_{j,t+1}$ is the excess return of security j at time $t+1$; equivalently said, it is the difference between the observed and the expected return projected at t, then

$$E(z_{j,t+1}|\Phi_t) = 0, \tag{2.5}$$

so z_j is a "fair game" with respect to Φ_t. Fama points out that this specification does not require serial covariance of returns to be zero. According to Fama (1970), the model can be interpreted as submartingale or simplified to a random walk model.

2.1.1 Random walk

The random walk model was the first concept used for efficient markets description, and Fama (1970) argues that the random walk model is the best extension of the expected value or "fair game" efficient markets model which allows description of the economic environment in greater detail. The random walk model was mainly used in the early studies of the efficient market hypothesis; it assumes that successive price changes are independent and identically distributed. The model can be written as:

$$f(r_{j,t+1}|\Phi_t) = f(r_{j,t+1}), \tag{2.6}$$

meaning that the conditional and marginal probability distributions of an independent random variable are identical. The density function f has to be the same for all t. However, random model respondents argue that investors spend a lot of money on securities analysis, which would be useless if random walk model was true. Further, random walkers suppose that investors quickly exploit unexpected patterns in security prices to make profits; hence, the unexpected patterns occur only in a short term. LeRoy (1989) says that supporters of the random walk model do not employ the condition that an irrational behaviour cannot consistently persist in the equilibrium. Random walk is more restrictive requirement than martingale; it rules out not only the dependence of conditional expectation of $(p_{t+1} - p_t)$ on information set, but also any higher conditional moments. Martingale describes the financial markets more precisely because security prices usually go through more and less volatile periods and the successive conditional variances are positively autocorrelated. This option is ruled out in the case of random walk model. However, the martingale hypothesis is difficult to test; hence, we use the random walk model as the main concept.

2.1.2 Submartingale

The submartingale model says that the expected value of next period's price based on the information set Φ_t is equal or greater than the current period price

$$E(p_{j,t+1}|\Phi_t) \geq p_{j,t}, \tag{2.7}$$

or equivalently in terms of returns

$$E(r_{j,t+1}|\Phi_t) \geq 0. \tag{2.8}$$

Fama (1970) claims that the submartingale model implies that buying-and-holding strategy has better or equal returns than any other trading strategy based only on Φ_t. This statement is opposed by LeRoy (1989), who argues that it is easy to find examples of economies where security prices follow submartingale, but the buy-and-hold strategy is outperformed by other trading strategies. If the inequalities hold as equations, the price sequence follows martingale. The martingale model was introduced by Samuelson (1965). LeRoy (1989) proves that martingale and "fair game" models characterize the same equilibrium in financial markets and that the two terms are synonyms. The stochastic process $r_{j,t+1}$ is then called a "fair game".

We are interested in the weak-form of the efficient market hypothesis, which says that past returns should not influence future returns and the expected return model can be simplified to e.g. random walk, martingale or Brownian motion.

2.1.3 Conditions of market efficiency

Fama (1976) proposes a slightly different definition of efficient markets in which he states that the efficient market hypothesis does not require agents to be rational, but they must have rational expectations. Population has to be correct on average; however, each individual investor may underreact or overreact.

Sufficient market conditions consistent with the efficiency hypothesis are easy to name: current prices on markets with no transaction costs, information freely available to all investors, and investors agreement on effect of current information on current and future price of security; such markets surely "fully reflect" all available information. However, transaction costs, not freely available information to all investors, and disagreement among investors generally do not have to imply market inefficiency; although they are potential sources of inefficiency.

2.2 Literature overview

The efficient market hypothesis is still an attractive topic for empirical studies, and there are thousands of studies focusing on the theory. We present only a sample of the studies to illustrate the different results and approaches. The first researcher considering this topic is Bachelier (1900), then the topic was forgotten for a while, and the problem was rediscovered 60 years later when computers allowed more rigorous examination of the hypothesis.

The first studies test the weak-form of efficient market hypothesis using mostly an autocorrelation test which is easy to implement and understand.

Fama (1965) examines the first-order autocorrelation in the Dow Jones Industrial Indices, and he find out that there is a positive autocorrelation in daily returns for 23 out of 30 examined indices. Moreover, the daily returns are as far as 2 standard errors away from 0 for 11 indices. But the work, together with the other early studies, suffers from lack of statistical power.

The power of efficient market tests increases with data availability. In 1980's there were daily data on NYSE and AMEX stocks available, and the data stretched back to the year 1962, creating a sufficient dataset for statistically significant conclusions. Lo and MacKinlay (1988) group NYSE stocks according to size and find reliable positive autocorrelation in the series of weekly returns on portfolios; the autocorrelation is stronger for small portfolios. However, according to Fisher (1966) this effect can be caused by non-synchronous trading problem, which is more important for the portfolios of small stocks.

Conrad and Kaul (1988) try to deal with the non-synchronous problem by employing Wednesday-to-Wednesday returns, but the problem is not completely mitigated by this approach. Their conclusion is the same as the one by Lo and MacKinlay (1988): returns are positively autocorrelated and the autocorrelation is stronger for small stocks portfolios.

French and Roll (1986) point out that stock prices are more variable for open stock markets. As the variance during trading hours is 72 times higher than during weekend days, the difference is not negligible. Black (1986) argues that the higher variance is caused by noise trading of uniformed investors. Based on this hypothesis, the daily returns should be negatively autocorrelated. French and Roll find that the autocorrelation in returns is positive for the top 60% of individual stocks at NYSE, and negative for the others. After a further analysis, they are not able to conclude that noise trading results in market inefficiency.

To sum it up, the availability of data enables the researchers to reject the hypothesis of market efficiency with constant expected returns.

The papers mentioned above consider only a short-horizon of returns. Shiller et al. (1984) and Summers (1986) challenge this approach by arguing that even a small deviation in autocorrelation from 0 in a short time horizon may result in a big inefficiency in a long time horizon. Fama and French (1988) examine returns on diversified portfolios of NYSE stock for period 1926-1985 and find that there is a strong negative autocorrelation in 3- and 5-year returns, while the autocorrelation is close to 0 for short horizons. However, the tests again have a low statistical power.

Dickinson and Muragu (1994) use autocorrelation test to examine stock prices behaviour on Nairobi Stock Exchange; Kenya as a developing economy is considered to have inefficient market. Nevertheless, Dickinson and Muragu do not find any signs leading to inefficiency and propose further studying to draw a strong conclusion. African markets are later studied by e.g. Magnusson and Wydick (2002) using Random Walk 3 test described by Campbell and Andrew (1997), which examines Partial Auto-Correlation Function of random increments of past price information and its divergence from zero. Although they do not specifically investigate the Nairobi Stock Exchange, their result is surprising as the efficiency is not rejected for 6 out of 8 markets. That contradicts the theory of correlation between market efficiency and development of economics proposed by other studies.

Testing of efficient market hypothesis is not only a historical issue, it is examined in recent studies as well. Paper of Lim et al. (2008) examines the effect of Asian crises on efficiency of eight Asian stock markets. They use bicorrelation test statistics for 3 periods – pre-crisis, crisis and post-crisis. They find that higher inefficiency occurred during the crisis period and the most hit region was Hong Kong. They say that the result is not surprising because the financial environment is chaotic and investors overreact to both local and global news during crisis years.

Alexeev and Tapon (2011) say that the test of weak-form of efficient markets should be performed on a vast pool of individual stock rather than on stock market indices, as the indices themselves are not traded in the spot market. They simulate a series of trials using model-based bootstrap and then use a modified chart pattern recognition algorithm to stocks listed on Toronto Stock Exchange. They fail to reject the null hypothesis of weak form of market efficiency, but they find that some sectors of Canadian economy (e.g. Travel and Leisure, Electricity, Food Producers, Insurance) seem to be less efficient than others.

Chong et al. (2012) examine the Chinese stock market and build on the series of researches on Asian markets efficiency. The results of the studies differ and include both outcomes - rejecting and failing to reject the null hypothesis of market efficiency. Chong et al. use trading rules based on time series model and try to examine the efficiency in several sub-periods. The forecasts profitability is evaluated by self-exciting threshold autoregressive model, autoregressive model, and moving average model. The authors find that efficiency on Chinese stock market has increased after the State-owned enterprises reform.

Most of the studies consider only a specific market or a group of usually regionally connected markets, and they either reject or not reject the null hypothesis of market efficiency. Some studies find a positive correlation between the economic development and market efficiency, but it would be useful to have a ranking of individual markets based on the level of market efficiency.

Such ranking is provided by Kristoufek and Vosvrda (2013), who employ fractal dimension as a measure of efficiency, but the authors apply the measure to stock indices without further specifying the type of the estimator and analysing the reactions of fractal dimension to changes in autocorrelation structure of returns. The examination of appropriateness of the measure is left for further studies.

In our study, we comprehensively examine the power of fractal dimension in context of efficient markets using extensive Monte Carlo simulations and show that the fractal dimension is appropriate and accurate measure of market efficiency. This approach enables us to rank individual markets based on level of efficiency. A ranking of the chosen stock markets completes our study and illustrates the use of fractal dimension in practise. The principle of fractal dimension is described in the following section.

Chapter 3

Fractal dimension

In this section, we define fractal dimension and describe approaches used for its estimation. Fractal dimension measures roughness of a time series; or in other words, it provides a description of how much space the time series fills. Fractal dimension does not have a single general definition, it can be defined in many ways, and some of them are more satisfactory than others. Different definitions may result in different values of fractal dimension and they also may have different properties. The oldest and very important measure of fractal dimension is s-dimensional Hausdorff measures $\mathcal{H}^s$ on subsets of $\mathbb{R}^n$, where $0 < s < n$. It is defined for any set and it is mathematically convenient.

A smooth, differentiable series has fractal dimension equal to one. However, when there are some spikes in the graph and the surface is non-differentiable, fractal dimension is higher than one and rises with both the number of spikes and their magnitudes. Generally, fractal dimension is always equal to or higher than topological dimension n and it never exceeds $n+1$. In our study, we focus on time series, so fractal dimension takes values on the interval [1, 2). The main idea is that the fractal dimension will identify even a small inefficiency in a time series. Fractal dimension of random walk, Brownian motion, and martingale (time series corresponding to the theory of efficient markets) is 1.5. If the volatility of a time series is higher, the time series is rougher and fractal dimension is closer to 2. On the other hand, if there is some long time memory in a time series, fractal dimension is lower than 1.5.

3.1 Hausdorff measure

The Hausdorff dimension can be defined be the following expression

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{–cover of } F \right\}, \tag{3.1}$$

where s stays for a non-negative number, δ is greater than 0, U is a non-empty subset of n-dimensional Euclidean space, $\mathbb{R}^n$, F is a subset of $\mathbb{R}^n$, and $|U| = \sup |x - y| : x, y \epsilon U$ is the diameter of U marking the greatest distance apart two point in the set U. If $\{U_i\}$ is a finite collection of sets of diameter smaller or equal to δ covering F, i.e. $F \subset \cup_{i=1}^{n} U_i$ with $0 \leq |Ui| \leq \delta$ for each i, then $\{U_i\}$ is a δ–cover of F; $\mathcal{H}^s(F)$ is the s-dimensional Hausdorff measure of F.

We look for the minimal s power of the diameters for δ sufficiently close to zero. The set of possible covers of F is reduced by decreasing δ and hence the infimum increases. There is a unique value of s marked D for which holds that if $s < D$, then $\mathcal{H}^s(F) = \infty$, and if $s > D$, then $\mathcal{H}^s(F) = 0$. The unique value is called Hausdorff dimension of the set F. So if $s = \dim_H F$, $\mathcal{H}^s(F)$ satisfies $0 < \mathcal{H}^s(F) < \infty$. The problem of Hausdorff dimension is that it is nearly impossible to calculate for real data; hence, we will use the box-count dimension which is simple to estimate and calculate.

3.2 Box-count dimension

The box-count dimension is one of the most widely used ones. It is a special form of the Hausdorff dimension, the difference is that in case of box-count dimension, all cubes covering the set F have the same size, while the Hausdorff dimension gives up on this preconception. Under the weak regularity conditions, the box-count dimension is the same as the Hausdorff dimension. The basic idea of the box-count method is dividing a box to quadrants and leaving out parts not covering the time series, each quadrant is then divided to other four parts; this process continues until the box width equals the resolution of the data. F is a non-empty subset of $\mathbb{R}^n$, $N(\delta)$ is the smallest number of sets of diameter δ needed to cover the set F. The naive box-count estimator can be expressed as

$$\dim_B F = \lim_{\delta \to 0} \frac{\log(N(\delta))}{\log(1/\delta)}. \tag{3.2}$$

When employing the box-counting dimension we suppose that the sets are non-empty and bounded, otherwise we may have problems with log(0) or log(∞).

3.3 Estimation of fractal dimension

There are many ways to estimate fractal dimension. Gneiting et al. (2012) provide a comprehensive study of several approaches to fractal dimension estimation and examine their large sample behaviour considering efficiency and robustness. They also publish R code for estimation of fractal dimension using the studied methods, it is accessible in R package "fractaldim". We use their code in our study. Although Gneiting et al. recommend using of the madogram for measuring fractal dimension of time series, we examine more measures in our study and compare the results. Namely, we employ the box-count estimator, the Hall-Wood estimator, and the madogram, a special case of the variogram, in our study.

3.3.1 Box-count estimator

The basic idea of the box-count estimator is defined by the box-count dimension. Ordinary least square regression of $\log N_\delta(F)$ on $\log(\delta)$ is run and the box count estimator equals to the slope coefficient. Gneiting et al. (2012) suppose sample size to be power of 2, $n = 2^K$, then it is possible to write down the naive box-count estimator as

$$\hat{\dim}_B = -\left\{\sum_{k=0}^{K}(s_k - \bar{s})\log N(\delta_k)\right\}\left\{\sum_{k=0}^{K}(s_k - \bar{s})^2\right\}^{-1}, \tag{3.3}$$

where $s_k = \log \delta_k$ and $\bar{s}$ is the mean of $s_0, s_1, \ldots, s_K$. There are opinions that all scales in regression fit of $\log(N(\delta))$ on $log(\delta)$ may cause a problem; hence, some modifications of the naive box-count estimator were proposed. The model that we use for estimation excludes the smallest scales δ_k for which $N(\delta_k) > \frac{n}{5}$ and the two largest scales from the regression fit; this modification was proposed by Liebovitch and Toth (1989). Although the box-count dimension is defined for $\delta \to 0$ and this modification may thus seem unfortunate, the restriction actually improves statistical and computational efficiency of the estimator.

3.3.2 Hall-Wood estimator

The Hall-Wood estimator is a modified version of the box-count estimator which operates in the smallest observed scales. Hall and Wood (1993) define $A(\delta)$ as the total area of boxes that are needed to cover a graph of time series. The box-count estimator can be than reformulated as

$$\dim_B = 2 - \lim_{\delta \to 0} \frac{\log(A(\delta))}{\log(\delta)}. \tag{3.4}$$

Set scale $\delta_l = \frac{l}{n}$, where $l = 1, 2, \ldots, n$, the estimator of $A(l/n)$ is then

$$\hat{A}(l/n) = \frac{l}{n} \sum_{i=1}^{\lfloor n/l \rfloor} |X_{il/n} - X_{(i-1)l/n}|, \tag{3.5}$$

where X_j is j-th observation in time series.

The Hall-Wood estimator is based on ordinary least square regression fit of $\log \hat{A}(l/n)$ on $\log(l/n)$,

$$\hat{\dim}_{HW} = 2 - \left\{ \sum_{l=1}^{L} (s_l - \bar{s}) \log(\hat{A}(l/n)) \right\} \left\{ \sum_{k=1}^{L} (s_l - \bar{s})^2 \right\}^{-1}, \tag{3.6}$$

where $L \geq 2$, $s_l = \log(l/n)$ and $\bar{s}$ is the mean of $s_0, s_1, \ldots, s_L$. To minimize bias Hall and Wood recommend to use $L = 2$, the expression can be then adjusted to

$$\hat{\dim}_{HW} = 2 - \frac{log(\hat{A}(2/n)) - log(\hat{A}(1/n))}{log(2)}, \tag{3.7}$$

which is the standard implementation of the Hall-Wood estimator.

3.3.3 Variogram

As the madogram is a special case of the variogram, we base the description on the explanation of the general version. The variogram is applied by Burrough (1981) for the very first time; however, it started to be used in statistics more often in 1990's. It is intuitive and easy to implement. A stochastic process $\{X_t : t \in \mathbb{R}\}$ with stationary increments has the following variogram $\gamma(t)$ (or so called structure function):

$$\gamma_2(t) = \frac{1}{2}\mathbb{E}(X_u - X_{u+t})^2, \tag{3.8}$$

the variogram satisfies

$$\gamma_2(t) = |c_2 t|^\alpha + \mathcal{O}(|t|^{\alpha+\beta}) \tag{3.9}$$

as $t \to 0$,where $\alpha \,\epsilon\, (0, 2]$, $\beta \geq 0$, $c_2 > 0$, and $|\cdot|$ stays for Euclidean norm. Then according to Orey (1970), and Adler (1981) the graph of the path has almost surely the following fractal dimension

$$D = d + 1 - \frac{\alpha}{2}. \tag{3.10}$$

Moments estimator for the structure function $\gamma(t)$ at lag $t = l/n$ from time series is classically expressed as

$$\hat{V}_2(l/n) = \frac{1}{2(n-l)} \sum_{i=l}^{n} (X_{i/n} - X_{(i-l)/n})^2. \tag{3.11}$$

Based on relations above, the variogram estimator is

$$\hat{\dim}_{V,2} = 2 - \frac{1}{2} \left\{ \sum_{l=1}^{L} (s_l - \bar{s}) \ \log(\hat{V}_2(l/n)) \right\} \left\{ \sum_{l=1}^{L} (s_l - \bar{s})^2 \right\}^{-1}, \tag{3.12}$$

where $L \geq 2$, $s_l = \log(l/n)$ and $\bar{s}$ is the mean of $s_0, s_1, \ldots, s_L$. Several studies (Constantine and Hall 1994,Davies and Hall 1999, Zhu and Stein 2002) conclude that the bias is minimized for $L = 2$. Gneiting et al. (2012) thus use this setting in their implementation gaining

$$\hat{\dim}_{V,2} = 2 - \frac{\log(\hat{V}_2(2/n)) - log(\hat{V}_2(1/n))}{log(2)}. \tag{3.13}$$

The variogram estimator can be generalized to the variogram of order p of a stochastic process with stationary increments,

$$\gamma_p(t) = \frac{1}{2}\mathbb{E}(X_u - X_{u+t})^p, \tag{3.14}$$

for $p = 2$ we have variogram, $p = 1$ madogram, and $p = 1/2$ rodogram. All of the expressions above can be generalized in similar manner. And it is not necessary to state all of them here, for more details see Gneiting et al. (2012). The critical question is what value of p should be chosen. Hall and Roy (1994) show that the variogram fail easily when applied to non-Gaussian process. The madogram is particularly robust and inclusive which is showed by Bruno and

Raspa (1989) and Bez and Bertrand (2011), but it can fail for very irregular paths. The madogram is a more efficient version of the Hall-Wood estimator.

Chapter 4

Simulation process description

We suppose that fractal dimension, in contrary to autocorrelation tests, captures local inefficiencies. To verify the ability, we simulate efficient time series containing one or more inefficient parts, and we examine the reaction of fractal dimension to changes in the inefficiency.

We use three different types of time series simulations corresponding to the efficient market hypothesis - namely martingales, random walk and Brownian motion - as we want to provide a comprehensive examination of explanatory power of fractal dimension and verify robustness of results. Fractal dimension of the respective time series is 1.5. For the inefficient part we use ARFIMA time series; the magnitude of inefficiency is influenced by the parameters ϕ, θ and d in the ARFIMA simulation as described later in this section. Final fractal dimension of the whole time series depends on the length of the included inefficient time series and the power of the inefficiency. (ARFIMA (0.9, 0.45, 0) has lower fractal dimension then ARFIMA (0.1, 0.45, 0), similarly a time series with longer part of inefficiency has more biased fractal dimension than the one with short part.)

Each time series has 10,000 observations and we simulate a thousand of time series for each setting to make the results statistically significant. The time series consist of efficient and inefficient parts. We measure fractal dimension of each time series using several ways of measurement (the madogram, the box-count estimator and the Hall-Wood estimator) and calculate the mean and tolerance interval for each setting. Now we provide detailed description of used types of time series and their simulation codes.

4.1 Efficient time series

4.1.1 Random walk (RW)

In random walk, each step's direction and size is chosen randomly. Random walk is a non-stationary process, and stock prices are described by the random walk model without drift, which can be write down as

$$Y_t = Y_{t-1} + \epsilon_t, \tag{4.1}$$

where ϵ_t is normally distributed with zero mean and finite variance such that it is independent and identically distributed with $Corr(\epsilon_t, \epsilon_{t-s}) = 0$. The expected value of random walk does not depend on t, but the variance of random walk increases with time; it is actually a linear function of time. Further, the best predictor of any future value of Y is always the current value, no matter how far into the future we look. The change in stock price from one period to the next period is unpredictable.

The code to simulate random walk can be found in the Appendix D.

4.1.2 Brownian motion (BM)

Brownian motion is a continuous-time stochastic process. Similarly to random walk, each individual change is unrelated to the previous and future changes, and it is one of the ways how to describe efficient market. Brownian motion has the following properties: $W_0 = 0$, W_t is almost surely continuous and it has independent increments, and the difference W_t–W_s has normal distribution with zero mean and variance $(t - s)$ for $0 \leq s \leq t$. For simulating this time series we use package "sde" and specifically the function "bm" in R project.

4.1.3 Martingale (M)

As we noted before, LeRoy (1989) showed that martingale and "fair game" are synonyms. We have probability space $(\Omega, \mathcal{F}, P)$, where Ω is a set of market information, $\mathcal{F}$ is σ–algebra of the subsets of Ω, and P is probability measure on $\mathcal{F}$. A sequence $X_1, X_2, \ldots, X_n$ of random variables (represented by stock prices in case of capital market) and corresponding σ–algebras $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ is martingale if it satisfies the following properties:

1. Each X_i is an integrable random variable measurable with respect to the corresponding $\mathcal{F}_i$.
2. Expression $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ holds for every i.
3. Relation $E\{X_{i+1}|\mathcal{F}_i\} = X_i$ a.e. P hold for every $i \in [1, 2, \ldots, n-1]$.

In our case, the information set Ω contains only the historical prices, as we focus on the weak-form of efficient market hypothesis.

The code used for simulation can be found in the Appendix D.

4.2 Inefficient time series

4.2.1 ARFIMA

The model is based on autoregressive moving average model, ARMA. The AR part expresses the linear dependence of current value on previous value(s); sequences with higher AR coefficient are more persistent. The MA part relates to the idea that all variation in a time series is driven by current and past shocks. MA(1) process, which we use in our simulations, has short time memory and weak dynamics, so the MA parameter has a small influence on fractal dimension of a time series in contrast to the AR(1) model. This does not hold for the general model, MA(l) can capture richer dynamic patterns, and AR model can be rewritten to a MA model. Generally, AR(k) model has infinite memory, and MA(l) model has a memory of exactly l periods.[1]

ARMA(1 ,1) can be written as

$$X_t = \alpha_1 X_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t, \tag{4.2}$$

where both $|\alpha_1|$ and $|\theta_1|$ are required to be smaller than 1 to ensure stationarity and invertibility respectively. The constant is set to 0 because returns are generally close to 0.

Generalized model of ARMA(k, l) is then

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \ldots + \alpha_k X_{t-k} + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_l \epsilon_{t-l} + \epsilon_t. \tag{4.3}$$

[1]Note, that we use unusual notations of AR and MA orders - k and l, respectively. We save p and q symbols to mark ϕ and θ coefficients of AR and MA parameters in figures and graphs.

Or using lag operators

$$\left(1 - \sum_{i=1}^{k} \alpha_i L^i\right) X_t = \left(1 - \sum_{i=1}^{l} \theta_i L^i\right) \epsilon_t, \tag{4.4}$$

where L^i are lag operators. For stationarity and invertibility, inverses of all roots of $(1 - \sum_{i=1}^{k} \alpha_i L^i)$ and $(1 - \sum_{i=1}^{l} \theta_i L^i)$ have to be inside the unit circle. If at least one of the k roots of autoregressive lag operator polynomial is 1, the sequence has a unit root and can be written down as

$$\left(1 - \sum_{i=1}^{k} \phi_i L^i\right) (1 - L) X_t = \left(1 - \sum_{i=1}^{l} \theta_i L^i\right) \epsilon_t. \tag{4.5}$$

The sequence is not stationary. General model ARFIMA(k, d, l) is then

$$\left(1 - \sum_{i=1}^{k} \phi_i L^i\right) (1 - L)^d X_t = \left(1 - \sum_{i=1}^{l} \theta_i L^i\right) \epsilon_t, \tag{4.6}$$

where d is an integer order of differencing. However, the condition of d as an integer is too binding as there are time series with too much long-range dependence to be stationary $I(0)$, but they are still not truly non-stationary $I(1)$. The time series are represented by ARFIMA models, which allow d to take values in $-0.5 < d < 0.5$, for $d \geq 0.5$ the process is non-stationary. The process is said to be anti-persistence for $d \in (-0.5, 0)$. For $d \in [0, 0.5)$ the autocorrelation of ARFIMA process decays hyperbolically to zero, stationary ARMA processes decay to zero geometrically, which is faster.

For simulation of these series we use package "fracdiff" and to gain series with d higher than 0.5 ensuring non-stationarity, we use the cumsum function.

The whole process of simulation has the following steps:

1) We simulate the first part of random walk/Brownian motion/martingale.

2) We take the last point of the simulation and use it as the first one for ARFIMA simulation. ARFIMA simulations may have totally different scale than the efficient part and it may not fit to the simulated efficient part at all. To make the two parts as similar as possible, we adjust the ARFIMA's scale. We define that the difference between the maximum and minimum of ARFIMA series has to be equal to the difference between highest and lowest point of given number of last efficient series observations. We take 1.5 longer part than is the length of ARFIMA to alleviate the restriction, as ARFIMA series mostly

stretches over much wider range of values than the efficient simulation. In the end we shift the ARFIMA series to begin in the last point of efficient part simulation.
3) The last part of the simulations is again an efficient one. We take the last observation of ARFIMA part and use it as the starting point for simulation of random walk/Brownian motion/martingale.

Now the simulated series is complete, and it consists of two efficient and one inefficient part. To see how the individual parts fit to each other we enclose few graphs of random walk with ARFIMA parts, see Figure B.7 in the Appendix. The code used for simulations can be found in the Appendix D.

4.3 Simulations setting

Each time series has 10,000 observations and we simulate a thousand of time series for each setting to ensure statistical significance of results. The time series consist of efficient and inefficient parts. The efficient parts are created by simulations of random walk (the core of simulations in Sections 5.4.1 and 5.4.2), Brownian Motion, and martingales and outputs of the three types of simulations are compared (the results can be found in Section 5.4.10). The inefficient parts are based on ARFIMA simulations. We need to limit the study to a manageable scope, so the autoregressive and moving average order is fixed to 1 and we change only the coefficients. Even so we estimate nearly 700 different settings of ARFIMA. Details about ARFIMA simulation are presented in Sections 4.2.1 and 5.2.

Most of the time series simulations include only one inefficient part as we want to track the effect of changes in inefficiency as accurately as possible. To explore the impact of the length of time series, we use inefficient time series with 300, 500 and 1,000 observations. The effect of observations numbers is examined in more details for chosen inefficiency settings in Section 5.4.3. Time series with more than one inefficient part are studied in Sections 5.4.6 and 5.4.7 and we specifically focus on inclusion of parts with contradictory effect on fractal dimension and on gradual estimation of fractal dimension of time series composed of parts with slightly different efficiency structure. This approach examines the ability of fractal dimension to track small changes in the efficiency structure.

Fractal dimension is estimated by three different estimators, namely the madogram, the box-count estimator, and the Hall-Wood estimator. The most efficient estimator is the madogram; hence, it is used as a core estimator. The other estimators are investigated in less detail and the results are compared. As the estimators depend on number of lags used for estimations, the study of lags' impact on resulting fractal dimension is examined (Section 5.1) and estimators using 5, 20 and 100 lags are used in the core study.

The coefficients ϕ and θ are marked as p and q in all tables and graphs; we do not mix the notation in text. Moreover, the expression ARFIMA(0.6, 0.2, 0.4) means that the ϕ, d, and θ coefficient are equal to 0.6, 0.2 and 0.4, respectively. Results are presented in tables, 2D graphs and 3D graphs. We use fractal dimensions of the thousands simulations for each setting and calculate average fractal dimension and standard error for each setting. The values of means and standard errors are listed in tables, most of the tables have fixed θ coefficient, ϕ coefficient changes in rows and d in columns. Shading represents trends in fractal dimension with changing either based on the column or row variable.

Further, the results are plotted in 2D figures including average fractal dimension and confidence intervals. Fractal dimension is assigned to y-axis, while the x-axis shows values of d coefficient. Coefficients ϕ and θ are fixed. 3D graphs provide a more complex view on the relationship between fractal dimension and individual parameters of ARFIMA because it allows two coefficients to differ. The graphs contain only average fractal dimensions to keep the graphs as clear as possible. Fractal dimension is assigned to vertical axes, other axes denote chosen parameters (ϕ, θ, d; one is always fixed); shading stresses the shape of graph.

Chapter 5

Results of simulations

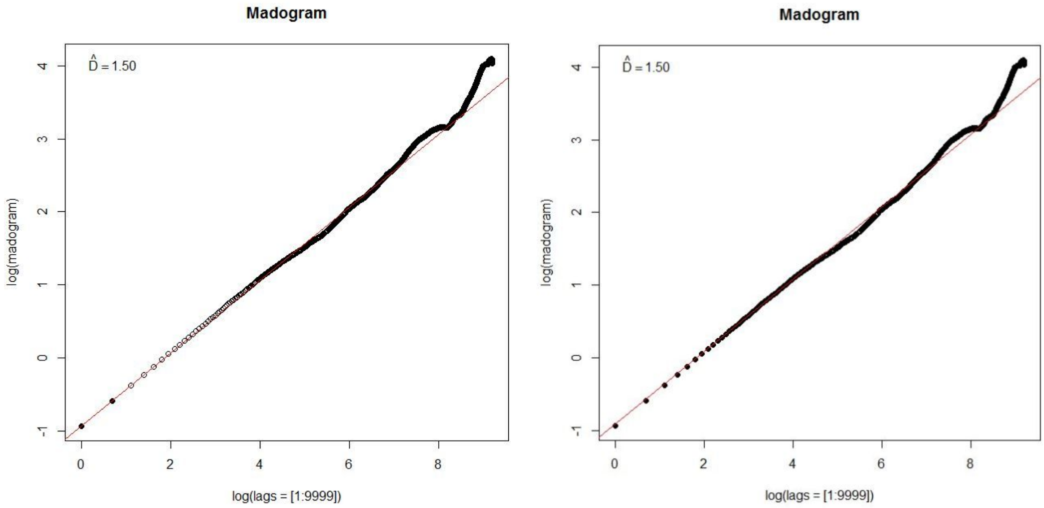
We now turn to the practical part of the thesis and present results of the simulations in this section. At first, we are challenged by a problem of number of lags. Secondly, we study fractal dimension of ARFIMA time series and then we focus on the explanatory power of fractal dimension in the context of efficient markets.

5.1 Problem of number of lags

The estimators of fractal dimension work with the number of lags determining the number of points used for fitting the line in the log-log regression as illustrated in Figure 5.1; only the black points are used for fitting the line.

The fractal dimension package by Gneiting et al. (2012) enables us to choose "auto" number of lags which corresponds to the theoretically "best" value – the theoretically optimal number of lags for the madogram is two, and the same holds for the Hall-Wood estimator. The box-count estimator determines the smallest m for which $n \leq 2^m$, where n is the number of observations; value $n_{eff} = 2^{m-1} + 1$ is calculated to define data points used for estimation: $x_1, x_2, ..., x_{n_{eff}}$. The "auto" option considers box sizes ϵ_k for $k = j, j+1, ..., m-2$, where $N(\epsilon_i) > \frac{n_{eff}}{5}$ holds for all $i < j$; this restriction corresponds to the modification by Liebovitch and Toth (1989) and eliminates two largest and very small boxes. The time needed for estimation is much higher for the automatic option than for filling in a specific number (approximately 40 times higher), so we do not use the "auto" option for the madogram and Hall-Wood estimators and rather fill in the optimal number of lags. The optimal number of lags is determined as the one with the smallest mean squared error (MSE); Davies

Figure 5.1: Log-log regression for 2 and 100 lags



and Hall (1999) prove the statement about the optimal number of lags for the Hall-Wood and madogram estimators numerically.

Gneiting et al. (2012) use the root mean squared error (RMSE) for comparison of individual estimators; we are inspired by their approach when examining influence of the number of lags on resulting fractal dimension. Simulation of Gaussian process is used for this purpose. Further, we examine the impact of number of lags for simulated combinations of random walk and ARFIMA series to find out how the individual estimators respond to changes in the ARFIMA setting. In that case, we focus on means and standard errors of the estimated fractal dimensions.

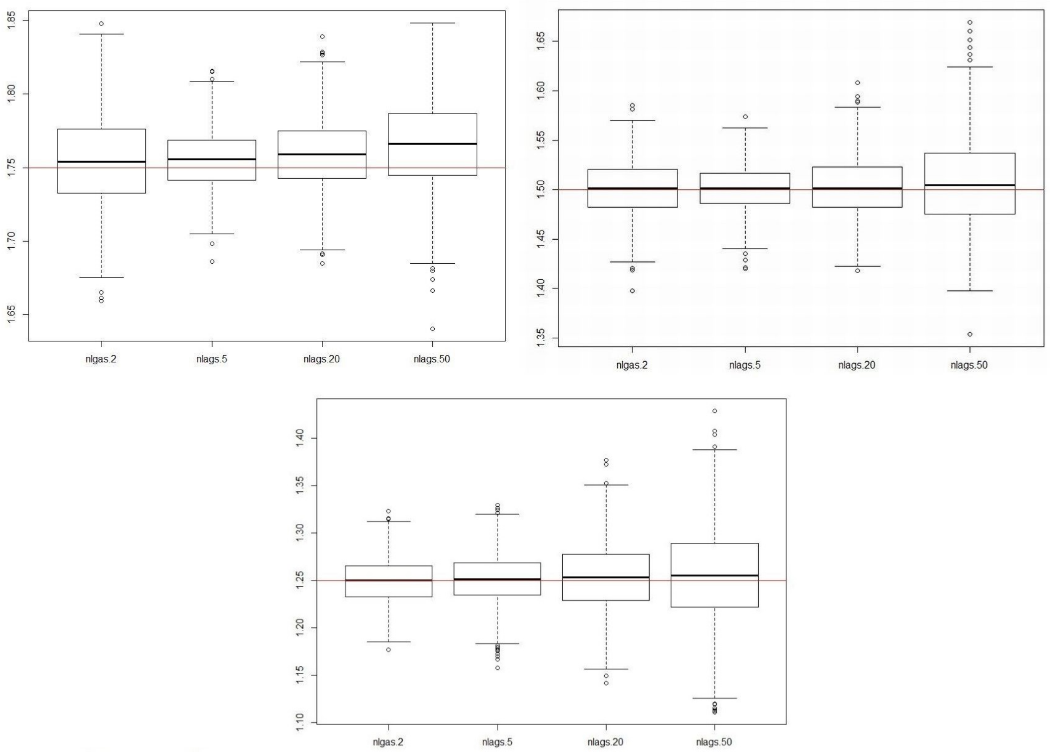
5.1.1 Gaussian process

We simulate Gaussian process with the powered exponential covariance function:

$$\sigma(t) = exp(-|t|^{\alpha}), \tag{5.1}$$

and use fractal index, α, equal to 0.5, 1 and 1.5. Adler (1981) says that the corresponding values of fractal dimension, D, are then given by $D = 2-\alpha/2$, so the estimated fractal dimensions for the chosen fractal indices should be equal to 1.75, 1.5 and 1.25, respectively. The sample size of our simulation is 1024 as in the study by Gneiting et al. (2012); the number of replicates for each α is 1,000, and we use 2, 5, 20, and 50 lags. In case of the box-count estimator,

Figure 5.2: Box plot of madogram estimations of fractal dimensions for α equal to 1.5, 1, and 0.5 implying fractal dimension 1.75, 1.5, and 1.25



we included the "auto" option as well. The results for all estimators, numbers of lags and α are presented in Figures 5.2, and B.1 and B.2 in the Appendix.

As can be observed, we confirm the results of Gneiting et al. (2012), who mark the madogram estimator as a more efficient version of the Hall-Wood estimator; both of the estimators are quite accurate especially for small number of lags. The box-count estimator underestimates actual fractal dimension. Nevertheless, we use it on a small sample of series to be able to compare the results of the three estimators.

Although the optimal number of lags in the madogram and Hall-Wood estimators is, according to Davies and Hall (1999), equal to 2, we obtain the smallest RMSE for the 5 lags estimators. The RMSE then increases with the number of lags, but the means of the fractal dimensions stay accurate. We conclude that both the madogram and Hall-Wood estimators capture fractal dimension well, even though that the number of observations is limited. The box-count estimator tends to significantly underestimate fractal dimension for all examined number of lags and settings of the simulated Gaussian process.

The estimates of series with fractal dimension equal to 1.5 are very accurate for all number of lags including the big ones. This fact is very important, as the efficient markets are characterized by fractal dimension equal to 1.5, and we want to determine the efficient markets flawlessly. For the other settings, the madogram and Hall-Wood estimators tend to slightly overestimate fractal dimension. However, in these cases, we are not interested in the exact value of fractal dimension - we rather care about the ranking, so it is important that the estimated fractal dimension reflects all changes in the series autocorrelation structure accurately. This property of the estimators is examined in the following sections. Later in this chapter, we show that estimators with higher number of lags have some more desirable results, but we are aware of the inaccuracies connected with them, so we use both small and large numbers of lags for estimation of fractal dimensions and compare the results.

We examine not only the effect of number of lags, but we provide an comparison of the efficiency of the different estimators as well. Figure 5.3 shows the box plot of the estimated fractal dimensions for the fractal index equal to 1 obtained through the madogram estimator with 5 lags, the Hall-Wood estimator with 5 lags and the box-count estimator with the “optimal” number of lags. Based on the lags analysis discussed above, these numbers of lags reflect actual fractal dimension most accurately.

5.1.2 Series with efficient and inefficient parts

Time series are simulated using the process described in Section 4.3. In this analysis, we operate with the random walk simulations and the madogram estimator because it is the most efficient estimator and random walk is easy and quick to simulate. We are not able to numerically determine exact fractal dimension of a combination of random walk and ARFIMA series, so we cannot use RMSE as the criterion for estimators' evaluation. But we can check how the estimations using different numbers of lags respond to changes in setting of ARFIMA. Further, we observe standard error of the estimations, as we are interested in the least volatile estimator.

The process of fractal dimension estimation is following:

1. Simulation of 1,000 time series for each setting. (Each setting has 10,000 observations.)
2. Fractal dimension estimation of each time series.
3. Calculation of the average fractal dimension and standard error for each

Figure 5.3: Comparison of estimators - madogram (5 lags), Hall-Wood (5 lags) and box-count (“auto” lags)

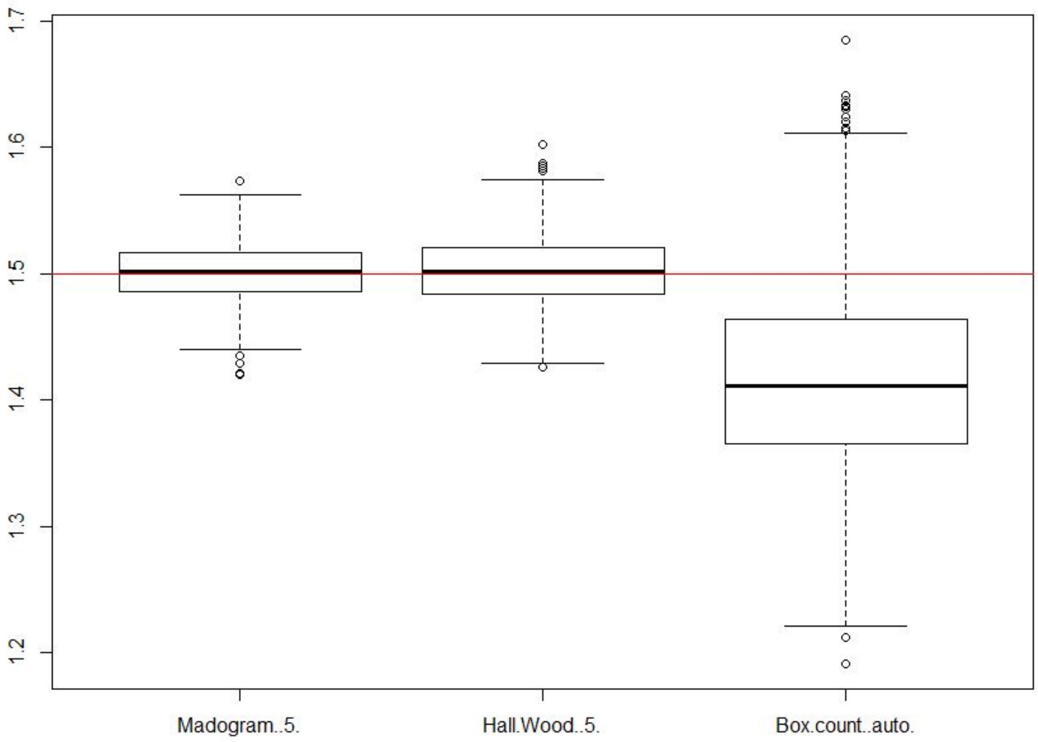


Table 5.1: Fractal dimensions of RW with ARFIMA(0, d, 0) in (20001:3000) estimated by madogram with changing number of lags

ARFIMA(0, d, 0) in (2001:3000), mean

nlags/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
2	1.5684	1.5441	1.5228	1.5095	1.4998	1.4955	1.4932	1.4933	1.4941
3	1.5694	1.5447	1.5235	1.5099	1.4998	1.4951	1.4930	1.4928	1.4938
4	1.5696	1.5451	1.5240	1.5102	1.4998	1.4949	1.4927	1.4925	1.4934
5	1.5696	1.5452	1.5244	1.5105	1.4997	1.4947	1.4925	1.4922	1.4931
6	1.5695	1.5452	1.5247	1.5106	1.4997	1.4946	1.4924	1.4919	1.4928
7	1.5693	1.5451	1.5248	1.5107	1.4997	1.4944	1.4922	1.4916	1.4926
8	1.5690	1.5450	1.5249	1.5108	1.4998	1.4943	1.4920	1.4914	1.4924
9	1.5686	1.5448	1.5250	1.5109	1.4998	1.4942	1.4918	1.4912	1.4922
10	1.5683	1.5447	1.5251	1.5109	1.4998	1.4941	1.4917	1.4911	1.4920
15	1.5667	1.5441	1.5251	1.5111	1.5000	1.4938	1.4912	1.4905	1.4911
20	1.5652	1.5435	1.5250	1.5112	1.5001	1.4936	1.4909	1.4901	1.4906
50	1.5588	1.5405	1.5240	1.5120	1.5000	1.4930	1.4906	1.4881	1.4880
100	1.5532	1.5379	1.5226	1.5126	1.4997	1.4934	1.4903	1.4866	1.4857
500	1.5398	1.5310	1.5215	1.5136	1.5020	1.4959	1.4928	1.4844	1.4832
1000	1.5364	1.5284	1.5205	1.5160	1.5070	1.4993	1.4975	1.4871	1.4853

setting.

The estimated fractal dimensions of Gauss process suggest using smaller number of lags; hence, we focus on comparing the results of the estimations done with small number of lags, but we also use a few estimators with higher number of lags to analyse their behaviour. The higher number of observations (10,000 instead of 1024 used in the previous section) allows us to examine wider range of number of lags. Specifically, we use the following numbers of lags: 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 50, 100, 500, and 1,000.

