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DIPLOMOVÁ PRÁCE

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Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

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Abstrakt: V této práci podáme tři různé definice projektivní struktury na varietě a pokusíme se dokázat jejich ekvivalenci. Za tímto účelem vyvineme prostředky pro práci s vyššími frame-bandly a parabolickými geometriemi. Uvedeme standardní tractorový bandl a jeho duál pro projektivní parabolickou geometrii a taky velkou třídu diferenciálních operátorů, které jsou invariantní ve smyslu klasické definice projektivní struktury, ale ne ve smyslu projektivní parabolické geometrie.
Klíčová slova: projektivní struktura, parabolická geometrie, diferenciální operátory

Title: Geometry of projective structures
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Abstract: In this thesis we give three different definitions of a projective structure on some manifold M and try to prove their equivalence. For this purpose we develop the machinery of higher frame bundles and parabolic geometries. We introduce the standard tractor bundle for projective parabolic geometry and its dual, and a very large family of differential operators for projective structures, which are invariant in the sense of classical definition, but not in the sense of projective parabolic geometry.
Keywords: projective structure, parabolic geometry, differential operators

Chapter 1

Preliminaries

1.1 Fibre bundles

We shall define basic notions we will need later.

Definition 1.1.1. C^∞ fibre bundle consists of 3 C^∞ manifolds E, F, M such that:
a) There exists a C^∞ submersion $\pi : E \to M$
b) There exists an open covering U_α of M such that $\forall \alpha \; \exists$ a C^∞ map $\psi : \pi^{-1}(U_\alpha) \to F$ such that $\pi^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha \times F$ via (π, ψ).
E is called *total space*, F *standard fibre* and M the *base manifold.* (π, ψ) is called *bundle chart* on E over $U_\alpha \subset M$.

Definition 1.1.2. A *morphism* F *of fibre bundles* $\pi : E \to M$ and $\pi' : E' \to M'$ over $f : M \to M'$ is a C^∞ map $E \to E'$ such that $\pi' \circ F = f \circ \pi$

Definition 1.1.3. Let $\pi : E \to M$ be a fibre bundle. A vector Xtangent to E is called *vertical* if and only if $\pi_* X = 0$.

Definition 1.1.4. Let $p : P \to M$ be a C^∞ fibre bundle, which standard fibre is a Lie group G. Then P is called a *principal fibre bundle over* M *with group* G, if there is given a C^∞ right action of G on P such that for every bundle chart (p, ψ) on P over $U \subset M$

$$\psi(b.g) = \psi(b)g, \quad b \in p^{-1}(U), \quad g \in G$$

If P is a principal bundle over M with group G, then $M = P/G$ and p is the canonical projection.

Definition 1.1.5. A *morphism of principal fibre bundles* $p : P \to M$ with group G and $p' : P' \to M'$ with group G' according to a homomorphism $f : G \to G'$ is a morphism of fibre bundles $F : P \to P'$ which commutes with the principal right action of G and G', i.e. such that

$$F(bg) = F(b)f(g)$$

Definition 1.1.6. Let $p : P \to M$ be a principal bundle over M with group G and let F be a C^∞ manifold endowed with a left action of G. Define the bundle $P \times_G F$ as $(P \times F)_\sim$, where

$$(bg, \xi) \sim (b, g\xi), \quad \text{that is} \quad (b, \xi) \sim (bg, g^{-1}\xi)$$

for $(b, \xi) \in P \times F, g \in G$. Denote the equivalence class of (b, ξ) by $[b, \xi]$ and define $\pi([b, \xi]) = p(b)$. Then $\pi : P \times_G F \to M$ is a *fibre bundle over M with fibre F associated to P with the given action of G on F.* If F is a vector space and the given left action is a representation of G, then we get a *vector bundle.*

Remark 1.1.1. If F carries any other structure then C^∞ structure, which is G-invariant, then every fibre of $P \times_G F$ also inherits this structure.

Definition 1.1.7. Let P be a principal bundle over M with group G and E a fibre bundle over M with standard fibre S associated to P. Then every element $u \in P$ induces a diffeomorphism $\underline{u}$ of S onto the fibre of E over $p(u)$ defined by

$$\underline{u}(s) = [u, s]$$

This we will use very often when speaking about parabolic geometries.

1.2 Connections on fibre bundles

Here we will explain basic ideas of connection theory on fibred bundles.

Definition 1.2.1. Let P be a principal bundle over M with group G. *Fundamental vector field* A^* on P for $A \in \mathfrak{g}$ is the vector field $(\lambda_b)_* A$, where $\lambda_b g = bg$.

Proposition 1.2.1. *Let P be a principal bundle over M with group G. Let X be a fundamental vector field on P corresponding to $A \in \mathfrak{g}$. Then $(R_g)_* X$ is a fundamental vector field corresponding to $Ad(g^{-1})A$ for $g \in G$.*

Proof. Let $g_t = exp(tA)$ be a curve in G. Then $dg_t/dt = A$ and

$$(R_g)_* X_b = d/dt(bgg^{-1}g_t g) = (Ad(g^{-1})A)^*_{bg}$$

□

Definition 1.2.2. Let E be a fibre bundle over M. A *connection* on E is a C^∞ distribution $\mathcal{H}$ on E, which is horizontal, i.e. complementary to the vertical bundle of E. A connection on TM is usually called a connection on M.

Definition 1.2.3. Let P be a principal fibre bundle over M with group G. A *principal connection* on P is a connection on P, which is right invariant in the sense that $(R_g)_* \mathcal{H}_u = \mathcal{H}_{ug}, u \in P, g \in G$.

Definition 1.2.4. Let P be a principal fibre bundle over M with group G and let $\mathcal{H}$ be a principal connection on P. A *connection form of principal connection* $\mathcal{H}$ is a 1-form ω on P with values in Lie algebra $\mathfrak{g}$ of G such that:
a) The kernel of ω is exactly $\mathcal{H}$
b) ω reproduces generators of fundamental vector fields, that is $\omega(A^*) = A$

Lemma 1.2.1. *Let P be a principal fibre bundle over M with group G, $\mathcal{H}$ a principal connection on P and ω a connection form of $\mathcal{H}$. Then $\forall g \in G$ $(R_g)^*\omega = Ad(g^{-1}) \circ \omega$.*

Proof. It is an immediate consequence of the fact that ω is on vertical vector fields the inverse of fundamental vector field map. □

Remark 1.2.1. We will often refer to ω as to connection on P.

Definition 1.2.5. Let P be a principal fibre bundle over M with group G and ω a connection form of some principal connection on P. The *curvature* Ω of the connection ω is a 2-form on P with values in $\mathfrak{g}$ given by

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$$

Lemma 1.2.2. *If X is a fundamental vector field corresponding to $A \in \mathfrak{g}$ and Y is a horizontal vector field, then $[X, Y]$ is horizontal.*

Proof. Let $g_t = \exp(tA)$ be a curve in G. Then $A = dg_t/dt$. We have

$$[X, Y] = \lim_{t \to 0} \frac{1}{t}(Y - (R_{g_t})_* Y)$$

So $[X, Y]$ is horizontal. □

Proposition 1.2.2. *Let P be a principal fibre bundle over M with group G, ω a connection on P and Ω its curvature. Then Ω is horizontal, i.e. it vanishes upon inserting one vertical vector field.*

Proof. It is sufficient to prove the proposition for constant vector fields, i.e. for vector fields such that $\omega(X)$ is constant. First let us consider that X and Y are fundamental vector fields corresponding to $A, B \in (g)$. Then we have

$$\Omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) + [\omega(X), \omega(Y)]$$

But $\omega(X)$ and $\omega(Y)$ are constant functions on P, so we have $\Omega(X, Y) = 0$.
Now assume that X is fundamental vector field corresponding to $A \in \mathfrak{g}$ and Y is constant horizontal. Then we have

$$\Omega(X, Y) = -\omega([X, Y])$$

since $\omega(X)$ and $\omega(Y)$ are constant. But $[X, Y]$ is horizontal, so we have $\Omega(X, Y) = 0$. □

Definition 1.2.6. Let P be a principal fibre bundle over M with group G, $\mathcal{H}$ a principal connection on P. Let $E = P \times_G F$ be a fibred bundle over M associated to P. Then $\mathcal{H}$ induces an *associated connection* on E defined by

$$\mathcal{H}_{[u,v]} = \pi_*(\mathcal{H}_u \times 0_s)$$

where $\pi : P \times F \to P \times_G F$.

1.3 Cartan connections

Here we shall develop some basic facts about Cartan connections, which will be needed later.

Definition 1.3.1. Let G be a Lie group, H a closed subgroup of G and P a principal H-bundle over M, $\dim M = \dim G/H$. A *Cartan connection on* P is a 1-form ω on P with values in Lie algebra $\mathfrak{g}$ such that:

a $\forall A \in \mathfrak{h}\ \omega(A^*) = A$

b $\forall h \in H\ (R_h)^*\omega = Ad(h^{-1}) \circ \omega$

c $\forall b \in P\ \omega_b$ induces a linear isomorphism of T_bP onto $\mathfrak{g}$

Definition 1.3.2. Let G be a Lie group, H a closed subgroup of G and P a principal H-bundle over M, $\dim M = \dim G/H$. Let ω be a Cartan connection on P. The *curvature* Ω of Cartan connection ω is a 2-form on P with values in $\mathfrak{g}$ given by

$$\Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)]$$

Proposition 1.3.1. *Let G, H, P and M be as above. Let ω be a Cartan connection on P and Ω its curvature. Then Ω is horizontal in the sense that it vanishes upon inserting one vertical vector field.*

Proof. It suffices to prove it for constant vector fields, i.e. vector fields such that $\omega(X)$ is constant on P. So assume that X is fundamental vector field corresponding to $A \in \mathfrak{h}$ and $Y = \omega^{-1}(B)$, $B \in \mathfrak{g} \setminus \mathfrak{h}$. We have

$$\Omega(X,Y) = -\omega([X,Y]) + [\omega(X), \omega(Y)]$$

Let $h_t = \exp(tX)$ be a curve in H. Then $dh_t/dt = X$ and

$$[X,Y] = \lim_{t\to 0} \frac{1}{t}(Y - (R_{h_t})_*Y) = \lim_{t\to 0} \frac{1}{t}(\omega^{-1}A - \omega^{-1}(Ad(h_t^{-1})B)) =$$

$$\lim_{t\to 0} \frac{1}{t}\omega^{-1}(A - Ad(h_t^{-1})B) = \omega^{-1}([A,B])$$

So we see that $\Omega(X,Y) = 0$.
Since the Cartan connection trivializes TP, any differential form on P is determined by its values on the constant vector fields $\omega^{-1}(X)$. □

Definition 1.3.3. Let $P \to M, \omega$ be a Cartan geometry and Ω its curvature form. Then the *curvature function* $\kappa : P \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ is defined by

$$\kappa(u)(X,Y) = \Omega(\omega^{-1}(X)(u), \omega^{-1}(Y)(u))$$

or, equivalently

$$\kappa(u)(X,Y) = [X,Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)](u)).$$

Since we know that Ω is horizontal, we can view κ as a function on P with values in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$.

1.4 Linear frame bundle and linear connections

Here we shall define the linear frame bundle of M and develop some theory about linear connections on M.

Definition 1.4.1. A *linear frame bundle* LM of M is a principal bundle over M with group $GL(n;R)$, whose fibre over $x \in M$ consists of all bases of T_xM.
A connection on LM induces a connection on every tensor bundle on M. These connections are usually referred to as *linear connections*.

Definition 1.4.2. Let ω be a connection on LM and X a vector field on M. The *horizontal lift* X^h of X is a unique horizontal vector field on LM such that $\pi_* X^h = X$.

Definition 1.4.3. Let E be a vector bundle on M associated to LM, φ a local cross section of E, x_t a curve in the domain U of definition of φ, $X = \dot{x}_t(0) \in T_{x_0}M$ and $f = \underline{u}^{-1}\varphi$ a function on $p^{-1}U$. Then the *covariant derivative* $\nabla_{\dot{x}_t(0)}\varphi = \nabla_X\varphi$ is defined by

$$\nabla_X\varphi = \underline{u}(X^h f)$$

Definition 1.4.4. Let E be as above, φ a cross section of E defined on M and X a vector field on M. Then the *covariant derivative* $\nabla_X\varphi$ of φ in the direction of (or with respect to) X is defined by

$$(\nabla_X\varphi)(x) = \nabla_{X_x}\varphi$$

Proposition 1.4.1. *Let $X, Y \in T_xM$ and let φ and ψ be cross sections of E defined in a neighbourhood of x. Then*

1. $\nabla_{X+Y}\varphi = \nabla_X\varphi + \nabla_Y\varphi$
2. $\nabla_X(\varphi + \psi) = \nabla_X\varphi + \nabla_X\psi$
3. $\nabla_{\lambda X}\varphi = \lambda.\nabla_X\varphi$, *where* $\lambda \in R$
4. $\nabla_X\lambda\varphi = \lambda(x)\nabla_X\varphi + (X\lambda).\varphi$, *where λ is an R-valued function defined on a neighbourhood of x.*

Proof. It follows directly from the definition. □

Similar proposition holds for covariant derivatives with respect to vector fields.

Proposition 1.4.2. *Let $\mathfrak{I}M$ be the algebra of tensor fields on M. Then the covariant differentiation has the following properties:*

1. $\nabla_X : \mathfrak{I}M \to \mathfrak{I}M$ *is a type-preserving derivation*
2. ∇_X *commutes with every contraction*
3. $\nabla_X f = Xf$ *for every function f on M*

4. $\nabla_{X+Y} = \nabla_X + \nabla_Y$

5. $\nabla_{fX} K = f.\nabla_X K$ *for every function* f *on* M *and* $K \in \mathfrak{I}M$

Proof. See [8] □

Definition 1.4.5. The *canonical form* θ of LM is a R^n-valued 1-form, which is defined by

$$\theta(X) = \underline{u}^{-1}\pi_* X, \quad X \in T_u LM$$

Definition 1.4.6. The *torsion* of linear connection ω is a 2-form Θ on LM with values in R^n defined by

$$\Theta(X,Y) = d\theta(X,Y) + \omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)$$

Definition 1.4.7. Let ω be a linear connection on M with curvature Ω and torsion Θ.
(a) The *torsion tensor field or torsion* is a tensor field defined on M by

$$T(X,Y) = \underline{u}\Theta(X^h, Y^h)$$

for $X, Y \in T_x M$ and u any point of LM with $\pi(u) = x$
(b)The *curvature tensor field or curvature* is a tensor field defined on M by

$$R(X,Y)Z = \underline{u}\Omega(X,Y)(\underline{u}^{-1}Z)$$

Proposition 1.4.3. *In terms of covariant differentiation the torsion* T *and the curvature* R *can be expressed as follows:*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

Proof. See [8] □

Let U be a coordinate neighbourhood in M with a local coordinate system $x^1, \dots, x^n$. We denote by X_i the vector field $\partial/\partial x^i, i = 1, \dots, n$, defined in U. Every linear frame at a point $x \in U$ can be uniquely expressed by

$$(\sum_i X_1^i (X_i)_x, \dots, \sum_i X_n^i (X_i)_x)$$

where $det(X_j^i) \neq 0$. We take (x^i, X_k^j) as a local coordinate system in $\pi^{-1}U \subset LM$. Let Y_k^j be the inverse matrix of X_k^j.
We can express the canonical form θ in terms of the local coordinate system introduced above. Let $e_1, \dots, e_n$ be the natural basis of R^n and set

$$\theta = \sum_i \theta^i e_i$$

Proposition 1.4.4. *In terms of the local coordinate system (x^i, X^j_k), the canonical form $\theta = \sum_i \theta^i e_i$ can be expressed as follows:*

$$\theta^i = \sum_j Y^i_j dx^j$$

Proof. See [8] □

Let ω be a connection form of a linear connection on M. With respect to the basis (E^j_i) of $\mathfrak{gl}(n; R)$, we write

$$\omega = \sum_{i,j} \omega^i_j E^j_i$$

Definition 1.4.8. Let σ be the cross section of LM over U, which assigns to each $x \in U$ the linear frame $((X_1)_x, \ldots, (X_n)_x)$. We set

$$\omega_U = \sigma^* \omega$$

Then ω_U is a $\mathfrak{gl}(n; R)$-valued 1-form on U. We define n^3 functions $\Gamma^i_{jk}, i, j, k = 1, \ldots, n$ on U by

$$\omega_U = \sum_{i,j,k} (\Gamma^i_{jk} dx^j) E^k_i$$

The functions Γ^i_{jk} are called *components* or *Christoffel's symbols* of linear connection ω with respect to the local coordinate system $x^1, \ldots, x^n$.

Proposition 1.4.5. *Let ω be a linear connection on M. Let Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$ be the components of ω with respect to local coordinate systems $x^1, \ldots, x^n$ and $\bar{x}^1, \ldots, \bar{x}^n$ respectively. In the intersection of the two coordinate neighbourhoods, we have*

$$\overline{\Gamma}^\alpha_{\beta\gamma} = \sum_{i,j,k} \Gamma^i_{jk} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^i} + \sum_i \frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{\partial \bar{x} \alpha}{\partial x^i}$$

Proof. See [8] □

Proposition 1.4.6. *Let $x^1, \ldots, x^n$ be a local coordinate system in M with a linear connection ω. Set $X_i = \partial / \partial x^i, i = 1, \ldots, n$. Then the components Γ^i_{jk} of ω with respect to $x^1, \ldots, x^n$ are given by*

$$\nabla_{X_j} X_i = \sum_k \Gamma^k_{ji} X_k$$

Proof. See [8] □

Proposition 1.4.7. *Assume that a mapping $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ denoted by $(X, Y) \to \nabla_X Y$, is given so as to satisfy the conditions on covariant derivative. Then there is a unique linear connection ω of M such that $\nabla_X Y$ is the covariant derivative of Y in the direction of X with respect to ω.*

Proof. See [8] □

Proposition 1.4.8. *Assume that, for each local coordinate system $x^1, \ldots, x^n$, there is given a set of functions $\Gamma^i_{jk}, i, j, k = 1, \ldots, n$, in such a way that they satisfy the transformation rule for Christoffel's symbols. Then there exists a unique linear connection ω whose components with respect to $x^1, \ldots, x^n$ are precisely the given functions Γ^i_{jk}. Moreover, the connection form $\omega = \sum_{i,j} \omega^i_j E^j_i$ is given in terms of the local coordinate system (x^i, X^j_k) by*

$$\omega^i_j = \sum_k Y^i_k (dX^k_j + \sum_{l,m} \Gamma^i_{lm} X^l_j dx^m)$$

Proof. See [8] □

Chapter 2

Definitions of projective structures

2.1 Classical definition

In this section we will give the classical definition of the projective structure.

Definition 2.1.1. Let ∇ be a linear connection on a C^∞ manifold M, $x(t)$ a curve in M. Then $x(t)$ is called a *geodesic* of the linear connection ∇ if and only if

$$\nabla_{\dot{x}(t)}\dot{x}(t) = 0$$

for all t in the domain of definition of $x(t)$.

