Charles University in Prague

Faculty of Social Sciences

Institute of Economic Studies

BACHELOR THESIS

Prospect Theory and Inertia in a Heterogeneous Agent Model

Author: **Jan Polách**

Supervisor: **PhDr. Jiří Kukačka**

Academic Year: **2014/2015**

Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature. The thesis was not used to obtain an academic degree before.
The author grants to Charles University permission to reproduce and to distribute copies of this thesis document in whole or in part.

Prague, May 12, 2015

Signature

Acknowledgments

Foremost, I would like to express my sincere gratitude to PhDr. Jiří Kukačka, the supervisor of this thesis, for his continuous support, patience, motivation, and enthusiasm. The supervisor's guidance helped me tremendously during the time of writing of this thesis.

Abstract

Using the Heterogeneous Agent Model framework, we develop and incorporate an extension based on Prospect Theory into a popular agent-based asset pricing model. The extension covers the phenomenon of *loss aversion* manifested mainly in risk aversion and asymmetric treatment of gains and losses. Additionally, we explore a special case of the model's intrinsic dynamics termed Asynchronous Updating that affects agents' selection of trading strategies and mimics the *investor inertia* effect. Using Monte Carlo methods, we investigate behavior and statistical properties of the extended versions of the model and assess relevance of the extensions with respect to empirical data and stylized facts of financial time series. We find that the Prospect Theory extension is feasible, that it keeps the essential underlying mechanics of the model intact, and that it changes the model's dynamics considerably. Moreover, the extension shifts the model closer to the behavior of real-world stock markets. Contrarily, the Asynchronous Updating feature does not produce statistically different empirical distributions of most of the main variables. However, it dramatically increases chances of fundamentalists to survive in the market even when changes to more profitable strategies are increasingly facile.

Bibliographic record

POLÁCH, Jan (2015): "Prospect Theory and Inertia in a Heterogeneous Agent Model." *Bachelor thesis.* Institute of Economic Studies, Faculty of Social Sciences, Charles University in Prague. 74 pages; 23481 words. Supervisor: PhDr. Jiří Kukačka.

JEL Classification	C1, C61, D84, G12
Keywords	Heterogeneous agent model, Prospect Theory, Behavioral Finance, Stylized facts

Author's e-mail `polach.honza@gmail.com`
Supervisor's e-mail `jiri.kukacka@gmail.com`

The thesis was typeset with LaTeX 2_ε using the template of Tomáš Havránek.

Abstrakt

V bakalářské práci je vytvořeno rozšíření populárního heterogenního agentního modelu oceňování kapitálových aktiv založené na Prospect Theory. Toto rozšíření zahrnuje zejména averzi ke ztrátám, která způsobuje averzi k riziku a asymetrické reakce vůči ziskům a ztrátám. Součástí práce je také posouzení speciálního případu výběru tržních strategií agenty, který ovlivňuje celkovou dynamiku modelu, označovaného jako Asynchronní obnovování. Toto rozšíření má za cíl modelovat setrvačnost investorů (investor inertia). Následně je pomocí metod Monte Carlo zkoumáno chování modelu s těmito rozšířeními—zejména pak statistické vlastnosti výstupních dat—a posouzena relevantnost těchto rozšíření ve vztahu k empirickým datům a stylizovaným faktům. Výsledkem je zjištění, že rozšíření modelu založené na Prospect Theory je proveditelné a umožňuje zachování původní jednoduchosti modelu, aniž by došlo ke ztrátě jeho důležitých charakteristik. Rozšíření dále mění kvalitativní chování modelu, produkuje statisticky odlišná rozdělení nejdůležitějších proměnných a posouvá model blíže k reálné dynamice trhů. Asynchronní obnovování na druhou stranu ve většině případů nevytváří statisticky odlišná empirická rozdělení proměnných modelu, avšak znatelně ovlivňuje poměry tržních strategií, zejména pak zvyšuje tržní podíl fundamentalistů a jejich šance udržet se na trhu i za situací, ve kterých je přechod k ziskovějším strategiím stále jednodušší.

Bibliografický záznam

Polách, Jan (2015): "Heterogenní agentní model s Prospect Theory a setrvačností." *Bakalářská práce.* Institut ekonomických studií, Fakulta sociálních věd, Univerzita Karlova v Praze. 74 stran; 23481 slov. Vedoucí práce: PhDr. Jiří Kukačka.

Klasifikace JEL	C1, C61, D84, G12
Klíčová slova	Heterogenní agentní model, Prospect Theory, Behaviorální finance, Stylizovaná fakta

E-mail autora	`polach.honza@gmail.com`
E-mail vedoucího práce	`jiri.kukacka@gmail.com`

Tato práce byla vysázena za použití LaTeX 2_ε dle šablony Tomáše Havránka.

Contents

List of Tables

List of Figures

Acronyms

AAM Autonomous Agent Model

ABM Agent-based Model

ABS Adaptive Belief System

ACE Agent-based Computational Economics

ACF Agent-based Computational Finance

AIC Akaike Information Criterion

ARMA Autoregressive Moving Average

AU Asynchronous Updating

BIC Bayesian Information Criterion

CARA Constant Absolute Risk Aversion

CBS Continuous Belief System

CRRA Constant Relative Risk Aversion

EMH Efficient Market Hypothesis

GA Genetic Algorithm

GARCH Generalized Autoregressive Conditional Heteroskedasticity

HAM Heterogeneous Agent Model

JB Jarque–Berra

LTL Large Type Limit

NYSE New York Stock Exchange

PDF Probability Density Function

PT Prospect Theory

RE Rational Expectations

SFI-ASM Santa Fe Institute Artificial Stock Market

Bachelor Thesis Proposal

Author	Jan Polách
Supervisor	PhDr. Jiří Kukačka
Proposed topic	Prospect Theory and Inertia in a Heterogeneous Agent Model

Topic characteristics The study of behavioral aspects of decisions of individual economic agents has aroused great interest in last decades. One of the most distinguished theories in this field is undoubtedly the Prospect Theory created by the economists Kahneman and Tversky (1979) which is, in spite of its simplicity and comprehensibility, widely accepted. The aim of this thesis is to extend a heterogeneous agent model (HAM) by using Prospect–Theoretical features and, with its help, try to better explain the roots of some of the classical financial stylized facts, e.g. fat tails of distributions of returns, weak autocorrelation in returns or volatility clustering. The HAM in this work will be based on the original model developed by the economists Brock and Hommes (1998) and besides the elements of Prospect Theory, a modification with asynchronous updating will be proposed. At the end of the thesis, both alternatives will be assessed on the basis of data generated by the respective models and compared to the original model by Brock and Hommes (1998).

Charakteristika práce v češtině Studium behaviorálních aspektů v rozhodování ekonomických agentů se v posledních desetiletích těší zvýšenému zájmu. Jednou z nejznámějších teorií v této oblasti je nepochybně tzv. Prospect Theory vytvořená ekonomy Kahnemanem a Tverským (1979), jež se i přes svou jednoduchost stala od svého vzniku v oboru široce uznávanou. Cílem této práce je rozšířit heterogenní agentní model (HAM) o prvky vycházející z Prospect Theory a pokusit se na něm lépe vysvětlit původ některých klasických stylizovaných fakt, např. takzvaných tlustých konců rozdělení výnosů, slabé autokorelace výnosů či shlukování volatility. HAM v této práci bude částečně vycházet z původního modelu vyvinutého ekonomy Brockem a Hommesem (1998) a kromě elementů z Prospect Theory bude doplněna i alternativa obsahující navíc prvek asynchronního obnovování. V závěru práce budou obě varianty vyhodnoceny na základě dat, která z obou modelů vyplynou, a porovnány s originálním modelem od Brocka a Hommese (1998).

Hypotheses

1. The original HAM by Brock and Hommes (1998) can be consistently extended by Prospect-Theoretical features.
2. Incorporation of Prospect Theory helps better explain some classical financial stylized facts.
3. Asynchronous updating modification changes the dynamics of the model considerably.
4. Asynchronous updating feature shifts the model closer to the real market dynamics.
5. The models with proposed behavioral features depict the empirical findings more accurately than models without them do.

Outline

1. Introduction
2. Theoretical background
3. The Model with Prospect–Theoretical features
4. Asynchronous updating
5. Comparison and assessment
6. Conclusion

Core bibliography

1. Barberis, N., M. Huang & T. Santos (2001): "Prospect Theory and Asset Prices." *The Quarterly Journal of Economics* **116**: pp. 1–53.
2. Brock, W. A. & C. H. Hommes (1998): "Heterogeneous beliefs and routes to chaos in a simple asset pricing model." *Journal of Economic Dynamics and Control* **22**: pp. 1235–1274.
3. Ghoulmie, F., R. Cont & J. P. Nadal (2005): "Heterogeneity and feedback in an agent-based market model." *Journal of Physics: Condensed Matter* **17**: S1259.

4. Hommes, C. H. (2006): *Handbook of Computational Economics, Agent-Based Computational Economics*, chapter Heterogeneous Agent Models in Economics and Finance: pp. 1109–1186. Elsevier Science B.V.

5. Hommes, C. H. (2011): "The heterogeneous expectations hypothesis: Some evidence from the lab." *Journal of Economic Dynamics and Control* **35**: pp. 1–24.

6. Kahneman, D. & A. Tversky (1979): "Prospect Theory: An Analysis of Decision under Risk." *Econometrica, The Econometric Society* **47**: pp. 263–291.

7. Shimokawa, T., K. Suzuki & T. Misawa (2007): "An agent–based approach to financial stylized facts." *Physica A: Statistical Mechanics and its Applications* **379**: pp. 207–225.

Author Supervisor

Chapter 1

Introduction

This thesis introduces the phenomena of *loss aversion* and *gain–loss asymmetry* into the popular Brock & Hommes (1998) asset pricing model and explores relevance and impacts of this extension. Our work is based on findings of the iconic Prospect Theory of Kahneman & Tversky (1979) which describes the way people choose between probabilistic alternatives that involve risk and is per se a critique of other, rather prescriptive decision-making economic theories. Already in 1979, Kahneman & Tversky found that the actual behavior of human beings might be very dissimilar to what major economic theories assumed, namely in terms of risk and attitude towards losses. According to Prospect Theory, people decide in terms of gains and losses rather than of the final outcome—the extension that we develop in the thesis therefore aims to account for these empirically observed irrationalities. Throughout the years, Prospect Theory has become one of the most influential works that merged psychology with economics. As Belsky & Gilovich (2010, p. 52) aptly remark, *"If Richard Thaler's concept of mental accounting is one of two pillars upon which the whole of behavioral economics rests, then Prospect Theory is the other."* The Kahneman & Tversky's (1979) paper is the most cited paper ever to appear in Econometrica (Chang *et al.*, 2011, p. 30).

The primary objective of this thesis is thus to extend the original model with features of Prospect Theory and, at the same time, keep the intrinsic mechanics intact in order to preserve the stylized, simple nature of the model. Additionally, we provide an in-depth analysis of a special case of the model's intrinsic dynamics termed Asynchronous Updating, a technical feature which causes inactivity of a certain number of traders. The concept of Asynchronous Updating aims to mimic the phenomenon of *investor inertia* present in real-world markets. This empirically observed behavior of investors is manifested in wrong investment decisions, especially in incorrect 'timing' of investment actions; for instance, holding onto a losing stock too long and selling a winning stock too soon—this behavior of some investors is called the disposition effect. Secondary focus of the thesis is hence exploration of the model's behavior under Asynchronous Updating. The two main topics of the thesis—Prospect Theory and Asynchronous Updating—are related: the disposition effect based on the Prospect Theory's value function might in reality be one of the important sources of the investor inertia phenomenon. Our analysis consists in using Monte Carlo methods to investigate behavior and statistical properties of the extended versions of the model and assess relevance of the extensions with respect to empirical data and stylized facts of financial time series.

The thesis is structured as follows: following the Introduction, Chapter 2 summarizes current literature on computational economics, agent-based modeling, and Prospect Theory and Chapter 3 describes mathematical structure and underlying mechanics of the original Brock & Hommes (1998) model. Chapter 4 and Chapter 5 represent the core of the thesis: Chapter 4 develops the behavioral extension based on Prospect Theory and introduces the Asynchronous Updating feature while Chapter 5 describes the numerical simulations using Monte Carlo methods and illustrates the main differences related to the proposed extensions along with the results of statistical tests and their implications. Chapter 6 highlights main results of the simulations and compares the model's behavior with empirical, real-world data. Additionally, introductory hypotheses specified in the Thesis proposal are assessed and summarized in this chapter. The Conclusion highlights the most important results of the analysis and hence concludes the thesis.

Chapter 2

Literature Review

In the following chapter, important milestones in the history of heterogeneous agent modeling (and, more generally, of agent-based modeling) and Prospect Theory are presented to the interested reader. First section of this chapter is focused on the former, second section offers some insight into the most essential proceeds and findings of the latter. Encompassing a vast amount of literature, both fields are described using chiefly the works of the most influential authors; within the scope of this thesis, the current writer cannot hope to cover all the available literature.

2.1 Agent-based Modeling

Generally, agent-based modeling deals with construction of models which consist of several *autonomous* agents who interact with one another and thus create micro-level patterns. Such agents might represent not only individuals, but also various collective entities whose interactions the model maker wishes to study; these interactions subsequently cause emergence of certain macro-level phenomena which are of ultimate interest but which cannot be deduced simply by *aggregating* the properties of the agents (Axelrod & Tesfatsion, 2006). As Page (2008) aptly remarks, *"Agent-based models allow us to consider richer environments that include micro features with greater fidelity than do existing techniques."* Agent-based Model (ABM) approaches are used in several scientific disciplines, for instance in economics, ecology, demography, and traffic planning to mention but a few. Gustafsson & Sternad (2010) list four fundamental properties of *micro-level* ABMs:

1. The *non-negative (integer) quantity* of the entities.

2. The *continuous nature of time*, which should at least be sufficiently well approximated in the model.

3. The *structural and temporal relations* creating the dynamics of the system.

4. The *irregularly occurring events* of the system which have to be characterized by an appropriate probabilistic representation in the model as they cannot be described in detail.

To illustrate the considerably wide scope of possible usage of ABMs, the present writer refers the interested reader to the work of Joshua M. Epstein, a pioneer in agent-based computational modeling of biomedical and social phenomena. Epstein (2002) developed an ABM of civil violence—in this model, a central authority tries to eradicate decentralized riots and communal violence between rival ethnic groups—visual representation can be found online.

ABMs described and used in this thesis have naturally economic or financial nature. The fields of economics and finance which try to make use of such models are called Agent-based Computational Economics (ACE) and Agent-based Computational Finance (ACF), respectively. One of the first research efforts to use the pure ACE framework can be found in Marks (1992) in which the author makes use of a *genetic algorithm* to investigate behavior of firms in an oligopolistic market and address the issue of market self-organization. This work is especially contributive since the agents are assumed to be 'only' boundedly rational. As a clarifying note on terminology, ACE is de facto agent-based modeling specialized to computational economics purposes. A comprehensible survey on methodology and aspects of ACE provides Tesfatsion (2006) who highlights four primary strands of the research in the field:

1. *Empirical understanding* investigates why certain global regularities have evolved and persisted, in spite of absence of centralized planning and control that could support such regularities.

2. *Normative understanding* attempts to use ABMs in a process of discovery of good economic designs.

3. *Qualitative insight* puts emphasis on changes in dynamical behavior of economic systems with modified initial conditions.

4. *Methodological advancement* tries to provide scholars in the field with superior methods and tools they need to make their work as rigorous as possible.

One of the most important stimuli which induced development of ABMs in economics was certainly an erosion of trust in the Efficient Market Hypothesis (EMH)—the EMH asserts, in Eugene Fama's words, that *"...security prices at any time 'fully reflect' all available information ..."* (Fama, 1970, p. 383)—and in Rational Expectations (RE) theory in the late 1970s and early 1980s which was due to increased focus on study of several *stylized empirical facts*—according to Cont (2001, p. 224), *"The seemingly random variations of asset prices do share some quite nontrivial statistical properties. Such properties, common across a wide range of instruments, markets and time periods are called stylized empirical facts."* The most essential difference between natural sciences and economics is arguably the fact that decisions of economic agents are determined by their *expectations* of the future and contingent on them—hence, the study of how these *beliefs* are formed plays a vital part of any economic theory.

Several scholars have published papers which confronted the EMH with empirical data mainly from the perspective of non-normal returns,[1] systematic deviations of asset prices from their fundamental value, and presumably excessive amount of stock price volatility—it was impossible to attribute these phenomena to the EMH or explain them within the RE framework. Offering an insightful survey on the volatility issue at that time, West (1988) summarizes and interprets literature related to this field. The author finds out that neither rational bubbles nor any standard models for expected returns adequately explain stock price volatility and emphasizes the necessity to introduce alternative models which would offer better explanation of the apparent contradiction between the EMH, RE theory, and empirical findings.

As for the normality of logarithmic returns assumed by the EMH, the interested reader is referred to Mandelbrot (1963) who offers a detailed theoretical and empirical discussion about the topic. For illustrative purposes, Table 2.1 summarizes daily logarithmic returns of the S&P 500 stock market index for four different periods in the past century. The Jarque–Berra (JB) ALM test for normality of distribution was used to test whether the objection of non-normality is relevant (or, more precisely, *was*—at the respective time periods).

[1] According to Ehrentreich (2007, p. 56), at the time when the foundations of the EMH were laid, logarithmic asset returns were thought to be normally distributed.

Clearly, the JB test strongly rejects the null hypothesis of normally distributed returns in all six periods.

Table 2.1: S&P 500 Summary statistics and test statistic of the JB test.

Period	Mean	St. dev.	Min.	Max.	Skew.	Kurt.	JB
1960–1965	0.000	0.007	-0.067	0.046	-0.536	15.86	8832
1970–1975	0.000	0.009	-0.037	0.050	0.384	5.532	372
1980–1985	0.000	0.009	-0.039	0.048	0.307	4.317	112
1990–1995	0.000	0.007	-0.037	0.037	-0.014	5.182	255
2000–2005	0.000	0.012	-0.058	0.057	0.194	4.821	185
2010–2015	0.001	0.010	-0.066	0.047	-0.380	7.449	1082

In Figure A.1 of the Appendix A, an estimated Probability Density Function (PDF) of S&P 500 returns, used in Table 2.1, is plotted along with the theoretical PDF of normally distributed random variable with the same mean and variance as the S&P 500 returns.

2.2 Agents' design and taxonomy

It appears that there is no generally accepted consensus on financial ABMs' taxonomy. LeBaron (2006) provides categorization using historical approach by presenting gradually improved models as time passes, while Hommes (2006) uses only narrower scope and focuses chiefly on Adaptive Belief System (ABS) models.[2] The Brock & Hommes (1998) model is of central interest to this thesis and hence for the purposes of ABMs' taxonomy, we will use a different, more general segmentation offered by Chen *et al.* (2012). In this work, using *complexity* as the main selection criterion, the authors categorize economic ABMs chiefly by *number of trading strategies* and level of agents' *autonomy*—two cardinal groups are ultimately distinguished: *N-type models* and *Autonomous Agent Models*. Furthermore, the authors specify three main elements which contribute to the overall complexity: level of heterogeneity, learning, and interactions.

[2] Here the term ABS corresponds to the behavioral strategy switching system introduced in Brock & Hommes (1997; 1998) which uses simple, stylized models of market with heterogeneous beliefs. Heterogeneity as such is naturally also present in the models described in Subsection 2.2.1 Subsection 2.2.2.

2.2.1 N-type models

N-type models represent the first of the two main groups of economic ABMs. The most essential property of this group is the fact that, as opposed to Autonomous Agent Models, all the types and rules of agents are specified at the beginning of the design—hence, the heterogeneity of agents consists in switching of among predefined strategies, not in development of new trading rules.

The simplest category of N-type models (in terms of the three aforementioned main elements) that Chen *et al.* (2012) give is *two- and three-type designs.* As the name suggest, this segment offers the most rudimentary version of heterogeneity—two-type models are characterized by presence of mere two traders or trading strategies, three-type models offer an additional strategy. Usually, a precise behavioral rule is associated with each type of strategy; subsequently, in every time period, all agents have to choose one of these types. For every agent, the choice is then evaluated in the next time period and the agent chooses a strategy according to the result of this evaluation process, which is commonly based on a binary choice model. Hence, switching among strategies is possible and desirable and it is the source of heterogeneity in the model.

Traditionally, one of the strategy types is called a *fundamentalist.* Fundamentalists are inclined to think that the price of an asset is determined namely by underlying economic fundamentals and to predict that the asset price will move in the direction of its fundamental value. In the two- and three-type models, the second and third strategy is typically called a *chartist*, a *technical analyst*, or a *noise trader* (Hommes, 2013). Agents of this type are of the opinion that prices are not determined by fundamentals only, but that they can be predicted by unsophisticated technical trading rules based upon observed patterns in past prices, such as trends or cycles. In practice, a chartist is modeled to be either a *trend follower* or a *contrarian*—the former tries to make use of a persistent trend while the latter intentionally trades against the ongoing market sentiment. To illustrate the difference between fundamentalists and chartists, Chen *et al.* (2012) specify simple forecasting rules for fundamentalists and chartists regarding future price of an asset as follows: led to be either a *trend follower* or a *contrarian*—the former tries to make use of a persistent trend while the latter intentionally trades against the ongoing market sentiment. To illustrate the difference between fundamentalists and chartists, Chen *et al.* (2012) specify simple forecasting rules for fundamentalists and chartists regarding future price of an asset as follows:

$$\mathbf{E}_{f,t}\left(p_{t+1}\right) = p_t + \alpha_f \cdot \left(p_t^f - p_t\right), \quad 0 \leqslant \alpha_f \leqslant 1 \tag{2.1}$$

$$\mathbf{E}_{c,t}\left(p_{t+1}\right) = p_t + \alpha_c \cdot \left(p_t - p_{t-1}\right), \tag{2.2}$$

where the former corresponds to fundamentalists and the latter to chartists (with $0 \leqslant \alpha_c$ to trend followers and $\alpha_c \leqslant 0$ to contrarians). α_f is a mean-reverting coefficient—fundamentalists assume that any price $p_t \neq p_t^f$ of an asset will sooner or later converge to the fundamental price p_t^f. On the other hand, chartists are characterized by an extrapolating coefficient α_c which they use to predict the future price with the assistance of past and present prices. Magnitudes of both of the coefficients naturally measure the intensity of mean-reversion and extrapolation, respectively.

To capture the real-world complexity of financial markets, it is often necessary to modify the basic two- and three-type design in an appropriate way. Most frequent generalizations of this category rely on inclusion of *memory* to the model; the aim of this effort is to mimic certain psychological features known to have been present in human behavior—these are chiefly phenomena called 'law of small numbers' and 'representativeness'. These terms mean, in layman's terms, that people hope to infer generally valid characteristics even if they only have a small sample of observations at hand (Kahneman & Tversky, 1972). This relates to financial markets by the fact that some investors incorrectly consider short-term above-average profits to be permanent also in the long run. Another feature that model makers often include in the model is more sophisticated *adaptive behavior*. Such an element can be incorporated into the design by modeling e.g. potential profits from a strategy rather than realized profits, as is often the case in simpler models. The interested reader is referred to Chen *et al.* (2012, pp. 192–194) for a more detailed discussion about two- and three-type models generalizations. In the same paper, one can find several other appealing N-type models, e.g. the Large Type Limit (LTL) or the Continuous Belief System (CBS). For the LTL, see e.g. Brock & Hommes (2001) or Brock *et al.* (2005), for the CBS e.g. Diks & Weide (2005). ling e.g. potential profits from a strategy rather than realized profits, as is often the case in simpler models. The interested reader is referred to Chen *et al.* (2012, pp. 192–194) for a more detailed discussion about two- and three-type models generalizations. In the same paper, one can find several other appealing N-type models, e.g. the LTL or the CBS. For the LTL, see e.g. Brock & Hommes (2001)

or Brock *et al.* (2005), for the CBS e.g. Diks & Weide (2005).

Last part of this subsection introduces two of the three most fundamental N-type designs which the subsequently developed ABMs more or less strictly follow: so-called *Ant model* and *Lux's model*. The third major scheme, ABS model developed by Brock & Hommes (1997) and Brock & Hommes (1998), is described in more detail in Chapter 3.

Kirman's 'Ant model' Kirman (1993) developed a model which aimed to explain apparently 'strange' behavior of ant colonies which the author fittingly compares to similarly irrational (at first glance) examples of human conduct. One such peculiarity might be choosing a restaurant. Becker (1991) gives an example of two restaurants with similar food, prices, service, and other amenities which are located just across the street. Yet, one of the restaurants is always full while the other always empty. Kirman (1993) observed a similar behavioral pattern in ant colonies—in spite of being offered two identical food sources, the ants would at first focus on only one of those but, after some time, *"... the ants switched their attention to the source they had previously neglected."* (Kirman, 1993, p. 137).

The 'Ant model' differs from the general setting given in the second paragraph of Subsection 2.2.1 by the switching mechanism—in this case, the behavior of agents is driven by a herding mechanism rather than financial success; behavior of the majority is the main 'recruitment' determinant. The author characterizes the switching potential of an individual by two parameters—*probability of self-conversion* and *probability of being converted*. The former refers to a chance of the individual of selecting the other 'strategy' without being influenced by other agents, the latter to a chance of the individual of being persuaded to select the strategy of a different agent with whom the individual is randomly matched. In terms of financial markets, the self-conversion might be viewed as a reaction of a trader to arrival of exogenous news while the persuasion could be considered a replacement of one trader by another one who share a different opinion than the original.