In the number of lag analysis, we use cumulative summations of ARFIMA (0, d, 0) and ARFIMA(0.4, d, 0.4), d in (-0.5, 0.5), which clearly demonstrate problems of smaller number of lags and simultaneously allow us to study the estimators performance in case of random walk (ARFIMA(0, 0, 0)). A thousand of random walk observations is replaced by an ARFIMA series. We track mean fractal dimension and volatility (standard error) of fractal dimension for changing d.

The means of the estimated fractal dimensions and the related standard errors for ARFIMA(0, d, 0) are presented in Tables 5.1 and 5.2, respectively.[2]

[2]The notation ARFIMA(0, d, 0) means that $\phi = 0$ and $\theta = 0$. The k and l parameters

Table 5.2: Standard errors of fractal dimension in Table 5.1

ARFIMA(0, d, 0) in (2001:3000), sd

nlags/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
2	0.0186	0.0151	0.0113	0.0089	0.0086	0.0090	0.0093	0.0093	0.0095
3	0.0179	0.0141	0.0104	0.0082	0.0077	0.0080	0.0086	0.0086	0.0086
4	0.0177	0.0140	0.0102	0.0081	0.0074	0.0078	0.0084	0.0086	0.0084
5	0.0176	0.0140	0.0102	0.0081	0.0074	0.0077	0.0085	0.0087	0.0085
6	0.0176	0.0140	0.0103	0.0081	0.0076	0.0078	0.0087	0.0088	0.0087
7	0.0177	0.0140	0.0104	0.0082	0.0077	0.0080	0.0089	0.0090	0.0089
8	0.0177	0.0141	0.0105	0.0083	0.0079	0.0082	0.0091	0.0091	0.0091
9	0.0178	0.0142	0.0106	0.0084	0.0081	0.0084	0.0093	0.0093	0.0094
10	0.0179	0.0143	0.0108	0.0085	0.0082	0.0086	0.0096	0.0095	0.0096
15	0.0179	0.0148	0.0115	0.0093	0.0090	0.0095	0.0105	0.0102	0.0104
20	0.0180	0.0152	0.0121	0.0101	0.0098	0.0103	0.0112	0.0110	0.0113
50	0.0188	0.0172	0.0152	0.0134	0.0141	0.0137	0.0143	0.0145	0.0157
100	0.0211	0.0198	0.0191	0.0178	0.0186	0.0179	0.0186	0.0189	0.0201
500	0.0387	0.0386	0.0389	0.0380	0.0377	0.0401	0.0384	0.0375	0.0384
1000	0.0557	0.0539	0.0551	0.0543	0.0538	0.0561	0.0555	0.0539	0.0530

We find that the average fractal dimension decreases with the number of lags. For high number of lags, the fractal dimension captures higher volatility in the simulated time series very weakly, the mean of the estimated fractal dimensions of time series with the most volatile ARFIMA, ARFIMA(0,-0.4, 0), is 1.536 for 1,000 lags, whereas it is 1.568 for 2 lags.

On the other hand, positive autocorrelation in differences is more significantly captured by the estimations with higher number of lags, the average fractal dimension for setting ARFIMA (0, 0.4, 0) is 1.485 for 1,000 lags and 1.494 for 2 lags. This difference is not as striking as for negative d parameters: the estimators with smaller number of lags capture the ARFIMA part in random walk series with wider range of values.

Looking at ARFIMA (0, 0, 0), we can see that the mean fractal dimension is very close to 1.5 for all estimators; 1,000 lags estimation with the fractal dimension equal to 1.507 deviates from the desired 1.5 value the most, while the other average fractal dimensions differ by less than 0.003.

The standard errors of the fractal dimensions increase with the number of lags. The standard errors of the fractal dimensions obtained by the 2 lags estimator are in the interval (0.008, 0.021), but for 1,000 lags the standard errors increase above 0.05, meaning that the estimations are less accurate.

are set to 1 in the whole thesis, so we use this type of notation to simplify the labelling. The shading captures trends in the fractal dimensions.

Table 5.3: Fractal dimensions of RW with ARFIMA(0.4, *d*, 0.4) in (20001:3000) estimated by madogram with changing number of lags

ARFIMA(0.4, d, 0.4) in (2001:3000), mean

nlags/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
2	1.4595	1.4619	1.4671	1.4715	1.4770	1.4830	1.4873	1.4909	1.4934
3	1.4701	1.4692	1.4715	1.4741	1.4780	1.4832	1.4871	1.4902	1.4929
4	1.4785	1.4751	1.4752	1.4764	1.4791	1.4835	1.4870	1.4899	1.4925
5	1.4851	1.4800	1.4785	1.4784	1.4802	1.4839	1.4869	1.4896	1.4921
6	1.4906	1.4841	1.4813	1.4803	1.4812	1.4842	1.4870	1.4894	1.4918
7	1.4952	1.4876	1.4838	1.4819	1.4821	1.4846	1.4870	1.4892	1.4914
8	1.4991	1.4906	1.4859	1.4833	1.4829	1.4849	1.4870	1.4890	1.4912
9	1.5024	1.4931	1.4878	1.4845	1.4837	1.4852	1.4870	1.4889	1.4909
10	1.5052	1.4954	1.4894	1.4856	1.4844	1.4855	1.4870	1.4887	1.4907
15	1.5152	1.5033	1.4953	1.4898	1.4870	1.4866	1.4872	1.4881	1.4898
20	1.5211	1.5082	1.4990	1.4926	1.4889	1.4874	1.4873	1.4877	1.4891
50	1.5326	1.5196	1.5080	1.5001	1.4935	1.4893	1.4871	1.4869	1.4863
100	1.5362	1.5239	1.5124	1.5042	1.4961	1.4908	1.4873	1.4862	1.4841
500	1.5443	1.5343	1.5232	1.5123	1.5026	1.4938	1.4853	1.4819	1.4736
1000	1.5404	1.5348	1.5264	1.5152	1.5084	1.5000	1.4893	1.4858	1.4757

Table 5.4: Standard errors of fractal dimension in Table 5.3

ARFIMA(0.4, d, 0.4) in (2001:3000), sd

nlags/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
2	0.0136	0.0132	0.0133	0.0132	0.0129	0.0114	0.0106	0.0102	0.0096
3	0.0109	0.0114	0.0117	0.0120	0.0117	0.0105	0.0097	0.0096	0.0088
4	0.0094	0.0103	0.0108	0.0113	0.0112	0.0102	0.0096	0.0096	0.0087
5	0.0085	0.0096	0.0101	0.0108	0.0108	0.0102	0.0096	0.0097	0.0088
6	0.0081	0.0092	0.0096	0.0104	0.0105	0.0102	0.0097	0.0098	0.0090
7	0.0080	0.0090	0.0092	0.0102	0.0103	0.0102	0.0098	0.0099	0.0092
8	0.0080	0.0089	0.0090	0.0100	0.0101	0.0102	0.0099	0.0100	0.0094
9	0.0082	0.0089	0.0089	0.0099	0.0100	0.0103	0.0100	0.0101	0.0097
10	0.0084	0.0090	0.0089	0.0099	0.0100	0.0103	0.0101	0.0102	0.0099
15	0.0099	0.0097	0.0093	0.0101	0.0100	0.0107	0.0108	0.0110	0.0108
20	0.0111	0.0105	0.0100	0.0107	0.0103	0.0111	0.0116	0.0117	0.0116
50	0.0153	0.0144	0.0137	0.0138	0.0133	0.0144	0.0154	0.0152	0.0156
100	0.0192	0.0183	0.0186	0.0184	0.0179	0.0189	0.0197	0.0200	0.0204
500	0.0353	0.0361	0.0385	0.0384	0.0407	0.0398	0.0381	0.0399	0.0394
1000	0.0522	0.0520	0.0546	0.0531	0.0577	0.0569	0.0548	0.0564	0.0573

In case of ARFIMA(-0.4, d, 0.4), the results are skewed, especially for small numbers of lags, where the fractal dimension increases with d parameter instead of decreasing. This bias diminishes with higher number of lags and the 100 lags estimator returns the expected result. But as the AR coefficient grows, skewness starts to appear in estimations using large number of lags as well. We do not expect extremely strong autocorrelation in the price differences to occur in real markets, so this bias should not influence our efficiency ranking.

As the estimators with higher number of lags give more accurate results in specific cases, we do not restrict our analysis to small number of lags; we try to estimate fractal dimension with higher number of lags up to 100 as well being aware of higher standard errors.

5.2 Simulated ARFIMA

In our study, we simulate stock market prices following the random walk / martingale process consistent with the efficient market theory. These processes are defined by no autocorrelation in returns specified as differences. To verify the power of fractal dimension, we include parts of time series that have autocorrelated first differences – both negatively and positively. For this purpose we use simulations of ARFIMA, specifically its cumulative summation.

We manage the level of autocorrelation through only coefficients of autoregressive and moving average parameters and not through the orders of autoregressive and moving average parts; otherwise the simulations would be too extensive, so all simulated ARFIMA are ARFIMA(1, d, 1).

The financial applications usually consider ARFIMA(1, d, 0) as a representative of inefficient time series; the inclusion of moving average parameter is less common. As we want to provide as comprehensive analysis as possible, we include the moving average parameter as well and track the influence of its coefficient on fractal dimension.

As the orders of both autoregressive and moving average parts are always equal to one, we change the usual notation mentioned in previous sentence to present the values of coefficients instead of orders; hence, ARFIMA(p, d, q)= =ARFIMA(ϕ, d, θ) to note the coefficients of autoregressive and moving average part, so ARFIMA (0.6, 0.4, 0.2) stays for $(1-0.6L)(1-L)^{0.4}(X_t - X_{t-1})=$ $=(1-0.2L)\epsilon_t$. In tables and figures, we use p and q to denote the coefficients, not the orders.

In simulations, parameter d takes values between -0.45 and 0.45 changing by 0.05 (19 different settings), ϕ coefficient of autoregressive parameter takes values between 0 and 0.95 changing by 0.2 except of the last step (6 different settings), and θ coefficient of moving average parameter changes in the same manner as the autoregressive parameter coefficient (6 different settings). By combining these three parameters, we get $19 \cdot 6 \cdot 6 = 684$ different settings examined in our study. Further, we test the ability of fractal dimension to capture the length of included inefficiency; we include 300, 500 and 1,000 inefficient observations.

The negative parameter d and the positive coefficients ϕ and θ may have contradictory impact on fractal dimension and the resulting fractal dimension of the whole time series may be close to 1.5 value for other settings than ARFIMA(0, 0, 0) as well. We can show this using the ARFIMA equation:

$$\left(1 - \sum_{i=1}^{k} \phi_i L^i\right)(1 - L)^d Y_t = \left(1 - \sum_{i=1}^{l} \theta_i L^i\right) .\epsilon_t \tag{5.2}$$

As we use the cumulative summation, Y_t stays for $(X_t - X_{t-1})$, where X_t is stock market price at time t.

In our specifications we have k ad l always equal to 1:

$$(1 - \phi L)(1 - L)^d (X_t - Xt - 1) = (1 - \theta L)\epsilon_t, \tag{5.3}$$

where

$$(1 - L)^d = \sum_{i=0}^{\infty} \left(\begin{array}{c} d \\ k \end{array}\right) (-L)^d = 1 - dL + \frac{d(d-1)}{2!}L^2 - \frac{d(d-1)(d-2)}{3!}L^3 + \dots \tag{5.4}$$

As we include only one part of moving average (l parameter is always equal to one), the moving average part has only a weak effect as described in the previous chapter. Hence, we focus on the opposite effects of d and ϕ coefficients.

We get random walk also if $(1 - \phi L)(1 - L)^d = 1$. The expression can be rewritten as:

$$(1-\phi L)(1-L)^d = 1 - dL + \frac{d(d-1)}{2!}L^2 - \frac{d(d-1)(d-2)}{3!}L^3 + \ldots - \phi L +$$
$$+ \phi dL^2 - \phi\frac{d(d-1)}{2!}L^3 - \ldots$$
$$= 1 - (\phi + d)L1 + (\frac{d(d-1)}{2!} + \phi d)L^2 - (\frac{d(d-1)(d-2)}{3!} +$$
$$+ \phi\frac{d(d-1)}{2!})L^3 + \ldots \tag{5.5}$$

The infinite summation is probably never equal to one; however, for negative d, it may be close to one. Table A.1 in the Appendix shows the coefficients of back-shift operators; the random walk has all of the coefficients equal to zero; for $\phi = 0.1$ and $d = -0.1$, the coefficient of L is equal to 0 and the coefficients of other back-shift operators are close to 0 and diminishing, which means that the effect is very weak and fractal dimension should be close to 1.5.

If the coefficients d and ϕ have opposite signs and same magnitude, there is no first order autocorrelation and the autocorrelations of higher orders decreases, as the coefficients converge to 0. The coefficients are positive, meaning that the positive autocorrelation effect of ϕ is outweighed by the negative autocorrelation effect of d, and fractal dimension should be larger than 1.5. According to the theoretically calculated coefficients, fractal dimension should be a bit more sensitive to the changes in d parameter than to the changes in ϕ coefficient, as shown in Tables A.2 and A.3. When we keep constant ϕ coefficient and change the d coefficient starting in ARFIMA(0.1, -0.4, 0) and ending in ARFIMA(0.1, 0.4, 0), we end up with slightly higher positive numbers than in case of keeping d constant and changing ϕ coefficient, starting in the same point and ending in ARFIMA(0.9, -0.4, 0).[3]

The graphs of time series give a clue about fractal dimension of the time series. The box-count estimator of fractal dimension uses the number of boxes that are crossed by the graph, so the visual appearance of graphs indicates fractal dimension; higher volatility implies higher fractal dimension. An overview of ARFIMA graphs can be found in the Appendix, Figures B.3 and B.4. The graphs give an idea how the ARFIMA series visually vary for different coefficients. The graphs show that the most visible changes are caused by the differencing parameter; the impact of changes in the autoregressive coefficient

[3]The ϕ coefficient is marked as p in all tables.

is visually perceptible as well; the least noticeable changes are those caused by moving average coefficient.

We estimate fractal dimension for series with 500 observations and for each setting we simulate 1,000 series; the ARFIMA analysis is done by the 5 lags madogram estimator. The estimated fractal dimensions correspond to the theory and reflect the changes in ARFIMA setting as expected and suggested by the Figures B.3 and B.4. We illustrate the fact by tables of the average fractal dimensions and 3D graphs.

Graphs in Figure 5.4 are smooth, indicating good performance of the fractal dimension estimators, which capture all changes in the setting of ARFIMA series on average. Decrease in the fractal dimension is steeper for the differencing coefficient than for the moving average and autoregressive coefficients. While the difference between the differencing parameter and the autoregressive coefficient is small, the fractal dimension decreases with the moving average coefficient much slower than with the differencing parameter; this is due to the weak effect of the moving average parameter described in the Section 3.

Further details including standard error of the estimated fractal dimensions can be found in Table 5.5. The standard errors are small with average of 0.0220 indicating that all measured fractal dimensions are close to the mean.

Detailed fractal dimension results presented in this section are estimated using the madogram with 5 lags. The trends in the estimated fractal dimensions are the same for estimators with all number of lags, but the values exhibit some peculiarities shown in Table 5.6, the shaded settings are referred in text.

As described above, it is difficult to strictly say which ARFIMA setting should have fractal dimension lower than 1.5 because of the prevailing positive autocorrelation and vice versa. For example in settings such as $\phi = 0.4$ and $d = -0.35$, there is a positive first order autocorrelation and a negative autocorrelation of higher orders. To make the computation of lag coefficients in ARFIMA time series easier, we focus on ARFIMA(ϕ, d, 0) settings.

Fractal dimension of ARFIMA(0.3, -0.4, 0) should be higher than 1.5, as negative autocorrelation is present in all orders and the coefficients of the backshifts operators decreases only very slowly, the function can be written as

$$(1 + 0.1L + 0.16L^2 + 0.14L^3 + 0.1232L^4 + 0.11043L^5 + \dots)X_t = \epsilon_t. \quad (5.6)$$

The estimated fractal dimensions for all examined number of lags are higher than 1.5, but whereas the fractal dimension for 2 lags is only 1.536, for 100

Figure 5.4: 3D graph of mean fractal dimensions of ARFIMA time series, one coefficient is fixed, the others are assigned to x and y axes, madogram

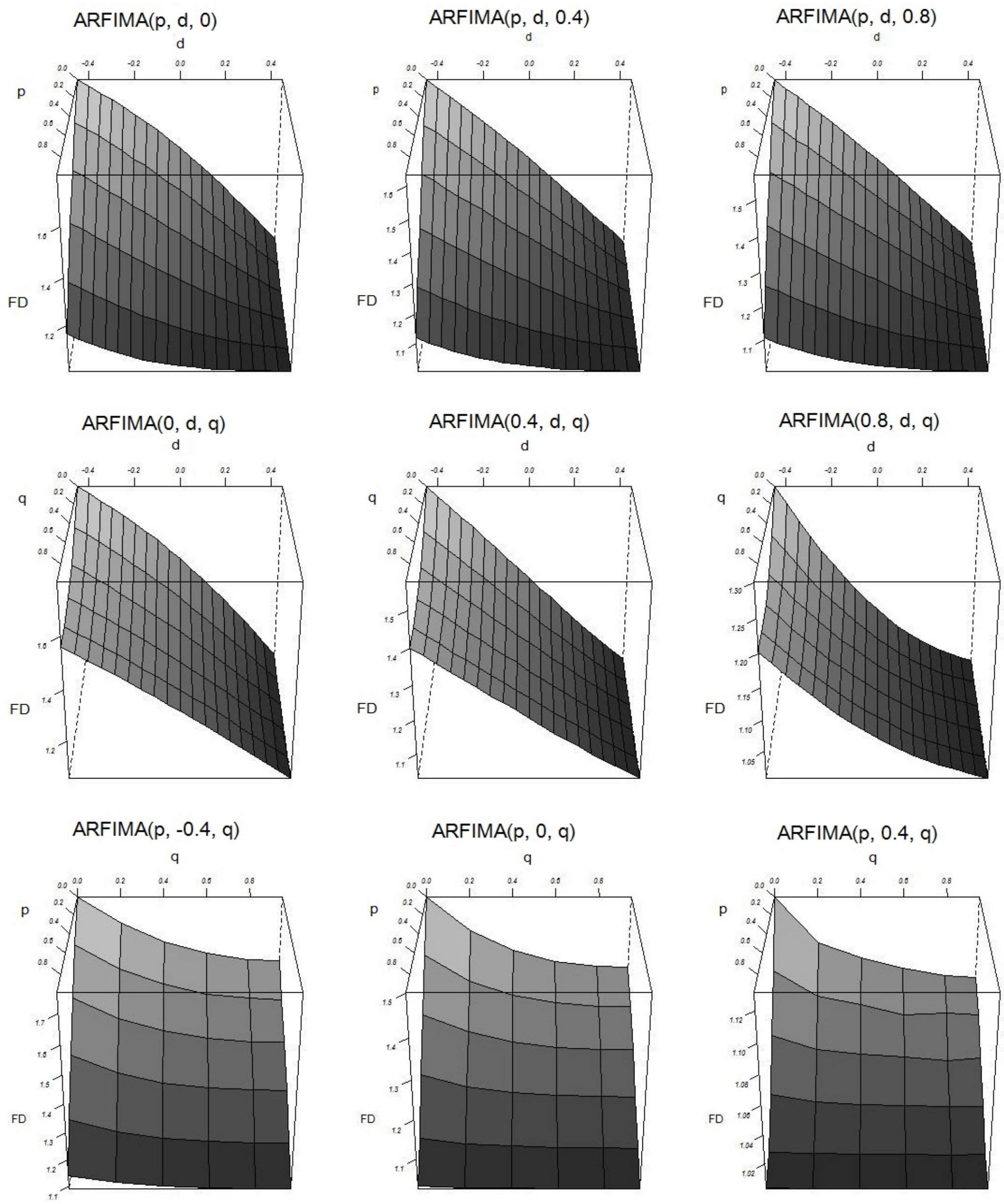


Table 5.5: Means (1st value) ad standard errors (2nd values) of fractal dimensions for ARFIMA, 5 lags madogram

ARFIMA(p, d, 0)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.76792	1.71068	1.64604	1.57812	1.50279	1.41721	1.32775	1.23344	1.13100
	0.03125	0.03111	0.03138	0.03389	0.03395	0.03450	0.03690	0.04408	0.05308
0.4	1.54454	1.48450	1.42377	1.36158	1.29918	1.23973	1.17686	1.11824	1.06239
	0.03221	0.03043	0.02977	0.02924	0.02743	0.02728	0.02723	0.02853	0.02902
0.8	1.27491	1.22334	1.17734	1.13554	1.09961	1.06982	1.04625	1.02734	1.01319
	0.02773	0.02403	0.02227	0.02005	0.01732	0.01447	0.01262	0.01119	0.00899

ARFIMA(p, d, 0.4)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.60574	1.55042	1.49007	1.42821	1.36537	1.29812	1.22903	1.15479	1.08897
	0.03185	0.03156	0.03146	0.03001	0.02975	0.03019	0.03051	0.03467	0.03846
0.4	1.42326	1.37160	1.32464	1.27493	1.22601	1.17958	1.13281	1.08828	1.04865
	0.03057	0.02775	0.02663	0.02666	0.02323	0.02205	0.02249	0.02266	0.02433
0.8	1.20434	1.16782	1.13263	1.10460	1.07773	1.05584	1.03721	1.02377	1.01166
	0.02248	0.02058	0.01865	0.01603	0.01409	0.01187	0.01064	0.00947	0.00795

ARFIMA(p, d, 0.8)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.53965	1.48918	1.43664	1.37999	1.32315	1.26359	1.20208	1.13747	1.07576
	0.03182	0.03137	0.03049	0.02929	0.02819	0.02846	0.02850	0.03118	0.03428
0.4	1.38131	1.33674	1.29439	1.25142	1.20756	1.16357	1.12320	1.08116	1.04356
	0.02830	0.02794	0.02528	0.02359	0.02279	0.02160	0.02112	0.02208	0.02282
0.8	1.18714	1.15293	1.12359	1.09611	1.07300	1.05193	1.03563	1.02229	1.01111
	0.02132	0.01911	0.01727	0.01491	0.01321	0.01141	0.01028	0.00943	0.00783

Table 5.6: Differences among fractal dimensions estimated using different number of lags, madogram, shaded settings are referred in text

ARFIMA(0.3, d, 0)

nlags/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
2	1.5359	1.4826	1.4266	1.3697	1.3112	1.2495	1.1878	1.1253	1.0658
5	1.6045	1.5442	1.4818	1.4168	1.3496	1.2809	1.2112	1.1422	1.0756
15	1.6932	1.6260	1.5576	1.4833	1.4056	1.3266	1.2472	1.1677	1.0906
20	1.7130	1.6445	1.5749	1.4982	1.4186	1.3371	1.2557	1.1737	1.0944
50	1.7653	1.6948	1.6187	1.5362	1.4525	1.3648	1.2794	1.1902	1.1062
100	1.7942	1.7223	1.6428	1.5565	1.4712	1.3799	1.2946	1.2025	1.1158

ARFIMA(0.6, d, 0)

nlags/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
2	1.3584	1.3063	1.2550	1.2072	1.1610	1.1195	1.0815	1.0500	1.0238
5	1.4159	1.3588	1.3039	1.2500	1.1991	1.1516	1.1069	1.0682	1.0344
15	1.5352	1.4720	1.4092	1.3443	1.2826	1.2214	1.1615	1.1085	1.0579
20	1.5693	1.5048	1.4401	1.3725	1.3071	1.2422	1.1777	1.1204	1.0651
50	1.6645	1.5991	1.5269	1.4531	1.3780	1.3025	1.2253	1.1561	1.0888
100	1.7192	1.6533	1.5773	1.4988	1.4198	1.3398	1.2553	1.1803	1.1058

number of lags the fractal dimension reaches 1.794; this difference suggests that some of the estimators overestimate fractal dimension. From the previous Gauss process analyses, we know that the estimators with high number of lags tend to do this; however, we should not generalize this finding without further analysis. If the estimator precisely identifies an efficient time series with the fractal dimension equal to 1.5 and correctly captures the changes in ARFIMA setting, the overestimation for inefficient parts would not cause a problem as we are not looking for exact fractal dimension of time series; we rather want to make a ranking based on inefficiencies. Nevertheless, the example illustrates that we cannot compare fractal dimensions estimated by estimators with different number of lags.

For ARFIMA (0, 0, 0), all estimators return values very close to 1.5, but the standard errors are increasing with the number of lags, so we obtain wider confidence intervals. But as we shown above there are other settings whose fractal dimension should be close to 1.5 and results for these are not so clear.

For example, ARFIMA(0.3, -0.3, 0) has zero first order autocorrelation and weak negative autocorrelation of higher orders, so we expect fractal dimension to be slightly higher than 1.5. The fractal dimension estimated by the madogram with 2 lags is 1.483. On the other hand, the 100 lags estimator returns 1.722 which seems to be very high. The hypothesis that the madogram with higher number of lags overestimates fractal dimension is supported by ARFIMA(0.3, -0.1, 0) results. The coefficient of the first back-shift operator is -0.2, meaning that there is a positive first order autocorrelation; the other back-shift operators' coefficients are decreasing from 0.025 for L^2 and reaching values lower than 0.01 for L^9 and higher powers, so the negative autocorrelation in higher orders is much weaker and decreasing. The coefficients suggest that fractal dimension should be lower than 1.5, and most of the estimators support this hypothesis except for the estimators with 50 and 100 lags.

We obtain the same coefficient of the first back-shift operator also for ARFIMA(0.6, -0.4, 0), so the power of the positive first order autocorrelation is the same. The coefficients of the other back-shift operators are higher, so the negative autocorrelation of higher orders should be stronger, and fractal dimension should be higher than for ARFIMA(0.3, -0.1, 0). This holds for all estimators except for those with 2 lags and 5 lags; the difference for 5 lags is much smaller than for 2 lags. As we can see, none of the estimators is 100% accurate and it is better to compare the results using estimators with several different number of lags keeping in mind imperfections of each of them.

Graphs capturing the estimated fractal dimension showing confidence and tolerance intervals for the 5 lags and 100 lags madogram estimator can be found in the Appendix as Figures B.5 and B.6. The confidence and tolerance intervals broaden with the used number of lags. The estimator with 5 lags results in slightly concave line of the fractal dimensions decreasing with d for ARFIMA(0, d, 0), linearly decreasing line for ARFIMA(0.4, d, 0.4) and slightly convex line for ARFIMA(0.8, d, 0.8); the estimator with 100 lags gives lines that always decrease linearly.

Graphs recording visual appearance of random walk time series with several observations replaced by different types of ARFIMA are placed in the Appendix as Figure B.7. Longer parts of ARFIMA can be distinguished by eye, but the shorter ones are less evident especially thanks to the fitting we use to adjust the starting point and value range of individual parts.

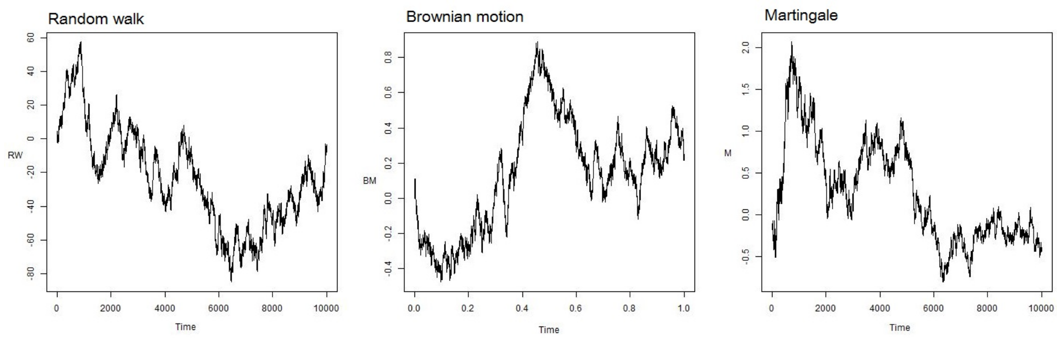
5.3 Comparison of random walk, Brownian motion and martingale

Random walk and Brownian motion are special cases of martingale processes; hence, in terms of generality, it would be appropriate to use the martingale

Table 5.7: Comparison of fractal dimensions of random walk, Brownian motion and martingale simulations, madogram

nlags	5	20	100
RW	1.5001	1.4999	1.5002
	0.0076	0.0099	0.0185
BM	1.5000	1.5001	1.5012
	0.0073	0.0099	0.0179
M	1.6025	1.5286	1.5088
	0.0072	0.0098	0.0192

Figure 5.5: Comparison of random walk, Brownian motion and martingale simulations



simulations. However, the simulation of martingales is a non-trivial problem. There is not any R package for simulation of martingales, only unofficial codes proposals - for example the one designed by Robert (2011) on his blog. This simulation has problems described in the Appendix D; as illustrated in Table 5.7, it shows the biggest deviation from desired 1.5 fractal dimension.

The mean fractal dimensions of random walk and Brownian motion are both very close to 1.5, and the differences in standard errors are minor. Graphs of all three types of efficient market simulation are similar and it is not possible to recognize which graph belongs to the random walk, Brownian motion and martingale simulation without further analysis as shown in Figure 5.5. Since the random walk simulation is more straightforward, we decide to use the random walk simulation for the core comprehensive analysis. We employ the other types of simulations in less comprehensive analysis and compare the results.

5.4 Simulations results

We start with a very simple set using random walk simulation with t observations inter-spaced by ARFIMA simulations. We make an extensive simulation of each combination of θ, ϕ and d coefficients and set t equal to 300, 500 and 1,000. For the core estimations, we use the madogram as the most efficient estimator and report the results for 5, 20 and 100 number of lags.

Further, we analyse the effect of changes in the number of observations on fractal dimension in more details for specific ARFIMA settings, verify whether the influence of inefficient time series placement on fractal dimension, and examine series with two inefficient parts having contradictory impact on fractal dimension. Theoretically, these two powers should go against each other with fractal dimension around 1.5. We check whether this statement is true, and try to find a solution to avoid the mistake. Later, we conduct shorter analysis of Brownian motion and martingale simulations, and we examine the results for different estimators, namely the Hall-Wood estimator and the box-count estimator.

5.4.1 Five hundred of ARFIMA observations

The results confirm our hypothesis that fractal dimension reflects all changes in autocorrelation structure of a time series in case of weak autocorrelation (small coefficients ϕ and θ), but as the ϕ coefficient increases, the results of fractal dimension start to show unexpected behaviour, especially for estimators with small number of lags. Tables A.4, A.5 and A.6 in the Appendix contain means and standard errors of the fractal dimensions estimated with the use of different number of lags, namely 5, 20 and 100. The fractal dimension reacts on changes in ϕ, d and θ coefficients in the combined time series equally as in the pure ARFIMA time series - the changes with the θ coefficient are much smaller than changes with the ϕ and d coefficients.

The 5 lags estimator exhibits expected results only for ARFIMA(0, d, 0); for ARFIMA(0.4, d, 0) and ARFIMA(0, d, 0.4), the fractal dimension starts to increase with higher values of d, and for ARFIMA(0.8, d, θ) we get a totally inverse relation between the fractal dimension and the d coefficient. This inaccuracy decreases with the number of lags as depicted in Figure 5.6 and Figures B.8 and B.9 in the Appendix.

Figure 5.6 contains graphs with fixed θ coefficient and changing ϕ and d

Figure 5.6: 3D graph of mean fractal dimensions of RW with 500 observations of ARFIMA, q is fixed, p and d are assigned to x and y axes, madogram

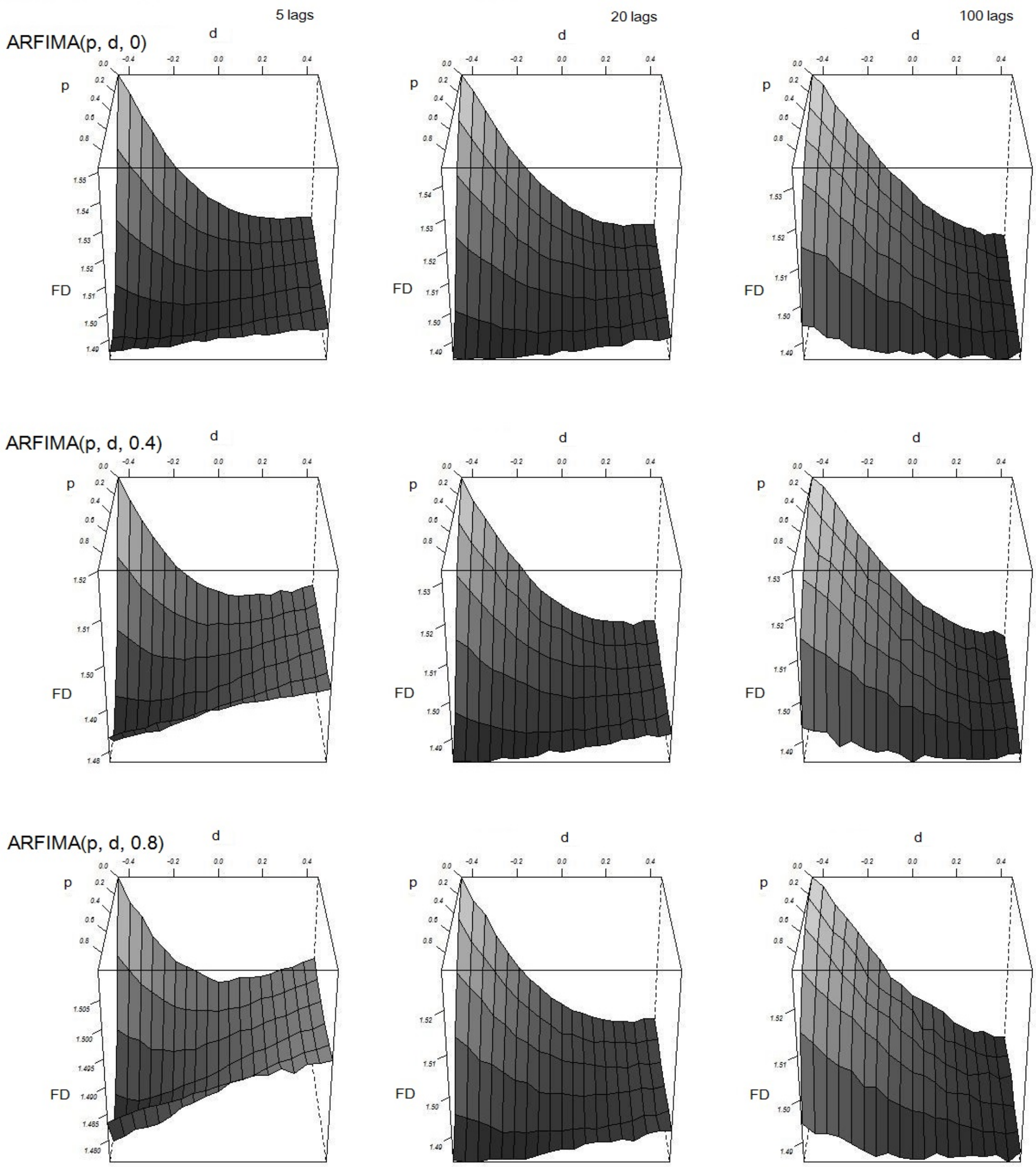
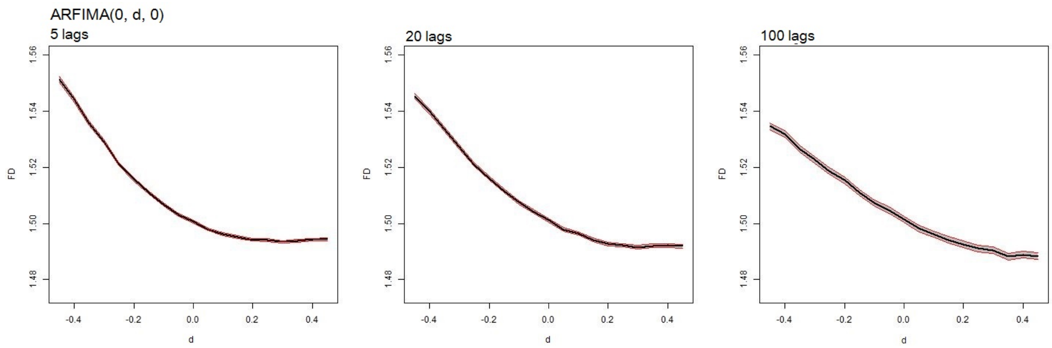


Figure 5.7: Mean fractal dimensions and confidence intervals of RW with 500 observations of ARFIMA(0, d, 0), d assigned to x-axis, changing number of lags, madogram



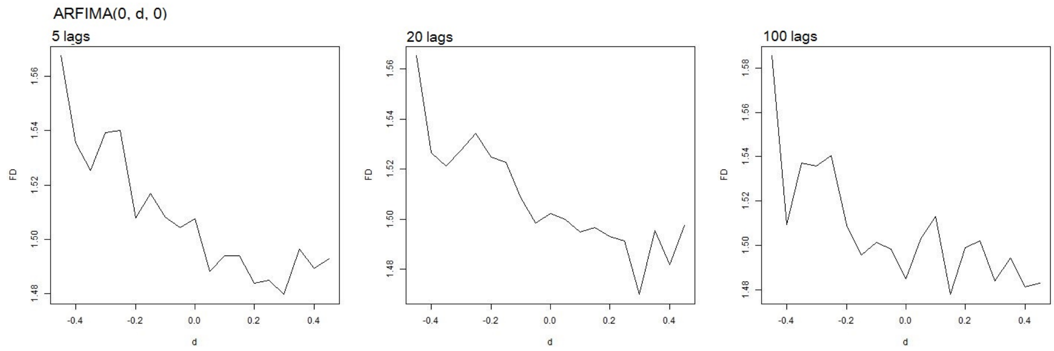
coefficients assigned to x and y axis; mean fractal dimension is on z axis.[4] In case of 5 lags, the fractal dimension decreases most rapidly with the ϕ coefficient and an inaccuracies are visible already in the first graph where the θ coefficient equals to 0; the graph with $\theta = 0.8$ is far from the expected shape. Although the graphs obtained through the 100 lags estimators are not as smooth as for the 5 lags estimators, the trends do not change the direction even for high ϕ and θ coefficients. Other 3D graphs of the fractal dimensions show similar behaviour; graphs for ARFIMA(ϕ, 0.4, θ) are less smooth, and even higher number of lags does not fix the peculiarities.

The use of estimators with higher number of lags is associated with some disadvantages as well, particularly with higher standard errors of the estimates. The 5 lags estimator returns mostly fractal dimensions with standard errors lower than 0.01, but for the 100 lags estimator the standard errors increase to values close to 0.02. The problem of increasing standard errors is shown in Figure 5.7; except for widening of the confidence interval, the figures show that the trend is more linear for higher numbers of lags compared to the convex trend for the 5 lags estimator.

The difference in standard errors can be tracked in volatility of the fractal dimensions for single simulation where d coefficient in ARFIMA(0, d, 0) increases. The mean fractal dimensions are represented as smoothly declining line, when the fractal dimension of single simulation moves along the trend as in Figure 5.8; the fractal dimensions estimated with 100 lags are more volatile than those estimated with 5 lags.

[4]In all graphs, the ϕ and θ coefficients are labelled as p and q, respectively. Further, we use the notation ARFIMA(ϕ, d, θ).

Figure 5.8: Fractal dimensions of RW with ARFIMA(0, d, 0) for a sample simulations, madogram



We provide graphs of the mean fractal dimensions with confidence intervals for changing ϕ, d, and θ coefficients and 5 and 100 lags. The Figures for changing d parameters are presented in the text as 5.9 and 5.10, while the other graphs can be found in the Appendix as Figures B.10 to B.13. The confidence intervals are generally narrow indicating that the estimation of fractal dimension should identify inefficiencies in real markets quite accurately.

Graphs of the fractal dimensions estimated with 5 lags turn with increasing ϕ coefficient from a convex line decreasing with d to linearly increasing line, whereas the fractal dimensions estimated with 100 lags are represented by a very similar line for all settings.

While the shape of the fractal dimension lines turns from convex line decreasing with θ to horizontal linear line as depicted in Figure B.10, all lines of the fractal dimensions are horizontal with the θ coefficient for 100 lags, and the shape does not differs across the ARFIMA settings as exhibited in Figure B.11.

Figure B.12 shows the relation between the fractal dimension and the ϕ coefficient for the 5 lags estimator; the line of the fractal dimensions is convex in ϕ for negative d and then turns to a horizontal linear line for higher d parameters. On the contrary, Figure B.13 of 100 lags estimations shows a concave line of the fractal dimensions decreasing with ϕ for negative d parameters which turns to horizontal line for increasing d.

