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DIPLOMOVÁ PRÁCE

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Algebraický přístup k CSP
(The Algebraic Approach to CSP)

Katedra algebry

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Abstrakt: Nechť $\mathbb{A}$ je konečná relační struktura. Problém splňování omezení s šablonou $\mathbb{A}$, CSP($\mathbb{A}$), rozhoduje, zda vstupní struktura $\mathbb{X}$ je homomorfní $\mathbb{A}$. Hypotéza o dichotomii CSP Federa a Vardiho říká, že CSP($\mathbb{A}$) je vždy buď v P nebo NP-úplný. V první části představíme algebraický přístup k CSP a shrneme známé výsledky o CSP pro orientované grafy, tzv. $\mathbb{H}$-barvení. Ve druhé části se zabýváme jistou třídou orientovaných stromů, tzv. speciálními polyádami. Pomocí algebraického přístupu potvrdíme dichotomickou hypotézu pro speciální polyády. V polynomiálním případě poskytneme jemnější popis a zkonstruujeme speciální polyádu $\mathbb{T}$ takovou, že CSP($\mathbb{T}$) je v P, ale $\mathbb{T}$ nemá šířku 1 ani žádné near-unanimity polymorfismy.

Klíčová slova: Problém splňování omezení, barvení grafů, konečná šířka, speciální triáda.

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Abstract: For a finite relational structure $\mathbb{A}$, the Constraint Satisfaction Problem with template $\mathbb{A}$, or CSP($\mathbb{A}$), is the problem of deciding whether an input relational structure $\mathbb{X}$ admits a homomorphism to $\mathbb{A}$. The CSP dichotomy conjecture of Feder and Vardi states that for any $\mathbb{A}$, CSP($\mathbb{A}$) is either in P or NP-complete. In the first part we present the algebraic approach to CSP and summarize known results about CSP for digraphs, also known as the $\mathbb{H}$-coloring problem. In the second part we study a class of oriented trees called special polyads. Using the algebraic approach we confirm the dichotomy conjecture for special polyads. We provide a finer description of the tractable cases and give a construction of a special polyad $\mathbb{T}$ such that CSP($\mathbb{T}$) is tractable, but $\mathbb{T}$ does not have width 1 and admits no near-unanimity polymorphisms.

Keywords: Constraint satisfaction problem, graph coloring, bounded width, special triad.

Chapter 0

Introduction

Let $\mathbb{A}$ be a fixed finite relational structure. The Constraint Satisfaction Problem with template $\mathbb{A}$, or CSP($\mathbb{A}$) for short, is the following decision problem:

INPUT:	A relational structure $\mathbb{X}$ (of the same type as $\mathbb{A}$).
QUESTION:	Is there a homomorphism from $\mathbb{X}$ to $\mathbb{A}$?

This class of problems has recently recieved a lot of attention, mainly because of the work of Feder and Vardi [13] from 1998. In this article the authors conjectured a large natural class of NP decision problems avoiding the complexity classes between P and NP-complete (assuming that P$\neq$NP). Many natural decision problems, such as k-SAT, graph k-colorability or solving systems of linear equations over finite fields belong to this class. The following conjecture which became the most famous open question in the study of CSP is the central theme of this thesis:

The CSP **dichotomy conjecture.** *For each relational structure* $\mathbb{A}$*,* CSP($\mathbb{A}$) *is in P or NP-complete.*

For brevity, we sometimes say that a relational structure $\mathbb{A}$ is tractable if CSP($\mathbb{A}$) is tractable and NP-complete if CSP($\mathbb{A}$) is NP-complete.

The algebraic approach to CSP was invented by Jeavons, Cohen and Gyssens [19] and later refined by Bulatov, Jeavons and Krokhin [8]. It led to an immediate breakthrough in the study of CSP and brought a rapid development of the subject and a plenty of new results heading towards the dichotomy conjecture (see [2], [18], [7] and a survey [9]). The revelation of a very strong connection between CSP and universal algebra allowed to apply deep algebraic tools, namely tame congruence theory.

At the core of the algebraic approach to CSP lies the concept of *compatible operation* or *polymorphism*, a generalization of homomorphism, and the fact that the complexity of CSP($\mathbb{A}$) depends only on the polymorphisms of $\mathbb{A}$. If a structure has "nice" polymorphisms, then the corresponding CSP is tractable.

A relational structure $\mathbb{A}$ is said to have *bounded width* if CSP($\mathbb{A}$) can be solved by a certain polynomial-time algorithm called *Local consistency checking* (see [13]). In [21], Larose and Zádori conjectured a full characterization of relational strucutres of bounded width. This conjecture was recently confirmed by Barto and Kozik [2]. Our work relies heavily on their result that relational structures with compatible *weak near-unanimity operations* of almost all arities have bounded width (see Theorem 3.6); as far as we know, our proof of the CSP dichotomy for special polyads (see Chapter 5) was the first application of the above result.

In this work we concentrate on CSPs whose template structures are digraphs. For a digraph $\mathbb{H}$, CSP($\mathbb{H}$) is also known as $\mathbb{H}$-coloring problem. The complexity of $\mathbb{H}$-coloring has been extensively studied in graph theory for almost 40 years. In [13], Feder and Vardi proved that each relational structure can be encoded into a digraph so that the corresponding CSPs are equivalent; hence the dichotmy conjecture for digraphs implies the general case.

The dichotomy was established for a number of special cases, including oriented paths (which are all tractable) [14], oriented cycles [12], undirected graphs [16] and many others. Using the algebraic approach, Barto, Kozik and Niven [4] established the CSP dichotomy for *smooth* digraphs (i.e., digraphs such that each vertex has an incoming and an outgoing edge).

In the class of all digraphs, oriented trees are in some sense very far from smooth digraphs. Except the oriented paths, the simplest class of oriented trees are the *triads* (i.e., oriented trees with one vertex of degree 3 and all other vertices of degree 1 or 2). Though even for triads the dichotomy conjecture remains open, it was confirmed by Barto, Kozik, Maróti and Niven [3] for the so-called *special triads*, a certain class of triads possessing sufficient structure to make the problem amenable. It turned out that each special triad is either NP-complete, or has width 1, or admits a compatible majority (ternary near-unanimity) operation.

In this work we establish the CSP dichotomy conjecture for *special polyads*; a straightforward generalization of special triads. We prove that each special polyad is either NP-complete or has bounded width.

Moreover, we characterize special polyads of width 1 as those whose core admits a binary idempotent commutative polymorphism.

We concentrated on special polyads for several reasons. Although special polyads do possess the same kind of structure as special triads, allowing us to apply some of the techniques used in [3], it was not obvious whether the results from [3] can be extended to them. We were also interested in the following question: Will every tractable special polyad be tractable for a "simple" reason, by which we mean satisfying some strong conditions ensuring tractability (e.g., possessing a compatible majority or near-unanimity operation or having width 1)? The answer to this question is negative. We construct a tractable core special polyad $\mathbb{T}$ which does not have width 1 and admits no near-unanimity polymorphisms (and thus, by a recent result of Barto [1], the variety generated by the algebra of polymorphisms of $\mathbb{T}$ is not congruence distributive).

The first part of this thesis serves as a brief summary of the basics of the Constraint Satisfaction Problem and the algebraic approach to CSP. In Chapter 1 we define several notions and notation used throughout the text. Chapter 2 introduces the Constraint Satisfaction Problem and the dichotomy conjecture. We provide several examples of problems expressible as CSPs and then discuss the notion of *bounded width* and the *Local consistency checking* algorithm. Chapter 3 presents elements of the algebraic approach to CSP and the algebraic tools which will be needed later in the second part. Chaper 4 summarizes known results on CSP for digraphs and puts into context *special polyads* treated in Part II.

In the second part we study special polyads. Chapter 5 contains the proof of the dichotomy and characterization of width 1. In Chapter 6 we present a method of constructing special polyads with certain desired properties, using the techniques developed for the proof of the dichotomy. We apply this method to obtain a tractable core special polyad $\mathbb{T}$ which does not have width 1 and admits no near-unanimity polymorphisms.

Part I

The Constraint Satisfaction Problem

Chapter 1

Preliminaries

In this chapter we define several notions and notation which will be used throughout the text.

1.1 Relational structures

An r-ary *relation* R on a set A is a subset $R \subseteq A^r$; r is called the *arity* of R and denoted $\operatorname{ar}(R)$. The definition of the Constraint Satisfaction Problem is based on the notion of *relational structure.* All relational structures in this thesis are assumed to be finite.

Definition 1.1. A (finite) *relational structure* is a tuple

$$\mathbb{A} = \langle A, R_1, \dots, R_n \rangle$$

where A is a finite set and R_1,…,R_n are relations on A.

Two relational structures $\mathbb{A} = \langle A, R_1, \dots, R_n \rangle$, $\mathbb{B} = \langle B, S_1, \dots, S_m \rangle$ are of the same *type* if $n = m$ and $\operatorname{ar}(R_i) = \operatorname{ar}(S_i)$ for every i. In such situation, a mapping $f : A \to B$ is a *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ if it preserves all the relations, i.e., for every i and every tuple $\langle a_1, \dots, a_{\operatorname{ar}(R_i)} \rangle \in R_i$ we have $\langle f(a_1), \dots, f(a_{\operatorname{ar}(R_i)}) \rangle \in S_i$. We say that $\mathbb{A}$ is *homomorphic* to $\mathbb{B}$ if there exists a homomorphism $\mathbb{A} \to \mathbb{B}$.

A relational structure $\mathbb{A}$ is a *core* if every homomorphism $\mathbb{A} \to \mathbb{A}$ is bijective (i.e., an *isomorphism*). For each relational structure $\mathbb{A}$ there exists a unique (up to isomorphism) core structure $\mathbb{A}'$ such that $\mathbb{A} \leftrightarrow \mathbb{A}'$. Such a structure $\mathbb{A}'$ is called the *core of* $\mathbb{A}$ and denoted $\operatorname{core}(\mathbb{A})$. For any relational structure $\mathbb{X}$, $\mathbb{X} \to \mathbb{A}$ if and only if $\mathbb{X} \to \mathbb{A}'$.

For $C \subseteq A$, the structure $\mathbb{A}[C] = \langle C, R_1 \cap C^{\operatorname{ar}(R_1)}, \dots, R_n \cap C^{\operatorname{ar}(R_n)} \rangle$ is the *substructure induced by* C.

