Charles University in Prague

Faculty of Social Sciences
Institute of Economic Studies

Bachelor Thesis

Construction of a quantum finance model of option premia

Author: **Pavel Irinkov**

Supervisor: **PhDr. Ladislav Krištoufek, Ph.D.**

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Declaration of Authorship

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Abstract

Last twenty years have seen a tremendous growth of the financial markets both in trading volumes and in sophistication of instruments. This ever-increasing complexity of the market structure necessitates use of mathematically advanced models from the side of market participants. So far, the prevalent paradigm for these models has been the stochastic analysis as a branch of applied mathematics. In the last few years however, there has been an influx of purely physical concepts and methodology, constituting nascent field of econophysics. To what extent this new approach is useful remains, however, an open question. In my bachelor thesis I will focus on one subfield of econophysics, namely quantum finance. First, I will give an overview of both stochastic analysis and the new quantum finance paradigm. Then using the framework of quantum theory and quantum field theory I will construct a model of European stock options.

JEL Classification	F12, F21, F23, H25, H71, H87
Keywords	econophysics, quantum finance, option pricing, Black-Scholes model

Author's e-mail	`pavelir@seznam.cz`
Supervisor's e-mail	`kristoufek@ies-prague.org`

Abstrakt

V posledních dvaceti letech došlo k převratnému vývoji finančních trhů jak z hlediska objemu obchodu, tak i sofistikovanosti používaných nástrojů. Tato stále narůstající složitost tržní struktury s sebou nese potřebu pokročilých modelů ze strany účastníků trhu. Doposud převládajícím paradigmatem těchto modelů byla stochastická analýza, jakožto odvětví aplikované matematiky. V posledních několika letech se ovšem objevily snahy o využití čistě fyzikálních konceptů a metodologie, vytvařejíce tak nový obor ekonofyziky. Do jaké míry je tento novy přístup efektivní zůstává přesto otevřenou otázkou. Ve své bakalářské práci se zaměřím na jeden podobor ekonofyziky, tzv. kvantové finance. Nejdříve nabídnu přehled jak stochastické analýzy, tak kvantových financí. Poté s pomocí aparátu kvantové teorie a kvantové teorie pole odvodím model evropských akciových opcí.

Klasifikace JEL	F12, F21, F23, H25, H71, H87
Klíčová slova	ekonofyzika, kvantové finance, oceňování opcí, Blackův-Scholesův model

E-mail autora	pavelir@seznam.cz
E-mail vedoucího práce	kristoufek@ies-prague.org

Contents

List of Figures

Chapter 1

Introduction

The main focus of this thesis is so-called quantum finance approach to modelling options price evolution. The very term "quantum finance" needs a bit of elaboration, since, being a rather new concept, it is not entirely clear at this point what it means. According to Baaquie (2007), the term quantum finance denotes "a synthesis of concepts, methods and mathematics of quantum theory with the field of theoretical and applied finance". As such, it does not refer to efforts to reformulate the first principles of theoretical finance in the framework of quantum physics, rather, it stands for a compendium of quantum mechanical toolkit applied to problems in finance. The beginnings of this approach can be traced back to seminal papers Baaquie (1998), Belal E. Baaquie (2002) where the interest rates and option prices are treated as quantum field and quantized degree of freedom respectively. As it is clear not only from intuition, trying to marry these two fields is quite a formidable task, whose characteristics fall under the auspices of a research area that gradually came to be called econophysics. Despite its sparse beginnings in the last decade of the twentieth century and a questionable status within the mainstream physical research the field has continued to attract attention both from the physics and economics community. Today, there is a journal specially devoted this research program and a conference is held anually to gather participants from various parts of the globe. Deep conceptual issues however remain unresolved. Relevance of both econophysics and quantum finance in particular in the contemporary economic, econometric and physical research and their potential to enrich the methodological toolbox of either of the "parental" science disciplines, remain by and large an open question. The economists particularly have stood unimpressed over what seemed to be just another attempt at overly mathematicizing their subject of inquiry - a grave reminder of this fact is that to this day most of the research papers dealing with econophysics are being published in exclusively physical and not economical reviews.

Options as a particular example of derivative securities have been used ever since antiquity as an instrument of speculation on olive harvest (Abraham 2010). It was not until the 1970's however, that their widespread use for the purpose of hedging and speculation has created considerable demand for their pricing models. For the bird's eye view of the plethora of pricing models, reader is referred to (Haug 2006).

The goal of this work is to give a derivation of a model of European option prices that would correct the deficiencies of one of the most popular option pricing models - the model of Black and Scholes using the new econophysical paradigm. In order to do so, it is organized as follows. The second chapter gives an exposition of the prevailing methodological paradigm of the option pricing models - stochastic analysis. We introduce the basic terminology of the options markets and then we give a self-contained pedagogical review of all its basic notions. Stochastic processes, Markov chains, martingales, the Beownian motion, Ito lemma, stochastic integration and stochastic differential equations are given due treatment. With this knowledge in mind, the Black-Scholes model is then derived using the original argument of its creators. The last part of the second chapter gives a list of assumptions that were made in the derivation.

The third chapter begins with an enumeration of the shortcomings that the Black-Scholes model despite its popularity and ubiquity posesses and which are to be corrected. To this end a quantum mechanical paradigm is proposed - after covering a necessary minimum of the quantum theory, a comparison between physical and financial systems within the context of option pricing is given. The construction of a new model is then made in two steps - in the first step, the Black-Scholes model is reformulated in the language of quantum theory and in the second step, using an empirical insight, a correction to the original model is derived.

Chapter 2

Basic notions of options market and stochastic analysis

In this chapter, we give a thorough review of the work's two underlying themes - options and econophysics. We try to make the exposure as pedagogical as possible, not only in order the ease reader's digestion of new ideas, but also for the sake of future reference. Much of what is written in subsequent chapter makes heavy use of mathematical framework exposed in this chapter. As a logical consequence, we make no attempt at originality of the content of this chapter and much of what follows is based on accounts given in (Hull 2009), (Jeffrey O. Katz 2005), (Wolfgang Paul 2010) and (Baaquie 2007). Every now and then, however, we extend the discussion a bit and make use of both fields' inherent richness to expound on the parallels one can identify when economics and physics are put side by side.

2.1 Options market

As mentioned in the introduction, options have been used ever since antiquity. Along with futures and forwards, they constitute the content of the term *derivative security*. Unlike both futures contract and a forward, what an option carries with itself is a right, not an obligation. Particular form of this right depends on a specific type of the option, but in general we can distinguish two types of an option contract depending on general characteristic of an underlying right. A **call option** entitles its holder to buy the underlying asset by a certain date for a certain price. A **put option**, to the contrary, entitles its holder to sell the underlying asset by a certain date for a certain price. In case the holder chooses to do so, the option is said to be **exercised**. The price in the contract is consequently known as the *exercise price* or the *strike price* and the date in the contract is known as the *expiration date* or *maturity*. Depending on whether one can exercise the option anytime until maturity or only at maturity, one can further classify the option respectively as either being **American** or **European**.

As it is quite clear, strike price is not the only parameter that determines holder's gain in case of exercising the option. This complete information is encoded in the **payoff function**. One basic thing that can be told stright away from the form of a payoff function is whether the option is path dependent or path independent. In the latter case, holder's gain only depends on a value of the underlying security at the time of maturity. That is, payoff function is independent on how the security arrived at its final price. European option is an example of this kind of option. In the former case, holder's gain depends on the entire path the security takes before the option expires. This dependence can take various forms. In case of an American option, path that the price of the security takes clearly influences whether or not the option at a particular instant is exercised. In case of an Asian option, payoff function depends on average value of underlying security during the whole period of its duration, from the time it is written until the time it expires.

In order to illustrate aforementioned concepts on a practical example, a graphical representation of a payoff function of call option is given in Figure 2.1, where $g(S)$ denotes gain of the holder in case they decide to exercise the option and S denotes price of underlying security. K stands for strike price. Complete discussion of various types option contracts can take can be found in (Jeffrey O. Katz 2005).

2.2 Stochastic calculus

In this section we give brief and succinct treatment of the prevalent financial mathematics paradigm - Ito stochastic calculus. The reason for doing so is twofold. First, it will allow us to derive the Black-Scholes pricing formula that will form the bedrock of a model to be developed in the next chapter and second, perhaps more important, it will allow us to compare the structure of both paradigms ("mathematical" and "econophysical") hence hopefully draw at the end some conclusions about the usefulness of the latter. Nevertheless, any treatment of a deep mathematical theory within several pages is bound to

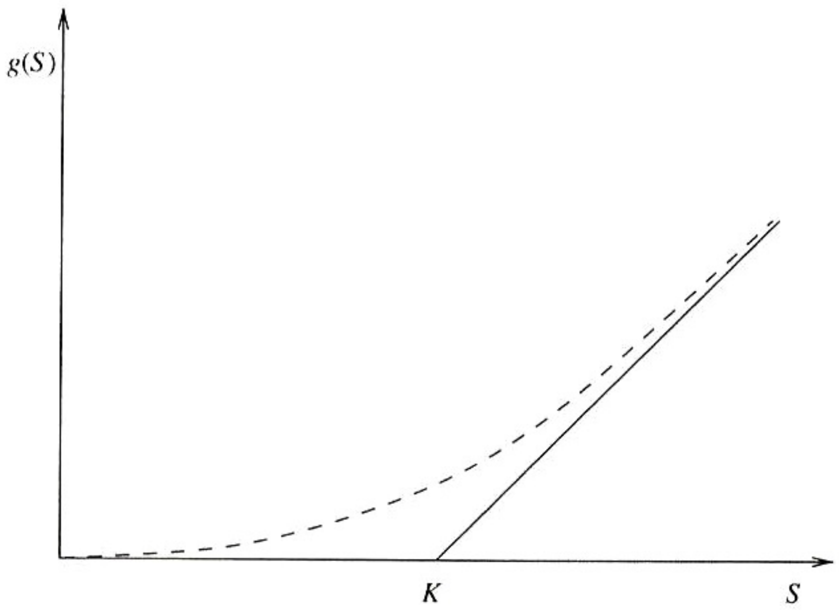

Figure 2.1: Payoff function for call option. The dashed line represents possible values of the option at a given time before maturity

Source: Baaquie (2007)

be incomplete and superficial. Thus, reader interested in rigorous treatment is referred to any of the books (Oksendal 2003) or (Steele 2001).