Remark 2.1.1. In terms of Christoffel's symbols this definition reads as

$$\frac{\partial^2 x^i(t)}{\partial t^2} + \sum_{i,j,k} \Gamma^i_{jk} \frac{\partial x^j(t)}{\partial t} \frac{\partial x^k(t)}{\partial t} = 0$$

This definition clearly depends on the parametrization of the curve in question. If $x(t)$ is one parametrization of curve c such that $x(t)$ is a geodesic of ∇, then all other such 'geodesic' parametrizations of c are given by $t' = at + b$, where a and b are arbitrary real numbers, see [8].

Definition 2.1.2. Let M be a C^∞ manifold. A *projective structure* on M is a class of torsion-free linear connections on M, which have the same geodesics up to parametrization.

Proposition 2.1.1. *Two torsion-free connections on C^∞ manifold M have same geodesics up to parametrization if and only if there exists a one-form α on M such that*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X$$

Proof. See [11] $\square$

Remark 2.1.2. In terms of Christoffel's symbols this reads as

$$\widetilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k} + \alpha_i \delta_j^k + \alpha_j \delta_i^k$$

Here we would like to show how to compute the covariant derivative of a tensor in local coordinates, because we will use it later. The convention is that contravariant indices are the upper indices and covariant indices are the lower indices. Here, T^a for instance means $T(dx^a)$ and T_a means $T(\partial/\partial x^a)$. Similarly, ∇_a denotes $\nabla_{\partial/\partial x_a}$.
First, consider a contravariant tensor $T^{a_1 \dots a_n}$:

$$\nabla_a T^{a_1 \dots a_n} = \partial_a T^{a_1 \dots a_n} + \sum_{i=1}^{n} \Gamma_{ac}^{a_i} T^{a_1 \dots c \dots a_n}$$

Next, consider a covariant tensor $T_{a_1 \dots a_n}$:

$$\nabla_a T_{a_1 \dots a_n} = \partial_a T_{a_1 \dots a_n} - \sum_{i=1}^{n} \Gamma_{a a_i}^{c} T_{a_1 \dots c \dots a_n}$$

Now we would like to see how this covariant derivative behaves under the projective change of connection. Here we are only interested in terms, which form the difference between the old and the new covariant derivative, so all other terms we will write as $\nabla_a T^{\dots}$ or $\nabla_a T_{\dots}$. So first for the contravariant tensor $T^{a_1 \dots a_n}$:

$$\hat{\nabla}_a T^{a_1 \dots a_n} = \nabla_a T^{a_1 \dots a_n} + n\alpha_a T^{a_1 \dots a_n} + \sum_{i=1}^{n} \delta_a^{a_i} \alpha_c T^{a_1 \dots c \dots a_n}$$

Here the terms $\delta_i^k \alpha_j$ in the expression for $\tilde{\Gamma}_{ij}^k$ form the terms $\delta_a^{a_i} \alpha_c T^{a_1 \dots c \dots a_n}$ and the terms $\delta_j^k \alpha_i$ the others.
For covariant tensor $T_{a_1 \dots a_n}$:

$$\hat{\nabla}_a T_{a_1 \dots a_n} = \nabla_a T_{a_1 \dots a_n} - n\alpha_a T_{a_1 \dots a_n} - \sum_{i=1}^{n} \alpha_{a_i} T_{a_1 \dots a \dots a_n}$$

Here the terms $\delta_i^k \alpha_j$ form the terms $\alpha_{a_i} T_{a_1 \dots a \dots a_n}$ and the terms $\delta_j^k \alpha_i$ form the others.

2.2 Projective structure as second order structure

In this section we will see how we can view projective structure as second order structure. But first we shall see how we can realize projective structure on real projective space.

2.2.1 Projective space and projective groups

Let M be a real projective space of dimension n with homogeneous coordinate system $\xi^0, \xi^1, \ldots, \xi^n$. Here we will view M as space of lines through origin in R^{n+1} and we will consider the group $SL(n+1; R)$ acting on M by projective transformations induced by its action on R^{n+1}. Let $x^1, \ldots, x^n$ be the inhomogeneous coordinate system on M defined by $x^i = \xi^i/\xi^0, i = 1, \ldots, n$. If $(s^\alpha_\beta)_{\alpha,\beta=0,\ldots,n} \in SL(n+1; R)$, then the induced projective transformation is given, in terms of the inhomogeneous coordinate system $x_1, \ldots, x_n$ by the following linear fractional transformation:

$$y^i = (s^i_0 + \sum_j s^i_j x^j)/(s^0_0 + \sum_j s^0_j x^j), \quad i = 1, \ldots, n \tag{2.1}$$

If $s^0_0 \neq 0$, then we set

$$a^i = s^i_0/s^0_0, a^i_j = s^i_j/s^0_0, a_i = s^0_i/s^0_0, \quad i, j = 1, \ldots, n$$

The preceding linear fractional transformation is then given by

$$y^i = (a^i + \sum_j a^i_j x^j)/(1 + \sum_j a_j x^j), \quad i = 1, \ldots, n$$

If $n = 2k$ for some $k \in N$, then we shall take $(a^i; a^i_j; a_j), i, j = 1, \ldots, n$ as a local coordinate system in the neighbourhood of identity of $SL(n+1; R)$ defined by $s^0_0 \neq 0$, else we must restrict ourselves to $s^0_0 > 0$.

Proposition 2.2.1. *Let $\omega^i, \omega^i_j, \omega_j, i, j = 1, \ldots, n$ be the left invariant 1-forms on $SL(n+1; R)$ which coincide with da^i, da^i_j, da_j at the identity. Then the equations of Maurer-Cartan of $SL(n+1; R)$ are given by*

$$d\omega^i = -\sum_k \omega^i_k \wedge \omega^k \tag{2.2}$$

$$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega_j + \delta^i_j \sum_k \omega_k \wedge \omega^k \tag{2.3}$$

$$d\omega_j = -\sum_k \omega_k \wedge \omega^k_j \tag{2.4}$$

Proof. If we set $(\bar\omega^\alpha_\beta) = s^{-1}ds$, where $s = (s^\alpha_\beta)$, then we have

$$\omega^i = \bar\omega^i_0, \quad \omega^i_j = \bar\omega^i_j - \delta^i_j \bar\omega^0_0, \quad \omega_j = \bar\omega^0_j$$

Our proposition now follows from the following formula:

$$(d\bar\omega^\alpha_\beta) = d(s^{-1}ds) = -s^{-1}.ds.s^{-1} \wedge ds = -s^{-1}ds \wedge s^{-1}ds = -(\sum_{\gamma=0}^{n} \bar\omega^\alpha_\gamma \wedge \bar\omega^\gamma_\beta)$$

□

The dual of this proposition may be formulated as follows. Let $\mathfrak{m} = R^n$ and $\mathfrak{m}^*$ its dual; an element of $\mathfrak{m}$ will be a column vector and an element of $\mathfrak{m}^*$ will be a row vector. Let $\mathfrak{gl}(n; R)$ be the space of all $n \times n$ matrices. Then the Lie algebra $\mathfrak{sl}(n+1; R)$ of $SL(n+1; R)$is the direct sum:

$$\mathfrak{sl}(n+1; R) = \mathfrak{m} + \mathfrak{gl}(n; R) + \mathfrak{m}^*,$$

with the following bracket operation:If $u, v \in \mathfrak{m}, u^*, v^* \in \mathfrak{m}^*$ and $U, V \in \mathfrak{gl}(n; R)$, then
$[u, v] = 0, \quad [u^*, v^*] = 0$
$[U, u] = Uu, \quad [u^*, U] = u^*U$
$[U, V] = UV - VU$
$[u, u^*] = uu^* + u^*uI_n$
where I_n denotes the identity matrix of degree n. There is an isomorphism between this representation of $\mathfrak{sl}(n+1; R)$and the standard one:

$$\begin{pmatrix} -trA & Y \\ X & A \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & Y \\ X & A + I_n trA \end{pmatrix} \tag{2.5}$$

Let o be the point of the projective space with homogeneous coordinates $(1, 0, \ldots, 0)$ or inhomogeneous coordinates $(0, \ldots, 0)$. Let H be the isotropy subgroup of $SL(n+1; R)$ at o so that $M = SL(n+1; R)/H = G/H$, when n is even and let H be the isotropy subgroup of $SL(n+1; R) \cap \tilde{E}SL(n+1; R)$ at o so that $M = (SL(n+1; R) \cap \tilde{E}SL(n+1; R)/H = G/H)$, when n is odd and $\tilde{E}$ is a diagonal matrix with ± 1 entries and number of -1 is odd. In terms of the local coordinate system $(a^i; a^i_j; a_j)$ of $SL(n+1; R)$ which is valid in a neighbourhood containing H (n is even), the subgroup H is defined by $a^i = 0. i = 1, \ldots, n$. In the case when n is odd, our 'coordinate system' (if we want the 'coordinate neighbourhood' to contain H, so we can't restrict to $s^0_0 > 0$) is a 2-fold covering (because I_n and $-I_n$ have same 'coordinates' in G), but H is still given by $a^i = 0$. The linear fractional transformation induced by an element of H is therefore given by an equation of the form:

$$y^i = (\sum_j a^i_j x_j)/(1 + \sum_j a_j x^j) = \sum_j a^i_j x^j - \sum_{j,k} (a^i_j a_k + a^i_k a_j) x^j x^k / 2 + \ldots \tag{2.6}$$

2.2.2 Jets and frames of higher order contact

Here we shall construct bundles of frames of higher order contact, especially the bundle $P^2(M)$ and its canonical form.

Definition 2.2.1. Let M be a manifold of dimension n. Let U and V be neighbourhoods of the origin 0 in R^n. Two mappings $f : U \to M$ and $g : V \to M$ give rise to the same *r-jet* at 0 if they have the same partial derivatives up to order r at 0. The r-jet given by f is denoted by $j^r_o(f)$.

Definition 2.2.2. If f is a diffeomorphism of a neighbourhood of 0 onto an open subset of M, then the r-jet $j^r_0(f)$ at 0 is called an *r-frame* at $x = f(0)$.

Let $G^r(n)$ be the set of r-frames $j_0^r(g)$ at $0 \in R^n$, where g is a diffeomorphism from a neighbourhood of 0 in R^n onto a neighbourhood of 0 in R^n. Then $G^r(n)$ is a group with multiplication defined by the composition of jets:

$$j_0^r(g).j_0^r(g') = j_0^r(g \circ g').$$

Definition 2.2.3. The *r-frame bundle* $P^r(M)$ of M is a set of r-frames of M with natural projection π, $\pi(j^r0(f)) = f(0)$ endowed with right action of $G^r(n)$ defined by

$$j_0^r(f).j_0^r(g) = j_0^r(f \circ g), \quad \text{for} \quad j_0^r(f) \in P^r(M) \quad \text{and} \quad j_0^r(g) \in G^r(n)$$

We can easily see that $P^r(M)$ is a principal bundle over M with group $G^r(n)$. As a special case, we have the so-called bundle of linear frames $P^1(M)$ with structure group $G^1(n) = GL(n; R)$.
From now on we shall be mainly interested in $P^2(M)$ and $P^1(M)$. Let $A(n; R)$ be the affine group acting on R^n. Considering $A(n; R)$ as a principal fibre bundle over $R^n = A(n; R)/GL(n; R)$ with structure group $GL(n; R)$, we have a natural bundle isomorphism between $A(n; R)$ and $P^1(R^n)$ covering the identity of R^n. Under this isomorphism, the identity e of $A(n; R)$ corresponds to $j_0^1(id)$, where id denotes the identity transformation of R^n. We shall therefore denote $j_0^1(id)$ by e. The tangent space of $P^1(R^n)$at e will be identified with that of $A(n; R)$at e, that is, with the Lie algebra

$$\mathfrak{a}(n; R) = R^n + \mathfrak{gl}(n; R) \quad \text{of} \quad A(n; R)$$

. We shall now define a 1-form on $P^2(M)$ with values in $\mathfrak{a}(n; R)$. First, we observe that $j_0^2(f) \to j_0^1(f)$ defines a homomorphism of the bundle $P^2(M)$ onto the bundle $P^1(M)$. Let X be a vector tangent to $P^2(M)$ at $j_0^2(f)$. Denote by X' the image of X under the homomorphism $P^2(M) \to P^1(M)$; it is a vector tangent to $P^1(M)$ at $j_0^1(f)$. Since f is a diffeomorphism of a neighbourhood of $0 \in R^n$ onto a neighbourhood of $f(0) \in M$, it induces a diffeomorphism of neighbourhood of $e \in P^1(R^n)$ onto a neighbourhood of $j_0^1(f) \in P^1(M)$ given by $j_0^1(g) \to j_0^1(f \circ g)$. The latter induces an isomorphism of the tangent space $\mathfrak{a}(n; R)$ of $P^1(R^n)$ at e onto the tangent space of $P^1(M)$ at $j_0^1(f)$; this isomorphism will be denoted by $\bar{f}$.

Definition 2.2.4. The *canonical form* θ on $P^2(M)$ is defined by

$$\theta(X) = \bar{f}^{-1}(X')$$

Since $\bar{f}$ depends only on $j_0^2(f)$, $\theta(X)$is well defined. The 1-form θ takes its values in $\mathfrak{a}(n; R)$.
Let $j_0^2(g) \in G^2(n)$ and $j_0^1(f) \in P^1(R^n)$. The mapping of a neighbourhood of $e \in P^1(R^n)$ onto a neighbourhood of $e \in P^1(R^n)$ defined by

$$j_0^1(f) \to j_0^1(g \circ f \circ g^{-1})$$

induces a linear isomorphism of the tangent space $\mathfrak{a}(n; R)$of $P^1(R^n)$ at e onto itself. This linear isomorphism depends only on $j_0^2(g)$ and will be denoted by $ad(j_0^2(g))$.

Proposition 2.2.2. *Let θ be the canonical form of $P^2(M)$. Then*

$$\theta(A^*) = A', \quad A \in \mathfrak{g}^2(n) \tag{2.7}$$

where $A' \in \mathfrak{gl}(n; R)$ is the image of A under the homomorphism

$$\mathfrak{g}^2(n) \to \mathfrak{g}^1(n) = \mathfrak{gl}(n; R);$$

and

$$(R_a)^*(\theta) = ad(a^{-1}) \circ \theta, \quad a \in G^2(n) \tag{2.8}$$

Proof. 2.7 Let $j_0^2(g_t)$ be a curve in $G^2(n)$ such that $j_0^2(g_0) = e$. Then $j_0^1(g_t)$ is a curve in $G^1(n)$ such that $j_0^1(g_0) = e$. Then

$$\theta(d/dt(j_0^2(f \circ g_t))) = \bar{f}^{-1}(d/dt(j_0^1(f \circ g_t))) = d/dt(j_0^1(g_t)) = A'$$

2.8 Let in addition $g \in G^2(n)$.Then

$$((R_g^*)\theta)(d/dt(j_0^2(f \circ g_t))) = \theta(d/dt(j_0^2(f \circ g_t \circ g))) =$$
$$= \bar{f}^{-1}(j_0^1(f \circ g \circ g^{-1} \circ g_t \circ g)) = (ad(g^{-1}) \circ \theta)(d/dt(j_0^2(f \circ g_t)))$$

□

Proposition 2.2.3. *Let M and M' be manifolds of the same dimension n and let θ and θ' be the canonical forms on $P^2(M)$ and $P^2(M')$ respectively. Let $f : M \to M'$ be a diffeomorphism and denote by the same letter f the induced bundle isomorphism $P^2(M) \to P^2(M')$. Then*

$$f^*(\theta)' = \theta$$

Conversely, if $F : P^2(M) \to P^2(M')$ is a bundle isomorphism such that $F^\star(\theta') = \theta$, then F is induced by a diffeomorphism f of the base manifolds.

Proof. See [7] □

We shall now express the canonical form of $P^2(M)$ in terms of the local coordinate system of $P^2(M)$ which arises in a natural way from a local coordinate system of M. For this purpose we may restrict ourselves to the case $M = R^n$. Let $e_1, \dots, e_n$ be the natural basis for R^n and $(x_1, \dots, x_n)$the natural coordinate system in R^n. Each 2-frame u in R^n has a unique polynomial representation $u = j_0^2(f)$ of the form

$$f(x) = \sum_i (u^i + \sum_j u^i_j x^j + (\sum_{j,k} u^i_{jk} x^j x^k)/2) e_i$$

where $x = \sum_i x^i e_i$ and $u^i_{jk} = u^i_{kj}$. We take $(u^i; u^i_j; u^i_{jk})$ as the natural coordinate system in $P^2(R^n)$. Restricting $(u^i_j; u^i_{jk})$to $G^2(n)$ we obtain the natural coordinate system in $G^2(n)$, which will be denoted by $(s^i_j; s^i_{jk})$. The action of $G^2(n)$ on $P^2(R^n)$is then given by

$$(u^i; u^i_j; u^i_{jk})(s^i_j; s^i_{jk}) = (u^i; \sum_p u^i_p s^p_j; \sum_p u^i_p s^p_{jk} + \sum_{q,r} u^i_{qr} s^q_j s^r_k)$$

In particular, the multiplication in $G^2(n)$ is given by

$$(\bar{s}^i_j; \bar{s}^i_{jk})(s^i_j; s^i_{jk}) = (\sum_p \bar{s}^i_p s^p_j; \sum_p \bar{s}^i_p s^p_{jk} + \sum_{q,r} \bar{s}^i_{qr} s^q_j s^r_k)$$

Similarly, we can introduce a coordinate system $(u^i; u^i_j)$in $P^1(R^n)$ and a coordinate system (u^i_j) in $G^1(n)$ so that the homomorphisms $P^2(R^n) \to P^1(R^n)$ and $G^2(n) \to G^1(n)$ are given by $(u^i; u^i_j; u^i_{jk}) \to (u^i; u^i_j)$ and $(s^i_j; s^i_{jk}) \to (s^i_j)$ respectively. Let $E_i; E^j_i$ the basis for $\mathfrak{a}(n; R)$ defined by

$$E_i = (\partial/\partial u_i)_e, \quad E^j_i = (\partial/\partial u^i_j)_e$$

We set

$$\theta = \sum_i \theta^i E_i + \sum_{i,j} \theta^i_j E^j_i$$

Let $j^2_0(f) = (u^i; u^i_j; u^i_{jk}) \in P^2(M)$ and $j^1_0(g_t) = (c^i t; \delta^i_j + c^i_j t) \in P^2(R^n)$. Then

$$\bar{f}(d/dt|_{t=0}(j^1_0(g_t))) = d/dt|_{t=0}(j^1_0(f \circ g_t))$$

Here we will use the representation of $j^2_0(f) = \sum_i (u^i + \sum_j u^i_j x^j + (\sum_{j,k} u^i_{jk} x^j x^k)/2) e_i$ and $j^1_0(g_t) = \sum_i (c^i t + (\delta^i_j + c^i_j t) x^j) e_i$and composition of jets will correspond to composition of these polynomials. Because we are differentiating it by t, we shall consider only those terms, which contain t only in the first power. To compute coefficient of E_i, we are only interested in terms, which contain no x^i, to compute coefficient of E^j_i, we are only interested in terms containing x^i in first power:

$$d/dt|_{t=0}(j^1_0(f \circ g_t)) = \left(\sum_i (u^i_j c^j) E_i + \sum_{i,j} \left(\sum_k u^i_k c^k_j \right) E^j_i \right)$$

From the definition of the canonical form θ, we obtain the following formulae:

$$du^i = \sum_j u^i_j \theta^j \tag{2.9}$$

$$du^i_j = \sum_k u^i_k \theta^k_j + \sum_h u^i_{hj} \theta^h \tag{2.10}$$

Let (v^i_j) be the inverse matrix of (u^i_j). Then

$$\theta^i = \sum_k v^i_k du^k \tag{2.11}$$

$$\theta^i_j = \sum_k v^i_k du^k_j - \sum_{h,k,l} v^i_k u^k_{hj} v^h_l du^l \tag{2.12}$$

From these formulae we obtain the following important equation:

Proposition 2.2.4. *Let $\theta = (\theta^i; \theta^i_j)$ be the canonical form of $P^2(M)$. Then*

$$d\theta^i = -\sum_k \theta^i_k \wedge \theta^k$$

Proof. We know that $d(du^i) = 0$. So we get

$$-\sum_j du^i_j \wedge \theta^j = \sum_j u^i_j d\theta^j \Rightarrow -\sum_{j,k} u^i_j \theta^j_k \wedge \theta^k = \sum_j u^i_j d\theta^j \Rightarrow$$

$$\Rightarrow -\sum_{i,j,k} v^l_i u^i_j \theta^j_k \wedge \theta^k = \sum_{i,j} v^l_i u^i_j d\theta^j \Rightarrow -\sum_k \theta^l_k \wedge \theta^k = d\theta^l$$

□

2.2.3 Projective structures and projective connections

The subset $H^2(n)$ of $G^2(n)$ consists of elements $(s^i_j; s^i_{jk})$ with $s^i_{jk} = -(s^i_j s_k + s^i_k s_j)$ forms a subgroup of dimension $n^2 + n$. Let 0 be the point of the projective space with homogeneous coordinates $(1, 0, ..., 0)$ and let H be the isotropy group at 0 of $SL(n+1; R)$ acting on the projective space.