The development of the system is described by a simple Markov chain. Suppose there are two sources of food, say black and white, and N ants—this translates to economic terms as existence of two trading strategies and N distinct traders. Then the state of the system at every moment is described by k, a number of ants feeding at the black source, and the system evolves as follows: two ants meet at random and the first one is converted to the belief of the other one with probability $1-\delta$. The system, starting at state k, develops either to

$$k+1 \quad \text{with prob. } p_1 = \left(1-\tfrac{k}{N}\right)\cdot\left(\varepsilon + (1-\delta)\cdot\tfrac{k}{N-1}\right) \tag{2.3}$$

or to

$$k-1 \quad \text{with prob. } p_2 = \tfrac{k}{N}\cdot\left(\varepsilon + (1-\delta)\cdot\tfrac{N-k}{N-1}\right), \tag{2.4}$$

where ε is the self-conversion probability. The system might also remain unchanged (i.e. at state k) with probability $1-p_1-p_2$.

The 'Ant model' is relevant for economics and finance as it tries to explain the herding behavior—a well documented process supported by numerous empirical findings (see e.g. Devenow & Welch, 1996). One of the first authors to point out this tendency was Keynes (1936). In this work, Keynes suggested that the stock market was similar to a beauty contest in which the judges picked who they thought other judges would select, rather than who they regarded as the most beautiful. Such an observation is consistent with Avery & Zemsky (1998) who define the herding behavior as occurring when individuals trade against their initial beliefs and follow the majority trend in previous trade instead. For a more detailed analysis and specific examples of herding in financial markets, such as bubbles and crashes, the interested reader is referred to Brunnermeier (2001).

Lux's model Lux (1995) introduced a model with a herding element as well. In this model however, rather than by pairwise micro interactions, each trader is influenced by prevailing market sentiment, which is assumed to be represented by an average of all other traders' influence. As opposed to the models described above, Lux's model employs *continuous* time. This characteristic makes the model naturally mathematically very different from the Ant model or the ABS model. Yet, similarly to the Ant model, the total population of traders is segmented into 2 major groups: fundamentalists and noise traders. Lux & Marchesi (2000) consider a total number of N traders who are either fundamentalists or chartists. The latter group of traders is further divided into optimistic and pessimistic—or, as Lux (2008) remarks, bullish and bearish—subgroups. Suppose the number of fundamentalists is n_f, number of chartists n_c, number of optimistic chartists n_+, and number of pessimistic chartists n_-. Hence,

$N = n_f + n_c$ and $n_c = n_+ + n_-$. What follows is a brief analysis of 'contagion' and switching between the two subgroups within the chartist group, as well as between fundamentalist and chartist belief.

The prevailing 'market sentiment' of chartists takes the following form:

$$x = \frac{n_+ - n_-}{n_c}. \tag{2.5}$$

It is useful to note that $x \in \langle -1, 1 \rangle$—$x = 0$ corresponds to a situation of balanced opinions among the noise traders in the market, while the extreme cases $x = -1$ and $x = 1$ to a situation in which pessimism or optimism, respectively, is the sole opinion among them.

It is assumed by Lux (1995; 1998) that all agents have an influence of the same strength—the overall 'aggregate' impact subsequently leads to migration between the two subgroups of optimists and pessimists. These transitions are modeled by Poisson processes with rates p^{+-} and p^{-+} for an individual to switch to the optimistic belief and to the pessimistic belief, respectively.

Lux & Marchesi (2000) use the following setting:

$$\begin{aligned} p^{+-} &= \upsilon_1 \cdot \left(\frac{n_c}{N} \cdot \exp\left(U_1\right) \right) \\ p^{-+} &= \upsilon_1 \cdot \left(\frac{n_c}{N} \cdot \exp\left(-U_1\right) \right), \end{aligned} \tag{2.6}$$

where υ_1 is a parameter which characterizes the general inclination to switching opinions,[3] $U_1 = \alpha_1 x + \alpha_2 \frac{p'}{\upsilon_1}$ a function of x, the prevailing market sentiment, and $p' = dp/dt$, the change in the price of the asset. α_1 is a parameter which captures the effect and significance of herding behavior and α_2 a parameter which measures the importance placed on actual price development by the chartists.

If we denote Δt a certain time interval, then the probability of an agent to switch from one subgroup to the other converges to $\Delta t \cdot p^{+-}$ (or $\Delta t \cdot p^{-+}$) for $\Delta t \to 0$. Provided that the Poisson processes specified in Equation 2.6 are indeed identical for each member of the two subgroups, the transition rates for subgroup occupation numbers can be derived from conditional probabilities $w\left(n_+ + 1, t + \Delta t | n_+, t\right)$ and $w\left(n_- + 1, t + \Delta t | n_-, t\right)$ as fol-

[3] This parameter is somewhat analogical to the *intensity of choice* parameter β used in ABS models by e.g. Brock & Hommes (1998); Hommes (2006; 2013).

lows (Lux, 2008, pp. 40–41):

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{w\left(n_{+}+1,t+\Delta t|n_{+},t\right)}{\Delta t}&\equiv w\left(n_{+}+1|n_{+},t\right)=n_{-}\cdot p^{+-}\\ \lim_{\Delta t\to 0}\frac{w\left(n_{-}+1,t+\Delta t|n_{-},t\right)}{\Delta t}&\equiv w\left(n_{-}+1|n_{-},t\right)=n_{+}\cdot p^{-+}.\end{aligned}\tag{2.7}$$

Furthermore, the agents might also switch between the fundamentalist and chartist belief based on myopic comparison of excess profits from the respective strategies with probability which depends on the profits' difference. Again, the transition probabilities are modeled similarly as in Equation 2.6; we refer the interested reader e.g. to Lux & Marchesi (2000, p. 683) or to Lux (2008) who offer an in-depth survey on the mechanics of the model. To summarize, the herding behavior only matters for switches within the noise traders' group while profit differential is the key determinant for fundamentalists—chartists transitions.

2.2.2 Autonomous Agent Models

As opposed to those based on N-type designs, agents present in an Autonomous Agent Model (AAM) are, as time goes by, able to *discover* and *develop* new strategies. This possibility arguably allows this category of ABMs to resemble the reality more accurately. Having merely the option to select from a class of pre-determined strategies, the agents in N-type designs are severely restricted in the level of autonomy (Chen *et al.*, 2012); AAM's agents are on the other hand allowed to behave more like financial agents of the real world.

Santa Fe Institute Artificial Stock Market (SFI-ASM) At the outset of the 1990s, *"...rational expectations approach to economic theory has been challenged from several quarters, and increasing interest has been shown in an alternative evolutionary economics viewpoint."* (Palmer *et al.*, 1994, p. 264). The SFI-ASM, an example of the evolutionary approach, is the most prominent representative of AAMs.

SFI-ASM uses a lot of trading strategies which are gradually enhanced as (discrete) time passes. It is the emphasis on learning and on development of strategies that is essential for this model and that makes it unique. The structure is built as follows: there are N traders initially endowed with certain amount of money. Each period, the traders allocate their funds between *cash*, which bears a risk-free return r_f, and a *risky stock*, which pays stochastic dividend generated by a mean-reverting autoregressive process $d_{t+1} = \bar{d} + \rho \cdot \left(d_t - \bar{d}\right) + u_{t+1}$ in which $\bar{d}$ is the dividend mean, ρ a strength of the mean-reversion, and u_{t+1}, $u_{t+1} \sim N\left(0, \sigma_u^2\right)$, a random shock. All agents maximize the same Constant Absolute Risk Aversion (CARA) utility function of expected total wealth of the form

$$U\left(W_{i,t+1}\right) = -\exp\left(-\lambda \cdot W_{i,t+1}\right), \tag{2.8}$$

where λ represents the extent of risk aversion, and $W_{i,t+1}$ the expected wealth of ith agent in the subsequent period. Maximization of individual agents' utility is subject to a budget constraint. Specifically,

$$W_{i,t+1} = x_{i,t} \cdot \left(p_{t+1} + d_{t+1}\right) + \left(1 + r_f\right) \cdot \left(W_{i,t} - p_t x_{i,t}\right), \tag{2.9}$$

where $x_{i,t}$ is the amount of stock agent i holds in period t, and p the stock's price. Provided the stock returns are normally distributed,[4] the optimal amount of stock the ith agent desires to hold is then

$$\tilde{x}_{i,t} = \frac{\mathbf{E}_{i,t}\left(p_{t+1} + d_{t+1}\right) - p_t \cdot \left(1 + r_f\right)}{\lambda \cdot \sigma^2_{t,p+d}}, \tag{2.10}$$

where $\mathbf{E}_{i,t}\left(p_{t+1} + d_{t+1}\right)$ is ith agent's expectation formed at time t about the sum of next period's dividend and price of the stock, and $\sigma^2_{t,p+d}$ the variance of the stock's combined dividend and price time series. When all agents decide upon the optimal amount of stock, a market specialist then tries to balance the market by setting a clearing price. In SFI-ASM, the heterogeneity of agents is captured in the term $\mathbf{E}_{i,t}\left(p_{t+1} + d_{t+1}\right)$; while it is true that all agents do have the same utility function specified in Equation 2.8, they differ in derivation of the expectation term. Such forecasts are conducted using *individual* trading rules of the basic form

$$condition\ met \rightarrow derive\ forecast. \tag{2.11}$$

More specifically, each agent is equipped with 100 trading rules—jth rule

[4]Ehrentreich (2007) and Arthur *et al.* (1997) point out that this assumption might not hold in the absence of homogeneous RE equilibrium. See the respective works for details and further discussion.

of ith agent then takes the following form:

$$\text{rule}_{i,j} = \left\{(\text{cond. part}), (\text{predictor}), \Phi_{t,i,j}, \upsilon^2_{t,i,j}\right\}, \tag{2.12}$$

where $\Phi_{t,i,j}$ is a measure of the forecast's fitness, and $\upsilon^2_{t,i,j}$ is the forecast's accuracy. Simply put, the 'condition part' may contain either value 1 or 0—then, in order for the condition to be met, a market descriptor[5] must also contain either 1 or 0, respectively—or value # which means that the rule ignores a particular market descriptor. Usually, condition parts of several rules satisfy the market descriptor. Agents then choose the best rule according to the value of accuracy measure $\upsilon^2_{t,i,j}$ of each rule—this measure captures how well the rule predicts the actually realized sum of price and dividend of the stock. Finally, the term $\mathbf{E}_{i,t}(p_{t+1} + d_{t+1})$ is formed as $a_{t,i,j} \cdot (p_t + d_t) + b_{t,i,j}$ where $a_{t,i,j}$ and $b_{t,i,j}$ are real-valued parameters contained in the 'predictor' part of each rule.

Additionally, agents gradually improve their prediction rules by altering them and possibly replacing the poorly performing ones using a so-called Genetic Algorithm (GA). New trading rules might be formed either by *mutation* or by *crossover*. For details about the GA, we refer the reader to Holland (1975); Goldberg (1989); Eshelman (2000), or Ehrentreich (2007, chap. 4).

Numerical simulations of the SFI-ASM are characterized by two distinct 'modes': in the first one, one can identify periods of lower volatility with prices close to the fundamental price[6] while in the second one, considerable deviations from the fundamental price and excessive volatility characterize the market which is inevitably dominated by chartists.

2.3 Prospect Theory

Created by Kahneman & Tversky (1979), Prospect Theory (PT) is an alternative to expected utility theory as a decision making model under risk. Some of the most fundamental foundations of the expected utility theory are inconsistent with real-world human behavior, *"In particular, people underweight out-*

[5] Market descriptor is a simple true-or-false statement coded either as 1 or 0; such descriptor might be e.g. whether the stock's instantaneous price is greater than its fundamental value.

[6] Essentially, the EMH holds in this first mode.

comes that are merely probable in comparison with outcomes that are obtained with certainty." (Kahneman & Tversky, 1979, p. 263). Moreover, according to PT, people decide in terms of gains and losses rather than of the final outcome; the term 'utility' is replaced by the term 'value' which is associated with the relative changes in wealth.

In the 1979 paper, Kahneman and Tversky—primarily psychologists—asses and criticize the then-mainstream paradigm of expected utility using a questionnaire survey. The respondents, surprisingly, often gave answers which were in sharp contrast to what they were supposed to answer according to the widely accepted theory; presented with a pair of problems, the respondents answered the second problem contradictorily to the first one, although the problems were intrinsically identical. One such example goes as follows (Kahneman & Tversky, 1979, pp. 265–266): there are two problems (1 and 2) and the respondents are, in each problem, asked to select one from two possible 'prospects', **A** or **B**. The term 'prospect' here refers to a gamble—respondents obtain certain sum of money with some probability.

- Problem 1:

$$\begin{aligned} \mathbf{A}: &\quad 2500 \quad \text{with. prob. } 0.33 \\ &\quad 2400 \quad \text{with. prob. } 0.66 \\ &\quad 0 \quad \text{with. prob. } 0.01 \\ \mathbf{B}: &\quad 2400 \quad \text{with. prob. } 1 \end{aligned} \tag{2.13}$$

- Problem 2:

$$\begin{aligned} \mathbf{A}: &\quad 2500 \quad \text{with. prob. } 0.33 \\ &\quad 0 \quad \text{with. prob. } 0.67 \\ \mathbf{B}: &\quad 2400 \quad \text{with. prob. } 0.34 \\ &\quad 0 \quad \text{with. prob. } 0.66 \end{aligned} \tag{2.14}$$

While 82 % of respondents chose prospect **B** in Problem 1, 83 % of them selected prospect **A** in Problem 2. However, letting $u(\bullet)$ be the utility, the former preference implies $u(2400) > 0.33u(2500) + 0.66u(2400)$ or $0.34u(2400) > 0.33u(2500)$, while the latter the opposite thus violating the expected utility theory. Listing several more problems, Kahneman & Tversky (1979) discover the similar contradictory patterns and distinguish three major effects that emerge from them:

Certainty Effect The certainty effect corresponds to the set of problems given in Equation 2.13 and Equation 2.14 and asserts that people tend to *over-*

weight outcomes regarded as certain relative to outcomes which are only probable.

Reflection Effect The reflection effect addresses diametrically different attitudes of people towards gains and losses. While arguably most people are risk-averse in the 'positive domain' (i.e. when choosing among prospects which offer gains), they are mostly risk-seeking in the negative domain. Letting (x, p) denote a prospect that pays x units of cash with probability p and 0 units of cash with probability $1 - p$, then while most people prefer e.g. $(3000, 1)$ to $(4000, 0.8)$, majority of them also prefers $(-4000, 0.8)$ to $(-3000, 1)$ although higher expected value is associated with the other prospect in both cases. In more recent economic literature, the reflection effect is often termed *gain and loss asymmetry.*

Isolation Effect The isolation effect refers to a situation in which alternative prospects share a common component—people often do not take such a component into account and only focus on the *differences* between the alternatives. However, prospects can usually be decomposed in more than one way and ignorance of this fact produces inconsistent preferences.

According to PT, selection process consists of two parts: *editing* and *evaluation.* In the former, the individual conducts a preliminary analysis of the available prospects in order to facilitate the selection, and in the latter, the individual evaluates the edited prospects, assigns a *value* to each of them, and makes the final decision. The interested reader might find details about the editing phase in Kahneman & Tversky (1979, pp. 274–275), here we will present the most essential properties of the evaluation phase.

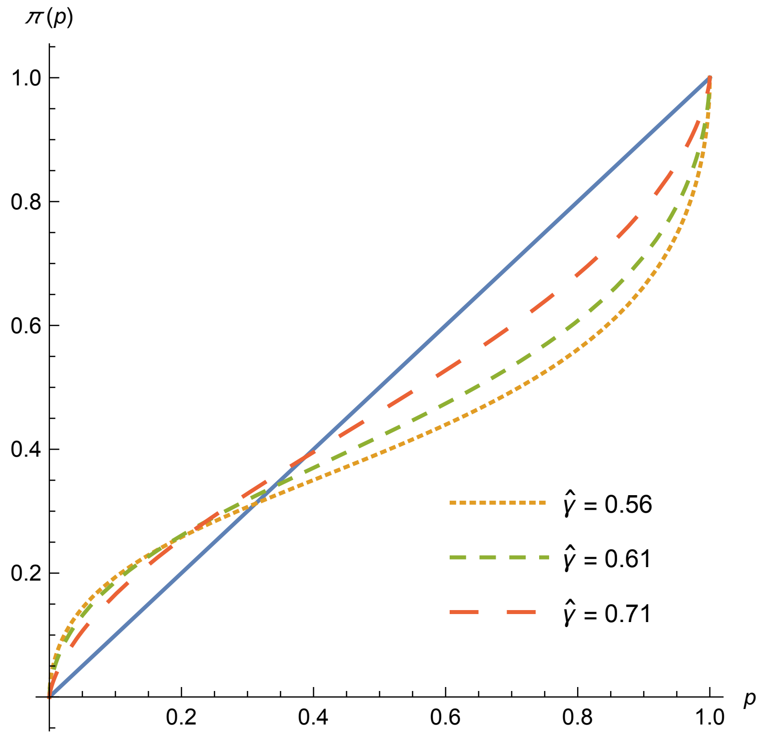
The overall *value* V of an edited prospect is formulated in terms of $\pi(\bullet)$ and $v(\bullet)$. The former is called a *weighting function* and it is a function of probability of the prospect's respective outcomes, the latter is called a *value function* and it assigns a number $v(\bullet)$ to each outcome. Letting $(x, p; y, q)$ denote a prospect which pays x, y, or 0 with probability p, q, and $1 - p - q$, respectively, the basic equation which assigns value to a *regular* prospect[7] is then given as follows:

$$V(x, p; y, q) = \pi(p) \cdot v(x) + \pi(q) \cdot v(y), \tag{2.15}$$

[7]Regular prospect is a prospect such that either $p + q < 1$, $x \geqslant 0 \geqslant y$, or $x \leqslant 0 \leqslant y$. Evaluation of prospects which are not regular follows a different rule—details are provided in Kahneman & Tversky (1979, p. 276).

where it is assumed that $v(0) = 0$, $\pi(0) = 0$, and $\pi(1) = 1$. It is important to note that the weighting function is *not* a probability measure and typical properties of probability need not be valid here, and that the value function is defined with respect to a *reference point* which is usually given as $x = 0$, i.e. the point in which a *gain* changes to a *loss* and vice versa.

Figure 2.1: Estimates of the weighting function $\pi(p)$.



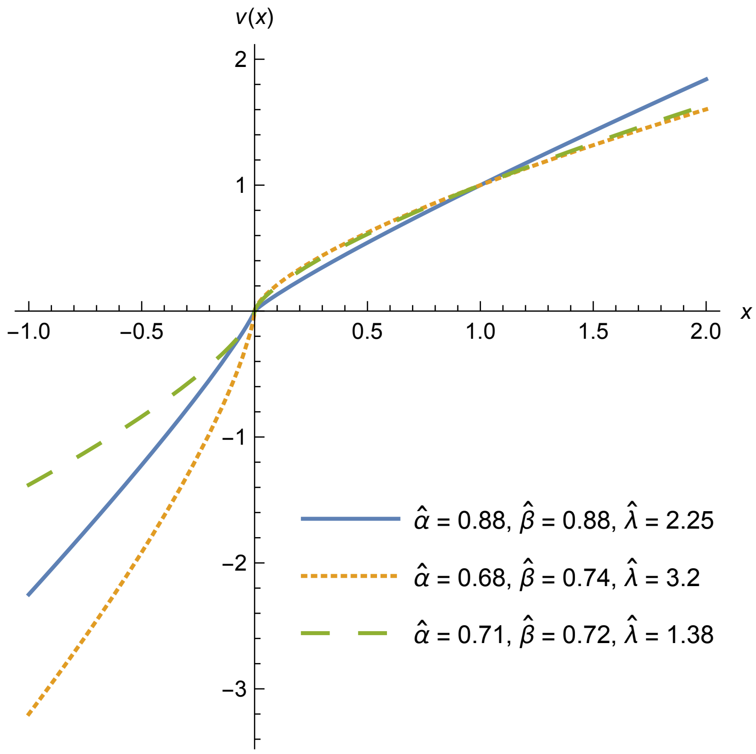
Source: Author's computations using results of Tversky & Kahneman (1992), Camerer & Ho (1994), and Wu & Gonzalez (1996).

Kahneman & Tversky (1979) defined the weighting function $\pi(p)$ relatively vaguely (from the mathematical point of view, not from the psychological one) as an increasing function of p, overweighting 'small' probabilities and 'under-weighting' large ones. Moreover, the function was discontinuous near $p = 0$ and $p = 1$ to reflect that there is a limit to how little a decision weight can be associated with an event. Several attempts have been made to estimate the weighting function; Tversky & Kahneman (1992) fitted a model of the form

$$\frac{p^\gamma}{\left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}}, \tag{2.16}$$

where γ is a parameter that controls for curvature of the weighting function, and obtained $\hat{\gamma} = 0.61$. Camerer & Ho (1994) used the same framework and reported $\hat{\gamma} = 0.56$, and Wu & Gonzalez (1996) gave $\hat{\gamma} = 0.71$, using again the model specified in Equation 2.16. The graphs of the weighting function with the parameter γ specified by these three results are plotted in Figure 2.1.

Figure 2.2: Estimates of the value function $v(x)$.



Source: Author's computations using results of Tversky & Kahneman (1992), Harrison & Rutström (2009), and Tu (2005).

The value function $v(x)$ satisfies the following properties: it is increasing $\forall x$, i.e. $v'(x) > 0$ always holds, convex below the reference point, i.e. $v''(x) > 0$ for $x < 0$, and concave above it, i.e. $v''(x) < 0$ for $x > 0$. Additionally, the value function is usually thought to be steeper for losses than for gains. Several scholars have estimated the value function, too, most often using a piecewise power function proposed by Tversky & Kahneman (1992). The function is of the following form:

$$v(x) = \begin{cases} x^{\alpha} & x \geqslant 0 \\ -\lambda \cdot (-x)^{\beta} & x < 0, \end{cases} \tag{2.17}$$

where the parameters α and β determine curvature of the value function for gains and for losses, respectively, relative to the reference point of $x = 0$, and λ is a parameter understood as loss aversion characterization.

Estimating the Equation 2.17, Tversky & Kahneman (1992) reported $\hat{\alpha} = 0.88, \hat{\beta} = 0.88$, and $\hat{\lambda} = 2.25$, Tu (2005) $\hat{\alpha} = 0.68, \hat{\beta} = 0.74$, and $\hat{\lambda} = 3.2$, and, e.g., Harrison & Rutström (2009) $\hat{\alpha} = 0.71, \hat{\beta} = 0.72$, and $\hat{\lambda} = 1.38$; all versions are plotted in Figure 2.2.

An interesting application of PT is offered by Barberis *et al.* (2001) who construct an economy in which investors are loss averse and derive their utility not only from mere instantaneous consumption, but also from variations in the value of their total wealth. Here the degree of loss aversion is given by previous investment successes or failures of an investor and hence captures the fact that *"After prior gains, the investor becomes less loss averse: the prior gains will cushion any subsequent loss, making it more bearable. Conversely, after a prior loss, he becomes more loss averse: after being burned by the initial loss, he is more sensitive to additional setbacks."* (Barberis *et al.*, 2001, p. 2).

To find more about relevance of PT for financial markets and its applications thereat, the interested reader might consult Subsection 4.1.1 of Chapter 4.

Chapter 3

Framework of the model

In the following chapter, general features and framework of the original Heterogeneous Agent Model (HAM) proposed by Brock & Hommes (1998) will be described. The main characteristic of the model is undoubtedly presence of *heterogeneity*—as was already pointed out in Chapter 2, the differences among the agents' beliefs make up an essential part of the model's design and play a vital role in the emergence of inter-agent interactions.[1]

In today's economic theory, there is little doubt that economic agents are heterogeneous to some extent. Frankel & Froot (1990) attribute the apparent divergence of US dollar interest rate from the then macroeconomic fundamentals at the beginning of the 1980s to the existence of speculative traders, Vissing-Jorgensen (2004) conducts a thorough analysis of chiefly qualitative data[2] on US stock markets from 1998 to 2002 and concludes that there is significant disagreement among the investors regarding expected profits, and, for instance, Hommes (2011) provides 'evidence from the lab' of presence of heterogeneous expectations in an experimental financial market.

3.1 Evolutionary selection

Before the financial market application, 'preliminary' research in the field of heterogeneous beliefs was done by Brock & Hommes (1997). In this paper, the authors proposed a system used by agents to switch among different expec-

[1]Heterogeneity is indeed crucial for the models listed in Chapter 2 and Chapter 3, generally, however, ABMs need not have such property.

[2]Data was collected during telephone surveys and respondents were asked questions of qualitative type, e.g. *"What overall rate of return do you think the stock market will provide investors during the coming twelve months?"* (Vissing-Jorgensen, 2004, p. 6).

tation rules according to the relative profitability of these rules—the system was termed *evolutionary selection* or *reinforcement learning.* Being founded on cobweb model,[3] the switching system was not restricted to sole economic applications but was rather intended as a general framework for description of formation of expectations under heterogeneous beliefs. In a simple supply–demand cobweb model with such beliefs, producers select from H different *forecasting rules* of the form $p^e_{h,t} = f_h(\bullet)$, where $p^e_{h,t}$ is a price predicted by rule h, and $f_h(\bullet)$ a forecasting function, $1 \leqslant h \leqslant H$. Furthermore, $n_{h,t}$ denotes the *fraction* of agents using the forecasting function h for prediction. The switching among the forecasting functions is then driven by a *performance* or *fitness measure* specific and unique for each forecasting function, e.g. realized net profits attained with the respective forecasting function.