First question any system of ideas needs to address is why it should be used in the first place. In case of stochastic calculus with respect to its financial applications, the question translates into why should there be any need of stochastic calculus given that we have such otherwise successful tool - real analysis - at our disposal. The answer to this question seems to be that the functions we encounter in finance surpass the scope of applicability of real analysis. The latter paradigm deals with continuous functions (at least C^1) of one or more real variables and that have finite variation as these unction suit most problems encountered in natural sciences. Functions encountered in economic sciences such as the interest rates curve or price development of a security are however *nowhere* C^1 continuous functions and moreover have unbounded variation (this means for a function of one variable that the distance along the direction of y-axis traveled by point moving along the graph does not have a finite value with analogous defnition applying to functions of more variables).

To see that this is indeed the case, let's consider a scenario where time development of security where random, but with continuous first derivative and bounded variation. Then, from the first property, one would be able to make a sure bet on a future development examining infinitesimal neighborhood of the asset price and thus violating the principle of no arbitrage. From the second property, it would be possible to generate huge profits by generating path-dependent options, which would be traded in the market at high premiums and almost zero costs. Both these possibilities are contradiction to reality. For an interesting discussion of these issues, reader is referred to the book (Sondermann 2006).

Knowing we need stochastic calculus in finance, the next pragraphs give an outline of the theory as it stands. There are basically two approaches to the field, a rigorous one, predominantly espoused by the mathematical community and an intuitive one, upheld mainly by natural sciences and economics researchers. Because of the elementary character of our exposition, we cling more to the second approach. Whenever possible, we underscore the parallel between physical systems and finance.

2.2.1 Stochastic process

The first important concept standing at the basis of the stochastic calculus is that of a **stochastic process**. Any variable whose values change over time in an uncertain way is said to follow a stochastic process. Mathematically speaking, it is a parametrized collection of random variables

$$X_t, t \in T \tag{2.1}$$

defined on a probability space (Ω, F, P), where Ω is the sample space (space of al elementary outcomes), F is a σ-algebra on Ω and P is a probability measure on F. Depending on the nature of th index set T, process is called discrete time, in case T has countable number of elements, or continuous time, in case T is uncountable. By a similar criterion on Ω, the process is said to be either discrete variable or continuous variable.

Most processes in both finance and physics are continuous time, continuous variable. As an example from economics, we might consider the time dependence of the rate of return $R(T)$ from holding certain security for a period T which is given by

$$R(t) = \frac{S(t+T) - S(t)}{S(t)}, \tag{2.2}$$

where we assumed for the sake of simplicity that there were no dividends being paid out over the holding period T. One well known example of the stochastic process in physics is position of a particle in a fluid subjected to random collision with the molecules of the environment. One then solves a stochastic differential equation (to which we will come in more detail later) of the form

$$X''(t) = -\lambda X'(t) + \eta(t), \tag{2.3}$$

where $\eta(t)$ denotes stochastic forces per unit mass and λ is viscous damping coefficient per unit mass.

2.2.2 Markov and martingale properties

Out of the class of all stochastic processes, two subgroups are of particular interest in terms of their applicability in mathematical finance. As we shall see, both are closely interrelated. The first one of them is so-called **Markov process** - or equivalently a stochastic process having Markov property. We say that a stochastic process has a Markov property, if the conditional distribution of future states of the process (conditional on both past and present values) depends only upon the present state, not on the sequence of events that preceded it. In mathematical notation,

$$P(X_n = x_n | X_{n-1} = x_{n-1} ... X_0 = x_0) = P(X_n = x_n | X_{n-1} = x_{n-1}), \tag{2.4}$$

where for simplicity we assumed sample space S to be a discrete set. From this it follows that the expectation value of the future states depends only on the present state as well. Thus, it can be said that Markov processes represent systems with the important quality of having no memory. Most models in matematical finance are constructed with this assumption. Using a bit of thought one can easily see that yet another example of process with no memory beyond the present is *random walk*, where in each turn one step is taken in either positive or negative direction. The second important example of a Markov process, albeit of somewhat peculiar nature, is the deterministic evolution of a physical system in a phase space as given by the Hamilton equations, the peculiarity here being that all the conditional distributions are singular. Markov property is then equivalent to the trajectories of the system not bifurcating along the path anywhere.

The second important group of processes is so-called **martingales** - or equivalently stochastic processes having martingale property. From a historical point of view, the incentive to study this kind of process came from the area of gambling. Let's use this historical example to elucidate its importance. Let's imagine a gambler tossing a coin and betting one dollar (or another fixed amount) at each toss on either heads or tails. Let S_i denote his or her winnings after i-th toss. Then, if the coin is fair, we have

$$E[S_i|S_j, j < i] = S_j. \tag{2.5}$$

This is the martingale property. Thus, for a martingale, the best estimator for the next value, taking into account all the information all the past values of the process, is the present value. The expected value of the increments is then zero

$$E[dS] = 0 \tag{2.6}$$

and the process has, in financial parlance, no drift.

Both these properties happen to be important characteristics of one process that lies at the core of most financial models. Its name is Brownian motion as it was studied for the first time by a Scottish botanist Robert Brown on pollen grains suspended in a liquid. The next paragraph gives its full treatment.

2.2.3 Brownian motion

Brownian motion, also called Wiener process in honor of Norbert Wiener, who contributed significantly to studying its properties, has many different manifestations, the movement of pollen grains in a liquid and the process drivivng stock price evolution being only two of them. The precise understanding of the second example and its ramifications for the pricing of securities is the goal of this paragraph.

One way to derive Brownian motion is via limiting procedure of random walk as the timesteps go to zero. In order to do this, let's recall the the gambler flipping the coin and betting on its outcomes from the previous section. Let's suppose now that the time t allowed for a certain number of tosses, e.g. n, is restricted. Also, the bet the gambler makes each round is no longer 1 dollar, or some arbitrary amount, but $\sqrt{t/n}$. As one can easily see, the Markov and martingale propeties are retained and moreover, we have an important result

$$\sum_{j=1}^{n}(S_j - S_{j-1})^2 = n \times (\sqrt{\frac{t}{n}})^2 = t, \tag{2.7}$$

where, as in the previous section, S_j denotes the winnings of the gambler after the j-th toss. First part of the equation (2.7) defines so-called **quadratic variation**, so that the equation equation can be interpreted as saying, that the quadratic variation of a random walk under the condition defined in this paragraph equals time t. The only thing that needs to be done in order to get a Wiener process is to let n go to infinity and all the properties of the random walk are retained in this limit case. Graphical depiction of this limit process is given in Figure 2.2.

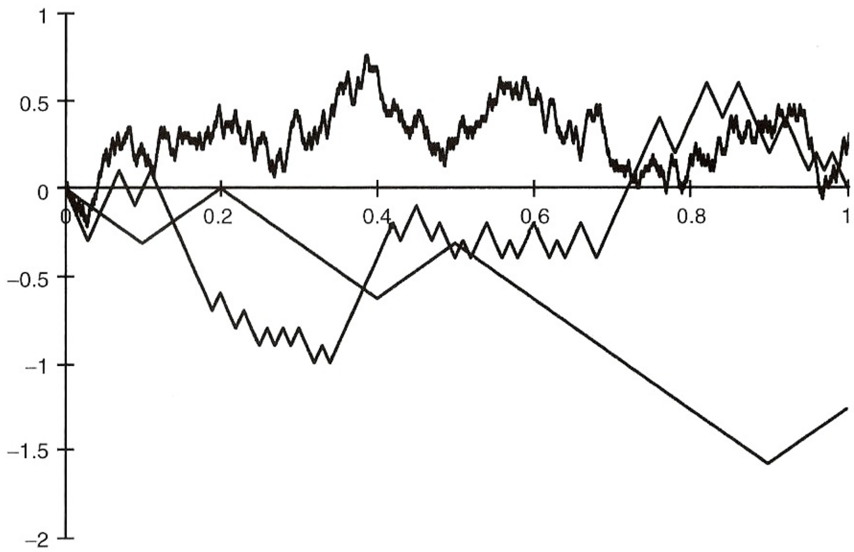

Figure 2.2: A series of random walks, the limit of which is Brownian motion
Source: Wilmott (2000)

Wiener process can be given axiomatic definition as well. In this, the properties that need to be satified in order for a given process to classify as a Wiener are:

- $S_0 = 0$
- The function $t \to S_t$ (i.e. the trajectory of the process) is almost surely everywhere continuous.

- The change ΔS during a period of time Δt is

$$\Delta S = \epsilon\sqrt{\Delta t}, \tag{2.8}$$

  where ϵ has a standard normal distribution $N(0,1)$.

- The values of ΔS for different disjunct time intervals Δt_1, Δt_2 are independent.

From this definition and a derivation given at the beginning, one can easily see these properties that are given for the sake of future reference:

- *Continuity*: Individual trajectories of the Wiener process are continuous. Brownian motion is thus continuous-time limit of the discrete random walk process.

- *Finiteness*: Values of $S(t)$ are smaller than infinity for all finite times t. This is because of the special choice of scaling $\sqrt{t/n}$ of the bet made at each round.

- *Markov property*: Limiting procedure preserves Markov property of the random walk. The conditional distribution of $S(t)$ given information up until $\tau < t$ depends only on $S(\tau)$.

- *Martingale property*: Given information up until $\tau < t$ the conditional expectation of $S(t)$ is $S(\tau)$.

- *Normality*: Over finite time intervals Δt, the increments of the process are normally distributed with mean zero and variance Δt. Hence,

$$p(\Delta S) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{\Delta S^2}{2\Delta t}}, \tag{2.9}$$

  where p denotes unconditional probability density function.