Proposition 2.2.5. *For each element $a \in H$, let f be the linear fractional transformation of R^n induced by a. Then in the case of even n $a \to j^2_0(f)$ gives an isomorphism of H onto $H^2(n)$. Moreover, if $a \in H$ has coordinates $(a^i; a^i_j; a_j)$, where $a^i = 0$, with respect to the local coordinate system in $SL(n+1; R)$ introduced above, then the corresponding element of $H^2(n)$ has coordinates $(a^i_j; -(a^i_j a_k + a^i_k a_j))$. If n is odd, the mapping $a \to j^2_0 f$ is a 2-fold covering of H onto $H^2(n)$. Moreover H is isomorphic to $Z_2 \times H^2(n)$. The other statement is also true in this case, if we consider the 'coordinate system' in G.*

Proof. The coordinates of $j^2_0(f)$ are evident from the explicit expression of the linear fractional transformation f given by equation 2.6. Now in the case of even n we rewrite an element of H using the local coordinate system in $SL(n+1; R)$:

$$\begin{pmatrix} \det^{-1}(g^i_j) & g_j \\ 0 & g^i_j \end{pmatrix} \to \begin{pmatrix} 1 & \det(g^i_j) g_j \\ 0 & \det(g^i_j) g^i_j \end{pmatrix} \tag{2.13}$$

From this expression it is evident that there is an isomorphism $H \to H^2(n)$ given by $a \to j^2_0(f)$. If n is odd, then using the 'coordinate system' we get the same way a homomorphism, which is a 2-fold covering $H \to H^2(n)$. The kernel of this homomorphism is the subgroup $-I_{n+1}, I_{n+1}$ of $SL(n+1; R)$, which is a normal subgroup of G contained in H. □

Definition 2.2.5. A *projective structure* $\bar{P}$ on manifold M is a subbundle P of $P^2(M)$ with structure group $H^2(n)$ if n is even, and $P \coprod P$ with structure group H, if n is odd.

Definition 2.2.6. Let $\bar{P}$ be a projective structure on manifold M and $\bar{P}'$ a projective structure on M'. A mapping $f : M \to M'$ is called *projective*, if, prolonged to $P^2(M)$, maps $\bar{P}$ to $\bar{P}'$.

Remark 2.2.1. If $\dim M$ is odd, then since $H = Z_2 \times H^2(n)$, we can prolong f uniquely to $\bar{P}$. So the definition makes sense also in this case.

Definition 2.2.7. Let $\theta = (\theta^i; \theta^i_j)$ be the canonical form on $P^2(M)$. Given a projective structure on M, let $(\bar{\theta}^i; \bar{\theta}^i_j)$be the restriction of $(\theta^i; \theta^i_j)$ to P. A *projective connection* associated with a projective structure P is a Cartan connection $\omega = (\omega^i; \omega^i_j; \omega_j)$ in P such that $\omega^i = \bar{\theta}^i$.

Remark 2.2.2. In the case, when $\dim M$ is odd, this also defines ω^i on $\bar{P}$ by equivariancy.

Lemma 2.2.1. *Let $a \in H$ and take the corresponding $j^2_0(f)$. Then*

$$ad(j^2_0(f)) : \mathfrak{a}(n; R) \to \mathfrak{a}(n; R)$$

and

$$ad(a^{-1}) : \mathfrak{m} + \mathfrak{gl}(n; R) \to \mathfrak{m} + \mathfrak{gl}(n; R)$$

where the latter is the mapping $\mathfrak{sl}(n+1; R)/\mathfrak{m}^* \to \mathfrak{sl}(n+1; R)/\mathfrak{m}^*$ induced by

$$ad(a^{-1}) : \mathfrak{sl}(n+1; R) \to \mathfrak{sl}(n+1; R)$$

coincide.

Proof. Let $j^2_0(g) = (u^i_j; u^i_{jk}) \in H^2(n)$ and $j^1_0(f_t) = (c^i t; \delta^i_j + c^i_j t) \in P^1(R^n)$. We will use here the polynomial representation of jets. So we have

$$ad(j^2_0(g^{-1}))(d/dt(j^1_0(f_t))) = d/dt(j^1_0(g^{-1} \circ f_t \circ g)) = \sum_{i,j} v^i_j c^j E_i + \sum_{i,j,k,l} v^i_l c^l_k u^k_j E^j_i$$

We see that it is exactly $ad(a^{-1})$ as can be easily computed by matrix conjugation. □

Proposition 2.2.6. *For each projective structure $\bar{P}$ of a manifold M, there is a unique projective connection $\omega = (\omega^i; \omega^i_j; \omega_j)$ such that*

$$\omega^i = \bar{\theta}^i \quad (2.14)$$

$$\omega^i_j = \bar{\theta}^i_j \quad (2.15)$$

$$\sum_i \Omega^i_i = 0 \quad (2.16)$$

$$\sum_i K^i_{jil} = 0 \quad (2.17)$$

where $\Omega^i_j = \sum_{k,l} K^i_{jkl} \omega^k \wedge \omega^l$.

Proof. See [7]. □

Definition 2.2.8. The unique projective connection described in the proposition above is called *normal projective connection.*

2.2.4 Equivalence of the two definitions

The group $G^1(n) = GL(n; R)$can be considered as the subgroup of $G^2(n)$consisting of elements $(s^i_j; s^i_{jk})$ with $s^i_{jk} = 0$. Thus $G^1(n) \subset H^2(n) \subset G^2(n)$. Since $G^2(n)$ acts on $P^2(M)$, the subgroups $G^1(n)$ and $H^2(n)$ act on $P^2(M)$. We consider the associated bundles $P^2(M)/G^1(n)$ and $P^2(M)/H^2(n)$ with fibres $G^2(n)/G^1(n)$ and $G^2(n)/H^2(n)$ respectively.

Proposition 2.2.7. *(1)The cross sections $M \to P^2(M)/G^1(n)$ are in one-to-one correspondence with the affine connections without torsion of M.*
(2)The cross sections $M \to P^2(M)/H^2(n)$ are in one-to-one correspondence with the projective structures of M.

Proof. Let $(u^i; u^i_j; u^i_{jk})$ be the local coordinate system in $P^2(M)$ induced from a local coordinate system $x^1, \ldots, x^n$ in M. We introduce a local coordinate system $z^i; z^i_{jk}$ in $P^2(M)/G^1(n)$ in such a way that the natural mapping $P^2(M) \to P^2(M)/G^1(n)$ is given by the equations

$$z^i = u^i, \quad z^i_{jk} = \sum_{p,q} u^i_{pq} v^p_j v^q_k, \quad (v^i_j) = (u^i_j)^{-1}$$

Then a cross section $\Gamma : M \to P^2(M)/G^1(n)$ is given, locally, by a set of functions

$$z^i_{jk} = -\Gamma^i_{jk}(x^1, \ldots, x^n), \quad \Gamma^i_{jk} = \Gamma^i_{kj}$$

Now consider the action of the group $G^2(n)$ on the fibre $G^2(n)/G^1(n)$:

$$\sum_{p,q} u^i_{pq} v^p_j v^q_k \to \sum_{l,m,n,p,q} s^i_l u^l_{mn} v^m_p v^n_q \bar{s}^p_j \bar{s}^q_k + \sum_{p,q} s^i_{pq} \bar{s}^p_j \bar{s}^q_k$$

where $(s^i_j; s^i_{jk}) \in G^2(n)$ and $(\bar{s}^i_j)$ is the inverse matrix of (s^i_j). So we have

$$\overline{\Gamma}^i_{jk} = \sum_{p,q,r} \frac{\partial \bar{x}^i}{\partial x_p} \Gamma^p_{qr} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} - \sum_{p,q} \frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^q} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k}$$

We must realize that

$$\sum_p \frac{\partial^2 \bar{x}^p}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \bar{x}^p} + \sum_{p,q} \frac{\partial^2 x^i}{\partial \bar{x}^p \partial \bar{x}^q} \frac{\partial \bar{x}^p}{\partial x^j} \frac{\partial \bar{x}^q}{\partial x^k} = 0$$

what we can obtain by differentiating equation

$$\frac{\partial x^i}{\partial x^k} = \sum_p \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial \bar{x}^p}{\partial x^k}$$

So we see that the functions Γ^i_{jk} behave under the change of coordinate system as Christoffel's symbols must do. This proves (1).
Since the reductions of structure group to $H^2(n)$ and the cross sections $M \to P^2(M)/H^2(n)$ are in one-to-one correspondence, (2) is evident.([7]) □

Every affine connection without torsion $\Gamma : M \to P^2(M)/G^1(n)$, composed with the natural mapping $P^2(M)/G^1(n) \to P^2(M)/H^2(n)$, gives a projective structure $M \to P^2(M)/H^2(n)$. Even in the case, when $\dim M$ is odd, knowing P is equivalent to knowing $\bar{P}$.

Definition 2.2.9. An affine connection Γ without torsion is said to *belong to a projective structure* $\bar{P}$, if Γ induces $\bar{P}$ in the manner described above.

Definition 2.2.10. We say that two affine connections without torsion are *projectively related* if they belong to the same projective structure.

Proposition 2.2.8. *The above definition of projectively related connections is equivalent to the classical one.*

Proof. An element $(a^i_j; -(a^i_j a_k + a^i_k a_j))$ of $H^2(n)$ induces the transformation of $P^2(M)$ given by

$$(u^i; u^i_j; u^i_{jk}) \to \left(u^i; \sum_p u^i_p a^p_j; -\sum_p u^i_p(a^p_j a_k + a^p_k a_j) + \sum_{q,r} u^i_{qr} a^q_j a^r_k\right)$$

It induces a transformation of $P^2(M)/G^1(n)$ given by

$$(z^i; z^i_{jk}) \to (z^i; -(\delta^i_j \tilde{a}_k + \delta^i_k \tilde{a}_j))$$

where $\tilde{a}_i = \sum_p b^p_i a_p$, $(b^i_j) = (a^i_j)^{-1}$. Our assertion is now clear. $\square$

Let Γ be the affine connection without torsion. It corresponds, in a natural manner, to a reduction of the structure group to $G^1(n)$. In other words, it induces an isomorphism of $P^1(M)$ into $P^2(M)$ given by

$$\gamma : (u^i; u^i_j) \to \left(u^i; u^i_j; -\sum_{p,q} \Gamma^i_{pq} u^p_j u^q_k\right) \tag{2.18}$$

Thus, an affine connection Γ without torsion belongs to a projective structure $\bar{P}$ if and only if the corresponding subbundle of $P^2(M)$ with structure group $G^1(n)$ is contained in P. If $\dim M$ is odd, then we extend $\gamma : P^1(M) \to P^2(M)$ to $\gamma : P^1(M) \coprod P^1(M) \to P^2(M) \coprod P^2(M)$.

Proposition 2.2.9. *Let Γ be an affine connection without torsion and $\gamma : P^1(M) \to P^2(M)$ the corresponding isomorphism. Let $(\theta^i; \theta^i_j)$ be the canonical form of $P^2(M)$. Then $\gamma^*\theta^i$ is the canonical form of $P^1(M)$ and $\gamma^*\theta^i_j$ is the connection form of the affine connection Γ.*

Proof. Consider a curve $s(t) = (u^i + c^i t; u^i_j + c^i_j t)$ in $P^1(M)$. Then

$$\gamma(s(t)) = \left(u^i + c^i t; u^i_j + c^i_j t; -\sum_{p,q} \Gamma^i_{pq}(u^p_j + c^p_j t)(u^q_k + c^q_k t)\right)$$

So

$$(\gamma^*\theta^i)ds(t)/dt = \theta^i\left(\sum_i c^i\partial/\partial u^i + \sum_{i,j} c^i_j\partial/\partial u^i_j - \sum_{i,j,k,p,q}\Gamma^i_{pq}(c^p_j u^q_k + u^p_j c^q_k)\partial/\partial u^i_{jk}\right)$$

and similarly

$$(\gamma^*\theta^i_j)ds(t)/dt = \theta^i_j\left(\sum_i c^i\partial/\partial u^i + \sum_{i,j} c^i_j\partial/\partial u^i_j - \sum_{i,j,k,p,q}\Gamma^i_{pq}(c^p_j u^q_k + u^p_j c^q_k)\partial/\partial u^i_{jk}\right)$$

So we see that

$$\gamma^*\theta^i = \sum_{i,k} v^i_k du^k \quad (2.19)$$

$$\gamma^*\theta^i_j = \sum_{i,j,k} v^i_k du^k_j + \sum_{i,j,k,l,m} v^i_k \Gamma^k_{lm} u^m_j du^l \quad (2.20)$$

what is exactly what we wanted to prove. $\square$

Chapter 3

Projective structures as parabolic geometries

3.1 Parabolic geometries

We start with a $|k|$-graded semisimple Lie algebras.

Definition 3.1.1. A $|k|$-graded semisimple Lie algebra $\mathfrak{g}$ is a Lie algebra $\mathfrak{g}$ with decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we set $\mathfrak{g}_i = 0$ for $|i| > k$.

This gradation induces a filtration on $\mathfrak{g}$: $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. Every filtration component $\mathfrak{g}^i$ is a subalgebra of $\mathfrak{g}$ by grading property. We denote $\mathfrak{g}^0$ by $\mathfrak{p}$ and $\mathfrak{g}^1$ by $\mathfrak{p}_+$. From the grading property it is evident that $\mathfrak{p}_+$ is an ideal of $\mathfrak{p}$. We also denote by $\mathfrak{g}_-$ the subalgebra $\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$.
Let now G be any Lie group with Lie algebra $\mathfrak{g}$. We define

$$G_0 = \{g \in G, Ad(g)\mathfrak{g}_i \subset \mathfrak{g}_i\}$$

and

$$P = \{g \in G, Ad(g)\mathfrak{g}^i \subset \mathfrak{g}^i\}$$

Now we are ready to define parabolic geometry of type (G, P):

Definition 3.1.2. Let G be a semisimple Lie group with $|k|$-graded Lie algebra $\mathfrak{g}$, G_0 and P the groups given by the $|k|$-grading. A parabolic geometry of type (G, P) on manifold M is a Cartan geometry of type (G, P) on M, i.e. a principal P-bundle $\mathcal{G}$ on M endowed with a Cartan connection with values in $\mathfrak{g}$.

We know that $\mathfrak{g}_-$ and $\mathfrak{p}_+$ are subalgebras of $\mathfrak{g}$. By invariance of the Killing form of $\mathfrak{g}$, they are dual as $\mathfrak{g}_0$-modules and if we consider $\mathfrak{g}_-$ as $\mathfrak{g}/\mathfrak{p}$, then also as $\mathfrak{p}$-modules. Now we can identify $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ and TM using the Cartan connection on $\mathcal{G}$. We define a map $\mathcal{G} \times \mathfrak{g}/\mathfrak{p} \to TM$ by $(u, X) \to T_u p.\omega^{-1}(X)$, where $p : \mathcal{G} \to M$ is a bundle projection. By equivariancy of ω this factors to a bundle map $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \to TM$, which restricts to an isomorphism on each fibre and thus a vector bundle isomorphism. We see that TM is associated to Cartan bundle $\mathcal{G}$. Similarly, we can realize the cotangent bundle and all tensor bundles as associated bundles.

Representing TM as associated to the Cartan bundle we get an induced filtration on it: $T^iM \cong \mathcal{G} \times_P \mathfrak{g}^i/\mathfrak{p}$. This induces an algebraic bracket on $\Gamma(TM)$. Similarly, we can define $T^i\mathcal{G} = p^{-1}T^iM = \{X \in T\mathcal{G};\ \omega X \in \mathfrak{g}_i\}$.

3.2 Regular parabolic geometries.

Now we want to define regular and normal parabolic geometries. First, we have to look on the structure of P.

Proposition 3.2.1. *Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$, G_0 and P the groups induced by the grading. Then P is a semidirect product of G_0 and vector group, i.e. for every $g \in P$ there exists a unique $g_0 \in G_0$ and unique $Z_1, \dots, Z_k$; $Z_i \in \mathfrak{g}_i$ such that*

$$g = g_0 \exp(Z_1) \dots \exp(Z_k)$$

Proof. See [1] □

Let us denote the vector group from the latter proposition by P_+. Now consider the action of P_+ on $\mathcal{G}$. We can form the orbit space $\mathcal{G}_0 = \mathcal{G}/P_+$. Since P_+ is a normal subgroup of P, we can define on $\mathcal{G}_0$ a free right action of G_0 simply by pulling down the action of G_0 on $\mathcal{G}$. So we see that $\mathcal{G}_0$ is a principal G_0-bundle over M. But since G_0 is a subgroup of P, we can also associate TM to $\mathcal{G}_0$. But this induces a gradation on TM, which is compatible with filtration induced by P: $T_iM \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$. Analogous to the filtration on $T\mathcal{G}$, we can define gradation on $T\mathcal{G}_0$. Now the gradation on TM induces an algebraic bracket on TM.