1.2 Digraphs

A *directed graph*, often abbreviated as *digraph*, can be viewed as a relational structure with just one relation, the binary *edge* relation. (In this context, the usual combinatorial notion of *(undirected) graph* means a symmetric digraph without loops, i.e., a digraph such that its edge relation is symmetric and irreflexive.)

Definition 1.2. A *digraph* $\mathbb{G} = (G, E)$ is a set of *vertices* G together with a binary relation $E \subseteq G^2$, the *edge* relation. For $\langle a, b\rangle \in E$ we write $a \xrightarrow{\mathbb{G}} b$ or simply $a \to b$ when there is no danger of confusion.

The definition of *homomorphism* and *core* for digraphs is similar as for general relational structures:

Let $\mathbb{G}$ and $\mathbb{H}$ be digraphs. A mapping $f : G \to H$ is a *(digraph) homomorphism* from $\mathbb{G}$ to $\mathbb{H}$, if it preserves the edges, i.e., for all $a, b \in \mathbb{G}$ such that $a \xrightarrow{\mathbb{G}} b$ we have $f(a) \xrightarrow{\mathbb{H}} f(b)$. A digraph $\mathbb{G}$ is a *core*, if every homomorphism $\mathbb{G} \to \mathbb{G}$ is bijective. Again, we denote by $\mathrm{core}(\mathbb{H})$ the unique (up to isomorphism) core digraph $\mathbb{H}'$ such that $\mathbb{H} \leftrightarrow \mathbb{H}'$. The digraph in the figure below contains two isomorphic copies of its core, marked by $\circ$'s and $*$'s:

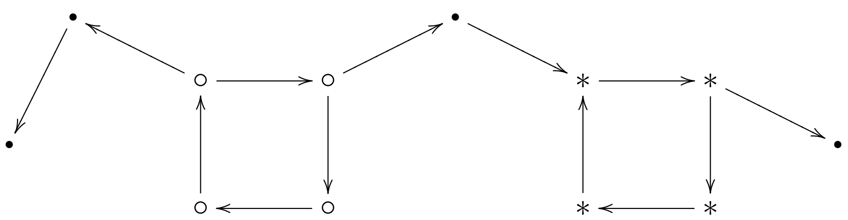


Figure 1.1: The core of a digraph.

A digraph $\mathbb{G}'$ is a *subgraph* of $\mathbb{G}$ (we write $\mathbb{G}' \subseteq \mathbb{G}$) if $G' \subseteq G$ and $E' \subseteq E$. If $E' = E \cap G'^2$, then $\mathbb{G}'$ is an *induced subgraph* of $\mathbb{G}$ (or a subgraph *induced by* G'), denoted by $\mathbb{G}[G']$.

Let $\mathbb{G}_1, \dots, \mathbb{G}_n$ be digraphs. The *product* of $\mathbb{G}_1, \dots, \mathbb{G}_n$ is the digraph $\prod_{i=1}^n \mathbb{G}_i = (G_1 \times \dots \times G_n, E)$ where $\langle \bar{a}, \bar{b}\rangle \in E$ iff $\langle a_i, b_i\rangle \in E_i$ for each $i = 1, \dots, n$.

An *oriented path* of length n is a digraph $\mathbb{P} = (P, E)$ with pairwise distinct vertices $P = \{v_0, v_1, \dots, v_n\}$ and edges $E = \{e_0, e_1, \dots, e_{n-1}\}$ such that $e_i \in \{\langle v_i, v_{i+1}\rangle, \langle v_{i+1}, v_i\rangle\}$ for each i. The vertex v_0 is called the *initial vertex*, denoted by $\mathrm{init}(\mathbb{P})$, and v_n is called the *terminal vertex*, denoted by $\mathrm{term}(\mathbb{P})$.

$$\operatorname{init}\mathbb{P} = v_0 \longrightarrow v_1 \longrightarrow v_2 \longleftarrow v_3 \longrightarrow v_4 \longleftarrow \quad \cdots \quad \longrightarrow v_n = \operatorname{term}\mathbb{P}$$

Figure 1.2: An oriented path.

A *directed path* is an oriented path such that $v_{i-1} \to v_i$ for $i = 1, \dots, n$. An *oriented cycle* is a digraph which can be obtained from an oriented path by identifying its initial and terminal vertex. A *circle* (or *directed cycle*) is a digraph which can be obtained from directed path by identifying its initial and terminal vertex.

Let $\mathbb{G} = (G, E)$ be a digraph and $a, b \in G$. We say that a is *connected* to b in $\mathbb{G}$ via a path $\mathbb{P}$ if $\mathbb{P} \subseteq \mathbb{G}$, $a = \operatorname{init}(\mathbb{P})$ and $b = \operatorname{term}(\mathbb{P})$. By the *distance* of two connected vertices a, b (denoted $\operatorname{dist}_{\mathbb{G}}(a, b)$) we mean the minimal length of an oriented path connecting a to b. The relation of connectedness is an equivalence relation on G. Its classes are called *components of connectivity.* $\mathbb{G}$ is *connected* if each two vertices $a, b \in G$ are connected.

For a vertex $a \in G$, The indegree ($\deg^-(a)$), outdegree ($\deg^+(a)$) and degree ($\deg(a)$) are defined as follows:

$$\begin{aligned} \deg^-(a) &= |\{b : \langle b, a\rangle \in E\}, \\ \deg^+(a) &= |\{b : \langle a, b\rangle \in E\}, \\ \deg(a) &= \deg^-(a) + \deg^+(a). \end{aligned}$$

A vertex $a \in G$ is a *source* if $\deg^-(a) = 0$ and a *sink* if $\deg^+(a) = 0$. A digraph is *smooth* if it does not have any sources or sinks.

1.3 Oriented trees

A digraph $\mathbb{T} = (T, E)$ is called an *oriented tree* if for each $a, b \in T$ there exists precisely one path connecting a to b. (Alternatively, an oriented tree is a digraph which can be obtained from an undirected tree, i.e., connected undirected graph without cycles, by orienting its edges.)

Let $\mathbb{T} = (T, E)$ be an oriented tree. There exists a unique mapping $\operatorname{lvl} : T \to \mathbb{N} \cup \{0\}$ satisfying the following conditions:

(i) If $a \to b$, then $\operatorname{lvl}(b) = \operatorname{lvl}(a) + 1$.

(ii) There exists a vertex $a \in T$ with $\operatorname{lvl}(a) = 0$.

For $a \in T$, $\mathrm{lvl}(a)$ is called the *level* of a. The *height* of $\mathbb{T}$, denoted by $\mathrm{hgt}(\mathbb{T})$, is the highest level of a vertex in $\mathbb{T}$. For any $i \geq 0$ we define the set

$$\mathrm{Level}_{\mathbb{T}}(i) = \{a \in T : \mathrm{lvl}(a) = i\}$$

(dropping the index when $\mathbb{T}$ is known from the context).

The following notion plays a crucial role in Chapter 5.

Definition 1.3. An oriented path $\mathbb{P}$ is *minimal* if it satisfies the following:

(i) $\mathrm{lvl}(\mathrm{init}(\mathbb{P})) = 0$,

(ii) $\mathrm{lvl}(\mathrm{term}(\mathbb{P})) = \mathrm{hgt}(\mathbb{P})$,

(iii) $0 < \mathrm{lvl}(v) < \mathrm{hgt}(\mathbb{P})$ for all $v \in P \setminus \{\mathrm{init}(\mathbb{P}), \mathrm{term}(\mathbb{P})\}$.

For an illustration of the definition see Figure 1.3 below.

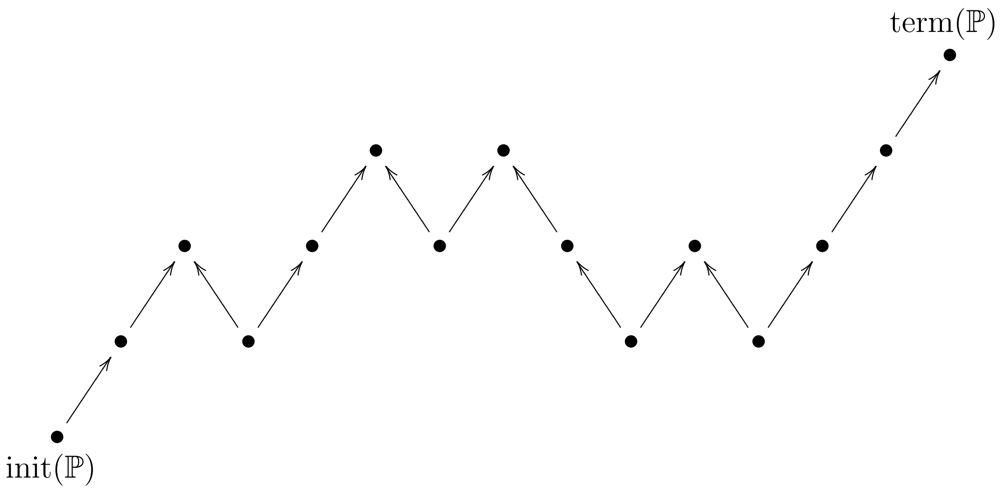


Figure 1.3: A minimal path of height 4.

Lemma 1.4. *Let $\mathbb{P}_1, \dots, \mathbb{P}_n$ be minimal paths of the same height l. There exists a minimal path $\mathbb{Q}$ of height l homomorphic to all the paths $\mathbb{P}_1, \dots, \mathbb{P}_n$.*

Proof. The proof is easy (see [15], Lemma 2.36). □

1.4 Algebras, operations

An r*-ary operation* on a set A is a mapping $f : A^r \to A$. By an *algebra* we mean a structure $\mathbf{A} = \langle A, F \rangle$, where A is a nonvoid set (the *universe* of $\mathbf{A}$) and F, the set of *basic operations* of $\mathbf{A}$, is a set of finitary operations on A (i.e., for each $f \in F$ there exists $r \geq 0$ such that $f : A^r \to A$).

A *term operation* of an algebra $\mathbf{A}$ is any operation which can be obtained by composing basic operations of $\mathbf{A}$ and the *projection operations* (i.e., the operations p_r^i $(r \in \mathbb{N}, 0 \leq i < r)$ satisfying $p_r^i(x_0, \ldots, x_{r-1}) = x_i$).