It is interesting that there is a difference between the impact of numbers of lags on fractal dimension for pure ARFIMA simulations and ARFIMA incorporated in efficient time series. In case of pure ARFIMA, the fractal dimensions estimated by the madogram with 100 lags are higher than those estimated by

Figure 5.9: Mean fractal dimensions and confidence intervals of RW with 500 observations of ARFIMA, d assigned to x-axis, 5 lags madogram

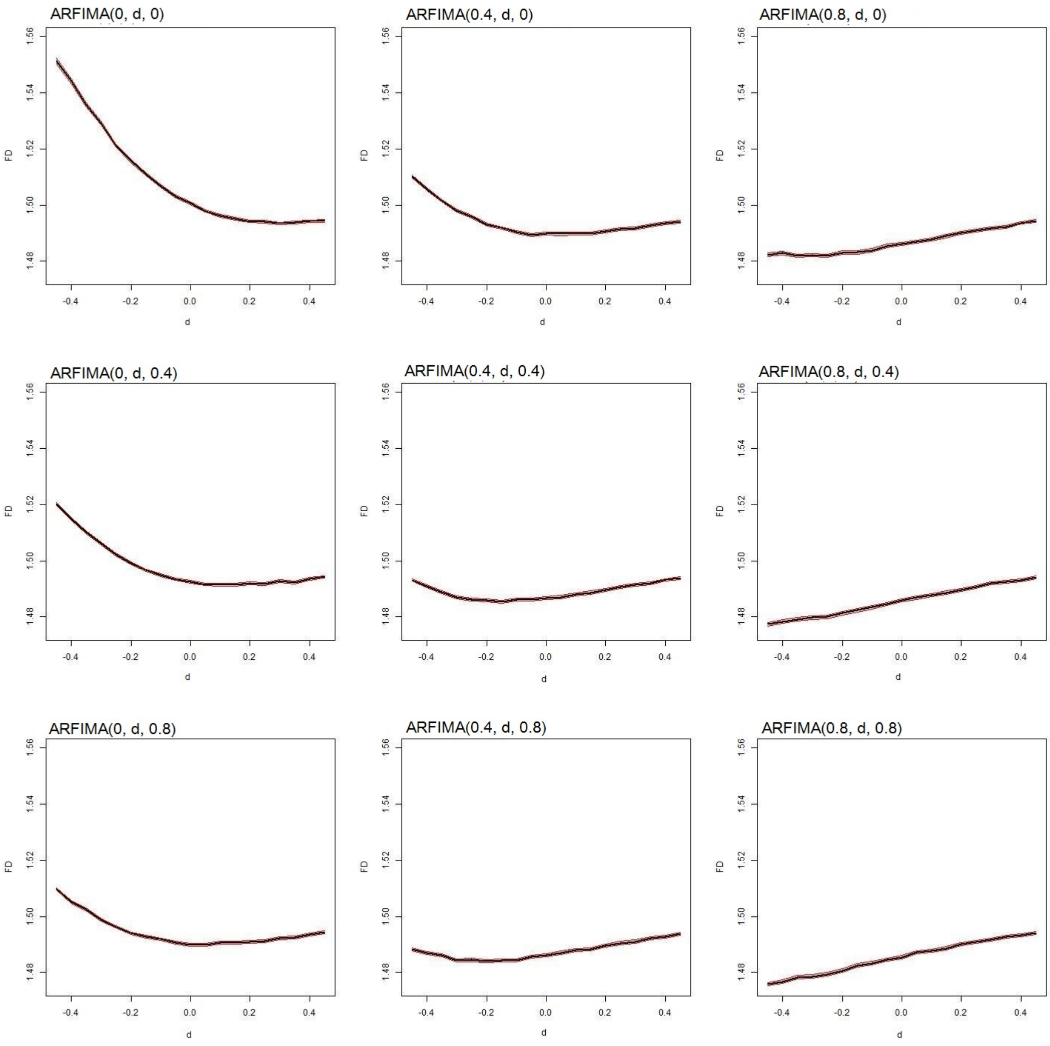
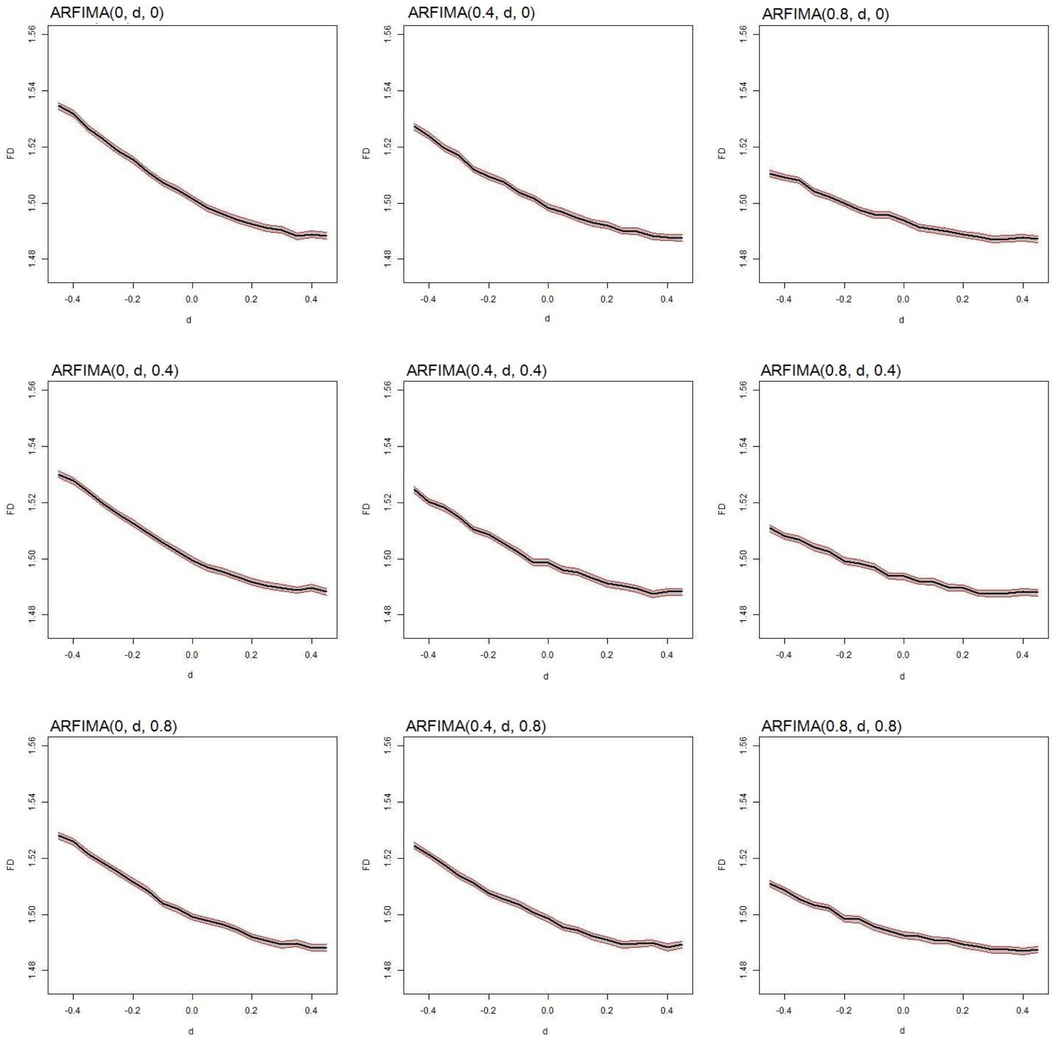


Figure 5.10: Mean fractal dimensions and confidence intervals of RW with 500 observations of ARFIMA, d assigned to x-axis, 100 lags madogram



the madogram with 5 lags; in case of ARFIMA(0, d, 0) incorporated to a series of random walk, it is the other way around - with the 100 lags estimator we get a lower fractal dimension than with the 5 lags estimator. In case of ARFIMA with higher values of ϕ ad θ coefficients, the 100 lags estimator returns higher fractal dimensions, and it is clear that especially for negative d parameter the estimator overestimates fractal dimension, as it returns fractal dimensions higher than 1.5 even for ARFIMA simulations with $\phi = 0.8$.

As we noted in the previous section, we do not suppose inefficiencies in real markets to be as extreme as e.g. ARFIMA(0.8, 0, 0.4), so the inaccuracies of estimators with small number of lags should not have an influence on the results of real markets analysis and compilation of the efficiency ranking.

Regression of fractal dimensions

The differences in the estimated fractal dimensions may not be caused only by imperfections in fractal dimension estimations but also by differences in simulated time series, as the simulations are not completely exact. We try to distinguish between these two imperfections by regressing the fractal dimension of the whole series on the fractal dimension of both random walk parts and ARFIMA part. We find that the coefficients of all these variables are significant and the average Adjusted R^2 of the regressions is 70% for the 5 lags estimators and 80% for the 20 lags estimators. We use simple OLS regressions and test the assumptions; all assumptions are fulfilled. The results are summed up in Table 5.8; the significance and value of coefficients of random walk parts suggests that there is a difference between the two time series reflecting in their fractal dimension; the coefficients' values correspond to the number of observations occupied by each of the parts. The coefficients of ARFIMA part depend on the model settings.

5.4.2 Three hundred and one thousand of ARFIMA observations

We now evaluate the fractal dimensions estimated on time series with 300 and 1,000 ARFIMA observations and compare the result with the previous section; we expect the results to be very similar, only the ARFIMA effect should be stronger in the case of 1,000 observations and weaker for 300 observations.

The outcomes are presented in the the Appendix. Figures B.14 and B.15 show 3D graphs of the fractal dimension with fixed θ parameter estimated by

Table 5.8: Results of regression of final series' fractal dimension on fractal dimensions of individual parts of the series

5 lags	RW1	RW2	ARFIMA
% observations	30%	65%	5%
Average coefficent	0.3163	0.645	depends on setting
Average p-value	0	0	0.0495
R^2		0.7343	

20 lags	RW1	RW2	ARFIMA
% observations	30%	65%	5%
Average coefficent	0.3104	0.641	depends on setting
Average p-value	0	0	0.0313
R^2		0.8498	

the 5 and 100 lags madogram; the bias in the fractal dimensions of the series with 300 observations is smaller than of the series with 1,000 observations as supposed. Even though the bias is smaller, it is still present and the shape of graphs is same for all numbers of ARFIMA observations. As the effect of more lags is same as in the case of 500 observations, we exhibit only the results for the 5 lags estimators. The weaker effect of ARFIMA in series with 300 observation and stronger effect in series with 1,000 observations is noticeable in the Tables A.7 and A.8 in the Appendix. The standard errors are higher in the case of 1,000 ARFIMA simulations, especially for ARFIMA series with higher ϕ coefficient, which is caused by the longer part of ARFIMA and its probably lower exactness of simulations. Confidence intervals for the 5 lags estimator are presented in Figures B.16 and B.17 in the Appendix. We can conclude that the results confirm our expectations about the effect of number of ARFIMA observations.

5.4.3 Detailed analysis of changing number of ARFIMA observations

We examine the influence of changing number of ARFIMA observation on the final series' fractal dimension in a greater detail. We estimate fractal dimension for 100, 200, 300, 400, 500, 750, 1,000, 1,500 and 2,000 ARFIMA observations in random walk series and find that the longer the ARFIMA part is, the stronger is the effect, which confirms our expectations; even the difference between 100 and 200 observations is noticeable. The standard error of estimations increases with the number of ARFIMA observations which can be found in Table A.9 in the Appendix containing means and standard errors of the fractal dimensions; the shading stresses the trends with changing number of observations. As the behaviour of fractal dimension in context of changing number of lags is described in previous sections of the thesis, we present the results for only the 5 lags estimator.

Figure B.18 in the Appendix depicts the relation between means and confidence intervals of the fractal dimensions and increasing number of observations of three selected types of ARFIMA. The reaction of the fractal dimension on growing number of ARFIMA observations is very similar for the estimators with all examined numbers of lags. In case of ARFIMA(0, -0.4, 0), the fractal dimension increases with the number of ARFIMA observations, and there are no differences among the estimators. The other examined types of included ARFIMA have positively autocorrelated returns, so the fractal dimension decreases with the number of ARFIMA observations; for ARFIMA(0.4, 0, 0.4) the estimator with 5 lags responds to the changes in number of observation more significantly than the estimator with 100 lags; for ARFIMA(0.8, 0.4, 0.8) it is the other way around: the 100 lags estimator is affected more than the 5 lags estimator.

Table 5.9 includes the mean fractal dimensions for different number of lags estimators and the shading stresses problem of the bias for different number of observation.

The problem of fractal dimension estimations bias is present in all 5 lags estimations and in none of the 100 lags estimations; most interesting are the results of the 20 lags estimator, where the fractal dimensions exhibit normal behaviour for 100 observations, start to be a bit distorted for 200 observation, and the skewness grows with the number of observations.

As both strength and length of ARFIMA observations influence fractal dimension, we are unable to distinguish between e.g. 1,000 observations of ARFIMA(0, 0.2, 0) and 400 observations of ARFIMA(0.8, 0, 0.8). For this reason we recommend to not rely on fractal dimension of whole series but try to estimate the series by smaller parts as described in one of the following sections.

Table 5.9: Mean fractal dimensions for different number of ARFIMA(0.8, d, 0.8) observations, 5, 20, and 100 lags madogram

ARFIMA(0.8, d, 0.8), 5 lags

d/t	100	200	300	400	500	750	1000	1500	2000
-0.4	1.4965	1.4922	1.4870	1.4824	1.4767	1.4610	1.4498	1.4222	1.3945
-0.2	1.4967	1.4927	1.4890	1.4853	1.4807	1.4691	1.4596	1.4377	1.4152
0	1.4970	1.4943	1.4915	1.4886	1.4854	1.4777	1.4712	1.4580	1.4415
0.2	1.4972	1.4955	1.4933	1.4916	1.4900	1.4852	1.4810	1.4747	1.4661
0.4	1.4973	1.4964	1.4952	1.4942	1.4933	1.4905	1.4898	1.4852	1.4811

ARFIMA(0.8, d, 0.8), 20 lags

d/t	100	200	300	400	500	750	1000	1500	2000
-0.4	1.4982	1.4968	1.4944	1.4920	1.4895	1.4818	1.4767	1.4645	1.4517
-0.2	1.4981	1.4944	1.4920	1.4892	1.4861	1.4773	1.4706	1.4558	1.4398
0	1.4977	1.4942	1.4921	1.4886	1.4862	1.4786	1.4727	1.4597	1.4447
0.2	1.4970	1.4947	1.4922	1.4900	1.4878	1.4825	1.4776	1.4697	1.4598
0.4	1.4964	1.4946	1.4931	1.4923	1.4903	1.4864	1.4854	1.4792	1.4741

ARFIMA(0.8, d, 0.8), 100 lags

d/t	100	200	300	400	500	750	1000	1500	2000
-0.4	1.5018	1.5020	1.5057	1.5066	1.5085	1.5134	1.5170	1.5270	1.5357
-0.2	1.5010	1.4996	1.4993	1.5002	1.4984	1.4984	1.4975	1.4970	1.4937
0	1.4990	1.4976	1.4970	1.4940	1.4926	1.4884	1.4853	1.4762	1.4687
0.2	1.4979	1.4957	1.4936	1.4912	1.4893	1.4830	1.4797	1.4699	1.4611
0.4	1.4966	1.4942	1.4911	1.4899	1.4869	1.4809	1.4786	1.4709	1.4637

5.4.4 Placement of ARFIMA observations

The placement of an ARFIMA part in the efficient time series should not have an influence on the resulting fractal dimension, but we rather empirically verify the statement. The estimators with different number of lags exhibit some peculiarities, so it is desirable to check how fractal dimension reacts to ARFIMA part placed at different positions in the final time series. We place the ARFIMA part to the first, second, third, and last quarters, use 300, 500, and 1,000 ARFIMA observations and estimate fractal dimension with 5, 20, and 100 lags.

A part of the result is depicted in Table A.10 in the Appendix. We detect that the placement does not influence the estimated fractal dimension; differences in the fractal dimensions for various placements are negligible.[5] Finally, we try to split 1,000 observations to 500 observations in the first quarter and 500 observations in the third quarter of the time series; the fractal dimension of this time series differs more significantly from the rest of the series containing 1,000 ARFIMA observations; the difference happens to be larger than 0.003. This is another finding that lead us to estimating fractal dimension of individual parts of the whole series.

5.4.5 Estimation of fractal dimension by parts

We estimate not only fractal dimensions of whole time series, but also fractal dimensions of individual parts – the first part of random walk, the second part of ARFIMA, and the third part of random walk, so we can compare the results of individual parts and the whole time series. The fractal dimensions of random walk parts are unsystematically distributed around 1.5, and the changes in fractal dimensions of ARFIMA are consistent with the theory of fractal dimension changes copying the changes in ARFIMA parameters; however, when we combine the series, some imperfections in the fractal dimension estimations turn up. Hence, we try to avoid the peculiarities by estimating fractal dimensions of each part of the series separately. Our hypothesis is that the imperfections in the estimated fractal dimension disappear if ARFIMA series covers a sufficient part of the examined number of observations.

Two settings of ARFIMA are chosen for the analysis of performance of the 5 lags and 20 lags estimators. Specifically, we select ARFIMA(0.4, d, 0.4) and ARFIMA(0.8, d, 0.8); each series contain 1,000 observations of ARFIMA. We

[5] The differences are smaller than 0.001.

estimate fractal dimension of the whole time series; then the time series is divided to 2, 4, 5, and 10 equally long parts, and we estimate fractal dimension of each part separately. The results are presented in Table A.11 in the Appendix. As we know from the previous analysis, both estimators return skewed results for the selected settings of ARFIMA. Looking at the 5 lags results, we can see that the skewness is quite persistent and starts to diminish when ARFIMA series covers at least the half of observations. A similar behaviour can be observed for the 20 lags madogram.

We conduct a more detailed analysis of reactions of fractal dimension to ratio of observations covered by ARFIMA time series; simulations containing from 10% to 100% of ARFIMA observations are executed and their fractal dimension is measured. We employ two different approaches: in the first one, we fix the total number of observation to 10,000 and change the number of ARFIMA observations, while in the second approach we fix the number of ARFIMA observations and change the total number of observations. Again the 20 lags estimator is analysed on ARFIMA(0.8, d, 0.8) and the 5 lags estimator on ARFIMA(0.4, d, 0.4). The results are shown in Tables 5.10 and 5.11.

In case of fixed number of total observations, skewness is persistent and the difference between 10% of ARFIMA and 90% of ARFIMA is very small. For fixed number of ARFIMA observations, the difference is more significant, but the results remains biased even for 90% of ARFIMA observations. As we know from the ARFIMA analysis and as it is shown in the tables with results again, the fractal dimensions of pure ARFIMA simulation do not exhibit any type of bias, but if an examined series contains a mixture of efficient and inefficient parts, the results are always skewed for stronger autocorrelation structures and estimators with lower number of lags. The ratio between efficient and inefficient observations plays only a minor role, which is showed by the difference between time series with 100% and 90% of ARFIMA observations.

Further, the efficient part has a higher than expected impact on the value of fractal dimension. The fractal dimension of an efficient time series is 1.5 and the fractal dimension of ARFIMA(0.4, 0.2, 0.4) is 1.1155. One would suppose that fractal dimension of a combination of these two series would be linearly dependent on their proportion and fractal dimensions, so a combination containing 90% of ARFIMA observations and 10% of efficient observations should have fractal dimension close to 1.154.[6] Nevertheless, the estimated fractal dimension of this combination is 1.35 and the fractal dimension than

[6] $0.9 \cdot 1.1155 + 0.1 \cdot 1.5 = 1.154$.

Table 5.10: Means and standard errors of fractal dimensions for RW with changing ratio of ARFIMA(0.4, d, 0.4) observations, 5 lags madogram

ARFIMA(0.4, d, 0.4), 5 lags, x% of ARFIMA observations										
d/x%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
a) # total observations 10 000, # ARFIMA observations changing										
-0.4	1.4872	1.4630	1.4504	1.4416	1.4374	1.4347	1.4315	1.4290	1.4265	1.4200
	0.0081	0.0102	0.0100	0.0096	0.0085	0.0082	0.0077	0.0072	0.0068	0.0205
-0.2	1.4815	1.4416	1.4178	1.3953	1.3868	1.3781	1.3693	1.3602	1.3496	1.3183
	0.0091	0.0155	0.0166	0.0164	0.0154	0.0144	0.0136	0.0123	0.0102	0.0189
0	1.4844	1.4481	1.4224	1.3970	1.3844	1.3722	1.3586	1.3385	1.3152	1.2160
	0.0095	0.0191	0.0221	0.0261	0.0284	0.0292	0.0284	0.0281	0.0268	0.0186
0.2	1.4889	1.4655	1.4488	1.4314	1.4219	1.4087	1.3977	1.3806	1.3492	1.1155
	0.0091	0.0174	0.0220	0.0281	0.0312	0.0334	0.0355	0.0398	0.0457	0.0206
0.4	1.4934	1.4819	1.4730	1.4624	1.4579	1.4526	1.4476	1.4367	1.4190	1.0377
	0.0088	0.0130	0.0181	0.0215	0.0233	0.0271	0.0298	0.0349	0.0431	0.0183
b) # total observations changing, # ARFIMA observations 1000										
-0.4	1.4872	1.4664	1.4556	1.4475	1.4433	1.4420	1.4387	1.4364	1.4320	1.4218
	0.0081	0.0128	0.0148	0.0161	0.0179	0.0186	0.0184	0.0200	0.0205	0.0208
-0.2	1.4815	1.4471	1.4250	1.4060	1.3956	1.3900	1.3813	1.3727	1.3618	1.3230
	0.0091	0.0168	0.0192	0.0218	0.0218	0.0232	0.0229	0.0236	0.0249	0.0187
0	1.4844	1.4475	1.4229	1.3955	1.3832	1.3711	1.3572	1.3420	1.3205	1.2261
	0.0095	0.0196	0.0252	0.0291	0.0314	0.0331	0.0347	0.0358	0.0390	0.0170
0.2	1.4889	1.4603	1.4388	1.4155	1.4001	1.3861	1.3693	1.3503	1.3231	1.1318
	0.0091	0.0197	0.0278	0.0332	0.0376	0.0439	0.0461	0.0515	0.0607	0.0160
0.4	1.4934	1.4757	1.4605	1.4419	1.4314	1.4201	1.4088	1.3869	1.3579	1.0464
	0.0088	0.0173	0.0248	0.0317	0.0368	0.0419	0.0488	0.0569	0.0753	0.0195

Table 5.11: Means and standard errors of fractal dimensions for RW with changing ratio of ARFIMA(0.8, d, 0.8) observations, 20 lags madogram

ARFIMA(0.8, d, 0.8), 20 lags, x% of ARFIMA observations

d/x%	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
a) # total observations 10 000, # ARFIMA observations changing										
-0.4	1.4859	1.4528	1.4318	1.4142	1.4067	1.3996	1.3925	1.3869	1.3795	1.3606
	0.0113	0.0145	0.0155	0.0145	0.0139	0.0131	0.0119	0.0113	0.0108	0.0290
-0.2	1.4822	1.4404	1.4097	1.3844	1.3704	1.3576	1.3454	1.3315	1.3140	1.2589
	0.0122	0.0183	0.0218	0.0224	0.0221	0.0209	0.0212	0.0198	0.0173	0.0269
0	1.4834	1.4458	1.4179	1.3903	1.3775	1.3630	1.3459	1.3254	1.2965	1.1635
	0.0120	0.0212	0.0256	0.0301	0.0309	0.0328	0.0325	0.0345	0.0337	0.0252
0.2	1.4871	1.4583	1.4405	1.4194	1.4085	1.3977	1.3845	1.3624	1.3299	1.0867
	0.0118	0.0203	0.0266	0.0321	0.0358	0.0385	0.0406	0.0439	0.0495	0.0247
0.4	1.4907	1.4728	1.4629	1.4503	1.4450	1.4379	1.4316	1.4192	1.3934	1.0310
	0.0109	0.0173	0.0217	0.0254	0.0274	0.0330	0.0354	0.0413	0.0531	0.0172
b) # total observations changing, # ARFIMA observations 1000										
-0.4	1.4859	1.4576	1.4418	1.4270	1.4207	1.4149	1.4086	1.4032	1.3948	1.3633
	0.0113	0.0172	0.0213	0.0232	0.0245	0.0269	0.0276	0.0302	0.0325	0.0299
-0.2	1.4822	1.4450	1.4189	1.3935	1.3828	1.3719	1.3619	1.3523	1.3363	1.2653
	0.0122	0.0213	0.0255	0.0289	0.0301	0.0315	0.0331	0.0362	0.0387	0.0264
0	1.4834	1.4458	1.4155	1.3906	1.3754	1.3610	1.3455	1.3279	1.3070	1.1761
	0.0120	0.0228	0.0310	0.0348	0.0381	0.0402	0.0442	0.0457	0.0534	0.0240
0.2	1.4871	1.4549	1.4279	1.4018	1.3886	1.3718	1.3578	1.3349	1.3051	1.0977
	0.0118	0.0244	0.0311	0.0407	0.0427	0.0474	0.0532	0.0593	0.0692	0.0221
0.4	1.4907	1.4678	1.4492	1.4240	1.4126	1.3998	1.3825	1.3598	1.3339	1.0353
	0.0109	0.0207	0.0303	0.0379	0.0404	0.0478	0.0557	0.0650	0.0791	0.0207

only slowly increase with the proportion of efficient observations to 1.49. This oddity may be caused by the fitting of ARFIMA series; we address the problem later in this section.

Standard errors increase with declining number of total observations and with the growing number of ARFIMA observation as well, which suggests that the simulation of ARFIMA is less exact than the simulation of random walk. We find that estimation by parts does not represent a solution for skewness of some results and it is better to use estimators with more numbers of lags and compare the results to avoid mistakes in the evaluation of a market; however, there are other reasons to employ estimation of fractal dimension of separate parts, e.g. a time series may contain parts with both positive and negative autocorrelation which could counteract each other.

5.4.6 More contradictory ARFIMA parts

To analyse the problem of two inefficient parts with contradictory effects mentioned in the previous section, we simulate a time series with two inefficient parts: the former part consists of ARFIMA(0, -0.4, 0) with negatively correlated returns and the latter part by ARFIMA(0.4, 0.2, 0.4) with positively correlated returns. The graphs of the time series can be found in the Appendix as Figure B.19. We place the ARFIMA(0, -0.4, 0) part to the first half of the time series and ARFIMA(0.4, 0.2, 0.4) to the second half; then we switch the order to check robustness of results.

We estimate fractal dimension for the whole time series and for the first and second part separately; results are noted in Table 5.12. The fractal dimension of the whole series confirms irrelevance of ARFIMA part placement and takes a value relatively close to 1.5. The separate estimation enables us to distinguish the part with positive and the part with negative autocorrelation. Although both of the ARFIMA parts occupy the same number of observations, the final fractal dimension is not equal to the average of fractal dimensions of the individual parts. It is in fact slightly higher, indicating that the more volatile ARFIMA part little overweights the other one.

As real markets consist of orderless sequence of more and less efficient parts, we recommend to examine the sequence structure by parts.

Table 5.12: Means and standard errors of fractal dimensions for RW with two contradictory ARFIMA parts, madogram

ARFIMA(0, -0.4, 0) ARFIMA(0.4, 0.2, 0.4)	in (3001:3500) in (7001:7500)		in (7001:7500) in (3001:3500)	
	1 part	2 parts	1 part	2 parts
5 lags	1.5338	1.5691	1.5337	1.4866
	0.0139	0.0200	0.0135	0.0129
		1.4867		1.5693
		0.0131		0.0199
average		1.5279		1.5279
20 lags	1.5299	1.5648	1.5299	1.4875
	0.0140	0.0198	0.0136	0.0157
		1.4871		1.5639
		0.0158		0.0204
average		1.5260		1.5257
100 lags	1.5223	1.5526	1.5217	1.4894
	0.0197	0.0256	0.0192	0.0269
		1.4872		1.5513
		0.0270		0.0263
average		1.5199		1.5203

5.4.7 Gradually estimated fractal dimension of complex time series

To examine the power of fractal dimension on time series that are closer to real markets, we simulate a complex time series composed of different types of ARFIMA and various length of random walk. We estimate fractal dimension of the whole series and of each tenth of the series separately. Then we apply a new approach - gradual estimation of fractal dimension. We begin with estimation of fractal dimension of the first 500 observations, add 10 observations in each step and always re-estimate fractal dimension to see whether small changes in autocorrelation structure influence resulting fractal dimension.

We begin with a time series composed of random walk and ARFIMA simulation that are both positively and negatively correlated. In spite of the fact that strong autocorrelation and rapid changes are inconsistent with real markets, we include ARFIMA series with relatively strong autocorrelation to stress out the effect. The composition of simulated time series is described by the chart in Figure 5.11 and a graph of a sample simulated time series can be found in Figure 5.12; the vertical grey lines signal changes in used types of simulation. The strong positive and negative autocorrelation structures are visually noticeable in the graph. Let us focus on fractal dimension to see whether the changes in the structure show up in fractal dimension as well.

Figure 5.11: Composition of simulated time series, 1st setting

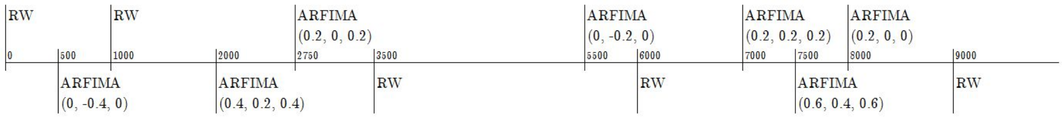
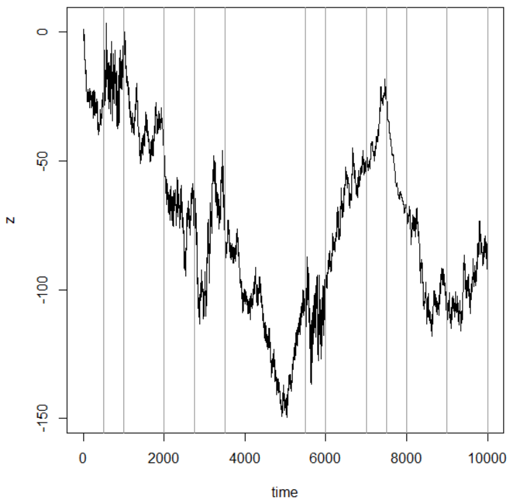


Figure 5.12: Example of the complex time series, 1st setting



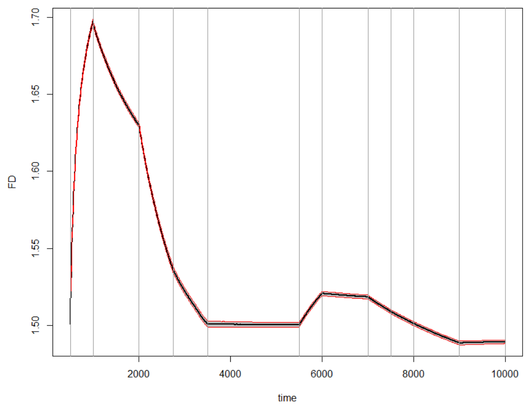
The means and standard errors of the fractal dimensions of the whole series and each tenth of the series are shown in Table 5.13; we use the madogram with 5, 20 and 100 lags. The first column indicates the range of observations used in the estimation of fractal dimension, whereas the second column contains information about the types of time series creating the separate parts. The fractal dimension of the whole time series is not very informative; the 5 lags estimator claims that there is slight positive autocorrelation, while the 20 and 100 lags estimators return fractal dimensions slightly higher than 1.5, suggesting negative correlation in the returns. The fractal dimensions of individual tenths of the time series capture the differences among the individual parts well, but standard errors are high especially for the estimators with large number of lags. Despite high standard errors, the fractal dimensions of individual parts give much better insight in the structure of the time series than the fractal dimension of the whole series.

Finally, we get to the gradual growth in the number of observations. As noted before, we estimate fractal dimension for the first 500 observations, then for the first 510, 520 and so on up to 10,000. We end up with 951 different fractal dimensions that should record the changes in the underlying time se-

Table 5.13: Means and standard errors of fractal dimensions for the whole complex series and its tenths, 1st setting, madogram

Observations		5 lags	20 lags	100 lags
(1:10000)	All	1.4895	1.5145	1.5248
		0.0190	0.0169	0.0221
(1:1000)	RW	1.6974	1.7068	1.7030
	(0, -0.4, 0)	0.0274	0.0291	0.0464
(1000:2000)	RW	1.4996	1.5013	1.5040
		0.0233	0.0309	0.0586
(2000:3000)	(0.4, 0.2, 0.4)	1.3256	1.4749	1.5773
	(0.2, 0, 0.2)	0.0200	0.0316	0.0570
(3000:4000)	(0.2, 0, 0.2)	1.4113	1.4574	1.4915
	RW	0.0261	0.0316	0.0586
(4000:5000)	RW	1.4994	1.5010	1.5062
		0.0234	0.0309	0.0605
(5000:6000)	RW	1.6023	1.6116	1.6173
	(0, -0.2, 0)	0.0272	0.0321	0.0547
(6000:7000)	RW	1.5006	1.5034	1.5095
		0.0229	0.0308	0.0610
(7000:8000)	(0.2, 0.2, 0.2)	1.1434	1.1873	1.2287
	(0.6, 0.4, 0.6)	0.0321	0.0443	0.0740
(8000:9000)	(0.2, 0, 0.2)	1.4001	1.4502	1.4855
		0.0221	0.0307	0.0574
(9000:10000)	RW	1.5002	1.5022	1.5068
		0.0222	0.0312	0.0613

Figure 5.13: Means and standard errors of fractal dimensions for graduate increase in number of observation complex time series, 1st setting, 5 lags madogram



ries. Disregarding standard errors, the results for the estimators with 5, 20 and 100 lags are very similar, so we report only those for 5 lags. The mean fractal dimensions and related confidence interval are depicted in Figure 5.13. The graph shows that average fractal dimension reacts to the changes in the time series exactly as expected and the reaction is nearly immediate. New observations influence total fractal dimension in the direction of the fractal dimension value of the just added part. The stronger is the autocorrelation, the sharper is the change in fractal dimension. The confidence interval of the fractal dimensions is narrow, so the approach is expected to work well not only in averages but for a single time series as well. This is illustrated in Figure B.20 in the Appendix. Although the line is more volatile than for means, it tracks the changes correctly; the volatility of the fractal dimensions increases with number of lags.

The previous results are promising; to make the simulations more similar to real market prices development, we simulate another time series and use smaller changes in the autocorrelation structure. Fractal dimension is expected to register these changes as well. The chosen settings are presented in Figure 5.14. The changes in the settings of time series are less visually noticeable than in the previous case as shown in Figure 5.15.

We conduct the same analysis as for the previous time series and find that the even such small changes are well reflected in fractal dimension as demonstrated in Figure 5.16. Even though the volatility in sample time series' fractal dimensions increases as shown in Figure B.21 in the Appendix, the trends still correspond to the changes in the underlying time series. The fractal dimensions

Figure 5.14: Composition of simulated time series, 2nd setting

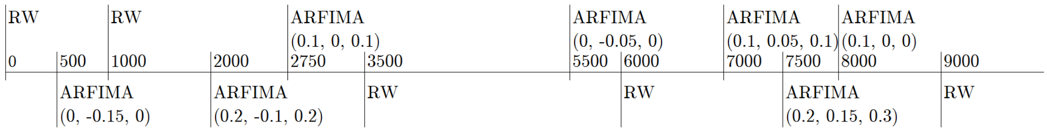
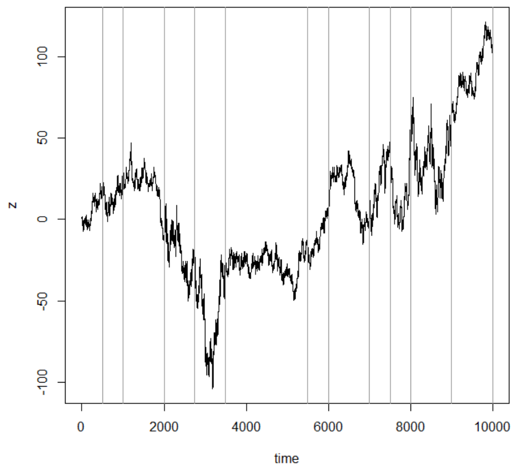


Figure 5.15: Example of the complex time series, 2nd setting



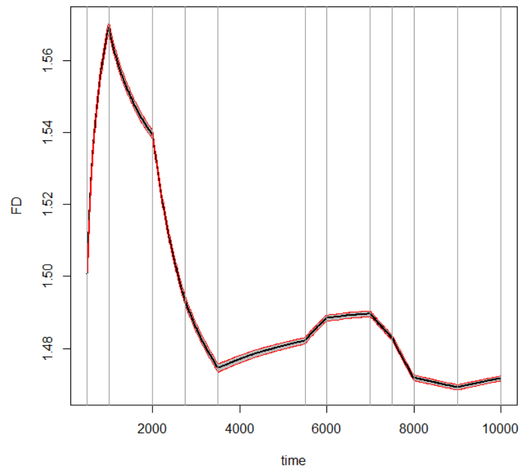
of the whole series and each tenth of the series can be found in Table A.12 in the Appendix.

These findings may be utilized in trading to identify changes in autocorrelation structure of returns. Together with analysis of slope of the line depicting the estimated fractal dimensions and a study of individual parts of prices time series, it can help to reveal the level of autocorrelation currently occurring on the market and its changes. The practical usage of fractal dimension would require a deeper study of e.g. the optimal length of time series used to tracking the changes in fractal dimension (and so in the autocorrelation structure) or of the connection between relative changes in the autocorrelation structure and fractal dimension. Such analyses would be beyond the scope of our thesis and are left to be examined by other studies. Although our work does not solve the problem, it represents the first steps of fractal dimension usage in analysis of market efficiency.

5.4.8 Impact of ARFIMA fitting

In the simulation of efficient and inefficient parts we use fitting of the series to keep the amplitude more or less constant. This fitting may influence the

Figure 5.16: Means and standard errors of fractal dimensions for graduate increase in number of observation, 2nd setting, 5 lags madogram



resulting fractal dimension by diminishing the effect of ARFIMA part. Figure B.22 in the Appendix depict ARFIMA simulation, fitted ARFIMA simulation, inclusion of not fitted ARFIMA part to random walk and inclusion of fitted ARFIMA part. The effect of fitting on ARFIMA series itself is minimal - it only influences the scale; the values range of ARFIMA simulations is usually much higher than values range of random walk and inclusion of such a series to random walk would result in a very unrealistic sequence of observations such as the one shown in the third graph. Therefore, a fitting is needed as it adjusts the values range to correspond to random walk and the resulting time series than look like the fourth graph.

The fitting influences fractal dimension of the time series as described in Table 5.14. The fractal dimensions of the time series with unfitted ARFIMA part are much smaller than those of time series with fitted ARFIMA, so we can say that fitting decreases the effect of ARFIMA part on fractal dimension of the whole series. Fractal dimension of ARFIMA series itself is not affected.

This finding explains the jump between the fractal dimensions of time series with 90% and 100% of ARFIMA observations in Section 5.4.5.

5.4.9 Effect of number of observations on standard error

The fractal dimension estimated by the madogram reflects both the number of lags and the number of observations as described in Equations 3.11 and

Table 5.14: Effect of fitting on fractal dimension of time series, madogram

ARFIMA(0.4, 0.2, 0.4) in (2001:3000), 3000 observations				
		fit.	RW +	RW + fit.
nlags/type	ARFIMA	ARFIMA	ARFIMA	ARFIMA
5	1.1322	1.1322	1.3061	1.4283
	0.0158	0.0158	0.0145	0.0303
20	1.2141	1.2141	1.3326	1.4338
	0.0319	0.0319	0.0246	0.0287
100	1.2801	1.2801	1.3567	1.4401
	0.0671	0.0671	0.0505	0.0383

3.12. We suppose that higher number of observations makes estimations more accurate, and standard error of the estimated fractal dimension thus decreases.

The statement is confirmed by the results in Table 5.15, showing the mean fractal dimensions and standard errors for given numbers of random walk observations and lags. The standard error of 5 lags estimation decreases from 0.0335 for 500 observations to 0.0023 for 100,000 observations; further, the estimated fractal dimensions are closest to 1.5 for 5,000 and more observations.

We are now want to find whether we can use higher number of lags for time series with more observations without increasing the standard error. To examine this topic, we fix the ratio between numbers of lags and observations to 1% and estimate the fractal dimension of 500 random walk observations with the 5 lags estimator, 1,000 observations with the 10 lags estimators, and so on up to using the 1,000 lags estimator for random walk with 100,000 observations. We discover that the estimations of fractal dimension are most precise for 2,000 observations and the 20 lags estimator, 5,000 observations and the 50 lags estimator, and 10,000 observations and the 100 lags estimator. The estimators with higher numbers of lags overestimate the fractal dimension, but the standard errors decrease with the number of observation, meaning that the fractal dimension estimated by the madogram with more lags on a bigger sample are less volatile; to keep the standard error unchanged, we may use even more number of lags than 1% of observations.

If it is possible, one should use at least 2,000 observations for estimation of fractal dimension to make standard errors small and estimations more precise. However, we have only daily data and this limitation would be very striking, the recommendation is thus applicable rather for high frequency data.

Table 5.15: Means and standard errors of fractal dimensions for RW with increasing numbers of observations and lags, madogram

Random Walk								
# observations	500	1 000	2 000	5 000	10 000	200 000	500 000	100 000
nlags	5	10	20	50	100	200	500	1 000
mean	1.5010	1.5022	1.4997	1.5004	1.4998	1.5016	1.5010	1.5007
sd	0.0330	0.0261	0.0216	0.0201	0.0185	0.0181	0.0167	0.0167
nlags	5	5	5	5	5	5	5	5
mean	1.5010	1.4988	1.4992	1.5003	1.5001	1.5001	1.5001	1.4999
sd	0.0335	0.0235	0.0163	0.0101	0.0075	0.0052	0.0033	0.0023

5.4.10 Other results

This subsection contains results for Brownian motion and martingale simulations in combination with the madogram, Hall-Wood, and box-count estimators. The outcomes description is divided to paragraphs for each type of estimator; these include description of results for all kinds of efficient time series simulation and comparison of the results. The background for the comments can be found in Appendix, where we presents both tables with means and standard errors of the fractal dimensions and 3D graphs. To save the simulation time, we simulate only 500 pieces of each setting for the Hall-Wood and box-count estimators. The results of individual estimators are discussed in the following order: the madogram, Hall-Wood, and box-count estimators.

There are only small differences between the results for random walk and Brownian motion using the madogram. The fractal dimensions estimated on Brownian motion cover wider range of values and their standard errors are mostly a bit higher than for simulations of random walk. Martingale simulation leads to overestimation of fractal dimension for the small lag estimators as described in the section discussing differences between random walk, Brownian motion, and martingale simulations. In case of the 100 lags estimator, the results are much closer to the random walk outcomes. All types of simulations exhibit the same bias for the 5 lags estimators and stronger positive autocorrelation, and the bias disappears for the 100 lags estimators. The results are summed up in Tables A.13 to A.15 in the Appendix. Further, we provide 3D graphs for all free types of simulation including ARFIMA(p, d, 0.4) inefficient part for 5, 20 and 100 lags in Figure B.23 in the Appendix. Figures for random walk and Brownian motion simulation are nearly identical, the martingale figures are more skewed and moved up for small number of lags, but the difference is reduced with higher numbers of lags. The dissimilarities are caused by less exact martingale simulation as described in one of the previous sections. We can conclude that there is not a significant difference between the results of the three types of simulations.

The results of the Hall-Wood estimators exhibit the same behaviour as in case of the madogram estimators; characteristics of the outcome are same so we do not repeat them here. The means and standard errors are presented in Tables A.16 to A.18 in the Appendix and 3D graphs can be found in Figure B.24 in the Appendix. For this estimator, we present only results for 5 and 20 lags.

The outcomes for the box-count estimator are completely different. Number of lags is set to the "auto" option as it is the most efficient estimator. The 3D graphs depicting the mean fractal dimension can be found in the Appendix in Figure B.25. The results for random walk and Brownian motion simulations keep the assumed trend but the fractal dimensions are underestimated. Further, the impact of p parameter diminishes and the differences between simulations with ARFIMA(0, d, 0.4) and ARFIMA(0.8, d, 0.4) are much smaller than in case of other estimators. Moreover, the graphs exhibit higher volatility, so trends are not so clear. The differences between random walk and Brownian motion can be characterized by the same description as provided for the madogram and Hall-Wood estimations. Martingale simulation stands out as it seems to be totally wrong. The fractal dimensions suffer from a big overestimation and the trend in both parameters is reversed. The results are probably caused by the problematic martingale simulation, which tends to overestimate fractal dimension also for the other estimators; however, the impact is stressed a lot for the box-count estimator. The means and standard errors for all types of simulation are presented in the Appendix in Tables A.19 to A.21.

To point out differences between individual estimators, we include Figure B.26 with results of the madogram, Hall-Wood and box-count estimators for random walk simulations. As noted before, the results of the madogram and Hall-Wood estimators are nearly identical while the box-count results stand out with the underestimated fractal dimension and diminished impact of p parameter.

5.5 Conclusion

Based on the conducted analysis we conclude that fractal dimension accurately captures the autocorrelation structure in returns and distinguishes between autocorrelations of different strength. We recommend using of the madogram estimator as it is the most efficient estimator. Although the 5 lags estimators have the lowest standard errors, we think that it is better to examine estimations with both lower and higher numbers of lags, because the small lags estimators show some peculiarities for fractal dimension of time series with stronger autocorrelation, which disappear with the increasing number of lags.

The problem of fractal dimension of a whole time series is that a time series may contain positive and negative autocorrelations parts which could counteract with each other and thus distort the efficiency result. For this reason, we suggest to divide the time series to several parts and estimate fractal dimensions for these parts separately to avoid distortions. If it is possible, the individual parts should have at least 2,000 observations. To track changes in a time series, one can estimate fractal dimension of the first part of the series and then gradually add the other observations, re-estimating fractal dimension after each extension of the time series.

Changes in fractal dimension together with the size of the changes may create a useful tool for identification of market efficiency and tracking the changes in the efficiency. A deeper study of the utilization of fractal dimension in trading is left for other studies as it is beyond the scope of this thesis. Now we use these findings in comparison of individual markets and create their efficiency ranking.

Chapter 6

Results of real market analysis

We analyse efficiency of 28 stock market indices listed in Table 6.1. The dataset has been obtained from finance.yahoo.com public database. We examine all indices with a reasonable number of observations available in the database; our dataset includes indices from North and South America, Europe, Asia and Oceania. The study utilizes all observations downloadable in the databases at the beginning of February 2014, and the number of observations is noted in Table 6.1. The exact analysed period of each individual stock can be found in the Appendix in the section of indices' analyses, where we describe the indices in a greater detail. Examined periods of most of the indices cover the years of stable growth, DotCom bubble burst and subsequent stable decrease, and the 2007-2008 financial crisis.

Note, that security prices can be viewed as simple price or a logarithmic price. Logarithmic prices approach is the more standard one, so we apply it in our thesis. Basic descriptive statistics of the logarithmic close/close returns are presented in Table 6.2. To make the statistics comparable, we omit the early observations and examine statistics for years 2000-2013 if available. Basic descriptive statistics of the logarithmic close/close returns for all observations are recorded in Table C.1 in the Appendix. According to the KPSS test and Augmented Dickey Fuller test, all returns are stationary; the ADF test strictly rejects the unit root for all indices, so we do not present the results.