It is important to note, that the independence of the increments universally implies their having normal distribution. Here lies the conceptual importance of the Wiener process and it is also a response to the question whether there could not be some different processes without Gaussian distribution. Of course there could be, but only if the increments would be no longer independent.

There is another way of thinking about the Brownian motion - one that is heuristical but deserves mentioning nevertheless because it involves another useful notion. One can think about the differential dB as a product

$$dB = R \cdot dt, \tag{2.10}$$

where dt is time differential and R is so-called **white noise**. White noise is a stochastic process with mean zero, constant variance and which is serially uncorrelated. As such, it finds many applications in engineering, particularly signal processing, but also in econometrics, where one often assumes that the data have "deterministic" and white noise part. Because of the expression (2.10), it makes sense to call white noise a generalized derivative of the Wiener process, however we shall not delve into these somewhat technical issues and refer reader to the discussion in (Oksendal 2003).

2.2.4 Ito lemma and stochastic integration

Having defined the properties of the Wiener process in the last subsection, we are one step closer to deriving the Black-Scholes model within the stochastic framework. However, in order to that, one crucial ingredient is still needed. As was mentioned in previous section, functions encountered on the financial markets seem to have peculiar properties, infinite variation over finite length interval being one of them. Thus, as was argued at the very beginning of section 2.2, an extension of the normal real variable calculus is needed.

One straightforward way to achieve this extension is to allow functions to be integrated not to depend only on real or complex variables, but on stochastic variables as well. Since we will want to integrate these new functions with respect to the stochastic variables, a new notion of integral is needed. The new integral is called **stochastic integral** and is defined as

$$W(t) = \int_0^t f(\tau) dX(\tau) = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{t-1})), t_j = \frac{jt}{n}, \tag{2.11}$$

where the function f can of course depend on the stochastic process X and the argument τ is given for the sake of clarity. Unlike Riemann integral, the limit at the right-hand side of the expression (2.12) is to be understood as a *mean square limit*. This technically means that we do not require pointwise convergence as in the Riemannian case but instead require that the expected value of the squared differences go to zero:

$$\lim_{n\to\infty} W_n = W \Leftrightarrow \lim_{n\to\infty} E\left[(W_n - W)^2\right] = 0. \tag{2.12}$$

It is also very important to note that the function f which is integrated is evaluated at the summation at the left-hand point. This ensures that the process $W(t)$ is a martingale and the integration is called **non-anticipatory** which means that that $W(t)$ is statistically independent of $X(s) - X(t)$ for all $s > t$. If all these conditions are met, we particularly speak about the **Ito integral**. Choice $f(\frac{t_{j-1}+t_j}{2})$ is also possible and popular and the resulting integral is called **Stratonovich**. However, intuitively speaking, because the function f is evaluated at time occuring later than time at which the stochastic differential is taken at, the process $W(t)$ is no longer martingale and therefore does not correspond to the situation, where no information about the future development is known. This choice can still be of use in theoretical physics, particularly statistical mechanics, but is no longer relevant in financial applications. For a detailed discussion of these issues, see (Oksendal 2003).

Having seen the expression for the Ito integral (2.11) one is led to ponder what implications do the assumptions of the previous paragraph have for the theory of differential equations. To see this in full detail, let us consider an ordinary differential equation describing exponential decay of radioactive isotope

$$N' = -\lambda N, \tag{2.13}$$

where N denotes number of yet undecaid atoms in the sample and λ is the decay constant. This equation can be put into equivalent form by multiplying both sides by time differential dt

$$dN = -\lambda N dt. \tag{2.14}$$

The whole trick of the stochastic differential equation then is to allow the differential dN to depend on differential of some stochastic process, ie in our example (2.14) to add terms proportional to dX. It what follows and for the reasons listed in the paragraph 2.2.3, we will always consider dX to be the differential of Brownian motion dB. Hence, in our example taking into account random influences from the environment, we might get an equation

$$dN = -\lambda N dt + \mu N dB. \tag{2.15}$$

Speaking in terms of solutions, graph of the function N from the equation (2.15) is a “randomized” version of the graph of the function N from the equation (2.14).

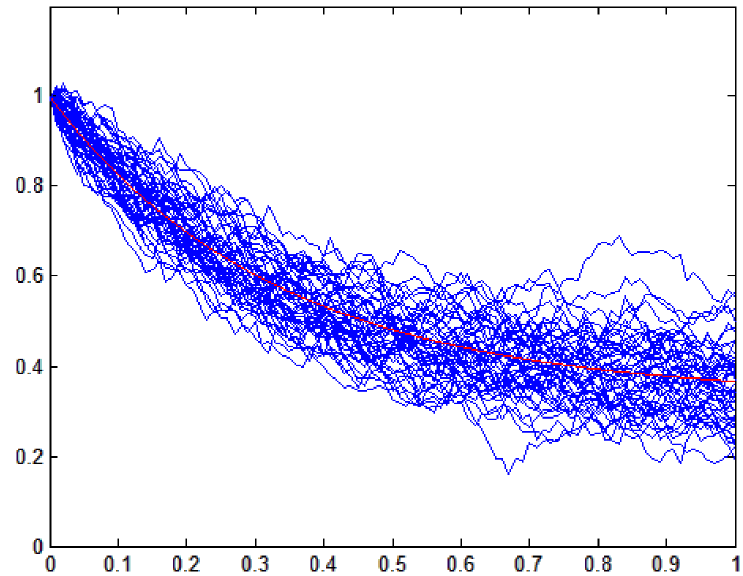

Figure 2.3: Fifty solutions of the stochastic differential equation $dX = (-3X + 1)dt + \sigma dB$ with $X(0) = 1$ and $\sigma = 0.2$ and an equation without the random term, vertical axis denotes X, horizontal t. Qualitative features of both solutions are visible.

Source: http://lis.epfl.ch

In physics, stochastic differential equations come in three, albeit somewhat vaguely differentiated, types. The first one of them by virtue of its historical precedence is so-called Langevin equation of which one example was given at the beginning of this chapter. There, the underlying stochastic process is a position of a particle subjected to random fluctuations of the environment and the equation took the form (2.3) but the term applies equally well to random evolution of any subset of degrees of freedom over time. Generalizing from this example, one finds second type of stochastic differential equation. It is characterized by the fact that it can be written in the form

$$dX_t = F(X_t, t)dt + G(X_t, t)dB, \tag{2.16}$$

where $F(X_t, t)$, $G(X_t, t)$ are sufficiently bounded functions in order for a unique solution to exist (for details and a proof see Oksendal (2003). The example (2.16) is in fact general enough to cover all equation we will deal with. Third type of equation is not strictly speaking stochastic differential equation but rather it forms a bridge between an equation of type (2.16) and partial differential equations. It is called **Fokker-Planck equation** and to see how it works, let us consider an equation (2.16) in slightly disguised form

$$dX_t = \mu(X_t, t)dt + \sqrt{2D(X_t, t)}dB_t. \tag{2.17}$$

Here the terminology follows from the fact, that this type of equation is used to describe diffusion type of processes. Hence, D is called diffusion coefficient and μ is called drift, random variable X_t can then be thought of as a trajectory of a particle of diffusing medium. Fokker-Planck equation for a process (2.17) then gives a time evolution of the probability density function of the random variable X_t for a fixed t. In this concrete example it has a form

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t) f(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D(x,t) f(x,t)\right]. \tag{2.18}$$

For a detailed derivation of this result, reader is referred to Allison Kolpas (2006). The similarity with the Schrodinger equation is evident, with one important difference being that equation (2.18) gives a time evolution of the probability distribution itself, whereas Schrodinger equation only for the probability amplitude.

In economics, stochastic differential equations are almost exclusively realized as the second type of equations from the previous paragraph, that is, equation of type (2.16). Out of these, one particular example stands out in terms of its ubiquity. It a stochastic differential equation describing evolution of an asset price over time and as an pricing model is used is widely used in equities, currencies, commodities and indices. It reads

$$dS = \mu S dt + \sigma S dB, \tag{2.19}$$

where S is the asset price, μ is the drift term which is in this case equal to the expected return on the asset and σ is its volatility, both these parameters are considered to be constant. One useful way of thinking about this equation is that in an infinitesimally small time interval δt, the asset price S changes its value by an amount that is normally distributed with expectation $\mu\delta t$ and variance $\sigma^2\delta t$ and is independent of the past behavior of the price. We assume the equation (2.20) to be valid in this work as well. Its solution will be given in the next section as an illustration of the power of stochastic framework.

Having discussed the stochastic integration and stochastic differential equations, one needs a computational tool to evaluate them. When integrating a function of real or complex variables, we seldom make use of the definition of Riemann or Lebesgue integral, similarly, this definition is of no use when finding a solution of an ordinary differential equation. What is almost invariably revoked at these situations is the fundamental theorem of calculus

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2.20}$$

given that $F'(x) = f(x)$. By a slight abuse of notation and in accordance with how we have expressed the equations (2.14) - (2.19), this is equivalent to

$$dF = fdx. \tag{2.21}$$

Let us now derive analogous formula for functions of stochastic variables. The derivation given here will be more heuristic than rigorous, we will for example completely omit a proof of convergence.