Throughout the text, we will occasionally mention some notions standard in universal algebra. For definitions and an elaborate treatment see for example [10].

Chapter 2

Constraint satisfaction problem

In this chapter we will give the definition of the Constraint satisfaction problem and provide several examples. Then we will discuss its computational complexity, introducing the famous CSP dichotomy conjecture, and present two algorithmical approaches that solve certain large classes of CSPs in polynomial time. The last section is devoted to the notion of *bounded width* which will play an important role later.

2.1 The definition(s)

As Constraint Satisfaction Problems naturally arise in various fields of mathematics and computer science, there are several equivalent ways to define them. We will use the following so-called "combinatorial" definition of CSP:

Definition 2.1 (**The Constraint Satisfaction Problem).** Let $\mathbb{A}$ be a relational structure. The Constraint satisfaction problem with *template* $\mathbb{A}$, CSP($\mathbb{A}$) for short, is the following decision problem:

INPUT:	A relational structure $\mathbb{X}$ (of the same type as $\mathbb{A}$).
QUESTION:	Is there a homomorphism from $\mathbb{X}$ to $\mathbb{A}$?

In the context of CSP, such a homomorphism is often called a *solution*. The above definition is the most suitable for our purposes, namely for the algebraic approach to CSP.

The term "constraint satisfaction" comes from the following so-called "Variable-Value" definition which originated in computer science, namely in the field of artificial intelligence.

The Variable-Value definition of CSP**.** Let D be a finite set (the *domain*; elements of D are *values*) and Γ a finite collection of relations on D (the *basis*). CSP(D, Γ) is the following decision problem:

INPUT:	V – a finite set of *variables*, $\mathcal{C} = \{C_1, \ldots, C_m\}$ – a finite set of *constraints*; each constraint C_i is a tuple $(\bar{s}_i, R_i)$, where $\bar{s}_i$ is a k_i-tuple of variables (the *scope* of C_i) and $R_i \subseteq D^{k_i}$ is a relation from Γ.
QUESTION:	Is there a *solution*, i.e., $\varphi : V \to D$ such that $\varphi(\bar{s}_i) \in R_i$?

Informally speaking, in the above definition CSP(D, Γ) asks if there exists a way to evaluate the variables without violating any constraints; in each constraint the list of permitted evaluations must come from the basis (which is fixed in advance).

Another approach to define CSP is via mathematical logic. The next definition was motivated by database theory (*conjunctive queries*).

The definition of CSP **via logic.** Let L be a first-order language constisting of finitely many relational symbols and let Γ be an L-structure. Then CSP(Γ) is the following decision problem:

INPUT:	A primitive positive L-sentence φ (i.e., an existentially closed conjunction of predicates).
QUESTION:	Does φ hold in Γ?

It is not hard to prove that the above three definitions are equivalent.

2.2 Examples

In this section we provide a few examples of decision problems which can be formulated as CSPs. For each example we give an idea how to encode the problem into the language of CSP. The list below is by no means complete, the problems were chosen to demonstrate the diversity of the problems expressible as CSPs.

Boolean formula satisfiability

Example (k-SAT). Let k be fixed.

INPUT:	A propositional formula φ in conjunctive normal form such that each clause has at most k literals.
QUESTION:	Is φ satisfiable?

It is easy to construct a structure $\mathbb{A}$ such that the above problem is equivalent to CSP($\mathbb{A}$). Its base set will be $\{0, 1\}$ and for each type of clause it will have one k-ary relation, consisting of just one tuple: the only possible evaluation of that clause.

Solving linear equations over a finite field $\mathbf{F}$

Example (k-SysLinEq). Let k and $\mathbf{F}$ be fixed.

INPUT:	A finite system of linear equations over $\mathbf{F}$ in k variables.
QUESTION:	Is there a solution?

This problem is equivalent to CSP($\mathbb{A}$) for a relational structure $\mathbb{A}$ whose base set is the universe of $\mathbf{F}$ and whose relations are all affine subspaces of $\mathbf{F}^k$.

Graph coloring

Example (k-COL).

INPUT:	A graph $\mathcal{G}$.
QUESTION:	Is there a way of coloring the vertices of $\mathcal{G}$ so that no two adjacent vertices have the same color?

A graph $\mathcal{G}$ is k-colorable if and only if it is homomorphic to $\mathcal{K}_k$ (the *complete graph* on k vertices).

2.3 Complexity of CSP: the dichotomy

The key question in the study of CSPs is their computational complexity. It is easily seen that each CSP is in NP:

Observation. *For each relational structure* $\mathbb{A}$*,* CSP($\mathbb{A}$) *is in NP.*

Proof. Given a relational structure $\mathbb{X}$ (of the same type as $\mathbb{A}$) and a mapping $f : X \to A$, it can be easily verified whether f is a homomorphism in a time polynomial in the size of the encoding of $\mathbb{X}$. $\square$

Notice the following fact, which follows directly from the definition of core. It implies that when investigating CSPs, we can restrict ourselves to core structures.

Observation. *For each relational structure* $\mathbb{A}$*,* CSP($\mathbb{A}$) *is equivalent to* CSP(core($\mathbb{A}$))*. In particular, they have the same computational complexity.*

In their celebrated paper [13], Feder and Vardi formulated the following conjecture which became the most famous open question in the study of CSP (and which is a central theme of this thesis):

The CSP **dichotomy conjecture.** *For each relational structure* $\mathbb{A}$*,* CSP($\mathbb{A}$) *is in P or NP-complete.*

So far, the dichotomy has been confirmed in many special cases: for CSPs over two-element [23] and three-element [5] domains, *conservative* CSPs [7] (i.e., such that the template contains all unary relations, which allows us to restrict possible values for every variable; such problems are also called *list homomorphism problems*) and many more. The known partial results in the case of CSP($\mathbb{H}$) where $\mathbb{H}$ is a digraph (also known as $\mathbb{H}$-coloring problem) will be discused in Chapter 4.

As for the examples from Section 2.2,

- k-SAT is tratable for $k = 1, 2$ and NP-complete else; by a famous result of Cook (1971) and Levin (1973),
- k-SysLinEq is tractable for every k (Gaussian elimination works in polynomial time),
- k-COL is tractable for $k = 1, 2$ and NP-complete else.

There are two main polynomial-time algorithms (or algorithmical approaches) both of which solve large classes of CSPs. One of them generalizes the Gaussian elimination and can be used for CSPs with so-called "few subpowers". A relational structure $\mathbb{A}$ has *few subpowers* if there exists a polynomial $p(x)$ such that the algebra of compatible operations of $\mathbb{A}$ (see Section 3.1) has for each $n > 0$ at most $2^{p(n)}$ subalgebras (see [18] for details). A typical problem solvable by the "few subpowers" algorithm is k-SysLinEq.

The other one, the Local consistency checking algorithm, will be treated in the next section. It is widely believed that all tractable CSPs can be solved by a certain combination of these two algorithms. All tractable CSPs that we encounter in this thesis are, in fact, solvable by the Local consistency checking algorithm.

2.4 Bounded width

Bounded width can be defined in several ways (bounded tree-width duality, solvability in Datalog, pebble games). We will introduce the approach via (k, l)*-strategies.*

Let $\mathbb{A} = \langle A, R_1, \ldots, R_n \rangle$ and $\mathbb{X} = \langle X, S_1, \ldots, S_n \rangle$ be relational structures of the same type and let $L \subseteq X$. A mapping $f : L \to A$ is a *partial homomorphism* from $\mathbb{X}$ to $\mathbb{A}$ if it is a homomorphism from the induced substructure $\mathbb{X}[L]$ to $\mathbb{A}$.

Definition 2.2. Let $k \leq l$ be positive integers. A nonempty family

$$\mathcal{F} = \bigcup_{L \subseteq X, |L| \leq l} \mathcal{F}_L$$

of partial homomorphisms from $\mathbb{X}$ to $\mathbb{A}$ is called a (k, l)-*strategy* for $(\mathbb{X}, \mathbb{A})$ if it satisfies the following:

(S0) $\mathrm{dom}(f) = L$ for each $f \in \mathcal{F}_L$.

(S1) For any $f \in \mathcal{F}_L$ and $K \subseteq L$ the function $f|_K$ belongs to $\mathcal{F}_K$.

(S2) If $K \subseteq L \subseteq X$ with $|K| \leq k$, $|L| \leq l$ and $f \in \mathcal{F}_K$, then there exists $g \in \mathcal{F}_L$ such that $g|_K = f$.

It is easy to see that if there exists a homomorphism from $\mathbb{X}$ to $\mathbb{A}$, then there exists a (k, l)-strategy for $(\mathbb{X}, \mathbb{A})$. $\mathbb{A}$ is said to have *bounded width* if the converse is true for some $k \leq l$:

Definition 2.3. Let $\mathbb{A}$ be a relational structure.

- $\mathbb{A}$ has *width* (k, l) if the following is true: For each $\mathbb{X}$, if there exists a nonempty (k, l)-strategy for $(\mathbb{X}, \mathbb{A})$, then $\mathbb{X}$ is homomorphic to $\mathbb{A}$.
- $\mathbb{A}$ has *bounded width* if it has width (k, l) for some $k \leq l$.
- $\mathbb{A}$ has *width 1* if it has width $(1, k)$ for some $k \geq 1$.

Informally speaking, a (k, l)-strategy is a family of ”locally consistent” partial solutions and $\mathbb{A}$ has width (k, l) if we can recover a solution from each (k, l)-strategy targeting $\mathbb{A}$.

The notion of width (k, l) (bounded width, width 1) cannot distinguish between homomorphically equivalent structures; if there exists a nonempty (k, l)-strategy for $(\mathbb{X}, \mathbb{A})$ and $\mathbb{A}$ is homomorphic to $\mathbb{B}$, then there also exists a nonempty (k, l)-strategy for $(\mathbb{X}, \mathbb{B})$. We will use this fact for cores:

Observation. *Let $\mathbb{A}$ be a relational structure. $\mathbb{A}$ has width (k, l) if and only if* $\mathrm{core}(\mathbb{A})$ *has width (k, l).*

We will now introduce the Local consistency checking algorithm, which solves CSPs of bounded width in polynomial time. Let $\mathbb{A}$ be a relational structure of width (k, l). The idea is simple: Take all partial homomorphisms from $\mathbb{X}$ to $\mathbb{A}$ with at most l-element domain. Then throw away one by one those which falsify conditions (S1) or (S2). We end up with the biggest (k, l)-strategy, which is nonempty if and only if there exists a homomorphism from $\mathbb{X}$ to $\mathbb{A}$.