Proposition 3.2.2. *Let $(p : \mathcal{G} \to M.\omega)$ be a parabolic geometry of type (G, P) corresponding to the $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_k$ of the Lie algebra $\mathfrak{g}$ of G. Let $(p_0 : \mathcal{G}_0 \to M)$ be the underlying G_0-principal bundle. Then for each $i = -k, \dots, -1$ the Cartan connection ω descends to a smooth section ω_i^0 of the bundle $L(T^i\mathcal{G}_0, \mathfrak{g})_i$. For each $u \in \mathcal{G}_0$ and $i = -k, \dots, -1$ the kernel of $\omega_i^0 : T_u^i\mathcal{G}_0 \to (g)_i$ is exactly $T_u^{i+1}\mathcal{G}_0$ and each ω_i^0 is equivariant in the sense that for $g \in G_0$ we have $(R_g)^*\omega_i^0 = Ad(g^{-1}) \circ \omega_i^0$.*

Proof. See [1] □

Definition 3.2.1. (1) An *infinitesimal flag structure* of type G, P on a smooth manifold M is given by

(i) A filtration $TM = T^{-k}M \supset \dots \supset T^{-1}M$ of the tangent bundle of M such that the rank of T^iM equals the dimension of $\mathfrak{g}^i/\mathfrak{p}$ for all $i = -k, \dots, -1$.

(ii) A principal G_0-bundle $p : E \to M$.

(iii) A collection $\theta = (\theta_{-k}, \dots, \theta_{-1})$ of smooth sections $\theta_i \in \Gamma(L(T^iE, \mathfrak{g}_i))$ which are G_0-equivariant in the sense that $(R_g)^*\theta_i = Ad(g^{-1}) \circ \theta_i$ for all $g \in G_0$ and such that for each $u \in E$ and $i = -k, \dots, -1$ the kernel of $\theta_i(u) : T^i_uE \to \mathfrak{g}_i$ is exactly $T^{i+1}_uE \subset T^i_uE$.

(2)Let M and $\tilde{M}$ be smooth manifolds endowed with infinitesimal flag structures $\{T^iM\}, p : E \to M, \theta$ and $\{T^i\tilde{M}\}, \tilde{p} : \tilde{E} \to \tilde{M}, \tilde{\theta}$ of type (G, P). Then a *morphism of infinitesimal flag structures* is a principal bundle homomorphism $\Phi : E \to \tilde{E}$, which covers a local diffeomorphism $f : M \to \tilde{M}$ such that Tf is filtration preserving and $\Phi^*\tilde{\theta}_i = \theta_i$ for all $i = -k, \dots, -1$.

Definition 3.2.2. A filtered manifold is a smooth manifold M together with a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of its tangent bundle by smooth subbundles, which is compatible with the Lie bracket of vector fields in the sense that $[\xi, \eta] \in \Gamma(T^{i+j}M)$ for any $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$.

Now assume that (M, T^iM) is a filtered manifold and for each $i = -k, \dots, -1$ let us denote by $q_i : T^iM \to T_iM$ the natural projection, and consider the operator $\Gamma(T^iM) \times \Gamma(T^jM) \to \Gamma(T_{i+j}M)$ defined by $(\xi, \eta) \to q_{i+j}([\xi, \eta])$. For a smooth function $f \in C^\infty(M, R)$ we have $[\xi, f\eta] = (\xi.f)\eta + f[\xi, \eta]$. Since $i \leq -1$, we see that $T^jM \subset T^{i+j+1}M$, so the first term lies in the kernel of q_{i+j}. Hence the mapping defined above is bilinear over smooth functions, so it is given by a bundle map $T^iM \times T^jM \to T_{i+j}M$. Moreover, for $\xi \in T^{i+1}M$ and $\eta \in T^jM$ we have $[\xi, \eta] \in T^{i+j+1}M$, so again this lies in the kernel of q_{i+j}, so this map further descends to a bundle map $T_iM \times T_jM \to T_{i+j}M$. Taking all these maps together, we obtain a bundle map $\mathcal{L} : gr(TM) \times gr(TM) \to (T)M$, which is compatible with the grading.

Definition 3.2.3. (1) For a filtered manifold (M, T^iM) the tensorial map $\mathcal{L} : gr(TM) \times gr(TM) \to gr(TM)$ induced by the Lie bracket of vector fields as described above is called the *(generalized) Levi bracket.*
(2) An infinitesimal flag structure $(\mathcal{G}_0 \to M, \theta)$ is called *regular*, if the algebraic bracket coincides with the Levi bracket $\mathcal{L}$. (3) A parabolic geometry is called regular, if the underlying infinitesimal flag structure is regular.

Definition 3.2.4. Multilinear mapping $f : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathfrak{g}$ is said to have *homogeneous degree* k, if f maps $\mathfrak{g}^{i_1} \times \cdots \times \mathfrak{g}^{i_n}$ to $\mathfrak{g}^{i_1 + \cdots + i_n + k}$. The k-th homogeneous component of f will be denoted by $f^{(k)}$.

Proposition 3.2.3. *Parabolic geometry $(\mathcal{G} \to M, \omega)$ is regular, if and only if the curvature κ of ω satisfies $\kappa^{(i)} = 0$ for $i \leq 0$.*

Proof. See [1] □

3.3 Normal parabolic geometries.

Before we define normal parabolic geometries, we need to look a bit on Lie algebra cohomology with values in some representation. The chain spaces are defined to be $C^k(\mathfrak{g}, V) = \Lambda^k(\mathfrak{g}^* \times V)$ and the differential $\partial : C^k(\mathfrak{g}, V) \to C^{k+1}(\mathfrak{g}, V)$ is defined by

$$\begin{aligned} \partial(f)(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i \rho(X_i) f(X_0, \dots, \hat{X}_i, \dots, X_k) \\ &+ \sum_{i \neq j} (-1)^{i+j} f([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned} \tag{3.1}$$

where $f : \Lambda^k \mathfrak{g} \to V$ and ρ is a representation of $\mathfrak{g}$ on V. By direct computation we get that $\partial^2 = 0$.

We know that the Killing form B of $\mathfrak{g}$ induces an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P-modules. Thus we can identify $C^j(\mathfrak{p}_+, \mathfrak{g}) = \Lambda^j \mathfrak{p}_+ \times \mathfrak{g}$ with the dual of P-module $\Lambda^j(\mathfrak{g}/\mathfrak{p})^* \times \mathfrak{g}$. From the definition it is obvious that $\partial_{\mathfrak{p}} : C^j(\mathfrak{p}_+, \mathfrak{g}) \to C^{j+1}(\mathfrak{p}_+, \mathfrak{g})$ is a P-homomorphism and we know that $\partial_{\mathfrak{p}}^2 = 0$. Dualizing this homomorphism, we obtain a P-homomorphism $\partial^* : \Lambda^j(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \Lambda^{j-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ which satisfies $\partial^* \circ \partial^* = 0$. This homomorphism is called the *Kostant codifferential*.

We now obtain a formula for ∂^* on decomposable elements. We can write a decomposable element of $\Lambda^{n+1}(\mathfrak{g}/\mathfrak{p})^* \times \mathfrak{g}$ as $Z_0 \wedge \dots \wedge Z_n \times A$ with $Z_i \in \mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ and $A \in \mathfrak{g}$. The pairing of $\psi \in C^{n+1}(\mathfrak{p}_+, \mathfrak{g})$ with that element is given by $B(\psi(Z_0, \dots, Z_n), A)$. Thus for $\phi \in C^n(\mathfrak{p}_+, \mathfrak{g})$ the pairing of $\partial\phi$ with our element is given by

$$\sum_{i=0}^{n} (-1)^i B([Z_i, \phi(Z_0, \dots, \hat{Z}_i, \dots, Z_n)], A) +$$
$$\sum_{i<j} (-1)^{i+j} B(\phi([Z_i, Z_j], Z_0, \dots, \hat{Z}_i, \dots, \hat{Z}_j, \dots, Z_n), A)$$

Using invariance of B, we may rewrite each of the summands in the first sum as $(-1)^{i+1} B(\phi(Z_0, \dots, \hat{Z}_i, \dots, Z_n), [Z_i, A])$. We get

$$\begin{aligned} \partial^*(Z_0 \wedge \dots \wedge Z_n \otimes A) &= \sum_{i=0}^{n} (-1)^{i+1} Z_0 \wedge \dots \hat{Z}_i \dots \wedge Z_n \otimes A \\ &+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_0 \wedge \dots \hat{Z}_i \dots \hat{Z}_j \dots \wedge Z_n \otimes A \end{aligned} \tag{3.2}$$

Definition 3.3.1. The parabolic geometry $p : \mathcal{G} \to M, \omega$ is called *normal*, if its curvature κ satisfies $\partial^*(\kappa) = 0$.

3.4 Adjoint tractor bundle.

We now return a bit to general Cartan geometries.

Definition 3.4.1. A tractor bundle for Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) is a vector bundle associated to $\mathcal{G}$ via a restriction to P of some representation of G.

We will use this definition with two different representations. In this paragraph we will consider the adjoint representation of G on $\mathfrak{g}$.

Definition 3.4.2. Let $p : \mathcal{G} \to M, \omega$ be a parabolic geometry of type G, P. The *adjoint tractor bundle* for geometry $\mathcal{G} \to M, \omega$ is the vector bundle $\mathcal{G} \times_P \mathfrak{g}$, where $\mathfrak{g}$ is considered as a P-module via adjoint action. We will denote it by $\mathcal{A}M$ and call $\mathcal{G}$ the adapted frame bundle for $\mathcal{A}M$.

Definition 3.4.3. For every section s of a bundle associated to $\mathcal{G}$ we define a function f_s on $\mathcal{G}$ by $f_s(u) = \underline{u}^{-1}(s(x)), u \in \mathcal{G}, p(u) = x$.

A function on $\mathcal{G}$, which corresponds to a section of some associated bundle, is P-equivariant, i.e. $f(ug) = g^{-1}f(u)$.
Since Lie bracket on $\mathfrak{g}$ is $Ad(P)$-invariant, we can define an *algebraic bracket* on $\mathcal{A}M$ by $f_{\{s,t\}}(u) = [f_s(u), f_t(u)]$. This algebraic bracket makes $\mathcal{A}M$ into a bundle of filtered Lie algebras, so we have a filtration on $\mathcal{A}M$defined by $\mathcal{A}^iM = \mathcal{G} \times_P \mathfrak{g}^i$, which is compatible with algebraic bracket of adjoint tractors. Since $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ and $T^*M \cong \mathcal{G} \times_P \mathfrak{p}_+$, we see that $\mathcal{A}M$ contains T^*M as a subbundle and TM is a quotient of $\mathcal{A}M$.
We can also consider the adjoint tractor bundle as associated to $\mathcal{G}_0$. Then we have an induced gradation on it: $\mathcal{A}_iM = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$. In this situation we can embed in it both tangent and cotangent bundle in an obvious way. On the graded version of $\mathcal{A}M$ the Killing form of $\mathfrak{g}$ induces a pairing such that $\mathcal{A}_i^*M = \mathcal{A}_{-i}M$ and the algebraic codifferential ∂^* defines natural algebraic mappings

$$\partial^* : \Lambda^{k+1}\mathcal{A}^1 \otimes \mathcal{A} \to \Lambda^k\mathcal{A}^1 \otimes \mathcal{A}$$

Similarly as in the case of algebraic bracket we can define the algebraic action of $\mathcal{A}M$ on any tractor bundle by $f_{s\bullet t} = \rho(f_s)f_t$, where s is a section of $\mathcal{A}M$ and $t$is a section of any tractor bundle. Visibly, the algebraic action of $\mathcal{A}M$ on itself is the algebraic bracket.

3.5 Existence and uniqueness of normal Cartan connection.

First we need that any infinitesimal flag structure that comes from a parabolic geometry, comes from a normal parabolic geometry.

Proposition 3.5.1. *Let $p : \mathcal{G} \to M, \omega$ be a regular parabolic geometry with curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ and suppose that $\partial^*(\kappa) \in \Omega^1(M, \mathcal{A}M)^{(l)}$ for some $l \geq 1$. Then there is a normal Cartan connection $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $\tilde{\omega} - \omega \in \Omega^1(M, \mathcal{A}M)^l$. In particular, there is always a normal Cartan connection $\tilde{\omega}$ which induces the same underlying infinitesimal flag structure as ω.*

Proof. See [1] □

Proposition 3.5.2. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$ and let $G_0 \subset P \subset G$ be the subgroups given by the $|k|$-grading. Then any regular infinitesimal flag structure or type (G, P) on a smooth manifold M is induced by a normal parabolic geometry of type (G, P).*

Proof. See [1] □

Now we want to have some result about uniqueness of normal parabolic geometry up to isomorphism. For that purpose we will need the following lemma:

Lemma 3.5.1. *Let $p : \mathcal{G} \to M, \omega$ be a regular parabolic geometry of type (G, P), let $\tilde{\omega}$ be another Cartan connection and put $\Phi := \tilde{\omega} - \omega \in \Omega^1(M, \mathcal{A}M)$. Then the Cartan connections ω and $\tilde{\omega}$ induce the same filtration of TM if and only if $\Phi \in \Omega^1(M, \mathcal{A}M)^0$ and they induce the same underlying infinitesimal flag structure if and only if $\Phi \in \Omega^1(M, \mathcal{A}M)^1$.*

Proof. See [1] □

Proposition 3.5.3. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra such that $H^1(\mathfrak{g}_-, \mathfrak{g})^l = 0$ for some $l \geq 1$. Let G be a Lie group with Lie algebra $\mathfrak{g}$, $G_0 \subset P \subset G$ the subgroups defined by the grading, and let $(p : \mathcal{G} \to M, \omega)$ be a normal regular parabolic geometry of type (G, P). Then the following folds:*
If $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that for each $i = -k, \ldots, -1$ the difference $\tilde{\omega} - \omega$ maps $T^i\mathcal{G}$ to $\mathfrak{g}^{i+l}$, then there is an automorphism Ψ of the principal bundle $\mathcal{G}$, which induces the identity on the underlying infinitesimal flag structure such that $\Psi^*\tilde{\omega} = \omega$.

Proof. See [1] □

3.6 Weyl structures.

Now we will define the main notion needed to see that projective parabolic geometry is equivalent to classical projective geometry. In this subsection we will consider the adjoint tractor bundle as a bundle of graded Lie algebras, i.e. as associated to $\mathcal{G}_0$. From now on, we will also identify the section of some bundle associated to $\mathcal{G}$ or to $\mathcal{G}_0$ and the corresponding equivariant functions.

Definition 3.6.1. Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P) on a smooth manifold M, and consider the underlying principal G_0-bundle $p_0 : \mathcal{G}_0 \to M$ and the canonical projection $\pi : \mathcal{G} \to \mathcal{G}_0$. A *Weyl structure* for $(\mathcal{G}, \omega)$ is a global G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of π, where equivariancy means that $\sigma(ug) = \sigma(u)g$ for $g \in \mathcal{G}_0$.

Proposition 3.6.1. *For any parabolic geometry $p : \mathcal{G} \to M, \omega$ there exists a Weyl structure. Moreover, if σ and $\hat{\sigma}$ are two Weyl structures, then there is a unique smooth section $\Upsilon = (\Upsilon_1, \dots, \Upsilon_k)$ of $\mathcal{A}_1 M \oplus \dots \oplus \mathcal{A}_k M$ such that*

$$\hat{\sigma}(u) = \sigma(u) \exp(\Upsilon_1(u)) \dots \exp(\Upsilon_k(u))$$

Finally, each Weyl structure σ and each section Υ define another Weyl structure $\hat{\sigma}$ by the above formula.

Proof. See [2] □

Consider the pullback $\sigma^*\omega$ of the Cartan connection ω along the section σ. Then $\sigma^*\omega$ is a $\mathfrak{g}$-valued one-form on $\mathcal{G}_0$, which by construction is G_0-equivariant, i.e.$(R_g)^*\sigma^*\omega = Ad(g^{-1}) \circ \sigma^*\omega$ for all $g \in G_0$. Since $Ad(g^{-1})$ preserves the grading of $\mathfrak{g}$, each component $\sigma^*\omega_i$ of $\sigma^*\omega$ is a G_0-equivariant one-form with values in $\mathfrak{g}_i$.
Now consider a vertical tangent vector on $\mathcal{G}_0$, i.e. the value $A^*(u)$ of a fundamental vector field corresponding to some $A \in \mathfrak{g}_0$. Since σ is G_0-equivariant, we conclude that $\sigma_* A^*_u = A^*_{\sigma(u)}$, where the second fundamental vector field is on $\mathcal{G}$. Thus we have $\sigma^*\omega(A^*) = \omega(A^*) = A \in \mathfrak{g}_0$. We see that for $i \neq 0$ the form $\sigma^*\omega_i$ is horizontal, while $\sigma^*\omega_0$ reproduces the generators of fundamental vector fields. So for $i \neq 0$ the form $\sigma^*\omega_i$ descends to a smooth one-form on M with values in $\mathcal{A}_i M$, which we denote by the same symbol, while $\sigma^*\omega_0$ defines a principal connection on $\mathcal{G}_0$.

Definition 3.6.2. The principal connection $\sigma^*\omega_0$ on $\mathcal{G}_0$ is called the *Weyl connection* of the Weyl structure σ. The form $\mathrm{P} = \sigma^*\omega_+$ is a one-form on M with values in T^*M and is called the *Rho-tensor* of σ.

The form $\sigma^*\omega_- = (\sigma^*\omega_{-k}, \dots, \sigma^*\omega_{-1})$ induces an isomorphism

$$TM \cong \mathcal{A}_{-k}M \oplus \dots \oplus \mathcal{A}_{-1}M \cong gr(TM)$$

We will denote this isomorphism by

$$\xi \to (\xi_{-k}, \dots, \xi_{-1}) \in \mathcal{A}_{-k}M \oplus \dots \oplus \mathcal{A}_{-1}M$$

for $\xi \in TM$. In particular, each fixed $u \in \mathcal{G}_0$ provides the identification of $T_{p_0(u)}M \cong \mathfrak{g}_-$ compatible with the grading. Thus, the choice of a Weyl structureσ provides a reduction of the structure group of TM to G_0 (via the soldering form $\sigma^*\omega_-$ on $\mathcal{G}_0$), the linear connection on M (the Weyl connection $\sigma^*\omega_0$), and the Rho-tensor P.
Now we will also need some more notation. By $\underline{j}$ we denote a sequence $(j_1, \dots, j_k)$ of nonnegative integers, and we put $||\underline{j}|| = j_1 + 2j_2 + \dots + kj_k$. Moreover we define $\underline{j}! = j_1! \dots j_k!$ and $(-1)^{\underline{j}} = (-1)^{j_1 + \dots + j_k}$, and we define $(\underline{j})_m$ to be the subsequence $(j_1, \dots, j_m)$ of $\underline{j}$. By 0 we denote sequences of any length consisting entirely of zeros.