Let us assume we have a function $g(x,t)$ twice continuously differentiable and so-called **Ito process** S_t given by

$$dX_t = \mu dt + \sigma dB_t. \tag{2.22}$$

Let us consider a new process $F_t = g(X_t, t)$. What we are interested in is an increment dF_t over infinitesimally small time dt. To obtain it, since $g(x,t)$ is twice continuously differentiable, we can expand it into Taylor series

$$dF = \frac{\partial g}{\partial t}(X_t,t)dt + \frac{\partial g}{\partial x}(X_t,t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t,t)\cdot(dX_t)^2 + \ldots \tag{2.23}$$

Now we substitute for dX_t from equation (2.22) and get:

$$dF = \frac{\partial g}{\partial t}(X_t,t)dt + \frac{\partial g}{\partial x}(\mu dt + \sigma dB_t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\cdot(\mu^2 dt^2 + 2\mu\sigma dt dB_t + \sigma^2 dB_t^2) + \ldots \tag{2.24}$$

If we now let dt go to zero, the terms $\mu^2 dt^2$ and $2\mu\sigma dt dB_t$ are both of order dt^2 and disappear but the last term does not because from the equation (2.9) we have

$$E[dB_t^2] = dt \tag{2.25}$$

and for $dt \to 0$ dB_t^2 converges to its expected value. Hence we get

$$dF = \left(\frac{\partial g}{\partial t}(X_t, t) + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial x^2}(X_t, t) \right) dt + \frac{\partial g}{\partial x}(X_t, t) dX_t \tag{2.26}$$

which is a shorthand general form of Ito lemma. The corresponding full form is obtained by integrating both sides of (2.26)

$$F(X(t)) = F(X(0)) + \int_0^t \left(\frac{\partial g}{\partial t}(X_t, t) + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial x^2}(X_t, t) \right) d\tau + \int_0^t \frac{\partial g}{\partial x}(X_t, t) dX_t \tag{2.27}$$

so that the value F_t at time t contains a sum of two integrals - Lebesgue with the differential $d\tau$ and Ito with the differential dX_t.

It is important to note that what Ito lemma establishes is an integration theory on subclass of all stochastic processes. One has no differentiation theory, thus the relation invoked in equation (2.10) must be understood only in a symbolic sense. However, as we shall see, that is sufficient to solve a large class of problems in stochastic analysis.

It is also important to note that by making a rather special choice of the underlying Ito process (2.22) we do not restrict the validity of the result (2.26) and (2.27). This choice in the derivation was made for the sake of clarity. For more involved dependence on the Brownian differential, one just needs to replace σ^2 in Ito lemma by the square of a relevant factor. It is also worthy of pointing out that one can always work with functions of Brownian motion only by putting $\mu = 0$.

As a proof of utility of Ito lemma let us determine the value of integral

$$I = \int_0^t B_s dB_s. \tag{2.28}$$

One would suggest, based on real variables calculus, that the value is $\frac{1}{2}B_t^2$ which is however not, as we shall see, the case. Let us put $\mu = 0$, $\sigma = 1$ in equation (2.22) and let us choose $g(x,t) = \frac{1}{2}x^2$. Then

$$F_t = g(B_t, t) = \frac{1}{2}B_t^2. \tag{2.29}$$

Then by Ito formula (2.26)

$$dF_t = \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right) dt + \frac{\partial g}{\partial x} dB_t = \frac{1}{2}dt + B_t dB_t. \tag{2.30}$$

Hence

$$d\left(\frac{1}{2}B_t^2\right) = B_t dB_t + \frac{1}{2}dt \tag{2.31}$$

and

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t. \tag{2.32}$$

Thus, contrary to the estimate given above, deterministic nature of the second moment of Brownian motion causes the integral (2.28) to aquire a time-dependent second term.

Let's now solve the equation (2.19) governing the evolution of asset price over time. Let's consider change of variable $x(t) = \log(S(t))$, then by straightforward application of Ito lemma for the function $\log x$ we get

$$d \log S = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB. \tag{2.33}$$

Integrating both sides of (2.33) yields

$$S(T) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma(B(T) - B(0))}. \tag{2.34}$$

Now, according to section 2.2.3 on Brownian motion, the random variable $B(T) - B(0)$ has a normal distribution with mean zero and variance $\sqrt{T}$. We can thus finally rewrite (2.34) as

$$S(T) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N}, \tag{2.35}$$

where N has normal distribution with zero mean and unit variance. One realization of the solution (2.35) is given in Figure 2.4.

2.2.5 Derivation of the Black-Scholes model

Previous paragraphs provide us with sufficient amount of methods and tools to finally derive the Black-Scholes model. However, prior to its derivation within the framework of stochastic calculus, general remarks and a historical digression are in order. Black-Scholes model, also called Black-Scholes-Merton model, is the most important derivative pricing model in terms of its histori-

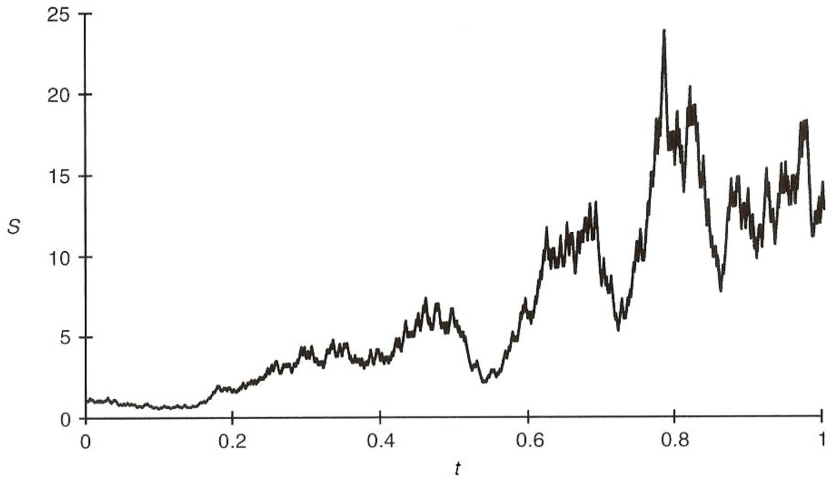

Figure 2.4: A realization of a solution of $dS = \mu S dt + \sigma S dB$.
Source: cite Wilmott

cal precedence, analytical solubility and ubiquity. It was published in by two American economists, Fischer Black and Myron Scholes, in 1973 in paper "The pricing of options and corporate liabilities" published in the Journal of Political Economy where the general idea behind its derivation was laid out. Few years later, another American economist, Robert Merton, published a paper expanding the mathematical treatment of the model and coining the name of the model. Scholes' and Merton's work was awarded by 1997 Nobel prize in Economics as only these two men were alive at the time.

From a mathematical point of view, the solution of the differential equation that constitutes the heart of the model gives the price of the contract (i.e. its **premium**) V as a function of the price of the underlying stock S, time to maturity $T - t$,volatility of the stock returns σ, drift rate of the stock price μ, strike price K and annualized risk-free interest rate r. Thus, we can write the option value as

$$V(S, t; \sigma, \mu; K, T; r),$$

where semicolons separate different types of variables and parameters:

- S and t are variables,
- σ and μ are parameters associated with the underlying stock price,
- K and T are parameters associated with the particular contract,

- r is a parameter associated with the currency in which the underlying stock is quoted.

We shall derive both the explicit form of the equation and its solution in the subsequent paragraphs and we shall proceed in two steps: first, we make use special method of eliminating risk to construct a portfolio whose price evolution over time is fully deterministic and second we use a fundamental principle of finance - principle of no arbitrage to get the Black-Scholes differential equation.

It is clear that what we are trying to do is to get a deterministic differential equation for the price of an option given that we are given securities whose price evolution is described by a stochastic differential equation and therefore is inherently indeterministic (as can be seen e.g. in Figure 2.1). One possible way out of this predicament is to construct a portfolio consisting of several securities where *somehow* the stochastic indeterministic terms get cancelled out. Indeed, given assumptions to be listed later, we are free to construct arbitrary portfolio and we choose the securities so that their values are *correlated* giving a clearer meaning to the use of word "somehow" in the previous sentence.

So let's consider a portfolio Π of one long position in an option and short position in some quantity Δ of the underlying stock S

$$\Pi = V(S,t) - \Delta S \tag{2.36}$$

where the minus sign is accounted for by the fact that quantity Δ of stock is being sold. Now let us consider that the price of the stock obeys equation

$$dS = \mu S dt + \sigma S dB, \tag{2.37}$$

i.e. it follows a **lognormal random walk**. Then we can finally use the celebrated result of previous subsections, Ito lemma, to write the expression for the infinitesimal change of the value of the portfolio Π over time increment dt:

$$d\Pi = dV - \Delta dS, \tag{2.38}$$

where

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \tag{2.39}$$

which put together yields

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta dS. \tag{2.40}$$

Here, the change of the value of the portfolio is given by terms of two types, deterministic dt and stochastic dS. If we pretend for a moment that we know the value V and its derivatives, we have a complete information about the future price development of the value of the portfolio Π except for the value of dS. Now we make use of the correlation between price increment of the option and underlying stock price dV and dS, in other word the fact that both the dSes in the equation (2.40) are the same and pertain to the same quantity, and set

$$\Delta = \frac{\partial V}{\partial S}. \tag{2.41}$$

The randomness is then reduced to zero and the evolution of the portfolio Π is fully deterministic. Any procedure of this kind is called **hedging** and this special case, where the correlation beween two instrument was exploited, is particularly called **delta hedging**. Because of the continous nature of this strategy, the amount of stock S needs to be continually rebalanced as the value of $\frac{\partial V}{\partial S}$ changes over time, delta hedging is said to be example of a *dynamic hedging strategy*. We have thus concluded the first step of deriving the Black-Scholes equation, we have constructed, using rules of Ito calculus, a portfolio whose price development is free of any stochastic disturbances. In the next paragraph, we make use of this result and introduce yet another notion - the notion of principle of no arbitrage - to complete the argument and arrive at a solvable differential equation.

It is often argued, and empirical evidence seems to support the statement that there no such thing as free lunch. In the financial setting this statement is equivalent to the impossibility of riskless profit above the risk-free rate of interest. This statement, the principle of no arbitrage, has important ramifications one of which we are now going to explore.