Two types of analysis are employed: an overall analysis ranking the markets according to market efficiency and more detailed study of main groups of markets based on the efficiency development of markets.

Table 6.1: List of analysed indices

Ticker	Index	Country	# obs.
AEX	Amsterdam Exchange Index	The Netherlands	5435
ASE	Athens Stock Exchange General Index	Greece	6746
ASX	Australian Securities Exchange Index	Australia	3120
ATX	Austrian Traded Index	Austria	5254
BEL20	Euronext Brussels Index	Belgium	5777
BSE	Bombay Stock Exchange Index	India	4104
CAC	Euronext Paris Bourse Index	France	6063
DAX	Deutscher Aktien Index	Germany	5875
ESTX	EURO STOXX 50 Index	Eurozone	6953
FTSE	Financial Times Stock Exchange 100 Index	UK	7541
HSI	Hang Seng Index	Hong Kong	6471
IBOSP	Ibovespa Brasil Sao Paulo Stock Exchange Index	Brazil	5141
IPC	Indice de Precios y Cotizaciones	Mexico	5563
IPSA	Santiago Stock Exchange Index	Chile	2593
JKSE	Jakarta Composite Index	Indonesia	4033
KLSE	Bursa Malaysia Index	Malaysia	4975
KS11	KOSPI Composite Index	Korea	4093
MERVAL	Mercado de Valores Index	Argentina	4270
NASD	NASDAQ Composite Index	USA	10847
NIKKEI	NIKKEI 225 Index	Japan	7402
NZX	New Zealand Exchange 50 Gross Index	New Zealand	2355
SPX	Standard & Poor's 500 Index	USA	16128
SSEC	Shanghai Composite Index	China	5935
SSMI	Swiss Market Index	Switzerland	5871
STRAITS	Straits Times Index	Singapore	6546
TSE	Toronto Stock Exchange TSE 300 Index	Canada	7420
TWSE	Taiwan Stock Exchange Weighted Index	Taiwan	4082
XAO	All Ordinaries Index	Australia	7467

Table 6.2: Descriptive statistics of analysed indices, close-close returns, observations for years 2000-2013

Index	Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
AEX	-0.0001	-0.0959	0.1003	0.0152	-0.0651	9.0820	0.1844	> 0.05
ASE	-0.0004	-0.1021	0.1343	0.0176	-0.0132	7.1399	0.1743	> 0.05
ASX	0.0002	-0.0870	0.0563	0.0105	-0.4471	8.8732	0.1130	> 0.05
ATX	0.0002	-0.1025	0.1202	0.0150	-0.3027	10.1011	0.2398	> 0.05
BEL20	0.0000	-0.0832	0.0933	0.0132	0.0445	8.8571	0.1216	> 0.05
BSE	0.0004	-0.1181	0.1599	0.0162	-0.1818	9.4239	0.1529	> 0.05
CAC	-0.0001	-0.0947	0.1059	0.0154	0.0262	7.6637	0.1371	> 0.05
DAX	0.0001	-0.0743	0.1080	0.0158	0.0233	7.1487	0.2505	> 0.05
ESTX	-0.0001	-0.0821	0.1044	0.0156	0.0177	7.2505	0.1402	> 0.05
FTSE	0.0000	-0.0926	0.0938	0.0126	-0.1243	8.9375	0.1804	> 0.05
HSI	0.0001	-0.1358	0.1341	0.0158	-0.0667	10.8159	0.1219	> 0.05
IBOSP	0.0003	-0.1210	0.1368	0.0187	-0.0898	6.7837	0.1523	> 0.05
IPC	0.0005	-0.0827	0.1044	0.0141	0.0242	7.5342	0.1108	> 0.05
IPSA	0.0005	-0.0724	0.2788	0.0120	4.8278	120.9241	0.4298	> 0.05
JKSE	0.0005	-0.1095	0.0762	0.0148	-0.6766	8.9110	0.2101	> 0.05
KLSE	0.0002	-0.1925	0.1986	0.0111	-0.2474	95.5659	0.1401	> 0.05
KS11	0.0002	-0.1280	0.1128	0.0170	-0.5296	8.3126	0.1350	> 0.05
MERVAL	0.0007	-0.1295	0.1612	0.0213	-0.1223	7.7794	0.0844	> 0.05
NASD	0.0000	-0.1017	0.1325	0.0175	0.0457	7.9047	0.4790	> 0.04
NIKKEI	0.0000	-0.1211	0.1323	0.0158	-0.4203	9.2486	0.2949	> 0.05
NZX	0.0003	-0.0494	0.0581	0.0074	-0.3604	7.8701	0.2265	> 0.05
SPX	0.0001	-0.0947	0.1096	0.0132	-0.1753	10.7004	0.2634	> 0.05
SSEC	0.0001	-0.0926	0.0940	0.0156	-0.0891	7.5811	0.1465	> 0.05
SSMI	0.0000	-0.0811	0.1079	0.0122	0.0156	9.1551	0.1346	> 0.05
STRAITS	0.0001	-0.0922	0.0753	0.0123	-0.4240	9.1606	0.1692	> 0.05
TSE	0.0001	-0.0979	0.0937	0.0119	-0.6422	11.7667	0.0653	> 0.05
TWSE	0.0000	-0.0994	0.0652	0.0150	-0.2380	5.7584	0.1416	> 0.05
XAO	0.0002	-0.0855	0.0536	0.0100	-0.6056	9.4036	0.0868	> 0.05

6.1 Overall analysis

In this section, we compare efficiency of the indices. We focus on the period 2000-2013 and estimate fractal dimension of each index using the madogram with 5, 20, and 100 lags. Reasons for using these types of estimators are described in the conclusion of the previous chapter. Recall that we have to be aware of possible inaccuracy of the 100 lags estimator associated with the limited number of observations.

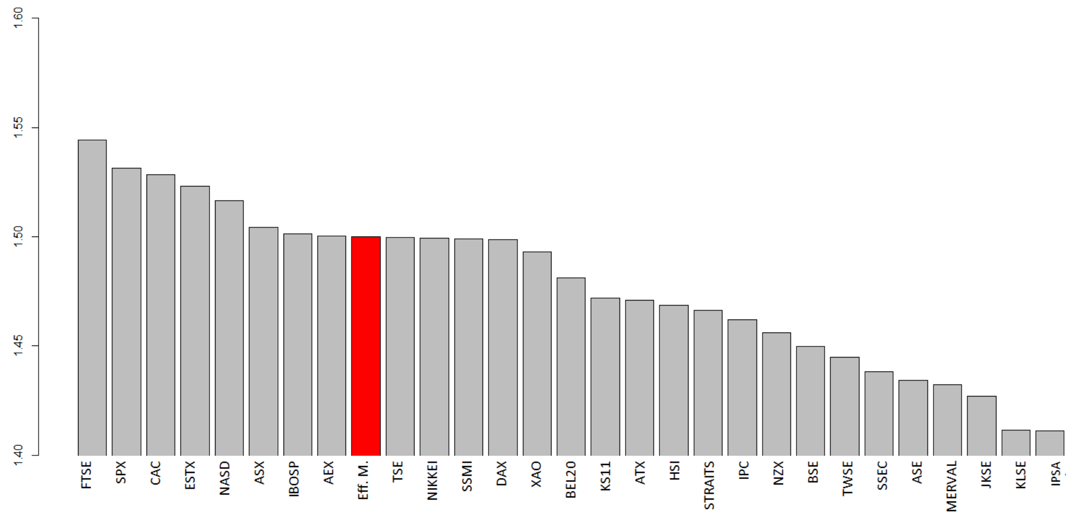
The fractal dimensions for period 2000-2013 are presented in Figure 6.1 showing the results for the 5 lags estimator; results for the 20 lags estimator can be found in the Appendix as Figures C.1. The fractal dimension of an efficient market is marked by red colour. There are 5 indices characterized by the dimension significantly higher than 1.5 indicating negative autocorrelation in returns (and higher volatility in prices) compared to an efficient market. Other 7 indices have the fractal dimensions very close to 1.5, implying structures close to efficient market, and the fractal dimensions of the rest of the indices are lower than 1.5, which is probably caused by local or global persistences in the data.

The index with the highest value of the fractal dimension is FTSE index of British Financial Times Stock Exchange, followed by American SPX, French CAC, European ESTX, and American NASD; all of them have the fractal dimension higher than 1.5. Australian ASX, Brazilian IBOSP, Austrian AEX, Canadian TSE, Japanese NIKKEI, Swiss SSMI, and German DAX have the fractal dimensions very close to 1.5. The least efficient markets are marked by the fractal dimension lower than 1.45; these markets probably experience strong or long term trend and thus are at least partially predictable. Chilean IPSA, Malaysian KLSE, Indonesian JKSE, Argentine MERVAL, and Greek ASE belong among the 5 least efficient markets.

Figure C.1 in the Appendix shows result for the madogram with 20 lags, which slightly overestimates fractal dimension, and it marks more indices as highly volatile. The two estimators are not consistent even in the ranking of indices; to identify the reasons of inconsistency, we have to analyse individual indices in more details, as some patterns in data are strongly captured by the 20 lags estimators, while others by the 5 lags estimators. Hence, we provide a brief description of each index in Appendix.

We do not present the overall results for the 100 lags madogram as they may be significantly skewed due to the low number of observations.

Figure 6.1: Fractal dimension for world stock indices, 5 lags madogram



In the Appendix, we provide an overview of statistics and results for each index: descriptive statistics for all observations and years 2000-2013, plot of the index, fractal dimension of all observations, years 2000-2013 and each thousand of observations separately, graph of the fractal dimensions gradually estimated for increasing number of observations, and a brief comment. Even though the tables of the the fractal dimensions contain values for the 5, 20 and 100 lags estimators, the results for the 100 lags madogram are probably biased due to low number of observations, this problem is illustrated in the previous chapter on the standard errors of the 100 lags estimates. For the last graph, we use the 5 lags values.

The estimated fractal dimensions of indices presented in our thesis are slightly higher than those introduced in the paper by Kristoufek and Vosvrda (2013) which may be caused by employment of different estimators, lags specifications or the time range; we use data from years 2000-2013, whereas Kristoufek and Vosvrda analyse time period January 2000 - August 2011. This can lead to the differences in ranking as well.

6.2 Main groups of indices

This section examines results of a separate analysis of each index, which includes estimation of fractal dimension for every 1,000 observations of the time series and re-estimation of fractal dimension for a gradually increasing sample, the re-estimated fractal dimensions are called "cumulative" fractal dimension in the rest of the study. These two approaches enable us to see when an index was more and less efficient and when the changes in efficiency occurred. Based on the analysis, the indices can be divided into four groups: very efficient markets, low performing markets, markets that improved their structure and reached efficiency but experienced a drop recently, and markets that steadily improve their efficiency. Main characteristics of each group are provided in the following sections. Even though we present only the graphs of "cumulative" fractal dimension here, the characteristics of individual groups are based on complex analysis of each index.

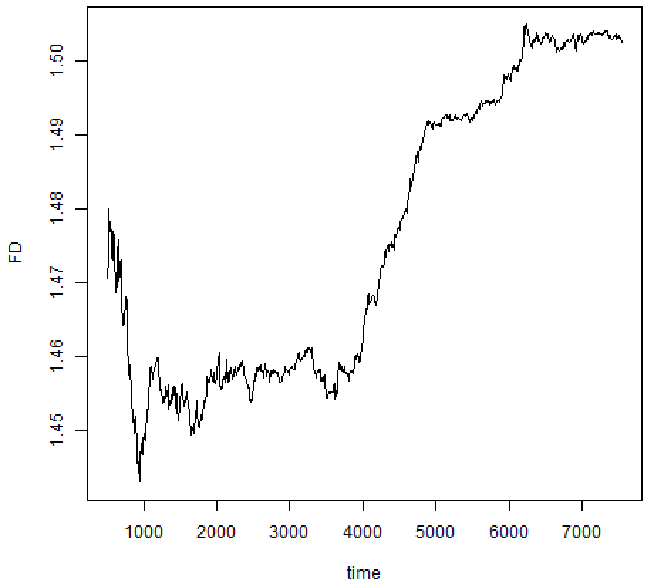
6.2.1 High volatility markets

This group of markets is characterized by anti-persistent periods when fractal dimensions of all included markets exceeded 1.5. Although at the time of each index introduction, the markets suffered from positive autocorrelation in returns and fractal dimensions of the indexes was not higher than 1.45, the inefficiencies disappeared, and fractal dimensions increased and approached 1.5 value. In the pre-crisis and the early crisis period, volatility increased above the level desired by efficiency markets, and fractal dimension reached values higher than 1.53. However, as the markets were hit by financial crisis, volatility has decreased to the martingale level, and fractal dimension returned to values close to 1.5. This development of fractal dimension is characteristic for highly developed markets.

Focusing on the graphs of gradual estimation of the fractal dimensions, we can see a rapid increase referring to the high volatility period. Let us describe the characteristic features of the group on Figure 6.2 depicting the fractal dimensions of a representative of the group - FTSE index. The observations 1 to 4,000 record the early existence of the index and the fractal dimension oscillating along the 1.46 value indicates low efficiency of the market at that time. Then there is a rapid growth of the "cumulative" fractal dimension, typical for a period with high volatility. At the end of the sample, the growth stops and the fractal dimension starts to stagnate with values close to 1.5 signalling an efficient market.

The group contains the following indices: CAC, ESTX, FTSE, NASD, NIKKEI, and SPX. Statistics and graphs relating to the indices are presented in Appendix C.

Figure 6.2: FTSE - graph of gradually estimated fractal dimensions



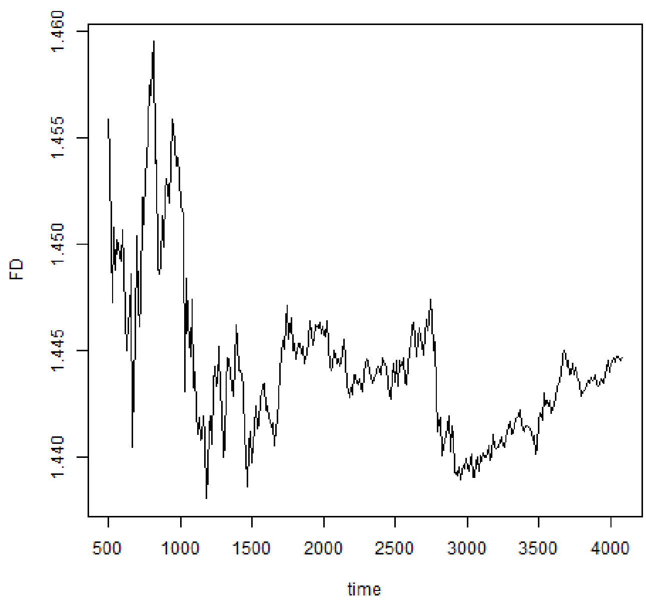
6.2.2 Inefficient markets

This group includes markets with stagnating low level of efficiency accompanied by low fractal dimension; these markets provide an opportunity for investors to gain higher than normal returns. The group is characterized by local or global persistences and contains mostly developing financial markets.

The members of this groups may have two different types of graphs - the former ones clearly show that fractal dimension is stabilized and takes values close to a given level, whereas the later ones can be misleading as "cumulative" fractal dimension increases in spite of the stagnation of fractal dimension. See for example the graph of STRAITS Section C.24 in the Appendix. The ongoing growth is caused by very low initial values of the fractal dimension; even though the fractal dimension keeps value close to 1.47 for the last 2,000 observations, the graphs depict and increasing curve. To register the reason, one has to focus on the values on y-axis. At the end of the sample, the "cumulative" fractal dimension reaches 1.42, a value significantly lower than 1.46. The first described type of graphs usually shows higher visual volatility in "cumulative" fractal dimensions as depicted in Figure 6.3 of the TWSE index fractal dimensions.

Indices belonging to this group are: ASE, BSE, IPSA, KLSE, KS11, MERVAL, SSEC, STRAITS, and TWSE. Further information about these indices can be found in the Appendix C.

Figure 6.3: TWSE - graph of gradually estimated fractal dimensions



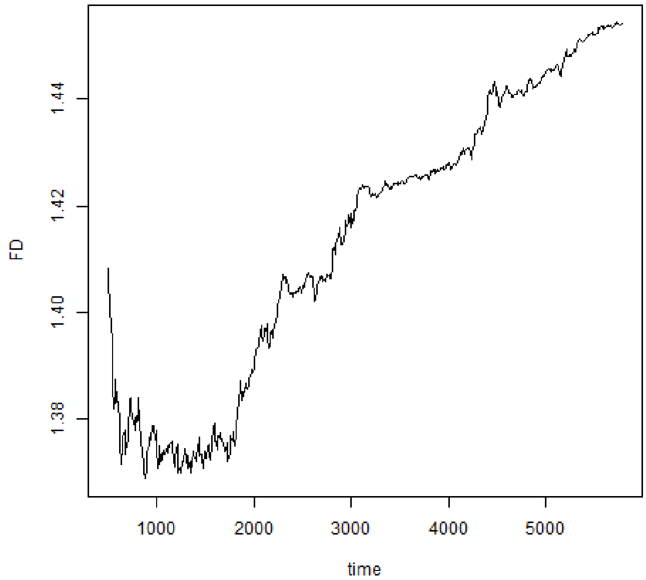
6.2.3 Constantly improving markets

Constantly improving markets group compiles indices that were not hit by the crisis and whose fractal dimensions continue to growth and approaches the level of 1.5. Most of the efficient markets were influenced by the recent crisis and their fractal dimensions have decreased, which implies a drop in inefficiency. However, there are few exceptions whose fractal dimensions continue to growth despite the crisis. Although one can observes local slowdown in the growth or even a slight decrease, fractal dimensions of individual parts of the indices show growth, and fractal dimension of the last 1,000 observations is very close to 1.5 for most of the group members. So this group, together with the "Very efficient markets", represent currently efficient markets.

Figure 6.4 is a representative illustration of the "cumulative" fractal dimension's development of BEL20 index. You can see a sequence of less and more steep increases. The graph shows that the "cumulative" fractal dimension approaches 1.45 value, but the fractal dimension of the last 1,000 observations is 1.50, signalling high efficiency of the market. This example explains why "cumulative" fractal dimension is a good indicator of fractal dimension development but a bad indicator of the state of fractal dimension.

The group includes the following indices: ATX, BEL20, IPC, JKSE, TSE. More detailed information about these indices are located in the Appendix C, which includes statistics and graphs of individual indices.

Figure 6.4: BEL20 - graph of gradually estimated fractal dimensions



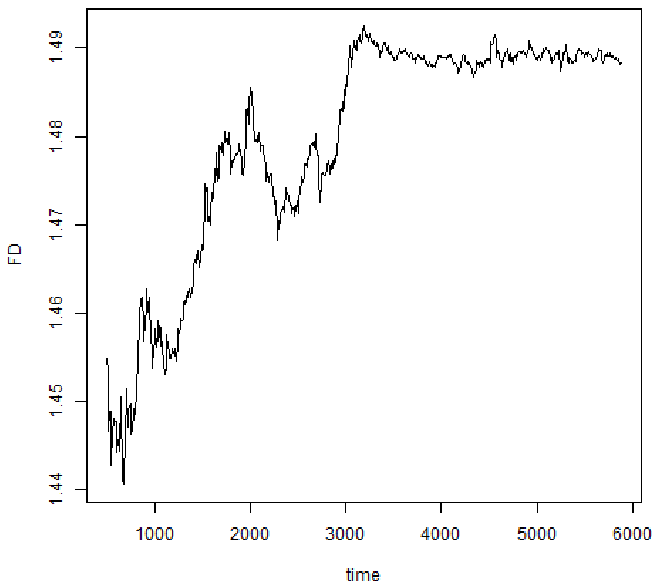
6.2.4 Markets hit by the recent crisis

As noted in the preceding subsection, most of the previously efficient markets were hit by the crisis that caused arising of local inefficiencies and positive autocorrelation in returns. All indices recently experiencing a drop are contained in this group, most of them were previously very close to efficiency and 1.5 value of fractal dimension, while now their fractal dimensions take values close to 1.48.

The decrease in efficiency may not be visible in graphs of “cumulative” fractal dimensions or it can appear as a slowdown in increase, which is caused by the impact of previous observations; hence, the main indicator of a drop is fractal dimension of the last 1,000 observations. We may show the effect on Figure 6.5 of DAX index. The “cumulative” fractal dimension grows to 1.49 and then it starts to slightly decrease with a drop in the fractal dimension from 1.5 to 1.485.

The following indices were categorized as recently efficient but affected by crisis: ASX, AEX, DAX, HSI, IBOSP, SSMI, XAO. Statistics and graphs regarding individual indices are presented in the Appendix C.

Figure 6.5: DAX - graph of gradually estimated fractal dimensions



Chapter 7

Conclusion

The efficient market hypothesis and its testing are important not only for researches but also for investors, because it is impossible to systematically achieve abnormal returns on efficient markets. The currently used approaches to testing the efficient market hypothesis have several drawbacks: primarily, they do not capture local inefficiencies; further, they either reject or not reject the null hypothesis of market efficiency, but they do not evaluate the level of efficiency; finally, the tests are unable to sufficiently track small changes in the efficiency structure. However, investors are interested not only in the presence of inefficiencies, but in their size and changes as well. Kristoufek and Vosvrda (2013) suggest utilization of fractal dimension as an efficiency measure, but they apply the tool in practise without testing its ability to capture the efficiency on simulated data. Hence, we provide an extensive examination of fractal dimension's explanatory power in the context of efficient markets.

Fractal dimension measures roughness of a time series, taking values in the interval [1,2). An efficient time series is supposed to follow martingale, which has fractal dimension equal to 1.5. Inefficiencies are characterized by positive/negative autocorrelation in returns, i.e. a decrease in prices is followed by decrease/increase with a probability higher than 0.5. Autocorrelation structures differing from martingales affect the graph of a times series - positive autocorrelation in returns makes it smoother and fractal dimension of time series containing persistences should thus be lower than 1.5; on the other hand, higher volatility should increase fractal dimension above 1.5. To test these hypotheses, we employ an extensive Monte Carlo simulation, insert different types of inefficiencies into efficient time series, measure their fractal dimensions and compare the results.

The main advantage of the fractal dimension analysis compared to other efficiency tests is its ability to track local persistences. To verify the hypothesis we use a simulation of efficient time series where we include a short part of inefficient observations; the level of inefficiency and the number of observations vary and the effect of the changes is studied. Each examined time series consists of 10,000 observations and we simulate 1,000 time series of each setting to ensure statistical significance. The efficient parts are obtained through random walk, Brownian motion, and martingale simulation, while the inefficient parts are based on ARFIMA (1, d, 1) simulations; the level of inefficiency is managed by changes in the coefficients; in total, we use nearly 700 different settings of ARFIMA. In addition to the analysis of series with one inefficient part, we examine also the impact of time series with two inefficient parts with reverse effect on fractal dimension and reaction of fractal dimension to gradual adding of observations with given settings to the examined sample. The gradual estimation enables us to evaluate the reaction of fractal dimension on small changes in the time series.

Fractal dimension is estimated by the madogram, the Hall-Wood and Box-count estimators. Based on our analysis of estimators' efficiency, we recommend to use the madogram with 5 lags as it is the most efficient and accurate estimator. However, the estimator distorts results for extreme conditions characterized by strong autocorrelation in returns. We suggest using the madogram with higher number of lags (20, 100) to validate the results, but one should be aware of the fact that estimators with higher number of lags tend to overestimate fractal dimension and should be used only for ranking.

Based on the simulation results, we conclude that fractal dimension is a good measure of market efficiency accurately reflecting all normal changes in efficiency structure. The results tend to be skewed for extreme inefficiencies, but we do not assume such strong autocorrelation in returns to be present in real markets, so it does not influence usage of fractal dimension in practice.

Further, we simulate a series consisting of ARFIMA simulations with slightly different settings - we estimate fractal dimension of first 500 observations and then gradually add more observations and re-estimate the fractal dimension each time. This study shows that fractal dimension quickly and accurately reflects all changes in efficiency structure of the examined time series.

We complete the work with an example of practical usage of the fractal dimension measure. Fractal dimensions of 28 stock indices are estimated and the indices are then ranked based on the fractal dimension results. We find that some indices such as FTSE and SPX have fractal dimensions higher than 1.5 which implies anti-persistence. All indices characterized by higher volatility are from developed countries. On the other hand, the lowest value of fractal dimension was obtained for some developing countries such as Chile and Malaysia that have fractal dimensions close to 1.4. Dutch and Canadian indices have fractal dimensions closest to 1.5, so these two are the most efficient ones.

The thesis proves that fractal dimension is a good measure of market efficiency for common levels of autocorrelation structure of returns, as it reflects all changes in ARFIMA setting accurately. It can be used for determination of the efficiency size of examined series and creation of a ranking of the studied markets. Further, fractal dimension very quickly shows even small changes in structure of observations that are gradually added to the time series. Therefore, fractal dimension can be used for examination of market efficiency development in different economics, and investors can employ it to track changes in market efficiency.

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Appendix A

Tables

The ϕ and θ coefficients are marked as p and q in all tables and figures.

Table A.1: Coefficients of back-shift operators for different values of d and p

	p	d	p	d	p	d	p	d	p	d	p	d
	0.1	-0.1	0.2	-0.2	0.3	-0.3	0.4	-0.4	0.3	-0.1	0.6	-0.4
L	0.0000	-	0.0000	-	0.0000	-	0.0000	-	0.2000	-	0.2000	-
L^2	0.0450	+	0.0800	+	0.1050	+	0.1200	+	0.0250	+	0.0400	+
L^3	-0.0330	-	-0.0640	-	-0.0910	-	-0.1120	-	-0.0220	-	-0.0560	-
L^4	0.0260	+	0.0528	+	0.0785	+	0.1008	+	0.0183	+	0.0560	+
L^5	-0.0215	-	-0.0451	-	-0.0691	-	-0.0914	-	-0.0155	-	-0.0533	-
L^6	0.0184	+	0.0394	+	0.0619	+	0.0838	+	0.0135	+	0.0503	+
L^7	-0.0160	-	-0.0351	-	-0.0562	-	-0.0776	-	-0.0119	-	-0.0474	-
L^8	0.0143	+	0.0318	+	0.0516	+	0.0724	+	0.0106	+	0.0448	+
L^9	-0.0129	-	-0.0291	-	-0.0479	-	-0.0680	-	-0.0097	-	-0.0425	-
L^{10}	0.0117	+	0.0268	+	0.0447	+	0.0643	+	0.0088	+	0.0405	+
L^{11}	-0.0108	-	-0.0249	-	-0.0420	-	-0.0610	-	-0.0081	-	-0.0387	-

Table A.2: Changes in coefficients of back-shift operators for increasing d

	p	d	p	d	p	d	p	d	p	d
	0.1	-0.4	0.1	-0.2	0.1	0	0.1	0.2	0.1	0.4
L	-0.3000	-	-0.1000	-	0.1000	-	0.3000	-	0.5000	-
L^2	0.2400	+	0.1000	+	0.0000	+	-0.0600	+	-0.0800	+
L^3	-0.1960	-	-0.0760	-	0.0000	-	0.0400	-	0.0520	-
L^4	0.1680	+	0.0616	+	0.0000	+	-0.0288	+	-0.0352	+
L^5	-0.1485	-	-0.0521	-	0.0000	-	0.0222	-	0.0258	-
L^6	0.1340	+	0.0453	+	0.0000	+	-0.0179	+	-0.0200	+
L^7	-0.1228	-	-0.0403	-	0.0000	-	0.0149	-	0.0161	-
L^8	0.1137	+	0.0363	+	0.0000	+	-0.0127	+	-0.0133	+
L^9	-0.1063	-	-0.0331	-	0.0000	-	0.0110	-	0.0113	-
L^{10}	0.1000	+	0.0305	+	0.0000	+	-0.0097	+	-0.0097	+
L^{11}	-0.0946	-	-0.0283	-	0.0000	-	0.0087	-	0.0085	-

Table A.3: Changes in coefficients of back-shift operators for increasing p

	p	d	p	d	p	d	p	d	p	d
	0.1	-0.4	0.3	-0.4	0.5	-0.4	0.7	-0.4	0.9	-0.4
L	-0.3000	-	-0.1000	-	0.1000	-	0.3000	-	0.5000	-
L^2	0.2400	+	0.1600	+	0.0800	+	0.0000	+	-0.0800	+
L^3	-0.1960	-	-0.1400	-	-0.0840	-	-0.0280	-	0.0280	-
L^4	0.1680	+	0.1232	+	0.0784	+	0.0336	+	-0.0112	+
L^5	-0.1485	-	-0.1104	-	-0.0724	-	-0.0343	-	0.0038	-
L^6	0.1340	+	0.1005	+	0.0670	+	0.0335	+	0.0000	+
L^7	-0.1228	-	-0.0926	-	-0.0625	-	-0.0323	-	-0.0022	-
L^8	0.1137	+	0.0862	+	0.0586	+	0.0310	+	0.0034	+
L^9	-0.1063	-	-0.0808	-	-0.0553	-	-0.0298	-	-0.0043	-
L^{10}	0.1000	+	0.0762	+	0.0524	+	0.0286	+	0.0048	+
L^{11}	-0.0946	-	-0.0722	-	-0.0498	-	-0.0275	-	-0.0051	-

Table A.4: Means (1st values) and standard errors (2nd values) of fractal dimensions for RW with ARFIMA, 500 observations, 5 lags madogram, shading underlines the trends

ARFIMA(p, d, 0)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5442	1.5293	1.5156	1.5067	1.5007	1.4961	1.4942	1.4934	1.4942
	0.0148	0.0114	0.0096	0.0079	0.0076	0.0075	0.0078	0.0079	0.0081
0.4	1.5056	1.4981	1.4930	1.4903	1.4898	1.4899	1.4907	1.4917	1.4934
	0.0084	0.0077	0.0079	0.0080	0.0088	0.0087	0.0087	0.0085	0.0084
0.8	1.4829	1.4822	1.4831	1.4838	1.4861	1.4878	1.4901	1.4917	1.4937
	0.0094	0.0098	0.0102	0.0098	0.0093	0.0093	0.0083	0.0085	0.0082

ARFIMA(p, d, 0.4)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5149	1.5062	1.4991	1.4948	1.4925	1.4913	1.4919	1.4928	1.4935
	0.0093	0.0078	0.0077	0.0076	0.0080	0.0085	0.0082	0.0082	0.0081
0.4	1.4910	1.4870	1.4859	1.4861	1.4867	1.4880	1.4897	1.4915	1.4932
	0.0085	0.0088	0.0088	0.0093	0.0094	0.0092	0.0088	0.0084	0.0081
0.8	1.4782	1.4797	1.4813	1.4834	1.4859	1.4878	1.4896	1.4918	1.4931
	0.0104	0.0107	0.0106	0.0098	0.0095	0.0091	0.0089	0.0085	0.0080

ARFIMA(p, d, 0.8)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5052	1.4988	1.4940	1.4918	1.4897	1.4905	1.4908	1.4923	1.4936
	0.0083	0.0077	0.0077	0.0084	0.0084	0.0085	0.0085	0.0084	0.0083
0.4	1.4868	1.4843	1.4841	1.4844	1.4861	1.4881	1.4895	1.4910	1.4928
	0.0087	0.0092	0.0089	0.0090	0.0093	0.0094	0.0087	0.0088	0.0082
0.8	1.4767	1.4785	1.4807	1.4834	1.4854	1.4877	1.4900	1.4916	1.4933
	0.0103	0.0100	0.0103	0.0097	0.0094	0.0093	0.0086	0.0085	0.0084

Table A.5: Means and standard errors of fractal dimensions for RW with ARFIMA, 500 observations, 20 lags madogram

ARFIMA(p, d, 0)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5399	1.5274	1.5161	1.5074	1.5011	1.4965	1.4926	1.4914	1.4921
	0.0146	0.0131	0.0113	0.0102	0.0102	0.0100	0.0102	0.0107	0.0108
0.4	1.5192	1.5106	1.5023	1.4969	1.4938	1.4919	1.4909	1.4910	1.4911
	0.0117	0.0103	0.0103	0.0100	0.0105	0.0105	0.0109	0.0106	0.0107
0.8	1.4920	1.4894	1.4870	1.4858	1.4867	1.4867	1.4883	1.4890	1.4910
	0.0105	0.0109	0.0108	0.0115	0.0115	0.0114	0.0113	0.0109	0.0108

ARFIMA(p, d, 0.4)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5266	1.5174	1.5088	1.5012	1.4968	1.4935	1.4918	1.4915	1.4918
	0.0124	0.0113	0.0101	0.0099	0.0103	0.0107	0.0105	0.0106	0.0108
0.4	1.5125	1.5052	1.4992	1.4950	1.4924	1.4909	1.4903	1.4902	1.4910
	0.0102	0.0102	0.0098	0.0102	0.0104	0.0107	0.0105	0.0108	0.0105
0.8	1.4900	1.4877	1.4863	1.4861	1.4863	1.4866	1.4878	1.4896	1.4899
	0.0105	0.0111	0.0119	0.0116	0.0118	0.0107	0.0114	0.0112	0.0106

ARFIMA(p, d, 0.8)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5229	1.5143	1.5060	1.5005	1.4960	1.4935	1.4917	1.4911	1.4919
	0.0119	0.0110	0.0101	0.0103	0.0097	0.0103	0.0104	0.0103	0.0108
0.4	1.5108	1.5037	1.4983	1.4944	1.4920	1.4913	1.4902	1.4901	1.4909
	0.0108	0.0099	0.0102	0.0102	0.0105	0.0107	0.0106	0.0112	0.0106
0.8	1.4895	1.4874	1.4861	1.4863	1.4862	1.4870	1.4878	1.4889	1.4903
	0.0108	0.0106	0.0113	0.0112	0.0118	0.0113	0.0119	0.0113	0.0108

Table A.6: Means and standard errors of fractal dimensions for RW with ARFIMA, 500 observations, 100 lags madogram

ARFIMA(p, d, 0)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5320	1.5227	1.5154	1.5074	1.5015	1.4962	1.4926	1.4905	1.4888
	0.0189	0.0186	0.0189	0.0176	0.0180	0.0189	0.0183	0.0190	0.0188
0.4	1.5240	1.5172	1.5095	1.5039	1.4984	1.4946	1.4920	1.4899	1.4877
	0.0185	0.0185	0.0187	0.0186	0.0181	0.0186	0.0188	0.0189	0.0186
0.8	1.5091	1.5040	1.5000	1.4958	1.4937	1.4905	1.4888	1.4871	1.4876
	0.0177	0.0179	0.0172	0.0187	0.0186	0.0191	0.0188	0.0195	0.0185

ARFIMA(p, d, 0.4)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5278	1.5197	1.5126	1.5057	1.4994	1.4954	1.4918	1.4896	1.4897
	0.0190	0.0186	0.0189	0.0184	0.0187	0.0189	0.0189	0.0196	0.0196
0.4	1.5202	1.5149	1.5086	1.5023	1.4987	1.4952	1.4911	1.4893	1.4881
	0.0183	0.0188	0.0183	0.0184	0.0180	0.0188	0.0180	0.0191	0.0189
0.8	1.5081	1.5041	1.4992	1.4971	1.4937	1.4917	1.4896	1.4875	1.4882
	0.0177	0.0196	0.0183	0.0191	0.0188	0.0191	0.0187	0.0197	0.0195

ARFIMA(p, d, 0.8)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5259	1.5184	1.5116	1.5038	1.4990	1.4964	1.4921	1.4892	1.4881
	0.0188	0.0180	0.0185	0.0181	0.0180	0.0182	0.0176	0.0180	0.0178
0.4	1.5213	1.5140	1.5075	1.5037	1.4985	1.4944	1.4908	1.4895	1.4882
	0.0183	0.0183	0.0179	0.0187	0.0181	0.0187	0.0190	0.0191	0.0195
0.8	1.5085	1.5034	1.4984	1.4956	1.4926	1.4908	1.4893	1.4874	1.4869
	0.0181	0.0183	0.0184	0.0186	0.0191	0.0187	0.0193	0.0190	0.0200

Table A.7: Means and standard errors of fractal dimensions for RW with ARFIMA, 300 observations, 5 lags madogram

ARFIMA(p, d, 0)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5249	1.5157	1.5091	1.5041	1.4999	1.4974	1.4966	1.4956	1.4959
	0.0108	0.0093	0.0079	0.0078	0.0076	0.0078	0.0079	0.0076	0.0077
0.4	1.5032	1.4989	1.4963	1.4940	1.4940	1.4938	1.4941	1.4947	1.4950
	0.0076	0.0075	0.0077	0.0077	0.0076	0.0081	0.0081	0.0076	0.0078
0.8	1.4905	1.4898	1.4899	1.4903	1.4916	1.4925	1.4935	1.4945	1.4953
	0.0080	0.0083	0.0085	0.0084	0.0080	0.0080	0.0079	0.0080	0.0074

ARFIMA(p, d, 0.4)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5082	1.5034	1.4994	1.4974	1.4957	1.4937	1.4944	1.4944	1.4952
	0.0079	0.0073	0.0075	0.0077	0.0076	0.0076	0.0076	0.0081	0.0080
0.4	1.4952	1.4929	1.4921	1.4918	1.4918	1.4927	1.4933	1.4942	1.4953
	0.0078	0.0080	0.0079	0.0081	0.0075	0.0079	0.0078	0.0078	0.0079
0.8	1.4880	1.4885	1.4895	1.4901	1.4913	1.4922	1.4930	1.4940	1.4955
	0.0084	0.0081	0.0083	0.0087	0.0082	0.0083	0.0080	0.0079	0.0080

ARFIMA(p, d, 0.8)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5034	1.4991	1.4968	1.4950	1.4941	1.4941	1.4943	1.4944	1.4951
	0.0077	0.0075	0.0074	0.0078	0.0079	0.0077	0.0077	0.0082	0.0076
0.4	1.4922	1.4915	1.4907	1.4910	1.4913	1.4921	1.4935	1.4942	1.4949
	0.0077	0.0078	0.0082	0.0083	0.0082	0.0079	0.0077	0.0079	0.0079
0.8	1.4872	1.4881	1.4892	1.4904	1.4913	1.4923	1.4931	1.4946	1.4952
	0.0088	0.0087	0.0086	0.0082	0.0082	0.0081	0.0080	0.0078	0.0078

Table A.8: Means and standard errors of fractal dimensions for RW with ARFIMA, 1000 observations, 5 lags madogram

ARFIMA(p, d, 0)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5891	1.5584	1.5329	1.5133	1.5000	1.4926	1.4896	1.4886	1.4902
	0.0202	0.0165	0.0118	0.0087	0.0075	0.0081	0.0088	0.0093	0.0093
0.4	1.5116	1.4961	1.4864	1.4808	1.4792	1.4805	1.4830	1.4862	1.4889
	0.0087	0.0080	0.0086	0.0096	0.0104	0.0112	0.0111	0.0107	0.0100
0.8	1.4611	1.4610	1.4640	1.4672	1.4722	1.4773	1.4822	1.4855	1.4894
	0.0131	0.0139	0.0142	0.0140	0.0134	0.0126	0.0117	0.0110	0.0094

ARFIMA(p, d, 0.4)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5312	1.5120	1.4985	1.4893	1.4848	1.4833	1.4847	1.4871	1.4896
	0.0112	0.0090	0.0075	0.0081	0.0092	0.0100	0.0102	0.0099	0.0094
0.4	1.4802	1.4735	1.4706	1.4714	1.4740	1.4779	1.4811	1.4860	1.4884
	0.0092	0.0101	0.0119	0.0125	0.0123	0.0121	0.0116	0.0110	0.0101
0.8	1.4514	1.4550	1.4598	1.4665	1.4718	1.4769	1.4813	1.4857	1.4888
	0.0152	0.0153	0.0149	0.0146	0.0133	0.0123	0.0117	0.0106	0.0097

ARFIMA(p, d, 0.8)

p/d	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
0	1.5124	1.4973	1.4879	1.4826	1.4807	1.4819	1.4835	1.4861	1.4890
	0.0089	0.0082	0.0084	0.0096	0.0108	0.0105	0.0110	0.0100	0.0093
0.4	1.4712	1.4674	1.4673	1.4695	1.4723	1.4766	1.4809	1.4849	1.4886
	0.0113	0.0111	0.0125	0.0128	0.0129	0.0121	0.0121	0.0108	0.0099
0.8	1.4490	1.4540	1.4590	1.4658	1.4720	1.4770	1.4808	1.4849	1.4898
	0.0162	0.0159	0.0151	0.0144	0.0136	0.0121	0.0120	0.0108	0.0092

Table A.9: Means and standard errors of fractal dimensions for RW with changing number of ARFIMA observations, 5 lags madogram

ARFIMA(0, d, 0)

d/t	100	200	300	400	500	750	1000	1500	2000
-0.4	1.5064	1.5149	1.5242	1.5334	1.5442	1.5698	1.5886	1.6258	1.6533
	0.0077	0.0090	0.0105	0.0124	0.0148	0.0179	0.0207	0.0220	0.0219
-0.2	1.5024	1.5059	1.5094	1.5131	1.5156	1.5256	1.5322	1.5476	1.5611
	0.0074	0.0076	0.0078	0.0088	0.0096	0.0109	0.0113	0.0137	0.0149
0	1.4999	1.5006	1.5006	1.5000	1.5007	1.5003	1.5004	1.5005	1.5000
	0.0072	0.0074	0.0074	0.0076	0.0076	0.0074	0.0070	0.0075	0.0077
0.2	1.4988	1.4972	1.4966	1.4954	1.4942	1.4918	1.4890	1.4845	1.4803
	0.0075	0.0072	0.0077	0.0075	0.0078	0.0082	0.0089	0.0099	0.0110
0.4	1.4986	1.4967	1.4961	1.4951	1.4942	1.4915	1.4898	1.4862	1.4830
	0.0073	0.0074	0.0075	0.0080	0.0081	0.0083	0.0091	0.0108	0.0114

ARFIMA(0.4, d, 0.4)

d/t	100	200	300	400	500	750	1000	1500	2000
-0.4	1.4990	1.4969	1.4949	1.4929	1.4910	1.4847	1.4804	1.4705	1.4622
	0.0078	0.0073	0.0077	0.0080	0.0085	0.0090	0.0098	0.0103	0.0102
-0.2	1.4971	1.4949	1.4921	1.4892	1.4859	1.4770	1.4704	1.4561	1.4417
	0.0074	0.0075	0.0079	0.0087	0.0088	0.0104	0.0118	0.0139	0.0153
0	1.4973	1.4946	1.4921	1.4895	1.4867	1.4795	1.4733	1.4601	1.4467
	0.0075	0.0080	0.0081	0.0087	0.0094	0.0110	0.0124	0.0154	0.0187
0.2	1.4971	1.4951	1.4939	1.4913	1.4897	1.4852	1.4810	1.4726	1.4656
	0.0073	0.0076	0.0084	0.0082	0.0088	0.0104	0.0113	0.0142	0.0177
0.4	1.4981	1.4961	1.4949	1.4936	1.4932	1.4904	1.4888	1.4847	1.4809
	0.0076	0.0072	0.0080	0.0083	0.0081	0.0091	0.0097	0.0118	0.0125

ARFIMA(0.8, d, 0.8)

d/t	100	200	300	400	500	750	1000	1500	2000
-0.4	1.4965	1.4922	1.4870	1.4824	1.4767	1.4610	1.4498	1.4222	1.3945
	0.0074	0.0081	0.0090	0.0096	0.0103	0.0138	0.0158	0.0204	0.0242
-0.2	1.4967	1.4927	1.4890	1.4853	1.4807	1.4691	1.4596	1.4377	1.4152
	0.0076	0.0076	0.0086	0.0095	0.0103	0.0123	0.0154	0.0203	0.0242
0	1.4970	1.4943	1.4915	1.4886	1.4854	1.4777	1.4712	1.4580	1.4415
	0.0074	0.0075	0.0083	0.0089	0.0094	0.0116	0.0143	0.0177	0.0216
0.2	1.4972	1.4955	1.4933	1.4916	1.4900	1.4852	1.4810	1.4747	1.4661
	0.0073	0.0079	0.0078	0.0088	0.0086	0.0109	0.0119	0.0139	0.0169
0.4	1.4973	1.4964	1.4952	1.4942	1.4933	1.4905	1.4898	1.4852	1.4811
	0.0074	0.0075	0.0075	0.0079	0.0084	0.0099	0.0098	0.0118	0.0134