Proposition 3.6.2. *Let σ and $\hat{\sigma}$ be two Weyl structures related by*

$$\hat{\sigma}(u) = \sigma(u) \exp(\Upsilon_1(u)) \dots \exp(\Upsilon_k(u)),$$

where $\Upsilon = (\Upsilon_1, \dots, \Upsilon_k)$ is a smooth section of $\mathcal{A}_1 M \oplus \cdots \oplus \mathcal{A}_k M$. Then we have:

$$\hat{\xi}_i = \sum_{||\underline{j}||+l=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} ad(\Upsilon_k)^{j_k} \circ \cdots \circ ad(\Upsilon_1)^{j_1}(\xi_l) \tag{3.3}$$

$$\begin{aligned} \hat{\mathrm{P}}_i(\xi) = & \sum_{||\underline{j}||+l=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} ad(\Upsilon_k)^{j_k} \circ \cdots \circ ad(\Upsilon_1)^{j_1}(\xi_l) + \\ & \sum_{||\underline{j}||+l=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} ad(\Upsilon_k)^{j_k} \circ \cdots \circ ad(\Upsilon_1)^{j_1}(\mathrm{P}_l(\xi)) + \\ & \sum_{m=1}^{k} \sum_{\substack{(\underline{j})_{m-1}=0 \\ m+||\underline{j}||=i}} \frac{(-1)^{\underline{j}}}{(j_m+1)\underline{j}!} ad(\Upsilon_k)^{j_k} \circ \cdots \circ ad(\Upsilon_m)^{j_m}(\nabla_\xi \Upsilon_m) \end{aligned} \tag{3.4}$$

where ad denotes the adjoint action with respect to algebraic bracket {-,-}. If E is an associated vector bundle to the principal bundle $\mathcal{G}_0$, then we have:

$$\hat{\nabla}_\xi s = \nabla_\xi s + \sum_{||\underline{j}||+l=0} \frac{(-1)^{\underline{j}}}{\underline{j}!} (ad(\Upsilon_k)^{j_k} \circ \cdots \circ ad(\Upsilon_1)^{j_1}(\xi_l)) \bullet s \tag{3.5}$$

where $\bullet$ denotes the map $\mathcal{A}_0 M \times E \to E$ induced by the action of $\mathfrak{g}_0$ on the standard fibre of E.

Proof. See [2] □

Example 3.6.1. For all |1|-graded parabolic geometries, the formulae from the proposition above become very simple. The grading of TM is trivial, the connection transforms as

$$\hat{\nabla}_\xi s = \nabla_\xi s - \{\Upsilon, \xi\} \bullet s \tag{3.6}$$

where Υ is a section of $\mathcal{A}_1 M = T^*M$, and the bracket of Υ and ξ is a field of endomorphisms of TM acting on s in an obvious way, because these are the only terms in the formula 3.5 which make sense. Next, the Rho-tensor transforms as

$$\hat{\mathrm{P}}(\xi) = \mathrm{P}(\xi) + \nabla_\xi \Upsilon + \frac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}. \tag{3.7}$$

Definition 3.6.3. The Weyl curvature of a Weyl structure σ is defined by

$$W(\xi, \eta) = d\sigma^*\omega(\xi, \eta) + [\sigma^*\omega(\xi), \sigma^*\omega(\eta)]$$

We see that W is exactly the pullback of the curvature Ω of Cartan connection ω. Since Ω is ∂^*-closed, we see that W is also ∂^*-closed.
Now we want to compare $W_{\leq 0}(\xi, \eta)$ with curvature and torsion of any Weyl connection, where $W_{\leq 0}$ is the part of W with values in $\mathfrak{g}_- \bigcup \mathfrak{g}_0$.

Definition 3.6.4. Let $\mathcal{G} \to M, \omega$ be a parabolic geometry of type (G, P), $\mathcal{G}_0$ the corresponding G_0-bundle and $\sigma : \mathcal{G}_0 \to \mathcal{G}$ any Weyl structure. The *total curvature* K of σ is defined by $K = K_{\leq 0} + K_+$, where

$$K_{\leq 0}(\xi^h, \eta^h) = d\sigma^*\omega_{\leq 0}(\xi^h, \eta^h) + [\sigma^*\omega_{\leq 0}(\xi^h), \sigma^*\omega_{\leq 0}(\eta^h)]$$

and

$$K_+(\xi^h, \eta^h) = d\sigma^*\omega_+(\xi^h, \eta^h) + [\sigma^*\omega_+(\xi^h), \sigma^*\omega_+(\eta^h)]$$

where ξ^h is the horizontal lift of ξ with respect to principal connection $\sigma^*\omega_0$.

Proposition 3.6.3. *Let $\mathcal{G} \to M, \omega$ be a parabolic geometry of type G, P, $\mathcal{G}_0$ the corresponding G_0-bundle, $\sigma : \mathcal{G}_0 \to \mathcal{G}$ any Weyl structure,* P *its Rho-tensor and $W(\xi, \eta)$ its Weyl curvature. Then*

$$W(\xi, \eta) = K(\xi, \eta) + \{\mathrm{P}(\xi), \eta\} - \{\mathrm{P}(\eta), \xi\}$$

Proof. For brevity we will denote $\sigma^*\omega$ by τ. By definition we have

$$W(\xi^h, \eta^h) = d\tau(\xi^h, \eta^h) + [\tau(\xi^h), \tau(\eta^h)]$$

Because h denotes the horizontal lift, the $\mathfrak{g}_0$-components of $\tau(\xi^h)$ and $\tau(\eta^h)$ are automatically zero, so we may write

$$\begin{aligned} [\tau(\xi^h), \tau(\eta^h)] &= [\tau_-(\xi^h), \tau_-(\eta^h)] + [\tau_+(\xi^h), \tau_-(\eta^h)] + \\ &+ [\tau_-(\xi^h), \tau_+(\eta^h)] + [\tau_+(\xi^h), \tau_+(\eta^h)] \end{aligned}$$

On the other hand we have

$$K(\xi^h, \eta^h) = d\tau(\xi^h, \eta^h) + [\tau_-(\xi^h), \tau_-(\eta^h)] + [\tau_+(\xi^h), \tau_+(\eta^h)]$$

Now $\tau_+(\xi^h)$ is exactly the function corresponding to $\mathrm{P}(\xi)$, while $\tau_-(\eta^h)$ corresponds to η. Since the algebraic bracket is induced by the Lie bracket on $\mathfrak{g}$, the formula for $W(\xi, \eta)$ is obvious. □

Example 3.6.2. Let us look on the case of $|1|$-graded parabolic geometries. Then we have

$$W_{-1}(\xi, \eta) = T(\xi, \eta)$$

where T is the torsion of the corresponding Weyl connection. Now

$$W_0(\xi, \eta) = R(\xi, \eta) + \{\mathrm{P}(\xi), \eta\} - \{\mathrm{P}(\eta), \xi\} \tag{3.8}$$

because the part of the total curvature K with values in $\mathfrak{g}_0$ is exactly the curvature of the corresponding Weyl connection.

3.7 Bundles of scales.

Here we introduce bundles of scales, which will be a very important ingredient in comparison of classical and parabolic definition of projective structure. These bundles will be principal R^+-bundles associated to $\mathcal{G}_0$. Clearly, they will be associated via homomorphism $\lambda : G_0 \to R^+$ with derivative $\lambda' : \mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}(\mathfrak{g}_0) \to R$, which automatically vanishes on the semisimple part.

Definition 3.7.1. An element E_λ of $\mathfrak{z}(\mathfrak{g}_0)$ is called a *scaling element* if and only if E_λ acts by a nonzero real scalar on each G_0-irreducible component of $\mathfrak{p}_+$. A *bundle of scales* is a principal R^+-bundle $\mathcal{L}^\lambda \to M$, which is associated to $\mathcal{G}_0$ via a homomorphism $\lambda : G_0 \to R^+$, whose derivative is given by $\lambda'(A) = B(E_\lambda, A)$ for some scaling element $E_\lambda \in \mathfrak{z}(\mathfrak{g}_0)$.

For any principal bundle $E \to M$ there is a bundle $QE \to M$, whose sections are exactly the principal connections on E, see [9].

Proposition 3.7.1. *Let $p : \mathcal{G} \to M$ be a parabolic geometry on M, and let $\mathcal{L}^\lambda \to M$ be a bundle of scales.*
(1) Each Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ determines the principal connection on $\mathcal{L}^\lambda$ induced by the Weyl connection $\sigma^*\omega_0$. This defines a bijective correspondence between the set of Weyl structures and the set of principal connections on $\mathcal{L}^\lambda$.
(2) There is a canonical isomorphism $\mathcal{G} \cong p_0^* Q\mathcal{L}^\lambda$, where $p_0 : \mathcal{G}_0 \to M$ is the projection. Under this isomorphism, the choice of a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is the pullback of the principal connection on the bundle of scales $\mathcal{L}^\lambda$, viewed as a section $M \to Q\mathcal{L}^\lambda$. Moreover, the principal action of G_0 is the canonical action on $p_0^* Q\mathcal{L}^\lambda$ induced from the action on $\mathcal{G}_0$, while the action of P_+ is described by equation 3.5.

Proof. See [2] □

3.8 Other tractor bundles.

Now we look a bit more closely on general tractor bundles and connections on them. We know that all tractor bundles are associated to $\mathcal{G}$ via restriction to P of some representation of G. We can evidently form a principal G-bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$ via left multiplication in G. We can also expand the Cartan connection ω on $\mathcal{G}$ by G-equivariancy to a principal connection on $\tilde{\mathcal{G}}$. This connection induces a linear connection on any associated vector bundle and thus on any tractor bundle on M. This connection will be called the *tractor connection.*
Now we will introduce another approach to parabolic geometries.

Definition 3.8.1. (1) Let M be a smooth manifold of the same dimension as $\mathfrak{g}/\mathfrak{p}$. An *adjoint tractor bundle* over M is a smooth vector bundle $\mathcal{A} \to M$, which is endowed with a decreasing filtration $\mathcal{A} = \mathcal{A}^{-k} \supset \mathcal{A}^{-k+1} \supset \cdots \supset \mathcal{A}^k$ by smooth subbundles and an algebraic Lie bracket $\{-,-\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, such that $\mathcal{A}$ is locally trivial bundle of filtered

Lie algebras modeled on $\mathfrak{g}$. This means that we have local trivializations $\mathcal{A}|_U \to U \times \mathfrak{g}$ for $\mathcal{A}$ which are compatible with the bracket.
(2) Let $\mathcal{A} \to M$ be an adjoint tractor bundle over M, and let G be a group with Lie algebra $\mathfrak{g}$ with the subgroups $G_0 \leq P \leq G$ defined by the grading of $\mathfrak{g}$. An *adapted frame bundle* for $\mathcal{A}$ corresponding to G is a smooth principal bundle $\mathcal{G} \to M$ with structure group P such that $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$, the associated bundle with respect to the adjoint representation of P on $\mathfrak{g}$.

Since we will only consider in applications the adjoint tractor bundles, which are constructed from the Cartan bundle of given parabolic geometry, we can also assume that we have a canonical adapted frame bundle for $\mathcal{A}$.

Definition 3.8.2. Let $\mathcal{A} \to M$ be an adjoint tractor bundle over M and let $\mathcal{G} \to M$ be an adapted frame bundle for $\mathcal{A}$ corresponding to a group G with Lie algebra $\mathfrak{g}$, and consider the subgroup $P \leq G$ as before. Let $\mathbb{V}$ be a finite-dimensional effective $(\mathfrak{g}, P)$-module, i.e. a P-module such that the infinitesimal action of $\mathfrak{p}$ on $\mathbb{V}$ extends to an effective action of $\mathfrak{g}$. We define the $\mathbb{V}$-*tractor bundle* $\mathcal{V}$ for $\mathcal{A}$ to be the associated bundle $\mathcal{G} \times_P \mathbb{V}$.

Let $\mathcal{V}$ be a $\mathbb{V}$-tractor bundle for $\mathcal{A}$. Let us denote by ρ the effective $(\mathfrak{g}, P)$-representation on $\mathbb{V}$. By definition, $\mathcal{V}$ is an associated bundle to $\mathcal{G}$, so we identify smooth sections of $\mathcal{V}$ with smooth maps $\mathcal{G} \to \mathbb{V}$, which are P-equivariant. Here, we will denote the function corresponding to t as $\tilde{t}$.
It is well known that there exists a unique element $E \in \mathfrak{g}$ called the grading element such that $[E, A] = jA$ holds for all elements $A \in \mathfrak{g}_j, j = -k, \dots, k$. Clearly, E is always contained in the centre of $\mathfrak{g}_0$ (since it preserves the grading and commutes with $\mathfrak{g}_0$). This implies that $Ad(b) \cdot E = E$ for each $b \in G_0$ and consequently E acts by some scalar on each irreducible G_0-module. Now we can split the space $\mathbb{V}$ according to eigenvalues of the action of E, and we denote by $\mathbb{V}_j$ the component corresponding to the eigenvalue j. Then the action of G_0 maps each $\mathbb{V}_j$ to itself, while the infinitesimal action of $\mathfrak{g}_i$ maps $\mathbb{V}_j$ to $\mathbb{V}_{i+j}$ for each $i = -k, \dots, k$.
Clearly, the decomposition $\mathbb{V} = \oplus_j \mathbb{V}_j$ is only G_0-invariant and not P-invariant. On the other hand, if we pass to associated filtration by putting $\mathbb{V}^j = \oplus_{j' \geq j} \mathbb{V}_{j'}$, then from the decomposition of P as a semidirect product of G_0 and P_+ we see that this endows $\mathbb{V}$ with a P-invariant decreasing filtration. Note that if $\mathbb{V}$ is irreducible as a $\mathfrak{g}$-module, then it is generated by a highest weight vector, so the possible eigenvalues$j$ lie in the set $\{j_0 - k, k \in \mathbb{N}\}$, where j_0 is an eigenvalue of highest weight vector. Thus, in the general finite-dimensional case, the eigenvalues lie in the union of finitely many sets of that type. Passing to the associated bundles, we see that for each eigenvalue j of E on $\mathbb{V}$, we get a smooth subbundle $\mathcal{V}^j \subset \mathcal{V}$ corresponding to the P-submodule $\mathbb{V}^j$ of $\mathbb{V}$, and these subbundles form a decreasing filtration of $\mathcal{V}$.
Suppose ∇ is a linear connection on $\mathcal{V}$. Consider a point $u \in \mathcal{G}$ and a tangent vector $\xi \in T_u\mathcal{G}$ and let $x = p(u)$. For a smooth section $t \in \Gamma(\mathcal{V})$ we have a well defined element $\nabla_{p_*\xi} t(x) \in \mathcal{V}_x$ and thus a point $\underline{u}^{-1}(\nabla_{p_*\xi} t(x)) \in \mathbb{V}$. On the other

hand, we also have the well defined element $\xi \cdot \tilde{t}(u) \in \mathbb{V}$. If f is a smooth real-valued function on M, then $\nabla_{p_*\xi} ft(x) = t(x)p_*\xi \cdot f(x) + f(x)\nabla_{p_*\xi}t(x)$. On the other hand, $\widetilde{ft} = (p^*f)\tilde{t}$ and thus $\xi \cdot \widetilde{ft}(u) = \xi \cdot (p^*f)(u)\tilde{t}(u) + (p^*f)(u)\xi \cdot \tilde{t}(u)$. But this implies that the difference $\underline{u}^{-1}(\nabla_{p_*\xi}t(x)) - \xi \cdot \tilde{t}(u)$ depends only on $t(x)$ and thus only on $\tilde{t}(u)$. Hence ξ induces a linear map $\Phi(\xi) : \mathbb{V} \to \mathbb{V}$, which is characterized by the fact that $\underline{u}^{-1}(\nabla_{p_*\xi}t(x)) - \xi \cdot \tilde{t}(u) = \Phi(\xi)(\tilde{t}(u))$, for each smooth section t of $\mathcal{V}$.

Definition 3.8.3. (1) A linear connection ∇ on $\mathcal{V}$ is called a $\mathfrak{g}$-*connection* if and only if for each tangent vector $\xi \in T_u\mathcal{G}$ the linear map $\Phi(\xi) : \mathbb{V} \to \mathbb{V}$ defined above is given by the action of some element of $\mathfrak{g}$.
(2) A linear connection on $\mathcal{V}$ is called *nondegenerate* if and only if for any point $x \in M$ and any nonzero tangent vector $\xi \in T_xM$ there exists a number i and a (local) smooth section t of $\mathcal{V}^i$ such that $\nabla_{p_*\xi}t(x) \notin \mathcal{V}^i_x$.
(3) A *tractor connection* on $\mathcal{V}$ is a nondegenerate $\mathfrak{g}$-connection.

Proposition 3.8.1. *Let $\mathcal{A} \to M$ be an adjoint tractor bundle, $\mathcal{G} \to M$ an adapted frame bundle corresponding to a choice of a group G with Lie algebra $\mathfrak{g}$, $\mathbb{V}$ an effective $(\mathfrak{g}, P)$-module and $\mathcal{V}$ the $\mathbb{V}$-tractor bundle for $\mathcal{A}$.*

1. *A tractor connection ∇ on $\mathcal{A}$ induces a Cartan connection on $\mathcal{G}$.*

2. *Conversely, a Cartan connection ω on $\mathcal{G}$ induces tractor connections on all tractor bundles for $\mathcal{A}$.*

Proof. See [4] □

Definition 3.8.4. Let $\mathcal{V}$ be a tractor bundle on M, ∇ a tractor connection on $\mathcal{V}$. The *curvature* of the connection ∇ is defined to be the End($\mathcal{V}$)-valued two-form R, which is characterized by $R(\xi, \eta)(t(x)) = (\nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]})t(x)$ for smooth vector fields ξ and η, and any smooth section t of $\mathcal{V}$.

Proposition 3.8.2. *Let ∇ be a tractor connection on $\mathcal{V}$. Then there is an $\mathcal{A}$-valued two-form κ on M, such that $R(\xi, \eta)(t) = \kappa(\xi, \eta) \bullet t$ for all $t \in \mathcal{V}$. Moreover, if $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is the Cartan connection induced by ∇, then the function $\mathcal{G} \to \mathfrak{g}$ representing $\kappa(\xi, \eta)$ is $d\omega(\bar{\xi}, \bar{\eta}) + [\omega(\bar{\xi}), \omega(\bar{\eta})]$, where $\bar{\xi}$ and $\bar{\eta}$ are lifts of ξ and η to smooth vector fields on $\mathcal{G}$.*

Proof. See [4] □

This proposition says that the curvature of tractor connection is given by an algebraic action of curvature of Cartan connection.

Definition 3.8.5. A tractor connection ∇ on a tractor bundle $\mathcal{V}$ is called *normal*, if the form $\kappa \in \Omega^2(M, \mathcal{A})$ representing its curvature has the property that $\partial^*(\kappa) = 0$.