Generally, the fitness measure takes the form[4]

$$\tilde{U}_{h,t} = U_{h,t} + \varepsilon_{i,h,t}, \tag{3.1}$$

where $U_{h,t}$ is the non-random, deterministic part, and $\varepsilon_{i,h,t}$ independent, identically distributed noise. Typically, the term $U_{h,t}$ depends on past realizations of a market indicator such as market price. Finally, the switching of rules is based on probabilities—associated with each rule—that an agent will select the rule. These probabilities are modeled using the multinomial logit model: led using the multinomial logit model:

$$n_{h,t} = \frac{\exp\left(\beta \cdot U_{h,t-1}\right)}{Z_{t-1}}, \tag{3.2}$$

where $Z_{t-1} \equiv \sum_{h=1}^{H} \exp\left(\beta \cdot U_{h,t-1}\right)$ is a normalization factor such that the fractions $n_{h,t}$ add up to 1, and β, $\beta \geqslant 0$,[5] is a parameter called *intensity of choice* which measures the agents' 'sensitivity' to the selection of optimal (i.e. the best-performing) forecasting rule. Two extreme cases may be distinguished—if $\beta = \infty$, all agents unerringly choose the best rule, while if $\beta = 0$, the fractions $n_{h,t}$ will remain constant over time and fixed to $1/H$, i.e. $n_{h,t} = 1/H \; \forall h, t$. The former extreme case corresponds to the situation in which there is no noise and thus all agents select the optimal strategy while the latter extreme case im-

[3] Cobweb model explains the origin of price fluctuations in a simple demand–supply model in which the there is a time lag between supply and demand decisions.

[4] The term $\varepsilon_{i,h,t}$ represents *individual* agents' forecasting errors.

[5] Value of β might generally be negative—such setting would, however, make little economic sense as the logic of the model would be shifted 'upside down' and unprofitable strategies would be preferred to profitable one.

plies presence of noise with infinite variance and inability of agents to switch strategies at all.

3.2 Adaptive Belief System

Adaptive Belief System (ABS), originally introduced in Brock & Hommes (1998) and slightly reformulated in Hommes (2006),[6] is a financial market application of the *Evolutionary selection* system of forecasting rules described in Section 3.1. Essentially, the ABS is a discounted value asset pricing model—extended to heterogeneous beliefs—in which the agents employ (myopic) mean-variance optimization while having the possibility to invest in either a risk-free or a risky asset. The agents are (only) boundedly rational and, in compliance with the Evolutionary selection system, select from a set of *predictors* based on the predictors' past performance.

3.2.1 Model mechanics

The risk-free asset pays a fixed rate of return r and is perfectly elastically supplied while the risky asset pays an uncertain dividend. Letting p_t and y_t denote the ex-dividend price of the risky asset and its random dividend process, respectively, and z_t the 'amount'[7] of risky asset the agent purchased at time t, each agent's *wealth dynamics* is of the following form:

$$\mathbf{W}_{t+1} = R \cdot W_t + z_t \cdot \left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t \right), \tag{3.3}$$

where R is the gross risk-free return rate equal to $1 + r$ and random variables are typeset in bold face. There are H forecasting rules—this fact implies existence of H different strategies or, equivalently, H distinct classes of agents. Let $E_{h,t}$ and $V_{h,t}$, respectively, denote the *belief* of agent who uses forecasting rule h about conditional mean and conditional variance[8] of wealth, $1 \leqslant h \leqslant H$. It is assumed that all agents maximize the same, exponential-type CARA utility function of wealth of the form $U(W) = -\exp(-a \cdot W)$, where a

[6] This reformulation consists in slightly different notation of terms which include the variable 'time'; the 'time' subscripts are shifted and the model becomes more comprehensible. For the purposes of this thesis, we will therefore use this 'shifted' formulation given in Hommes (2006).

[7] Such 'amount' might be, e.g., a number of shares of a certain stock.

[8] In contrast to Chapter 2, we will not use bold face to denote expectation in order to avoid confusion with random variables' denotation.

is a risk-aversion parameter. Given the mean-variance maximization, the optimal demand $z^*_{h,t}$ for the risky asset of agents of type h then solves the following maximization problem:[9]

$$\max_{z_{h,t}} \left\{ E_{h,t}\left(\mathbf{W}_{t+1}\right) - \frac{a}{2} \cdot V_{h,t}\left(\mathbf{W}_{t+1}\right) \right\}. \tag{3.4}$$

The demand $z^*_{h,t}$ is then

$$z^*_{h,t} = \frac{E_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t\right)}{a \cdot V_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t\right)}, \tag{3.5}$$

which, assuming that $V_{h,t} \equiv \sigma^2 \; \forall h, t$, simplifies to

$$z^*_{h,t} = \frac{E_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t\right)}{a \cdot \sigma^2}. \tag{3.6}$$

Denoting z^s the supply of outside risky shares per investor, and $n_{h,t}$ the fraction of agents using forecasting rule h, the demand–supply equilibrium is

$$\sum_{h=1}^{H} n_{h,t} \cdot \frac{E_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t\right)}{a \cdot \sigma^2} = z^s, \tag{3.7}$$

where, again, H is the total number of forecasting rules (i.e. strategies). In case of *zero* supply of outside shares, i.e. $z^s = 0$, Equation 3.7 becomes

$$R \cdot p_t = \sum_{h=1}^{H} n_{h,t} \cdot E_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}\right). \tag{3.8}$$

Now, should all traders be identical and their expectations homogeneous, we would obtain a simplified version of Equation 3.8 called *arbitrage market equilibrium* of the form

$$R \cdot p_t = E_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}\right). \tag{3.9}$$

Equation 3.9 asserts that this period's price of the risky asset is equal to the sum of next period's *expected* price and dividend, discounted by the gross risk-free interest rate. In this homogeneous-expectations case, provided that the transversality condition

$$\lim_{t \to \infty} \frac{E_t\left(\mathbf{p}_{t+k}\right)}{\left(1+r\right)^k} = 0 \tag{3.10}$$

[9]Proof of this result is given in Section A.1 of the Appendix A.

holds,[10] the *fundamental* price of the risky asset is given as

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t\left(\mathbf{y}_{t+k}\right)}{\left(1+r\right)^k}, \tag{3.11}$$

where E_t is the conditional expectation operator. The price p_t^* is the equilibrium price of the risky asset in a perfectly efficient market with fully rational traders and, as can be seen directly from Equation 3.11, it depends on the expectation of the stochastic dividend process $\mathbf{y}_t$, $E_t\left(\mathbf{y}_t\right)$. Assuming the dividend process $\mathbf{y}_t$ is independent, identically distributed with mean $\bar{y}$, the fundamental price p_t^* becomes constant $\forall t$ and is given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{\left(1+r\right)^k} = \frac{\bar{y}}{r}. \tag{3.12}$$

The *deviation* from the fundamental price is defined as follows:

$$x_t = p_t - p_t^*. \tag{3.13}$$

There are now two additional assumptions Brock & Hommes (1998) make:

1. Expectations about future dividends $\mathbf{y}_{t+1}$ are the same for all agents, regardless of the specific forecasting rule they use, and equal to the true conditional expectation. In other words, $E_{h,t}\left(\mathbf{y}_{t+1}\right) = E_t\left(\mathbf{y}_{t+1}\right)\ \forall h, t$.

2. Agents believe that the stock price might deviate from the fundamental price p_t^* by some function f_h which depends on previous deviations from the fundamental price, i.e. on $x_{t-1}, \ldots, x_{t-K}$. This assumption might be stated as

$$E_{h,t}\left(\mathbf{p}_{t+1}\right) = E_t\left(\mathbf{p}_{t+1}^*\right) + f_h\left(x_{t-1}, \ldots, x_{t-K}\right)\ \forall h, t. \tag{3.14}$$

It is now important to note two crucial facts: firstly, the assumption number one above implies that all agents have *homogeneous* expectations about future dividends, i.e. the heterogeneity of the model lies in the assumption number two. Secondly, the asset price in period $t+1$, p_{t+1}, is predicted using price realized in period $t-1$—not in period t—as the agents are yet unaware of

[10] Hommes (2013, p. 162) remarks that the Equation 3.9 is also satisfied by the so-called *rational bubble* solution of the form $p_t = p_t^* + (1+r)^t \cdot (p_0 - p_0^*)$. However, this solution does not satisfy the transversality (or 'no-bubbles') condition

the price p_t when they make the prediction. This fact follows directly from Equation 3.7.

Using the facts that $p_t = x_t + p_t^*$ and that the fundamental price p_t^* satisfies

$$R \cdot p_t^* = E_t\left(p_{t+1}^* + \mathbf{y}_{t+1}\right), \tag{3.15}$$

Equation 3.8 can be reformulated in deviations from the fundamental price by a substitution using Equation 3.14 as

$$R \cdot x_t = \sum_{h=1}^{H} n_{h,t} \cdot E_{h,t}\left(\mathbf{x}_{t+1}\right) \equiv \sum_{h=1}^{H} n_{h,t} \cdot f_h\left(x_{t-1}, \ldots, x_{t-K}\right). \tag{3.16}$$

The fractions $n_{h,t}$ are given by Equation 3.2. Next, Brock & Hommes (1998) define *realized excess return*, which, for the purpose of this thesis, will be denoted as $\mathcal{R}_{t+1} = \mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t$. The realized excess return over period t to period $t+1$ might be expressed in deviations from the fundamental value as follows:

$$\begin{aligned} \mathcal{R}_{t+1} &= p_{t+1} + y_{t+1} - R \cdot p_t = x_{t+1} + p_{t+1}^* + y_{t+1} - R \cdot x_t - R \cdot p_t^* \\ &= x_{t+1} - R \cdot x_t + \underbrace{p_{t+1}^* + y_{t+1} - E_t\left(p_{t+1}^* + y_{t+1}\right)}_{\delta_{t+1}} + \underbrace{E_t\left(p_{t+1}^* + y_{t+1}\right) - R \cdot p_t^*}_{=0} \\ &= x_{t+1} - R \cdot x_t + \delta_{t+1}, \end{aligned} \tag{3.17}$$

where the latter underbrace holds because Equation 3.15 is satisfied. The term δ_{t+1} is a Martingale Difference Sequence with respect to some information set $\mathcal{F}_t$, i.e. $E\left(\delta_{t+1} | \mathcal{F}_t\right) = 0 \; \forall t$.

Now, the *fitness measure* of strategy h, $U_{h,t}$, is defined as

$$U_{h,t} = \mathcal{R}_{t+1} \cdot z_{h,t}^* = \left(x_{t+1} - R \cdot x_t + \delta_{t+1}\right) \cdot z_{h,t}^*. \tag{3.18}$$

Two cases might be distinguished:

1. The case in which $\delta_{t+1} = 0$ corresponds to a *deterministic* nonlinear pricing dynamics with constant dividend $\bar{y}$ and, according to Hommes (2006) who uses slightly modified notation,[11] Equation 3.18, written in

[11] The notation difference consists in 'shifting' time subscripts of realized excess return by one period—for this reason, Equation 3.18 reduces to Equation 3.19 only after this shift.

deviations, reduces to

$$U_{h,t} = (x_t - R \cdot x_{t-1}) \cdot \frac{f_{h,t-1} - R \cdot x_{t-1}}{a \cdot \sigma^2}, \tag{3.19}$$

where $f_{h,t-1}$ is the forecasting function of type h.

2. The case in which dividend is given by a stochastic process $y_t = \bar{y} + \epsilon_t$ where ϵ_t is independent, identically distributed random variable with uniform distribution. In these circumstances, $\delta_{t+1} = \epsilon_{t+1}$.

For the formation of expectations, the functions $f_{h,t}$ are crucial. Brock & Hommes (1998) proposed simple forecasting rules of the form

$$f_{h,t} = g_h \cdot x_{t-1} + b_h \tag{3.20}$$

since, as Hommes (2013, p. 167) explains, *"...for a forecasting strategy to have any impact in real markets, it has to be simple. For a complicated forecasting rule it seems unlikely that enough traders will coordinate on that particular rule so that it affects market equilibrium."* The term g_h is a *trend* parameter indicating the strength of the particular's strategy trend following (or possibly reverting), and the term b_h is a *bias* parameter measuring the magnitude of the strategy's bias. For $g_h = b_h = 0$, the function $f_{h,t}$ reduces to $f_{h,t} \equiv 0$ and corresponds to the fundamentalist belief of no price deviations from the fundamental value. Additionally, if $g_h \neq 0$, then such a trader type is called a *chartist*. This class of traders can be further divided into four categories: the type is called a *pure trend chaser* if $0 < g_h \leqslant R$, a *strong trend chaser* if $g_h > R$, a *contrarian* if $-R \leqslant g_h < 0$, and a *strong contrarian* if $g_h < -R$. Finally, the term b_h determines the nature (if $b_h \neq 0$) of each agent class's *bias*—if $b_h < 0$, the bias is *downward*, while if $b_h > 0$, the bias is *upward*.

Provided the above-mentioned assumptions hold, the ABS is fully specified by the following three equations:

$$\begin{aligned} R \cdot x_t &= \sum_{h=1}^{H} n_{h,t} \cdot (g_h \cdot x_{t-1} + b_h) + \varepsilon_t, \\ n_{h,t} &= \frac{\exp(\beta \cdot U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta \cdot U_{h,t-1})}, \\ U_{h,t-1} &= (x_{t-1} - R \cdot x_{t-2}) \frac{g_h \cdot x_{t-3} + b_h - R \cdot x_{t-2}}{a \cdot \sigma^2}, \end{aligned} \tag{3.21}$$

where ε_t is a (small) noise term which represents natural uncertainty about the performance of economic fundamentals and replaces the term $\delta_t = \epsilon_t$ defined above.

Chapter 4

Extensions

In this chapter, we will introduce two major extensions of the original Brock & Hommes (1998) model, which is described in detail in Chapter 3, investigate several versions of them—mainly with respect to various different settings of parameters—and evaluate on each such version. The goal of this effort is to modify the mechanics of the model in a certain way but, at the same, try to leave the model's fundamental features intact. At the end of the chapter, addition of traders' *memory* to the model will be discussed. As the title of this thesis suggests, the two extensions are

1. features of Prospect Theory, and
2. Asynchronous Updating.

4.1 Features of Prospect Theory

The reason for the attempt to merge the original HAM with PT is the indisputable relevance of findings of PT for study of human decision making and choice theory.[1] For the convenience of the interested reader, we will briefly recall the main findings and results of PT; for a more in-depth treatment, please consult Section 2.3 of Chapter 2.

Proposed in the seminal paper of Kahneman & Tversky (1979), PT is a critique of then-mainstream expected utility theory. Using convincing evidence obtained from questionnaires, Kahneman & Tversky (1979) illustrate several issues with the expected utility and its applicability to real-life human decision making. The most critical objection consists in incapacity of the expected

[1] See Subsection 4.1.1.

utility theory to explain certain 'irrational'[2] choices of people. As a result, Kahneman & Tversky (1979)—and later Tversky & Kahneman (1992)—propose a brand new *descriptive*[3] theory which takes all such 'irrational' choices into account and explains them rigorously using (chiefly) so-called *value function* and *weighting function*. Graphical representations of both functions are provided in Figure 2.2 and Figure 2.1, respectively. The three major features of PT are

1. *Existence of a reference point.* PT suggests that people make decisions in terms of gains and losses with respect to some reference point, rather than in terms of final wealth.

2. *Differences in treatment of gains and losses.* While most people are risk-seeking towards losses, they are, at the same time, risk-averse towards gains. Moreover, most people are generally loss-averse which explains why the value function is steeper for losses than for gains. Figure 2.2 shows some notable estimates of the value function.

3. *Distorted understanding of probability.* According to PT, average person underweights large probabilities and overweights small probabilities. Given the proposed specification and shape of the weighting function, value is not linear in probability.

4.1.1 Relevance for financial markets

In this subsection we point out several interesting studies and papers that deal with applications of PT in financial markets. The main motivation is to show that inclusion of PT features into the original HAM is meaningful and that it may shift the model closer to the real-world markets.

Since the formulation of PT, several scholars have confirmed significant relevance of it for financial markets. One of the most cited applications of PT is an aid in explanation of so-called *disposition effect*. The term was first coined by Shefrin & Statman (1985) and refers to a tendency to *"... sell winners too early and ride losers too long,"* (Shefrin & Statman, 1985, p. 778) essentially meaning that traders tend to hold value-loosing assets too long and value-gaining assets too short. Already in 1985, the authors, using the PT's value function,

[2]The 'irrationality' is meant within the expected utility theory. It may be claimed that any possible choice individuals make is optimal—and hence rational—for them.

[3]PT is descriptive in a sense that it tries to capture the real-world decision making whereas the expected utility theory is *de facto* prescriptive—it models how people *are supposed* to decide.

explain the disposition effect for an investor who owns a loosing stock as a *gamble* between

1. selling the stock now and thereby realizing a loss, or

2. holding the stock for another period given, say, 50–50 chance between loosing some additional value or breaking even.

As the investor finds himself in the 'negative domain' with respect to the reference point given here as the break even point, the choice between the two above-mentioned options is associated with the convex part of the value function. This fact implies that the investor will select the second option and thus 'ride the loser too long'.

Li & Yang (2013) also attempt to explain the disposition effect using findings of PT. In this paper, the authors build a general equilibrium model and, besides the disposition effect, they also focus on trading volume and asset prices. The results suggest that loss aversion implied by PT tends to predict a reversed disposition effect and price reversal for stocks with non-skewed[4] dividends.

Yao & Li (2013), on the other hand, investigate trading patterns in the market with Prospect-theoretical investors who base their choices on the value and weighting functions and related features of PT. The authors find out that the three main features of PT (those described in the first part of this section) can be regarded as behavioral causes of negative-feedback trading. The authors subsequently construct a market populated by the PT traders and traders with Constant Relative Risk Aversion (CRRA) utility function and discover that individual PT preferences might cause *contrarian* noise trading.

Some other research efforts related to the study of PT traits in financial markets are made by Grüne & Semmler (2008) who try to attribute some of the most frequently observed asset price characteristics—yet unexplainable by 'standard' preferences—to the loss aversion feature of traders; Giorgi & Legg (2012) make use of the weighting function and show that dynamic models of portfolio choice might be consistently and meaningfully extended by the probability weighting; Zhang & Semmler (2009) further investigate properties of the model proposed by Barberis *et al.* (2001), which is shortly described at the end of Section 2.3, using time series data and conclude that models with PT features are able to better explain some financial 'puzzles', e.g. the

[4]'Non-skewness' is a property of a random variable's distribution and means that this distribution's skewness is close to that of normal distribution.

equity premium puzzle;[5] and, for instance, Giorgi *et al.* (2010) explore aspects of *Cumulative Prospect Theory*—a modification of the original PT developed by Tversky & Kahneman (1992)—and find out that financial markets' equilibria need not exist under assumptions of PT.

4.1.2 Inclusion into the model

As the relevance of PT is—according to current literature—highly topical, we may now proceed to the inclusion of certain PT features into the original HAM described in detail in Chapter 3. At first, it is important to remark that the current writer is not aware of any previous attempts to equip the original Brock & Hommes (1998) HAM 'as is' with PT features. Although there is an abundance of PT models—of which only a small minority was mentioned in Subsection 4.1.1—there are apparently no PT extensions of the original Brock & Hommes (1998) model.

The plausible reason for the absence of such ABM designs is relatively self-evident: the HAM developed by Brock & Hommes (1998) is populated with agents with CARA utility function and the most fundamental result—demand for the risky asset—is derived by maximization of *expected utility*. Such a characteristic is not unique for this particular model; generally, agent-based models with heterogeneous beliefs usually employ the exponential utility function (Shimokawa *et al.*, 2007, p. 208). As the origins of PT were based on critique of the expected utility theory and subsequent development of diametrically different approach to decisions under risk, the very basic component of the ABS—CARA utility function—seems incompatible with PT. Yet, although the authors did not use the original Brock & Hommes (1998) model, Shimokawa *et al.* (2007) proposed a relatively straightforward method to implement PT features into ABMs in which the agents have CARA preferences. This fact facilitates the research of PT features within the ABM framework as it essentially enables the researcher to keep the intrinsic mechanics of the model unchanged.

The basic structure of the model remains the same, i.e. all agents maximize a CARA utility function of wealth. However, as opposed to the original Brock & Hommes (1998) model, we will introduce features of PT into the model as

[5]The equity premium puzzle is a phenomenon that the average return on equity is far greater than return on a risk-free asset. Such a characteristic has been observed in many markets. The term was first coined by Mehra & Prescott (1985).

follows: PT traders maximize utility function of the general form

$$U_l(W) = -\exp(-a \cdot B \cdot W), \tag{4.1}$$

where we will term B the *loss aversion* parameter. Generally, the loss aversion parameter is distinct for each agent class and time period, therefore we will denote it as $B_{h,t}$ from now on. Furthermore, note that the subscript l distinguishes the utility function of these PT investors from that of 'standard' traders specified in the original model—we will refer to the PT traders as *loss-averse* traders since this characteristic is the main feature of PT which is possible to incorporate into the model using the utility function defined in Equation 4.1. Other letters present in Equation 4.1 have their usual meaning as given in Subsection 3.2.1 of Chapter 3. We assume that the wealth dynamics (i.e. the budget constraint) is of the same form as in Equation 3.3.

The crucial aspect of the utility function given in Equation 4.1 is the loss aversion parameter $B_{h,t}$ and its specification. Following the general idea proposed by Shimokawa *et al.* (2007, p. 211), we will define the parameter as follows:

$$B_{h,t} = \begin{cases} c_g, & E_{h,t}(\mathbf{p}_{t+1}) > \tilde{p}_t = \tilde{p}_t(p_{t-1}, \dots, p_{t-K}) \\ c_l, & E_{h,t}(\mathbf{p}_{t+1}) \leqslant \tilde{p}_t = \tilde{p}_t(p_{t-1}, \dots, p_{t-K}), \end{cases} \tag{4.2}$$

where c_g and c_l are *gain* and *loss* parameters, respectively, $0 < c_g < c_l$, and $\tilde{p}_t = \tilde{p}_t(p_{t-1}, \dots, p_{t-K})$ is a *reference point* as defined by PT.[6] It is important to emphasize that each agent might maximize either the original utility function $U(W) = -\exp(-a \cdot W)$ or the 'augmented' utility function U_l with the loss aversion parameter given in Equation 4.1, however, the term $E_{h,t}(\mathbf{p}_{t+1})$, i.e. a (loss-averse) agent's forecast about next period's price, is constructed essentially in the same way as in Equation 3.14 in Chapter 3 whether the agent is loss-averse or not.

Optimal demand $z^*_{l,t}$ of the loss-averse traders for the risky asset then solves the familiar maximization problem

$$\max_{z_{l,t}} \left\{ E_{h,t}(\mathbf{W}_{t+1}) - \frac{a \cdot B_{h,t}}{2} \cdot V_{h,t}(\mathbf{W}_{t+1}) \right\}, \tag{4.3}$$

where $V_{h,t}(\mathbf{W}_{t+1})$ is the (loss-averse) traders' belief about next period's condi-

[6] See, e.g., the beginning of Section 4.1 or Section 2.3.

tional variance of wealth, and is thus given as[7]

$$z^*_{l,t} = \frac{E_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t\right)}{a \cdot B_{h,t} \cdot \sigma^2}, \tag{4.4}$$

where we used the assumption that

$$V_{h,t}\left(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t\right) \equiv \sigma^2 \ \forall h, t, \tag{4.5}$$

i.e. the assumption that beliefs of all traders about the conditional variance, regardless of the agents' loss-aversion feature, are identical and equal to σ^2 for all time periods.[8]

Note that the basic structure of the model remains the same: there are H distinct trading (or, equivalently, price-forecasting) strategies (i.e. there are essentially H classes of agents or traders), and each agent maximizes a CARA utility function. Certain number of the H classes, say first L classes, $0 \leqslant L \leqslant H$, are endowed with the above-specified PT feature—optimal demand of agents of these L classes for the risky asset is given by Equation 4.4—while the agents of the $H - L$ remaining classes are 'standard' in terms of the original model's construction and do *not* exhibit PT behavior. The general specification of the optimal demand for the risky asset, $z^*_{h,t}$, $1 \leqslant h \leqslant H$, thus remains the same and is given by Equation 3.6 where, if hth class of agents has the PT feature (i.e. for $h \leqslant L$, $1 \leqslant h \leqslant H$), we use $z^*_{l,t}$ given by Equation 4.4 instead of $z^*_{h,t}$.