Table A.10: Means and standard errors of fractal dimensions for RW with ARFIMA on different places, madogram

ARFIMA(0, d, 0), 5 lags, t=300

location/d	-0.4	-0.2	0	0.2	0.4
2500	1.5238	1.5093	1.5000	1.4967	1.4960
	0.0100	0.0082	0.0073	0.0075	0.0076
5000	1.5246	1.5094	1.5003	1.4963	1.4955
	0.0105	0.0081	0.0074	0.0077	0.0075
7500	1.5242	1.5091	1.4996	1.4962	1.4960
	0.0104	0.0085	0.0075	0.0074	0.0076
10000	1.5247	1.5092	1.5006	1.4961	1.4960
	0.0111	0.0082	0.0074	0.0074	0.0078

ARFIMA(0.4, d, 0.4), 20 lags, t=500

location/d	-0.4	-0.2	0	0.2	0.4
2500	1.5120	1.4993	1.4928	1.4899	1.4916
	0.0103	0.0102	0.0103	0.0105	0.0109
5000	1.5130	1.4994	1.4926	1.4907	1.4909
	0.0107	0.0101	0.0104	0.0104	0.0107
7500	1.5123	1.4999	1.4923	1.4903	1.4911
	0.0108	0.0096	0.0101	0.0111	0.0102
10000	1.5124	1.4994	1.4931	1.4909	1.4917
	0.0109	0.0097	0.0103	0.0111	0.0109

ARFIMA(0.8, d, 0.8), 100 lags, t=1000

location/d	-0.4	-0.2	0	0.2	0.4
2500	1.5173	1.4981	1.4841	1.4788	1.4785
	0.0181	0.0182	0.0189	0.0204	0.0207
5000	1.5184	1.4977	1.4848	1.4779	1.4789
	0.0180	0.0181	0.0196	0.0199	0.0197
7500	1.5184	1.4979	1.4855	1.4790	1.4785
	0.0189	0.0188	0.0191	0.0197	0.0201
10000	1.5190	1.4988	1.4863	1.4791	1.4778
	0.0185	0.0190	0.0191	0.0190	0.0207
2x500					
2500, 7500	1.5149	1.4983	1.4845	1.4776	1.4745
	0.0183	0.0183	0.0194	0.0193	0.0195

Table A.11: Means and standard errors of fractal dimensions for RW with ARFIMA estimated for individual parts of the time series, only parts including the ARFIMA series and one efficient part are presented, madogram

ARFIMA(0.4, d, 0.4), 5 lags, t=1000 in (2001:3000)

1 part

part/d	-0.4	-0.2	0	0.2	0.4
1	1.4800	1.4705	1.4733	1.4815	1.4883
	0.0093	0.0114	0.0122	0.0115	0.0099

2 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.4661	1.4459	1.4474	1.4615	1.4750
	0.0127	0.0170	0.0205	0.0202	0.0174
2	1.4999	1.4999	1.4998	1.5001	1.4999
	0.0105	0.0101	0.0103	0.0104	0.0103

4 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.4677	1.4469	1.4478	1.4618	1.4741
	0.0174	0.0199	0.0231	0.0231	0.0209
2	1.4659	1.4459	1.4473	1.4611	1.4756
	0.0170	0.0196	0.0224	0.0228	0.0209
3	1.4998	1.5002	1.5001	1.5001	1.5005
	0.0147	0.0142	0.0147	0.0151	0.0145

5 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.4988	1.5000	1.5006	1.4997	1.5005
	0.0162	0.0159	0.0166	0.0163	0.0164
2	1.4412	1.3894	1.3701	1.3855	1.4205
	0.0167	0.0200	0.0296	0.0380	0.0376

10 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.5015	1.5003	1.5002	1.5008	1.5001
	0.0231	0.0230	0.0230	0.0236	0.0236
3	1.4206	1.3230	1.2264	1.1320	1.0466
	0.0204	0.0188	0.0165	0.0164	0.0196

ARFIMA(0.8, d, 0.8), 20 lags, t=1000 in (2001:3000)

1 part

part/d	-0.4	-0.2	0	0.2	0.4
1	1.4770	1.4704	1.4725	1.4776	1.4846
	0.0124	0.0141	0.0143	0.0138	0.0130

2 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.4574	1.4436	1.4449	1.4539	1.4669
	0.0179	0.0211	0.0229	0.0229	0.0218
2	1.5007	1.4997	1.5004	1.4999	1.5007
	0.0135	0.0140	0.0142	0.0139	0.0143

4 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.4592	1.4442	1.4458	1.4538	1.4670
	0.0223	0.0254	0.0275	0.0277	0.0273
2	1.4585	1.4450	1.4452	1.4542	1.4663
	0.0230	0.0244	0.0255	0.0256	0.0250
3	1.5011	1.4997	1.5001	1.5004	1.5007
	0.0189	0.0199	0.0203	0.0191	0.0197

5 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.5010	1.4998	1.5015	1.5029	1.5005
	0.0223	0.0215	0.0226	0.0219	0.0221
2	1.4131	1.3707	1.3615	1.3707	1.3961
	0.0228	0.0280	0.0345	0.0419	0.0415

10 parts

part/d	-0.4	-0.2	0	0.2	0.4
1	1.5015	1.5009	1.5012	1.5012	1.5010
	0.0311	0.0319	0.0318	0.0319	0.0325
3	1.3636	1.2644	1.1764	1.0982	1.0347
	0.0280	0.0262	0.0238	0.0218	0.0201

Table A.12: Means and standard errors of fractal dimensions for graduate increase in number of observation complex time series, 2nd setting, 5 lags madogram

Observations		5 lags
(1:10000)	All	1.4717
		0.0111
(1:1000)	RW	1.5693
	(0, -0.15, 0)	0.0252
(1000:2000)	RW	1.5002
		0.0240
(2000:3000)	(0.2, -0.1, 0.2)	1.4007
	(0.1, 0, 0.1)	0.0207
(3000:4000)	(0.1, 0, 0.1)	1.4492
	RW	0.0240
(4000:5000)	RW	1.5005
		0.0234
(5000:6000)	RW	1.5250
	(0, -0.05, 0)	0.0246
(6000:7000)	RW	1.4995
		0.0237
(7000:8000)	(0.1, 0.05, 0.1)	1.3077
	(0.2, 0.15, 0.3)	0.0282
(8000:9000)	(0.1, 0, 0)	1.4508
		0.0226
(9000:10000)	RW	1.4995
		0.0230

Table A.13: Means and standard errors of fractal dimensions for RW with ARFIMA, 500 observations, 5 and 100 lags madogram, 1000 simulations

ARFIMA(p, d, 0)			5 lags					100 lags		
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5442	1.5156	1.5007	1.4942	1.4942	1.5320	1.5154	1.5015	1.4926	1.4888
	0.0148	0.0096	0.0076	0.0078	0.0081	0.0189	0.0189	0.0180	0.0183	0.0188
0.4	1.5056	1.4930	1.4898	1.4907	1.4934	1.5240	1.5095	1.4984	1.4920	1.4877
	0.0084	0.0079	0.0088	0.0087	0.0084	0.0185	0.0187	0.0181	0.0188	0.0186
0.8	1.4829	1.4831	1.4861	1.4901	1.4937	1.5091	1.5000	1.4937	1.4888	1.4876
	0.0094	0.0102	0.0093	0.0083	0.0082	0.0177	0.0172	0.0186	0.0188	0.0185

ARFIMA(p, d, 0.4)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5149	1.4991	1.4925	1.4919	1.4935	1.5278	1.5126	1.4994	1.4918	1.4897
	0.0093	0.0077	0.0080	0.0082	0.0081	0.0190	0.0189	0.0187	0.0189	0.0196
0.4	1.4910	1.4859	1.4867	1.4897	1.4932	1.5202	1.5086	1.4987	1.4911	1.4881
	0.0085	0.0088	0.0094	0.0088	0.0081	0.0183	0.0183	0.0180	0.0180	0.0189
0.8	1.4782	1.4813	1.4859	1.4896	1.4931	1.5081	1.4992	1.4937	1.4896	1.4882
	0.0104	0.0106	0.0095	0.0089	0.0080	0.0177	0.0183	0.0188	0.0187	0.0195

ARFIMA(p, d, 0.8)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5052	1.4940	1.4897	1.4908	1.4936	1.5259	1.5116	1.4990	1.4921	1.4881
	0.0083	0.0077	0.0084	0.0085	0.0083	0.0188	0.0185	0.0180	0.0176	0.0178
0.4	1.4868	1.4841	1.4861	1.4895	1.4928	1.5213	1.5075	1.4985	1.4908	1.4882
	0.0087	0.0089	0.0093	0.0087	0.0082	0.0183	0.0179	0.0181	0.0190	0.0195
0.8	1.4767	1.4807	1.4854	1.4900	1.4933	1.5085	1.4984	1.4926	1.4893	1.4869
	0.0103	0.0103	0.0094	0.0086	0.0084	0.0181	0.0184	0.0191	0.0193	0.0200

Table A.14: Means and standard errors of fractal dimensions for BM with ARFIMA, 500 observations, 5 and 100 lags madogram, 1000 simulations

ARFIMA(p, d, 0)	5 lags					100 lags				
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5530	1.5199	1.5002	1.4928	1.4925	1.5382	1.5184	1.5017	1.4903	1.4841
	0.0157	0.0101	0.0076	0.0081	0.0088	0.0201	0.0186	0.0182	0.0194	0.0190
0.4	1.5075	1.4917	1.4871	1.4876	1.4918	1.5303	1.5125	1.4987	1.4908	1.4852
	0.0084	0.0085	0.0095	0.0093	0.0088	0.0187	0.0183	0.0195	0.0193	0.0193
0.8	1.4779	1.4775	1.4827	1.4871	1.4912	1.5126	1.4992	1.4914	1.4859	1.4834
	0.0108	0.0116	0.0103	0.0098	0.0089	0.0185	0.0190	0.0191	0.0196	0.0195

ARFIMA(p, d, 0.4)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5181	1.4990	1.4903	1.4890	1.4917	1.5326	1.5151	1.4995	1.4909	1.4857
	0.0098	0.0077	0.0087	0.0089	0.0085	0.0199	0.0187	0.0184	0.0194	0.0202
0.4	1.4885	1.4827	1.4832	1.4873	1.4910	1.5270	1.5097	1.4975	1.4891	1.4858
	0.0087	0.0095	0.0098	0.0098	0.0093	0.0186	0.0196	0.0193	0.0193	0.0191
0.8	1.4721	1.4763	1.4814	1.4875	1.4914	1.5111	1.4989	1.4912	1.4866	1.4836
	0.0115	0.0115	0.0106	0.0096	0.0090	0.0187	0.0192	0.0195	0.0190	0.0196

ARFIMA(p, d, 0.8)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5068	1.4926	1.4883	1.4890	1.4914	1.5327	1.5148	1.4993	1.4898	1.4858
	0.0090	0.0081	0.0091	0.0092	0.0092	0.0197	0.0182	0.0195	0.0193	0.0192
0.4	1.4831	1.4800	1.4826	1.4870	1.4906	1.5256	1.5099	1.4970	1.4893	1.4838
	0.0097	0.0103	0.0101	0.0098	0.0093	0.0190	0.0185	0.0194	0.0194	0.0195
0.8	1.4708	1.4759	1.4819	1.4865	1.4916	1.5112	1.4993	1.4907	1.4846	1.4836
	0.0118	0.0117	0.0104	0.0099	0.0088	0.0183	0.0190	0.0191	0.0187	0.0194

Table A.15: Means and standard errors of fractal dimensions for M with ARFIMA, 500 observations, 5 and 100 lags madogram, 1000 simulations

ARFIMA(p, d, 0)	5 lags					100 lags				
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.6277	1.6086	1.5993	1.5960	1.5978	1.5390	1.5248	1.5137	1.5049	1.5002
	0.0106	0.0079	0.0076	0.0090	0.0086	0.0189	0.0194	0.0186	0.0192	0.0186
0.4	1.5979	1.5910	1.5906	1.5934	1.5972	1.5333	1.5205	1.5095	1.5025	1.5008
	0.0083	0.0095	0.0099	0.0094	0.0088	0.0200	0.0190	0.0191	0.0198	0.0193
0.8	1.5821	1.5841	1.5889	1.5936	1.5975	1.5190	1.5113	1.5038	1.5013	1.4984
	0.0109	0.0113	0.0103	0.0094	0.0088	0.0189	0.0189	0.0186	0.0183	0.0201

ARFIMA(p, d, 0.4)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.6056	1.5949	1.5928	1.5938	1.5969	1.5366	1.5230	1.5110	1.5037	1.4998
	0.0082	0.0087	0.0089	0.0095	0.0088	0.0194	0.0181	0.0194	0.0203	0.0186
0.4	1.5860	1.5859	1.5888	1.5929	1.5964	1.5321	1.5194	1.5082	1.5029	1.4999
	0.0100	0.0106	0.0098	0.0097	0.0088	0.0194	0.0194	0.0188	0.0194	0.0197
0.8	1.5784	1.5834	1.5889	1.5938	1.5975	1.5199	1.5111	1.5043	1.5008	1.5001
	0.0116	0.0112	0.0104	0.0093	0.0088	0.0189	0.0191	0.0186	0.0202	0.0197

ARFIMA(p, d, 0.8)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5981	1.5913	1.5908	1.5940	1.5971	1.5355	1.5218	1.5111	1.5037	1.4997
	0.0086	0.0087	0.0095	0.0090	0.0088	0.0194	0.0191	0.0201	0.0193	0.0199
0.4	1.5832	1.5835	1.5877	1.5926	1.5971	1.5304	1.5187	1.5092	1.5022	1.4999
	0.0106	0.0105	0.0103	0.0096	0.0086	0.0187	0.0193	0.0198	0.0192	0.0197
0.8	1.5779	1.5832	1.5892	1.5938	1.5978	1.5193	1.5100	1.5050	1.4996	1.4998
	0.0119	0.0109	0.0100	0.0094	0.0086	0.0188	0.0196	0.0198	0.0200	0.0196

Table A.16: Means and standard errors of fractal dimensions for RW with ARFIMA, 500 observations, 5 and 20 lags Hall-Wood, 500 simulations

ARFIMA(p, d, 0)	5 lags					20 lags				
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5438	1.5162	1.5006	1.4944	1.4949	1.5414	1.5164	1.5010	1.4932	1.4933
	0.0156	0.0100	0.0089	0.0092	0.0090	0.0146	0.0111	0.0104	0.0112	0.0112
0.4	1.5055	1.4944	1.4906	1.4910	1.4936	1.5207	1.5031	1.4945	1.4918	1.4917
	0.0099	0.0089	0.0092	0.0102	0.0099	0.0123	0.0106	0.0105	0.0113	0.0112
0.8	1.4825	1.4829	1.4857	1.4905	1.4937	1.4925	1.4877	1.4866	1.4885	1.4907
	0.0107	0.0109	0.0106	0.0096	0.0095	0.0108	0.0116	0.0117	0.0116	0.0113

ARFIMA(p, d, 0.4)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5155	1.4994	1.4927	1.4917	1.4939	1.5277	1.5088	1.4969	1.4918	1.4921
	0.0103	0.0093	0.0092	0.0096	0.0094	0.0129	0.0106	0.0099	0.0105	0.0112
0.4	1.4912	1.4860	1.4877	1.4901	1.4930	1.5132	1.4999	1.4932	1.4897	1.4913
	0.0102	0.0097	0.0100	0.0100	0.0097	0.0116	0.0110	0.0109	0.0110	0.0109
0.8	1.4778	1.4813	1.4861	1.4903	1.4941	1.4897	1.4868	1.4864	1.4892	1.4919
	0.0106	0.0111	0.0109	0.0102	0.0095	0.0112	0.0119	0.0119	0.0115	0.0113

ARFIMA(p, d, 0.8)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5063	1.4945	1.4906	1.4914	1.4933	1.5241	1.5071	1.4949	1.4921	1.4910
	0.0101	0.0097	0.0096	0.0099	0.0094	0.0121	0.0111	0.0103	0.0105	0.0110
0.4	1.4867	1.4839	1.4867	1.4896	1.4931	1.5115	1.4984	1.4921	1.4906	1.4914
	0.0102	0.0108	0.0104	0.0101	0.0100	0.0109	0.0095	0.0112	0.0112	0.0108
0.8	1.4767	1.4809	1.4862	1.4901	1.4948	1.4896	1.4858	1.4865	1.4886	1.4923
	0.0117	0.0114	0.0105	0.0101	0.0091	0.0109	0.0116	0.0113	0.0117	0.0114

Table A.17: Means and standard errors of fractal dimensions for BM with ARFIMA, 500 observations, 5 and 20 lags Hall-Wood, 500 simulations

ARFIMA(p, d, 0)	5 lags					20 lags				
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5526	1.5206	1.5001	1.4930	1.4932	1.5490	1.5206	1.4999	1.4925	1.4903
	0.0173	0.0127	0.0091	0.0101	0.0099	0.0162	0.0134	0.0108	0.0117	0.0116
0.4	1.5071	1.4915	1.4864	1.4883	1.4915	1.5243	1.5045	1.4922	1.4894	1.4888
	0.0104	0.0096	0.0103	0.0109	0.0100	0.0131	0.0111	0.0112	0.0119	0.0115
0.8	1.4777	1.4787	1.4820	1.4871	1.4916	1.4897	1.4840	1.4825	1.4854	1.4877
	0.0115	0.0113	0.0121	0.0106	0.0098	0.0115	0.0121	0.0126	0.0123	0.0125

ARFIMA(p, d, 0.4)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5177	1.4988	1.4907	1.4899	1.4921	1.5339	1.5106	1.4952	1.4896	1.4898
	0.0110	0.0091	0.0102	0.0096	0.0103	0.0133	0.0113	0.0110	0.0111	0.0122
0.4	1.4890	1.4822	1.4830	1.4870	1.4912	1.5155	1.5002	1.4907	1.4875	1.4884
	0.0102	0.0109	0.0111	0.0109	0.0102	0.0113	0.0111	0.0108	0.0111	0.0121
0.8	1.4733	1.4765	1.4824	1.4877	1.4914	1.4887	1.4830	1.4834	1.4854	1.4887
	0.0126	0.0126	0.0115	0.0114	0.0108	0.0115	0.0122	0.0122	0.0135	0.0134

ARFIMA(p, d, 0.8)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5075	1.4938	1.4879	1.4891	1.4926	1.5289	1.5082	1.4951	1.4897	1.4891
	0.0106	0.0096	0.0108	0.0108	0.0099	0.0128	0.0110	0.0110	0.0115	0.0118
0.4	1.4843	1.4803	1.4824	1.4874	1.4916	1.5136	1.4989	1.4907	1.4873	1.4901
	0.0107	0.0116	0.0113	0.0104	0.0105	0.0118	0.0113	0.0118	0.0115	0.0107
0.8	1.4716	1.4765	1.4822	1.4871	1.4919	1.4878	1.4831	1.4824	1.4843	1.4889
	0.0130	0.0130	0.0113	0.0113	0.0101	0.0115	0.0124	0.0125	0.0131	0.0122

Table A.18: Means and standard errors of fractal dimensions for M with ARFIMA, 500 observations, 5 and 20 lags Hall-Wood, 500 simulations

ARFIMA(p, d, 0)	5 lags					20 lags				
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.6278	1.6097	1.5992	1.5967	1.5989	1.5622	1.5435	1.5289	1.5238	1.5225
	0.0123	0.0099	0.0095	0.0097	0.0104	0.0147	0.0122	0.0110	0.0114	0.0127
0.4	1.5984	1.5911	1.5917	1.5950	1.5981	1.5465	1.5317	1.5239	1.5214	1.5217
	0.0096	0.0108	0.0111	0.0108	0.0103	0.0119	0.0107	0.0113	0.0115	0.0120
0.8	1.5835	1.5855	1.5900	1.5941	1.5974	1.5230	1.5183	1.5169	1.5195	1.5215
	0.0122	0.0118	0.0118	0.0101	0.0108	0.0125	0.0121	0.0125	0.0123	0.0121

ARFIMA(p, d, 0.4)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.6054	1.5955	1.5938	1.5947	1.5973	1.5528	1.5366	1.5262	1.5225	1.5220
	0.0100	0.0102	0.0100	0.0105	0.0103	0.0121	0.0115	0.0110	0.0113	0.0117
0.4	1.5862	1.5863	1.5883	1.5938	1.5970	1.5404	1.5287	1.5220	1.5211	1.5221
	0.0110	0.0122	0.0113	0.0112	0.0096	0.0115	0.0107	0.0116	0.0116	0.0116
0.8	1.5784	1.5840	1.5899	1.5945	1.5982	1.5204	1.5171	1.5177	1.5193	1.5221
	0.0129	0.0119	0.0108	0.0109	0.0103	0.0117	0.0125	0.0126	0.0123	0.0121

ARFIMA(p, d, 0.8)										
p/d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
0	1.5983	1.5922	1.5920	1.5945	1.5967	1.5501	1.5348	1.5250	1.5224	1.5213
	0.0101	0.0104	0.0112	0.0102	0.0105	0.0124	0.0109	0.0114	0.0112	0.0116
0.4	1.5837	1.5851	1.5888	1.5935	1.5978	1.5392	1.5285	1.5218	1.5207	1.5224
	0.0123	0.0118	0.0112	0.0108	0.0103	0.0119	0.0110	0.0109	0.0124	0.0119
0.8	1.5774	1.5845	1.5895	1.5940	1.5985	1.5195	1.5172	1.5172	1.5194	1.5217
	0.0129	0.0128	0.0116	0.0110	0.0099	0.0122	0.0124	0.0124	0.0124	0.0121

Table A.19: Means and standard errors of fractal dimensions for RW with ARFIMA, 500 observations, “auto” lags box-count, 500 simulations

ARFIMA(p, d, 0)		"auto" lags			
p/d	-0.4	-0.2	0	0.2	0.4
0	1.4544	1.4457	1.4358	1.4250	1.4172
	0.0479	0.0482	0.0456	0.0452	0.0477
0.4	1.4526	1.4422	1.4367	1.4274	1.4208
	0.0480	0.0469	0.0468	0.0464	0.0490
0.8	1.4468	1.4411	1.4353	1.4278	1.4167
	0.0467	0.0470	0.0464	0.0477	0.0460

ARFIMA(p, d, 0.4)					
p/d	-0.4	-0.2	0	0.2	0.4
0	1.4558	1.4421	1.4342	1.4289	1.4185
	0.0465	0.0475	0.0470	0.0495	0.0468
0.4	1.4509	1.4421	1.4331	1.4271	1.4218
	0.0465	0.0485	0.0489	0.0470	0.0507
0.8	1.4468	1.4392	1.4365	1.4283	1.4170
	0.0458	0.0485	0.0473	0.0496	0.0461

ARFIMA(p, d, 0.8)					
p/d	-0.4	-0.2	0	0.2	0.4
0	1.4507	1.4405	1.4362	1.4274	1.4216
	0.0498	0.0478	0.0471	0.0532	0.0503
0.4	1.4558	1.4403	1.4403	1.4294	1.4243
	0.0444	0.0486	0.0490	0.0500	0.0489
0.8	1.4501	1.4374	1.4353	1.4228	1.4173
	0.0474	0.0488	0.0493	0.0467	0.0490

Table A.20: Means and standard errors of fractal dimensions for BM with ARFIMA, 500 observations, “auto” lags box-count, 500 simulations

ARFIMA(p, d, 0)		”auto” lags			
p/d	-0.4	-0.2	0	0.2	0.4
0	1.4588	1.4493	1.4366	1.4254	1.4141
	0.0461	0.0461	0.0517	0.0519	0.0486
0.4	1.4581	1.4452	1.4338	1.4265	1.4173
	0.0464	0.0469	0.0480	0.0489	0.0489
0.8	1.4490	1.4398	1.4284	1.4263	1.4083
	0.0488	0.0443	0.0492	0.0466	0.0492

ARFIMA(p, d, 0.4)					
p/d	-0.4	-0.2	0	0.2	0.4
0	1.4551	1.4454	1.4316	1.4257	1.4109
	0.0466	0.0465	0.0439	0.0482	0.0492
0.4	1.4566	1.4437	1.4325	1.4256	1.4146
	0.0462	0.0463	0.0483	0.0498	0.0480
0.8	1.4479	1.4407	1.4335	1.4195	1.4112
	0.0479	0.0501	0.0452	0.0478	0.0521

ARFIMA(p, d, 0.8)					
p/d	-0.4	-0.2	0	0.2	0.4
0	1.4550	1.4446	1.4327	1.4218	1.4133
	0.0462	0.0456	0.0476	0.0463	0.0493
0.4	1.4544	1.4456	1.4355	1.4260	1.4135
	0.0464	0.0462	0.0467	0.0524	0.0497
0.8	1.4473	1.4415	1.4300	1.4175	1.4162
	0.0478	0.0490	0.0454	0.0495	0.0497

Table A.21: Means and standard errors of fractal dimensions for M with ARFIMA, 500 observations, "auto" lags box-count, 500 simulations

ARFIMA(p, d, 0)			"auto" lags		
p/d	-0.4	-0.2	0	0.2	0.4
0	1.9536	1.9536	1.9659	1.9751	1.9841
	0.0253	0.0252	0.0244	0.0256	0.0232
0.4	1.9331	1.9458	1.9622	1.9772	1.9848
	0.0293	0.0300	0.0262	0.0238	0.0228
0.8	1.9442	1.9591	1.9699	1.9786	1.9862
	0.0289	0.0266	0.0246	0.0236	0.0226

ARFIMA(p, d, 0.4)					
p/d	-0.4	-0.2	0	0.2	0.4
0	1.9221	1.9412	1.9604	1.9748	1.9828
	0.0330	0.0295	0.0255	0.0236	0.0251
0.4	1.9211	1.9419	1.9592	1.9756	1.9837
	0.0305	0.0268	0.0271	0.0258	0.0232
0.8	1.9401	1.9570	1.9701	1.9785	1.9858
	0.0296	0.0267	0.0251	0.0242	0.0223

ARFIMA(p, d, 0.8)					
p/d	-0.4	-0.2	0	0.2	0.4
0	1.9095	1.9355	1.9566	1.9731	1.9821
	0.0333	0.0320	0.0281	0.0250	0.0244
0.4	1.9144	1.9403	1.9604	1.9732	1.9831
	0.0315	0.0293	0.0264	0.0242	0.0237
0.8	1.9409	1.9590	1.9710	1.9809	1.9852
	0.0280	0.0274	0.0251	0.0234	0.0240

Appendix B

Figures

Figure B.1: Box plot of Hall-Wood fractal dimension for α equal to 1.5, 1, and 0.5 implying fractal dimensions 1.75, 1.5, and 1.25

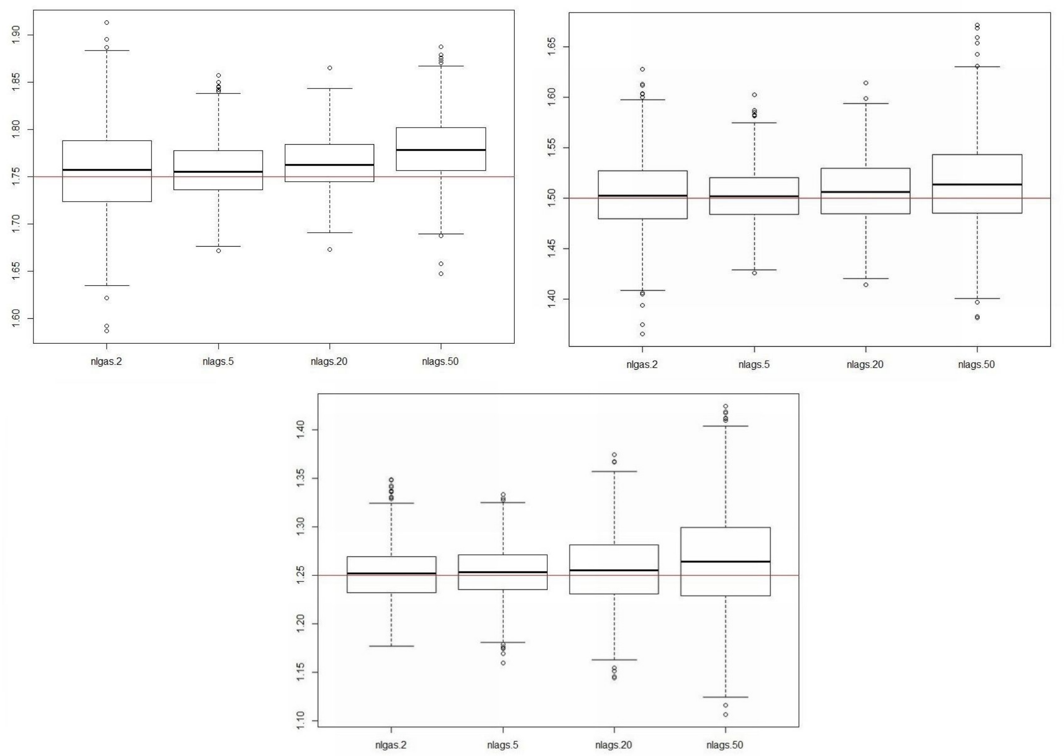


Figure B.2: Box plot of box-count fractal dimension for α equal to 1.5, 1, and 0.5 implying fractal dimension 1.75, 1.5, and 1.25

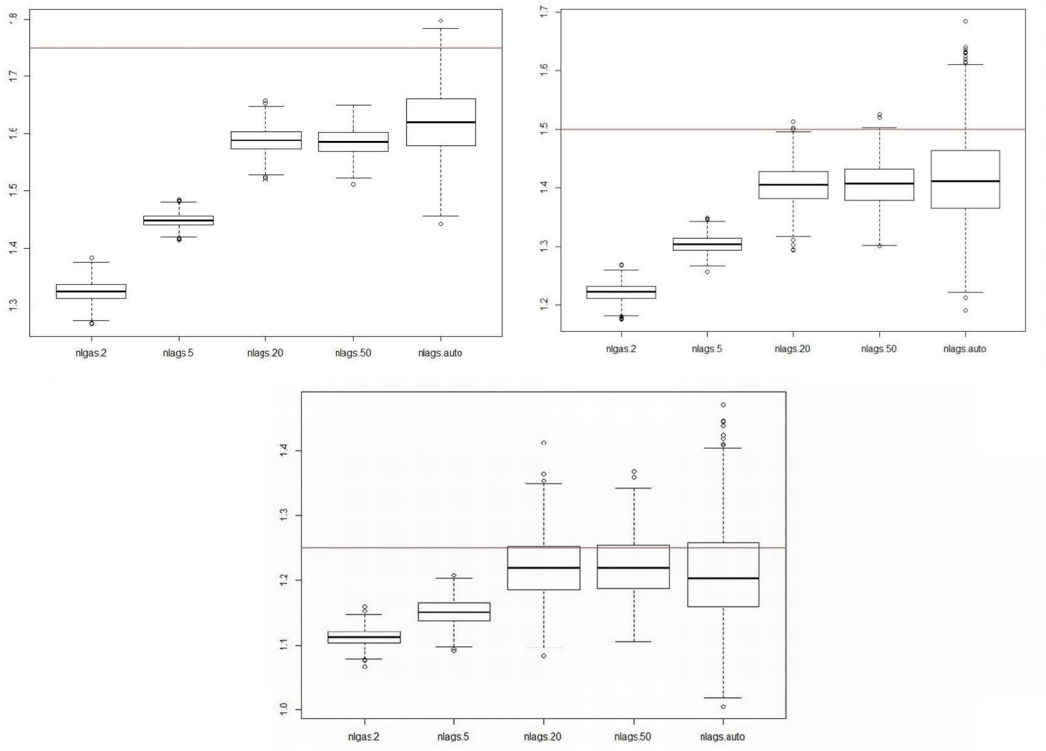


Figure B.3: Illustration of ARFIMA(p, d, 0) series for increasing d on x-axis and p on y-axis

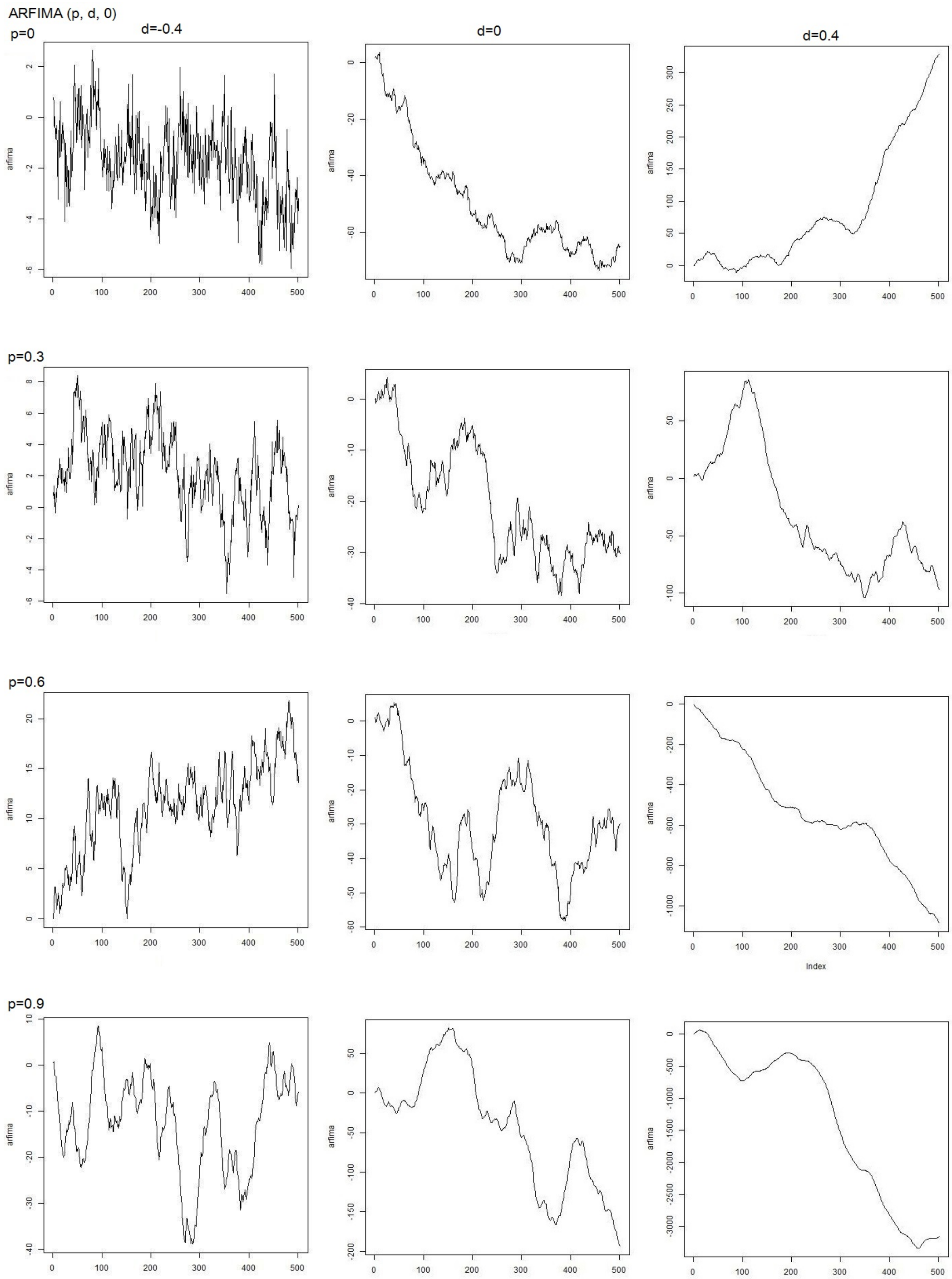


Figure B.4: Illustration of ARFIMA(p, d, 0.9) series for changing d on x-axis and p on y-axis

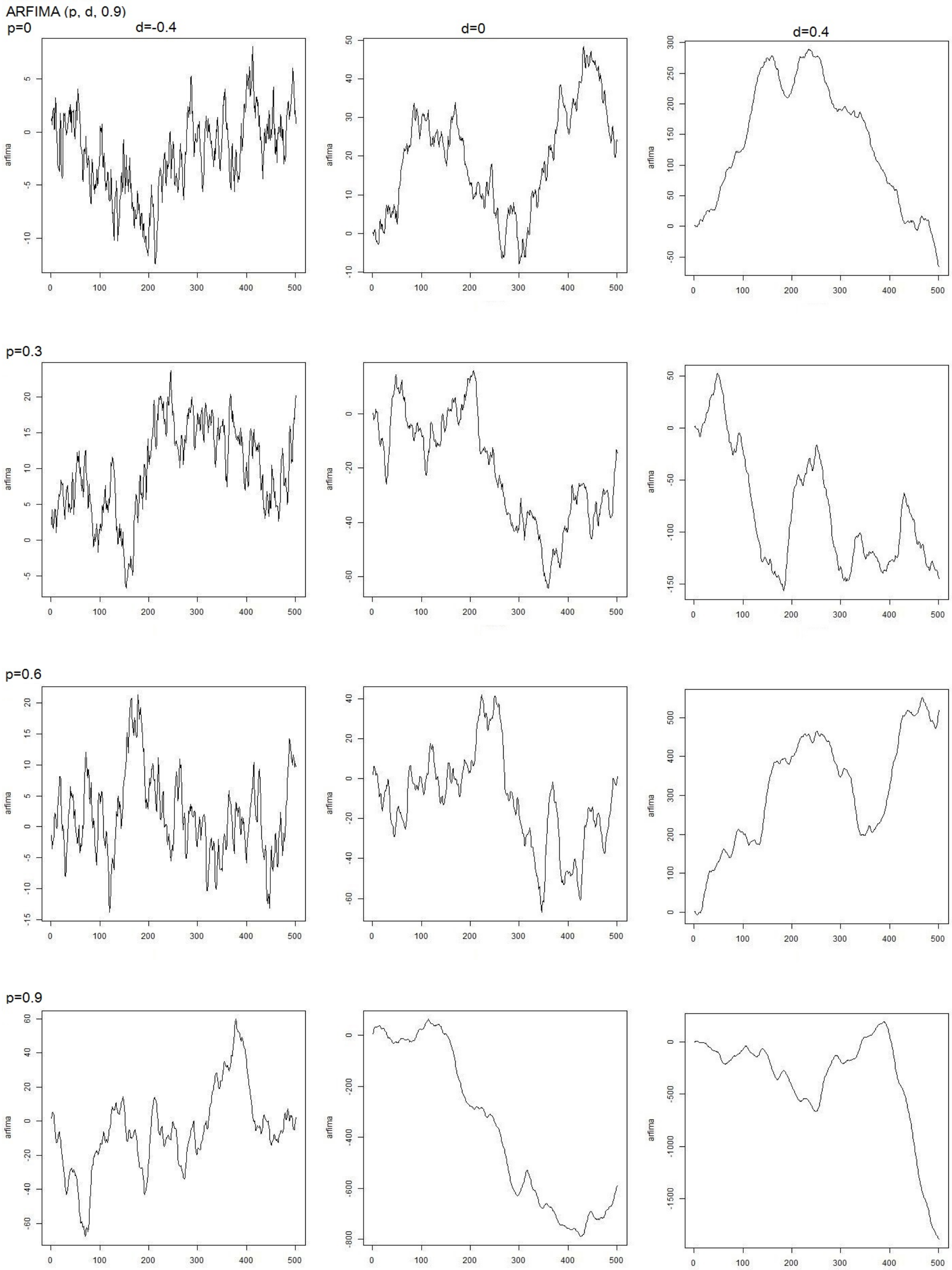


Figure B.5: Fractal dimension and confidence interval of ARFIMA time series for 5 and 100 lags madogram estimator

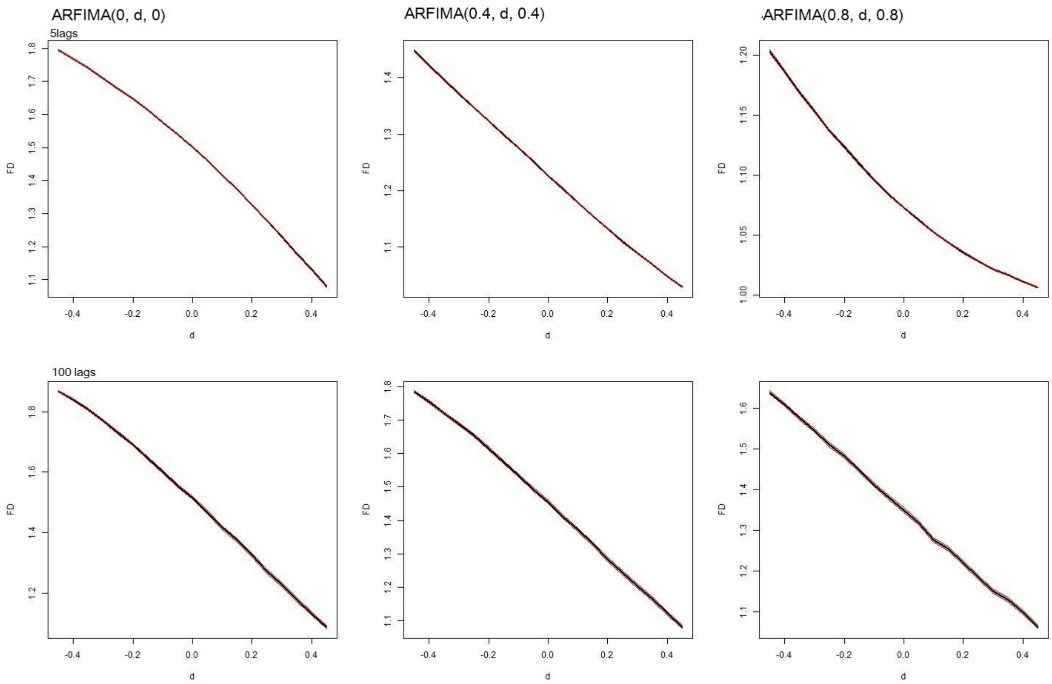


Figure B.6: Fractal dimension and tolerance interval of ARFIMA time series for 5 and 100 lags madogram estimator