Before the next proposition we restrict ourselves to $|1|$-graded geometries.

Proposition 3.8.3. *Suppose that $\mathcal{V} \to M$ is a vector bundle, and suppose that for each Weyl connection ∇ on M we can construct an isomorphism $\mathcal{V} \to \overrightarrow{\mathcal{V}} = \oplus_j \mathcal{V}_j$, which we write as $t \to \overrightarrow{t} = (\dots, t_j, t_{j+1}, \dots)$ both on the level of elements and of sections. Suppose farther that changing ∇ to $\hat{\nabla}$ with corresponding one-form Υ, this isomorphism changes to $t \to \widehat{\vec{t}} = (\dots, \hat{t}_j, \hat{t}_{j+1}, \dots)$, where*

$$\hat{t}_k = \sum_{i \geq 0} \frac{1}{i!} \rho(-\Upsilon)^i (t_{k-i}) \tag{3.9}$$

Then for a point $x \in M$ the set $\mathcal{A}_x$ of all linear maps $\phi : \mathcal{V}_x \to \mathcal{V}_x$ for which there exists an element $\vec{\phi} \in \vec{\mathcal{A}}_x = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ such that $\overrightarrow{\phi(t)} = \rho(\vec{\phi})(\vec{t})$ for all $t \in \mathcal{V}_x$ is independent of the choice of the Weyl connection ∇. The spaces $\mathcal{A}_x$ form a smooth subbundle $\mathcal{A}$ of $L(\mathcal{V}, \mathcal{V}) = \mathcal{V}^* \otimes \mathcal{V}$, which is an adjoint tractor bundle on M isomorphic to $\overrightarrow{\mathcal{A}}$. Then $\mathcal{V}$ is a $\mathbb{V}$-tractor bundle for an appropriate adapted frame bundle for $\mathcal{A}$. The expression (in the isomorphism corresponding to ∇)

$$\overrightarrow{\nabla^{\mathcal{V}}_{\xi} t} = \nabla_{\xi} \vec{t} + (\rho(\xi) + \rho(\mathrm{P}(\xi)))(\vec{t}) \tag{3.10}$$

for $\xi \in XM$ and $t \in \Gamma(\mathcal{V})$ defines a normal tractor connection on $\mathcal{V}$. Thus $\mathcal{V}$ is a normal tractor bundle on M corresponding to $\mathbb{V}$.
Finally, the curvature R of this connection is given by

$$\overrightarrow{R(\xi, \eta)(s)} = (T(\xi, \eta) + W_0(\xi, \eta) + CY(\xi, \eta)) \bullet \vec{s} \tag{3.11}$$

where T, W and CY are torsion, Weyl curvature and the exterior covariant derivative of the Rho-tensor.

Proof. See [3] □

3.9 Projective parabolic geometry.

Now we define the projective structure in terms of parabolic geometry and prove that this is equivalent to definitions given in the second chapter.

Definition 3.9.1. Let M be an n-dimensional smooth manifold. If n is even, we define the projective parabolic geometry on M to be the Cartan geometry on M of type (G, P), where $G = SL(n+1, R)$ and P is the isotropy subgroup of the line $(x, 0, \dots, 0)$ through origin in R^{n+1}. If n is odd, we define the projective parabolic geometry to be the Cartan geometry of type (G, P), where $G = Sl(n+1, R) \bigcup \tilde{E} \cdot SL(n+1, R)$, where $\tilde{E}$ was defined in the second chapter and P is the isotropy subgroup of the line $(x, 0, \dots, 0)$ through origin in R^{n+1}.

In both cases the Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(n+1, R)$ with the block decomposition $\begin{pmatrix} -trA & Y \\ X & A \end{pmatrix}$ where $X \in \mathfrak{g}_{-1}$, $Y \in \mathfrak{g}_1$, the rest is $\mathfrak{g}_0$ and the blocks have sizes 1 and n. Since we know that there is a unique (up to isomorphism) normal parabolic geometry of type (G, P), we will restrict ourselves to normal projective parabolic geometry.
First, we observe that the nonclassical definition of projective structure given in previous chapter is in fact the parabolic definition given above if we prove that the normal projective connection in the sense of previous chapter is the same as normal Cartan connection in the sense of parabolic geometry. For that purpose we analyze the condition on Cartan connection to be normal, i.e. the equation $\partial^*\Omega = 0$. Since projective geometry is $|1|$-graded, we see that the second term in the formula 3.2 vanishes, so it reads as

$$\sum_{i=1}^{n} \{\eta_i, \Omega(X, \xi_i)\} = 0$$

where $\{\xi_i\}$ is a basis of $\mathfrak{g}_-$ and $\{\eta_i\}$ is the dual basis of $\mathfrak{p}_+$ with respect to the Killing form of $\mathfrak{g}$. We will use the basis of matrices, which have only one entry equal 1 and all other entries are 0. Then when we order the basis of $\mathfrak{g}_-$ in such a way that first vector has entry 1 in the second row, second vector in third row,...,and the last vector in the last row. The corresponding ordering of the dual basis of $\mathfrak{p}_+$ is as follows: the first vector will have 1 in the second column, the second vector in the third column,...,the last vector in the last column. Next, we will need the inverse of transformation 2.5 given by

$$\begin{pmatrix} 0 & Y \\ X & A \end{pmatrix} \to \begin{pmatrix} -\frac{1}{n+1}trA & Y \\ X & A - \frac{1}{n+1}I_n trA \end{pmatrix} \tag{3.12}$$

Now we compute what gives the formula for normality. We rewrite the curvature Ω from the previous chapter in matrix as follows (here we will use the transformation above):

$$(\Omega^i, \Omega^i_j, \Omega_j) \to \begin{pmatrix} -\frac{1}{n+1}K_{ij}{}^k{}_k & K_{ijl} \\ K_{ij}{}^k & K_{ij}{}^k{}_l - \frac{1}{n+1}K_{ij}{}^k{}_k I_n \end{pmatrix} \tag{3.13}$$

Finally the computation:

$$\begin{aligned}
\textstyle\sum_{j=1}^{n} \Bigg[\begin{pmatrix} 0 & Y_j \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{n+1}K_{ij}{}^k{}_k & K_{ijl} \\ K_{ij}{}^k & K_{ij}{}^k{}_l - \frac{1}{n+1}K_{ij}{}^k{}_k I_n \end{pmatrix} - \\
- \begin{pmatrix} -\frac{1}{n+1}K_{ij}{}^k{}_k & K_{ijl} \\ K_{ij}{}^k & K_{ij}{}^k{}_l - \frac{1}{n+1}K_{ij}{}^k{}_k I_n \end{pmatrix} \begin{pmatrix} 0 & Y_j \\ 0 & 0 \end{pmatrix} \Bigg] = \\
= \textstyle\sum_{j=1}^{n} \begin{pmatrix} K_{ij}{}^j & K_{ij}{}^j{}_l - \frac{1}{n+1}K_{il}{}^k{}_k \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\frac{1}{n+1}K_{il}{}^k{}_k \\ 0 & K_{ij}{}^k \otimes Y_j \end{pmatrix} = \\
= \textstyle\sum_{j=1}^{n} \begin{pmatrix} K_{ij}{}^j & K_{ij}{}^j{}_l \\ 0 & -K_{ij}{}^k \otimes Y_j \end{pmatrix}
\end{aligned}$$

Since $K_{ij}{}^k \otimes Y_j$ is exactly the matrix of $\Omega^i(X, -)$, we see that the torsion must vanish. The only remaining condition is on Ω^i_j: $\sum_j K_{ijl}{}^j = 0$. We know that the normal projective connection from the previous chapter satisfies 2.14, so in particular it is normal.

Up to now we have seen that the projective structure with the normal projective connection as defined in the previous chapter is a normal projective parabolic geometry. Now we want to show that every normal projective parabolic geometry is isomorphic to some projective structure with normal projective connection.
First we prove that the bundle $\mathcal{G}_0$ for projective parabolic geometry is isomorphic to P^1M or $P^1M\coprod P^1M$, respectively. Consider the underlying infinitesimal flag structure for projective parabolic geometry. From the definition we see that ω^0_{-1} is a smooth section of $L(T\mathcal{G}_0,\mathfrak{g}/\mathfrak{p})$. Moreover, the kernel of ω^0_{-1} is exactly the subbundle $T^0\mathcal{G}_0$, i.e. the vertical subbundle of $\mathcal{G}_0\to M$. Hence for any point $u\in\mathcal{G}_0$ with $x=p_0(u)$ the form $\omega^0_{-1}(u)$ may be viewed as a linear isomorphism $T_u\mathcal{G}_0/T^0_u\mathcal{G}_0\cong T_xM\to\mathfrak{g}/\mathfrak{p}$. Since $\dim M=\dim\mathfrak{g}/\mathfrak{p}$, we may view the manifold M as being modeled on a vector space $\mathfrak{g}/\mathfrak{p}$. Associating to u the inverse of the above linear isomorphism gives a bundle map $\Phi:\mathcal{G}_0\to P^1M$. By construction this covers the identity on M and is equivariant for the homomorphism $Ad:G_0\to GL(\mathfrak{g}/\mathfrak{p})$. Since $\mathcal{G}_0$ has the structure group $GL(n,R)$ or $GL(n,R)\times Z_2$, respectively, and in the first case the homomorphism $Ad:G_0\to GL(\mathfrak{g}_-)$ is isomorphism and in the second case a two-fold covering, we see that $\mathcal{G}_0$ is really isomorphic to P^1M or to $P^1M\coprod P^1M$, respectively.
Now we prove that two isomorphic projective parabolic geometries induce the same Weyl connections on M. Consider two parabolic geometries $\mathcal{G}\to M,\omega$ and $\tilde{\mathcal{G}}\to M,\tilde{\omega}$ and consider the isomorphism $\Phi:\mathcal{G}\to\tilde{\mathcal{G}}$ such that $\Phi^*\tilde{\omega}=\omega$. From $\mathcal{G}$ and $\tilde{\mathcal{G}}$ we may form an orbit space $\mathcal{G}_0$ or $\tilde{\mathcal{G}}_0$, respectively. The isomorphism Φ factors to an isomorphism $\underline{\Phi}:\mathcal{G}_0\to\tilde{\mathcal{G}}_0$, such that $\underline{\Phi}\circ\pi=\tilde{\pi}\circ\Phi$, where $\pi:\}\to\mathcal{G}_0$ is the projection. Consider any Weyl structure $\sigma:\mathcal{G}_0\to\mathcal{G}$ and define $\tilde{\sigma}:\tilde{\mathcal{G}}_0\to\tilde{\mathcal{G}}$ by $\Phi\circ\sigma=\tilde{\sigma}\circ\underline{\Phi}$. They induce the Weyl connections on M: $\sigma^*\omega_0$ and $\tilde{\sigma}^*\tilde{\omega}_0$. Since they are $\underline{\Phi}$-related, we conclude that they induce the same connection on M.
Now we assume that $\mathcal{G}$ is a subbundle of P^2M or $P^2M\coprod P^2M\cdot\tilde{E}$. First we realize, what are the Weyl structures for such geometries. The maps $\gamma:P^1M\to P$ (or $\gamma:P^1M\coprod P^1M\to P\coprod P$) obviously satisfy the definition of Weyl structure: it is smooth section of $\mathcal{G}\to\mathcal{G}_0$, since P^1M or $P^1M\coprod P^1M$ is exactly the orbit space of the action of P_+ on $\mathcal{G}$ and it commutes with the action of G_0. (In fact, Weyl structure is an embedding of $\mathcal{G}_0$ into $\mathcal{G}$). To see that these are all Weyl structures, we need to compute how in the projective case changes the Weyl connection when we change the Weyl structure. We compute it from formula 3.6, where we must realize that the algebraic action is in fact algebraic bracket on adjoint tractor bundle:

$$\begin{aligned}\begin{pmatrix}0&\Upsilon\\0&0\end{pmatrix}\begin{pmatrix}0&0\\\xi&0\end{pmatrix}-\begin{pmatrix}0&0\\\xi&0\end{pmatrix}\begin{pmatrix}0&\Upsilon\\0&0\end{pmatrix}&=\\\begin{pmatrix}\Upsilon(\xi)&0\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\0&\xi\otimes\Upsilon\end{pmatrix}&=\\\begin{pmatrix}\Upsilon(\xi)&0\\0&-\xi\otimes\Upsilon\end{pmatrix}&\end{aligned}\tag{3.14}$$

So we have computed $\{\Upsilon, \xi\}$ and now we compute $\{\{\Upsilon, \xi\}, \eta\}$:

$$\begin{pmatrix} \Upsilon(\xi) & 0 \\ 0 & -\xi\otimes\Upsilon \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix}\begin{pmatrix} \Upsilon(\xi) & 0 \\ 0 & -\xi\otimes\Upsilon \end{pmatrix} = \tag{3.15}$$
$$\begin{pmatrix} 0 & 0 \\ -\Upsilon(\eta)\xi & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Upsilon(\xi)\eta & 0 \end{pmatrix} =$$
$$\begin{pmatrix} 0 & 0 \\ -\Upsilon(\eta)\xi - \Upsilon(\xi)\eta & 0 \end{pmatrix}$$

so we see that $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. So the Weyl connections of our Weyl structures are by 2.18 exactly the torsion-free connections from the classical definition of projective structures, because to any two such connections we can relate a one-form on M and having one such connection and any one-form Υ one gets another such connection, so they are parametrized by one-forms on M and there are no other Weyl structures.
We see that any connection on M is a Weyl connection for some projective parabolic geometry of special type-that of previous chapter. Now consider any projective parabolic geometry and its Weyl connections. They are clearly the Weyl connections of projective parabolic geometry of this special type, so we may hope that these two geometries are isomorphic. From the Proposition 3.7.1 we know that since they have isomorphic bundles $\mathcal{G}_0$ as we have seen above, they have also isomorphic Cartan bundles. The only thing we have to prove is that the normal Cartan connection on $\mathcal{G}$ satisfies $\sum_i \Omega^i_i = 0$. Since $W_0 = \sigma^*\Omega^i_j$, it will be sufficient to prove it for W_0: $\sum_c (W_0)_{ab}{}^c{}_c = 0$. This will follow from the computation below.
Now we want to write the Weyl connection W_0 in terms of the curvature of some Weyl connection and the associated Rho tensor. For this purpose we use 3.8. First, we compute $\{\mathrm{P}(\xi), \eta\}$:

$$\begin{pmatrix} 0 & \mathrm{P}(\xi) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{P}(\xi) \\ 0 & 0 \end{pmatrix} =$$
$$\begin{pmatrix} \mathrm{P}(\xi)(\eta) & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \eta\otimes\mathrm{P}(\xi) \end{pmatrix} =$$
$$\begin{pmatrix} \mathrm{P}(\xi)(\eta) & 0 \\ 0 & -\eta\otimes\mathrm{P}(\xi) \end{pmatrix}$$

Using the transformation 2.5 we get

$$W_0(\xi, \eta) = R(\xi, \eta) - \mathrm{P}(\xi)(\eta)Id + \mathrm{P}(\eta)(\xi)Id - \eta\otimes\mathrm{P}(\xi) + \xi\otimes\mathrm{P}(\eta)$$

what in the abstract index notation reads as

$$(W_0)_{ab}{}^c{}_d = R_{ab}{}^c{}_d - \mathrm{P}_{ab}\delta^c_d + \mathrm{P}_{ba}\delta^c_d - \mathrm{P}_{ad}\delta^c_b + \mathrm{P}_{bd}\delta^c_a \tag{3.16}$$

From this equation we easily get

$$(W_0)^c_{[ab\ d]cycl} = 0$$

what together with $(W_0)^c_{ac\ d} = 0$ and antisymmetry of W_0 in first two indices gives $(W_0)^c_{ab\ c} = 0$, where together with the abstract index notation we use the Einstein summation convention. Making the trace over indices c and d in equation 3.16 gives

$$0 = R_{ab\ c}^{\ \ c} - n\mathrm{P}_{ab} + n\mathrm{P}_{ba} - \mathrm{P}_{ab} + \mathrm{P}_{ba}$$

Making the trace over indices b and c in 3.16 gives

$$0 = R_{ac\ d}^{\ \ c} - \mathrm{P}_{ad} + \mathrm{P}_{da} - n\mathrm{P}_{ad} + \mathrm{P}_{ad}$$

Together we get

$$\mathrm{P}_{ab} = \frac{(n+1)R_{ac\ b}^{\ \ c} - R^c_{ab\ c}}{(n+1)(n-1)}$$

Finally, we can compute the Weyl curvature

$$\begin{aligned}(W_0)_{ab\ d}^{\ \ c} &= R_{ab\ d}^{\ \ c} - \mathrm{P}_{ab}\delta^c_d + \mathrm{P}_{ba}\delta^c_d - \mathrm{P}_{ad}\delta^c_b + \mathrm{P}_{bd}\delta^c_a = \\ &= R_{ab\ d}^{\ \ c} - \tfrac{(n+1)R^c_{ac\ b} - R_{ab\ c}^{\ \ c}}{(n+1)(n-1)}\delta^c_d + \tfrac{(n+1)R_{bc\ a}^{\ \ c} - R_{ba\ c}^{\ \ c}}{(n+1)(n-1)}\delta^c_d - \\ &\quad - \tfrac{(n+1)R_{ac\ d}^{\ \ c} - R_{ad\ c}^{\ \ c}}{(n+1)(n-1)}\delta^c_b + \tfrac{(n+1)R^c_{bc\ d} - R^c_{bd\ c}}{(n+1)(n-1)}\delta^c_a\end{aligned}$$

3.10 Standard tractor bundle and its dual.

In this section we will construct the standard tractor bundle and its dual. First, we will introduce some line-bundles associated to $\mathcal{G}_0$.

Definition 3.10.1. Bundle $\mathcal{E}[\omega]$ is defined to be the bundle $\mathcal{G}_0 \times_{G_0} R$ via representation $\rho(g_0)r = \det(Ad(g_0))^{\frac{\omega}{n+1}}$. The infinitesimal representation of $\mathfrak{g}_0$ is given by $\rho'(X)r = \frac{\omega}{n+1}r.tr(ad(X))$. Here Ad is the adjoint representation of G_0 on $\mathfrak{g}_-$ and ad its derivative.