By prudent choice of hedge (2.41) we have obtained a portfolio whose value changes over time as

$$d\Pi = \left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt. \tag{2.42}$$

Since this return is completely riskless, it must equal by the no arbitrage principle to the growth we would get if we deposited an equaivalent amount of money into risk-free interest-bearing account

$$d\Pi = r\Pi dt. \tag{2.43}$$

Putting (2.36), (2.42) and (2.43) together, we get

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt. \tag{2.44}$$

Dividing by dt and rearranging the terms leads to the celebrated Black-Scholes equation of the price of an option

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.45}$$

As with all differential equations, one needs to add relevant boundary and initial conditions in order to completely specify a problem. Boundary conditions tell us how the solution behaves at all times at certain values of the asset. In our case, we specify the behavior of the solution for $S = 0$ and $S \to \infty$

$$V(0,t) = 0\,\forall t, \quad V(S,t) \to S \quad \text{as} \quad S \to \infty. \tag{2.46}$$

As for the initial conditions, the nature of the problem in this case is that we know the value of the option at the time of its expiry (see e.g. Figure 2.1), that is, its payoff function $g(S) \equiv V(S, t = T)$. Under these circumstances it is more appropriate to talk about *final conditions* and these then completely specify particular type of the option we are dealing with. To be concrete, let's give some examples that we shall most closely deal with in the subsequent chapters. If we have a call option, the final condition is

$$V(S,T) = \max(S - K, 0) \tag{2.47}$$

and for a put option, we have

$$V(S,T) = \max(K - S, 0). \tag{2.48}$$

Solution of the Black-Scholes equation

How can we use information in equations (2.46) - (2.48) to find a solution of the Black-Scholes model? Fully satisfactory answer to this question is beyond the scope of this work and an interested reader can find it in Wilmott (2000). For our purposes, let's state without further discussion that among the plenty of methods that can be used the most computationally efficient is a transformation to constant coefficient diffusion equation and that other popular options are the method of Green's functions or a "Fourier-like" method supposing a solution in the form of a series expansion. All these methods give the same solutions that we will write down without explicit calculation.

For a call option one finds that the solution has the form

$$V(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2) \tag{2.49}$$

where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{2.50}$$

and

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{2.51}$$

For the put option we get

$$V(S,t) = -SN(-d_1) + Ee^{-r(T-t)}N(-d_2) \tag{2.52}$$

where d_1 and d_2 have the same meaning as above and N is a cumulative distribution function for the standardized normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}x'^2}dx'. \tag{2.53}$$

Plots of the value of a call and put options as functions of underlying asset price and time are given in Figures 2.5, 2.6, 2.7 and 2.8.

Assumptions of the Black-Scholes model

In the previous sections, we have given a derivation and a solution of the most widely used model of mathematical finance. Our treatment of it was terse yet as rigorous as possible. For the sake of completeness, and preparing ground for the discussion of model's shortcomings in the next chapter, we list all the assumptions that we have made along the way. These are:

- **The portfolio satisfies no arbitrage condition.** This translates into the impossibility of making a riskless profit on the markets where the asset and the option are traded. Of course, on real markets arbitrage

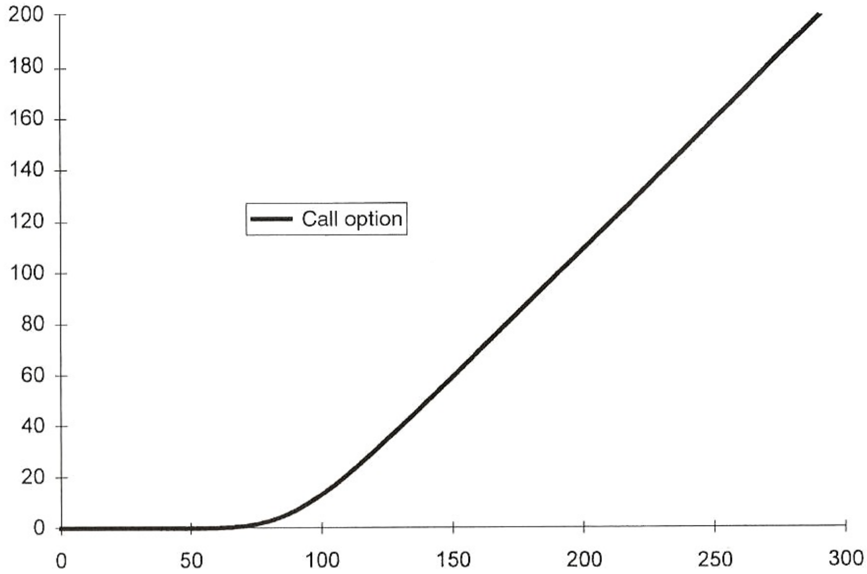

Figure 2.5: Value of a call option as a function of the underlying price S at a fixed time to expiry.
Source: Wilmott (2000)

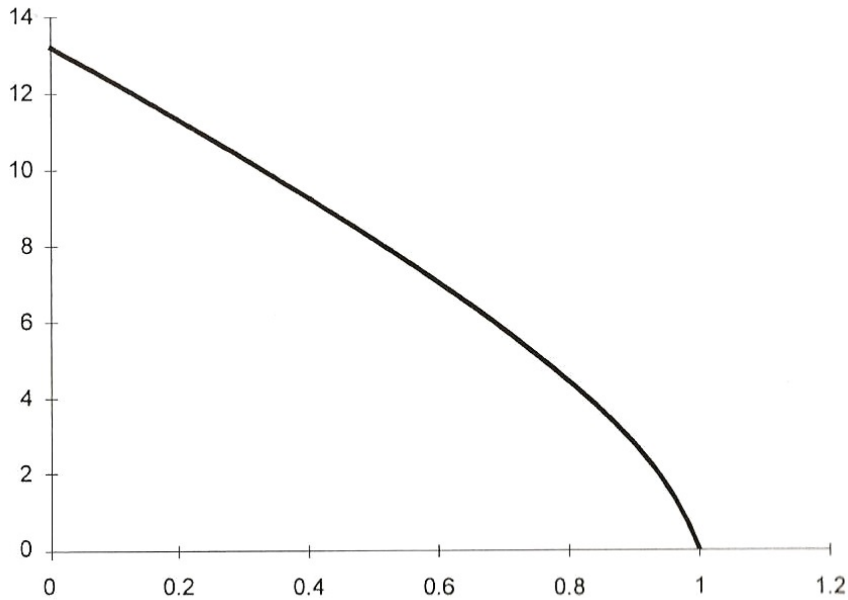

Figure 2.6: Value of a call option as a function of time, $S = K$.
Source: Wilmott (2000)

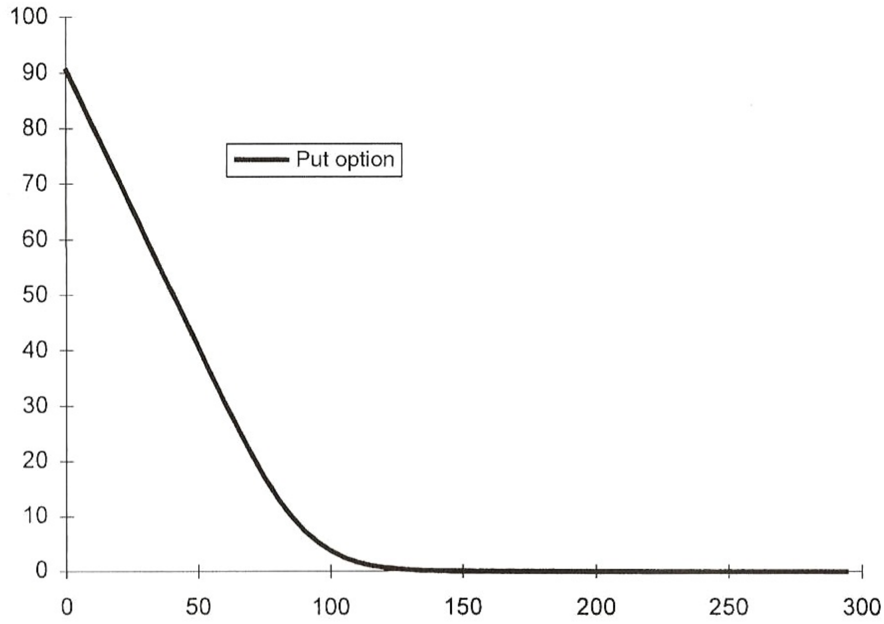

Figure 2.7: Value of a put option as a function of the underlying price S at a fixed time to expiry.

Source: Wilmott (2000)

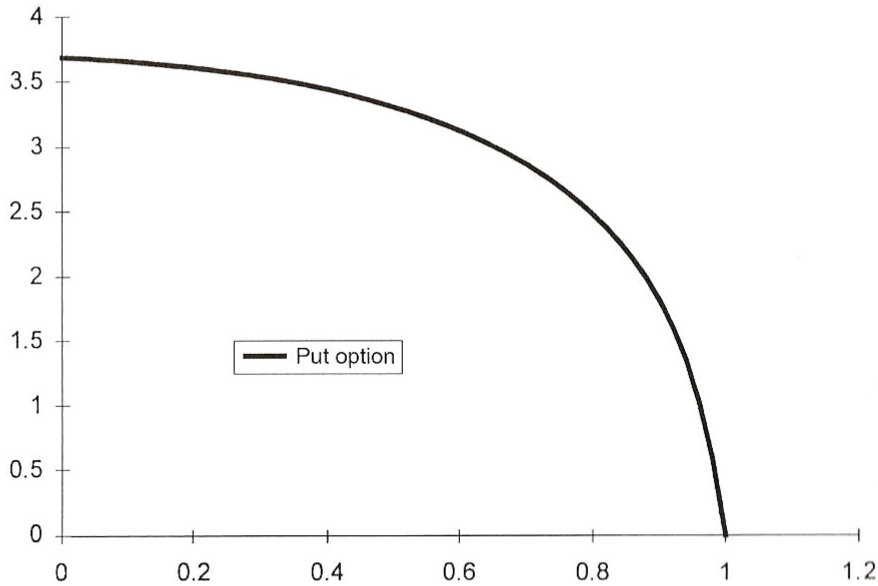

Figure 2.8: Value of a put option as a function time, $S = K$.

Source: Wilmott (2000)

opportunities exist so the no-arbitrage condition pertains only to the *model-dependent* arbitrage. From the point of view of economical theory, no arbitrage is a precondition for a market to be in general equilibrium and thus all arbitrage opportunities should be only short-term.