The (k, l)-consistency checking algorithm

Input:	A structure $\mathbb{X}$ of the same type as $\mathbb{A}$.
Initial step:	For each $L \subseteq X, \|L\| \leq l$ let $\mathcal{F}_L :=$ all partial homomorphisms from $\mathbb{X}$ to $\mathbb{A}$ with domain L.
Iteration step:	If there exist $f \in \mathcal{F}$ falsifying (S1) or (S2), remove f from $\mathcal{F}$.
Output:	If $\mathcal{F} = \emptyset$, return NO, else return YES.

Lemma 2.4. *The (k, l)-consistency checking algorithm runs in polynomial time.*

Proof. We have at most $|X|^l \cdot |A|^l = O(|X|^l)$ partial mappings. In the initial step we verify for each of them if it is a partial homomorphism; this can be done in polynomial time. The number of iterations is $O(|X|^l)$; as in each iteration we remove one partial homomorphism from $\mathcal{F}$. In every iteration step we simply go through $\mathcal{F}$ and for each $f \in \mathcal{F}$ check if (S1) and (S2) holds; this is tractable as well. □

More about bounded width and Local consistency checking can be found in [13], [21] and [2].

Chapter 3

Algebraic approach to CSP

In this chapter we introduce the "algebraic approach" to the Constraint Satisfaction Problem and present the algebraic tools which will be used later in the text. At the core of the algebraic approach to CSP lies the concept of *compatible operation*.

3.1 Compatible operations

In this section we define the notion of *compatible operation* (*polymorphism*). Note that the unary operations compatible with a relational structure $\mathbb{A}$ are precisely the endomorphisms $\mathbb{A} \to \mathbb{A}$. Recall that by an *$r$-ary operation* on a set A we wean a mapping $A^r \to A$.

Definition 3.1. Let R be a k-ary relation and f an r-ary operation on a set A. We say that f is *compatible* with R if whenever $\langle a_{1i}, \ldots, a_{ki} \rangle \in R$ for $i = 1, \ldots, r$ we have $\langle f(a_{11}, \ldots, a_{1r}), \ldots, f(a_{k1}, \ldots, a_{kr}) \rangle \in R$.

The above condition means that if we arrange elements of A into a matrix such that its columns are tuples from R and apply f on the rows of that matrix, the resulting column must belong to R as well.

Definition 3.2. Let $\mathbb{A} = (A, R_1, \ldots, R_n)$ be a relational structure and let f be an operation on A. We say that f is *compatible* with $\mathbb{A}$ (or f is a *polymorphism* of $\mathbb{A}$) if it is compatible with all the relations R_i, $i = 1, \ldots, n$.

Remark. Using the language of universal algebra, the above definition can be formulated as follows: an operation f is *compatible* with $\mathbb{A}$ if for each i, R_i is a subalgebra of the algebra $\langle A, f \rangle^{\mathrm{ar}(R_i)}$.

For digraphs the definition is somewhat simpler (see the diagram below the definition):

Definition 3.3. Let $\mathbb{H} = (H, E)$ be a digraph. An r-ary operation f on H is *compatible* with $\mathbb{H}$ if whenever $a_i \xrightarrow{\mathbb{H}} b_i$ for $i = 1, \ldots, r$ we have $f(a_1, \ldots, a_r) \xrightarrow{\mathbb{H}} f(b_1, \ldots, b_r)$.

$$\begin{array}{cccccc} f(a_1 & a_2 & \ldots & a_r) & = & a \\ \downarrow & \downarrow & & \downarrow & \Longrightarrow & \downarrow \\ f(b_1 & b_2 & \ldots & b_r) & = & b \end{array}$$

The fact crucial to the algebraic approach to CSP is that the computational complexity of CSP($\mathbb{A}$) is fully determined by the polymorphisms of $\mathbb{A}$ (up to log-space reductions). See [19], [8] and [20] for more details.

3.2 Weak near-unanimity

In this section we introduce some ”nice” polymorphisms connected to the complexity of CSP.

Definition 3.4. An r-ary operation f on a set A is *idempotent* if it satisfies $f(a, a, \ldots, a) = a$ for all $a \in A$.

- Let $r \geq 2$. An r-ary operation ω on A is called a *weak near-unanimity* operation (or a *weak-NU*), if it is idempotent and satisfies

  $$\omega(a, \ldots, a, b) = \omega(a, \ldots, a, b, a) = \cdots = \omega(b, a, \ldots, a)$$

  for all $a, b \in A$. We define the binary operation $\circ_\omega$ by setting

  $$a \circ_\omega b = \omega(a, \ldots, a, b).$$

- A weak-NU ν of arity ≥ 3 is called a *near-unanimity* operation (*NU*), if $a \circ_\nu b = a$ for all $a, b \in A$. A ternary NU is called a *majority* operation.

- An r-ary operation τ is *totally symmetric idempotent* (*TSI*), if it is idempotent and satisfies

  $$\tau(a_1, a_2, \ldots, a_r) = \tau(a'_1, a'_2, \ldots, a'_r)$$

  whenever $\{a_1, a_2, \ldots, a_r\} = \{a'_1, a'_2, \ldots, a'_r\}$. (Note that a totally symmetric idempotent operation is a weak-NU.)

Remark. It can be easily seen that an operation obtained by composing operations compatible with $\mathbb{A}$ is also compatible with $\mathbb{A}$. In particular, if ω is a weak-NU operation compatible with $\mathbb{A}$, then $\circ_\omega$ is also compatible with $\mathbb{A}$, as we can obtain it by composing ω with the projection operations (which are indeed compatible with $\mathbb{A}$).

3.3 Algebraic tools

There are many theorems connecting the computational complexity of CSP($\mathbb{A}$) with the existence or non-existence of certain types of polymorphisms. In this section we present a few of them; those that are to be used later in the text. As the algebraic approach to CSP is vividly evolving, there is no comprehensive list of such theorems. The survey [9] might serve as a good starting point.

Our tool to prove NP-completeness of CSPs is the following theorem, a combination of a result of Bulatov, Jeavons and Krokhin from [8] and a result of Maróti and McKenzie [22].

Theorem 3.5. *Let $\mathbb{A}$ be a relational structure. If* core($\mathbb{A}$) *admits no compatible weak-NU operations, then* CSP($\mathbb{A}$) *is NP-complete.*

The algebraic dichotomy conjecture – a strengthening of the conjecture of Feder and Vardi – states that the converse is also true. It can be formulated as follows:

The algebraic dichotomy conjecture. *Let $\mathbb{A}$ be a core relational structure. If $\mathbb{A}$ admits a compatible weak-NU operation, then* CSP($\mathbb{A}$) *is tractable, otherwise it is NP-complete.*

The "Bounded width conjecture" of Larose and Zádori which was recently proved by Barto and Kozik [2] states that a core relational structure $\mathbb{A}$ has bounded width if and only if the algebra of polymorphisms of $\mathbb{A}$ generates a congruence meet semi-distributive variety. By a result of Maróti and McKenzie from [22], the latter holds for an algebra if and only if it has weak-NU terms of almost all arities. Hence the following theorem from [2], which is our main tool to establish tractability of CSPs:

Theorem 3.6. *Let $\mathbb{A}$ be a core relational structure. The following conditions are equivalent:*

(i) A has bounded width.

(ii) A admits compatible weak-NU operations of almost all arities (i.e., there exists r_0 *such that for all* $r \geq r_0$ $\mathbb{A}$ *admits a compatible* r*-ary weak-NU).*

For a finer description of the tractable CSPs, we will use the following characterization of structures of width 1 by Dalmau and Pearson [11]:

Theorem 3.7. *Let* $\mathbb{A}$ *be a core relational structure. The following conditions are equivalent:*

(i) $\mathbb{A}$ *has width 1.*

(ii) $\mathbb{A}$ *admits compatible totally symmetric idempotent operations of all arities.*

Admitting a near-unanimity polymorphism also ensures tractability, the proof of the following theorem can be found in [13].

Theorem 3.8. *Let* $\mathbb{A}$ *be a relational structure. If* $\mathbb{A}$ *admits an* r*-ary compatible near-unanimity, then* $\mathbb{A}$ *has width* $(r, r+1)$.

Chapter 4

CSP and digraphs

In this chapter we will focus on Constraint Satisfaction Problems such that the template relational structure is a digraph. In graph theory, such problems are called $\mathbb{H}$-*coloring* problems and their computational complexity has been extensively studied since 1970s. There are several reasons to restrict to digraphs:

- Digraphs provide a good test field for hypotheses in CSP and an inspiration for ideas which can be usually generalized. They are much easier to deal with than general relational structures (one can "draw pictures") while preserving the difficulty and diversity of general CSPs.
- $\mathbb{H}$-coloring has the same "computational power" as the general CSP: Each relational structure can be encoded into a digraph so that the corresponding CSPs are polynomially equivalent (see [13] for proof). Therefore, proving the CSP dichotomy conjecture for digraphs would imply the general case.
- The $\mathbb{H}$-coloring itself is an interesting problem in combinatorics and theoretical computer science.

4.1 $\mathbb{H}$-coloring problem

Definition 4.1 (The $\mathbb{H}$-coloring problem). Let $\mathbb{H}$ be a digraph. The $\mathbb{H}$-*coloring problem* is the problem CSP($\mathbb{H}$), i.e., the following decision problem:

INPUT:	A digraph $\mathbb{G}$.
QUESTION:	Is there a homomorphism from $\mathbb{G}$ to $\mathbb{H}$?

The following reduction was proved by Feder and Vardi [13]:

Proposition 4.2. *For each relational structure* $\mathbb{A}$ *there exists a digraph* $\mathbb{H}$ *such that the problems* $\mathrm{CSP}(\mathbb{A})$ *and* $\mathrm{CSP}(\mathbb{H})$ *are polynomially equivalent.*

The above proposition implies that the CSP dichotomy conjecture is equivalent to the following:

Conjecture. *For each digraph* $\mathbb{H}$*,* $\mathrm{CSP}(\mathbb{H})$ *is tractable or NP-complete.*

Recall that for a digraph $\mathbb{H}$, $\mathrm{CSP}(\mathbb{H}) = \mathrm{CSP}(\mathrm{core}(\mathbb{H}))$; and so when studying CSP we can restrict ourselves to core digraphs.