In matrices this definition reads as follows: If $(a_{ij})^n_{i,j=0}$ is a matrix of g_0, then $\rho(g_0)r = (a_{00})^{-\omega}r$ and if $(A_{ij})^n_{i,j=0}$ is a matrix of X, then $\rho'(X)r = -\omega.A_{00}$. We recall that a matrix of g_0 or X has the property that $A_{0i} = A_{i0} = 0$ for $i = 1, \ldots, n$.
We start with dual of the standard tractor bundle - we will call it the *co-standard* tractor bundle. It will correspond to the dual of the defining representation of the Lie algebra $\mathfrak{g}$. The action of $\rho(\xi)$ and $\rho(\mathrm{P}(\xi))$ will be given by this representation. Consider the jet exact sequence:

$$0 \to \mathcal{E}_i[1] \to J^1(\mathcal{E}[1]) \to \mathcal{E}[1] \to 0 \tag{3.17}$$

where $\mathcal{E}_i[1] = \mathcal{E}_i \otimes \mathcal{E}[1]$, $\mathcal{E}_i = T^*M$ in the abstract index notation. Next, we will need the transformation law for Weyl connections on bundles $\mathcal{E}[\omega]$:

$$\hat{\nabla}_\xi \sigma = \nabla_\xi \sigma - \{\Upsilon, \xi\} \bullet \sigma = \nabla_\xi \sigma + \omega\Upsilon(\xi)\sigma \tag{3.18}$$

Definition 3.10.2. The *co-standard tractor bundle* $\mathcal{E}_I$ is the bundle $J^1\mathcal{E}[1]$.

Proposition 3.10.1. *For a Weyl connection ∇, the map*

$$j^1_x\sigma \to (\sigma(x), \nabla_i\sigma(x))$$

induces an isomorphism $\mathcal{E}_I \to \mathcal{E}[1] \oplus \mathcal{E}_i[1]$ of vector bundles. Moreover, changing ∇ to $\hat{\nabla}$ with the corresponding one-form Υ, we obtain a normal tractor bundle transformation as required in section 3.8, i.e.

$$\widehat{(\sigma, \mu)} = (\sigma, \mu_i + \Upsilon_i\sigma)$$

Proof. Clearly, the formula in the proposition defines a bundle map $J^1\mathcal{E}[1] \to \mathcal{E}[1] \oplus \mathcal{E}_i[1]$ which is injective. Since both bundles have the same rank, it is an isomorphism of vector bundles.
If $\hat{\nabla}$ is another Weyl connection corresponding to Υ, then the first component stays the same, while for the second component we get $\hat{\nabla}_i\sigma = \nabla_i\sigma + \Upsilon_i\sigma$ - so we ge the transformation law for the second component. □

Now we look on the tractor connection on the co-standard tractor bundle and its curvature. By 3.10 the tractor connection is given by

$$\overrightarrow{\nabla^{\mathcal{V}}_\xi t} = (\nabla_\xi\sigma - \xi^i\mu_i, \nabla_\xi\mu)$$

for $\vec{t} = (\sigma, \mu_i)$, which just means

$$\overrightarrow{\nabla^{\mathcal{V}}_i t} = (\nabla_i\sigma - \mu_i, \nabla_i\mu_j)$$

and by 3.11 its curvature is given by

$$\overrightarrow{R^{\mathcal{V}}(\xi, \eta)(t)} = (0, -\mu_j W(\xi, \eta)^j_i - \sigma CY(\xi, \eta))$$

The standard tractor bundle is simply the dual $\mathcal{E}^I$ of the co-standard tractor bundle. We could introduce it in a similar manner - starting with the splitting, but we don't know how to define it invariantly (the co-standard tractor bundle we defined as $J^1\mathcal{E}[-1]$).
As an interest we can define a very useful operator:

Definition 3.10.3. On a section f of $\mathcal{E}[w]$, the operator $D_I : \mathcal{E}[w] \to \mathcal{E}_I[w-1]$ is defined by

$$D_I f = (\nabla_i f \quad wf)$$

Chapter 4

Differential operators.

4.1 Invariant operators.

We develop some basic theory about invariant operators. First, we concentrate on the homogeneous case.

4.1.1 Homogeneous case

Homogeneous bundles

We have to introduce the concept of homogeneous vector bundles. Recall that on each homogeneous space $M = G/P$ we have the canonical left action of G denoted by $\ell : G \times M \to M$ given by $\ell(g, g'P) = gg'P$.

Definition 4.1.1. (1) A *homogeneous bundle* over $M = G/P$ is a locally trivial fibre bundle $\pi : E \to M$ together with a left action $\tilde{\ell} : G \times E \to E$, which extends the action on M, i.e. which satisfies $\pi(\tilde{\ell}(g, e)) = \ell(g, \pi(e))$.
(2) A *homogeneous vector bundle* over M is a homogeneous bundle $\pi : E \to M$, which is a vector bundle and such that for each element $g \in G$ the bundle map $\tilde{\ell}_g : E \to E$ is a vector bundle homomorphism, i.e. linear on each fibre.
(3) A *morphism* of homogeneous bundles (respectively homogeneous vector bundles) is a G-equivariant bundle map (respectively G-equivariant homomorphism of vector bundles) which covers the identity on M.

Proposition 4.1.1. *Let $M = G/P$ and let $o \in M$ be the distinguished element eP. Then the mappings $E \to E_o$ and $f \to f|_o$ induces equivalences between the category of homogeneous bundles on M and the category of manifolds endowed with left P-action and P-equivariant maps, as well as between the category of homogeneous vector bundles on M and the category of finite dimensional representations of P.*

Proof. See [1] □

Using this left action of G on E, we can easily define the action of G on the space $\Gamma(E)$ of all sections of E given by $g \cdot \sigma(x) = g(\sigma(g^{-1}x))$. This action is clearly linear, so we can view the space $\Gamma(E)$ as a representation of G.

Invariant operators

Definition 4.1.2. Let M be a homogeneous space G/P and let E, F be two homogeneous vector bundles over M. An *invariant differential operator* is a differential operator $D : \Gamma(E) \to \Gamma(F)$, which is equivariant for the G-action, i.e. such that $D(g \cdot s) = g \cdot D(s)$ for all $s \in \Gamma(E)$ and $g \in G$.

The first step towards an algebraic description of such an operator is to pass to jet prolongations. If M is a smooth manifold and $\mathcal{V} \to M$ is any vector bundle, then for $k \in N$ we have the k-jet prolongation $J^k\mathcal{V}$. The fibre of $J^k\mathcal{V}$ at $x \in M$ is the vector space of all k-jets at x of sections of $\mathcal{V}$ and [1] says that $J^k\mathcal{V}$ is a vector bundle over M. If $\mathcal{W}$ is another vector bundle over M, then a differential operator $D : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ is of order $\leq k$ if and only if for any two sections $s, t \in \Gamma(\mathcal{V})$ and any point $x \in M$, the equation $j^k s(x) = j^k t(x)$ implies $D(s)(x) = D(t)(x)$. If D is such an operator, then we get an induced bundle map $\tilde{D} : J^k\mathcal{V} \to \mathcal{W}$ over the identity on M, defined by $\tilde{D}(j^k s(x)) := D(s)(x)$, where s is any representative of the jet. Conversely, this formula associates to any bundle map $\tilde{D}$ a differential operator D, which is linear if and only if $\tilde{D}$ is a homomorphism of vector bundles.
In the special case of a homogeneous vector bundle $E \to G/P$, each J^kE is again a homogeneous vector bundle. The action of $g \in G$ is given by $g \cdot (j^k s(x)) := j^k(g \cdot s)(x)$. By construction, a differential operator D corresponding to the bundle map $\tilde{D} : J^kE \to F$ is invariant if and only if $\tilde{D}$ is a morphism of homogeneous vector bundles, i.e. G-equivariant. Hence we have reduced the determination of linear invariant differential operators to the determination of homomorphisms between homogeneous vector bundles.
For $l < k$ we have the obvious projection $\pi^k_l : J^kE \to J^lE$ defined by $\pi^k_l(j^k s(x)) = j^l s(x)$ for any homogeneous vector bundle E over M. This projection is a homomorphism of homogeneous vector bundles.

Definition 4.1.3. Let E, F be two homogeneous vector bundles over $M = G/P$ and let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator of order $\leq k$ corresponding to a bundle map $\tilde{D} : J^kE \to F$. Then the k-th order *symbol* of D is the vector bundle map $\sigma(D) := \tilde{D} \circ i_k : \ker \pi^k_{k-1} \to F$.

The kernel of π^k_{k-1} is by [1] isomorphic to $S^kT^*M \otimes \mathcal{V}$.
Now from the previous chapter we know that there is a correspondence between smooth sections of E and smooth P-equivariant maps $G \to V$, where V is the representation inducing E. Similarly, there is a correspondence between k-jets of such sections in $o = eP$ and k-jets of P-equivariant smooth functions $G \to V$ in $e \in G$. Next, we define the infinite jet prolongation $J^\infty E$ as the direct limit of the system $\cdots \to J^kE \to J^{k-1}E \to \dots$ of vector bundles, where the maps are just the canonical projections from k-jets to $k-1$-jets. In particular, we may view the fibre $J^\infty_o E$ over the base point o as the direct limit

of the fibres of the finite jet prolongations. The identification for finite jets used above then directly leads to the identification of $J^\infty_o E$ with the space $J^\infty_e(G,V)^P$ of infinite jets at e of P-equivariant smooth functions $f : G \to V$. Given $X \in \mathfrak{g}$, we consider the right invariant vector field R_{-X} on G with generator $-X$. For a smooth function $f : G \to V$, we can now consider $R_{-X} \cdot f$, and of course the infinite jet of this function in e depends only on the infinite jet of f in e. Since the flow of R_{-X} up to time t through a point $g \in G$ is given by $\exp(-tX)g$, we can compute $R_{-X} \cdot f(g)$ as the derivative in $t = 0$ of $t \mapsto f(\exp(-tX)g)$. On the one hand, together with the fact that P acts by linear maps on V, this implies that for a P-equivariant function f also $R_{-X} \cdot f$ is P-equivariant. On the other hand, we see that $f \mapsto R_{-X} \cdot f$ is simply the infinitesimal version of the canonical G-action on $C^\infty(G,V)^P$ corresponding to the action of G on $\Gamma(E)$, so it defines a representation of the Lie algebra $\mathfrak{g}$. Thus we have seen that $J^\infty_e(G,V)^P$ is a $(\mathfrak{g},P)$-module, that is a P-module with given action of $\mathfrak{g}$, whose restriction to $\mathfrak{p}$ is the infinitesimal version of the P-action.

Definition 4.1.4. A *generalized Verma module* $M_\mathfrak{p}(V^*)$ for $\mathfrak{p}$-module V is the vector space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^*$. This space is naturally a $(\mathfrak{g},P)$-module (via left multiplication and the corresponding left action).

Proposition 4.1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{p} \leq \mathfrak{g}$ a parabolic subalgebra, G a Lie group with Lie algebra $\mathfrak{g}$, P the parabolic subgroup corresponding to $\mathfrak{p}$ and V a finite dimensional representation of P with dual V^*. Then*

$$\langle Y_1 \cdots Y_n \otimes A, j^\infty_e f\rangle := A(L_{Y_1} \cdots L_{Y_n} \cdot f(e))$$

for $Y_j \in \mathfrak{g}$, $A \in V^*$ and a P-equivariant smooth map $f : G \to V$ induces a well defined pairing between the generalized Verma module $M_\mathfrak{p}(V^*) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^*$ and the space $J^\infty_e(G,V)^P$ of infinite jets of P-equivariant maps. This pairing is compatible with the natural $(\mathfrak{g},P)$-module structures on both spaces and it identifies $M_\mathfrak{p}(V^*)$ with the space of those linear maps $J^\infty_e(G,V)^P \to R$, which factorize over one of the spaces $J^k_e(G,V)^P$.

Proof. See [1] □

Corollary 4.1.1. *Let G/P be a homogeneous model of some parabolic geometry and let E and F be homogeneous vector bundles over G/P corresponding to the representations V and W of P. Then the space of finite order G-invariant differential operators $D : \Gamma(E) \to \Gamma(F)$ is isomorphic to the space $Hom_{(\mathfrak{g},P)}(M_\mathfrak{p}(W^*), M_\mathfrak{p}(V^*))$ of $(\mathfrak{g},P)$-module homomorphisms between generalized Verma modules.*

Proof. See [1] □

Hasse diagram

Now we want to define the Hasse diagram associated to a parabolic subalgebra $\mathfrak{p}$ in a semisimple Lie algebra $\mathfrak{g}$. All what follows is valid for complex Lie algebras, but since we will use it for the split form of a complex Lie algebra, there will be no complications and the results remain valid. First, we define $W_{\mathfrak{p}}$ to be the Weyl group of the semisimple part $\mathfrak{g}_0^{ss}$. This is naturally a subgroup of the Weyl group $W_{\mathfrak{g}}$ of $\mathfrak{g}$. The Hasse diagram associated to $\mathfrak{p}$ will be a set of distinguished representatives for the set $W_{\mathfrak{p}}\backslash W_{\mathfrak{g}}$ of right cosets. Denote by $\Delta^+(\mathfrak{g}_0)$ and $\Delta^+(\mathfrak{p}_+)$ the set of those positive roots such that the corresponding root spaces lie in the indicated subalgebra. Of course, Δ^+ is a disjoint union of these two subsets and both subsets are saturated. Now any root space lies in some $\mathfrak{g}_i$. If $\alpha \in \mathfrak{g}_i$, we put $ht(\alpha) = i$. Since roots are the weights of adjoint representation of $\mathfrak{g}$ on itself, this function is additive because of Jacobi identity. Next, define $\Phi_w := \{\alpha \in \Delta^+ : w^{-1}(\alpha) \in -\Delta^+\}$. In [1] it is shown that $w(\lambda)$ is $\mathfrak{p}$-dominant for any $\mathfrak{g}$-dominant weight λ if and only if $w^{-1}(\alpha) \in \Delta^+$ for any $\alpha \in \Delta^+(\mathfrak{g}_0)$, i.e. if and only if $\Phi_w \subset \Delta^+(\mathfrak{p}_+)$. Here $\mathfrak{p}$-dominance is simply dominance with respect to the semisimple part of $\mathfrak{g}_0$.

Definition 4.1.5. The *Hasse diagram* $W^{\mathfrak{p}}$ of the standard parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$ is the subset of $W_{\mathfrak{g}}$ consisting of all elements w such that $\Phi_w \subset \Delta^+(\mathfrak{p}_+)$ or equivalently such that $w(\lambda)$ is $\mathfrak{p}$-dominant for any $\mathfrak{g}$-dominant weight λ together with the structure of a directed graph induced from the structure on $W_{\mathfrak{g}}$.

To see the structure of a directed graph on $W_{\mathfrak{g}}$ we need a notion of length of an element of $W_{\mathfrak{g}}$. We know that the Weyl group of $\mathfrak{g}$ is generated by simple root reflections. That means that each $w \in W_{\mathfrak{g}}$ can be expressed as a composition of some simple root reflections. The minimal length of such an expression is the *length* of w. The graph structure is as follows: Vertices are the elements of $W_{\mathfrak{g}}$. For two elements w, w' of $W_{\mathfrak{g}}$ there is an arrow $w \to w'$ if and only if $\ell(w)+1 = \ell(w')$ and there exists a positive root α such that $w' = s_\alpha \circ w$, where $\ell(w)$ is the length of w and s_α is a reflection corresponding to α.

Recipe for determining the Hasse diagram

Now we give a recipe for general parabolics for determining the Hasse diagram.

(A) Determine the Dynkin diagram of the parabolic, i.e. the Dynkin diagram of $\mathfrak{g}$ with those simple roots crossed, whose root spaces are contained in $\mathfrak{g}_1$.

(B) Take the weight $\delta^{\mathfrak{p}}$, i.e. the weight which has coefficient 1 over the crossed nodes and zeros over the uncrossed nodes. Apply simple root reflections to this weight. Since we are going to apply this recipe only in the projective case, the only rule interesting for us is the following:

$$\cdots + a\varpi_{i-1} + b\varpi_i + c\varpi_{i+1} + \ldots \mapsto \cdots + (a+b)\varpi_{i-1} + (-b)\varpi_i + (b+c)\varpi_{i+1} + \ldots$$

In each step one only has to apply reflections corresponding to nodes with non-zero coefficients, and one should not apply the reflections that have led to the weight in the last step. Record the reflection by putting the number of the simple root over the arrow.

The resulting pattern gives all elements of the Hasse diagram and some of the arrows. The element corresponding to the weight obtained by applying first $s_{\alpha_{i_1}}$, then $s_{\alpha_{i_2}}$ and so on up to $s_{\alpha_{i_\ell}}$ to $\delta^{\mathfrak{p}}$ is given by $s_{\alpha_{i_\ell}} \dots s_{\alpha_{i_1}}$, so one has to reverse the order of composition. The length of this element is ℓ.

(C) For each element w in pattern, determine the corresponding set Φ_w of roots as well as the labels of the arrows determined so far.
Start with the empty set for the point corresponding to $\delta^{\mathfrak{p}}$. Having determined the sets for all elements of length $< \ell$ and the labels of the arrows leading to these sets, consider a point corresponding to an element of length ℓ in the original diagram determined in step (B). Choose a path of arrows leading from $\delta^{\mathfrak{p}}$ to the given point, take the simple root indicated on the last arrow in the path, and apply the simple reflections corresponding to the other arrows in the path going back to $\delta^{\mathfrak{p}}$. The resulting root has to be contained in $\Delta^+(\mathfrak{p}_+)$ and the set corresponding to the chosen point is given by adding this root to the set corresponding to the source of the last arrow. Now for any of the arrows determined so far which ends in the given element, the set corresponding to the source of the arrow has to be obtained by deleting one element from the set corresponding to the target of the arrow, and this element is the right label of the arrow.

D Determine the remaining arrows. Here we will not write how to obtain remaining arrows, because in the projective case we will obtain all arrows by the recipe above, since the Hasse diagram will consist only of one row and all arrows are between adjacent columns.

Justification of this algorithm can be found in [1].

BGG resolutions

It is known (see [1]) that isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{p}$ are in bijective correspondence with $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral highest weights λ, so generalized Verma modules are available only for such weights. Also, the infinitesimal character restricts the possibilities for the existence of nonzero homomorphisms to the affine Weyl orbit of a weight (see [1]). However, it suffices to restrict to the orbit under the Hasse diagram $W^{\mathfrak{p}} \subset W$:

Lemma 4.1.1. *Let λ be a weight for $\mathfrak{g}$ such that $\lambda + \delta$ is $\mathfrak{g}$-dominant. Then all $\mathfrak{p}$-dominant weights in the affine Weyl group orbit of λ are contained in the set $\{w \cdot \lambda : w \in W^{\mathfrak{p}}\}$.*

Proof. See [1] □

The homomorphisms of generalized Verma modules are of two types. First, generalized Verma modules are quotients of true Verma modules (corresponding to the Borel subalgebra), see [1]. In 1997 Lepowsky proved that any homomorphism of true verma modules descends naturally to the homomorphism of corresponding generalized Verma modules if they exist. These homomorphisms between generalized Verma modules are called *standard homomorphisms* and the corresponding operators are called *standard operators.* Any other homomorphism is called *nonstandard homomorphism* and the corresponding operators are the *nonstandard* ones. The BGG resolution is a diagram of standard homomorphisms between generalized Verma modules, which can be written as

$$\cdots \to \bigoplus_{w\in W^{\mathfrak{p}}:\ell(w)=i} M_{\mathfrak{p}}(w\cdot\lambda) \to \cdots \to \bigoplus_{w\in W^{\mathfrak{p}}:\ell(w)=1} M_{\mathfrak{p}}(w\cdot\lambda) \to M_{\mathfrak{p}}(\lambda) \to V \to 0$$

where we compose the standard homomorphisms with $\pm$ identity and add all these homomorphisms together.
Now if λ is $\mathfrak{g}$-dominant, we will get the full Hasse diagram of invariant operators. But if $\lambda+\delta$ is $\mathfrak{g}$-dominant and λ itself not, i.e. if $\lambda+\delta$ lies on the wall of the fundamental Weyl chamber, some weights in the Hasse diagram may coincide and they needn't be always $\mathfrak{p}$-dominant. In the first case, we say that λ is of *regular* infinitesimal character, while in the second case we say that λ is of *singular* infinitesimal character and the Hasse diagram is degenerated.
This correspondence between invariant differential operators and homomorphisms of generalized Verma modules is the reason why we will denote the natural bundles on any parabolic geometry (given by a representation of P) by the highest weight of the dual of the inducing representation. In this dual language it is a resolution of G-modules. The modules are spaces of sections of homogeneous bundles and homomorphisms are invariant differential operators.