- **The asset price S has a continuous-time evolution.** If the asset price followed a more general stochastic process that would include discontinuous jumps, it can be shown that the portfolio could not be perfectly hedged and the Black-Scholes analysis would be no longer applicable.
- **Delta hedging is done continuously.** Continuous rehedging is only a theoretical construct, in real markets only hedging is possible. The frequency of rehedging depends on level of transaction costs in the underlying asset, the higher the costs, the more frequent delta hedging is possible.
- **There are no transaction costs on the underlying.**
- **There are no dividends on the underlying.** This assumption simplifies the solution of the model. It can be dropped and the resulting equation will still be analytically solvable. For exact solution see Wilmott (2000).
- **The risk-free interest rate is a known function of time.** This assumption is a prerequisity so that we could find a explicit solution. In reality, the rate r is not known in advance and is itself stochastic.
- **In the hedged portfolio, the asset S is infinitely divisible and short-selling is possible.** This technical assumption is a precondition for a continuous delta hedging to be possible. Of course, the first of the conditions is never realized in real markets as the assets are traded in discrete quantities.

This then completes our treatment of the Black-Scholes model. As celebrated a model as it can be, it is not without serious critics. For a thoroughly negative review of the Black-Scholes model approach to the option pricing, reader is invited to consult (Espen G. Haug 2010). From our point of view, we will use it as a backbone for the model to be developed in the next chapter.

Chapter 3

The new econophysical framework

The main goal of this chapter is to give a derivation of a model of European options that will be based on econophysical framework and will in some sense correct the deficiencies of the Black-Scholes model which was introduced in the previous chapter. For the sake of greater clarity, this task will be done in three steps. First, we give an overview of the tests of empirical validity of the Black-Scholes model, giving a firmer ground to the criticisms that were mentioned in the previous chapter. Particular details of these test shall reveal a precise nature of the shortcomings of this model. Then, in the second step we will cover a sufficient amount of the quantum mechanics as a possible new framework for the particular problem of option pricing. Hilbert spaces, wavefunctions and the Schrodinger equation are given a due treatment. In the third step, we apply the knowledge of quantum mechanical framework and derive a simple model for the price of an European option.

3.1 Empirical shortcomings of the Black-Scholes model

On a more philosophical note, the question of whether a certain model is right or wrong in a certain sense lacks meaning because by constructing a simplified version of reality we always make some phenomenological reduction. Furthermore, it is almost always the case, that we impose conditions which are to be met for a relation between the model and reality to be representative. By virtue of this, the link beween reality *as it is* and a model is broken. Thus, much more appropriate question to ask is to what extent a given model is able to explain empirical observations and to what degree it is in the popperian sense falsifiable by them. In the particular case of the Black-Scholes model thanks to the ubiquity of the financial data the former question can be given a thoroughly definite answer that we shall try to convey in the subsequent paragraph.

In an econometrical setting which is in our case relevant, the question of judgment of empirical validity of a model at the end reduces to a judgement whether certain R^2-type statistics has a sufficiently high value. We shall not reproduce empirical studies aimed at resolving this question as this type of studies has been frequently done in the past and the results can be found in relevant papers. Suffice to say and to cite Wilmott (2000): "it must be emphasised how well the model has done in practice, how widespread is its use and how much impact it had on financial markets." Let us now find the limits to the aforementioned citation. As it turns out, the exposition of shortcomings of the Black-Scholes model to a large extent follows the list of assumption that was given at the end of the previous chapter:

- *Only discrete hedging is possible.* This is a clear contradiction of an assumption of continuous delta hedging. In presence of discrete steps at which hedging can be done, the Black-Scholes formula holds only *on average*. However, as we shall shortly see, breaking this assumption does not constitute a serious fundamental problem to the formula because of course in the limit of infinitesimal discrete hedging steps the hedging errors converge to zero. For the description of the precise nature of hedging error due to non-continuous trading and its implications see Takaki Hayashi (2005) and Toft (1994).

- *There are transaction costs.* The precise magnitude of the transaction costs depends on the particular market and can range from negligible to significant. One implication of the existence of the transaction costs is that for arbitrarily low positive transaction costs there exists a rehedging time step such that under these costs the perpetual hedging is no longer the optimal strategy. In the continuum limit, the total cost of hedging approaches infinity. Transaction costs as a result of bid-ask spread are especially significant in emerging markets stocks and equity derivatives.

- *Volatility is not constant neither deterministic known function.* The treatment of volatility in the Black-Scholes model is severely oversimplified as it supposes that the parameter σ in the equation is a constant. Empirical studies show that it is not constant or even predictable, so the best way to treat it is as a stochastic variable itself. Moreover, it is not even directly observable: from the equation (2.19) it can be seen that

$$\mathrm{Var}\left[\frac{dS}{S}\right] = \sigma^2 dt, \tag{3.1}$$

where we have used the properties of the Wiener process. Thus, σ is a standard deviation of a stock's logarithmic returns. But standard deviation can never be directly measured and it depends on which way we normalize it in equation (3.1), i.e. which period do we take into account for the calculation of returns. A standard choice is a one year period which is far from unique - different choices give different values for the σ. When we compute volatility in this manner we speak about **historical volatility**. An alternative approach is to consider market price of some instrument whose pricing formula depends on the volatility parameter σ. In case of the Black-Scholes model formulae (2.49) - (2.52) the option price V is monotonous in σ and thus these relations can be inverted to give a unique value of volatility σ for a given price V. The resulting value is called **implied volatility** because it is implicitly implied by the market price of a derivative contract. The values of historical and implied volatilities generally do not coincide.

- **The asset price do not follow geometrical Brownian motion.** This is perhaps the gravest defect of the Black-Scholes model. Numerous empirical investigations have shown that the distribution of the logarithmic returns is not normal as the model assumes but exhibits dependence on the time interval over which we compute the asset returns. Let us give a overview of the exact forms of the underlying non-Gaussian distributions for different time periods. Highest-frequency data where the log returns are recorded by one minute, exhibit purely exponential heavy tails that are best described by the Boltzmann distribution

$$B(z) = \frac{1}{2T} e^{-|z|/T}, \tag{3.2}$$

where we have taken $z \equiv \log S$ and T denotes temperature. Introduction of a new parameter T with intuitive interpretation and positive correlation with volatility σ poses a major convenience whose description can be found in Masud Chaichian (2001). Higher recording periods $\Delta T = 2\text{min}$,

$\Delta T = 3$min lead to a data that are best fitted by a Student-Tsallis distribution

$$D_\delta(z) = N_\delta \frac{1}{\sqrt{2\pi\sigma_\delta^2}} e_\delta^{-z^2/2\sigma_\delta^2} \tag{3.3}$$

where

$$N_\delta = \frac{\sqrt{\delta}\Gamma(1/\delta)}{\Gamma(1/\delta - 1/2)}, \quad \sigma_\delta = \sigma\sqrt{1 - 3\delta/2} \tag{3.4}$$

and

$$e_\delta^z = (1 - \delta z)^{-1/\delta} \tag{3.5}$$

is an approximation to the exponential function called δ-exponential. Another probability distribution whose applications range from high-frequency data to ΔT in the range of several days is the Levy distribution and truncated Levy distribution. These are defined by their Fourier transform

$$L_{\sigma^2}^{\lambda} \equiv \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipz} e^{\frac{-(\sigma^2 p^2)^{\lambda/2}}{2}} \tag{3.6}$$

in the former case and by slightly more involved Fourier transform formula

$$L_{\sigma^2}^{(\lambda,\alpha)}(z) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipz - H(p)} \tag{3.7}$$

in the later case with

$$H(p) = \frac{\sigma^2}{2} \frac{\alpha^{2-\lambda}}{\lambda(1-\lambda)} \left[(\alpha + ip)^\lambda + (\alpha - ip)^\lambda - 2\alpha^\lambda\right] \tag{3.8}$$

where σ^2 denotes the second moment of respective distribution and λ, α are parameters. In the limit $\Delta T \to \infty$, i.e. for the period T ranging from several weeks to years the underlying distribution approaches normal Gaussian. This corresponds to the limit $\lambda \to 2$ in equation (3.6). In simple terms, what the Black-Scholes model seriously underestimates is the probability of extreme events given by the “lean tails” of the Gauss distribution. For a somewhat popular critique of this fact, the reader is referred to Taleb (2010).

This list of reasons and particularly the last one reveal serious flaws at the conceptual foundations of the Black-Scholes model. While it is the case that wrong assumptions sometimes lead to correct predictions, it should serve only as a weak comfort. In the following, we take the approach akin to physical methodology. Particularly, we propose a new model using heuristic principles and insight from the experimantal data. But before giving an full account of the derivation of the econophysical model, we need to give an overview of the quantum mechanics with respect to its applications in economy.

3.2 Quantum mechanics

This section ought to give a succint treatment of one of the pillars of modern theoretical physics - quantum mechanics. Ever since its incarnation in the seminal papers in 1920's, the subject has not ceased to receive a relentless attention, mainly because of the philosophical interpretation of reality that it foists on us. We will try to convey the meaning of this new outlook on reality in the subsequent paragraphs because it is tangential to the subject of this work. It should be said right at the outset however that any exposition of quantum mechanics that fits into less than considerable amount of pages is necesarily an oversimplification and is doomed to be in some sense incomplete. The serious reader interested in more rigorous exposition is therefore referred to books (Ballentine 2003), (Claude Cohen-Tannoudji 1977). The outline of this section is as follows: first, we introduce two types of theories that the world around us is desribed with and the stepping bridge from one type to the other. This will lead us to the second point - the mathematical structure of the quantum theory. Third, the attention is focused on the analogies between physical and economic systems.It should be noted that because of the our language is somewhat more relaxed than what is the case in most of physical literature.