In presence of strategies with the PT feature, the fitness measure of strategy h, $U_{h,t}$, then becomes

$$U_{h,t} = \begin{cases} \left(x_t - R \cdot x_{t-1}\right) \frac{g_h \cdot x_{t-2} + b_h - R \cdot x_{t-1}}{a \cdot \sigma^2}, & h > L \\ \left(x_t - R \cdot x_{t-1}\right) \frac{g_l \cdot x_{t-2} + b_l - R \cdot x_{t-1}}{a \cdot B_{h,t-1} \cdot \sigma^2}, & h \leqslant L, \end{cases} \tag{4.6}$$

where we used the notation g_l and b_l to indicate the trend and bias parameters of the strategies with the PT feature. Note the timing of the loss aversion parameter $B_{h,t}$: for fitness measure in time t, it is important to delay the parameter and work with $B_{h,t-1}$ in order to use the correct reference point $\tilde{p}_{t-1}$ whose most recent deviation from the fundamental price occurs in time

[7]Proof of this result is very similar to that of the original model's demand, which is given in Section A.1 of the Appendix A.

[8]This assumption is identical to the one originally made by Brock & Hommes (1998) and serves for better analytical tractability of the model.

$t-2$—this period's price, p_{t-2}, is the 'freshest' piece of information about the price that the agents have when making the prediction about the price p_t as the price p_{t-1} is not revealed yet.

The definition of the parameter $B_{h,t}$ given in Equation 4.2 essentially enables us to mimic the first two of the three major features of PT listed in the beginning of Section 4.1, i.e. the loss aversion and biased treatment of gains and losses, and relation of decisions under risk to a reference point, by using an 'imitation' of the value function. In this application, however, we omit the third major feature of PT, the probability weighting and the weighting function, to keep the model within the stylized, simple framework proposed by Brock & Hommes (1998). Also the curvature of the value function *per se* will not be studied and incorporated into the model as it can well approximated by a linear function (see Figure 2.2).

The choice of specific numerical values of the gain and loss parameters c_g and c_l is relatively unfettered and will be discussed later during the description of the simulation process in Chapter 5—besides the positivity of both parameters, the only condition that has to be fulfilled is lower magnitude of the gain parameter—i.e. the inequality $c_g < c_l$ must always hold—in order to capture the loss aversion feature properly. The choice of $\tilde{p}_t$, the reference point, is more interesting. Note that the subscript t indicates the fact that the reference point is *updated* each time period to properly reflect the gain and loss treatment of PT traders. Generally, the reference point is given by a deterministic function of past performance of the model—one might make use of K previous realized prices of the risky asset, i.e. $p_{t-1}, p_{t-2} \ldots, p_{t-K}$, and define the reference point—as Shimokawa *et al.* (2007) suggests—as the moving average of the form

$$\tilde{p}_t = \frac{a_1 \cdot p_{t-1} + a_2 \cdot p_{t-2} + \ldots + a_K \cdot p_{t-K}}{a_1 + a_2 + \ldots + a_K}, \tag{4.7}$$

where $a_1, a_2, \ldots, a_K$ are constants $\in \mathbb{R}$ such that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_K \geqslant 0$ which allow for a stronger effect of the most recent prices of the risky asset. The interpretation of the definition of the parameter $B_{h,t}$ is straightforward in such a case: if the traders with the PT feature expect the next period's price to be higher than the moving average of previous K prices, they find themselves in the positive domain in terms of the gain–loss gamble and set the value $B_{h,t}$ to c_g; if, on the other hand, they expect the next period's price to be lower than the moving average, i.e. they expect a loss, the loss aversion of PT manifests itself by the parameter $B_{h,t}$ which is set to c_l. Definition of the reference point is not restricted to the (weighted) moving average of past prices only; one may calculate a cumulative moving average which reflects *all* past prices—in such a case, the reference point $\tilde{p}_t$ would be defined as

$$\tilde{p}_t = \frac{p_{t-1} + p_{t-2} + \ldots + p_2 + p_1}{t-1}, \tag{4.8}$$

where—alternatively—e.g. exponential weights can be included to reflect the effect of the most recent prices as in Equation 4.7.

4.2 Asynchronous updating

In this section we will evaluate on the so-called Asynchronous Updating (AU) feature which will be added to the original model developed by Brock & Hommes (1998). This feature—as the term suggests—affects the way in which the fractions of traders using trading strategy h at time t, $n_{h,t}$, $1 \leqslant h \leqslant H$, are updated. These fractions, given by Equation 3.2 of Chapter 3, represent the percentage of agents using the respective forecasting rule and hence might be interpreted as a 'snapshot' of the current market situation and prevailing market sentiment. The interactions and flow of the strategies, together with the switching among them is the main driving force of the model, and, as Hommes (2013, p. 23) points out, *"Interactions and evolutionary switching between these strategies cause complicated dynamical behavior."* Hence, how the fractions $n_{h,t}$ are defined at the beginning is the design's crucial aspect which determines the dynamics of the model thereafter.

In the original model, the fractions are specified as

$$n_{h,t} = \frac{\exp\left(\beta \cdot U_{h,t-1}\right)}{Z_{t-1}}, \tag{4.9}$$

where $U_{h,t-1}$ is a fitness measure of strategy h and Z_{t-1} a normalization factor such that the fractions $n_{h,t}$ add up to 1 for $1 \leqslant h \leqslant H$.[9] However, such definition presupposes that *all* agents in *all* time periods switch among the strategies. Yet, this assumption might not be realistic in the real world as several studies have provided strong evidence of what is called *investor inertia*—a finding which suggests that certain percentage of traders or investors are in reality inactive, passive, and slow in making investment decisions, i.e. buying or selling an asset.

[9] Consult, please, Section 3.1 of Chapter 3 for details.

At first glance, the phenomenon of investor inertia appears closely related to the disposition effect which was predicted by PT—the incorrect 'timing' of investment decisions, i.e. holding onto a losing stock too long and selling a winning stock too soon, is a well-documented fact.[10] A vast amount of economic literature deals with investor inertia in pension funds and individual employees' pension plans. Madrian & Shea (2000) investigated investment behavior of '401(k)' savers[11] and found that a considerable fraction of the 401(k) participants hired under automatic enrollment remained largely inactive and inert in a sense that they retained both the default contribution rate and allocation of their funds despite the fact that only negligible percentage of participants hired *without* the automatic enrollment had chosen this particular contribution rate and allocation. Madrian & Shea (2000, p. 1150) further remark that *"...procrastination is an extremely important factor in the widely perceived inadequacy of individual savings for retirement,"* and explain such behavior is due to high costs of gathering information and only a small short-run profit from making the subsequent adequate investment decision.

Dow & Werlang (1992) researched investor inertia in stock markets from the perspective of decisions under risk and found that there was a certain range of prices at which the traders did not trade at all—if the prices went up and beyond the range, the investors would buy hold a short position of an asset (i.e. *speculate* on the asset's price); on the other hand, if the prices decreased substantially below the range, the investors would hold a long position. Dow & Werlang (1992) attribute such trading patterns to so-called *uncertainty aversion* meaning that the investors do not trade unless they are sufficiently sure about the level of price, i.e. unless the price is below or above certain threshold values. Simonsen & Werlang (1990) report existence of similar prices ranges at which the traders are inactive.

Other evidence of investor inertia was documented on the New York Stock Exchange (NYSE) for instance. Summarized in the Shareownership2000 report,[12] the NYSE's survey of investors also reveals that many of them have very low levels of trading activity—the report finds that *"In 1998, for example, 23 percent of stockholders with brokerage accounts report no trading, while another 35 percent report trading only once or twice in the last year."* (Shareownership2000 report, p. 59).

[10] See Subsection 4.1.1 for definition and evidence of the disposition effect.

[11] '401(k)' is a pension plan of the United States under which retirement savings contributions are provided by an employer to its employees.

[12] Available online at http://www1.nyse.com/pdfs/shareho.pdf.

The investor inertia phenomenon suggests that the assumption—made by Brock & Hommes (1998)—that all fractions of all strategies update each period might not be very realistic. To shift the model closer to the real-world market structure, following the idea proposed by, e.g., Hommes (2013, p. 34) or Bolt *et al.* (2011, p. 9), we may define a parameter δ as the fraction of inert, inactive traders. Naturally, $\delta = 0$ represents the case in which all traders are active, while $\delta = 1$ represents the other extreme of no active traders.[13] Note that in the stylized framework of the model, the traders are inactive in terms of strategy-switching—they simply use the strategy they chose in the previous period also in this period; the inactivity does not mean the agents do not trade, they only, in economic terms, do not bother with gathering information about their strategy's profitability and 'inertly' choose the same strategy again.

The fractions $n_{h,t}$ are then in general form given as

$$n_{h,t} = \delta \cdot n_{h,t-1} + (1-\delta) \cdot \frac{\exp\left(\beta \cdot U_{h,t-1}\right)}{Z_{t-1}}, \tag{4.10}$$

where, for our application, we set $Z_{t-1} \equiv \sum_{h=1}^{H} \exp\left(\beta \cdot U_{h,t-1}\right)$ to guarantee that the fractions $n_{h,t}$ add up to 1 $\forall t$.

4.3 Memory

The specification of the original Brock & Hommes (1998) model, namely the fitness measure of strategies, $U_{h,t}$, takes into account only the very recent (past three periods', specifically) development of the market and does not allow for evaluation of the strategies' profitability over a greater number of past time periods. This attribute of the original model might not be very realistic in the real world. One could argue that at least chartists (i.e. noise traders) generally use more than one previous price realization for the technical analysis, which is based on various indicators of statistical character. Lo *et al.* (2000) provide an in-depth introduction to this area and list several methods that were traditionally frequently used in the financial markets by technical traders, e.g. head-and-shoulders, broadening tops, triangle tops, and double tops, to name a few. Other more recent popular indicators include e.g. the moving average convergence divergence, Aroon Oscillator, or Stochastic Oscillator. To be able

[13] The case $\delta = 1$ produces essentially the same behavior of the model as the case of $\beta = 0$ does—the fractions of strategies do not update at all.

to use these methods, the trader must have many previous price realizations at hand.

For the above-mentioned reasons, several authors have tried to extend the original model with *memory.* Vošvrda & Vácha (2002) consider a settings in which this period's fitness measure is a weighted average of past periods' fitness measures, namely

$$n_{h,t} = \frac{\exp\left(\beta \cdot \sum\limits_{p=1}^{M} \eta_{h,t} \cdot U_{h,t-p}\right)}{\sum\limits_{h=1}^{H} \exp\left(\beta \cdot \sum\limits_{p=1}^{M} \eta_{h,t} \cdot U_{h,t-p}\right)}, \tag{4.11}$$

where M is the memory length and $\eta_{h,t}$ memory weights. The authors also define the forecasting functions $f_{h,t}$ as $f_{h,t} = g_h \cdot \frac{1}{K_h} \cdot \sum_{p=1}^{K_h} x_{t-p} + b_h$. Vošvrda & Vácha (2003); Vácha & Vošvrda (2005) use similar settings while Barunik *et al.* (2009) consider only simple, unweighted average of past fitness measures.

Although we restrict our analysis to the original, 'memoryless' version and PT extended version of the model specified in the systems of Equations 3.21 and 5.1, respectively, note that certain form of memory is also present in our setup of the model—traders have to be able to calculate the reference point, which is based on previous prices of the risky asset, to be able to properly determine whether the next period's expected price of the asset will bring them a gain or a loss. The memory property of our model thus consists only in the agents' rudimentary analyses of the moving average of past prices and subsequent 'shift' of their reference points, not in the more advanced calculations of the strategies' fitness measures as specified in Equation 4.11.

Chapter 5

Simulations

In this chapter, we will investigate, compare, and interpret performance of the Brock & Hommes (1998) model and its extended versions (consult, please, Chapter 4) using Monte Carlo methods based on repeated random sampling. For this purpose, an algorithm was written in Wolfram Mathematica. The main motivation for our effort is to assess the impact of the PT and AU features on chiefly the qualitative behavior of the original version of the model—or, in other words, the extended model's capabilities to mimic real-world market characteristics and some stylized facts. The original model, as it is defined in Chapter 3, has already been studied minutely—see e.g. Brock & Hommes (1998); Hommes (2006); Hommes & Wagener (2009); Hommes (2013); Kukacka & Barunik (2013); Barunik *et al.* (2009); Vácha & Vošvrda (2005); Vošvrda & Vácha (2003).

We have already pointed out in Chapter 2 that one of the mainsprings of the development of ABMs and HAMs was undoubtedly the effort to explain some of the most common statistical properties of financial time series called *stylized facts*. Cont (2001) lists the following phenomena as the most frequent: absence of autocorrelations, heavy or fat tails,[1] volatility clustering, intermittency, gain–loss asymmetry, and several others. We will focus on the first three stylized facts as the original Brock & Hommes (1998) was found capable of explaining them soundly (Chen *et al.*, 2012).

[1]The two adjectives related to a distribution's tails—*heavy* and *fat*—are sometimes used loosely and interchangeably. While a heavy-tailed distribution is one whose tails are heavier than those of the exponential distribution, fat-tailed distributions make up a subclass of the heavy-tailed distributions. The fat-tailed distributions are characterized by larger skewness and/or kurtosis relative to the normal distribution. Additionally, the distribution of a random variable is said to have a fat tail if $P(X > x) \sim x^{-\alpha}$ as $x \to \infty$, $\alpha > 0$. Some fat-tailed distributions have *power law* decay in the tail of the distribution (i.e. they have the fat tail), but do not necessarily follow a power law elsewhere.

1. *Absence of autocorrelations.* This fact emphasizes that autocorrelations of returns of an asset are insignificant at most times and for most time scales, except for very small time scales of approximately 20 minutes in which micro structures may have an effect on the autocorrelations (Cont, 2001).

2. *Fat tails.* According to this fact, probability distributions of many assets' returns have large skewness or kurtosis relative to the normal distribution. Additionally, the distributions exhibit a power-law or Pareto-like tails, with a tail index $2 \leqslant \alpha \leqslant 5$ (Cont, 2001), i.e. the (upper) tail $P(X > x) = \bar{F}(x) = x^{-\alpha} \cdot G(x)$, where $G(x)$ is a slowly varying function (Haas & Pigorsch, 2009).

3. *Volatility clustering.* This fact means, in words of Mandelbrot (1963, p. 418), that *"... large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes."* More mathematically, an asset's absolute or squared returns are characterized by a significant, slowly decaying autocorrelation function, i.e. $\operatorname{corr}(|r_t|, |r_{t+\tau}|) > 0$ (or $\operatorname{corr}\left(r_t^2, r_{t+\tau}^2\right) > 0$), where τ, the time span, ranges from minutes to weeks or months (Cont, 2007).

5.1 General model setup

For the simulations, we will use the extended version of the original Brock & Hommes (1998) model with no additional memory.[2] To summarize the extensions presented in Chapter 4, the ABS extended with the PT loss aversion and the AU features becomes (in lines with the system of Equations 3.21)

$$
\begin{aligned}
R \cdot x_t &= \sum_{h=1}^{H} n_{h,t} \cdot (g_h \cdot x_{t-1} + b_h) + \varepsilon_t, \\
n_{h,t} &= \delta \cdot n_{h,t-1} + (1-\delta) \cdot \frac{\exp(\beta \cdot U_{h,t-1})}{\sum\limits_{h=1}^{H} (\beta \cdot U_{h,t-1})}, \\
U_{h,t-1} &= \begin{cases} (x_{t-1} - R \cdot x_{t-2}) \frac{g_h \cdot x_{t-3} + b_h - R \cdot x_{t-2}}{a \cdot \sigma^2}, & h > L \\ (x_{t-1} - R \cdot x_{t-2}) \frac{g_h \cdot x_{t-3} + b_h - R \cdot x_{t-2}}{a \cdot B_{h,t-2} \cdot \sigma^2}, & h \leqslant L, \end{cases}
\end{aligned}
\tag{5.1}
$$

[2] See Section 4.3 for a short discussion about the role of memory in our extended version of the original model. Note that memory is present in the original Brock & Hommes (1998) model in the fitness measure of strategies, $U_{h,t}$, for which the agents have to know previous price deviations up to x_{t-3}.

where first L of the H agent classes are endowed with the PT feature. The system of Equations 5.1 is in essence a generalization of the original ABS given in the system of Equations 3.21—for $L = 0$ and $\delta = 0$, one might obtain the original system which will be used as a 'benchmark' case for the simulations.

The inevitable 'downside' of the ABS (and of many other ABMs) is somewhat excessive leeway in choice of the parameters of the model, especially of β, g_h, b_h, and the distribution of the noise term ε_t. The possible variety of countless different settings of the model enables us to study many different aspects of it, at the same time, however, it presents a significant challenge of the proper selection of the parameters should we want to manipulate with only one or two remaining parameters. We will follow a number of previous studies—e.g. Kukacka & Barunik (2013); Vácha & Vošvrda (2005); Vošvrda & Vácha (2003)—and adopt the following settings:

1. Trend and bias parameters g_h and b_h will be drawn from the normal distributions $N(0, 0.16)$ and $N(0, 0.09)$, respectively—where the notation is $N(\mu, \sigma^2)$—unless we state otherwise. The fundamentalist strategy—if it is present in the model—is always the *first* of the H strategies (i.e. only the term $n_{1,t}$ may ever correspond to the fraction of fundamentalists in the market). Should we *ex ante* (i.e. before the start of the simulation) indicate presence of fundamentalists in the model, the algorithm sets both of the parameters g_1 and b_1 to 0.

2. The noise term for each time period, ε_t, will be drawn from the uniform distribution $U(-0.05, 0.05)$. Benčík (2010) investigated behavior of the model in which the noise term was drawn from different uniform distributions and concluded that the behavior was similar.

3. Other parameters will be set as follows: the gross risk-free return rate, $R = 1 + r$, to 1.0001 and the term $\frac{1}{a \cdot \sigma^2}$ to 1. The choice of the gross risk-free return rate allows us to compare results of the simulations with real-world market data since $1.0001^{250} \cong 1.025$. Annual interest rate of 2.5 % can be considered a realistic risk-free rate.

The simulation *per se* consists of several *runs*; each run is characterized by a distinct intensity of choice parameter β. In our case, there will be 11 runs and the parameter β will gradually take values from 5 to 505 in increments of 50. Additionally, there are a number of *cycles* in each run; for each cycle, the parameters g_h and b_h are changed (i.e. randomly drawn from the aforementioned distributions) to guarantee that the simulation results are robust. We will work with 1000 repeat cycles. Finally, there are 500 *ticks* or *iterations* in each cycle; the ticks represent trading days (or, e.g., months). To summarize, we will have $475 \cdot 1000 \cdot 11 = 5225000$ realizations of x_t and some multiple of 5225000 of realizations of $U_{h,t}$ and $n_{h,t}$ depending on H, the number of agent classes, at hand after the end of the simulation.

5.1.1 Benchmark simulation

We ran a benchmark simulation of the original model specified by the system of Equations 5.1 in which the parameter $\delta = 0$ and the parameter $L = 0$, i.e. the model with *no* PT feature and only *synchronous* updating; number of total strategies $H = 4$ and fundamentalists were present in the model. Results are summarized below.

Table 5.1: Benchmark simulation summary statistics and p–value of JB test for normality of distribution for x_t in 11 runs with different β. There are fundamentalists and three other strategies in the model, i.e. $H = 4$.

β	Mean	Var.	Skew.	Kurt.	Min.	Max.	Med.	JB
5	-0.0012	0.0224	-0.5389	7.1002	-1.6221	0.8397	0.0029	0.000
55	0.0059	0.1236	0.1880	5.9124	-1.9789	2.4209	0.0032	0.000
105	0.0119	0.1225	0.0306	4.5966	-1.7256	1.6157	0.0072	0.000
155	0.0036	0.1142	-0.0214	3.8254	-1.4534	1.5590	0.0068	0.000
205	0.0030	0.1005	0.0713	3.4390	-1.2670	1.2970	0.0018	0.000
255	0.0077	0.0979	-0.0210	3.2504	-1.2113	1.1175	0.0042	0.000
305	0.0002	0.0860	-0.0420	3.0802	-1.0447	1.0593	-0.0003	0.000
355	-0.0067	0.0845	-0.0942	3.1020	-1.0269	1.0099	-0.0003	0.000
405	0.0089	0.0766	0.0013	3.1255	-0.9395	0.9324	0.0059	0.000
455	-0.0026	0.0738	-0.0285	3.0397	-0.8961	0.8954	-0.0004	0.000
505	-0.0001	0.0673	-0.0380	2.9215	-0.8525	0.8532	0.0022	0.000

Table 5.1 shows selected descriptive statistics of the x_t time series obtained from the benchmark simulation. Note that in each repeat cycle, first 5 % of realizations of x_t were discarded as the model needed some initial time to 'stabilize'. Clearly, the distributions of the deviations from the fundamental price are statistically different from the normal distribution, as is indicated by small p–values of the JB test for all βs. For increasing values of β, the distributions exhibit sample kurtosis closer to that of the normal distribution. Apparently,

the behavior of the model was most dramatic for $\beta \in \{55, 105, 155\}$—values of sample variance are highest, the same is true for minima and maxima of x_t. Figure 5.3 shows estimated PDFs of four selected time series corresponding to the Table 5.1 along with its counterparts obtained from the simulation with the PT feature. Kernel density estimation was used, with the Epanechnikov kernel function and Silverman's rule[3] for bandwidth selection. Epanechnikov kernel function was used as it is the most efficient kernel function (Wand & Jones, 1994, p. 31).

Figure 5.1 shows 4 sample plots[4] of the x_t time series obtained from the benchmark simulation. The interested reader might notice the gradually increasing chaotic behavior of the series as β gets larger. Figure 5.2 shows, on a log–log scale, the complementary CDF $\bar{F}_{|x_t|}(y)$, $\bar{F}_{|x_t|}(y) = P(|x_t| > y)$, for the 150 largest absolute deviations $|x_t|$ corresponding, respectively, to the time series plotted in Figure 5.1, along with a regression-based linear fit. Although not rigorously, the slopes of the regression lines are, in absolute values, estimates of the respective tail indices. These are equal to 10.77 for $\beta = 5$ ($R^2 = 0.849$), 9.13 for $\beta = 105$ ($R^2 = 0.979$), 8.74 for $\beta = 305$ ($R^2 = 0.953$), and 5.16 for $\beta = 505$ ($R^2 = 0.948$).

Having only an informative character, the plots nonetheless show possible existence of a power law in tails of the empirical distribution of $|x_t|$. It is important to emphasize, however, that the power law apparently does not hold universally for the whole tail; most extreme observations, for which the imaginary curvature is relatively significant and the realizations clearly do not follow the linear pattern estimated for the complete collection of the 150 observations, might exhibit a different tail index than the remaining observations do—the 'break point' is evidently around $\bar{F}_{|x_t|}(y) = 0.05$.

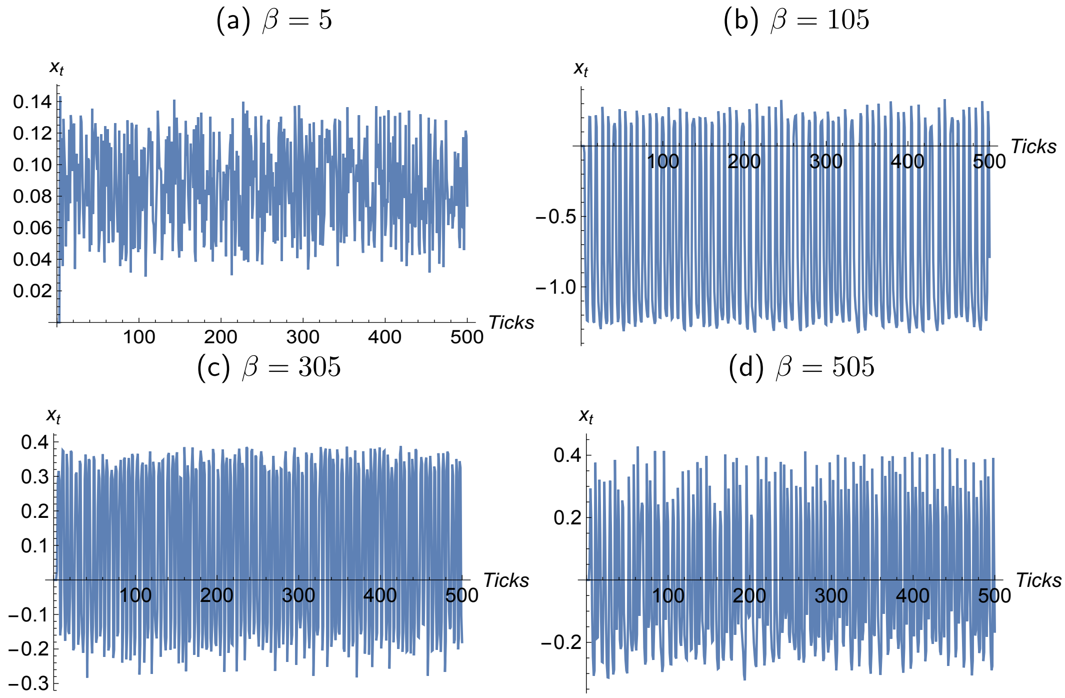
5.2 Employment of PT

This section summarizes our findings from a simulation with PT investors. Again, there are four trading strategies (i.e. $H = 4$), first of which is the fundamentalist strategy, and fractions of traders, $n_{h,t}$, update synchronously (i.e. $\delta = 0$). The simulation was run *together* with the benchmark case from Subsection 5.1.1 meaning that for each repeat cycle, exactly the same parameters

[3] Consult Silverman (1986) for details regarding the bandwidth selection rule.

[4] These particular 4 plots were selected 'randomly' from the pool of the 11000 plots which comprise one simulation.

Figure 5.1: Plots of sample x_t time series for one repeat cycle.