4.2 Known results

The CSP dichotomy conjecture has been confirmed for a number of classes of digraphs so far. In this section we will mention some of the most important results.

Oriented paths

All oriented paths are tractable (see [14]). Using the algebraic approach, the proof is quite easy:

Proposition 4.3. *Every oriented path has width 1; and thus is tractable.*

Proof. Let $\mathbb{P} = (P, E)$ be an oriented path. For $r \geq 1$ we define an r-ary operation τ_r on P by setting $\tau_r(a_1, \ldots, a_r)$ to be the vertex from $\{a_1, \ldots, a_r\}$ with minimal distance from the initial vertex of $\mathbb{P}$. It is easy to see that τ_r is a totally symmetric idempotent operation compatible with $\mathbb{P}$. The rest follows by Theorem 3.7. $\square$

Oriented cycles

The dichotomy for oriented cycles was proved by Feder in [12]. Each oriented cycle is either NP-complete or has bounded width.

Undirected graphs

The dichotomy for undirected graphs was established by Hell and Nešetřil in [16]: an undirected graph is tractable if and only if it is bipartite; otherwise it is NP-complete. In [6], Bulatov reproved their result using algebraic methods and confirmed that it agrees with the algebraic version of the dichotomy conjecture.

Smooth digraphs

A digraph is *smooth* if it does not have any sources or sinks. In [4], Barto, Kozik and Niven confirmed the CSP dichotomy for smooth digraphs, generalizing the above result of Hell and Nešetřil. The core of their proof is the following theorem:

Theorem 4.4. *Let $\mathbb{G}$ be a smooth digraph. If $\mathbb{G}$ admits a compatible weak-NU operation, then the core of $\mathbb{G}$ is a disjoint union of circles.*

A disjoint uninon of circles is tractable; it is not hard to see that it admits a compatible majority operation. The dichotomy for smooth digraphs now follows from Theorem 3.5.

4.3 CSP and oriented trees

In the class of all digraphs, oriented trees are in some sense very far from smooth digraphs; therefore once the dichotomy for smooth digraphs was proved it was logical to direct attention to oriented trees. Except the oriented paths (which are "too easy"), the simplest class of oriented trees are the *triads* (i.e., oriented trees with one vertex of degree 3 and all other vertices of degree 1 or 2). Even for triads the dichotomy remains an unsolved problem.

In [17], Hell, Nešetřil and Zhu identified a certain subclass of triads, which they called *special triads*, possessing enough structure to deal with at least some examples. They were able to construct an NP-complete special triad (with 45 vertices; at that time the smallest known NP-complete oriented tree). Using the most up to date algebraic machinery, Barto, Kozik, Maróti and Niven [3] confirmed the CSP dichotomy conjecture for special triads. They proved that for each special triad $\mathbb{T}$, one of the following is true (the definition of special triad can be found in Chapter 5):

- $\mathbb{T}$ admits a compatible majority operation,
- $\mathbb{T}$ admits compatible TSI operations of all arities,
- $\mathbb{T}$ admits no compatible weak-NU operation.

Moreover, they provided a structural description of the above three cases. And, as a by-product, an NP-complete special triad with 39 vertices which is very likely to be the smallest NP-complete oriented tree.

The rest of this thesis is devoted to a generalization of the above result to *special polyads* and problems related to them. A *polyad* is an oriented tree with at most one vertex of degree greater than 2. Special polyads are a straightforward generalization of special triads.

Part II

Special polyads

Chapter 5

Special polyads: the dichotomy

In this chapter we investigate *special polyads*, a certain class of oriented trees generalizing special triads treated in [3]. We establish the CSP dichotomy conjecture for special polyads, proving that every special polyad is either NP-complete or has bounded width. Moreover, we characterize special polyads of width 1 as those whose core admits a compatible binary idempotent commutative operation.

5.1 The definition

We start with the definition of the *special polyad*. An oriented tree is called a *polyad* if at most one of its vertices has degree greater than 2.

Definition 5.1. (i) By a *half-branch* we mean a minimal path, the *root* of the half-branch $\mathbb{P}$ is its initial vertex.

(ii) Let $\mathbb{P}$ and $\mathbb{P}'$ be two disjoint minimal paths of the same height. The *branch* $\langle \mathbb{P}, \mathbb{P}' \rangle$ is the oriented tree obtained by identifying the terminal vertices of $\mathbb{P}$ and $\mathbb{P}'$ into a single vertex. The *root* of the branch $\langle \mathbb{P}, \mathbb{P}' \rangle$ is the initial vertex of $\mathbb{P}$.

(iii) Let n, k be nonnegative integers, $n + k > 0$ and let $\langle \mathbb{P}_i, \mathbb{P}'_i \rangle$ $(1 \leq i \leq n)$ and $\mathbb{P}_{n+i}$ $(1 \leq i \leq k)$ be n branches and k half-branches of the same height (pairwise disjoint). The *special polyad* given by $\langle \mathbb{P}_1, \mathbb{P}'_1 \rangle, \dots, \mathbb{P}_{n+k}$ is the oriented tree $\mathbb{T}$ obtained by identifying the roots of $\langle \mathbb{P}_1, \mathbb{P}'_1 \rangle, \dots, \mathbb{P}_{n+k}$ into a single vertex, the *root*.

In the following, we will denote the root of $\mathbb{T}$ by 0, the initial vertex of $\mathbb{P}'_i$ by i and the top-level vertex of $\langle \mathbb{P}_i, \mathbb{P}'_i \rangle$ or $\mathbb{P}_i$ by $\widehat{i}$ (see the figure below, arrows indicate "direction" of paths). Let us also define

$$\begin{aligned}
\mathrm{Base}_{\mathbb{T}} &= \mathrm{Level}_{\mathbb{T}}(0) = \{0, 1, \dots, n\}, \\
\mathrm{Top}_{\mathbb{T}} &= \mathrm{Level}_{\mathbb{T}}(\mathrm{hgt}(\mathbb{T})) = \{\widehat{1}, \dots, \widehat{n+k}\} \\
\mathrm{Half}_{\mathbb{T}} &= \{\widehat{n+1}, \dots, \widehat{n+k}\}
\end{aligned}$$

and

$$\mathrm{Paths}_{\mathbb{T}} = \{\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_{n+k}, \mathbb{P}'_1, \mathbb{P}'_2, \dots, \mathbb{P}'_n\}$$

(we will usually drop the index $\mathbb{T}$).

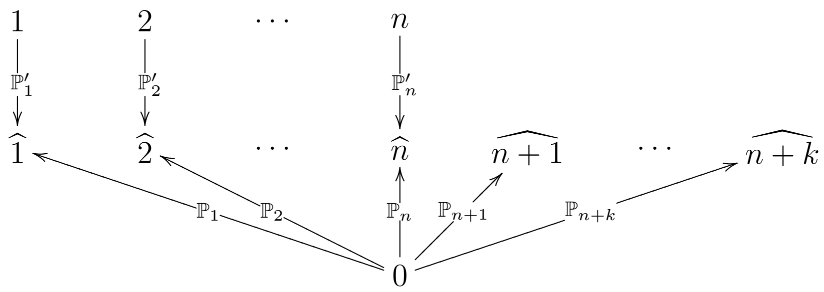


Figure 5.1: A special polyad.

In our terminology, a special triad from [3] is a special polyad with 3 branches and no half-branches.

5.2 The dichotomy theorem

The following theorem is the main result of this thesis:

Theorem 5.2. *For every special polyad* $\mathbb{T}$*,* $\mathrm{CSP}(\mathbb{T})$ *is either NP-complete or tractable. More specifically, let* $\mathbb{T}'$ *be the core of* $\mathbb{T}$*.*

(i) $\mathbb{T}$ *has bounded width, if and only if* $\mathbb{T}'$ *admits a compatible weak near-unanimity operation, otherwise* $\mathbb{T}$ *is NP-complete.*

(ii) $\mathbb{T}$ *has width* 1*, if and only if* $\mathbb{T}'$ *admits a compatible binary weak-NU (i.e., a binary idempotent commutative operation).*

Corollary 5.3. *The CSP dichotomy conjecture holds for special polyads.*

In order to prove Theorem 5.2, we will need several lemmata.

5.3 Preliminary results

In the following, for a positive integer n, let $[n] = \{1, \dots, n\}$.

First, we will reduce the problem to core special polyads. In the next two easy lemmata we prove that the core of a special polyad is still a special polyad and inherits its "nice" polymorphisms.

Lemma 5.4. *Let* $\mathbb{T}$ *be a special polyad with* n *branches and* k *half-branches. Then* $\mathrm{core}(\mathbb{T})$ *is a special polyad with* n' *branches and* k' *half-branches, where* $n' \leq n$ *and* $k' \leq k$*.*

Proof. It is easily seen that a homomorphism from a minimal path of height l to an oriented tree of height l maps the initial vertex to a vertex of level 0 and the terminal vertex to a vertex of level l. The rest follows directly from this fact. $\square$

Lemma 5.5. *Let* $\mathbb{H}$ *be a digraph. If* $\mathbb{H}$ *has a compatible* r*-ary weak-NU* ω*, then there exists an* r*-ary weak-NU* ω' *compatible with* $\mathrm{core}(\mathbb{H})$ *such that if* ω *is a NU, then* ω' *is also a NU and if* ω *is TSI, then* ω' *is also TSI.*

Proof. Let $f : \mathbb{H} \to \mathrm{core}(\mathbb{H})$ and $g : \mathrm{core}(\mathbb{H}) \to \mathbb{H}$ be homomorphisms. Then the homomorphism $f \circ g : \mathrm{core}(\mathbb{H}) \to \mathrm{core}(\mathbb{H})$ is bijective and since $\mathrm{core}(H)$ is finite, there exists $k > 0$ such that $(f \circ g)^k = \mathrm{id}_{\mathrm{core}(\mathbb{H})}$. For $\bar{x} \in \mathrm{core}(H)^r$ we define $\omega'(\bar{x}) = (f \circ (g \circ f)^{k-1})(\omega(g(x_1), \dots, g(x_r)))$. The rest is easy. $\square$

In the rest of this section we show that if an oriented tree $\mathbb{T}$ has a compatible partial weak-NU, NU or TSI operation defined for the tuples of vertices of the same level, it can be easily extended to a full weak-NU, NU or TSI operation, respectively.