As was mentioned in the previous paragraphs, the physical world around us seems to be described by two types of theories. These theories are **classical** and **quantum**. The definition of the classical theories seems to be negative and not very descriptive - they are a type of theories where one does not take into account the Heisenberg uncertainty principle, one of the hallmarks of the second type of theories. Newton's laws of motion, Maxwell's equations, special and general relativity all lead to theories whose description is classical. On the other hand and not surprisingly, the theories that *do* take the Heisenberg uncertainty principle into account are called quantum. These theories include quantum mechanics, quantum electrodynamics and the theory of nuclear forces.Theories of both types are not unrelated in the ideal case. The procedure of finding a quantum theory corresponding to a given classical one is called *quantization*, whereas from a given quantum theory to be consistent, one requires that in the *classical limit*, one recovers the corresponding classical theory. The criteria that determines which type of theory is suitable is the magnitude of the action of the system - for $S \gg \hbar$ classical theory is applied, for $S \approx \hbar$ quantum theory is relevant. The constant $\hbar$ that we have just introduced is called the reduced Planck constant. Its physical dimension is that of action or equivalently $E \cdot t$ and the above mentioned classical limit then logically corresponds to $\hbar \to 0$. For an explicit account of the limiting procedure on case of ordinary quantum mechanics, reader is invited to consult Ballentine (2003). Suffice to say that the classical theory generated in this way is always unique.

In case one has a classical theory and wants to obtain its quantum analogue, the situation is not that clear - there are numerous quantization procedures that lead to non-equivalent quantum theories. Historically the oldest approach is so-called *canonical quantization* which makes use of the Hamiltonian formulation of mechanics and a phase space parametrized by generalized coordinates q_i and their conjugate momenta p_i. The set of functions $f(q_i, p_i)$ can then by given a structure of an algebra by introducing special binary operation - **Poisson bracket** defined as

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{3.9}$$

The canonical quantization then consists in finding a map from the Poisson algebra (that is the set of functions on the phase space together with the binary operation of Poisson bracket) into the set of Hermitian operators on the Hilbert space such that

$$\{f, g\} \to \frac{1}{i\hbar} [\hat{A}, \hat{B}], \tag{3.10}$$

where A, B are operators corresponding to the functions f, g respectively. The precise meaning of terms will be elucidated on the next paragraphs. Other quantization options include the path integral quantization, geometric quantization, deformation quantization and others. Their full treatment goes beyond the scope of this work and can be found in for example in (Michael E. Peskin 1995).

What has been explained in the previous paragraph is how a certain special procedure called canonical quantization turns a classical theory into its quantum counterpart as it is the clearest example of the relation between classical and quantum theories. This on its own however does not answer the question of what quantum theory really is and what mathematical structures it makes use of. These issues are dealt with in the next subsection. Our exposition to a large extent follows (Masud Chaichian 2001) which can also be consulted for further details.

The axioms of quantum mechanics

In trying to derive the laws quantum mechanics one has several possibilities how to proceed. We take the axiomatic path, that is, we list the set of axioms that are sufficient and necessary to recover a quantum theory.

Proposition 3.1. *Quantum mechnical states are described by non-zero vectors of a complex separable Hilbert space H, two vectors describing the same state if they differ from each other only by a non-zero complex factor. To any observable, there corresponds a linear Hermitian operator on H.*

Hilbert space means that we work with a linear vector space that is complete with respect to the norm induced by the scalar product. Hermitian operator is an operator such that

$$\langle \hat{A}\phi|\psi\rangle = \langle \phi|\hat{A}\psi\rangle \tag{3.11}$$

i.e. it is symmetric and the domains of the operator acting to the left and acting to the right coincide. We have used the Dirac bra-ket notation for the scalar product. The space H that we have just postulated is called the *state space* and its elements are called *state vectors* or equivalently the *wavefunctions*. It is customary to suppose that all the vectors that we work with have unit norm because any multiples of the given state vector represent the same state.

The observables $A_1, \ldots, A_n$ are called simultaneously measurable if their values can be determined with arbitrary precision simultaneously, so that in any state $\psi \in H$, the random variables $A_1, \ldots, A_n$ have a joint probability density. Heisenberg uncertainty principle that was mentioned in the previous paragraph as an example of criteria between quantum and classical theories is then a special form of a statement that two observables are not simultaneously measurable

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \tag{3.12}$$

where σ is a standard deviation of a probability distribution of position or momentum when measured simultaneously. This fact can be given equivalent characterization in terms of the operators that represent the observables:

Proposition 3.2. *Observables are simultaneously measurable if the corresponding self-adjoint operators commute with each other. The joint probability density probability distribution of simultaneously measurable observables in a state $\psi \in H$ has the form*

$$w(\lambda_1, \ldots, \lambda_n) = \langle \psi_{\lambda_1,\ldots,\lambda_n} | \psi \rangle^* \langle \psi_{\lambda_1,\ldots,\lambda_n} | \psi \rangle \tag{3.13}$$

where * denotes complex conjugation and $\psi_{\lambda_1,\ldots,\lambda_n}$ are common eigenfunctions of the operators $\hat{A}_1, \ldots, \hat{A}_n$, i.e.

$$\hat{A}_i \psi_{\lambda_1,\ldots,\lambda_n} = \lambda_i \psi_{\lambda_1,\ldots,\lambda_n}, \quad i = 1, \ldots, n \tag{3.14}$$

Elementary theorem then states that for $\hat{A}$ Hermitian, the eigenvalues λ_i are real and the eigenfunctions ψ_i are orthogonal. Moreover, in case of Hermitian operators the set of eigenfunctions $\{\psi_a\}$ is complete in H so that its linear span is H. This means that any vector $\psi \in H$ can be represented by the series

$$\psi = \sum_a c_a \psi_a, \quad c_a \in \mathbb{C} \tag{3.15}$$

where the index a runs over the eigenvalues of $\hat{A}$. The coefficients of this expansion can then be expressed as

$$c_a = \langle \psi_a | \psi \rangle \, . \tag{3.16}$$

But this according to the proposition 2.2 gives a probability amplitude that a measurement of the observable A gives the value λ_a if the system is in the state represented ψ, the corresponding probability is then

$$w_\psi^a = |c_a|^2 = |\, \langle \psi_a | \psi \rangle \,|^2 \tag{3.17}$$

and the mean value of quantity A in the state ψ is

$$\langle \hat{A} \rangle_\psi \equiv \langle \psi | \, \hat{A} \, | \psi \rangle = \sum_a \lambda_a w_\psi^a = \sum_a \lambda_a |c_a|^2. \tag{3.18}$$

The variance reads

$$\mathrm{Var}_\psi A = \langle (\hat{A} - \langle \hat{A} \rangle_\psi)^2 \rangle_\psi. \tag{3.19}$$

Having exposed the static description of state in quantum mechanics we now turn to the question of dynamics i.e. how the state evolves over time.

Proposition 3.3. *Let a state of a system, at some tome t_0, be described by a vector $\psi(t_0)$. Then at any moment t, the state of a system is described by the vector*

$$\psi(t) = \hat{U}(t, t_0)\psi(t_0) \tag{3.20}$$

where

$$\hat{U}(t, t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)} \tag{3.21}$$

is so-called evolution operator. The wavefunction $\psi(t)$ is differentiable with respect to time if it lies in the domain of the operator $\hat{H}$, called the Hamiltonian operator, and in this case one has the relation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}\psi(t). \tag{3.22}$$

As one can see from the purely imaginary exponent in equation (3.21), the evolution operator is unitary. The quantum evolution is thus equivalent to the a rotation of the hypersphere of all possible states in a Hilbert space of infinite dimension - vectors of unit norm are mapped to vectors with unit norm. The Hamiltonian operator H represents the total energy of the system, the wealth of possibilities of how the rotation can actually take place is then equivalent to the wealth of possibilities of how the energy of a system can depend on the generalized coordinates and their conjugate momenta. The particular case of equation (3.14) for the case of a Hamiltonian operator is

$$\hat{H}\psi_i = E_i\psi_i. \tag{3.23}$$

In this case, the operator $\hat{H}$ in the equation (3.21) can be replaced with

E_i so that the time evolution given by equation (3.20) is reduced to multiplication by a complex amplitude which by virtue of postualte 3.1 represents the same state. This is why the equation (3.23) is called the time-independent Schrodinger equation. It fully characterizes the stationary state of the system under consideration.

Analogies between quantum and financial systems

The discussion of relation between classical and quantum theories and exposition of the basic structure of quantum mechanics definitely deserves a justification in a work supposed to deal mainly economic problems. This justification comes in two parts. We explicitly state the analogy between the quantities of interest in both fields. Then give a reformulation of the Black-Scholes equation in the language of quantum mechanics which will turn out to be just as natural as the one within the stochastic analysis.

First fact one notices when comparing the two theories with respect to the determinicity of their description of the time evolution is the parallel with the character of the equation (2.19) giving the description of geometric Brownian motion depending on the parameter σ. For $\sigma = 0$ one gets a deterministic ordinary differential equation

$$dS = \mu S dt \tag{3.24}$$

with a solution $S(t) = S(0)e^{\mu S t}$. For $\sigma > 0$ on the other hand the resulting equation is stochastic differential, the value of a solution at a given time t is indeterministic and given by the equation (2.34) and (2.35). Thus the first case corresponds to a classical theory with fully deterministic equations of motion. In the case $\sigma \neq 0$ the price of a asset price is a random variable in the same sense that the position of a system (e.g. a scalar particle) in a certain state ψ is a random variable. The classical limit $\hbar \to 0$ used to obtain a classical theory from a quantum one is equivalent to a limit $\sigma \to 0$ of vanishing volatility.

Another parallel concerns the way the state of the system $|\psi\rangle$ is treated in the quantum mechanics. It is an object of central importance in the theory. By the same token and by taking a look at the equation (2.45), one can see that the natural candidate for an object of central importance and an analogue of the state vector $|\psi\rangle$ in option pricing is the price of the option V as a function of the price of underlying security S. There are also important conceptual differences however, in quantum mechanics, the wavefunction ψ is unobservable and can be learned about only through the act of measurement. Option price V on the other hand is always directly observable and the quantity given by the equation (3.13) is no longer of importance.