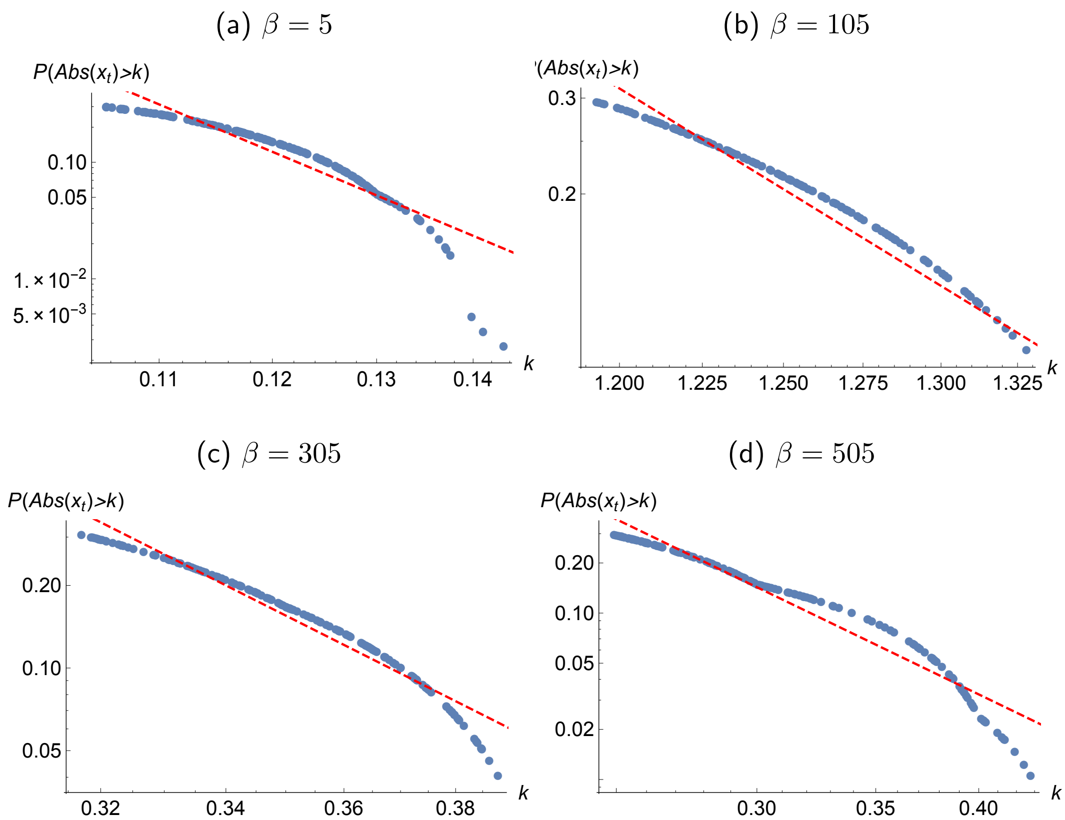
Source: Author's computations.

g_h and b_h, along with the noise term ε_t, were used—therefore *any* differences between the benchmark and the PT simulations can be attributed to the PT feature completely and unreservedly. The same approach is adopted in all simulations which compare the extended version of the model with its original predecessor. Important parameters, exclusive for the PT simulation, were given as follows:

1. The gain and loss parameters c_g and c_l were set to 1 and 2.5, respectively, to properly account for the gain–loss asymmetry. These particular numerical values were chosen based on the facts that *"... the disutility of giving something up is twice great as the utility of acquiring it,"* (Benartzi & Thaler, 1993), and that *"... losses hurt more than equal gains please; typically two to two-and-a-half times more."* (van Kersbergen & Vis, 2014, p. 163). Moreover, such setting of the respective parameters is well justified by Figure 2.2 which shows estimates of the PT value function. Initially, *all* strategies exhibit the PT feature, i.e. $L = 4$.

2. Length of the 'memory' used for the moving average of past prices, K, essential for determination of the reference point, was—initially—set to 10. Traders do *not* attach greater importance to the most recent past

Figure 5.2: Plots of the tails of sample x_t time series' empirical distributions and OLS fit. Benchmark case w/o the PT feature.



(a) $\beta = 5$

(b) $\beta = 105$

(c) $\beta = 305$

(d) $\beta = 505$

Source: Author's computations.

prices relative to more distant ones—i.e. the parameters $a_1, a_2, \ldots, a_K$ were all set to 1.

Under these circumstances, the reference point $B_{h,t}$, specified in Equation 4.2, becomes

$$B_{h,t} = \begin{cases} 1, & E_{h,t}\left(\mathbf{p}_{t+1}\right) > \tilde{p}_t = \tilde{p}_t\left(p_{t-1}, \ldots, p_{t-5}\right) \\ 2.5, & E_{h,t}\left(\mathbf{p}_{t+1}\right) \leqslant \tilde{p}_t = \tilde{p}_t\left(p_{t-1}, \ldots, p_{t-5}\right). \end{cases} \tag{5.2}$$

Table 5.2 summarizes descriptive statistics along with p–values of JB and Kruskal–Wallis tests of the x_t time series obtained from the simulation in which all trading strategies are endowed with the PT feature. The Kruskal–Wallis method tests, in layman's terms,[5] whether two samples originate from the same distribution—here we compare the empirical distributions of the x_t time series obtained from the PT simulation with those of the x_t time series obtained from the benchmark simulation with no PT traders (consult, please, Table 5.1). Again, the time series' distributions are statistically strongly different from the normal distribution. Moreover, addition of the PT feature causes, except for the case of $\beta = 5$ in which the p–value of the Kruskal–Wallis test is large, significant differences of the distributions with respect to those of the benchmark simulation. Notice especially the smaller variance of the time series with respect to the benchmark case (for lower values of β, the sample variance is about 25% smaller for the model with the PT feature than for the benchmark model without the feature), and also smaller extreme (i.e. maximum and minimum) values.

Figure 5.3 shows empirical PDFs of the x_t time series obtained from simulations of two models in question, i.e. the one with the PT feature and the one without it. Notice that although the Kruskal–Wallis test rejects the equality of the distributions for all values of β except for $\beta = 5$ (consult Table 5.2), the PDFs look relatively similar.

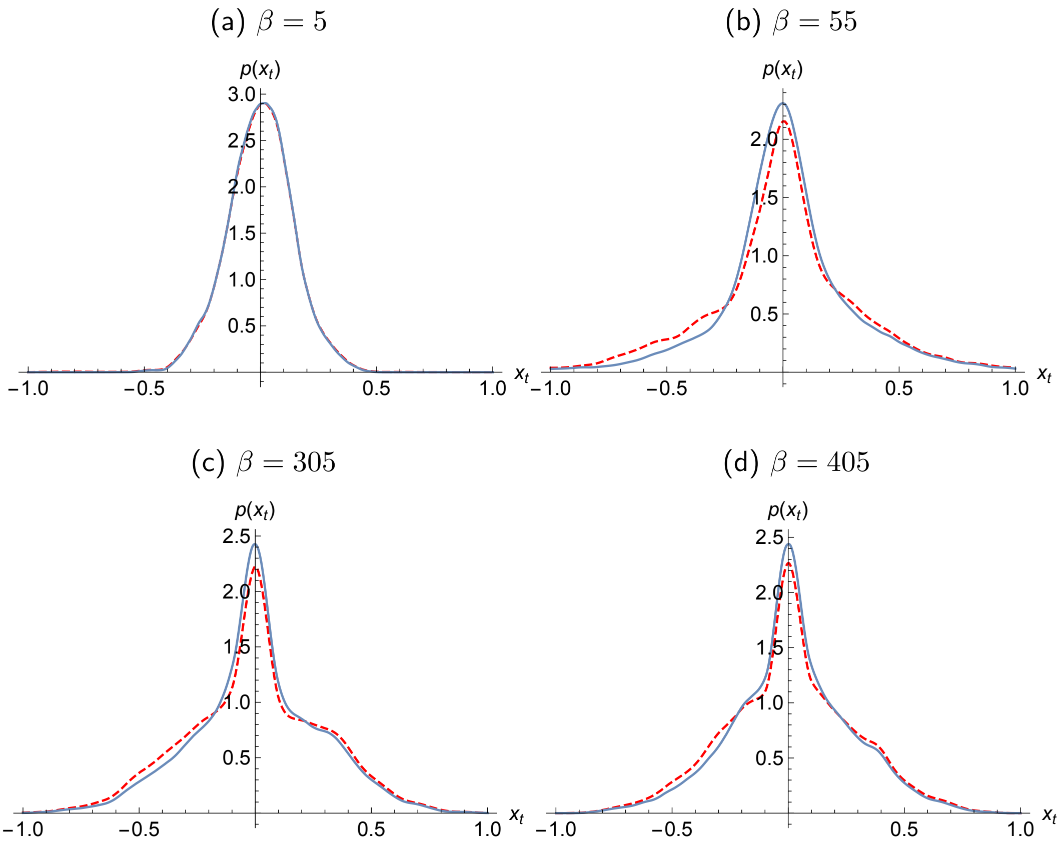
Figure 5.4 shows 4 sample plots of the x_t time series obtained from the PT simulation. The same parameters g_h, b_h, ε_t, and β were used as in the Figure 5.1. This fact means that any differences between the two figures are due solely to the PT feature. For $\beta \in \{305, 505\}$, notice the conspicuously lower sample variance of the time series and less dramatic appearance—the series is more 'well-behaved'.

[5] In statistics jargon, the Kruskal–Wallis test effectively performs a one-way analysis of variance on the ranks of the data. The test statistic is corrected for ties. The null hypothesis of the Kruskal–Wallis test states that the mean ranks of the groups are the same.

Table 5.2: PT simulation summary statistics of x_t and p–values of JB and Kruskal–Wallis tests in 11 runs with different β. There are fundamentalists and three other strategies in the model, i.e. $H = 4$, and all strategies have the PT feature, i.e. $L = 4$.

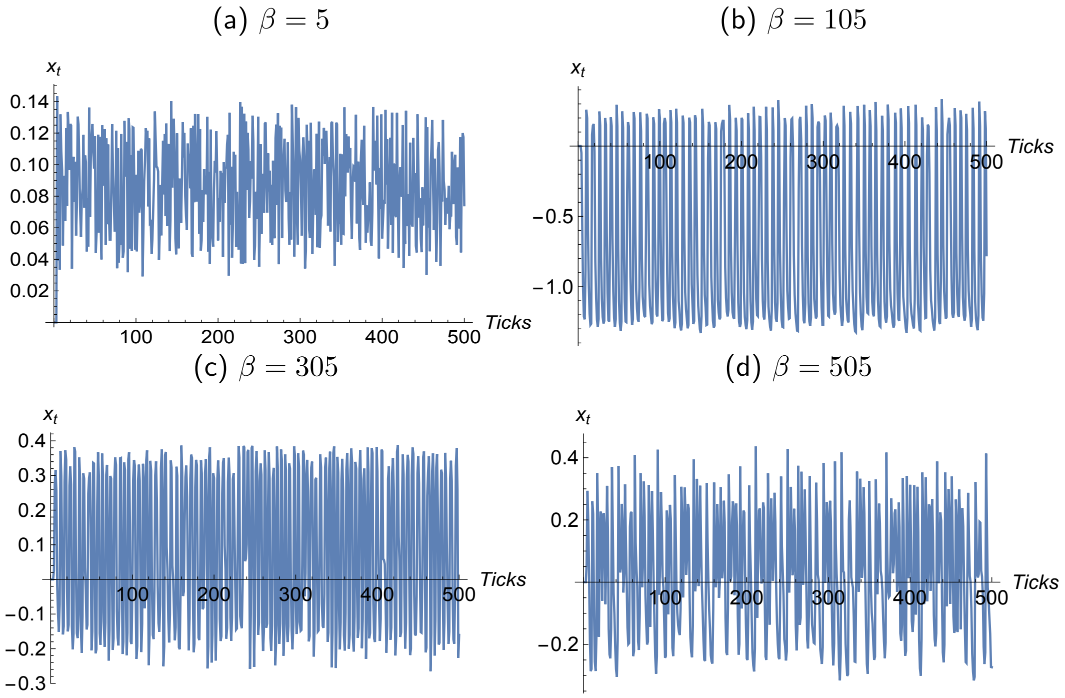
$\boldsymbol{\beta}$	**Mean**	**Var.**	**Skew.**	**Kurt.**	**Min.**	**Max.**	**JB**	**KW**
5	-0.0002	0.0201	-0.0691	3.2125	-0.7410	0.5000	0.000	0.747
55	0.0171	0.0933	0.0828	6.9381	-1.9744	2.3506	0.000	0.000
105	0.0178	0.0999	-0.0308	5.3802	-1.7316	1.6008	0.000	0.000
155	0.0109	0.0954	-0.0511	4.3116	-1.4001	1.4358	0.000	0.000
205	0.0129	0.0854	0.0808	3.8215	-1.2733	1.2907	0.000	0.000
255	0.0139	0.0841	-0.0224	3.5736	-1.2105	1.1040	0.000	0.000
305	0.0087	0.0742	-0.0307	3.3799	-1.0395	1.0301	0.000	0.000
355	0.0004	0.0731	-0.1115	3.4346	-1.0244	1.0022	0.000	0.000
405	0.0151	0.0670	0.0062	3.4075	-0.9194	0.9322	0.000	0.000
455	0.0032	0.0648	-0.0192	3.2903	-0.9092	0.8733	0.000	0.000
505	0.0072	0.0589	-0.0424	3.1610	-0.8529	0.8406	0.000	0.000

Figure 5.3: Plots of PDFs of the x_t time series obtained from simulation with (blue solid line) and without (red dashed line) the PT feature.



Source: Author's computations.

Figure 5.4: Plots of sample x_t time series of one repeat cycle with PT.



Source: Author's computations.

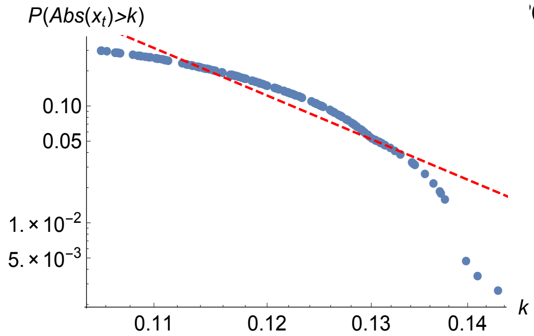
Figure 5.5 shows, on a log–log scale, the complementary CDF $\bar{F}_{|x_t|}(y)$ for the 300 largest absolute deviations $|x_t|$ corresponding, respectively, to the time series plotted in Figure 5.4, along with a regression-based linear fit. The estimates of the respective tail indices (i.e. the opposites of the estimated slope coefficients) are equal to 10.33 for $\beta = 5$ ($R^2 = 0.835$), 9.41 for $\beta = 105$ ($R^2 = 0.981$), 7.52 for $\beta = 305$ ($R^2 = 0.937$), and 5.28 for $\beta = 505$ ($R^2 = 0.934$). The OLS fits provide roughly the same R^2, although the most extreme observations do, again, exhibit considerable curvature and departure from *any* power law, mainly in the region for which $\bar{F}_{|x_t|}(y) < 0.05$

These findings are summarized in Figure 5.6 which merges Figure 5.2 and Figure 5.5 and shows, on a log–log scale, the complementary CDFs for largest 150 x_t observations for one repeat cycle with and without the PT feature. One might notice the similarity of the tails for the lowest value of β and the subsequent departure of the tails the value of β increases.

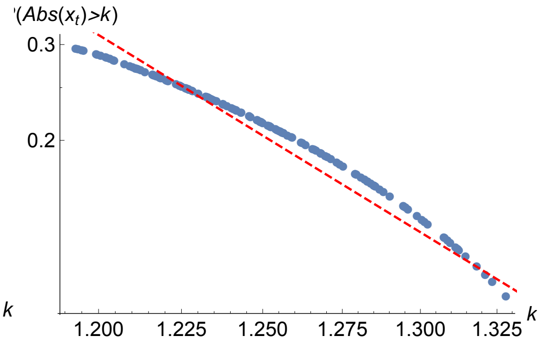
Figure 5.7 shows autocorrelation and partial autocorrelation functions of the empirical x_t series for one repeat cycle without the PT feature and with it. The same parameters g_h, b_h, ε_t, and β were used in both cases to maintain mutual comparability. The series with $\beta = 505$ were used in this figure, the interested reader might find the plots in Figure 5.1 and Figure 5.4. At first glance, no

Figure 5.5: Plots of the tails of sample x_t time series' empirical distributions with the PT feature employed and OLS fit.

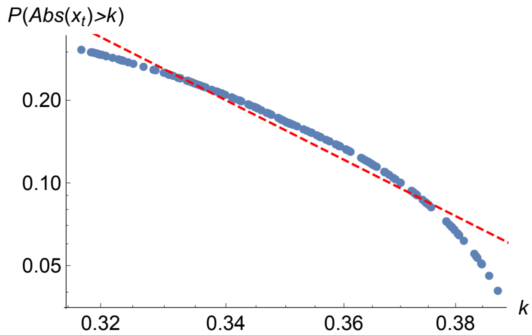
(a) $\beta = 5$



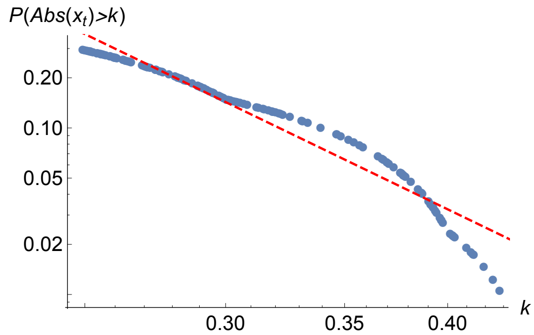
(b) $\beta = 105$



(c) $\beta = 305$

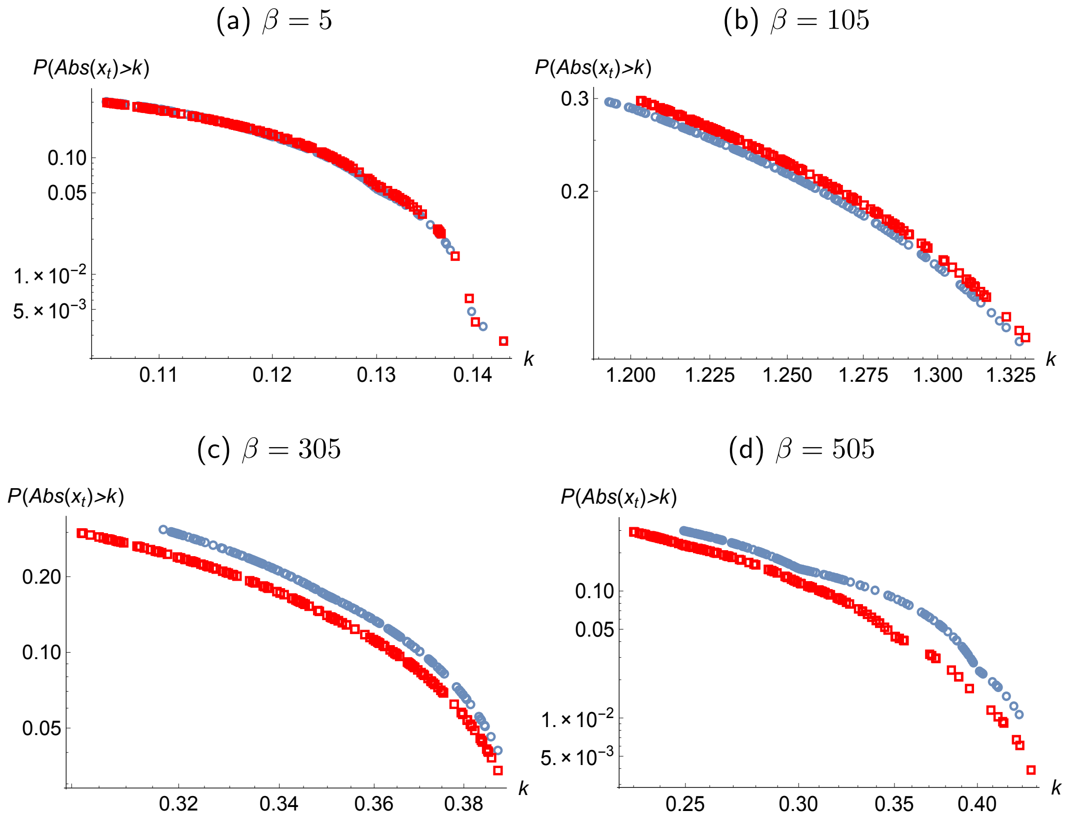


(d) $\beta = 505$



Source: Author's computations.

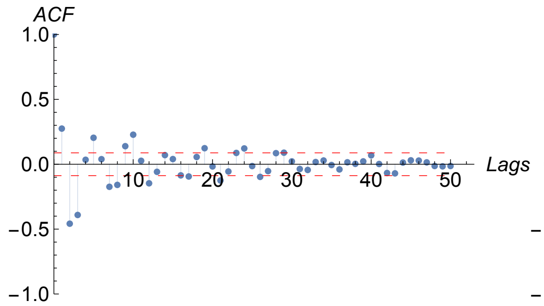
Figure 5.6: Plots of the tails of sample x_t time series' empirical distributions with the PT feature (red squares) and without it (blue circles).



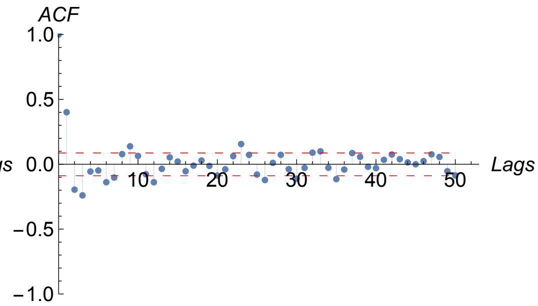
Source: Author's computations.

Figure 5.7: Plots of autocorrelation and partial autocorrelation functions the x_t time series for $\beta = 505$.

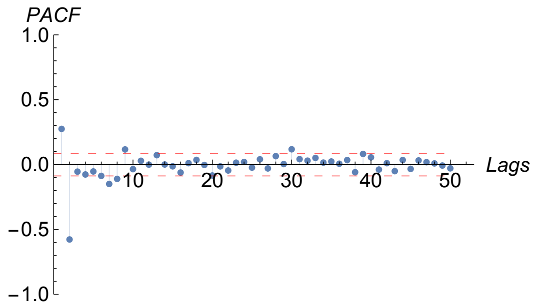
(a) No PT, ACF



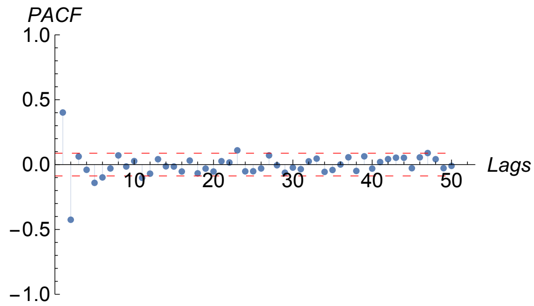
(b) PT, ACF



(c) No PT, PACF



(d) PT, PACF



Source: Author's computations.

apparent striking differences are noticeable—yet, a closer look reveals a slightly different structure for the ACFs for approximately first 5 lags, in which the time series without the PT feature exhibits considerably higher autocorrelation. For subsequent lags (i.e. for lags 6–50), absolute values of all autocorrelation and partial autocorrelation functions are very low and clearly within the 95% white noise band indicated by the red dashed lines.

5.2.1 Aggregate characteristics

This subsection summarizes aggregate qualitative characteristics of the price deviations time series obtained from the simulations. The main aim of this effort is to compare the model without the PT feature (i.e. the one in which $L = 0$) and the one in which *all* trading strategies have the feature (i.e. $L = 4$) namely in terms of time dependence (autocorrelation and moving average patterns) in the x_t time series and x_t^2 time series, and incidence of fat tails.

The method we used for assessment of time dependence using aggregate data was the following: in each repeat cycle and for all values of β, we fitted a time series model to the simulated x_t (or x_t^2) data, saved the respective coefficients, and—using the kernel density estimation[6]—constructed an empirical distribution of these coefficients. The optimal model was selected based on the Akaike Information Criterion (AIC)—the simulations showed that the data generally fitted an Autoregressive Moving Average (ARMA) model best; therefore the coefficients saved were always those from this model, i.e. $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q$ if the model is specified as[7]

$$X_t = c + \sum_{i=1}^{p} \alpha_i \cdot X_{t-i} + \sum_{i=1}^{q} \beta_i \cdot \varepsilon_{t-i} + \varepsilon_t. \tag{5.3}$$

Finally, we compared PDFs of the distributions using the Kruskal–Wallis test.

Time dependence of x_t

Table 5.3 summarizes expected values of the estimated distributions of α_1 (first autoregressive coefficient) and β_1 (first moving average coefficient) and p–values of the Kruskal–Wallis test applied to the x_t time series obtained from models with and without the PT feature. The distributions of the autoregressive coefficient are—except for $\beta = 305$—statistically significantly different. This fact further supports the finding that the PT extensions changes the behavior of the HAM. On the other hand, p–values of the test applied to the first moving average coefficient fail to reject the null hypothesis of equal distributions at a reasonable significance level. This fact indicates that the PT extensions affects the autoregressive structure of the x_t time series more than it does the moving average one. Notice that for most values of β, both coefficients tend to be larger for the PT extended model; the realizations of x_t seem to be slightly more dependent on previous realizations x_{t-1}.