Let A be any set and $K \subseteq A^r$. By a *partial* r*-ary operation* on a set A with *domain* K we mean a mapping $f : K \to A$. We define *partial weak-NU*, *partial NU* and *partial TSI* in an obvious fashion, restricting the conditions required in Definition 3.4 to tuples from the domain. The notion of compatibility generalizes to partial operations similarly:

Definition 5.6. Let $\mathbb{H} = (H, E)$ be a digraph and let f be a partial r-ary operation on H with domain K. We say that f is *compatible with* $\mathbb{H}$ if it satisfies the following condition: if $\bar{a}, \bar{b} \in K$ and $a_i \xrightarrow{\mathbb{H}} b_i$ for $i = 1, \dots, r$, then $f(\bar{a}) \xrightarrow{\mathbb{H}} f(\bar{b})$.

Lemma 5.7. *Let* $\mathbb{T}$ *be an oriented tree.*

(i) Each partial weak-NU operation compatible with $\mathbb{T}$ with domain $\bigcup_{k=0}^{\mathrm{hgt}\,\mathbb{T}} \mathrm{Level}(k)^r$ (i.e., tuples of vertices of the same level) can be extended to a weak-NU $\omega' \supseteq \omega$ compatible with $\mathbb{T}$ in such a way that if ω is a partial NU, then ω' is a NU.

(ii) Each partial TSI τ_r compatible with $\mathbb{T}$ with domain $\bigcup_{k=0}^{\mathrm{hgt}\,\mathbb{T}} \mathrm{Level}(k)^r$ can be extended to a TSI operation $\tau'_r \supseteq \tau_r$ compatible with $\mathbb{T}$.

Proof. To prove (i), we define ω' as follows (let $\bar{a} \in T^r$):

(1) If all the vertices a_i have the same level, then we put $\omega'(\bar{a}) = \omega(\bar{a})$.

(2) If there exists $i \in [r]$ such that $\mathrm{lvl}(a_j) = k$ for all $j \neq i$ and $\mathrm{lvl}(a_i) \neq k$, then

  (2a) if $r = 2$, we define $\omega'(a_1, a_2) = a_1$ if $\mathrm{lvl}(a_1) < \mathrm{lvl}(a_2)$ and $\omega'(a_1, a_2) = a_2$ else,

  (2b) if $r \geq 3$, we define $\omega'(\bar{a}) = a_2$ if $i = 1$ and $\omega'(\bar{a}) = a_1$ else.

(3) In all other cases we put $\omega'(\bar{a}) = a_1$.

First, we will prove that ω' is a weak-NU. We want to prove that for any $a, b \in T$, $\omega'(a, \dots, a, b) = \omega'(a, \dots, a, b, a) = \cdots = \omega'(b, a, \dots, a)$. Clearly, for all of these tuples the same case of the definition applies. In case (1) the equalities hold because ω is a weak-NU, while in case (2) the result is independent on the coordinate at which the 'b' occurs. Moreover, $a \circ_{\omega'} b = a$ in case (2b); and so ω' is a NU whenever ω is a partial NU.

To prove compatibility, choose $\bar{a}, \bar{b} \in T^r$ such that $a_i \to b_i$ for each i. The same case of the definition applies for both $\omega'(\bar{a})$ and $\omega'(\bar{b})$. From the compatibility of ω' (case (1)) and the fact that $a_i \to b_i$ (cases (2) and (3)) it follows that $\omega'(\bar{a}) \to \omega'(\bar{b})$ and (i) is proved.

In order to prove (ii), for $\bar{a} \in T^r$ let $a_{i_1}, \dots, a_{i_k}$ ($i_1 < \cdots < i_k$) be the vertices of minimal level among $\{a_1, \dots, a_r\}$. We define

$$\tau'_r(\bar{a}) = \tau_r(a_{i_1}, \dots, a_{i_k}, \underbrace{a_{i_k}, \dots, a_{i_k}}_{(r-k)\text{-times}}).$$

It is easy to check that τ'_r is TSI. The compatibility of τ'_r follows immediately from the compatibility of τ_r. □

5.4 Reduction to $\mathcal{A}(\mathbb{T})$

Let $\mathbb{T}$ be a special polyad. In this section we translate the question if $\mathbb{T}$ has a compatible r-ary weak-NU, NU or TSI operations of all arities into a question whether there exists a weak-NU, NU or TSI operations of all arities compatible with a certain family $\mathcal{A}(\mathbb{T})$ of digraphs on the set $\mathrm{Base} \cup \mathrm{Top}$. This translation significantly simplifies the proof of Theorem 5.2 and also allows us to construct special polyads with some desired properties such as the one in Section 6.2.

Definition 5.8.

(i) Let $\mathcal{I} \subseteq \mathrm{Paths}$ be nonempty. We define $\bigotimes_{\mathbb{S}\in\mathcal{I}} \mathbb{S}$ to be the component of connectivity of the digraph $\prod_{\mathbb{S}\in\mathcal{I}} \mathbb{S}$ containing the tuple $\langle \mathrm{init}(\mathbb{S}) : \mathbb{S} \in \mathcal{I}\rangle$. (Note that $\bigotimes$ is, up to isomorphism, associative and commutative.)

(ii) Let us denote by $\mathcal{R}$ the mapping from the set $\mathcal{P}(\mathrm{Paths})$ (the *power set* of Paths) to itself defined by

$$\mathcal{R}(\mathcal{I}) = \{\mathbb{P} \in \mathrm{Paths} : \bigotimes_{\mathbb{S}\in\mathcal{I}} \mathbb{S} \to \mathbb{P}\}$$

for $\mathcal{I} \neq \emptyset$; we put $\mathcal{R}(\emptyset) = \emptyset$.

We will need the following easy lemma.

Lemma 5.9. *Let* $\mathcal{I} = \{\mathbb{S}_1, \dots, \mathbb{S}_r\} \subseteq \mathrm{Paths}$ *be nonempty. Then the tuple of terminal vertices* $\langle \mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r)\rangle$ *belongs to* $\bigotimes_{i=1}^r \mathbb{S}_i$ *and any homomorphism* $\psi : \bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{T}$ *maps the tuple* $\langle \mathrm{init}(\mathbb{S}_1), \dots, \mathrm{init}(\mathbb{S}_r)\rangle$ *to a vertex of level* 0 *and* $\langle \mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r)\rangle$ *to a vertex of level* $\mathrm{hgt}(\mathbb{T})$*; the image of* $\bigotimes_{i=1}^r \mathbb{S}_i$ *under* ψ *is a minimal path from* Paths.

Proof. Let $\mathbb{Q}$ be a minimal path (of height $\mathrm{hgt}(\mathbb{T})$) homomorphic to all the paths $\mathbb{S}_1, \dots, \mathbb{S}_r$ via $\varphi_1, \dots, \varphi_r$, respectively. Consider the natural homomorphism $\varphi : \mathbb{Q} \to \prod_{i=1}^r \mathbb{S}_i$ defined by $\varphi(\bar{x}) = \langle \varphi_1(x_1), \dots, \varphi_r(x_r)\rangle$. Since $\mathbb{Q}$ is connected, it follows that $\varphi : \mathbb{Q} \to \bigotimes_{i=1}^r \mathbb{S}_i$; thus $\varphi(\mathrm{term}(\mathbb{Q})) = \langle \mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r)\rangle \in \bigotimes_{i=1}^r \mathbb{S}_i$. The homomorphism $\psi \circ \varphi : \mathbb{Q} \to \mathbb{T}$ maps $\mathbb{Q}$ onto a minimal path $\mathbb{P} \in \mathrm{Paths}$. Thus $\psi(\mathrm{init}(\mathbb{S}_1), \dots, \mathrm{init}(\mathbb{S}_r)) = (\psi \circ \varphi)(\mathrm{init}(\mathbb{Q})) = \mathrm{init}\,\mathbb{P}$ has level 0 and $\psi(\mathrm{term}(\mathbb{S}_1), \dots, \mathrm{term}(\mathbb{S}_r))$ has level $\mathrm{hgt}(\mathbb{T})$. The rest is obvious. $\square$

In the following lemma we prove that $\mathcal{R}$ is a *closure operator* on the set Paths.

Lemma 5.10. *The following statements hold:*

(i) $\mathcal{I} \subseteq \mathcal{R}(\mathcal{I})$ *for any* $\mathcal{I} \subseteq \text{Paths}$. *(extensivity)*

(ii) If $\mathcal{I} \subseteq \mathcal{J} \subseteq \text{Paths}$, *then* $\mathcal{R}(\mathcal{I}) \subseteq \mathcal{R}(\mathcal{J})$. *(monotonicity)*

(iii) $\mathcal{R}(\mathcal{R}(\mathcal{I})) = \mathcal{R}(\mathcal{I})$ *for all* $\mathcal{I} \subseteq \text{Paths}$. *(idempotency)*

Proof. In the following, let $\mathcal{I} = \{\mathbb{S}_1, \dots, \mathbb{S}_r\}$. The projection homomorphisms $\pi_j(\bar{x}) = x_j$ witness $\bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{S}_j$ for all j and (i) is proved.

To prove (ii), let $\mathbb{P} \in \mathcal{R}(\mathcal{I})$, $\varphi : \bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{P}$. By (i), for each i there exists a (projection) homomorphism $\pi_{\mathbb{S}_i} : \bigotimes_{\mathbb{S} \in \mathcal{J}} \mathbb{S} \to \mathbb{S}_i$. The mapping $\psi : \bigotimes_{\mathbb{S} \in \mathcal{J}} \mathbb{S} \to \mathbb{P}$ defined by $\psi(\bar{x}) = \varphi(\pi_{\mathbb{S}_1}(\bar{x}), \dots, \pi_{\mathbb{S}_r}(\bar{x}))$ is a homomorphism witnessing $\mathbb{P} \in \mathcal{R}(\mathcal{J})$.