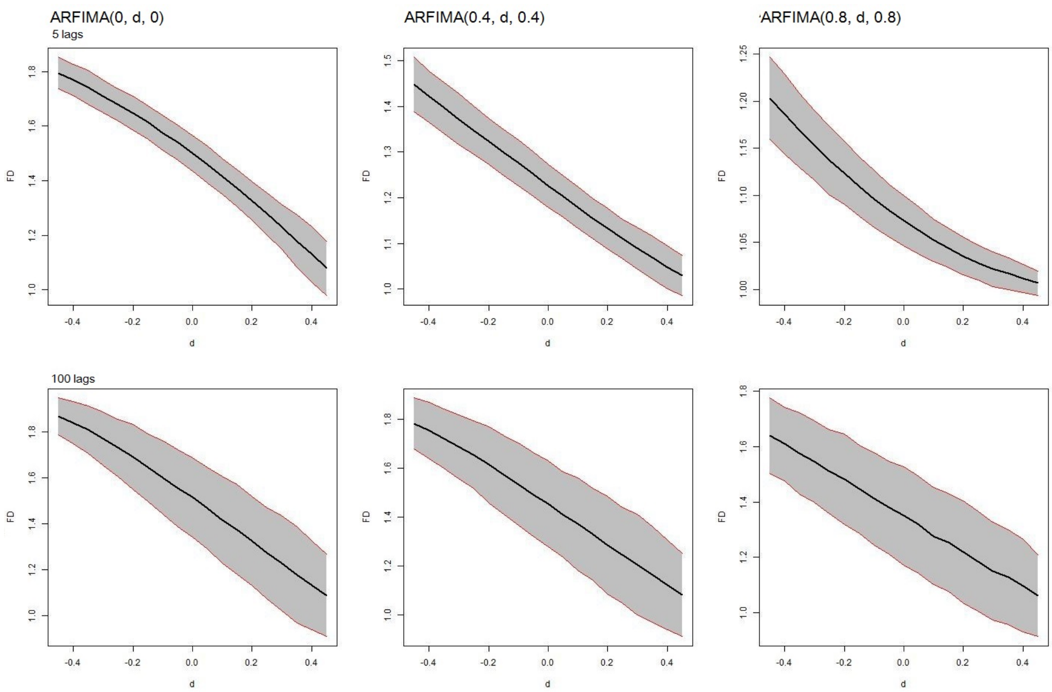


Figure B.7: Inclusion of ARFIMA observations to random walk time series

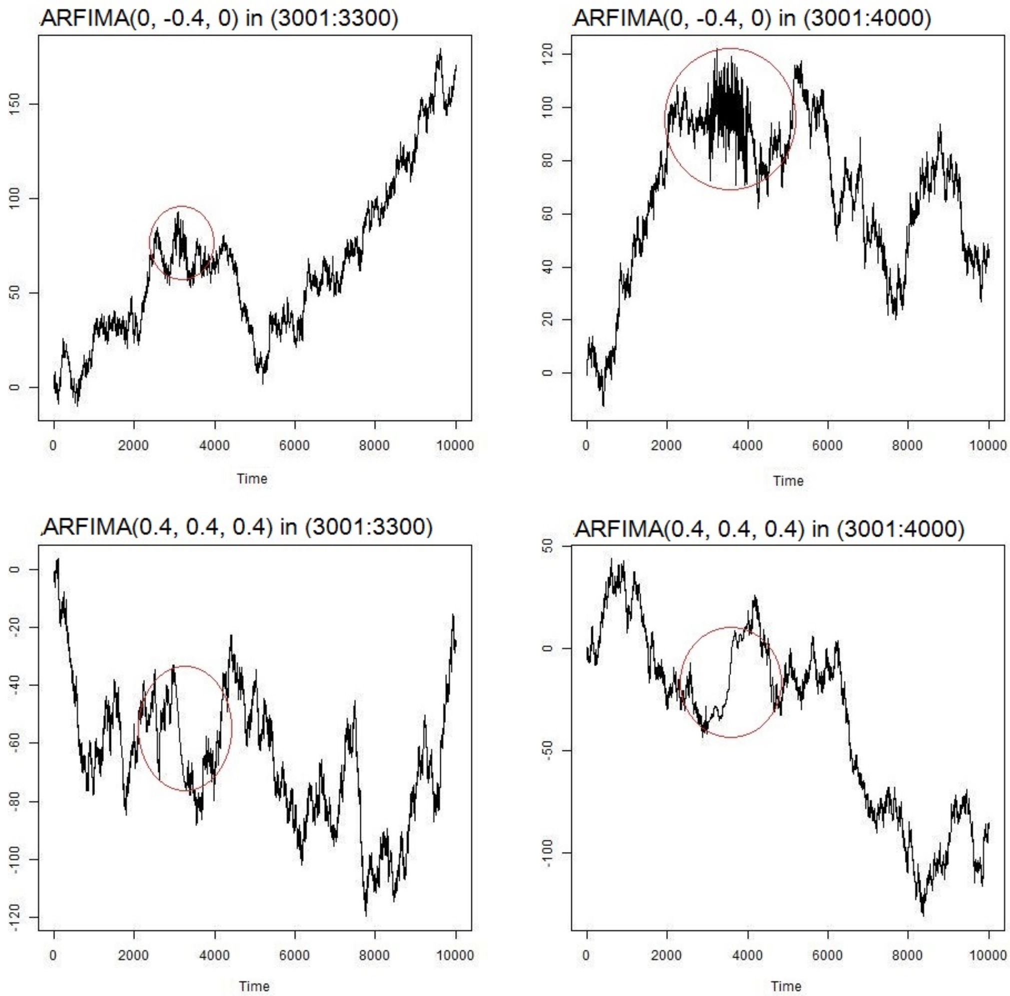


Figure B.8: 3D graph of mean fractal dimensions of RW with 500 observations of ARFIMA, p is fixed, d and q are assigned to x and y axes, madogram

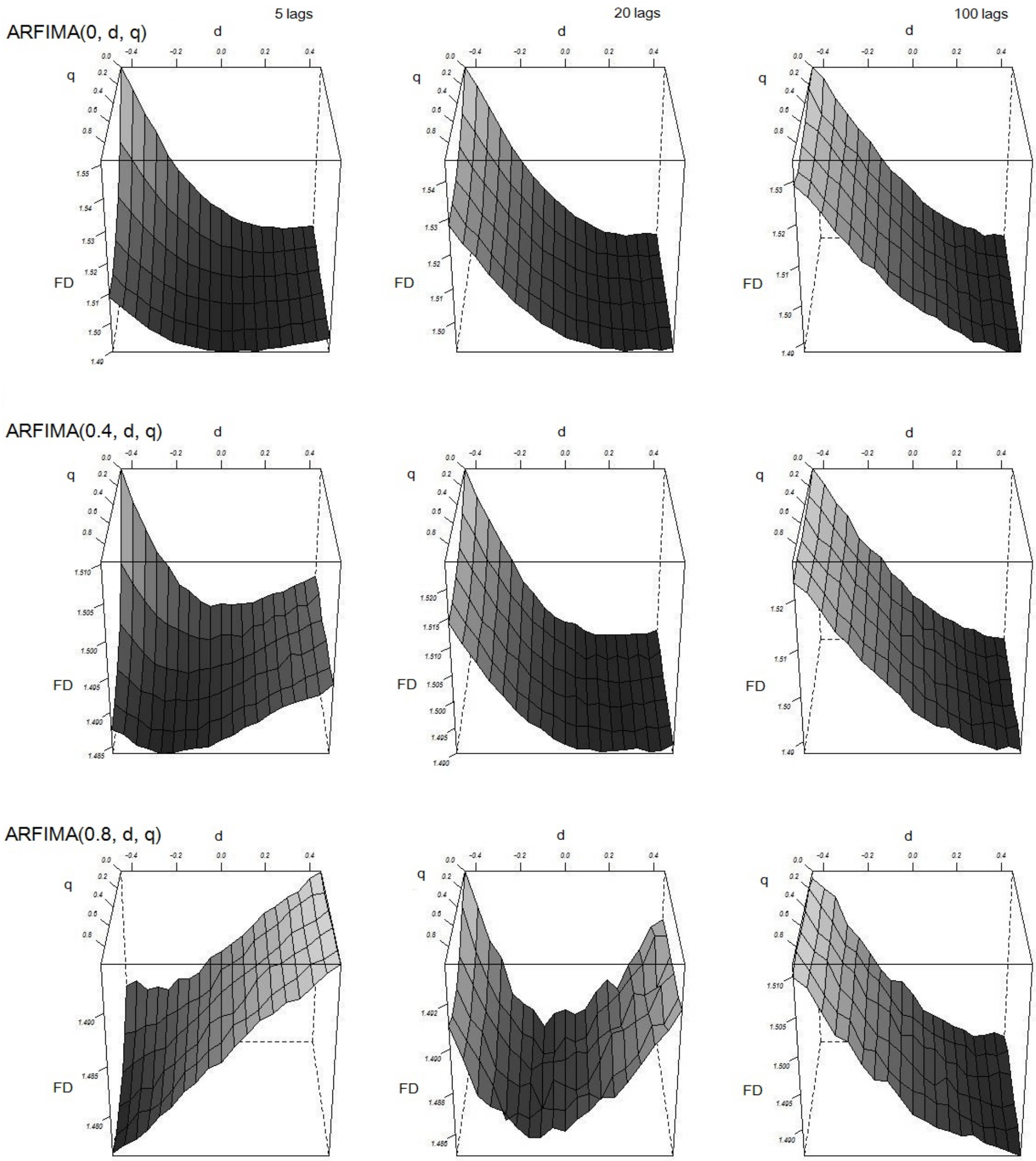


Figure B.9: 3D graph of mean fractal dimensions of RW with 500 observations of ARFIMA, d is fixed, p and q are assigned to x and y axes, madogram

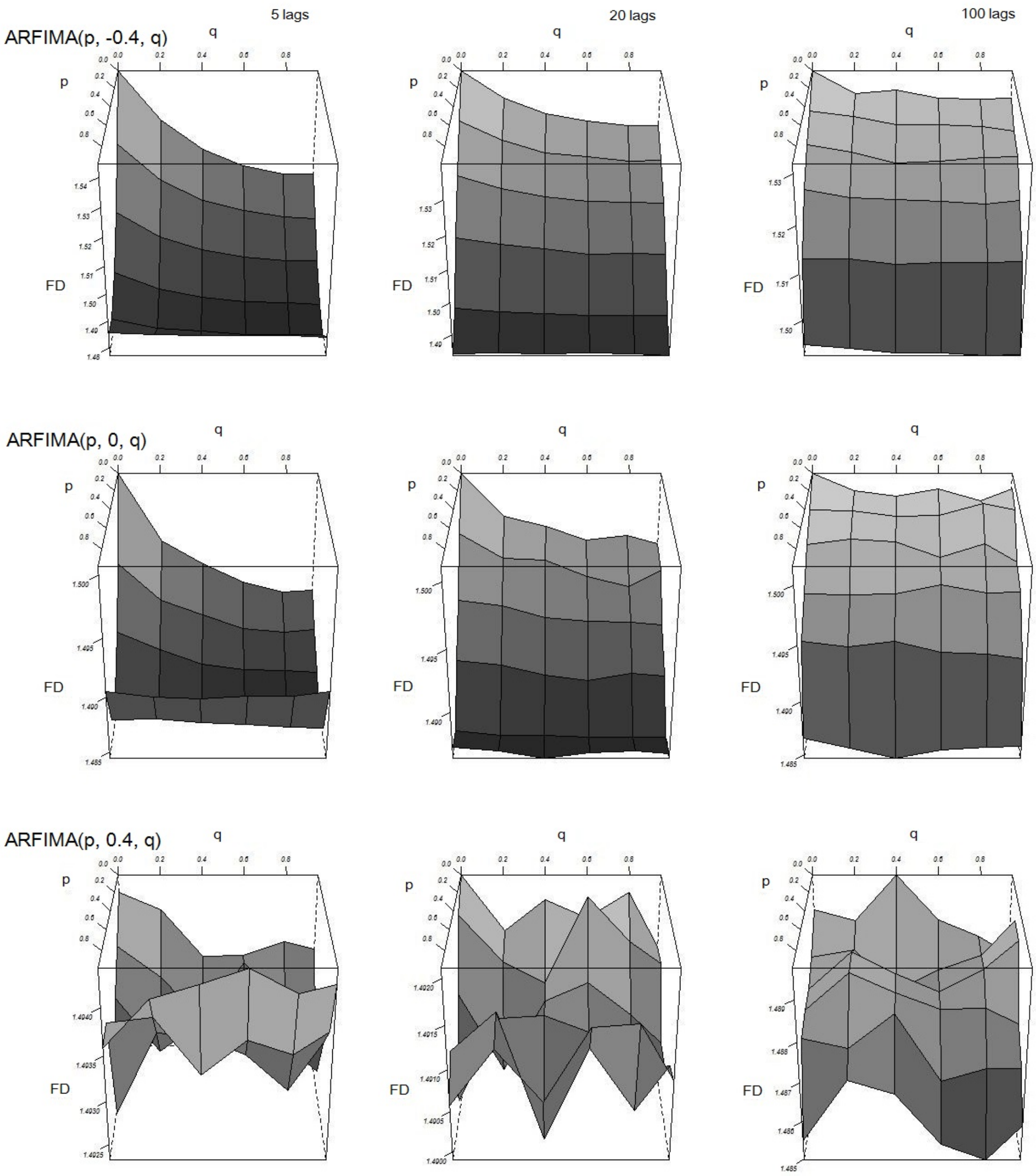


Figure B.10: Mean fractal dimensions and confidence intervals of RW with 500 observations of ARFIMA, q assigned to x-axis, 5 lags madogram

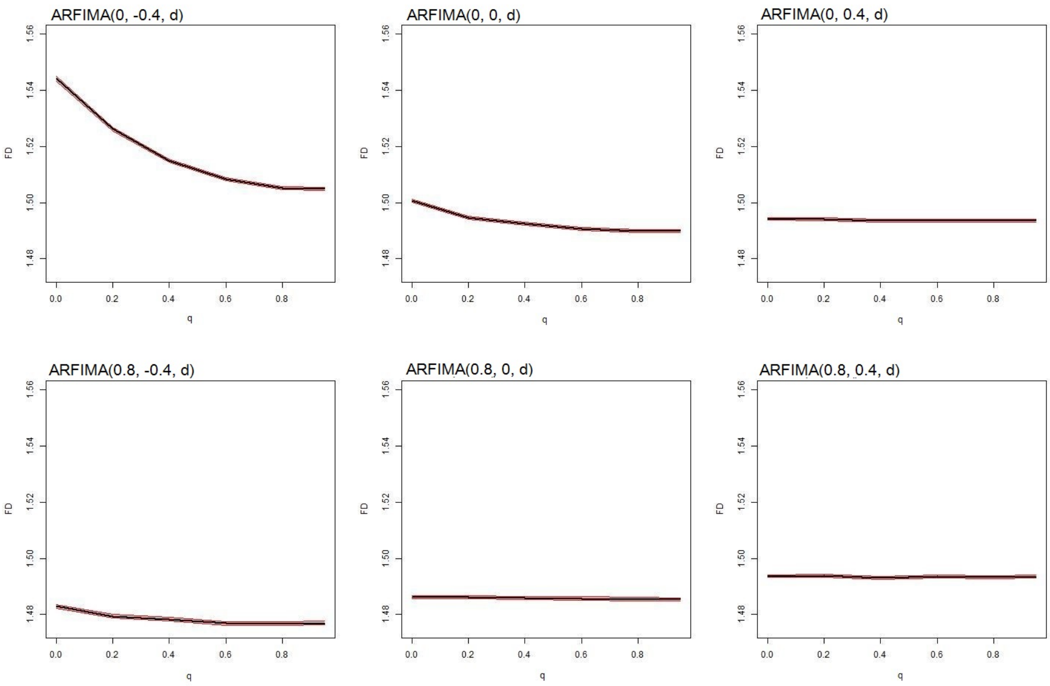


Figure B.11: Mean fractal dimensions and confidence intervals of RW with 500 observations of ARFIMA, q assigned to x-axis, 100 lags madogram

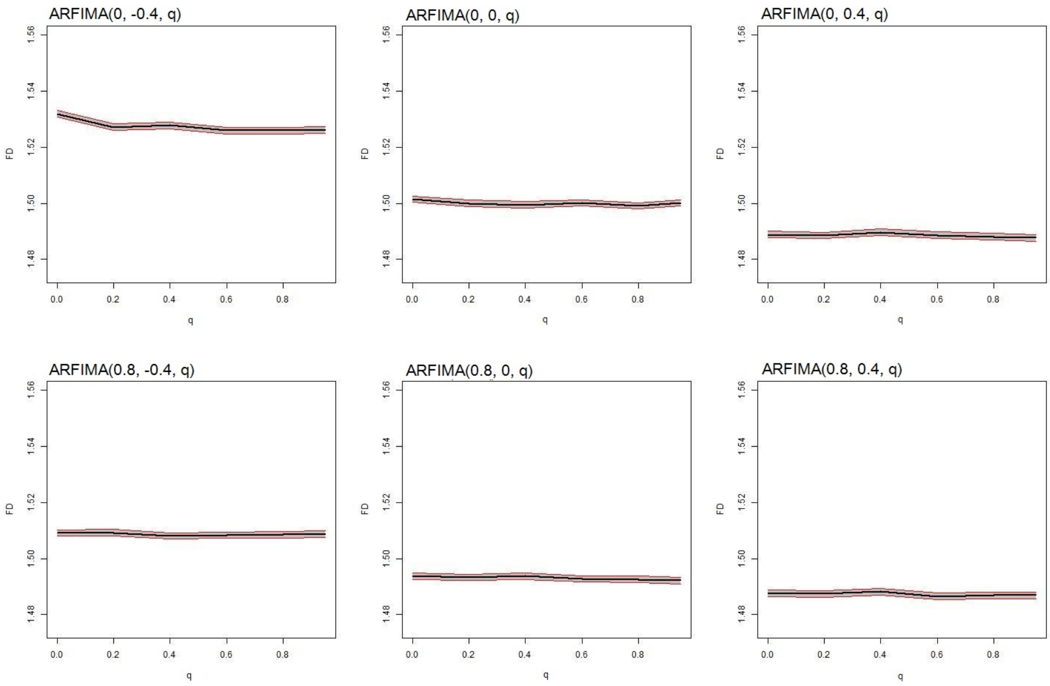


Figure B.12: Mean fractal dimensions and confidence intervals of RW with 500 observations of ARFIMA, p assigned to x-axis, 5 lags madogram

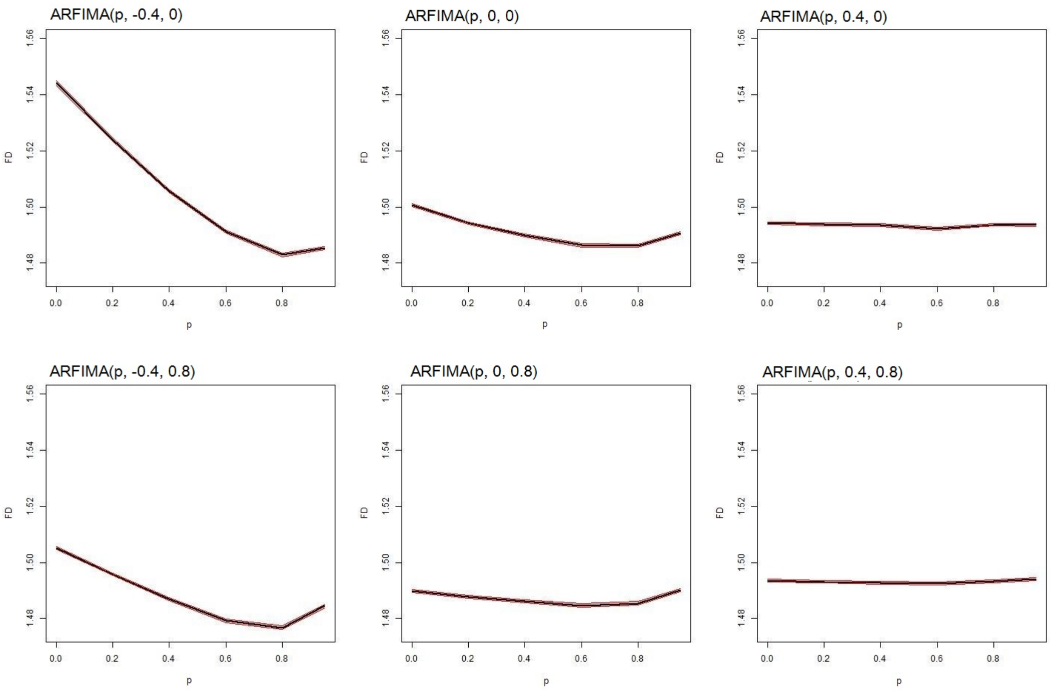


Figure B.13: Mean fractal dimensions and confidence intervals of RW with 500 observations of ARFIMA, p assigned to x-axis, 100 lags madogram

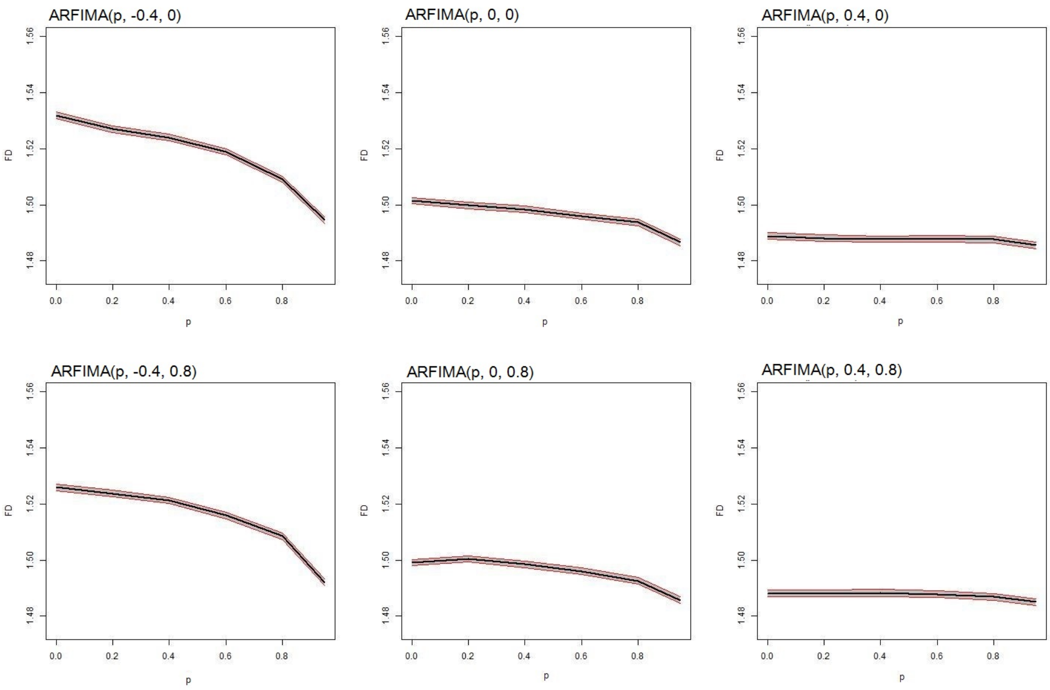


Figure B.14: 3D graph of mean fractal dimensions of RW with 300 observations of ARFIMA, q is fixed, p and d are assigned to x and y axes, madogram

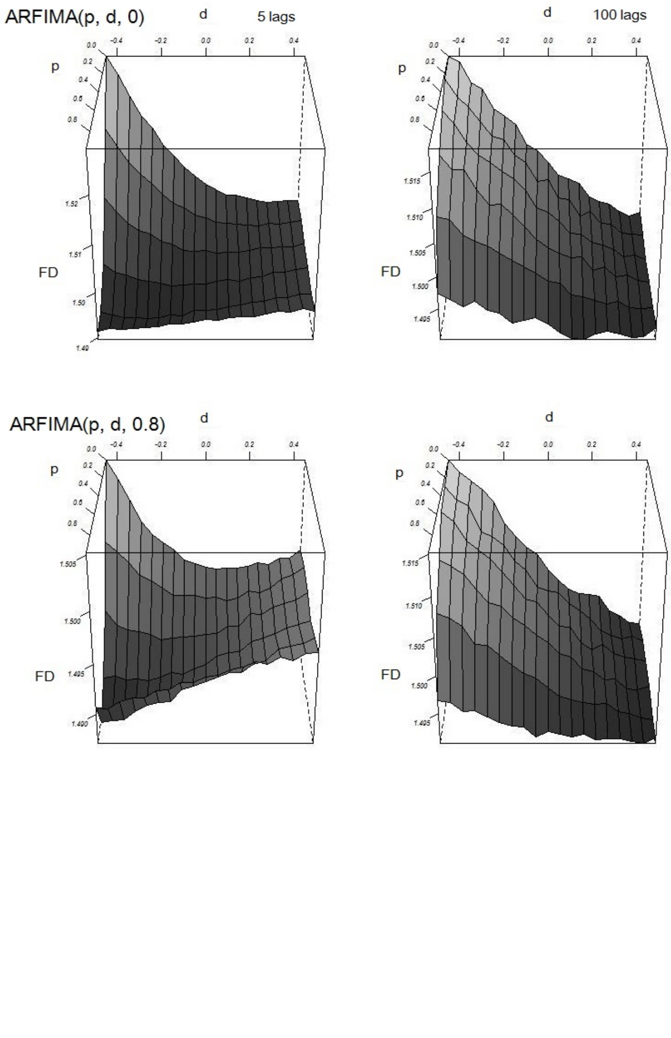


Figure B.15: 3D graph of mean fractal dimensions of RW with 1000 observations of ARFIMA, q is fixed, p and d are assigned to x and y axes, madogram

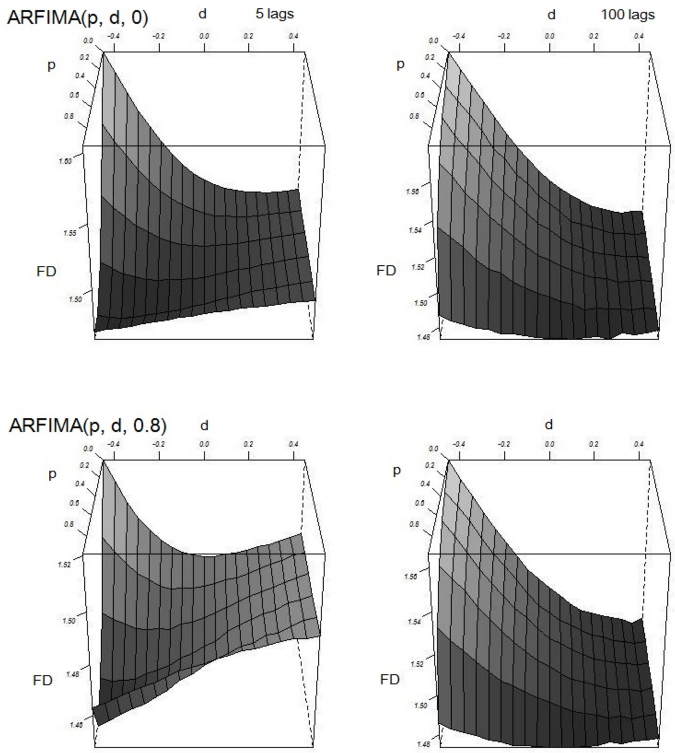


Figure B.16: Mean fractal dimensions and confidence intervals of RW with 300 observations of ARFIMA, d assigned to x-axis, 5 lags madogram

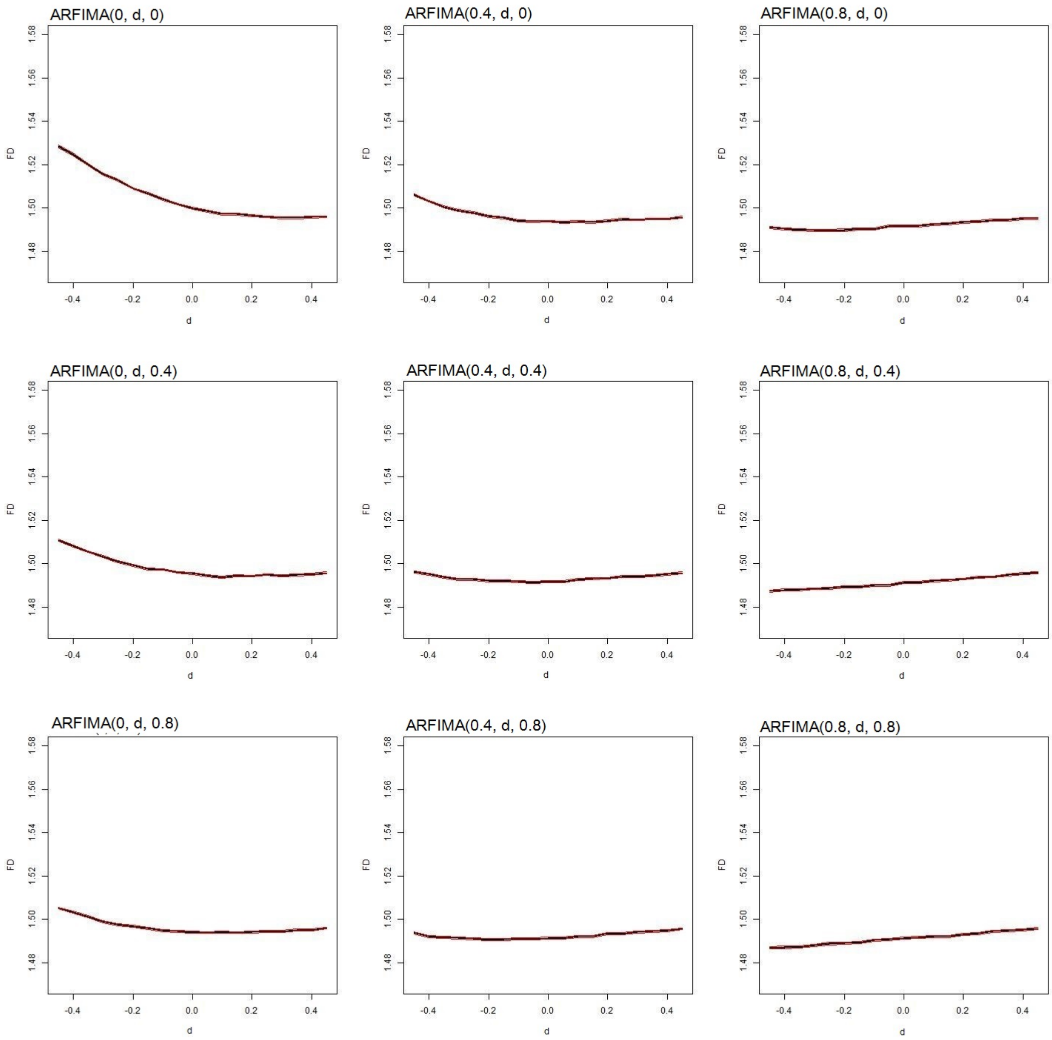


Figure B.17: Mean fractal dimensions and confidence intervals of RW with 1000 observations of ARFIMA, d assigned to x-axis, 5 lags madogram

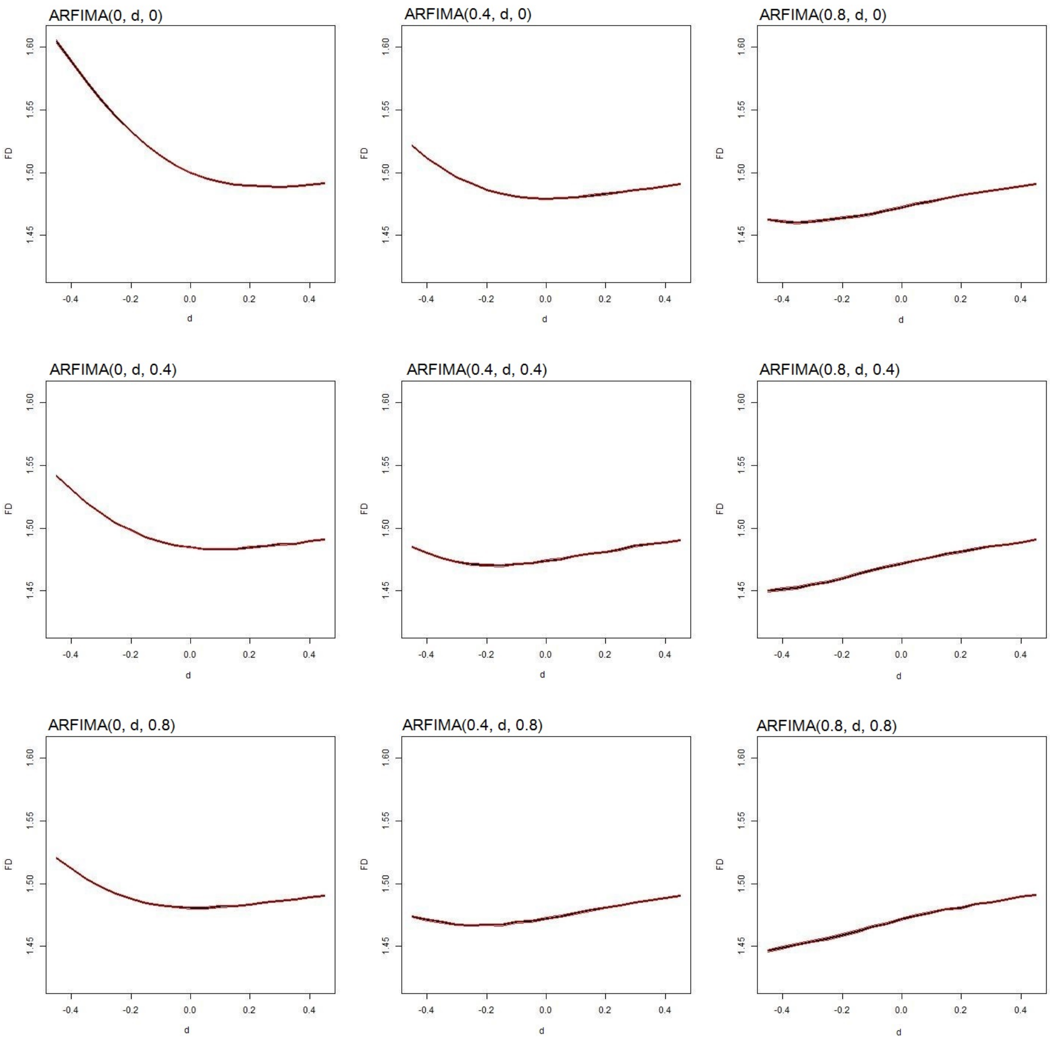


Figure B.18: Mean fractal dimensions and confidence intervals of RW with ARFIMA(0, -0.4, 0), ARFIMA(0.4, 0, 0.4), or ARFIMA(0.8, 0.4, 0.8), 5, 20 and 100 lags madogram

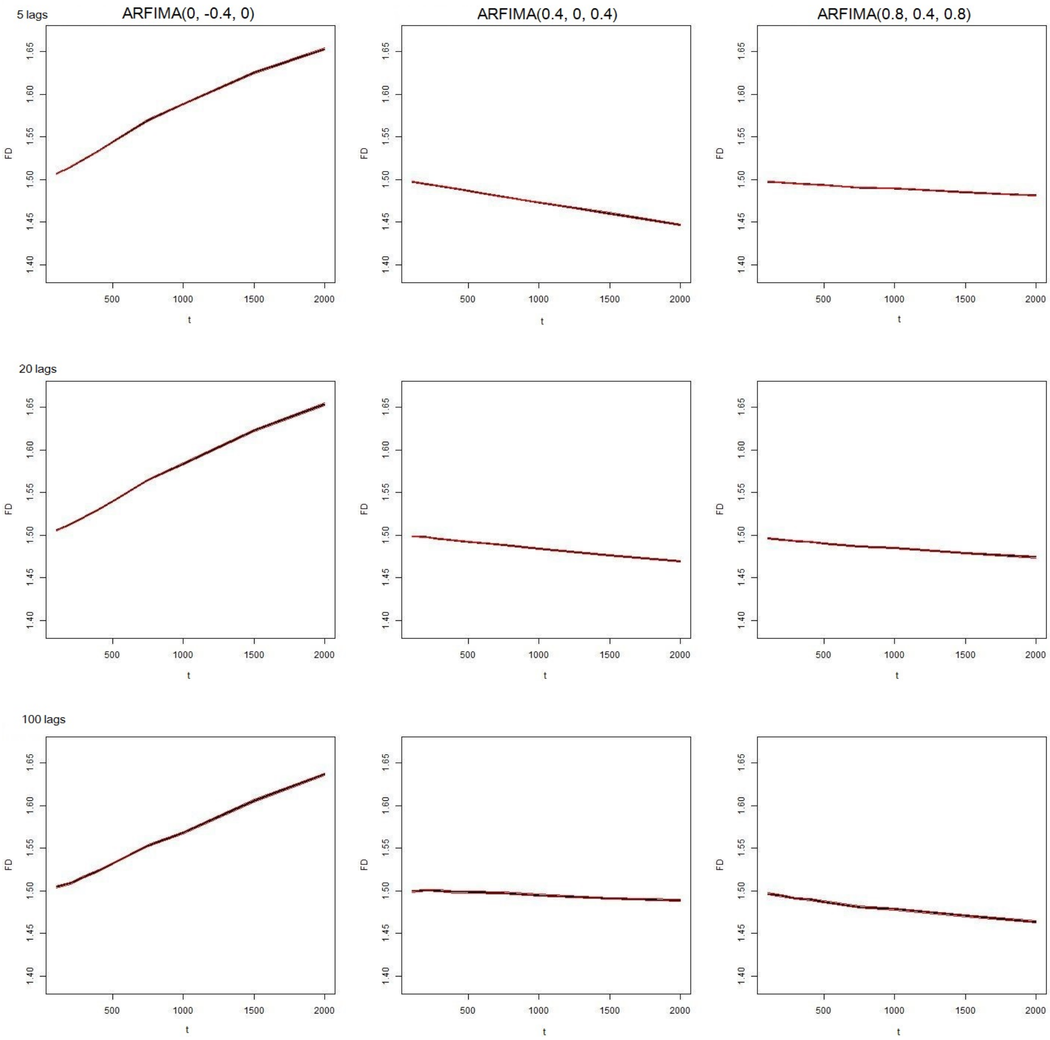


Figure B.19: Example of time series with two inefficient parts

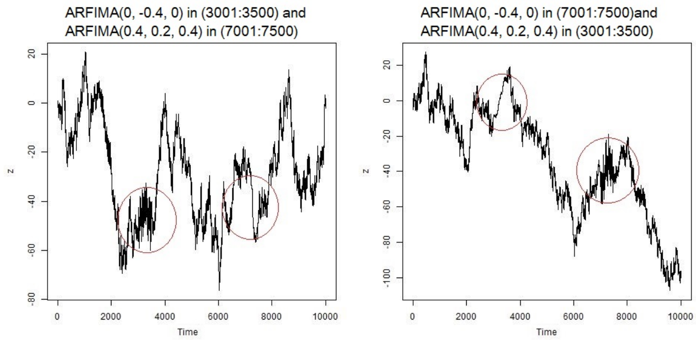


Figure B.20: Means and standard errors of fractal dimensions for graduate increase in number of observation, a sample time series, 1st setting, 5 lags madogram

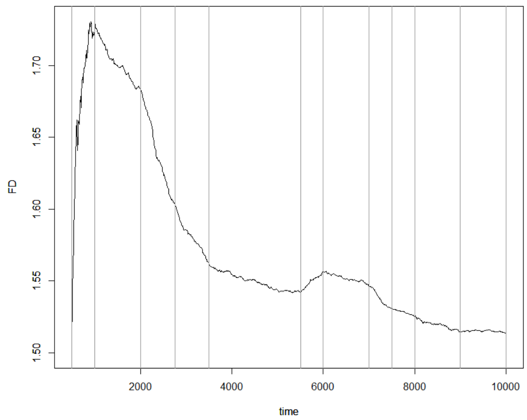


Figure B.21: Means and standard errors of fractal dimensions for graduate increase in number of observation, a sample time series, 2nd setting, 5 lags madogram

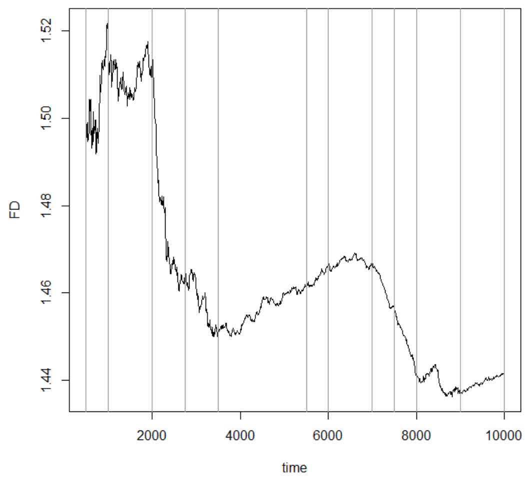


Figure B.22: Fitting of ARFIMA part

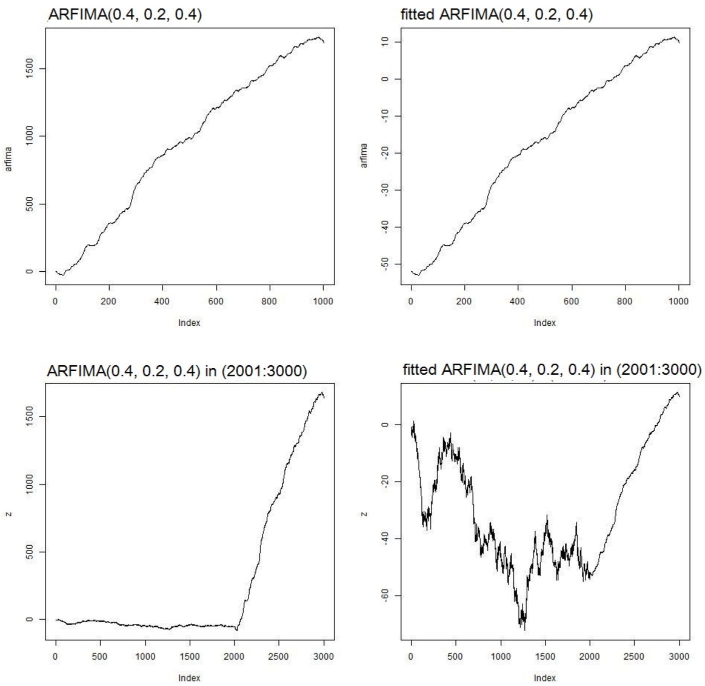


Figure B.23: 3D graphs of mean fractal dimensions of time series with 500 observations of ARFIMA, q coefficient fixed, RW, BM, and M, madogram

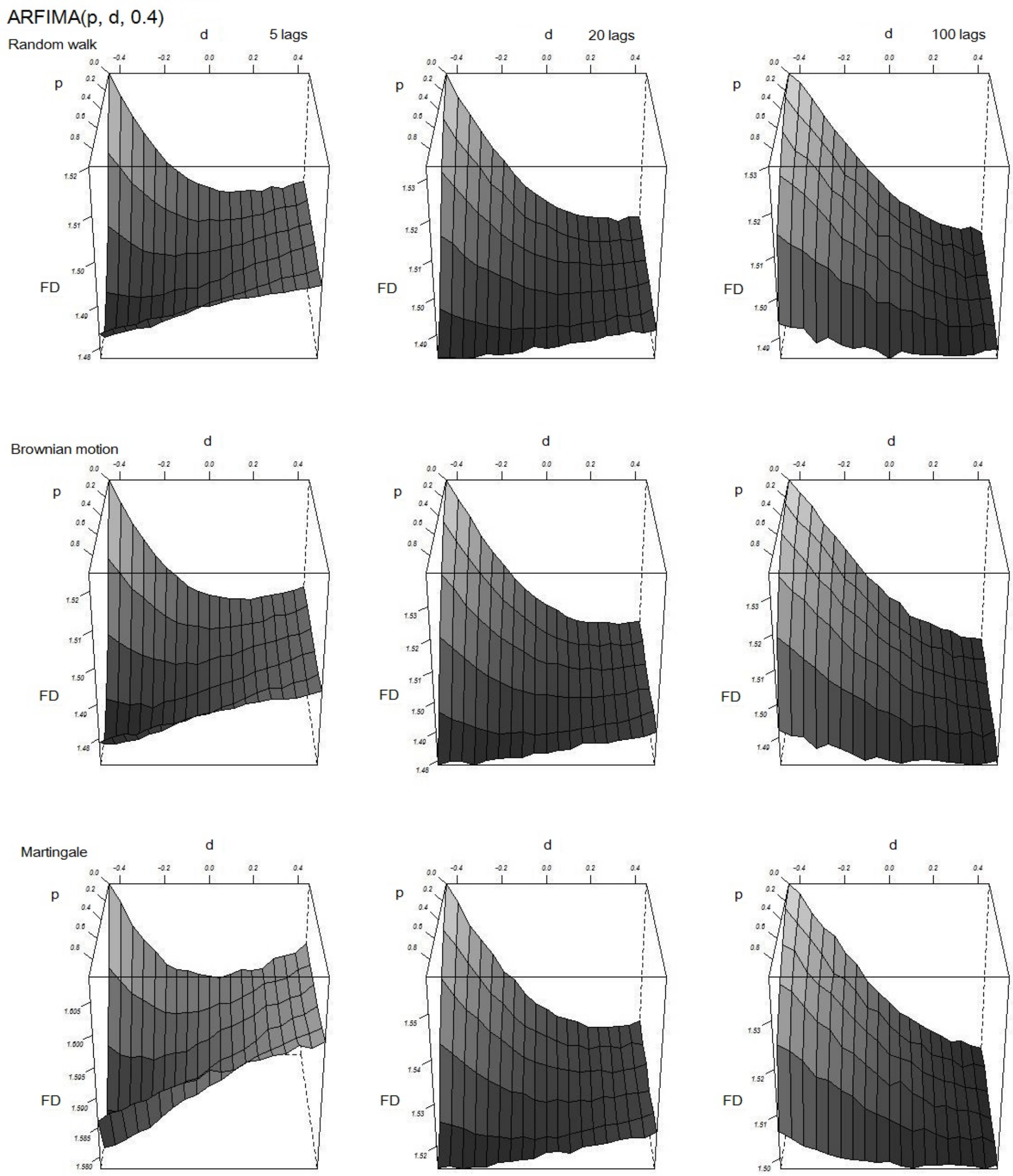


Figure B.24: 3D graphs of mean fractal dimensions of time series with 500 observations of ARFIMA, q coefficient fixed, RW, BM, and M, Hall-Wood estimator