Figure 5.8 shows estimated PDFs of the MA(1) coefficient β_1 from the model specified in Equation 5.3 for the x_t time series. Note that for different repeat cycles (and different values of β), the optimal models naturally exhibited different orders p and q—yet, the β_1 coefficient *always* corresponds to the moving average relationship of the first lag, regardless of the value of q. The same is true for the AR(1) coefficient α_1 and the value of p.[8] The figure suggests that

[6] Silverman's (1986) rule for bandwidth selection was used, along with the Epanechnikov kernel function.

[7] Or, alternatively, $X_t^2 = c + \sum_{i=1}^{p} \alpha_i \cdot X_{t-i}^2 + \sum_{i=1}^{q} \beta_i \cdot \varepsilon_{t-i} + \varepsilon_t$ if the squared deviations are in question.

[8] In other words, we let the AIC determine the best model. If we had instead chosen e.g. AR(1) or MA(1) model, we could have omitted a better model with lower AIC.

overall, the behavior of both models is relatively similar—yet, for $\beta \in \{5, 505\}$, the series exhibit somewhat less moving average dependence which is depicted by the higher peaks of the respective PDFs and higher expected values (consult, please, Table 5.3).

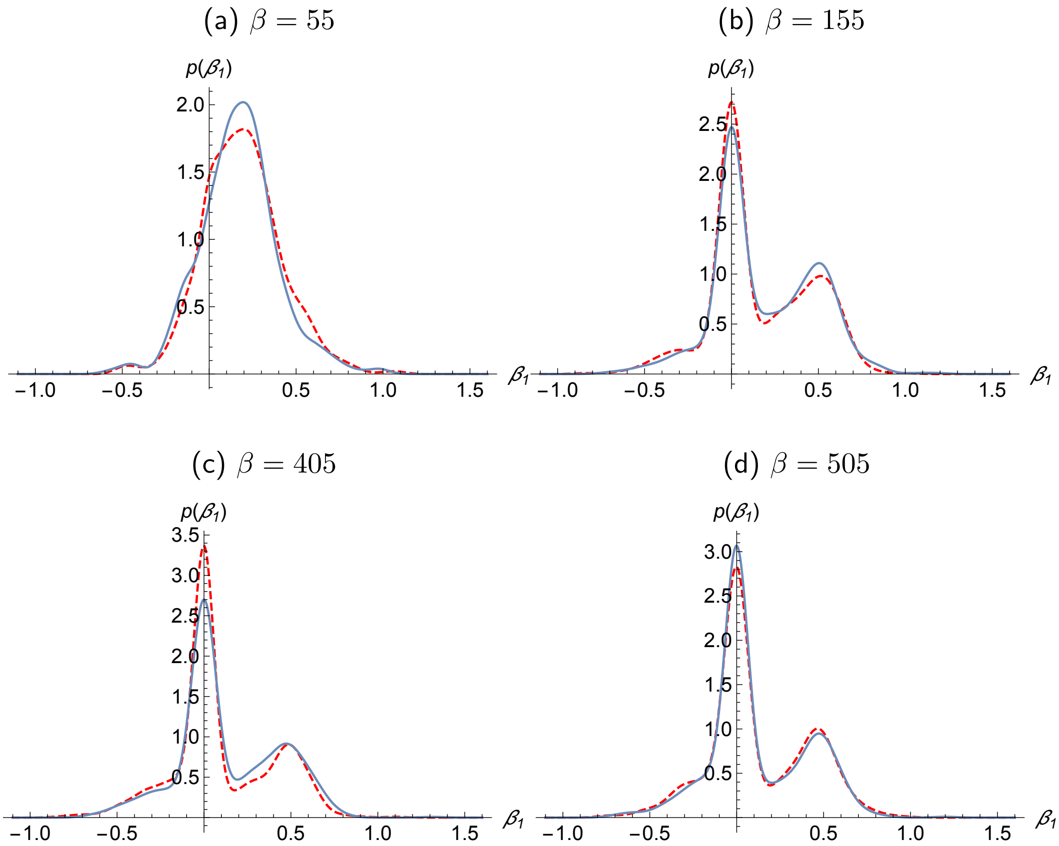
Table 5.3: Expected value of the empirical distributions of α_1 (AR) and β_1 (MA) coefficients and p–value of the Kruskal–Wallis test applied to x_t with and without the PT feature.

$\boldsymbol{\beta}$	**MA**	**MA^{PT}**	**MA^{KW}**	**AR**	**AR^{PT}**	**AR^{KW}**
5	0.1883	0.1691	0.2137	0.3068	0.2578	0.0159
55	0.1908	0.2272	0.0157	0.2858	0.3546	0.0003
105	0.1668	0.2041	0.0274	0.2278	0.2891	0.0011
155	0.1607	0.1848	0.1386	0.1997	0.2797	0.0000
205	0.1418	0.1711	0.0203	0.1947	0.2814	0.0000
255	0.1344	0.1579	0.0724	0.1901	0.2523	0.0022
305	0.1253	0.1441	0.2213	0.2173	0.2481	0.2745
355	0.1336	0.1602	0.0524	0.1906	0.2815	0.0000
405	0.0971	0.1331	0.0130	0.1986	0.2415	0.0184
455	0.1008	0.1229	0.0753	0.2138	0.2639	0.0158
505	0.1188	0.1152	0.9492	0.2097	0.2659	0.0050

Time dependence of x_t^2

Table 5.4 summarizes expected values of the estimated distributions of the AR(1) coefficient α_1 and MA(1) coefficient β_1 and p–values of the Kruskal–Wallis test applied to the x_t^2 time series obtained from models with and without the PT feature. The empirical distributions of β_1 are, again, not statistically different. Moreover, the p–values of the Kruskal–Wallis test are even higher. On the other hand, the distributions of α_1 obtained from the PT extended model are statistically different from their non-PT counterparts and their expected values are greater than those obtained from the non-PT model. This fact implies that the phenomenon of volatility clustering is more significant and recognizable in our extended version of the HAM; such a finding is consistent with real-world market data (Cont, 2001).

Figure 5.8: Plots of PDFs of the MA(1) coefficient β_1 of optimal ARMA models fitted to x_t time series with (blue line) and without (red dashed line) the PT feature.



Source: Author's computations.

Table 5.4: Expected value of the empirical distributions of α_1 (AR) and β_1 (MA) coefficients and p–value of the Kruskal–Wallis test applied to x_t^2 with and without the PT feature.

$\boldsymbol{\beta}$	**MA**	**MA^{PT}**	**MA^{KW}**	**AR**	**AR^{PT}**	**AR^{KW}**
5	0.1718	0.1601	0.7209	0.2582	0.2266	0.0586
55	0.1756	0.2219	0.0047	0.1825	0.2415	0.0016
105	0.1598	0.1665	0.5712	0.1038	0.1884	0.0000
155	0.1199	0.1554	0.0151	0.1052	0.1758	0.0001
205	0.1534	0.1545	0.9527	0.1117	0.1816	0.0002
255	0.1079	0.1299	0.2425	0.0846	0.1567	0.0000
305	0.1009	0.1038	0.9450	0.0871	0.1339	0.0136
355	0.0993	0.0999	0.9752	0.0875	0.1614	0.0000
405	0.0895	0.1299	0.0092	0.1045	0.1394	0.0469
455	0.1029	0.1207	0.1957	0.0911	0.1639	0.0014
505	0.1044	0.1159	0.4161	0.0967	0.1836	0.0000

Aggregate tails

Table 5.5 shows estimated tail indices of the x_t time series for 500 repeat cycles[9] with and without the PT feature; as many repeat cycles were used to generate the data (i.e. 500 different setups for g_h, b_h, and ε_t were used), the estimates are more robust than those for only one repeat cycle (shown e.g. in Figure 5.6). The values of R^2 can be considered relatively satisfactory for the power law fit—moreover, the PT extended model's tail indices are in most cases smaller than those of the non-extended model (and thus closer to the real-world ones—consult Section 6.3 of Chapter 6) and the coefficient of determination is higher. Nonetheless, it is not clear whether the power law is really the ideal model for this type of HAM as the coefficients of determination are smaller than those of the real-world indices (consult e.g. Cont, 2001 for a discussion of real-world tail indices).

Table 5.5: Estimated tail indices of the x_t time series along with R^2 for the original and PT extended versions of the model.

With PT			Without PT		
β	Tail	R^2	β	Tail	R^2
5	6.842	0.933	5	6.401	0.966
105	8.731	0.938	105	7.736	0.955
205	8.469	0.918	205	9.381	0.903
305	10.652	0.955	305	11.643	0.936
405	11.636	0.967	405	12.491	0.963
505	10.824	0.933	505	11.438	0.916

5.2.2 PT vs. non-PT traders

We may now relax the assumption that *all* trading strategies are endowed with the PT feature and examine behavior of the model by running additional simulations with $L \leqslant H$, i.e. simulations in which some of the trading strategies exhibit loss aversion and gain–loss asymmetry, and some do not. Additionally, more values of the parameter K, length of the moving average considered for reference point, can be inspected. Table 5.6 summarizes simulations with $L = 1$, $L = 2$, $L = 3$, and different values of K; fundamentalist strategy was present in the model as the *first* strategy—i.e. $L = 1$ corresponds to a situation in

[9]For technical reasons, we have used only 500 repeat cycle as the process of obtaining the tail index is difficult in terms of computational time which grows steadily with increasing number of repeat cycles.

the market in which there are PT fundamentalists and three other (non-PT) chartist strategies, i.e. $H = 4$. The Kruskal–Wallis test compares, in this case, the distributions obtained from the simulations with the PT feature with those obtained from a simulation without it, i.e. the one for which $L = 0$.[10] To maintain mutual comparability, the same parameters g_h, b_h, and ε_t were used for each value of $L \neq 0$ and for $L = 0$.

Table 5.6: P–value of the Kruskal–Wallis test for different L, K and β.

	$K = 1$ (L)				$K = 5$ (L)		
β	1	2	3	β	1	2	3
5	0.99637	0.47500	0.72023	5	0.00000	0.00000	0.00000
55	0.08591	0.00000	0.00000	55	0.90140	0.79120	0.01819
105	0.04846	0.00000	0.00000	105	0.29019	0.00000	0.00000
155	0.02313	0.00000	0.00000	155	0.04389	0.00000	0.00000
205	0.04640	0.00000	0.00000	205	0.05195	0.00000	0.00000
255	0.00436	0.00000	0.00000	255	0.01827	0.00000	0.00000
305	0.00050	0.00000	0.00000	305	0.00003	0.00000	0.00000
355	0.00000	0.00000	0.00000	355	0.00003	0.00000	0.00000
405	0.00000	0.00000	0.00000	405	0.00000	0.00000	0.00000
455	0.00000	0.00000	0.00000	455	0.00000	0.00000	0.00000
505	0.00000	0.00000	0.00000	505	0.00000	0.00000	0.00000
	$K = 10$ (L)				$K = 15$ (L)		
β	1	2	3	β	1	2	3
5	0.82531	0.66399	0.57620	5	0.94767	0.50089	0.04147
55	0.34511	0.00000	0.00000	55	0.90644	0.00000	0.00000
105	0.59452	0.00000	0.00000	105	0.64302	0.00000	0.00000
155	0.69513	0.00000	0.00000	155	0.81774	0.00000	0.00000
205	0.48258	0.00000	0.00000	205	0.13191	0.00000	0.00000
255	0.00112	0.00000	0.00000	255	0.00903	0.00000	0.00000
305	0.00896	0.00000	0.00000	305	0.01347	0.00000	0.00000
355	0.00102	0.00000	0.00000	355	0.00008	0.00000	0.00000
405	0.00004	0.00000	0.00000	405	0.00000	0.00000	0.00000
455	0.00000	0.00000	0.00000	455	0.00000	0.00000	0.00000
505	0.00000	0.00000	0.00000	505	0.00000	0.00000	0.00000

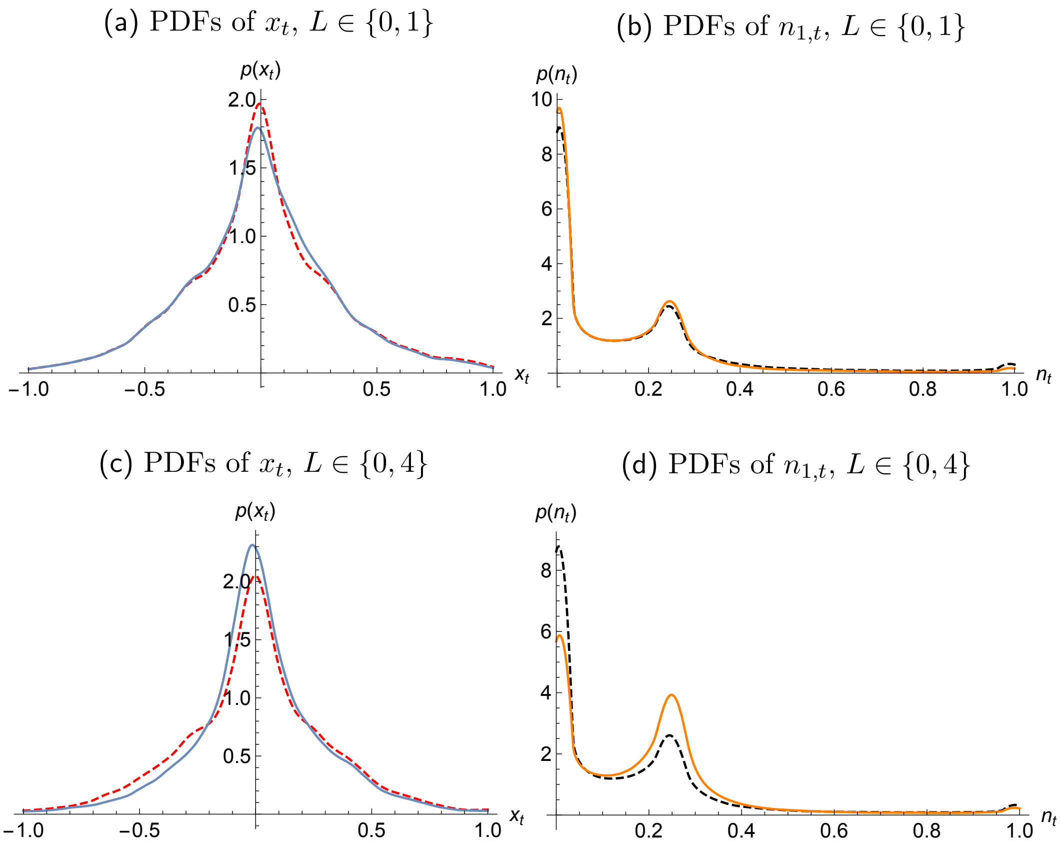
Figure 5.9 further examines, for $\beta = 105$ and $K = 15$, the cases in which $L = 1$ and $L = 4$, i.e. the situation in which only the first strategy (i.e. the

[10]We ran another 'benchmark' simulation of the model without the proposed extensions, i.e. for the KW test, we used *different* benchmark than that examined in Subsection 5.1.1.

fundamentalist strategy) has the PT feature and the one in which all strategies have the PT feature, respectively, and compares these situations to the benchmark case of $L = 0$. Estimated densities of the x_t time series are plotted in the left-hand side of the figure while the right-hand side of the figure shows estimated densities of the $n_{1,t}$ time series, i.e. of the fraction of traders using the fundamentalist strategy. Apparently—as can be also seen from Table 5.6—the behavior of the model for $L = 1$ is relatively similar to that of the benchmark case; Kruskal–Wallis test does not reject the null hypothesis and the estimated densities of x_t are very similar. Yet, PT fundamentalists are driven out of the market more strongly—this finding can be inferred from higher peak of the respective distribution around 0. The PT feature, manifested in significant loss aversion, poses a relatively heavy 'burden' for the fundamentalists when they face chartists who are not loss-averse. On the other hand, when all trading strategies have the PT feature, the behavior of the model is significantly different from the benchmark case—the PT feature *stabilizes* the market and rules out a fraction of extreme price deviations which were present in the benchmark case. Moreover, fundamentalists are able to *survive* in the market more easily around the equilibrium fraction of $n_{1,t} = 0.25$—such a finding is in contrast with the benchmark case in which fundamentalists were driven off the market by chartists more often (again, consider the 'peakedness' of the respective distributions around 0 and 0.25).

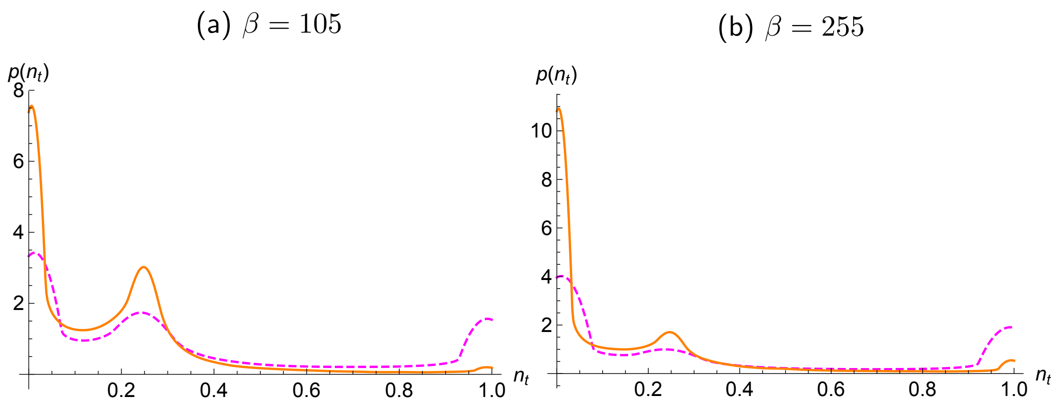
Figure 5.10 shows estimated densities of $n_{1,t}$ and $n_{4,t}$ for $L = 3$ and $K = 15$, i.e. fractions of PT fundamentalists and non-PT chartists in a model in which one chartist trading strategy does *not* have the PT feature, for different values of β. Notice that, as β gets larger, the non-PT chartist strategy becomes increasingly popular and dominates the market (i.e. $n_{4,t} \equiv 1$) at non-negligible amount of time. Moreover, fundamentalists are less likely to be able to survive in the market than they were when they faced only PT traders—this effect can be best inferred from the case in which $\beta = 105$, i.e. β is the same as it was in Figure 5.9—while for $L = 4$, $p(n_{1,t}) = 6$ for $n_{1,t} \to 0$, for $L = 3$ we have $p(n_{1,t}) = 7.6$ for $n_{1,t} \to 0$ and the relatively frequent disappearance of fundamentalists can thus be attributed the presence of non-PT chartists.

Figure 5.9: Behavior of the model for different L versus the benchmark case of $L = 0$; $\beta = 105$ and $K = 15$, $L = 0$ depicted by dashed lines.



Source: Author's computations.

Figure 5.10: Estimated densities of $n_{1,t}$ (orange solid line) and $n_{4,t}$ (dashed magenta line) for $L = 3$ and $K = 15$.



Source: Author's computations.

5.3 Employment of Asynchronous Updating

We may now proceed to simulations with the AU feature. Let us remind the interested reader that the fractions $n_{h,t}$ of trading strategies are given as

$$n_{h,t} = \delta \cdot n_{h,t-1} + (1-\delta) \cdot \frac{\exp\left(\beta \cdot U_{h,t-1}\right)}{\sum_{h=1}^{H} \exp\left(\beta \cdot U_{h,t-1}\right)} \tag{5.4}$$

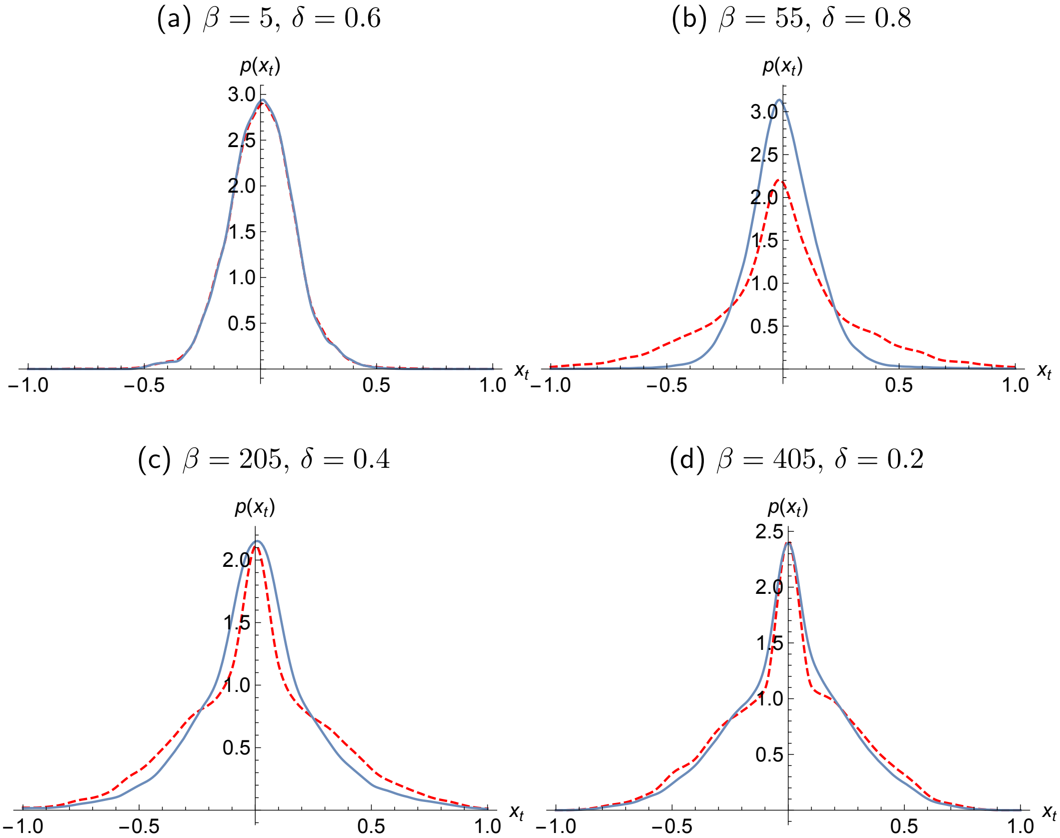
where δ will vary. All other parameters and settings remain the same as they were specified in Section 5.1.

Table 5.7 shows summary statistics and p–value of the Kruskal–Wallis test for the x_t time series obtained from simulations with varying δ. The Kruskal–Wallis test compares the models with the AU feature to those without it (i.e. to those in which $\delta = 0$). Apparently, the distributions are not significantly different in majority of cases—for $\delta \in \{0.2, 0.6\}$, empirical distribution of x_t are different from the that of the benchmark case for only one value of β out of eleven possible; on the other hand, for $\delta \in \{0.4, 0.8\}$ the distributions are statistically different for four values of β. The JB test produced p–values lower than 0.0000 in all cases, therefore the null hypothesis of normally distributed price deviations can be soundly rejected $\forall \delta, \beta$. Figure 5.11 shows estimated PDFs of the x_t time series whose statistics are summarized in Table 5.7 along with PDFs obtained from the benchmark simulation. Apparently, the PDFs are—from the visual perspective at least—somewhat less similar than they were in the PT extended models.

Although the 'aggregate' summary statistics do not show—in majority of cases—significant differences, some qualitative variation between the original version and that extended with AU certainly exists. Figure 5.12 shows plots of autocorrelation function for $\beta = 55$ and different values of δ. The slowly decaying, 'sine' pattern of the function is highly noticeable and striking—such phenomenon was not apparent either for the default model without any extensions or for the PT feature extended model (see Figure 5.7 for comparison) and is a manifestation of the AU feature.

Figure 5.13 shows, on a log–log scale, complementary CDFs for 150 largest absolute deviations $|x_t|$ obtained from *one* repeat cycle of simulations of the aforementioned models. Clearly, AU *stabilizes* the model to some extent, larger deviations from the fundamental price are somewhat less likely; such tendency is most apparent for higher values of β. The estimates of the respective tail indices (i.e. the opposites of the estimated slope coefficients) for the AU extended

Figure 5.11: Plots of PDFs of the x_t time series obtained from simulation with (blue solid line) and without (red dashed line) the AU feature.