It remains to prove (iii). The inclusion $\mathcal{R}(\mathcal{R}(\mathcal{I})) \supseteq \mathcal{R}(\mathcal{I})$ follows from (i). Let $\mathbb{P} \in \mathcal{R}(\mathcal{R}(\mathcal{I}))$ and let $\varphi : \bigotimes_{\mathbb{S} \in \mathcal{R}(\mathcal{I})} \mathbb{S} \to \mathbb{P}$. For each $\mathbb{S} \in \mathcal{R}(\mathcal{I})$ there exists a homomorphism $\varphi_{\mathbb{S}} : \bigotimes_{i=1}^r \mathbb{S}_i \to \mathbb{S}$. Similarly as before the composition $\psi(\bar{x}) = \varphi(\langle \varphi_{\mathbb{S}}(\bar{x}) : \mathbb{S} \in \mathcal{R}(\mathcal{I}) \rangle)$ is a homomorphism from $\bigotimes_{i=1}^r \mathbb{S}_i$ to $\mathbb{P}$, and the proof is finished. $\square$

Now we are ready to define the family $\mathcal{A}(\mathbb{T})$.

Definition 5.11. For any $\mathcal{I} \subseteq \text{Paths}$, let $\mathbb{T}(\mathcal{I})$ be the digraph on the set $\text{Base} \cup \text{Top}$ defined by the following condition:

$$a \xrightarrow{\mathbb{T}(\mathcal{I})} b \ \text{ iff } \ a \text{ is connected to } b \text{ via } \mathbb{P} \text{ for some } \mathbb{P} \in \mathcal{R}(\mathcal{I}).$$

Let us denote by $\mathcal{A}(\mathbb{T})$ the family of digraphs $\mathcal{A}(\mathbb{T}) = \{\mathbb{T}(\mathcal{I}) : \mathcal{I} \subseteq \text{Paths}\}$. We say that an operation on the set $\text{Base} \cup \text{Top}$ is *compatible with* $\mathcal{A}(\mathbb{T})$, if it is compatible with all the digraphs $\mathbb{T}(\mathcal{I}) \in \mathcal{A}(\mathbb{T})$.

Below is a figure of the digraph $\mathbb{T}(\text{Paths})$. From Lemma 5.10 it follows that all digraphs from $\mathcal{A}(\mathbb{T})$ are subgraphs of this digraph.

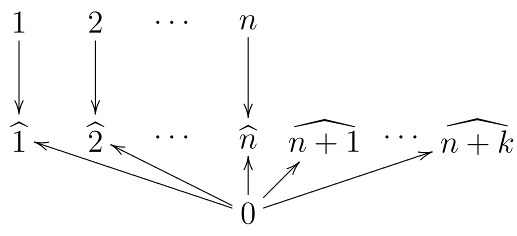


Figure 5.2: The digraph $\mathbb{T}(\text{Paths})$.

The following immediate corollary summarizes the connection between $\mathcal{R}$ and compatible operations of $\mathbb{T}$.

Corollary 5.12. *Let f be an r-ary operation compatible with $\mathbb{T}$ and $\mathcal{I} \subseteq$ Paths. If $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i$ for all $i = 1, \ldots, r$, then*

$$f(\bar{a}) \xrightarrow{\mathbb{T}(\mathcal{I})} f(\bar{b}).$$

Finally, we conclude this section with the "reduction" lemma, which allows us to look for compatible weak-NUs on $\mathcal{A}(\mathbb{T})$, a family of quite simple digraphs, instead of $\mathbb{T}$.

Lemma 5.13. *Let $\mathbb{T}$ be a special polyad. The following statements hold:*

(i) $\mathbb{T}$ admits an r-ary compatible weak-NU, if and only if $\mathcal{A}(\mathbb{T})$ admits an r-ary compatible weak-NU.

(ii) $\mathbb{T}$ admits an r-ary compatible NU, if and only if $\mathcal{A}(\mathbb{T})$ admits an r-ary compatible NU.

(iii) $\mathbb{T}$ admits an r-ary compatible TSI, if and only if $\mathcal{A}(\mathbb{T})$ admits an r-ary compatible TSI.

Proof. For an r-ary operation f compatible with $\mathbb{T}$, let f' be the restriction of f to the domain $\mathrm{Base}^r \cup \mathrm{Top}^r$. Choose arbitrary $\mathcal{I} \subseteq$ Paths, $\bar{a} \in \mathrm{Base}^r$ and $\bar{b} \in \mathrm{Top}^r$ such that $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i\,(1 \leq i \leq r)$. From the previous corollary it follows that the partial operation f' is compatible with $\mathcal{A}(\mathbb{T})$. The first implications now follow from Lemma 5.7 (which can be easily generalized to compatibility with a family of oriented trees on a set), as the properties of being weak-NU, NU or TSI are preserved by restriction.

It remains to prove the converse implications. For each $\mathcal{I} \subseteq$ Paths we fix an arbitrary $\mathbb{S}_{\mathcal{I}} \in \mathcal{I}$ and whenever $\bigotimes_{\mathbb{S}\in\mathcal{I}} \mathbb{S}$ is homomorphic to $\mathbb{P} \in$ Paths, we fix a homomorphism $\varphi_{\mathcal{I},\mathbb{P}} : \bigotimes_{\mathbb{S}\in\mathcal{I}} \mathbb{S} \to \mathbb{P}$ in such a way that if $\mathbb{P} \in \mathcal{I}$, then $\varphi_{\mathcal{I},\mathbb{P}}$ is the projection homomorphism.

To prove the converse implications of (i) and (ii), let ω' be an r-ary weak-NU compatible with $\mathcal{A}(\mathbb{T})$. We will define a partial operation ω on $\mathbb{T}$ with domain $\bigcup_{k=0}^{\mathrm{hgt}\,\mathbb{T}} \mathrm{Level}(k)^r$. Let $\bar{a} \in \mathrm{Level}(k)^r$. For $k \notin \{0, \mathrm{hgt}(\mathbb{T})\}$, let $\mathbb{S}_i \in$ Paths be such that $a_i \in \mathbb{S}_i$ and denote the set $\{\mathbb{S}_1, \ldots, \mathbb{S}_r\}$ by $\mathcal{I}$. For each i let a_i' be the vertex from $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i$ *second closest* to $\mathrm{init}(\mathbb{S}_i)$. (To be precise, if $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i = \{a_i\}$, then $a_i' = a_i$, else if a_j is the vertex from $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i$ with minimal distance from $\mathrm{init}(\mathbb{S}_i)$, then we define a_i' to be the vertex from $\{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_r\} \cap \mathbb{S}_i$ with minimal distance from $\mathrm{init}(\mathbb{S}_i)$. This is needed to ensure the NU property, i.e., that $a \circ_\omega b = a$, in the case that $a, b \in \mathbb{P}$ for some $\mathbb{P} \in$ Paths and b is closer to init(Paths) than a.)

(1) If $k = 0$ or $k = \mathrm{hgt}(\mathbb{T})$, we put $\omega(\bar{a}) = \omega'(\bar{a})$.

(2) Else, if $\bar{a} \in \bigotimes_{i=1}^{r} \mathbb{S}_i$, let $\mathbb{P} \in \mathrm{Paths}$ be the minimal path connecting $\omega'(\langle \mathrm{init}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$ to $\omega'(\langle \mathrm{term}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$. We put $\omega(\bar{a}) = \varphi_{\mathcal{I},\mathbb{P}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle)$.

(3) If $\bar{a} \notin \bigotimes_{i=1}^{r} \mathbb{S}_i$, then

  (3a) if $r \geq 3$ and there exist $i, j \in [r]$ such that $\{a_l : l \neq j\} \subseteq \mathbb{S}_i$, we put $\omega(\bar{a}) = a'_i$.

  (3b) if $r = 2$, we put $\omega(a_1, a_2) = a'_1$ if $\mathbb{S}_{\mathcal{I}} = \mathbb{S}_1$ and $\omega(a_1, a_2) = a'_2$ else.

  (3c) In all other cases we define $\omega(\bar{a}) = a_1$.

It is straightforward to verify that ω is a weak-NU and that if ω' is a NU, then ω is also a NU. To prove compatibility, choose any $\bar{a} \in \mathrm{Level}(k)^r$ and $\bar{b} \in \mathrm{Level}(k+1)^r$ such that $a_i \xrightarrow{\mathbb{T}} b_i$, $i = 1, \ldots, r$. We can assume that $\mathrm{hgt}(\mathbb{T}) > 1$ (otherwise $\omega = \omega'$). If $\omega(\bar{a})$ is defined by (1), then $\omega(\bar{b})$ is defined by (2). It is easily seen that in this case $\bar{b} = \bar{b}'$ and $\omega(\bar{a}) = \varphi_{\mathcal{I},\mathbb{P}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle) \xrightarrow{\mathbb{T}} \varphi_{\mathcal{I},\mathbb{P}}(\langle b'_i : \mathbb{S}_i \in \mathcal{I} \rangle) = \omega(\bar{b})$ follows from the fact that $\varphi_{\mathcal{I},\mathbb{P}}$ is a homomorphism. The proof is analogous for the case when $\omega(\bar{b})$ is defined by (1). Now assume that neither $\omega(\bar{a})$ nor $\omega(\bar{b})$ are defined by (1). In this situation, both $\omega(\bar{a})$ and $\omega(\bar{b})$ fall into the same case of the definition. Observe that $a'_i \to b'_i$, $i = 1, \ldots, r$, and the set $\mathcal{I}$ is the same for both $\bar{a}$ and $\bar{b}$. Now $\omega(\bar{a}) \xrightarrow{\mathbb{T}} \omega(\bar{b})$ follows from the fact that $\varphi_{\mathcal{I},\mathbb{P}}$ (case (2)) and projections (cases (3a)-(3c)) are homomorphisms. We extend ω using Lemma 5.7 and the proof of (i) and (ii) is finished.

To prove the converse implication of (iii) we slightly modify the construction. Assume that $\mathcal{A}(\mathbb{T})$ admits r-ary compatible TSI τ'_r. Similarly as before, we will construct a partial TSI operation τ_r compatible with $\mathbb{T}$ with domain $\bigcup_{k=0}^{\mathrm{hgt}\,\mathbb{T}} \mathrm{Level}(k)^r$. Let $\bar{a} \in \mathrm{Level}(k)^r$. For $k \notin \{0, \mathrm{hgt}(\mathbb{T})\}$, let $\mathbb{S}_i \in \mathrm{Paths}$ be such that $a_i \in \mathbb{S}_i$ and denote the set $\{\mathbb{S}_1, \ldots, \mathbb{S}_r\}$ by $\mathcal{I}$. For each i let a'_i be the vertex from $\{a_1, \ldots, a_r\} \cap \mathbb{S}_i$ with minimal distance from $\mathrm{init}(\mathbb{S}_i)$.