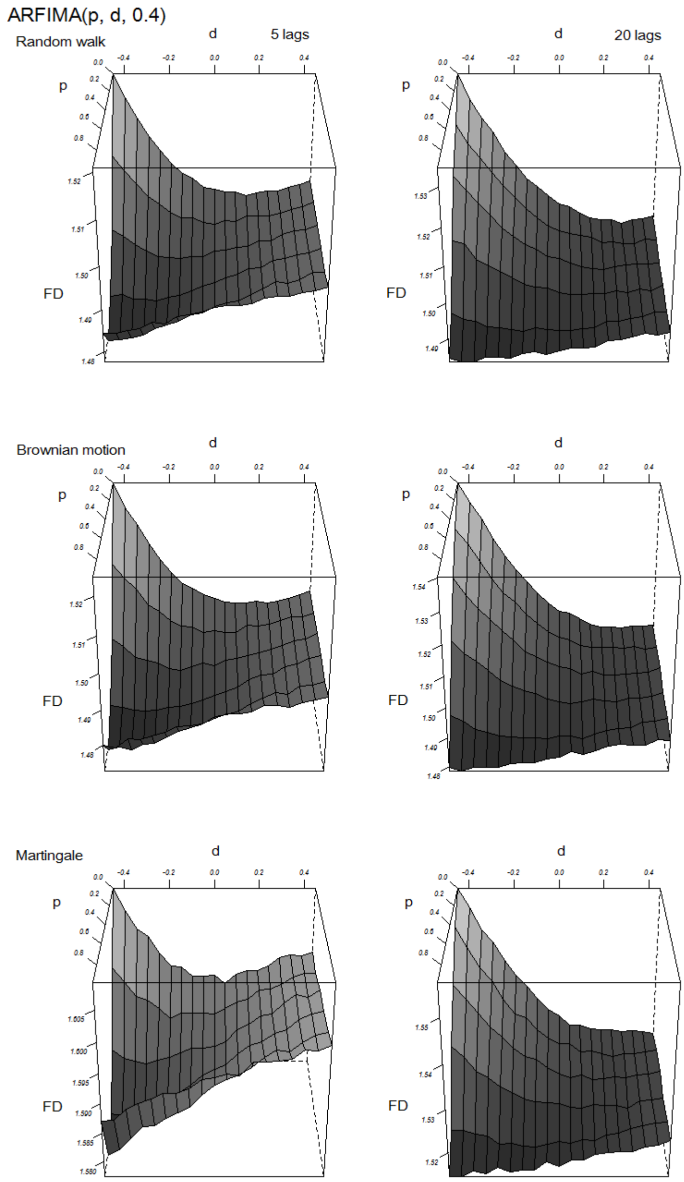


Figure B.25: 3D graphs of mean fractal dimensions of time series with 500 observations of ARFIMA, q coefficient fixed, RW, BM, and M, box-count estimator

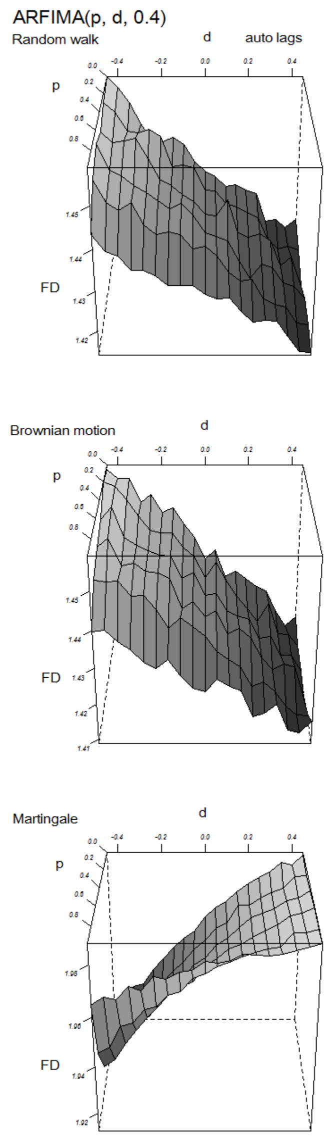
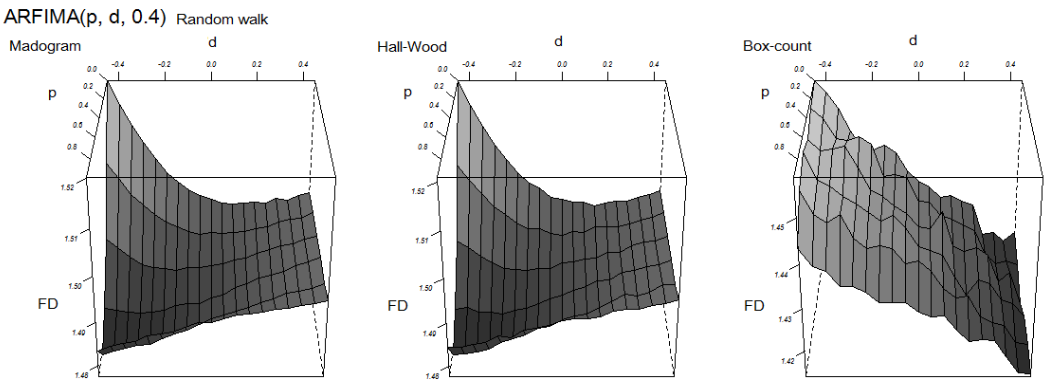


Figure B.26: 3D graphs of mean fractal dimensions of time series with 500 observations of ARFIMA, q coefficient fixed, RW, madogram, Hall-Wood and box-count estimators



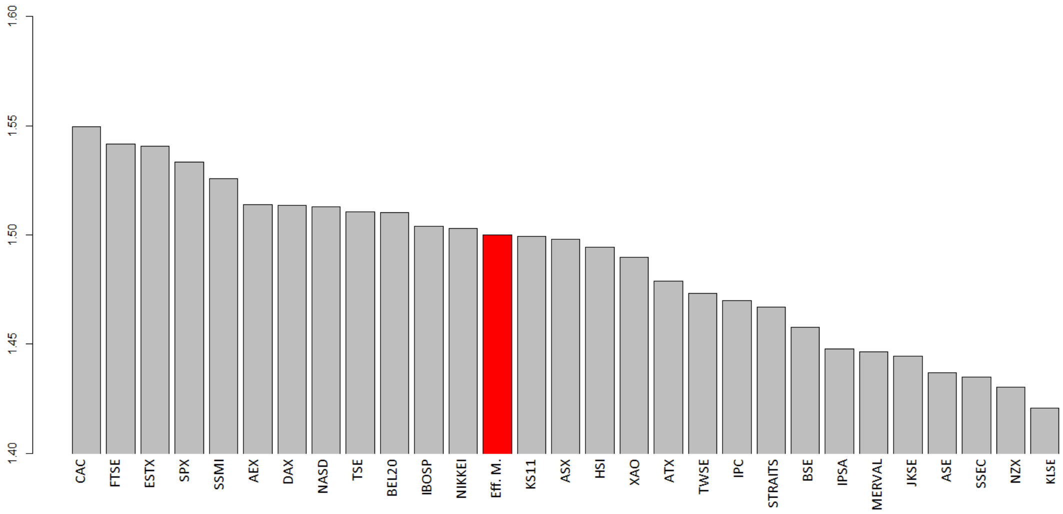
Appendix C

Indices

Table C.1: Descriptive statistics of analysed indices, close-close returns, all observations

Index	Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
AEX	0.0001	-0.7526	0.1003	0.0173	-15.2238	663.7096	0.1584	> 0.05
ASE	0.0002	-0.6695	0.6523	0.0212	-0.3032	284.1992	0.4289	> 0.05
ASX	0.0001	-0.0870	0.0563	0.0105	-0.4441	8.8710	0.1167	> 0.05
ATX	0.0002	-0.1025	0.1202	0.0137	-0.3835	10.5432	0.1347	> 0.05
BEL20	0.0002	-0.0832	0.0933	0.0117	-0.0095	9.7523	0.1655	> 0.05
BSE	0.0004	-0.1181	0.1599	0.0165	-0.0901	8.5820	0.1158	> 0.05
CAC	0.0001	-0.0947	0.1059	0.0142	-0.0286	7.4976	0.1159	> 0.05
DAX	0.0003	-0.0987	0.1080	0.0144	-0.1028	7.7368	0.0975	> 0.05
ESTX	0.0002	-0.0826	0.1044	0.0134	-0.1550	8.9242	0.1762	> 0.05
FTSE	0.0002	-0.1303	0.0938	0.0111	-0.3767	11.3630	0.1914	> 0.05
HSI	-0.0003	-0.1725	0.4054	0.0174	2.3857	59.9602	0.0887	> 0.05
IBOSP	0.0010	-2.2872	0.2883	0.0399	-36.5638	2105.1083	0.6145	> 0.02
IPC	0.0006	-0.1431	0.1215	0.0156	0.0243	8.7596	0.0522	> 0.05
IPSA	0.0005	-0.0724	0.2788	0.0120	4.7941	120.3318	0.4973	> 0.03
JKSE	0.0004	-0.1273	0.1313	0.0173	-0.1887	9.8524	0.2357	> 0.05
KLSE	0.0001	-0.2415	0.2082	0.0151	0.4074	56.7406	0.1605	> 0.05
KS11	0.0002	-0.1280	0.1128	0.0195	-0.1943	7.3659	0.0663	> 0.05
MERVAL	0.0005	-0.1476	0.1612	0.0217	-0.2867	8.3415	0.1427	> 0.05
NASD	0.0003	-0.1204	0.1325	0.0125	-0.2910	12.7884	0.0621	> 0.05
NIKKEI	0.0000	-0.1614	0.1323	0.0146	-0.2977	11.0927	0.1808	> 0.05
NZX	0.0003	-0.0494	0.0581	0.0074	-0.3519	7.8275	0.2287	> 0.05
SPX	0.0003	-0.2290	0.1096	0.0098	-1.0294	30.6599	0.0806	> 0.05
SSEC	0.0005	-0.1791	0.7192	0.0237	5.6100	159.2916	0.3435	> 0.05
SSMI	0.0003	-0.0838	0.1079	0.0117	-0.1259	8.9905	0.3877	> 0.05
STRAITS	0.0002	-0.1054	0.1287	0.0126	-0.1289	11.7824	0.0887	> 0.05
TSE	0.0002	-0.1200	0.0937	0.0099	-0.9272	17.5730	0.0454	> 0.05
TWSE	0.0000	-0.0994	0.0852	0.0153	-0.1605	5.7030	0.0990	> 0.05
XAO	0.0003	-0.2871	0.0607	0.0100	-3.7845	99.6413	0.1437	> 0.05

Figure C.1: Fractal dimension for world stock indices, 20 lags madogram



C.1 AEX

Amsterdam Exchange Index (The Neatherlands)
5435 observations (12.10.1992 - 6.2.2014)

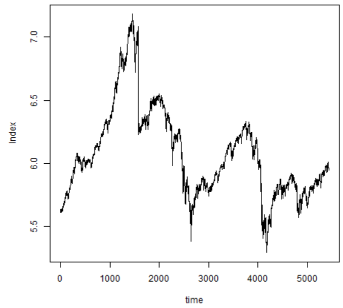
Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.7526	0.1003	0.0173	-15.2238	663.7096	0.1584	> 0.05

Statistics, years 2000-2013

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
-0.0001	-0.0959	0.1003	0.0152	-0.0651	9.0820	0.1844	> 0.05

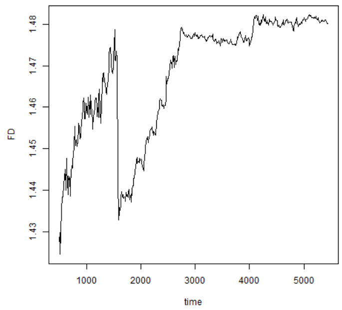
Graph, all observations



Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5435
5 lags	1.4802	1.5003	1.4597	1.4403	1.5175	1.4637	1.4938
20 lags	1.4800	1.5139	1.4394	1.4276	1.5279	1.4533	1.5156
100 lags	1.4446	1.4844	1.3901	1.3701	1.4867	1.4716	1.4873

Fractal dimensions for gradually increasing number of observations, all observations (5 lags)



There is an observable drop in the AEX index caused by the introduction of euro on the stock exchange in 1999 (1577th observation), which affected fractal dimension; the results show persistences in the data during the first 2000 observations (December 1992 - August 2000). Then the series exhibits several substantial changes in the autocorrelation; at first, the revenues became anti-persistent; then the market experienced slight local persistence between August 2004 - July 2008; thereafter the index seems to be nearly efficient. The variation is more strongly captured by the 20 lags estimator.

C.2 ASE

Athens Stock Exchange General Index (Greece)
6746 observations (5.2.1988 - 4.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.6695	0.6523	0.0212	-0.3032	284.1992	0.4289	> 0.05

Statistics, years 2000-2013

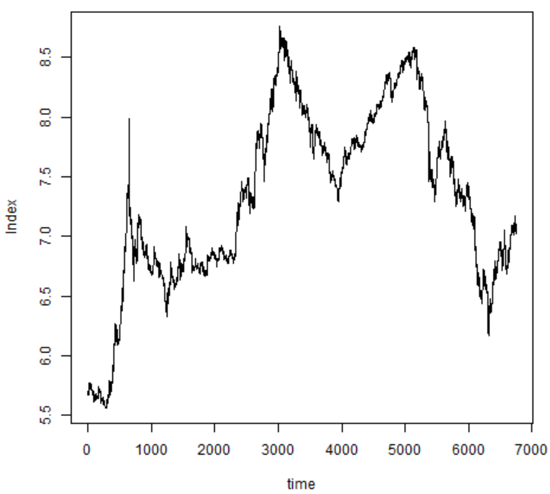
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
-0.0004	-0.1021	0.1343	0.0176	-0.0132	7.1399	0.1743	> 0.05

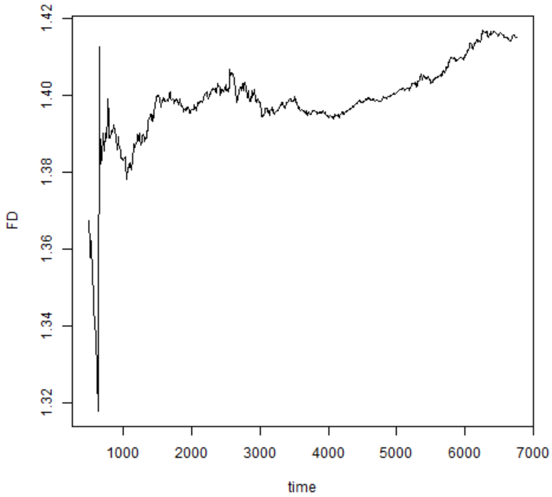
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4151	1.4342	1.3830	1.4138	1.3993	1.3857	1.4453
20 lags	1.4286	1.4369	1.3856	1.4169	1.4259	1.4555	1.4365
100 lags	1.3974	1.4149	1.3198	1.4384	1.3681	1.5097	1.3942

	5000-6000	6000-6746
5 lags	1.4478	1.4387
20 lags	1.4569	1.4210
100 lags	1.3921	1.3948

Index development and "cumulative" fractal dimension





Fractal dimensions of individual parts as well as the gradual re-estimation of fractal dimension show that there are long-term or repeating persistences on the market. Even though the strength of the persistences decreased since 2003, they are still present.

C.3 ASX

Australian Securities Exchange Index (Australia)
3120 observations (17.10.2001 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.0870	0.0563	0.0105	-0.4441	8.8710	0.1167	> 0.05

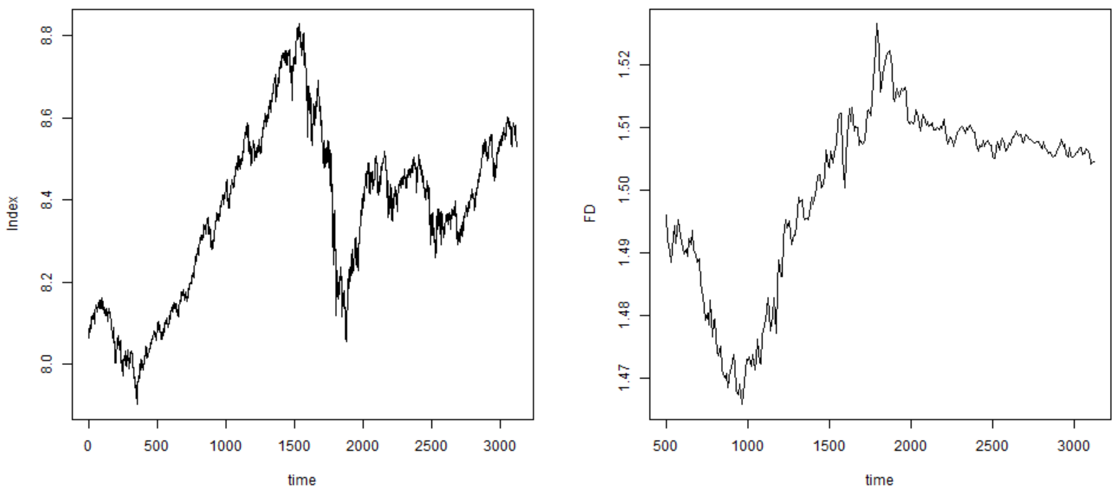
Statistics, years 2000-2013

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.0870	0.0563	0.0105	-0.4471	8.8732	0.1130	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3120
5 lags	1.5046	1.5042	1.4729	1.5284	1.4933
20 lags	1.4975	1.4980	1.4512	1.5097	1.5124
100 lags	1.4628	1.4613	1.4182	1.4303	1.5726

Index development and "cumulative" fractal dimension



The ASX index is relatively new, it was launched in 2000, and at the beginning it faced local persistences; its fractal dimension was lower than 1.47. Then the index went through a period of higher volatility to stabilize in price structure close to martingales.

C.4 ATX

Austrian Traded Index (Austria)

5254 observations (11.11.1992 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1025	0.1202	0.0137	-0.3835	10.5432	0.1347	> 0.05

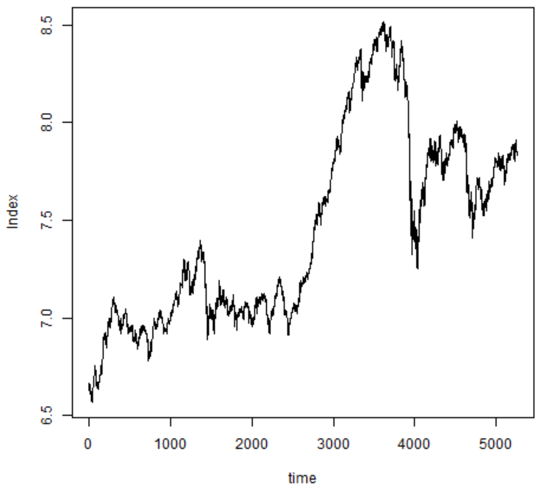
Statistics, years 2000-2013

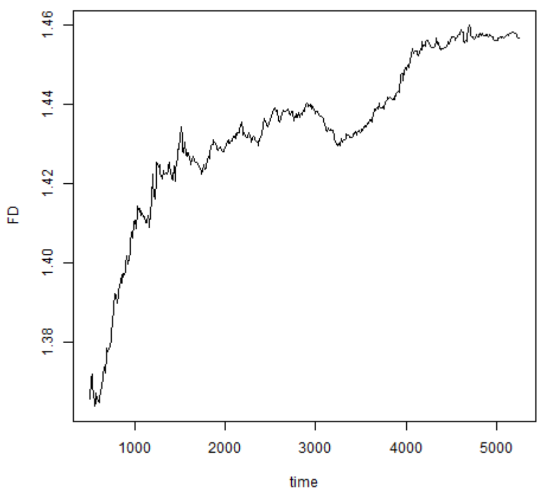
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1025	0.1202	0.0150	-0.3027	10.1011	0.2398	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5254
5 lags	1.4566	1.4710	1.4107	1.4432	1.4578	1.4687	1.4773
20 lags	1.4661	1.4787	1.4322	1.4564	1.3963	1.4662	1.5202
100 lags	1.4146	1.3995	1.4419	1.4741	1.3258	1.3793	1.4770

Index development and “cumulative” fractal dimension





The index has a strong positive autocorrelation structure in its beginning, since then fractal dimension is steadily growing (except for a short period in the second half of 2004, when the price rapidly increased). Fractal dimension of the index reaches 1.48 and seems to be stabilized, so some local persistence is still present.

C.5 BEL20

Euronext Brussels Index (Belgium)
5777 observations (9.4.1991 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.0832	0.0933	0.0117	-0.0095	9.7523	0.1655	> 0.05

Statistics, years 2000-2013

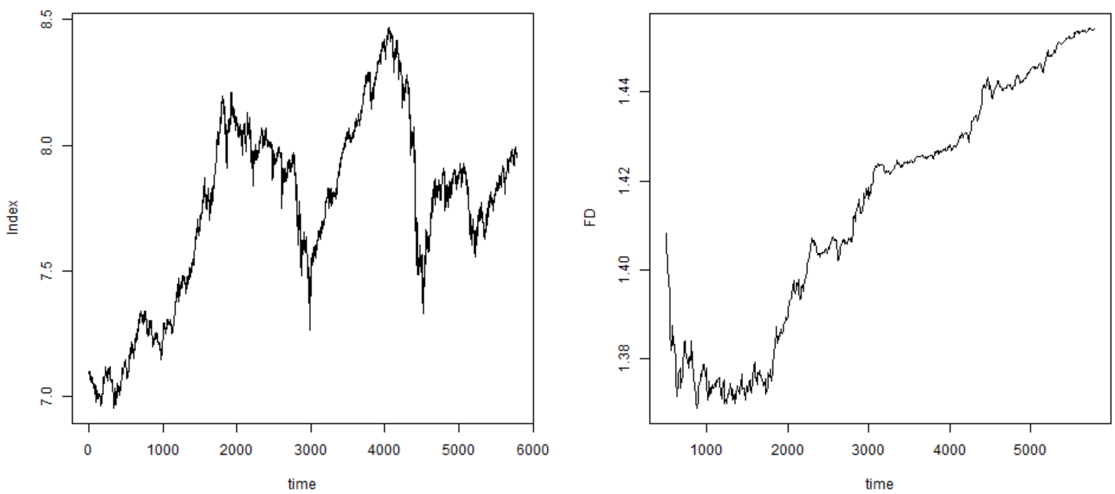
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.0832	0.0933	0.0132	0.0445	8.8571	0.1216	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4541	1.4813	1.3750	1.4020	1.4527	1.4748	1.4911
20 lags	1.4852	1.5101	1.4145	1.4517	1.5223	1.4567	1.5010
100 lags	1.4536	1.4570	1.4501	1.4064	1.5508	1.3416	1.4245

	5000-5777
5 lags	1.5072
20 lags	1.5497
100 lags	1.4869

Index development and "cumulative" fractal dimension



The story of BEL20 index is similar to ATX; it was very inefficient when it was launched, but its fractal dimension is constantly increasing, trends diminished and current fractal dimension is close to 1.5 meaning that the stock index is relatively efficient.

C.6 BSE

Bombay Stock Exchange Index (India)
4104 observations (1.7.1997 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0004	-0.1181	0.1599	0.0165	-0.0901	8.5820	0.1158	> 0.05

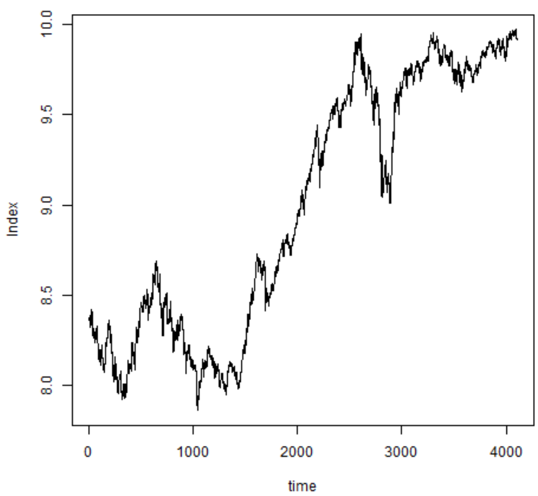
Statistics, years 2000-2013

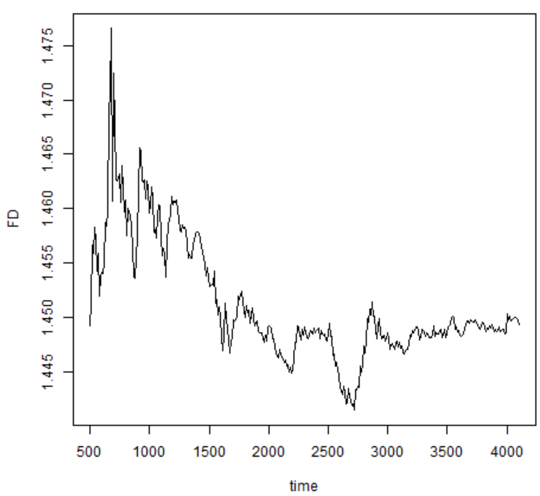
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0004	-0.1181	0.1599	0.0162	-0.1818	9.4239	0.1529	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4104
5 lags	1.4493	1.4499	1.4596	1.4332	1.4478	1.4552
20 lags	1.4579	1.4576	1.4575	1.4151	1.4571	1.4981
100 lags	1.4526	1.4492	1.4944	1.3955	1.4035	1.5996

Index development and “cumulative” fractal dimension





The BSE index experiences strong autocorrelation structure in its returns throughout the observed period. Its fractal dimension does not show any improvement in the efficiency.

C.7 CAC

Euronext Paris Bourse Index (France)
6063 observations (1.3.1990 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.0947	0.1059	0.0142	-0.0286	7.4976	0.1159	> 0.05

Statistics, years 2000-2013

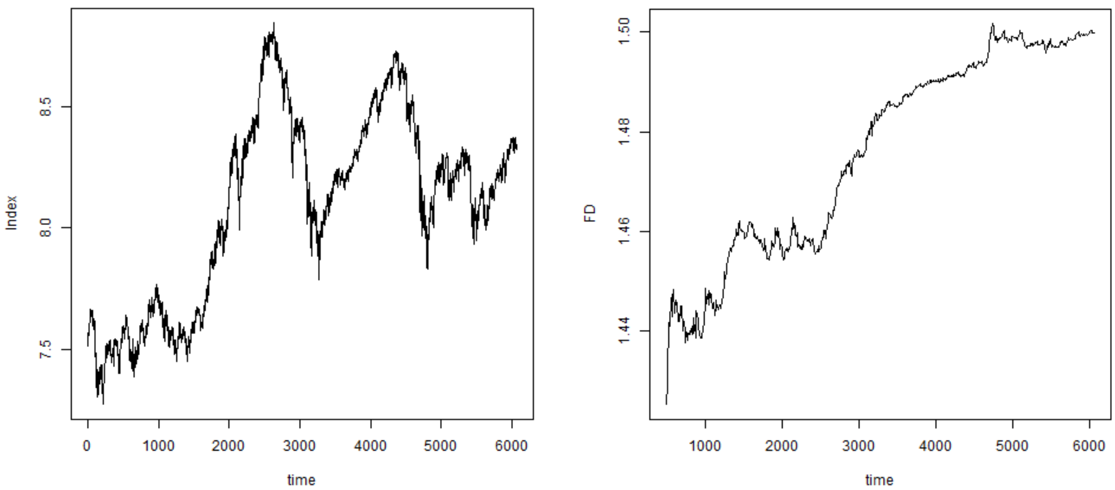
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
-0.0001	-0.0947	0.1059	0.0154	0.0262	7.6637	0.1371	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4999	1.5285	1.4465	1.4678	1.5068	1.5326	1.5301
20 lags	1.5273	1.5494	1.4756	1.5206	1.5291	1.5600	1.5396
100 lags	1.4928	1.5068	1.5042	1.5273	1.4741	1.4807	1.4597

	5000-6063
5 lags	1.5029
20 lags	1.5400
100 lags	1.5436

Index development and "cumulative" fractal dimension



Fractal dimension of the index is very close to 1.5 since 1998. In years 2002-2010, volatility of the index was higher than implied by martingale structure, which is strongly reflected in 20 lags fractal dimensions. Today, returns of index have slightly negatively autocorrelated structure, 5 lags fractal dimension is close to 1.5, but 20 lags fractal dimension reaches 1.54.

C.8 DAX

Deutscher Aktien Index (Germany)

5875 observations (26.11.1990 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.0987	0.1080	0.0144	-0.1028	7.7368	0.0975	> 0.05

Statistics, years 2000-2013

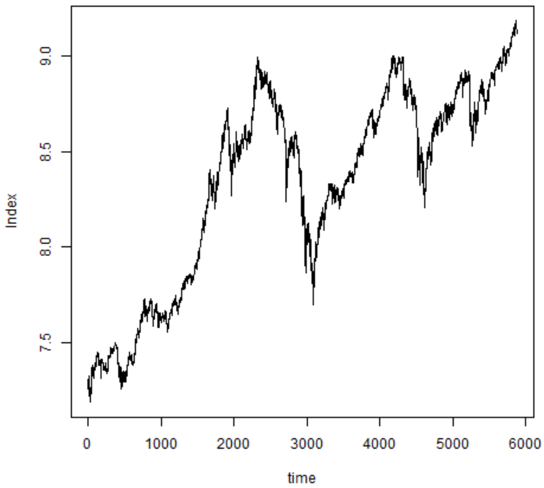
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.0743	0.1080	0.0158	0.0233	7.1487	0.2505	> 0.05

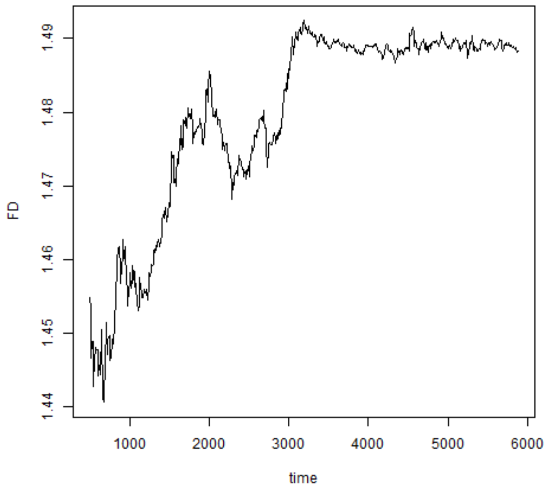
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4882	1.4988	1.4568	1.5099	1.4879	1.5010	1.4858
20 lags	1.5005	1.5134	1.4812	1.4597	1.5095	1.5306	1.5057
100 lags	1.4665	1.4789	1.4905	1.3925	1.4900	1.4654	1.4878

	5000-5875
5 lags	1.4855
20 lags	1.5012
100 lags	1.4634

Index development and "cumulative" fractal dimension





Disregarding the period 1990-1994, fractal dimension of DAX index moves between 1.486 and 1.509, so the index is quite long-term efficient. Nowadays, the fractal dimension stabilize at 1.486 according to the 5 lags estimator and 1.5 according to the 20 lags estimator.

C.9 ESTX

EURO STOXX 50 Index (Eurozone)

6953 observations (31.12.1986 - 31.1.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.0826	0.1044	0.0134	-0.1550	8.9242	0.1762	> 0.05

Statistics, years 2000-2013

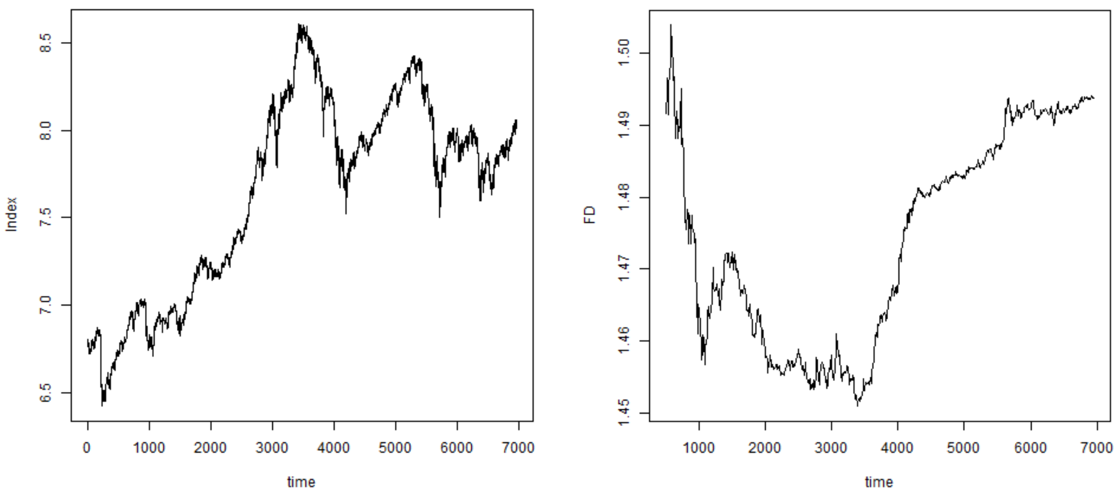
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
-0.0001	-0.0821	0.1044	0.0156	0.0177	7.2505	0.1402	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4937	1.5232	1.4644	1.4518	1.4527	1.4843	1.5358
20 lags	1.5047	1.5406	1.4283	1.4550	1.4744	1.5217	1.5609
100 lags	1.4588	1.5063	1.3788	1.4407	1.3684	1.5137	1.5381

	5000-6000	6000-6953
5 lags	1.5270	1.4988
20 lags	1.5240	1.5443
100 lags	1.4587	1.5329

Index development and "cumulative" fractal dimension



The autocorrelation structure of the index revenues can be divide to three parts; firstly it went through a time of strong positive correlation in 1987-2000; then it jumped to negatively correlated structure; finally the negative autocorrelation weakened and the series is now nearly efficient according to the 5 lags estimator, or the negative autocorrelation remains in the structure but it is weaker according to the 20 lags estimator.

C.10 FTSE

Financial Times Stock Exchange 100 Index (UK)
7541 observations (2.4.1984 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1303	0.0938	0.0111	-0.3767	11.3630	0.1914	> 0.05

Statistics, years 2000-2013

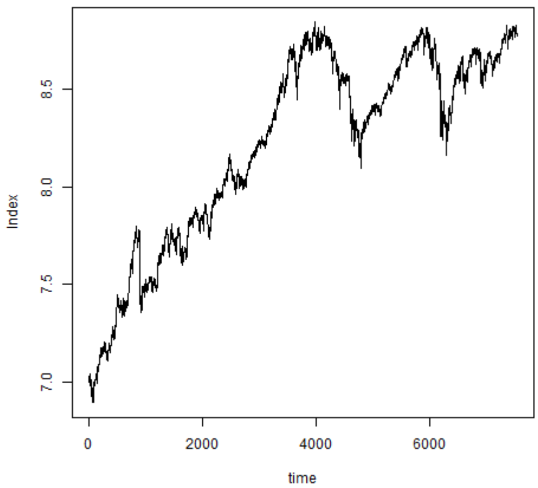
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.0926	0.0938	0.0126	-0.1243	8.9375	0.1804	> 0.05

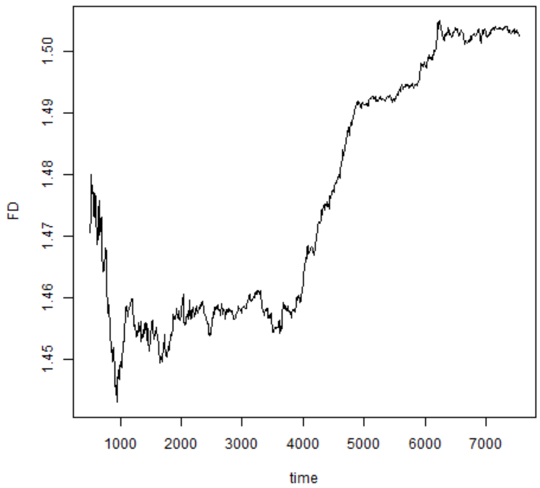
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.5025	1.5443	1.4497	1.4665	1.4568	1.4720	1.5800
20 lags	1.5049	1.5415	1.4305	1.4471	1.4676	1.5331	1.5700
100 lags	1.4993	1.5398	1.3839	1.4898	1.4641	1.4821	1.5484

	5000-6000	6000-7000	7000-7541
5 lags	1.5382	1.5216	1.5044
20 lags	1.5310	1.5366	1.4906
100 lags	1.5401	1.5185	1.5882

Index development and "cumulative" fractal dimension





The patterns in fractal dimension of FTSE index are very similar to ESTX patterns. Low fractal dimension indicating short term persistence abruptly changed to negatively correlated revenues and the correlation then weakened.

C.11 HSI

Hang Seng Index (Hong Kong)

6471 observations (31.12.1986 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
-0.0003	-0.1725	0.4054	0.0174	2.3857	59.9602	0.0887	> 0.05

Statistics, years 2000-2013

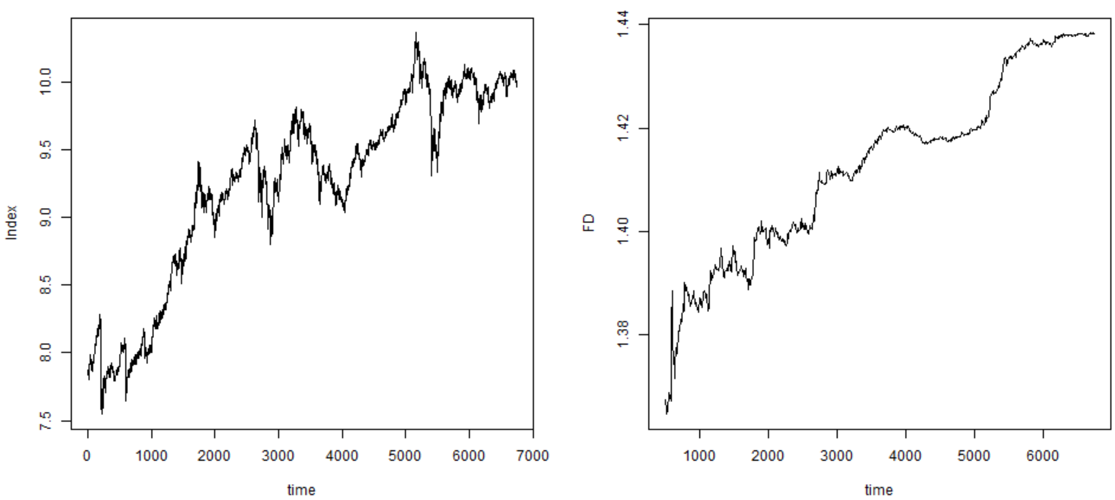
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.1358	0.1341	0.0158	-0.0667	10.8159	0.1219	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4382	1.4684	1.3870	1.4073	1.4330	1.4432	1.4172
20 lags	1.4605	1.4943	1.3906	1.4281	1.4799	1.4552	1.4874
100 lags	1.4570	1.4820	1.4118	1.4392	1.4475	1.5413	1.4219

	5000-6000	6000-6471
5 lags	1.5019	1.4528
20 lags	1.5008	1.4952
100 lags	1.4328	1.4926

Index development and “cumulative” fractal dimension



Fractal dimension of HSI index is increasing, meaning that the markets become more efficient. In the years 2007-2009, both fractal dimension estimators (5 and 20 lags) mark the market as efficient, but according to the 5 lags estimator the following period contains some short term persistences, while the 20 lags estimator indicates that its structure is close to an efficient market’s structure.

C.12 IBOSP

Ibovespa Brasil Sao Paulo Stock Exchange Index (Brazil)
5141 observations (27.4.1993 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0010	-2.2872	0.2883	0.0399	-36.5638	2105.1083	0.6145	> 0.02

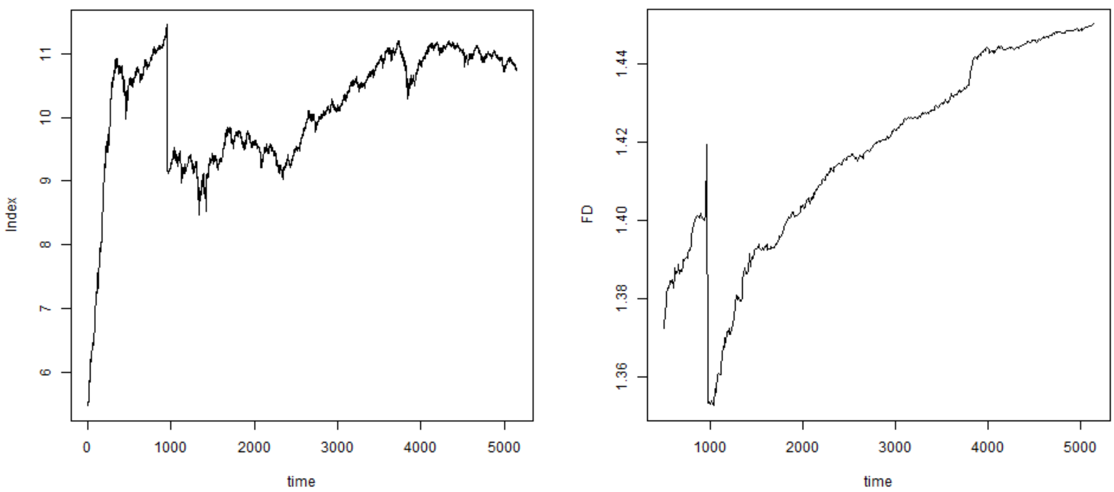
Statistics, years 2000-2013

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.1210	0.1368	0.0187	-0.0898	6.7837	0.1523	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5141
5 lags	1.4503	1.5014	1.3528	1.4681	1.4865	1.5303	1.4900
20 lags	1.4220	1.5039	1.2861	1.4819	1.4603	1.5217	1.5038
100 lags	1.3592	1.4713	1.2318	1.5040	1.4430	1.4313	1.5179

Index development and "cumulative" fractal dimension



The index was divided by 10 in 1997 which had a similar effect as introducing euro in the case of AEX - a big decrease in fractal dimension. However, since then the fractal dimension increases and is close to 1.5. The index went through a more volatile period in 2012, but the volatility has returned back to the martingale standards.