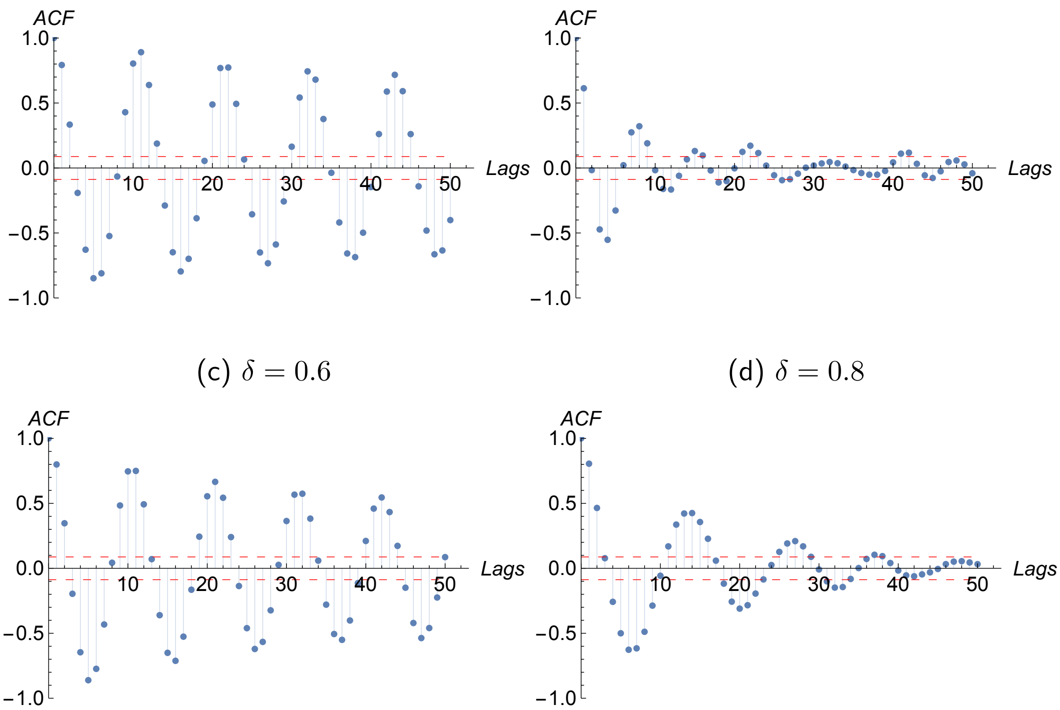


Source: Author's computations.

Figure 5.12: ACF plots for $\beta = 55$.

(a) $\delta = 0.2$ (b) $\delta = 0.4$

(c) $\delta = 0.6$ (d) $\delta = 0.8$



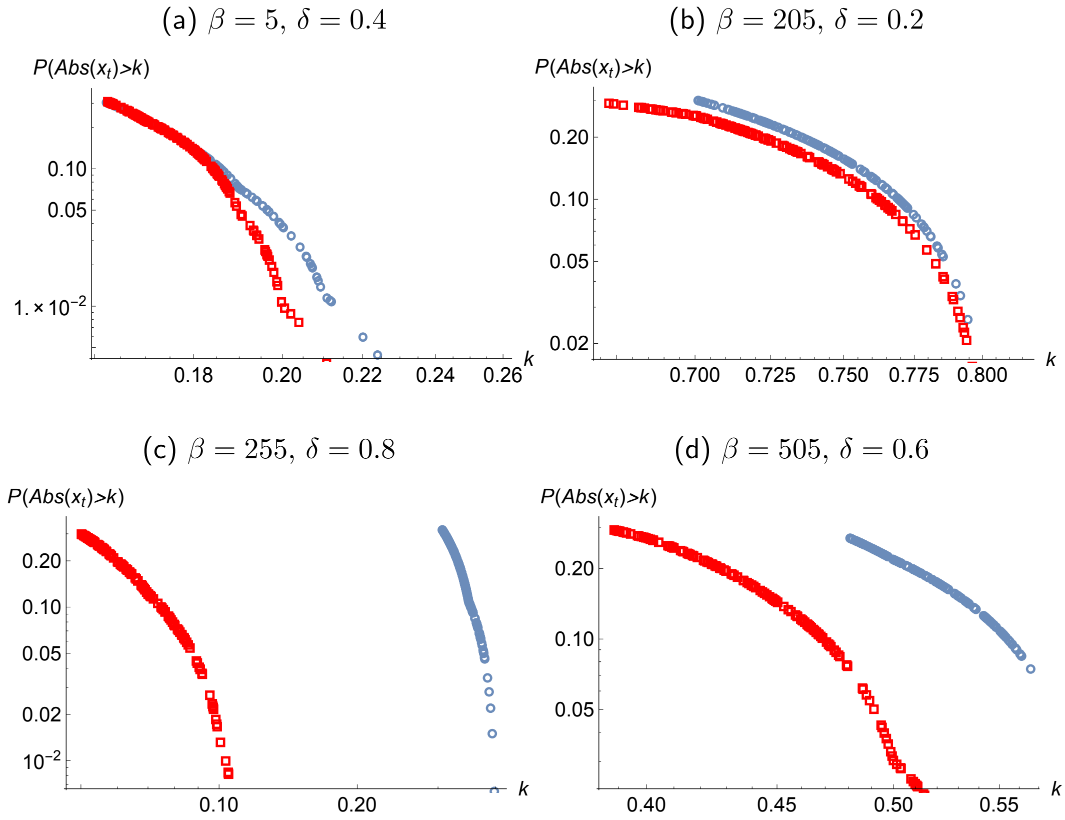
Source: Author's computations.

model (depicted with red squares) are equal to 15.32 for $\beta = 5$ ($R^2 = 0.907$), 16.8 for $\beta = 205$ ($R^2 = 0.786$), 4.19 for $\beta = 255$ ($R^2 = 0.908$), and 9.46 for $\beta = 505$ ($R^2 = 0.892$). While the tails of the two compared models are practically the same for $\beta \in \{5, 205\}$, there is a marked difference between them for $\beta \in \{255, 505\}$.

Most importantly, however, the simulations with the AU extended model revealed certain *economic* implications, namely related to evolution and incidence of the fundamentalist strategy $n_{1,t}$. While in the default model with no extensions and increasing β the fundamentalists survived in the market only with dramatically increasing 'difficulties',[11] they were able to realize profits (and thus remain in the market) in the AU extended model. This phenomenon is emphasized for larger values of δ. Figure 5.14 visually summarizes this finding; the interested reader may compare it with Figure 5.9 to gain some insight on how the AU extension differs from the PT extensions in this aspect.

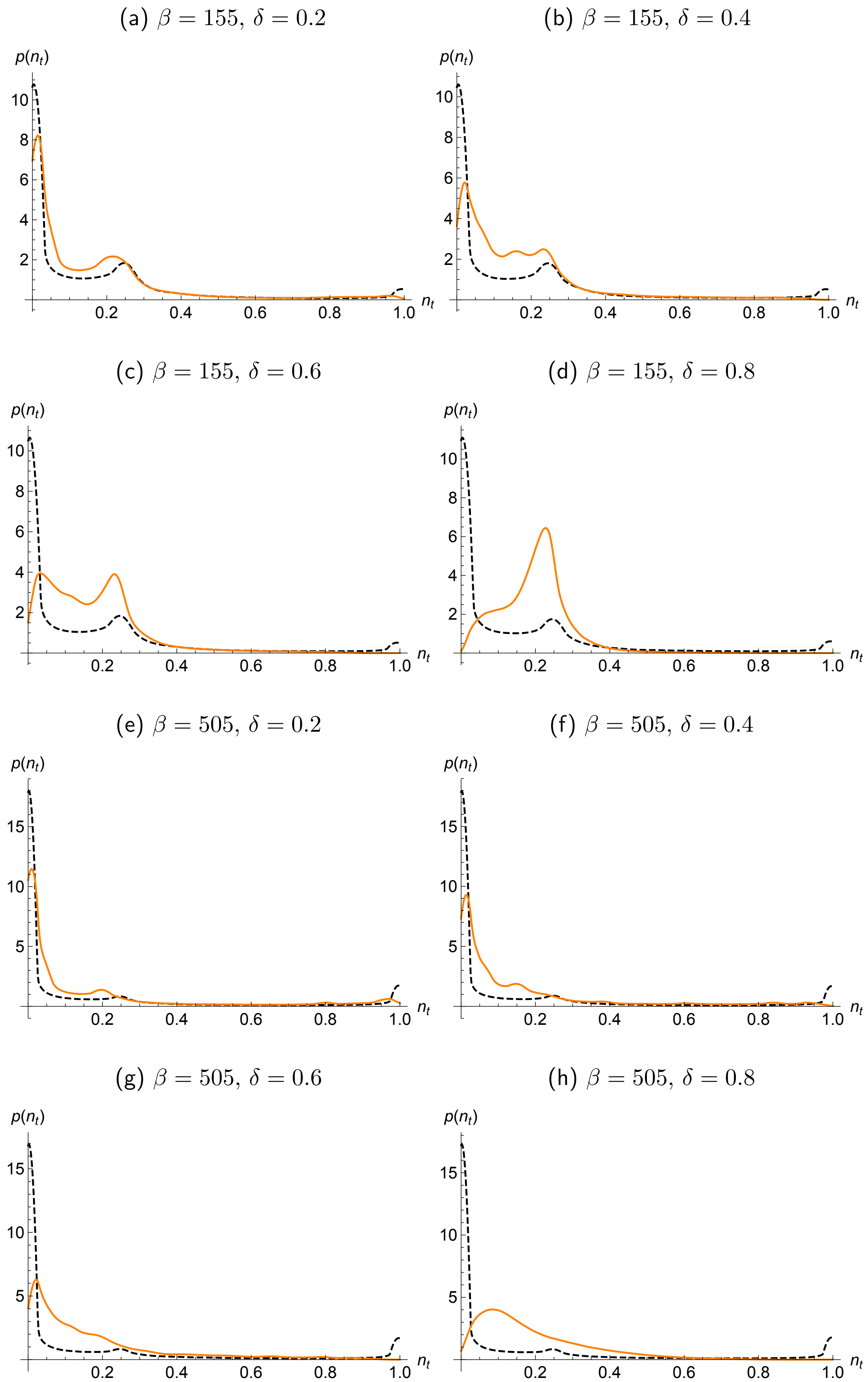
[11]By 'increasing difficulties' we mean the fact that the expected value of the empirical distributions of $n_{1,t}$ decreased with increasing β.

Figure 5.13: Plots of the tails of sample x_t time series' empirical distributions with AU (red squares) and and without it (blue circles) for one repeat cycle.



Source: Author's computations.

Figure 5.14: Incidence of the fundamentalist strategy $n_{1,t}$; estimated PDFs, benchmark model depicted with black dashed line.



Source: Author's computations.

Table 5.7: Summary statistics and p–value of the KW test for x_t series with AU.

$\delta = 0.2$

β	Var.	Skew.	Kurt.	Min.	Max.	KW
5	0.0202	-0.1498	3.3813	-0.7228	0.5494	0.8700
55	0.1005	0.6540	14.4917	-2.4994	5.0058	0.0759
105	0.1051	-0.2003	5.6205	-1.8740	1.7736	0.6231
155	0.0961	0.1240	4.2409	-1.4188	1.4835	0.0497
205	0.0848	0.0268	3.5895	-1.2143	1.2924	0.1984
255	0.0861	-0.4586	7.6114	-3.3343	1.6688	0.5937
305	0.0794	-0.5218	8.1653	-3.1963	1.8027	0.4952
355	0.0702	-0.0799	3.5361	-1.0125	1.0297	0.7967
405	0.0630	-0.0896	3.3924	-0.9634	0.9045	0.8908
455	0.0609	0.0048	3.4001	-0.9403	0.9015	0.3186
505	0.0581	0.0183	3.2907	-0.8186	0.8637	0.7478

$\delta = 0.4$

β	Var.	Skew.	Kurt.	Min.	Max.	KW
5	0.0238	-2.4778	48.5050	-3.3419	0.9075	0.0962
55	0.0845	0.3621	9.8956	-2.0048	2.2661	0.0430
105	0.0855	-0.0698	6.8941	-1.8385	2.4327	0.2396
155	0.0817	-0.0583	4.9833	-1.4885	1.4489	0.1601
205	0.0742	-0.1080	5.5369	-2.8937	1.3335	0.7427
255	0.0714	-0.1207	4.3089	-1.9444	1.1234	0.0015
305	0.0656	-0.0601	3.7862	-1.0831	1.1138	0.0442
355	0.0617	0.0421	3.7431	-0.9787	1.0099	0.1113
405	0.0582	0.2655	4.5886	-0.9407	1.7818	0.7284
455	0.0547	0.0753	3.8876	-0.8889	1.4084	0.0099
505	0.0502	-0.1539	4.8034	-2.7208	0.7873	0.4126

$\delta = 0.6$

β	Var.	Skew.	Kurt.	Min.	Max.	KW
5	0.0202	-0.0182	3.5142	-0.5695	0.6343	0.6700
55	0.0615	0.1839	13.0803	-2.0838	2.3070	0.8818
105	0.0726	0.0041	6.5930	-1.5471	1.5226	0.0504
155	0.0650	0.1866	6.3505	-1.4574	1.9262	0.3267
205	0.0682	-1.0721	20.4282	-4.6893	2.4899	0.0019
255	0.0553	-0.0564	4.5545	-1.1387	1.1098	0.0013
305	0.0533	-0.0021	4.4097	-1.4223	1.0843	0.7560
355	0.0474	-0.0016	4.1278	-1.3252	0.9713	0.0748
405	0.0468	-0.0612	4.3411	-0.9936	1.5137	0.5567
455	0.0407	-0.0373	3.8318	-0.8678	0.8642	0.6047
505	0.0415	0.0104	3.8907	-0.8288	0.8331	0.8233

$\delta = 0.8$

β	Var.	Skew.	Kurt.	Min.	Max.	KW
5	0.0197	-0.0515	3.2024	-0.5051	0.5427	0.5843
55	0.0282	0.8403	12.4256	-1.3554	2.0660	0.0669
105	0.0379	-0.2580	7.6272	-1.3525	1.5706	0.0000
155	0.0409	-0.2072	6.4109	-1.3374	1.3444	0.1336
205	0.0377	0.0011	6.0638	-1.5922	1.1882	0.5674
255	0.0349	0.2419	6.4722	-1.0375	2.2046	0.7592
305	0.0332	-0.1365	4.4883	-1.0058	0.9008	0.0000
355	0.0312	0.0836	4.4442	-0.8744	0.9416	0.0000
405	0.0292	0.0073	4.0630	-0.8793	0.8241	0.0692
455	0.0281	-0.0178	3.5547	-0.7447	0.8207	0.0031
505	0.0268	0.0021	3.5978	-0.7392	0.7562	0.8259

Chapter 6

Results and hypotheses

In this chapter, we will summarize and interpret results of the simulations from Chapter 5 and assess the introductory hypotheses from the Thesis Proposal. As we have already provided a relatively in-depth quantitative and qualitative analyses of the simulations of the model with the PT and asynchronous updating features in Chapter 5, the interpretation will namely consist in comparison of the results with real-world market data in order to tell whether the proposed extensions shift the model closer to reality.

6.1 PT feature

Implementation of the PT feature into the model evidently changes the behavior of the model *significantly.* Nonetheless, some of the key characteristics remain the same as the underlying mathematical structure of model is intact—the generated time series of the deviations from the fundamental price of the asset, x_t, exhibits decreased variance as the intensity of choice parameter β increases, extreme price deviations (i.e. minimal and maximal) are less 'extreme' for larger β, and, for instance, the deviations are still far from being normally distributed—this fact stems from minuscule p–values of the JB test. However—and most importantly—the differences are, too, considerable and non-negligible—as indicated by very low p–values of the Kruskal–Wallis tests as well. The main conclusions arising from the PT extension can be summarized as follows:

1. *Stability.* Probably the most noticeable change evident from the PT extended model's simulations—compared to the original version—is the overall increased stability. Summarized in Table 5.2, the descriptive

statistics of the x_t time series alone provide evidence of this phenomenon. The sample variance is usually 20 % to 30 % lower than it was in the benchmark case; we consider this finding robust to large extent since each sample contains 475000 realizations of x_t (1000 repeat cycles with 500 iterations each, initial 5 % discarded). Figure 5.4 supports this conclusion as well, less dramatic behavior of the x_t time series is apparent compared to the original (but analogical) plots in Figure 5.1—we remind the reader that the same bias and trend parameters b_h and g_h, respectively, were used, along with identical noise term ε_t. The difference in stability can therefore be attributed to the PT extension completely.

2. *Loss aversion matters.* Another apparent result of the simulations with the PT extended model is the fact that the number of strategies endowed with the PT feature, L, affects the performance of the model significantly. Summarizing the Kruskal–Wallis tests for different L, Table 5.6 shows that if only the fundamentalist strategy is loss-averse (i.e. $L = 1$), the empirical distributions of x_t are statistically different at a reasonable significance level from those obtained from the model with $L = 0$ only for higher values of β. On the other hand, for $L > 1$ the distributions are statistically (and visually) different from those of the benchmark, $L = 0$ case.

3. *Occurrence of fundamentalists is more extreme.* Not as striking as the previous two findings—yet probably of the strongest *economic* relevance—is the ambiguous status of the fundamentalist strategy. In the original model, fundamentalists were, with increasing β, less likely to survive in the market than they were for low values of β—we found that this phenomenon was even more emphasized for $L = 1$. The 'burden' of loss aversion presents significant hindrance for fundamentalists when they have to face a large number of non-PT strategies (chartists) and the cases in which $n_{1,t} = 0$ are more frequent than they were for $L = 0$. On the other hand, for $L = 4$ (i.e. when *all* strategies are loss-averse), fundamentalists are able to *survive* in the market more easily; this fact is in contrast with the findings of the original model; consult, please, Figure 5.9 for an illustration of this phenomenon. Moreover, fundamentalists survive in the market also for $L = 3$—although not as easily as they did for $L = 4$. Yet, the cases in which $n_{1,t} \neq 0$ are still very *frequent* even for high values of β—this finding might come as a surprise considering the fact that the

fundamentalists face non-PT chartists who do not have the burden of loss aversion. Figure 5.10 compares the occurrence of PT fundamentalists and non-PT chartists and thus illustrates this finding.

6.2 Asynchronous updating feature

The most important finding of the asynchronous updating extended model's simulations is the fact that the feature—in majority of cases—does *not* produce statistically different distributions of the x_t time series in comparison with the original, non-extended model. The interested reader is referred to Table 5.7 which summarizes, among other statistics, p–values of the Kruskal–Wallis test applied to the x_t time series with and without the AU feature. Nonetheless, there are some qualitative differences regarding the distributions of x_t—see, e.g., Figure 5.13 which compares tails of these distributions for data obtained from the AU extended model and benchmark model. The simulations also revealed implications of the AU feature with respect to autocorrelations in the x_t series—Figure 5.12 shows that a slowly decaying, 'sine' pattern is clearly visible—such a phenomenon was *not* found in the simulations with the non-extended model. Most importantly, however, the AU feature—designed to mimic the empirical finding of *investor inertia*—has dramatically improved prospects for fundamentalist who are now able to survive in the market more easily.

6.3 Hypotheses

This section presents evaluation of the five introductory hypotheses specified *ex ante* in the Thesis Proposal.

The original HAM by Brock & Hommes (1998) can be consistently extended by Prospect-Theoretical features.

The proceedings of Chapter 4, along with the simulations in Chapter 5, clearly show that this hypothesis is true. Using a general idea proposed by Shimokawa *et al.* (2007), we developed a straightforward extension of the original model. This extension enabled us to include the most important features of PT into the model, namely the loss aversion with reference point dependence, and distorted treatment of gains and losses. Subsequent simulations of the augmented model showed that the extensions were consistent with the original HAM and that they changed the behavior of it dramatically.

Incorporation of Prospect Theory helps better explain some classical financial stylized facts.

Answer to this hypothesis is somewhat intricate. We will try to assess the extended model's explanatory power chiefly with respect to the three stylized facts specified at the beginning of Chapter 5, i.e. *fat tails*, *absence of autocorrelation of returns*, and *volatility clustering*. The analysis will consist in comparison of the obtained time series and its properties with real-world market data, namely with four stock market indices—S&P 500, FTSE 100 (London), HSI (Hang Seng Index, Hong Kong), and Nikkei 225 (Tokyo).

The main characteristic of the indices that will be studied will be—contrarily to most research in the field—daily *close price differences*. The time period that the data comes from is January 1, 2009 to May 1, 2015. We do not choose the more common logarithmic returns as the price differences[1] better mimic the 'deviation' nature of the x_t time series from the HAM. Table 6.1 summarizes the most important statistics of the price differences times series of the four indices and the best-fit time series model from the ARMA model family determined by the AIC.[2] The '*' symbol indicates that the particular statistic refers to a standardized time series—the standardization consists in division of each price difference by mean value of the respective index to establish better mutual comparability among the indices, i.e. the standardized series are of the form

$$r_t = \frac{p_t - p_{t-1}}{\bar{p}}, \tag{6.1}$$

where p_t is the respective index's daily close price, and $\bar{p}$ is the arithmetic mean of daily close prices of the index for the time period in question. The time series models in Table 6.1 were fitted to *nonstandardized* price differences.

Apparently, two of the indices' price differences time series are best characterized by an autoregressive model—such a finding is in accordance with the best-fit models from our simulations. On the other hand, FTSE 100 and HSI are best described by a MA(0) process. This finding means that the time series

[1] We will nonetheless refer to these price differences as r_t in plot labels.

[2] We chose AIC as the model selection criterion since, in our opinion, the analysis is complex and exploratory-like. For this application, the AIC is optimal—unlike e.g. Bayesian Information Criterion (BIC) which is more useful in confirmatory analyses of models with lower dimension (Aho *et al.*, 2014).

Table 6.1: Summary statistics of real-world indices' price differences and best-fit time series model.

Ind.	Mean	Mean*	Var.	Var.*	Skew.	Kurt.	Model
S&P	0.739	0.00052	194	0.00010	-0.336	5.029	AR(1)
FTSE	1.555	0.00027	3240	0.00010	-0.203	4.489	MA(0)
HSI	8.238	0.00039	67835	0.00015	-0.092	4.187	MA(0)
N225	6.706	0.00058	26923	0.00020	-0.555	6.967	AR(2)

is essentially a white noise process—a comparison of this result with Figure 5.8 reveals the fact that also the HAM simulations produced MA(1) coefficient equal to 0 at non-negligible amount of times, although expected value of this coefficient is higher than 0. The estimated autoregressive coefficients for S&P 500 and Nikkei 225 are, respectively, equal to -0.068 and $\{-0.04, 0.04\}$—thus, they are on average either slightly negative or zero. Comparison with Table 5.3 reveals that neither of our simulations were able to replicate this finding with sufficient accuracy.

Table 6.2 shows best-fit models for *squared* price differences along with estimated α_1 and β_1, the AR(1) and MA(1) coefficients, respectively, and arithmetic averages of *all* autoregressive and moving average coefficients. Clearly, our simulations were able to replicate these findings in terms of the optimal model (consult Table 5.4); both non-PT and PT x_t^2 time series were best characterized by the same model family. Table 5.4 also reveals that the PT extended model's α_1 AR(1) coefficient is—for $\beta = 105$—very close to the HSI's estimated α_1 coefficient, and for $\beta = 55$ relatively close to the S&P 500's estimated α_1 coefficient. On the other hand, the moving average component of the real-world indices exhibits considerably lower 'expected' values than our simulated distributions of the β_1 coefficients do.

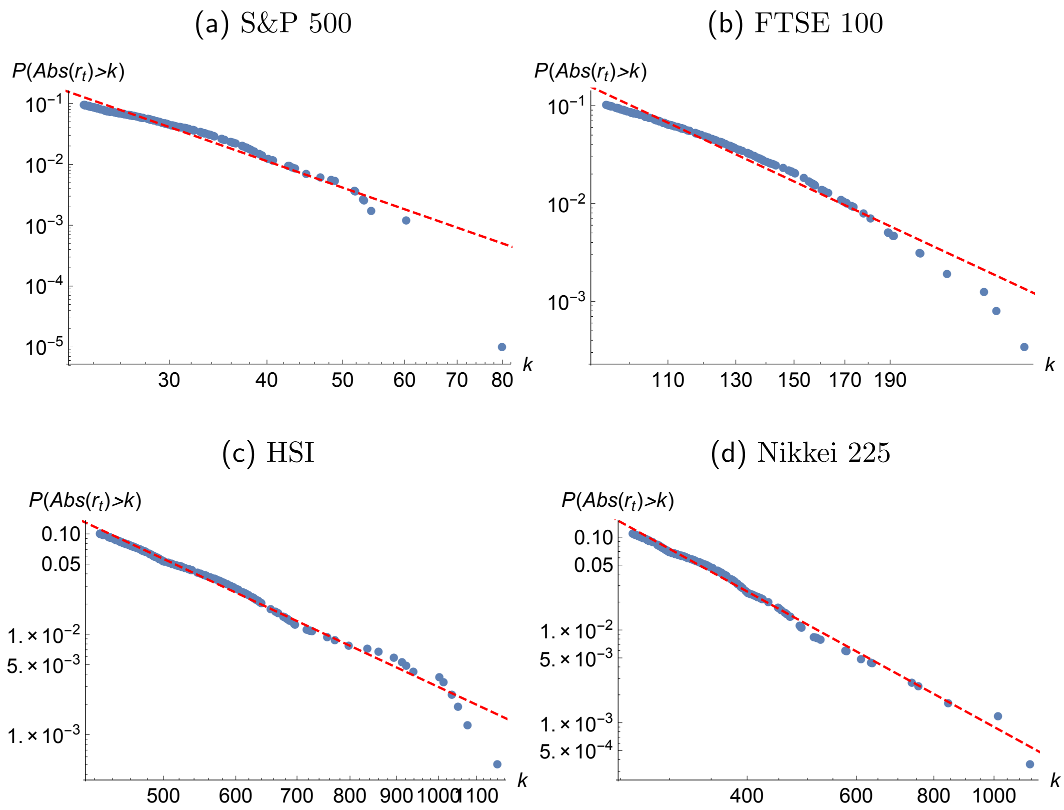
Table 6.2: Best-fit models of real-world indices' squared price differences.