Third common feature is the formal analogy of equations (2.45) and (3.22). It can be shown that the Black-Scholes differential equation can be reformulated as a special type of the Schrodinger equation for a particular choice of the Hamiltonian operator $\hat{H}$. Because of its importance in the construction of alternative model that would remedy some of the shortcomings of the Black-Scholes model, the whole next subsection is devoted to putting the equation (2.45) into form similar to (3.22) and exploring its ramifications.

The Black-Scholes model reformulated in the language of QM

The outline of this subsection is straightforward. One substitution in the Black-Scholes differential equation shall allow us to find an explicit form for an operator that can be interpreted as representing total energy of the system. Along the way some divergences from the case of quantum mechanics emerge.

The equation (2.45) can be written in more convenient form

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV. \tag{3.25}$$

Now let's consider the substitution $V = e^x$ for $x \in (-\infty, \infty)$. A computation then yields

$$\frac{\partial}{\partial S} = \frac{\partial x}{\partial S}\frac{\partial}{\partial x} = \frac{1}{S}\frac{\partial}{\partial x} = e^{-x}\frac{\partial}{\partial x},$$

$$\frac{\partial^2}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial}{\partial S}\right) = e^{-x}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial}{\partial x}\right) = -e^{-2x}\frac{\partial}{\partial x} + e^{-2x}\frac{\partial^2}{\partial x^2}.$$

Substituting into (3.25) gives

$$\frac{\partial V}{\partial t} = \left(-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r\right)V, \tag{3.26}$$

i.e. $\frac{\partial V}{\partial t} = \hat{H}_{BS}V$ with

$$\hat{H}_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r. \tag{3.27}$$

This is the Black-Scholes Hamiltonian. Its eigenvalues represent the values of a “generalized Black-Scholes energy”. The point of vast divergence from the quantum mechanical case is however, that these values are not generally real. The reason for this is that the operator (3.27) is not Hermitian, nor can it be made Hermitian by a coordinate transformation. As unusual as it seems, this fact does not hinder it however from providing a fresh new method for solving the Black-Scholes model by means of a momentum eigenfunctions. The detailed description of this method is beyond the scope of this work and can be found in (Baaquie 2007).

One remark is in order. The Black-Scholes equation and the Schrodinger equation in the form still differ in the factor $i\hbar$, a fact that has been skimmed over in the previous derivation. The reason for this is that the reduced Planck constant can be made equal to zero by a suitable coordinate transformation without changing the qualitative features of the solutions of the relevant equation and that both equations are of different types when it comes to the reality or complexity of its solutions - while the Schrodinger equation gives complex solutions, the Black-Scholes equation is real and admits only real-valued solutions. A purely formal way how to resolve this discrepancy would be by considering the equation (3.26) as a Schrodinger equation in “imaginary time”.

3.3 Construction of a new model of option price premia

The word *construction* in the title of this section can be slightly misleading because it hints at a possible usage of deductive reasoning when searching for a model explaining the behavior of real world data. In economic sciences as a part of the world of social science it is the case that universal principles a bit like universal constants are quite scarce. Concretely in the field of finance besides the principle of no arbitrage and the martingale condition, one has no universally agreed upon laws to base one’s deductive reasoning on. It would be therefore more suitable to call one’s efforts an educated proposal but we adhere to the widely held academic modus operandi and denote these endeavors as “construction”.

As section 3.1 on the empirical shortcomings of the Black-Scholes model should attest, the behavior of the real world market data diverge in some points significantly from what the model supposes. Not only are there transaction costs that show a strong variation across the markets, not only are there dividends and non-constant risk-free rates, but perhaps most significantly the very assumptions of Markov and martingale properties are not entirely correct, leading the distributions of the logarithmic returns to have substantial deviations from the Gaussian distributions. Vast corpus of literature exists on the application of memory processes in economic time series which by its definition breach both these assumptions. So let us take its existence as yet another hint for finding a another more fitting model.

In the following we shall make a case for a particular form of discrepancy and give its analytical treatment from quantum finance perspective. This form of discrepancy was not particularly pointed out in the discussion of the empirical shortcomings of the Black-Scholes model in the beginning of this chapter but it is consisent with it the distributions of the logarithmic returns. Numerous empirical investigations show (see for example Jeffrey O. Katz (2005)) that the distribution of the logarithmic returns of financial samples display small but consistent negative third central moment

$$\gamma_1 = E\left[\left(\frac{\log S - \mu}{\sigma}\right)^3\right] \tag{3.28}$$

where μ denotes the mean and σ standard deviation of the distribution, which is a clear contradiction of a Gaussian normal distribution where the skewness γ_1 is zero. This fact has a simple intuitive implications - negative skewness indicates distribution whose peak is tilted to the right, this means that in the stock prices movements should tend to display occasional sharp declines that are set against a background of frequent, but relatively frequent price gains.

Let us now finally derive a pricing model that takes all these considerations into account. A glance at the derivation of the Ito formula (2.26) immediately reveals that one cannot derive such a model by considering the Taylor expansion of the function $F_t = g(X_t, t)$ in equation (2.23) to the "second order", i.e. considering term proportional to dt^2, $dtdB_t$ and dB_t^2 a proceeding in a manner analogous to the original derivation of Black of Scholes. By virtue of a quantity dB_t^2 one would this way or the other get an equation that includes fourth derivatives with respect to the price S of the underlying asset. In other words, instead of addressing the issue of *skewness* of the underlying distribution, one would deal with its *kurtosis*. But the analysis of the Black-Scholes model recasted in the language of the quantum mechanics showed us a description that is in all aspects equivalent to the original formulation. In quantum mechanics, it is fortunately the case that depending on the specific parameters of the system under consideration, one is not constrained in any way in choosing the concrete form of the constituent terms in the Hamiltonian. One can therefore add a term corresponding to the third central moment of the underlying distribution without the need to invoke the argument containing Ito formula derivations and thus effectively avoiding the line of reasoning that in this particular case does not lead to an end. So let us add a term to the Black-Scholes Hamiltonian (3.27) proportional to the third derivative with respect to x and let us choose it in such a way that the corresponding term in the equation 2.45 is $kS^3\frac{\partial^3 V}{\partial S^3}$.Then continuing the computations of the section on reformulation of the Black-Scholes model in the language of quantum mechanics gives

$$\frac{\partial}{\partial S} = \frac{\partial x}{\partial S}\frac{\partial}{\partial x} = \frac{1}{S}\frac{\partial}{\partial x} = e^{-x}\frac{\partial}{\partial x},$$

$$\frac{\partial^2}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial}{\partial S}\right) = e^{-x}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial}{\partial x}\right) = -e^{-2x}\frac{\partial}{\partial x} + e^{-2x}\frac{\partial^2}{\partial x^2}.$$

$$\begin{aligned}\frac{\partial^3}{\partial S^3} &= \frac{\partial}{\partial S}\left(\frac{\partial^2}{\partial S^2}\right) = e^{-x}\frac{\partial}{\partial x}\left(e^{-2x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\right) \\ &= e^{-3x}\left(-2\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) \\ &= e^{-3x}\left(\frac{\partial^3}{\partial x^3} - 3\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x}\right).\end{aligned}$$

Putting this into equation we wish to obtain

$$\frac{\partial V}{\partial t} = kS^3\frac{\partial^3 V}{\partial S^3} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV \tag{3.29}$$

where $k \approx 0$ is a convenient non-zero constant account for the non-zero skewness, gives

$$\begin{aligned}\frac{\partial V}{\partial t} &= ke^{3x}\left(e^{-3x}\left(\frac{\partial^3 V}{\partial x^3} - 3\frac{\partial^2 V}{\partial x^2} + 2\frac{\partial V}{\partial x}\right) - \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial V}{\partial x} + rV\right) \\ &= k\frac{\partial^3 V}{\partial x^3} - \left(3k + \frac{\sigma^2}{2}\right)\frac{\partial^2 V}{\partial x^2} + \left(2k + \frac{\sigma^2}{2} - r\right)\frac{\partial V}{\partial x} + rV.\end{aligned} \tag{3.30}$$

We can see that for $k = 0$ the equation (3.30) yields (3.25) as expected. We can now read off the form of a Hamiltonian of a constructed model

$$H = k\frac{\partial^3}{\partial x^3} - \left(3k + \frac{\sigma^2}{2}\right)\frac{\partial^2}{\partial x^2} + \left(2k + \frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} + r. \tag{3.31}$$

Chapter 4

Conclusion

This thesis was meant to be a contribution to the ongoing discussion about the relevance of the econophysical approach to problems in economic sciences. To this end, we have investigated the area option pricing through the prism of quantum finance with the particular goal of deriving a model that would make up for the deficiencies of the model of Black an Scholes.

The first chapter dealt with the prevailing paradigm - stochastic analysis and its application in mathematical finance. It introduced the basic terminology of the option market and it gave a self-contained pedagogical review of the stochastic analysis. All the indispensable notions of mathematical finance were successively covered - stochastic processes, Markov chains and martingales, the Brownian motion, Ito lemma and stochastic integration and stochastic differential equations. This knowledge was then used to give an account of the derivation of the Black-Scholes model following its authors' original argument. In the final part, the set of assumption being made along the way was listed.

The second chapter dealt with the deficiencies of the Black-Scholes model and with what the quantum finance can offer to remedy them. It opened with a list of the points where the Black-Scholes model goes wrong. After providing a necessary background in quantum mechanics, a comparison of quantum and financial systems was given. This knowledge justified the reformulation of the Black-Scholes model in the quantum mechanical framework. It was shown that this approach is natural and fully equivalent to the original one. In the next section the heuristic principle for the construction of a new model for formulated and it was shown that the stochastic paradigm, unlike the quantum one, cannot easily accomodate it. Finally, the explicit form of the Hamiltonian driving the time evolution of the model was deduced.

In our future research we would like to address the problem of solving the derived model using econophysical methods as well as its econometrical testing on the European call options data.

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