C.13 IPC

Indice de Precios y Cotizaciones (Mexico)
5563 observations (8.11.1991 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0006	-0.1431	0.1215	0.0156	0.0243	8.7596	0.0522	> 0.05

Statistics, years 2000-2013

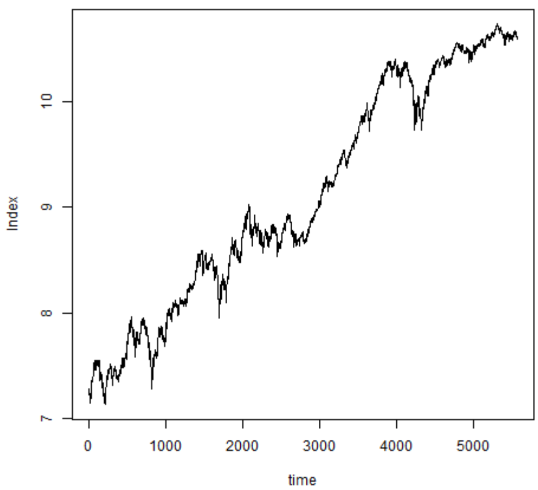
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0005	-0.0827	0.1044	0.0141	0.0242	7.5342	0.1108	> 0.05

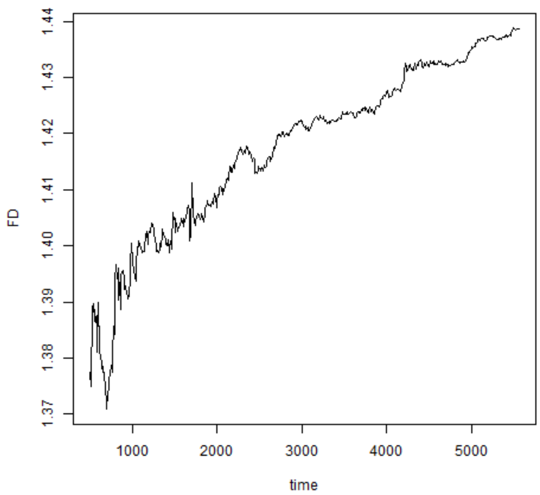
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4387	1.4621	1.3976	1.4203	1.4537	1.4497	1.4692
20 lags	1.4507	1.4698	1.4141	1.4477	1.4609	1.4351	1.4885
100 lags	1.4559	1.4669	1.4204	1.5046	1.5070	1.4074	1.4666

	5000-5563
5 lags	1.4914
20 lags	1.5412
100 lags	1.5726

Index development and "cumulative" fractal dimension





IPC index experienced strong trends in its beginnings such as BEL20, but the trend has been disappearing in a slower pace and the fractal dimension only approximates the 1.5 value.

C.14 IPSA

Santiago Stock Exchange Index (Chile)
2593 observations (10.1.2003 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0005	-0.0724	0.2788	0.0120	4.7941	120.3318	0.4973	> 0.03

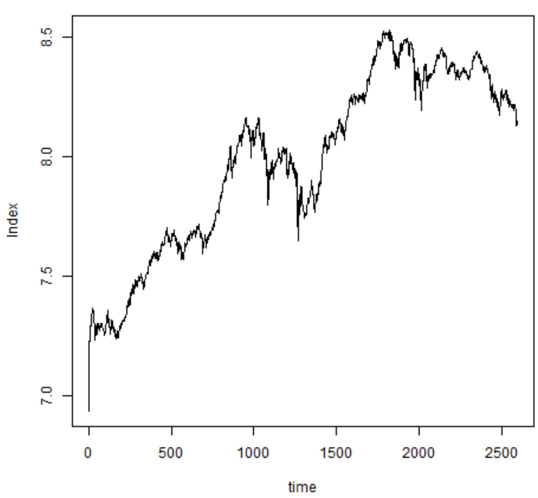
Statistics, years 2000-2013

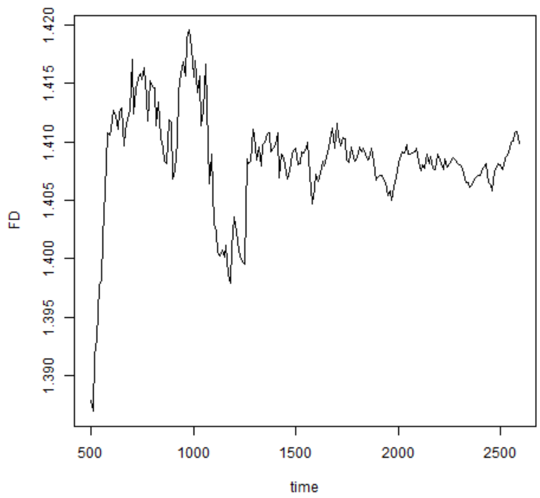
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0005	-0.0724	0.2788	0.0120	4.8278	120.9241	0.4298	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-2593
5 lags	1.4098	1.4109	1.4155	1.4029	1.4183
20 lags	1.4491	1.4477	1.4412	1.4459	1.4816
100 lags	1.4444	1.4433	1.4212	1.4385	1.4907

Index development and "cumulative" fractal dimension





IPSA is one of the least efficient indices. In the analysis, we are limited to narrow data range (2003-2013) of observations but the index shows to have fractal dimension very close to 1.41 all the time. Only the 20 lags estimators shows the series as a bit more efficient during last two years.

C.15 JKSE

Jakarta Composite Index (Indonesia)
4033 observations (1.7.1997 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0004	-0.1273	0.1313	0.0173	-0.1887	9.8524	0.2357	> 0.05

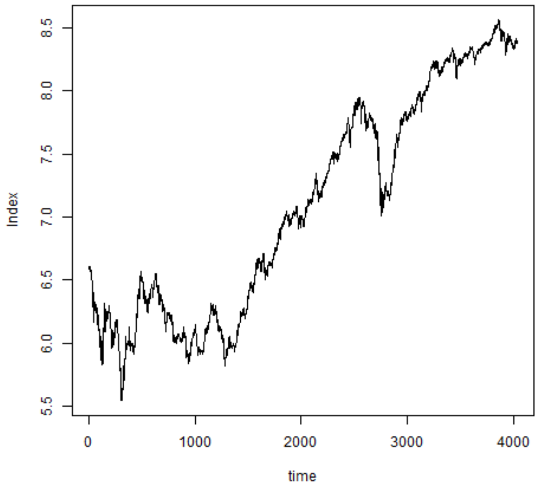
Statistics, years 2000-2013

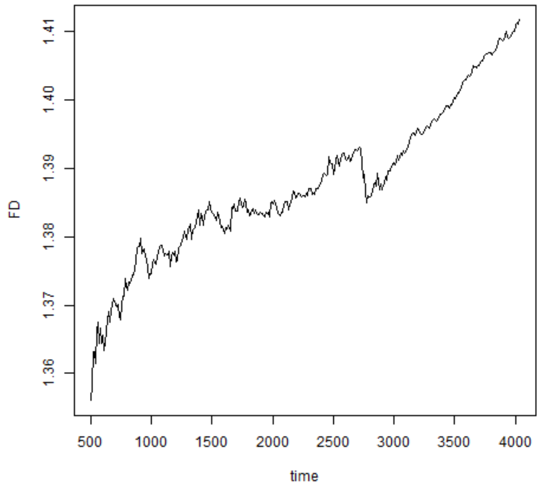
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0005	-0.1095	0.0762	0.0148	-0.6766	8.9110	0.2101	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4033
5 lags	1.4117	1.4270	1.3745	1.4027	1.4024	1.5050
20 lags	1.4418	1.4445	1.4271	1.4132	1.4320	1.5116
100 lags	1.4127	1.3939	1.4572	1.3561	1.3332	1.4863

Index development and "cumulative" fractal dimension





JKSE index is another example of indices experiencing persistences in their beginnings; at first, fractal dimension was moving around 1.4, but fractal dimension of period 2010-2013 is 1.5 indicating improvement of the market efficiency. However, this increase in fractal dimension does not significantly influence the fractal dimension of the whole 2000-2013 period, and the index is placed among the least efficient ones.

C.16 KLSE

Bursa Malaysia Index (Malaysia)

4975 observations (3.12.1993 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.2415	0.2082	0.0151	0.4074	56.7406	0.1605	> 0.02

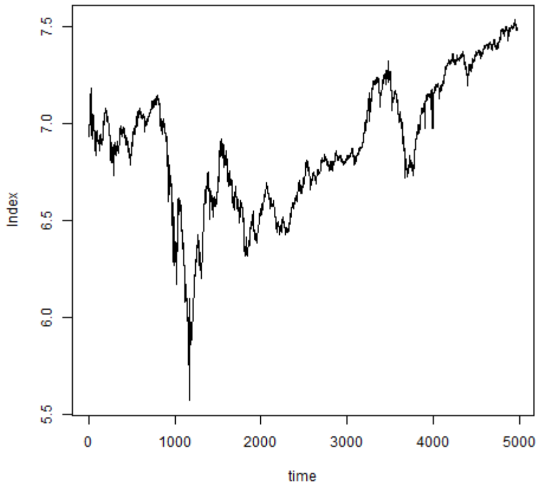
Statistics, years 2000-2013

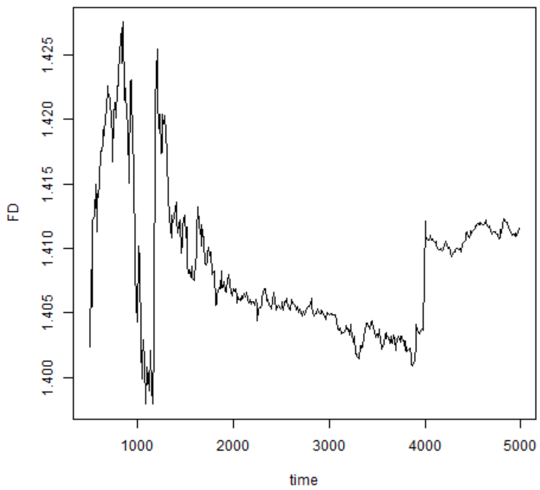
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1925	0.1986	0.0111	-0.2474	95.5659	0.1401	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-4975
5 lags	1.4116	1.4113	1.4043	1.4064	1.3942	1.4395	1.4141
20 lags	1.4266	1.4208	1.4837	1.3866	1.4036	1.4304	1.4430
100 lags	1.4118	1.4190	1.4481	1.3993	1.4374	1.3433	1.4869

Index development and "cumulative" fractal dimension





KLSE index fractal dimension behaves similarly as fractal dimension of IPSA index, it moves around the 1.4 value and does not show any significant improvement. Malaysia is one of the worst performing markets.

C.17 KS11

KOSPI Composite Index (Korea)
4093 observations (1.7.1997 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1280	0.1128	0.0195	-0.1943	7.3659	0.0663	> 0.05

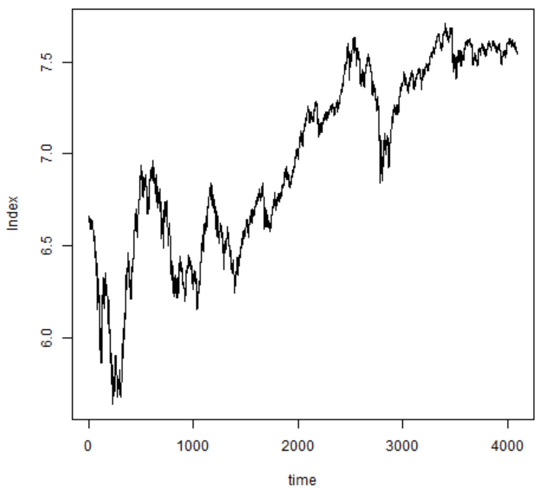
Statistics, years 2000-2013

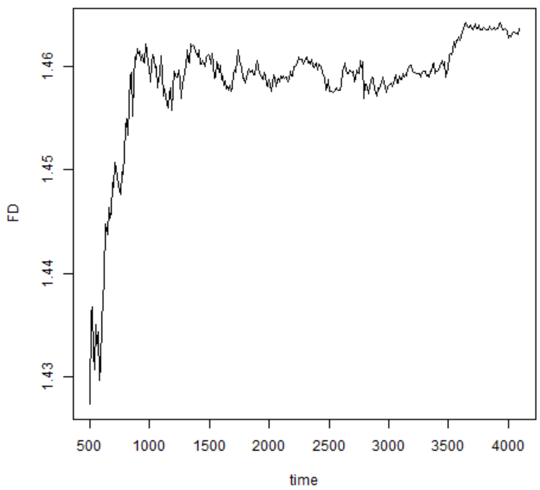
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1280	0.1128	0.0170	-0.5296	8.3126	0.1350	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4093
5 lags	1.4636	1.4719	1.4600	1.4574	1.4558	1.4906
20 lags	1.4885	1.4994	1.4805	1.4861	1.4799	1.5154
100 lags	1.4591	1.4772	1.4454	1.4272	1.4444	1.5742

Index development and "cumulative" fractal dimension





In the period 1997-2009, fractal dimension oscillated around 1.45 value, then we can observe an increase in the fractal dimension signalling an improvement in the market efficiency.

C.18 MERVAL

Mercado de Valores Index (Argentina)
4270 observations (8.10.1996 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0005	-0.1476	0.1612	0.0217	-0.2867	8.3415	0.1427	> 0.05

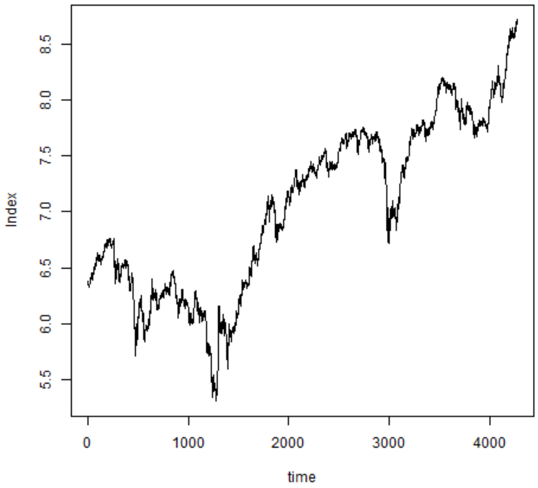
Statistics, years 2000-2013

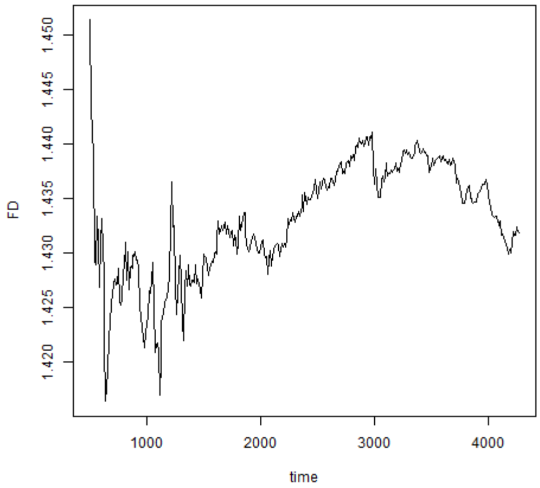
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0007	-0.1295	0.1612	0.0213	-0.1223	7.7794	0.0844	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4207
5 lags	1.4318	1.4324	1.4235	1.4384	1.4530	1.4217
20 lags	1.4496	1.4465	1.4462	1.4311	1.4839	1.4519
100 lags	1.4318	1.4226	1.4779	1.3865	1.5346	1.4042

Index development and "cumulative" fractal dimension





Argentina is one of the least efficient markets, fractal dimension of the index started on the 1.42 value, then it increased to 1.45 but market was not able to continue in the efficiency improvement; on the contrary, the fractal dimension felt to the initial values.

C.19 NASD

NASDAQ Composite Index (USA)

10847 observations (5.2.1971 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.1204	0.1325	0.0125	-0.2910	12.7884	0.0621	> 0.05

Statistics, years 2000-2013

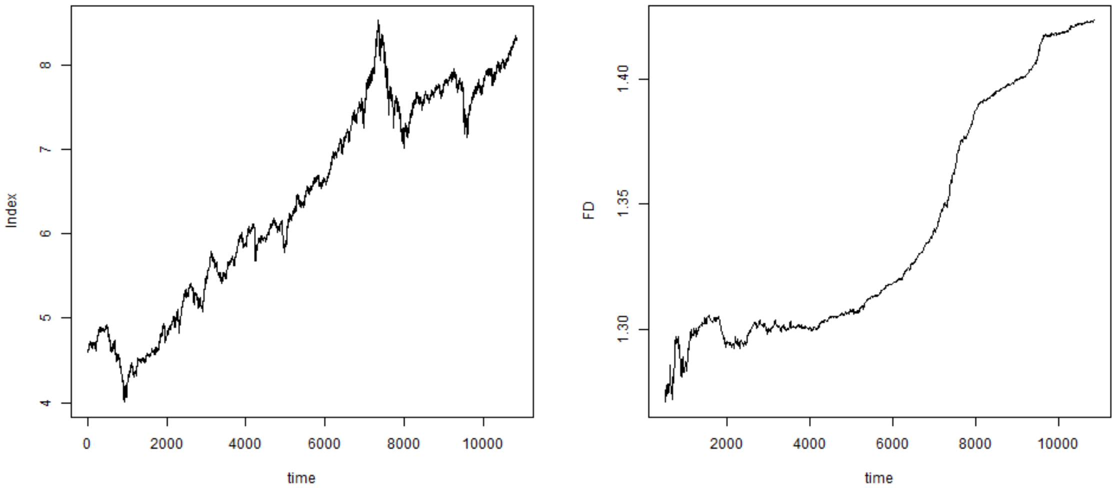
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.1017	0.1325	0.0175	0.0457	7.9047	0.4790	> 0.04

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4234	1.5165	1.2873	1.3100	1.3075	1.3050	1.3323
20 lags	1.4357	1.5130	1.3491	1.3744	1.3164	1.3351	1.3660
100 lags	1.4197	1.4833	1.3927	1.4174	1.3995	1.3500	1.3776

	5000-6000	6000-7000	7000-8000	8000-9000	9000-10000	10000-10847
5 lags	1.3723	1.4312	1.5106	1.4995	1.5345	1.4919
20 lags	1.4456	1.4556	1.4927	1.5253	1.5206	1.5187
100 lags	1.4043	1.4440	1.4468	1.4958	1.4634	1.4856

Index development and “cumulative” fractal dimension



Fractal dimension of NASDAQ index was very low until 1999, then it started to increase rapidly and reached the 1.5 value signalling that the market is close to an efficient one.

C.20 NIKKEI

NIKKEI 225 Index (Japan)

7402 observations (4.1.1984 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.1614	0.1323	0.0146	-0.2977	11.0927	0.1808	> 0.05

Statistics, years 2000-2013

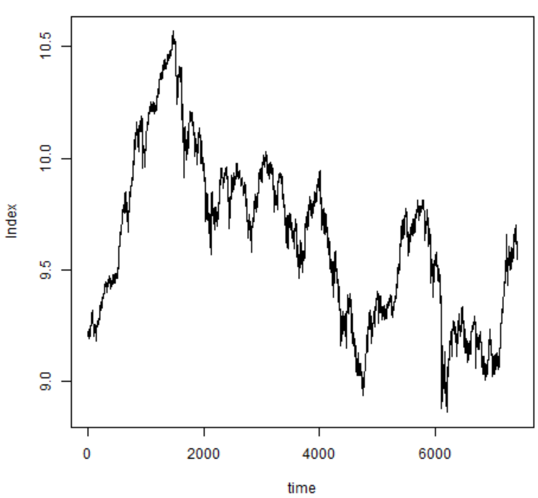
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.1211	0.1323	0.0158	-0.4203	9.2486	0.2949	> 0.05

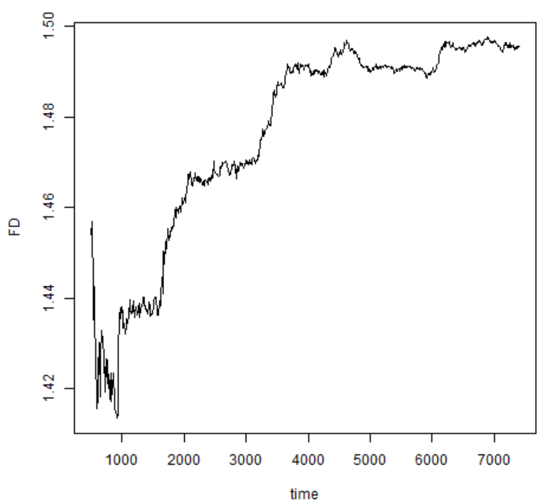
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4956	1.4994	1.4370	1.4812	1.4833	1.5409	1.4871
20 lags	1.4886	1.5029	1.4356	1.4636	1.4673	1.5261	1.5215
100 lags	1.4639	1.4747	1.3941	1.4339	1.5011	1.5108	1.4770

	5000-6000	6000-7402
5 lags	1.4879	1.5140
20 lags	1.5035	1.4889
100 lags	1.4839	1.4599

Index development and "cumulative" fractal dimension





The case of Japan market is very similar to the German one; except for the period 1984-1987, fractal dimension indicates that the market is quite long-term efficient.

C.21 NZX

New Zealand Exchange 50 Gross Index (New Zealand)
2355 observations (30.4.2004 - 5.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.0494	0.0581	0.0074	-0.3519	7.8275	0.2287	> 0.05

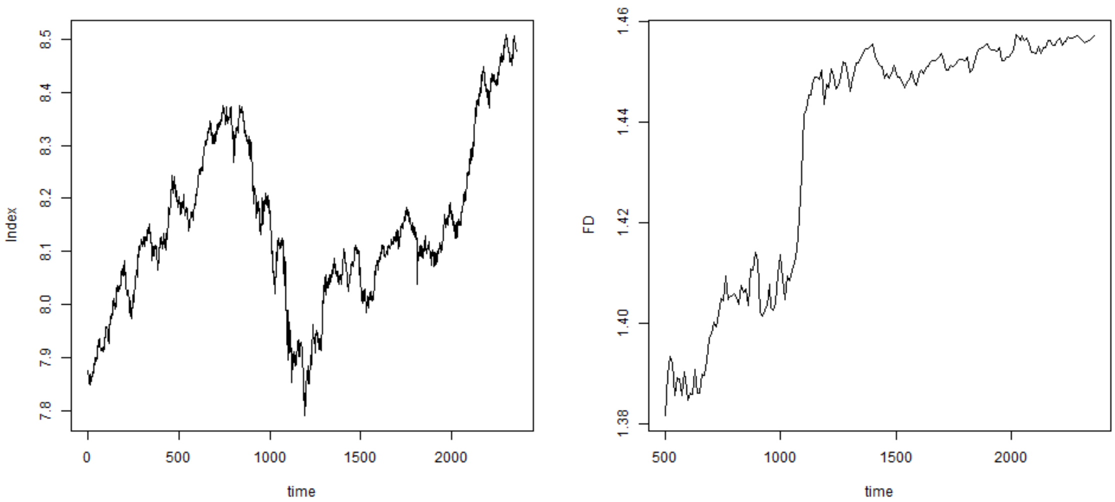
Statistics, years 2000-2013

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.0494	0.0581	0.0074	-0.3604	7.8701	0.2265	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2355
5 lags	1.4573	1.4561	1.4137	1.4873
20 lags	1.4300	1.4303	1.4297	1.4375
100 lags	1.4072	1.4054	1.3996	1.4178

Index development and "cumulative" fractal dimension



NZX index is relatively new and we are very limited in the number of observations (2004-2013), so we estimate fractal dimensions for only two 1,000 observations long periods (2004-2008, 2009-2013). The fractal dimension of the first one is low, but we can observe a significant increase in the second one with fractal dimension close to 1.49. The graph of fractal dimensions for gradually increasing number of observations confirms a steep increase in the fractal dimension at the beginning of 2009, since then the fractal dimension seems to keep a value close to 1.49.

C.22 SSEC

Shanghai Composite Index (China)

5935 observations (19.12.1990 - 30.1.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0005	-0.1791	0.7192	0.0237	5.6100	159.2916	0.3435	> 0.05

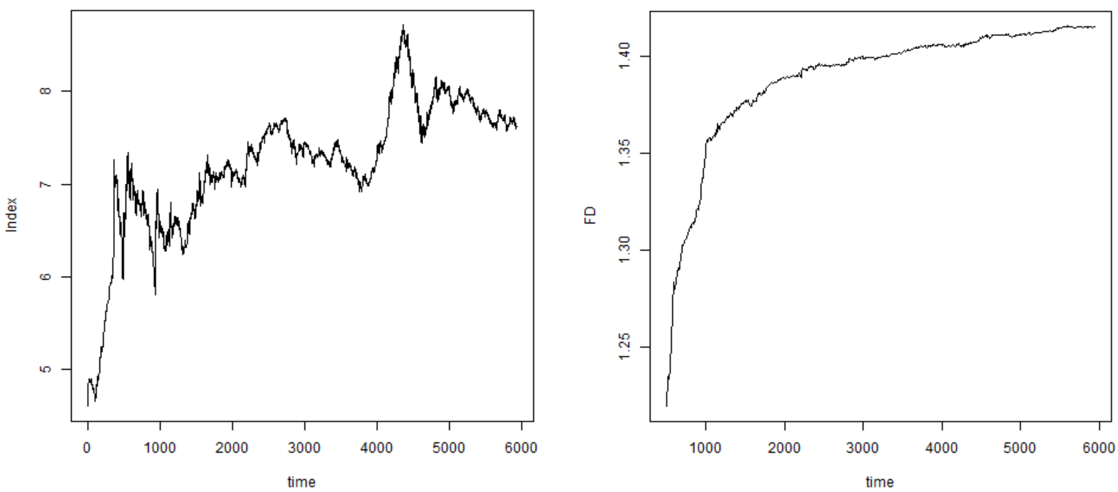
Statistics, years 2000-2013

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.0926	0.0940	0.0156	-0.0891	7.5811	0.1465	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000	5000-5935
5 lags	1.4149	1.4381	1.3530	1.4407	1.4388	1.4446	1.4340	1.4484
20 lags	1.4001	1.4350	1.3108	1.4505	1.4462	1.4435	1.4260	1.4585
100 lags	1.3989	1.4109	1.3338	1.4870	1.4314	1.4325	1.3346	1.5329

Index development and “cumulative” fractal dimension



The Shanghai’s market seems to steadily experience some persistences as fractal dimension of the index constantly moves around 1.44 without any significant changes.

C.23 SSMI

Swiss Market Index (Switzerland)

5871 observations (9.11.1990 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.0838	0.1079	0.0117	-0.1259	8.9905	0.3877	> 0.05

Statistics, years 2000-2013

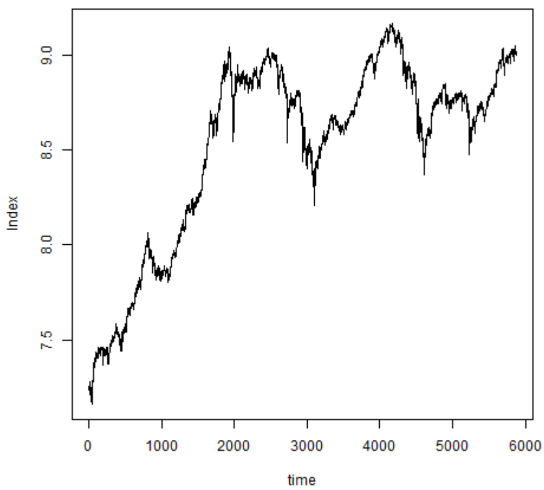
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.0811	0.1079	0.0122	0.0156	9.1551	0.1346	> 0.05

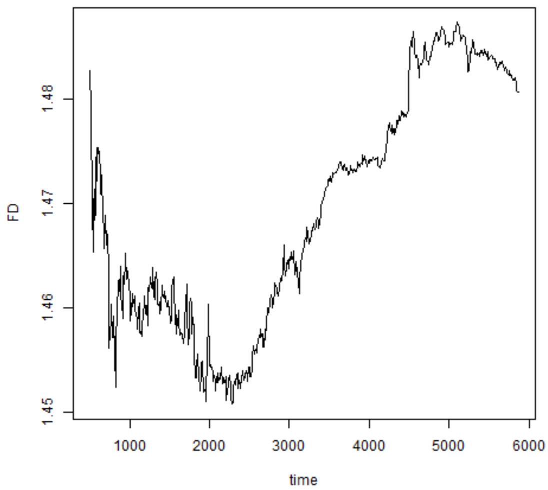
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4807	1.4991	1.4613	1.4490	1.4828	1.5160	1.5207
20 lags	1.5030	1.5257	1.4741	1.4372	1.5487	1.5156	1.5462
100 lags	1.4531	1.4651	1.4063	1.3732	1.5643	1.4198	1.4697

	5000-5871
5 lags	1.4487
20 lags	1.5091
100 lags	1.4775

Index development and "cumulative" fractal dimension





Low fractal dimension in period 1990-1998 implies presence of some persistence in this period, then the market became more efficient and the fractal dimension increased above 1.5. During the last period, some trend appeared in the market and fractal dimension significantly decreased.

C.24 STRAITS

Straits Times Index (Singapore)

6546 observations (28.12.1987 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1054	0.1287	0.0126	-0.1289	11.7824	0.0887	> 0.05

Statistics, years 2000-2013

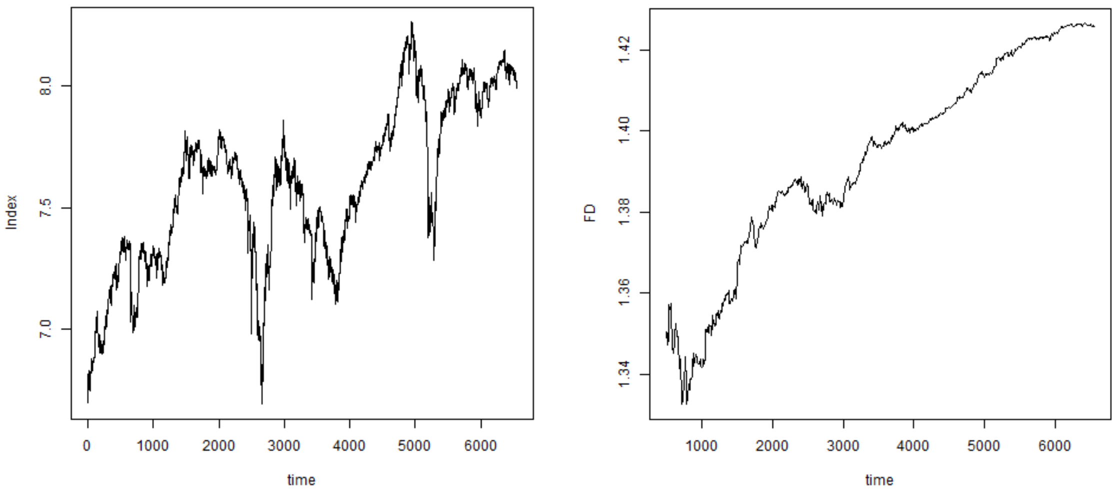
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.0922	0.0753	0.0123	-0.4240	9.1606	0.1692	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4258	1.4662	1.3417	1.4241	1.3857	1.4477	1.4885
20 lags	1.4493	1.4670	1.4182	1.4404	1.4339	1.4733	1.4871
100 lags	1.4219	1.4351	1.3679	1.4733	1.3807	1.4434	1.4467

	5000-6000	6000-6546
5 lags	1.4682	1.4647
20 lags	1.4725	1.4100
100 lags	1.4059	1.5479

Index development and "cumulative" fractal dimension



The Singapore market efficiency improved during the analysed period, but stronger trends start to appear recently; the fractal dimension was highest in 2004-20007, and then the fractal dimension started to decrease again.

C.25 TSE

Toronto Stock Exchange TSE 300 Index (Canada)
7420 observations (23.4.1984 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.1200	0.0937	0.0099	-0.9272	17.5730	0.0454	> 0.05

Statistics, years 2000-2013

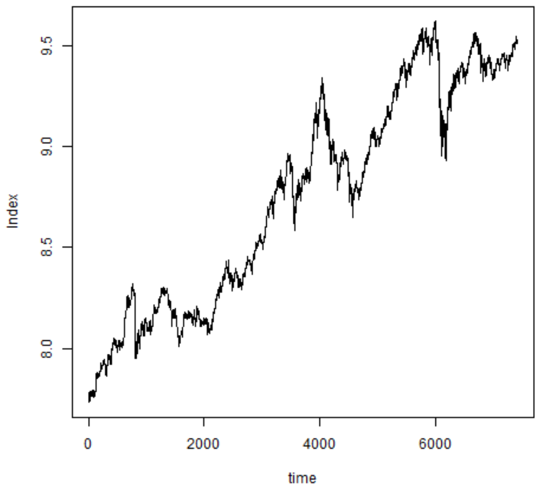
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.0979	0.0937	0.0119	-0.6422	11.7667	0.0653	> 0.05

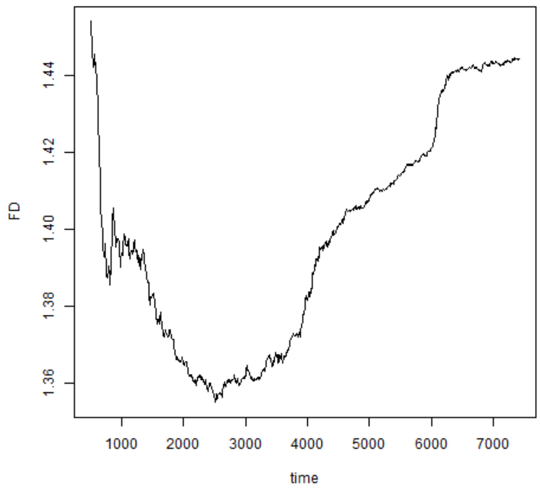
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4442	1.4996	1.3940	1.3250	1.3553	1.4156	1.4839
20 lags	1.4652	1.5106	1.4191	1.3914	1.3996	1.4515	1.4687
100 lags	1.4432	1.4658	1.3891	1.4438	1.4111	1.4164	1.4264

	5000-6000	6000-7420
5 lags	1.4850	1.5133
20 lags	1.4759	1.5471
100 lags	1.4990	1.4664

Index development and "cumulative" fractal dimension





Fractal dimension of TSE index behaves similarly as the one of BEL20 index, so the development of market efficiency in Canada and Belgium has been comparable.

C.26 TWSE

Taiwan Stock Exchange Weighted Index (Taiwan)
4082 observations (2.7.1997 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.0994	0.0852	0.0153	-0.1605	5.7030	0.0990	> 0.05

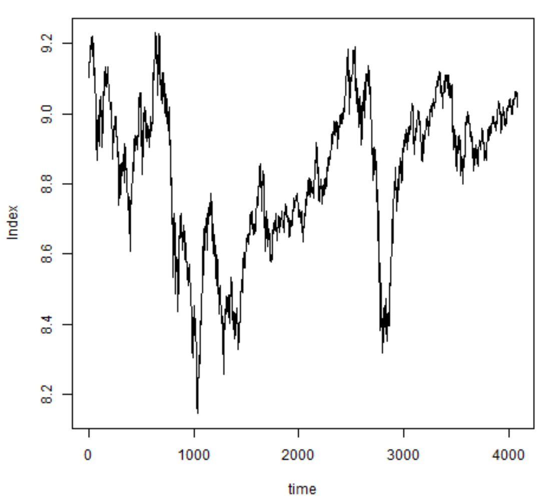
Statistics, years 2000-2013

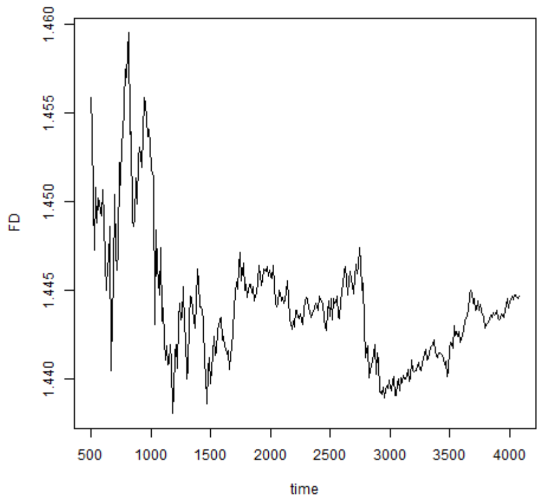
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0000	-0.0994	0.0652	0.0150	-0.2380	5.7584	0.1416	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4082
5 lags	1.4447	1.4448	1.4518	1.4386	1.4244	1.4680
20 lags	1.4680	1.4731	1.4655	1.4559	1.4585	1.4964
100 lags	1.4590	1.4502	1.4749	1.4672	1.3778	1.5687

Index development and "cumulative" fractal dimension





Based on the fractal dimension values, there can be found some relatively strong local persistences on the Taiwan market. Fractal dimension was decreasing in 1997-2008 with a rapid drop in August 2008. Recently, fractal dimension has increased and reached 1.468 for 2009-2013 period.

C.27 XAO

All Ordinaries Index (Australia)
7467 observations (3.8.1984 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.2871	0.0607	0.0100	-3.7845	99.6413	0.1437	> 0.05

Statistics, years 2000-2013

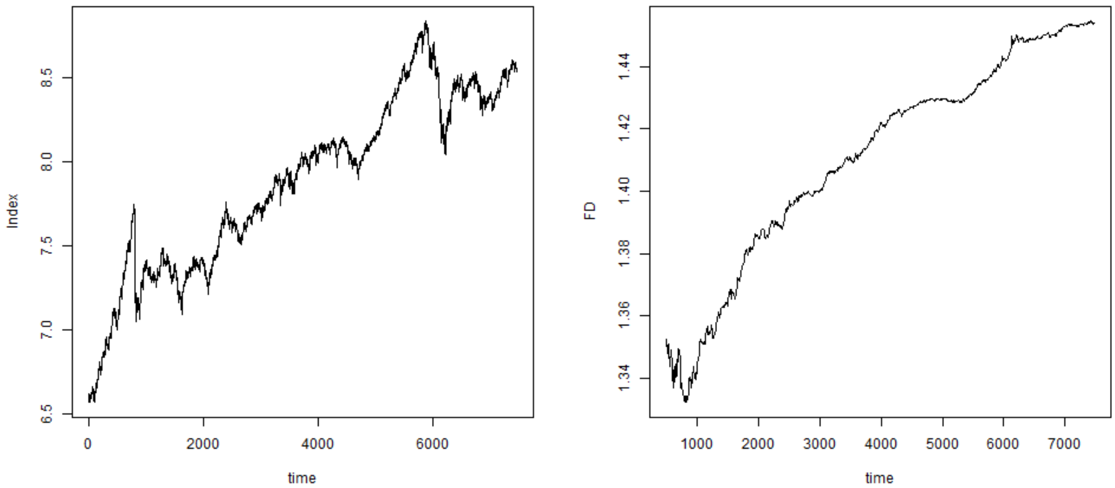
Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0002	-0.0855	0.0536	0.0100	-0.6056	9.4036	0.0868	> 0.05

Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4538	1.4931	1.3429	1.4388	1.4349	1.4887	1.4735
20 lags	1.4515	1.4896	1.3271	1.4297	1.4769	1.4975	1.4714
100 lags	1.4356	1.4641	1.2902	1.4787	1.4228	1.5360	1.5188

	5000-6000	6000-7420
5 lags	1.5069	1.4885
20 lags	1.4724	1.4985
100 lags	1.4065	1.4600

Index development and "cumulative" fractal dimension



XAO index is a typical example of index with low initial fractal dimension that gradually increased to values close to 1.5. It can be compared to IBOSP.

C.28 SPX

Standard & Poor's 500 Index (USA)

16128 observations (3.1.1950 - 6.2.2014)

Statistics, all observations

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0003	-0.2290	0.1096	0.0098	-1.0294	30.6599	0.0806	> 0.05

Statistics, years 2000-2013

Mean	Min	Max	SD	Skewness	Kurtosis	KPSS	p-value
0.0001	-0.0947	0.1096	0.0132	-0.1753	10.7004	0.2634	> 0.05

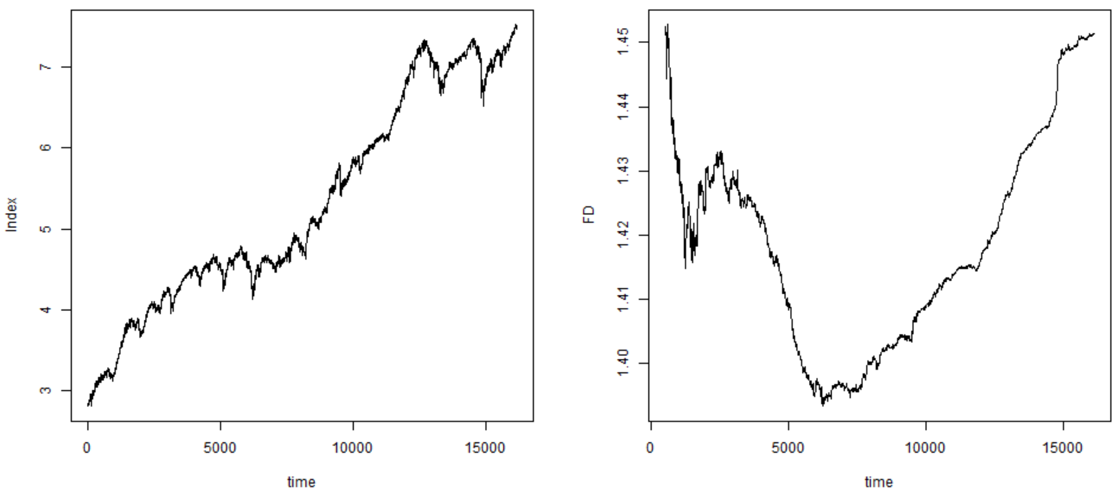
Fractal dimension

	all obs.	y. 00-13	0-1000	1000-2000	2000-3000	3000-4000	4000-5000
5 lags	1.4514	1.5314	1.4319	1.4276	1.4264	1.3966	1.3580
20 lags	1.4824	1.5333	1.4365	1.4287	1.4287	1.4026	1.3980
100 lags	1.4714	1.5019	1.5212	1.4047	1.4077	1.3767	1.4944

	5000-6000	6000-7000	7000-8000	8000-9000	9000-10000	10000-11000	11000-12000
5 lags	1.3488	1.3960	1.4190	1.4316	1.4401	1.4682	1.4451
20 lags	1.4364	1.4851	1.4576	1.5017	1.4702	1.5363	1.4688
100 lags	1.4628	1.4670	1.5338	1.4624	1.4207	1.5192	1.3835

	12000-13000	13000-14000	14000-15000	15000-16128
5 lags	1.5075	1.5249	1.5706	1.4868
20 lags	1.5510	1.5331	1.5436	1.5171
100 lags	1.5545	1.5199	1.4671	1.4924

Index development and "cumulative" fractal dimension



The SPX index is the one with most observations stretching to 1950. The market efficiency of the market was relatively low until 1997. Then it increased rapidly with fractal dimension reaching the value 1.5; between 2002 and 2009, the fractal dimension increased above 1.5 indicating some negative autocorrelation in revenues. Recently, the fractal dimension has decreased to the 1.487 value for 2010-2013 period.

Appendix D

R project code

D.1 Random walk

The first part of random walk is always simulated from 0; other random walk parts use the last observation of inefficient section as the starting point.

The first part:

```
time=3000
startingpoint=0
x=c(startingpoint, rnorm(time))
randomwalk=cumsum(x)
```

The second part:

```
n=6500
x2=c(z[3500],rnorm(n))
rw3=cumsum(x2)
z[3500:10000]=rw3
```

D.2 Martingale

No package for simulation of martingale process exists, Robert a professor of Statistics writing blog XI'AN'S OG proposes code presented in Appendices for simulation of martingales; the code is based on paper by Shafer et al. (2011) which links martingales and Bayes factor.

```
x=sample(0:1,10^4,rep=TRUE,prob=c(1-theta,theta))
s=cumsum(x)
```

```
ma=pbinom(s,1:10^4,.5,log.p=TRUE)-pbinom(s-1,1:10^4,.5,...
...log.p=TRUE,lower.tail=FALSE)
```

The problem of this code is that for θ different from 1/2 it produces a sequence going to infinity ,and it returns a time series with maximum and minimum observed in the first steps for $\theta = 1/2$. For this reason, we omit first 2000 observations.

D.3 ARFIMA

```
arfima.4=fracdiff.sim(501, ar=0.4, ma=-0.3, d=0.25)
arfima.3=arfima.4\$series$
arfima.2=cumsum(arfima.3)$

rw2=(1:750)
rw2[1:750]=rw[2251:3000]
drw=max(rw2)-min(rw2)

darfima=max(arfima.2)-min(arfima.2)

a=darfima/drw
arfima.1=arfima.2/a
arfima=arfima.1-arfima.1[1]+rw[3000]

z[1:3000]=rw
z[3000:3500]=arfima
```