Index	Model	$\boldsymbol{\alpha_1}$	$\boldsymbol{\beta_1}$	$\varnothing\boldsymbol{\alpha}$	$\varnothing\boldsymbol{\beta}$
S&P	ARMA(2,1)	0.329	-0.260	0.303	—
FTSE	AR(6)	0.042	—	0.079	—
HSI	ARMA(6,4)	0.182	-0.161	0.097	-0.079
N225	AR(6)	0.104	—	0.075	—

Figure 6.1 shows, on a log–log scale, tail plots of 10 % of largest absolute price differences of the four aforementioned indices for the period from January 1, 2009 to May 1, 2015 are plotted. The estimated tail indices are 4.488 (S&P

500, $R^2 = 0.89$), 4.456 (FTSE 100, $R^2 = 0.961$), 4.241 (HSI, $R^2 = 0.983$), and 3.669 (Nikkei 225, $R^2 = 0.991$). To the eye, the data fit the power law well, and the tails of Nikkei 225 are almost perfect power law fit which is manifested in extremely high value of R^2.

Figure 6.1: Tails of real stock market indices' close price differences distributions; log–log scale.



Source: Author's computations using Yahoo Finance data.

A comparison of there results with those shown in Table 5.5 shows that in most cases, the tail indices are lower than those of the simulated x_t time series. However, the same table provides evidence that the PT extension actually moves the model a little bit closer to reality in terms of the tail indices' magnitude for most values of β. Nonetheless, we again emphasize that it is not absolutely clear whether the power law is the ideal model for tails of the time series obtained from this type of HAM as the coefficients of determination are smaller than those of the real-world indices.

Asynchronous updating modification changes the dynamics of the model considerably.

The asynchronous updating extended model's simulations reject this hypothesis in most cases in terms of the distributions of price deviations x_t. We have showed that the empirical distributions of x_t when $\delta \neq 0$ are—in majority of cases—not statistically different from those of the cases in which $\delta = 0$, i.e. the distributions are 'similar'—nonetheless, especially for $\delta \in \{0.4, 0.8\}$, almost half of the distributions are statistically different from those of the benchmark case.

On the other hand—and most importantly—incidence of the fundamentalist strategy is dramatically different than it was in the non-extended model and is arguably closer to the real-world market fraction of fundamentalists (see e.g. Vissing-Jorgensen, 2004 for a survey of fractions of market strategies). This finding is of significant economic importance and supports the following hypothesis that the AU feature shifts the model closer to real market dynamics.

Asynchronous updating feature shifts the model closer to the real market dynamics.

According to Table 5.7, distributions of the x_t time series are not—in most cases—different from those of the benchmark x_t series. Yet, the AU extension shifts the model closer to reality in terms of incidence of the fundamentalist strategy, consider Figure 5.14. This finding is probably more interesting than the similarity of the x_t empirical distributions with respect to the original, non-extended model and suggests that the *investor inertia* phenomenon might actually be behind the empirically observed ability of fundamentalists to survive in the market (see Figure 5.14 for comparison of survival of fundamentalists in models with and without the AU extension).

The models with proposed behavioral features depict the empirical findings more accurately than models without them do.

The effect of both extensions (i.e. the PT and AU extension) must be assessed separately as they change the behavior of the original Brock & Hommes (1998) model differently. While the AU extension does not produce—for most values of β—significantly different distributions of x_t (with respect to the non-extended model) and the hypothesis is hence somewhat irrelevant, it is nonetheless able to better replicate the empirically observed fraction of fundamentalists in the market, as we have already pointed out before.

The PT extension, on the other hand, produces x_t time series that are statistically different from those obtained from the non-extended version of the model and evaluation of the hypothesis is therefore relevant. The three stylized facts we have been exploring are weak autocorrelations of deviations from the fundamental price,[3] volatility clustering (i.e. autocorrelations of *squared* price deviations), and fat tails. Results of Chapter 5 indicate that the PT extended model is able to *better* replicate the latter two facts and slightly worse the first fact, in comparison with the non-extended model. It is however important to emphasize that these differences in replication of empirical findings are not *dramatically* large.

[3]Normally, stylized facts are focused on time series of *returns*—the Brock & Hommes (1998) models nonetheless makes use of time series of deviations from the fundamental price.

Chapter 7

Conclusion

Extending the popular Brock & Hommes (1998) agent-based asset pricing model, we introduced features based on findings of Prospect Theory into the framework and assess the impact of Asynchronous Updating—technical setting of the model that aims to incorporate the empirically observed phenomenon of *investor inertia* which causes faulty timing of investment decisions. For comparison of the extended versions of the model with its original version, we have developed an algorithm which uses the methods of Monte Carlo simulations for random generation of parameters of the model and subsequent estimation of statistical distributions of the model's major variables.

The main contribution of this thesis is the finding that the original model can be consistently and meaningfully extended with the most relevant features of Prospect Theory and—at the same time—its intrinsic 'stylized' (i.e. simple) structure kept essentially intact. We find that distributions of the main variables are statistically different from those obtained from the non-extended version of model; moreover, the extension based on Prospect Theory shifts the original framework closer to real-world market dynamics in terms of two of the three stylized empirical facts that the thesis focuses on.

The inclusion of Asynchronous Updating typically does not produce statistically different distributions of the main variables. Yet, it does produce economically relevant implications regarding incidence of fundamentalists in the market that were clearly absent in the original, non-extended model.

As the Brock & Hommes (1998) model is *per se* characterized by 'many degrees of freedom' (i.e. countless possible options of settings of the main parameters) and the extensions bring even more options in this regard, future research might concentrate on exploration of other possible combinations of the parameters. Additionally, the extended model could be estimated using real-world empirical data to reveal the natural values of some parameters, e.g. of degree of loss aversion present in the markets. Other field that could be explored with respect to the extended version of the model is a more in-depth analysis of volatility structure of the x_t time series.

Bibliography

Aho, K., D. Derryberry, & T. Peterson (2014): "Model selection for ecologists: the worldviews of aic and bic." *Ecology* **95(3)**: pp. 631–636.

Arthur, W., B. LeBaron, & R. Palmer (1997): "Time Series Properties of an Artificial Stock Market." *Working papers 9725*, Wisconsin Madison - Social Systems.

Avery, C. & P. Zemsky (1998): "Multidimensional Uncertainty and Herd Behavior in Financial Markets." *American Economic Review* **88(4)**: pp. 724–48.

Axelrod, R. & L. S. Tesfatsion (2006): "A guide for newcomers to agent-based modeling in the social sciences." *Staff general research papers*, Iowa State University, Department of Economics.

Barberis, N., M. Huang, & T. Santos (2001): "Prospect theory and asset prices." *The Quarterly Journal of Economics* **116(1)**: pp. 1–53.

Barunik, J., L. Vacha, & M. Vosvrda (2009): "Smart predictors in the heterogeneous agent model." *Journal of Economic Interaction and Coordination* **4(2)**: pp. 163–172.

Becker, G. S. (1991): "A Note on Restaurant Pricing and Other Examples of Social Influences on Price." *Journal of Political Economy* **99(5)**: pp. 1109–16.

Belsky, G. & T. Gilovich (2010): *Why smart people make big money mistakes and how to correct them: Lessons from the life-changing science of behavioral economics.* Simon and Schuster.

Benartzi, S. & R. H. Thaler (1993): "Myopic loss aversion and the equity premium puzzle." *Technical report*, National Bureau of Economic Research.

Benčík, D. (2010): "Modeling financial markets using heterogeneous agent models." *Bachelor thesis*, Charles University.

Bolt, W., M. Demertzis, C. G. Diks, & M. Van der Leij (2011): "Complex methods in economics: an example of behavioral heterogeneity in house prices." .

Brock, W. & C. Hommes (2001): "Evolutionary dynamics in financial markets with many trader types." *Working papers 7*, Wisconsin Madison - Social Systems.

Brock, W. A. & C. H. Hommes (1997): "A rational route to randomness." *Econometrica: Journal of the Econometric Society* pp. 1059–1095.

Brock, W. A. & C. H. Hommes (1998): "Heterogeneous beliefs and routes to chaos in a simple asset pricing model." *Journal of Economic Dynamics and Control* **22**: pp. 1235 – 1274.

Brock, W. A., C. H. Hommes, & F. O. Wagener (2005): "Evolutionary dynamics in markets with many trader types." *Journal of Mathematical Economics* **41(1â€"2)**: pp. 7 – 42. Special Issue on Evolutionary Finance.

Brunnermeier, M. K. (2001): *Asset pricing under asymmetric information: bubbles, crashes, technical analysis, and herding.* Oxford University Press.

Camerer, C. F. & T. Ho (1994): "Violations of the betweenness axiom and nonlinearity in probability." *Journal of Risk and Uncertainty* **8(2)**: pp. 167–196.

Chang, C.-L., M. McAleer, & L. Oxley (2011): "Great expectatrics: Great papers, great journals, great econometrics." *Econometric Reviews* **30(6)**: pp. 583–619.

Chen, S.-H., C.-L. Chang, & Y.-R. Du (2012): "Agent-based economic models and econometrics." *The Knowledge Engineering Review* **27**: pp. 187–219.

Cont, R. (2001): "Empirical properties of asset returns: stylized facts and statistical issues." *Quantitative Finance* **1(2)**: pp. 223–236.

Cont, R. (2007): "Volatility clustering in financial markets: empirical facts and agent-based models." In "Long memory in economics," pp. 289–309. Springer.

Devenow, A. & I. Welch (1996): "Rational herding in financial economics." *European Economic Review* **40(3–5)**: pp. 603 – 615. Papers and Proceedings of the Tenth Annual Congress of the European Economic Association.

Diks, C. & R. V. D. Weide (2005): "Heterogeneity as a natural source of randomness 1."

Dow, J. & S. R. d. C. Werlang (1992): "Uncertainty aversion, risk aversion, and the optimal choice of portfolio." *Econometrica* **60(1)**: pp. pp. 197–204.

Ehrentreich, N. (2007): *Agent-based modeling: The Santa Fe Institute artificial stock market model revisited*, volume 602. Springer Science & Business Media.

Epstein, J. M. (2002): "Modeling civil violence: An agent-based computational approach." *Proceedings of The National Academy of Sciences* **99**: pp. 7243–7250.

Eshelman, L. J. (2000): "Genetic algorithms." *Evolutionary Computation* **1**: pp. 64–80.

Fama, E. F. (1970): "Efficient capital markets: A review of theory and empirical work." *The Journal of Finance* **25(2)**: pp. 383–417.

Frankel, J. A. & K. A. Froot (1990): "Chartists, fundamentalists, and trading in the foreign exchange market." *The American Economic Review* **80(2)**: pp. pp. 181–185.

Giorgi, E. D., T. Hens, & M. O. Rieger (2010): "Financial market equilibria with cumulative prospect theory." *Journal of Mathematical Economics* **46(5)**: pp. 633 – 651. Mathematical Economics: Special Issue in honour of Andreu Mas-Colell, Part 1.

Giorgi, E. G. D. & S. Legg (2012): "Dynamic portfolio choice and asset pricing with narrow framing and probability weighting." *Journal of Economic Dynamics and Control* **36(7)**: pp. 951 – 972.

Goldberg, D. E. (1989): *Genetic Algorithms in Search, Optimization and Machine Learning*. Boston, MA, USA: Addison-Wesley Longman Publishing Co., Inc., 1st edition.

Grüne, L. & W. Semmler (2008): "Asset pricing with loss aversion." *Journal of Economic Dynamics and Control* **32(10)**: pp. 3253–3274.

Gustafsson, L. & M. Sternad (2010): "Consistent micro, macro and state-based population modelling." *Mathematical Biosciences* **225(2)**: pp. 94 – 107.

Haas, M. & C. Pigorsch (2009): "Financial economics, fat-tailed distributions." In "Encyclopedia of Complexity and Systems Science," pp. 3404–3435. Springer.

Harrison, G. W. & E. E. Rutström (2009): "Expected utility theory and prospect theory: One wedding and a decent funeral." *Experimental Economics* **12(2)**: pp. 133–158.

Holland, J. H. (1975): *Adaptation in natural and artificial systems: an introductory analysis with applications to biology, control, and artificial intelligence.* U Michigan Press.

Hommes, C. (2011): "The heterogeneous expectations hypothesis: Some evidence from the lab." *Journal of Economic Dynamics and Control* **35(1)**: pp. 1 – 24.

Hommes, C. (2013): *Behavioral rationality and heterogeneous expectations in complex economic systems.* Cambridge University Press.

Hommes, C. & F. Wagener (2009): "Complex evolutionary systems in behavioral finance." *Handbook of financial markets: Dynamics and evolution* pp. 217–276.

Hommes, C. H. (2006): *Handbook of Computational Economics, Agent-Based Computational Economics*, chapter Heterogeneous Agent Models in Economics and Finance, pp. 1109–1186. Elsevier Science B.V.

Kahneman, D. & A. Tversky (1972): "Subjective probability: A judgment of representativeness." *Cognitive Psychology* **3(3)**: pp. 430 – 454.

Kahneman, D. & A. Tversky (1979): "Prospect theory: An analysis of decision under risk." *Econometrica* **47(2)**: pp. 263–291.

van Kersbergen, K. & B. Vis (2014): *Comparative Welfare State Politics: Development, Opportunities, and Reform.* Cambridge University Press.

KEYNES, J. M. (1936): "The general theory of employment, interest and money." *The General Theory of Employment, Interest and Money* .

KIRMAN, A. (1993): "Ants, Rationality, and Recruitment." *The Quarterly Journal of Economics* **108(1)**: pp. 137–56.

KUKACKA, J. & J. BARUNIK (2013): "Behavioural breaks in the heterogeneous agent model: The impact of herding, overconfidence, and market sentiment." *Physica A: Statistical Mechanics and its Applications* **392(23)**: pp. 5920–5938.

LEBARON, B. (2006): "Agent-based Computational Finance." In L. TESFATSION & K. L. JUDD (editors), "Handbook of Computational Economics," volume 2 of *Handbook of Computational Economics*, chapter 24, pp. 1187–1233. Elsevier.

LI, Y. & L. YANG (2013): "Prospect theory, the disposition effect, and asset prices." *Journal of Financial Economics* **107(3)**: pp. 715 – 739.

LO, A. W., H. MAMAYSKY, & J. WANG (2000): "Foundations of technical analysis: Computational algorithms, statistical inference, and empirical implementation." *The Journal of Finance* **55(4)**: pp. 1705–1770.

LUX, T. (1995): "Herd Behaviour, Bubbles and Crashes." *Economic Journal* **105(431)**: pp. 881–96.

LUX, T. (1998): "The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions." *Journal of Economic Behavior & Organization* **33(2)**: pp. 143–165.

LUX, T. (2008): "Stochastic Behavioral Asset Pricing Models and the Stylized Facts." *Kiel Working Papers 1426*, Kiel Institute for the World Economy.

LUX, T. & M. MARCHESI (2000): "Volatility clustering in financial markets: A microsimulation of interacting agents." *International Journal of Theoretical and Applied Finance* **03(04)**: pp. 675–702.

MADRIAN, B. C. & D. F. SHEA (2000): "The power of suggestion: Inertia in 401 (k) participation and savings behavior." *Technical report*, National bureau of economic research.

MANDELBROT, B. (1963): "The Variation of Certain Speculative Prices." *The Journal of Business* **36**: p. 394.

MARKS, R. E. (1992): "Breeding hybrid strategies: optimal behavior for oligopolists." *Journal of Evolutionary Economics* **2**: pp. 17–38.

MEHRA, R. & E. C. PRESCOTT (1985): "The equity premium: A puzzle." *Journal of monetary Economics* **15(2)**: pp. 145–161.

PAGE, S. E. (2008): "agent-based models." In S. N. DURLAUF & L. E. BLUME (editors), "The New Palgrave Dictionary of Economics," Basingstoke: Palgrave Macmillan.

PALMER, R., W. B. ARTHUR, J. H. HOLLAND, B. LEBARON, & P. TAYLER (1994): "Artificial economic life: a simple model of a stockmarket." *Physica D: Nonlinear Phenomena* **75(1–3)**: pp. 264–274.

SHEFRIN, H. & M. STATMAN (1985): "The disposition to sell winners too early and ride losers too long: Theory and evidence." *The Journal of Finance* **40(3)**: pp. pp. 777–790.

SHIMOKAWA, T., K. SUZUKI, & T. MISAWA (2007): "An agent-based approach to financial stylized facts." *Physica A: Statistical Mechanics and its Applications* **379(1)**: pp. 207 – 225.

SILVERMAN, B. W. (1986): *Density estimation for statistics and data analysis*, volume 26. CRC press.

SIMONSEN, M. H. & S. R. d. C. WERLANG (1990): "Subadditive probabilities and portfolio inertia." *Economics Working Papers (Ensaios Economicos da EPGE) 162*, FGV/EPGE Escola Brasileira de Economia e Finanças, Getulio Vargas Foundation (Brazil).

TESFATSION, L. (2006): "Agent-based computational economics: A constructive approach to economic theory." *Staff General Research Papers 12514*, Iowa State University, Department of Economics.

TU, Q. (2005): "Empirical analysis of time preferences and risk aversion." *Technical report*, School of Economics and Management.

TVERSKY, A. & D. KAHNEMAN (1992): "Advances in prospect theory: Cumulative representation of uncertainty."

Vácha, L. & M. Vošvrda (2005): "Dynamical agents' strategies and the fractal market hypothesis." *Prague Economic Papers* **14(2)**: pp. 172–179.

Vissing-Jorgensen, A. (2004): "Perspectives on behavioral finance: Does "irrationality" disappear with wealth? evidence from expectations and actions." In "NBER Macroeconomics Annual 2003, Volume 18," NBER Chapters, pp. 139–208. National Bureau of Economic Research, Inc.

Vošvrda, M. & L. Vácha (2002): "Heterogeneous agent model and numerical analysis of learning." *Bulletin of the Czech Econometric Society* **9**.

Vošvrda, M. & L. Vácha (2003): "Heterogeneous agent model with memory and asset price behaviour." *Prague Economic Papers* **2003(2)**.

Wand, M. P. & M. C. Jones (1994): *Kernel smoothing*. Crc Press.

West, K. D. (1988): "Bubbles, fads, and stock price volatility tests: A partial evaluation." *Working Paper 2574*, National Bureau of Economic Research.

Wu, G. & R. Gonzalez (1996): "Curvature of the probability weighting function." *Management science* **42(12)**: pp. 1676–1690.

Yao, J. & D. Li (2013): "Prospect theory and trading patterns." *Journal of Banking & Finance* **37(8)**: pp. 2793 – 2805.

Zhang, W. & W. Semmler (2009): "Prospect theory for stock markets: Empirical evidence with time-series data." *Journal of Economic Behavior & Organization* **72(3)**: pp. 835 – 849.

Appendix A

Supplements

A.1 Proof of optimal demand for risky asset

Given the utility function of wealth of the form $U(W) = -\exp(-a \cdot W)$, where a is the risk aversion parameter, and the wealth dynamics equation $\mathbf{W}_{t+1} = R \cdot W_t + z_{h,t} \cdot (\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t)$, which serves as a budget constraint, the optimal quantity demanded for a myopic, mean–variance investor of the risky asset, $z^*_{h,t}$, maximizes the expected utility $E(U(\mathbf{W}_{t+1}))$ of an agent:

$$\begin{aligned} E(U(\mathbf{W}_{t+1})) &= E(-\exp(-a \cdot \mathbf{W}_{t+1})) \\ &= -\exp\left(-a \cdot E_{h,t}(\mathbf{W}_{t+1}) + \frac{a^2}{2} \cdot V_{h,t}(\mathbf{W}_{t+1})\right), \end{aligned} \tag{A.1}$$

where we used the assumption that $\mathbf{W}_{t+1} \sim N(E_{h,t}(\mathbf{W}_{t+1}), V_{h,t}(\mathbf{W}_{t+1}))$, and, hence, the random variable $Y = -\exp(-a \cdot \mathbf{W}_{t+1})$ has a log-normal distribution.

Now, the maximization problem

$$\max_{z_{h,t}} \left\{-\exp\left(-a \cdot E_{h,t}(\mathbf{W}_{t+1}) + \frac{a^2}{2} \cdot V_{h,t}(\mathbf{W_{t+1}})\right)\right\} \tag{A.2}$$

is equivalent to the minimization problem

$$\min_{z_{h,t}} \left\{\exp\left(-a \cdot E_{h,t}(\mathbf{W}_{t+1}) + \frac{a^2}{2} \cdot V_{h,t}(\mathbf{W}_{t+1})\right)\right\}, \tag{A.3}$$

which is, moreover, equivalent to the minimization problem

$$\min_{z_{h,t}} \left\{-a \cdot E_{h,t}(\mathbf{W_{t+1}}) + \frac{a^2}{2} \cdot V_{h,t}(\mathbf{W_{t+1}})\right\}. \tag{A.4}$$

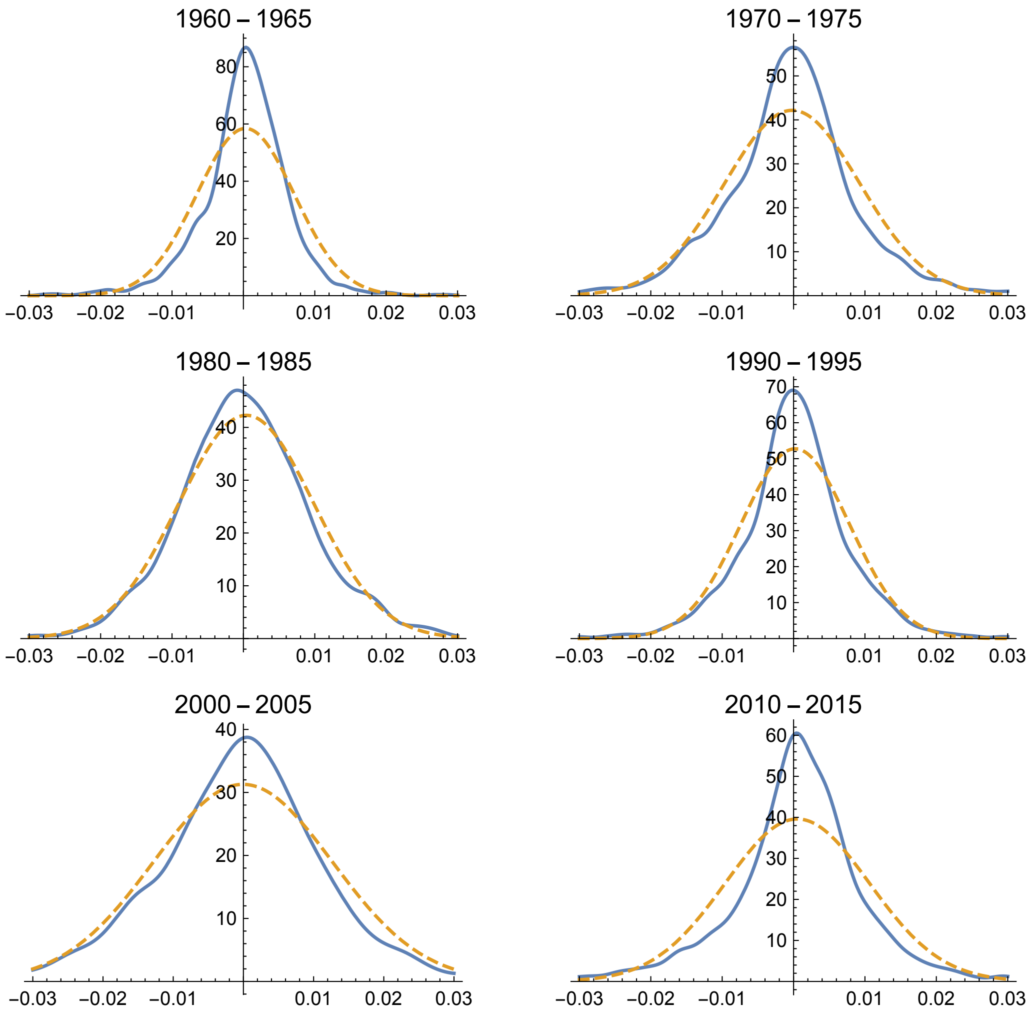
Using the facts that $E_{h,t}(\mathbf{W}_{t+1}) = R \cdot W_t + z_t \cdot E(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t)$ and $V_{h,t}(\mathbf{W}_{t+1}) = z_{h,t}^2 \cdot V_{h,t}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t) \equiv \sigma^2$, and taking the derivative of equation A.4 with respect to $z_{h,t}$, we obtain the desired result

$$z_{h,t}^* = \frac{E_{h,t}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - R \cdot p_t)}{a \cdot \sigma^2}. \tag{A.5}$$

□

A.2 PDFs of S&P 500 daily logarithmic returns

Figure A.1: S&P 500 daily logarithmic returns. Contrast the kurtosis of the empirical returns' distribution (blue solid line) to that of normal distribution with the same mean and variance (orange dashed line).



Source: Author's computations using Yahoo Finance data.

Appendix B

Contents of enclosed ZIP archive

There is a 'ZIP' archive[1] enclosed to this thesis which contains *Wolfram Mathematica* source codes ('.nb' file extension) and all graphs and diagrams obtained from the simulations.

- Folder 1: Mathematica source codes
- Folder 2: Graphs and diagrams

[1] Besides the official Charles University theses repository website, the archive—named 'Thesis_data.zip'—can be downloaded from OneDrive at http://1drv.ms/1dXbZd0.