4.1.2 General case

We have seen that in the homogeneous case there are many invariant differential operators. Now we want to have some analogue of them in the case of general parabolic geometry of given type. Given an invariant differential operator $D : \Gamma E \to \Gamma(F)$ on G/P, we say that the differential operator $\bar{D} : \Gamma(\bar{E}) \to \gamma(\bar{F})$ is a *curved analogue* of D, if it has the same symbol as D. The bar over E and F means that it is the 'same' natural bundle (induced by the same representation of P) but not over G/P.

Definition 4.1.6. Consider a fixed category of real parabolic geometries and two representations V and W of P. Let E and F be the corresponding natural vector bundles. A *natural linear operator* mapping sections of E to sections of F is defined to be a system of linear operators$D_{\mathcal{G},\omega} : \Gamma(EM) \to \Gamma(FM)$, where M is the base of $\mathcal{G}$ such that for any morphism $\Phi : (\mathcal{G},\omega) \to (\mathcal{G}',\omega')$ we have

$$\Phi^* \circ D_{(\mathcal{G}',\omega')} = D_{(\mathcal{G},\omega)} \circ \Phi^*$$

We will only be interested in natural differential operators.
For the next section we will need the following fact: In the case of $|1|$-graded geometries it is shown in [10] that naturality of (even non-linear) operators is equivalent to the possibility to express them by a universal formula in terms of all underlying affine connections (and their curvature). Any operator natural on all parabolic geometries of

given type is obviously natural on all flat parabolic geometries of that type. If it is a first-order operator, it is on flat geometries automatically strongly invariant (it can be written in the dual language as homomorphism of generalized semi-holonomic Verma modules - see [5]), so the corresponding weights which determine the Verma modules must lie on the same affine Weyl group orbit.

4.1.3 Projective case

First, we determine the Hasse diagram for projective parabolic geometry. We use the recipe for general parabolic geometries given above. By step (A) we see that the Dynkin diagram of $\mathfrak{p}$ is given by the Dynkin diagram of $\mathfrak{sl}(n+1)$ by crossing the first node in the diagram corresponding to the root $\varepsilon_1 - \varepsilon_2$. By step (B) we get a sequence

$$\overset{1}{\times}-\overset{0}{\circ}-\cdots-\overset{0}{\circ}-\overset{0}{\circ} \xrightarrow{\alpha_1} \overset{-1}{\times}-\overset{1}{\circ}-\overset{0}{\circ}-\cdots-\overset{0}{\circ} \xrightarrow{\alpha_2} \overset{0}{\times}-\overset{-1}{\circ}-\overset{1}{\circ}-\cdots-\overset{0}{\circ} \xrightarrow{\alpha_3}$$
$$\xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} \overset{0}{\times}-\overset{0}{\circ}-\cdots-\overset{-1}{\circ}-\overset{1}{\circ} \xrightarrow{\alpha_n} \overset{0}{\times}-\overset{0}{\circ}-\cdots-\overset{0}{\circ}-\overset{-1}{\circ}$$

where α_i is the simple root $\varepsilon_i - \varepsilon_{i+1}$.
By step (C) we determine the 'vertices' of the Hasse diagram. The first vertex is simply the empty set. The second is by (C) the simple root α_1. Next vertex is $\{\alpha_1, s_{\alpha_1}(\alpha_2)\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3\}$, since the simple root reflections act by transpositions $i \leftrightarrow i+1$. For the brevity we will denote the root $\varepsilon_i - \varepsilon_j$ by α_{ij}. Going on through the sequence, we see that in each step we will add the root α_{1i} for the i-th vertex. Indeed, applying $s_{\alpha_1} \cdots s_{\alpha_{i-2}}$ to α_{i-1} we get α_{1i}, since we start with $\alpha_{i-1,i}$ and each root reflection changes the first index by $j \mapsto j-1$ until $j = 1$. So the Hasse diagram for projective parabolic geometry looks like

$$\emptyset \xrightarrow{\alpha_{12}} \{\alpha_{12}\} \xrightarrow{\alpha_{13}} \cdots \xrightarrow{\alpha_{1n}} \{\alpha_{12}, \ldots, \alpha_{1n}\} \xrightarrow{\alpha_{1,n+1}} \{\alpha_{12}, \ldots, \alpha_{1,n+1}\}$$

Now when we have the Hasse diagram, we can determine the BGG resolutions (and the BGG sequences) for any $\mathfrak{g}$-dominant weight λ. Here we start with the representation of G, the dual of which has the highest weight λ. Consider the weight $\lambda = \overset{a_1}{\times}-\overset{a_2}{\circ}-\cdots-\overset{a_{n-1}}{\circ}-\overset{a_n}{\circ}$ with each a_i nonnegative. Then $\lambda = \sum_i (\sum_{j=i}^n a_j)\varepsilon_i$. Adding to λ the weight$\rho = \sum_i \varpi_i$, we get

$$\lambda + \rho = \sum_i (n + 1 - i + \sum_{j=i}^{n} a_j)\varepsilon_i$$

All weights on the $W^{\mathfrak{p}}$-orbit of $\lambda + \rho := \mu$ are obtained from μ by taking the first coefficient to ε_2, second to ε_3,..., $(i-1)$-th to ε_i and i-th toε_{i+1} for $i = 1, \ldots, n$. Indeed, the first reflection interchanges the first two coefficients. Combining with the second reflection, which interchanges the first and the third coefficient, we get the weight μ with the first coefficient at ε_2, the second at ε_3 and the third coefficient at ε_1. Generally, when we have μ with first i coefficients at $\varepsilon_2, \ldots, \varepsilon_{i+1}$ and $(i+1)$-th coefficient at ε_1, and now interchange the first and the $(i+2)$-th coefficient of this new weight, we get the weight μ with first $i+1$ coefficients at $\varepsilon_2, \ldots, \varepsilon_{i+2}$ and $(i+2)$-th coefficient at ε_1.
The first weight in the sequence is simply λ itself. To compute the second weight, we

have $s_{\alpha_{12}}(\mu) = (n-1+\sum_{j=2}^{n} a_j)\varepsilon_1 + (n+\sum_{j=1}^{n} a_j)\varepsilon_2 + \sum_{i=3}^{n+1}(n+1-i+\sum_{j=i}^{n} a_j)\varepsilon_i$. Subtracting the weight ρ we get $(-1+\sum_{j=2}^{n} a_j)\varepsilon_1 + (1+\sum_{j=1}^{n} a_j)\varepsilon_2 + \sum_{i=3}^{n+1}\sum_{j=i}^{n} a_j\varepsilon_i$. Translating it to fundamental weights, we get

$$\overset{-a_1-2}{\times} - \overset{a_1+a_2+1}{\circ} - \overset{a_3}{\circ} \cdots - \overset{a_n}{\circ}$$

To compute the last weight, we have $s_{\alpha_{n,n+1}}(\mu) = \sum_{i=2}^{n+1}(n+2-i+\sum_{j=i-1}^{n} a_j)\varepsilon_i$. Subtracting the weight ρ we get $-n\varepsilon_1 + \sum_{i=2}^{n+1}(1+\sum_{j=i-1}^{n} a_j)\varepsilon_i$. Translating it to fundamental weights, we get

$$\overset{-n-1-\sum_{j=1}^{n} a_j}{\times} - \overset{a_1}{\circ} \cdots - \overset{a_{n-1}}{\circ}$$

All other weights obtained from μ are of the form $(n+1-k+\sum_{j=k}^{n} a_j)\varepsilon_1 + \sum_{i=2}^{k}(n+2-i+\sum_{j=i-1}^{n} a_j)\varepsilon_i + \sum_{i=k+1}^{n}(n+1-i+\sum_{j=i}^{n} a_j)\varepsilon_i$ for $2 < k < n+1$. Subtracting the weight ρ we get $(1-k+\sum_{j=k}^{n} a_j)\varepsilon_1 + \sum_{i=2}^{k}(1+\sum_{j=i-1}^{n} a_j)\varepsilon_i + \sum_{i=k+1}^{n}(\sum_{j=i}^{n} a_j)\varepsilon_i$. Translating it to fundamental weights, we get

$$\overset{-k-\sum_{j=1}^{k-1} a_j}{\times} - \overset{a_1}{\circ} \cdots - \overset{a_{k-2}}{\circ} - \overset{a_{k-1}+a_k+1}{\circ} - \overset{a_{k+1}}{\circ} \cdots - \overset{a_n}{\circ}$$

As we have seen above, the above equation holds also for $k = 2, n+1$. Together we get

$$\overset{a_1}{\times} - \overset{a_2}{\circ} \cdots - \overset{a_{n-1}}{\circ} - \overset{a_n}{\circ} \longrightarrow \overset{-a_1-2}{\times} - \overset{a_1+a_2+1}{\circ} - \overset{a_3}{\circ} \cdots - \overset{a_n}{\circ} \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow \overset{-k-\sum_{j=1}^{k-1} a_j}{\times} - \overset{a_1}{\circ} \cdots - \overset{a_{k-2}}{\circ} - \overset{a_{k-1}+a_k+1}{\circ} - \overset{a_{k+1}}{\circ} \cdots - \overset{a_n}{\circ} \longrightarrow$$

$$\longrightarrow \cdots \longrightarrow \overset{-n-1-\sum_{j=1}^{n} a_j}{\times} - \overset{a_1}{\circ} \cdots - \overset{a_{n-1}}{\circ}$$

Special example of such a complex is the Calabi complex - for $\lambda = \varpi_2$. The first operator in this complex is the operator $D : \Gamma(T^*M \otimes \mathcal{E}[2]) \to \bigodot^2 T^*M \otimes \mathcal{E}[2]$ given by $\alpha_a \mapsto \nabla_{(a}\alpha_{b)}$, which inspired us for the next section. Another standard example is the de Rham complex

$$0 \to R \to \Lambda^1 T^*M \xrightarrow{d} \Lambda^2 T^*M \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n T^*M$$

Hypothesis 1. It seems that there are no nonstandard operators in the projective case, at least up to order 10.

4.2 One family of operators

Here we introduce a very large family of differential operators which are invariant under the change of Weyl connection but they are not invariant in the usual sense - we can't find them in any BGG sequence.
We start with an operator from [6] that inspired us. It is an operator from projectively

weighted vector fields on M to projectively weighted symmetric bilinear forms on M. Assume that M is a Riemannian manifold with Riemannian metric g. The Levi-Civita connection of g induces a projective structure on M. It is known that there is a projectively invariant operator $D' : T^*M \otimes \mathcal{E}[2] \to \odot^2 T^*M \otimes \mathcal{E}[2]$ given by $\alpha_i \to \nabla_{(i}\alpha_{j)}$. Now we compose it with g, so we get an operator

$$D : TM \otimes \mathcal{E}[2] \to \odot^2 T^*M \otimes \mathcal{E}[2]$$

given by

$$X^a \mapsto \nabla_{(a}(g_{b)c}X^c)$$

where we use the abstract index notation and Einstein summation convention. It is very easy to compute that it is invariant under the change of connection in the projective structure:

$$\begin{aligned}
&\hat{\nabla}_a(g_{bc}X^c) + \hat{\nabla}_b(g_{ac}X^c) = \\
&= (\hat{\nabla}_a g_{bc})X^c + g_{bc}\hat{\nabla}_a X^c + (\hat{\nabla}_b g_{ac})X^c + g_{ac}\hat{\nabla}_b X^c = \\
&= (\nabla_a g_{bc} - 2\Upsilon_a g_{bc} - \Upsilon_b g_{ac} - \Upsilon_c g_{ba})X^c + g_{bc}(\nabla_a X^c + 3\Upsilon_a X^c + \Upsilon_d X^d \delta^c_a) + \\
&+ (\nabla_b g_{ac} - 2\Upsilon_b g_{ac} - \Upsilon_a g_{bc} - \Upsilon_c g_{ab})X^c + g_{ac}(\nabla_b X^c + 3\Upsilon_b X^c + \Upsilon_d X^d \delta^c_b) = \\
&= (\nabla_a g_{bc})X^c + g_{bc}\nabla_a X^c + \nabla_b g_{ac} X^c + g_{ac}\nabla_b X^c = \\
&= \nabla_a(g_{bc}X^c) + \nabla_b(g_{ac}X^c)
\end{aligned}$$

We saw, that we didn't use the fact that g is a Riemannian metric on M, but only the fact that it is a bilinear form. So the first natural generalization is to replace the metric by any bilinear form on M.
But this operator is not invariant in the sense of parabolic geometry, because the weights of corresponding generalized Verma modules do not lie on same affine orbit of the Weyl group of $\mathfrak{g}$. Indeed, the corresponding weights are $3\varepsilon_1 - \varepsilon_{n+1}$ for the source space and $2\varepsilon_2$ for the target space, so shifting it by ρ gives $(n+3)\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n - \varepsilon_{n+1}$ for the source space and $(n+2)\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n$ for the target space. This two weights clearly don't lie on one orbit of the Weyl group, since in this notation the Weyl group acts by permutations.
Now we are going to construct a very large family of operators in a similar spirit. Above, we started with projectively weighted vector fields on M. The idea here is to start with any projectively weighted positive tensor power of vector fields on M.

Theorem 4.2.1. *Assume that n is some natural number. Consider the operator D : $\bigotimes^n TM \otimes \mathcal{E}[2] \to \bigodot^2 T^*M \otimes \mathcal{E}[2]$ given by*

$$X^{a_1\ldots a_n} \mapsto \nabla_{(a}(S_{b)a_1\ldots a_n}X^{a_1\ldots a_n})$$

where $S_{ba_1\ldots a_n}$ is some $(n+1)$-linear form on M. Then this operator is invariant under the projective change of connection but not in the sense of the previous section.

Proof. Invariance under the projective change of connection is easy:

$$
\begin{aligned}
&\hat{\nabla}_a(S_b a_1 \dots a_n X^{a_1\dots a_n}) + \hat{\nabla}_b(S_{aa_1\dots a_n} X^{a_1\dots a_n}) = \\
&= (\hat{\nabla}_a S_{ba_1\dots a_n}) X^{a_1\dots a_n} + S_{ba_1\dots a_n} \hat{\nabla}_a X^{a_1\dots a_n} + \\
&+ (\hat{\nabla}_b S_{aa_1\dots a_n}) X^{a_1\dots a_n} + S_{aa_1\dots a_n} \hat{\nabla}_b X^{a_1\dots a_n} = \\
&= (\nabla_a S_{ba_1\dots a_n} - (n+1)\Upsilon_a S_{ba_1\dots a_n} - \Upsilon_b S_{aa_1\dots a_n} - \sum_{i=1}^{n} \Upsilon_{a_i} S_{ba_1\dots a\dots a_n}) X^{a_1\dots a_n} + \\
&+ S_{ba_1\dots a_n}(\nabla_a X^{a_1\dots a_n} + (n+2)\Upsilon_a X^{a_1\dots a_n} + \sum_{i=1}^{n} \delta_a^{a_i} \Upsilon_c X^{a_1\dots c\dots a_n}) + \\
&+ (\nabla_b S_{aa_1\dots a_n} - (n+1)\Upsilon_b S_{aa_1\dots a_n} - \Upsilon_a S_{ba_1\dots a_n} - \sum_{i=1}^{n} \Upsilon_{a_i} S_{aa_1\dots b\dots a_n}) X^{a_1\dots a_n} + \\
&+ S_{aa_1\dots a_n}(\nabla_b X^{a_1\dots a_n} + (n+2)\Upsilon_b X^{a_1\dots a_n} + \sum_{i=1}^{n} \delta_b^{a_i} \Upsilon_c X^{a_1\dots c\dots a_n}) = \\
&= (\nabla_a S_{ba_1\dots a_n}) X^{a_1\dots a_n} + S_{ba_1\dots a_n} \nabla_a X^{a_1\dots a_n} + \\
&+ (\nabla_b S_{aa_1\dots a_n}) X^{a_1\dots a_n} + S_{aa_1\dots a_n} \nabla_b X^{a_1\dots a_n} = \\
&= \nabla_a (S_{ba_1\dots a_n} X^{a_1\dots a_n}) + \nabla_b (S_{aa_1\dots a_n} X^{a_1\dots a_n})
\end{aligned}
$$

Now we prove that this operator is not invariant in the sense of parabolic geometry. Indeed, the target space is determined by $\lambda = -2\varpi_1 + 2\varpi_2$, so $\lambda + \rho = -\varpi_1 + 3\varpi_2 + \sum_{i=3}^{n} \varpi_i$. Here, ϖ_i are the fundamental weights for $\mathfrak{sl}(n+1; R)$. Next, the complete list of weights for TM (in fact, for $\mathfrak{g}_-$) is given - in our dual notation - by $2\varpi_1 - \varpi_2; \varpi_1 + \varpi_2 - \varpi_3; \varpi_1 + \varpi_3 - \varpi_4; \dots; \varpi_1 + \varpi_{n-1} - \varpi_n; \varpi_1 + \varpi_n$. Now the weights for n-th tensor power of TM are exactly the nonnegative linear combinations of weights for TM, since the n-th tensor power of the dual of some representation V is the dual of the n-th tensor power of V and weights of any positive tensor power of V are exactly the nonnegative linear combinations of weights of V. Especially, all weights (also highest weights for irreducible pieces) for $\bigotimes^n TM$ have positive coefficient at ϖ_1. Tensoring with $\mathcal{E}[2]$ adds 2 to this coefficient and adding ρ to any such weight adds 1 to it. So this coefficient will be always positive and at least 3. On the other hand, from the previous section we know that any weight on the affine Weyl group orbit of $-2\varpi_1 + 2\varpi_2$ has nonpositive coefficient at ϖ_1, so shifting it by ρ makes it ≤ 1. So when we restrict D to any irreducible subrepresentation of $\bigotimes^n TM \otimes \mathcal{E}[2]$, we don't get an invariant operator in the sense of parabolic geometry, because the weights determining the source and target of this operator don't lie on the same affine Weyl group orbit. $\square$

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