(1) If $k = 0$ or $k = \mathrm{hgt}(\mathbb{T})$, we put $\tau_r(\bar{a}) = \tau'_r(\bar{a})$.

(2) Else, if $\bar{a} \in \bigotimes_{i=1}^{r} \mathbb{S}_i$, let $\mathbb{P} \in \mathrm{Paths}$ be the minimal path connecting $\tau'_r(\langle \mathrm{init}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$ to $\tau'_r(\langle \mathrm{term}(\mathbb{S}_i) : 1 \leq i \leq r \rangle)$. We put $\tau_r(\bar{a}) = \varphi_{\mathcal{I},\mathbb{P}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle)$.

(3) If $\bar{a} \notin \bigotimes_{i=1}^{r} \mathbb{S}_i$, then $\tau_r(\bar{a}) = a'_i$, where i is such that $\mathbb{S}_i = \mathbb{S}_{\mathcal{I}}$.

It is not hard to verify that τ_r is a TSI operation, just note that if $\{a_1, \ldots, a_r\} = \{b_1, \ldots, b_r\}$, then the set $\mathcal{I}$ and the paths $\mathbb{P}$ (case (2)) and $\mathbb{S}_{\mathcal{I}}$ (case (3)) are the same for both $\bar{a}$ and $\bar{b}$. The argumentation to verify compatibility is similar as before. We conclude the proof by extending τ_r using Lemma 5.7. $\square$

5.5 $\mathcal{A}(\mathbb{T})$ and compatible weak-NUs

In this section we will be constructing operations compatible with $\mathcal{A}(\mathbb{T})$. The main goal is to prove that if $\mathcal{A}(\mathbb{T})$ admits a compatible r-ary weak-NU, then it admits a compatible $(r+1)$-ary weak-NU.

Lemma 5.14. *If $\mathcal{A}(\mathbb{T})$ admits a compatible binary weak-NU (i.e., a commutative idempotent operation), then $\mathcal{A}(\mathbb{T})$ admits compatible TSI operations of all arities.*

Proof. Let $\star$ be a binary weak-NU compatible with $\mathcal{A}(\mathbb{T})$. First, we will prove that the following holds:

$$(\exists z \in \mathrm{Base})(\forall a \in \mathrm{Base}, a \neq z)\ a \star 0 = 0.$$

Let $z, z' \in \mathrm{Base}$ be such that $z \star 0 \neq 0$, $z' \star 0 \neq 0$. Since $\star$ is compatible with the digraph $\mathbb{T}(\mathrm{Paths})$ in which $a \to \widehat{a}$ and $0 \to \widehat{a}$ for all $a \neq 0$, it follows that $a \star 0 \to \widehat{a} \star \widehat{a} = \widehat{a}$; and so $a \star 0 \in \{0, a\}$ for all $a \in \mathrm{Base}$. Therefore $z \star 0 = z$ and $z' \star 0 = z'$. But as $z \star 0 \to \widehat{z} \star \widehat{z'}$ and $z' \star 0 = 0 \star z' \to \widehat{z} \star \widehat{z'}$ in $\mathbb{T}(\mathrm{Paths})$, we conclude that $z = z'$.

Now fix $z \in \mathrm{Base}$ with the above property. We will define a partial order on the set $\mathrm{Base} \cup \mathrm{Top}$ and then use $\star$ to "compare the incomparable" elemets. For all $\widehat{a} \in \mathrm{Top}$, $\widehat{a} \neq \widehat{z}$ we put $z \prec \widehat{z} \prec 0 \prec \widehat{a}$ and if $\widehat{a} \notin \mathrm{Half}$, then also $\widehat{a} \prec a$. We define $\preceq$ to be the partial order generated by these relations. Let us fix an arbitrary linear order $\leq$ on the set $\mathrm{Top} \setminus \{\widehat{z}\}$. (We can assume without loss of generality that $z = 1$ and $\mathrm{Top} \setminus \{\widehat{z}\} = \{\widehat{2} < \widehat{3} < \cdots < \widehat{n+k}\}$.)

For each $i > 0$ we denote by t_i the i-ary operation defined in the following way (note that all these operations are compatible with $\mathcal{A}(\mathbb{T})$):

$$\begin{aligned}
t_1(x) &= x, \\
t_2(x_1, x_2) &= x_1 \star x_2, \\
&\vdots \\
t_i(x_1, \ldots, x_i) &= t_{i-1}(x_1, \ldots, x_{i-1}) \star x_i.
\end{aligned}$$

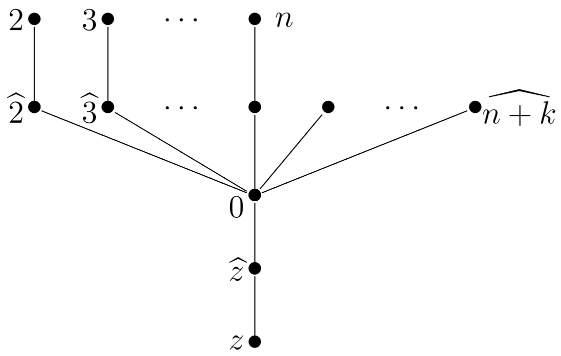


Figure 5.3: The partial order $\preceq$.

For each $\widehat{c} \in \mathrm{Top}$ we define the set $R(\widehat{c})$ as follows: we put $R(\widehat{c}) = \{\widehat{c}\}$ if $\widehat{c} \in \mathrm{Half}$ and $R(\widehat{c}) = \{\widehat{c}, c\}$ else.

Now we are ready to define the TSI operations. For each $r \geq 1$, we define an r-ary operation τ_r in the following way: For any $\bar{a} \in (\mathrm{Base} \cup \mathrm{Top})^r$ let $S(\bar{a})$ be the smallest subset of $\mathrm{Base} \cup \mathrm{Top}$ containing $\{a_1, \ldots, a_r\}$ and closed under the operation $\star$ (i.e., $c \star c' \in S(\bar{a})$ whenever $c, c' \in S(\bar{a})$).

(1) If $S(\bar{a})$ has the least element with respect to $\preceq$, we define $\tau_r(\bar{a})$ to be that element,

(2) else let $\{\widehat{c_1} < \widehat{c_2} < \cdots < \widehat{c_m}\}$ be the set of all $\widehat{c} \in \mathrm{Top} \setminus \{\widehat{z}\}$ such that $S(\bar{a}) \cap R(\widehat{c}) \neq \emptyset$. Note that $m \geq 2$. For $i = 1, \ldots, m$ we denote by a'_i the $\preceq$-least element of $S(\bar{a}) \cap R(\widehat{c_i})$. Finally, we put $\tau_r(\bar{a}) = t_m(a'_1, a'_2, \ldots, a'_m)$.

It is easy to check that τ_r is totally symmetric and idempotent. To verify compatibility, choose $\mathcal{I} \subseteq \mathrm{Paths}$, $\bar{a} \in \mathrm{Base}^r$ and $\bar{b} \in \mathrm{Top}^r$ such that $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i$, $i = 1, \ldots, r$. If $\tau_r(\bar{a})$ and $\tau_r(\bar{b})$ are defined by the same case, then it is not hard to see that $\tau_r(\bar{a}) \xrightarrow{\mathbb{T}(\mathcal{I})} \tau_r(\bar{b})$.If $\bar{a}$ falls into case (2), then so does $\tau_r(\bar{b})$. Thus it only remains to investigate the case when $\tau_r(\bar{a})$ is defined by (1) and $\tau_r(\bar{b})$ by (2). In this case, we have that $\tau(\bar{a}) = 0$ and $\tau(\bar{b}) = t_m(\widehat{c_1}, \ldots, \widehat{c_m})$ for some $m \geq 2$ and $\widehat{c_i} \in \mathrm{Top} \setminus \{\widehat{z}\}$.

For each i, let $c'_i \in S(\bar{a})$ be $\preceq$-minimal such that $c'_i \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c_i}$ ($c'_i = 0$ if $\widehat{c_i} \in \mathrm{Half}$ and $c'_i \in \{0, c_i\}$ else.) Since $0 \in S(\bar{a})$, there exists j such that $c'_j = 0$. We will prove that $t_m(c'_1, \ldots, c'_m) = 0$. Then the proof will be concluded, as we will have that

$$\tau_r(\bar{a}) = 0 = t_m(c'_1, \ldots, c'_m) \xrightarrow{\mathbb{T}(\mathcal{I})} t_m(\widehat{c}_1, \ldots, \widehat{c}_m) = \tau_r(\bar{b}).$$

Since the $\preceq$-least element of $S(\bar{a})$ is 0 and $S(\bar{a})$ is closed under $\star$, it follows that $t_{j-1}(c'_1, \ldots, c'_{j-1}) \neq z$; and so $t_j(c'_1, \ldots, c'_{j-1}, c'_j) = t_{j-1}(c'_1, \ldots, c'_{j-1}) \star 0 = 0$. Now we have that

$$t_{j+1}(c'_1, \ldots, c'_{j+1}) = t_j(c'_1, \ldots, c'_j) \star c'_{j+1} = 0 \star c'_{j+1}$$

and since $c'_{j+1} \neq z$, it follows that $t_{j+1}(c'_1, \ldots, c'_{j+1}) = 0$. We can proceed by induction, proving that $t_m(c'_1, \ldots, c'_m) = 0$. $\square$

The following lemma plays a key role in our proof of Theorem 5.2.

Lemma 5.15. *If $\mathcal{A}(\mathbb{T})$ admits an r-ary weak-NU ω, then it admits an $(r+1)$-ary weak-NU ω'.*

Proof. First, let us consider the case when there exists $z \in \text{Base}, z \neq 0$ such that $0 \circ_\omega z = z$. We will prove that then $\mathcal{A}(\mathbb{T})$ admits a binary idempotent commutative operation $\star$; and thus by Lemma 5.14 also an $(r+1)$-ary weak-NU (even totally symmetric) operation.

Let $\preceq$, $\leq$ and $R(\widehat{c})$, $\widehat{c} \in \text{Top}$ be the same as in the proof of Lemma 5.14. We will define $\star$ for $\langle a, b\rangle \in \text{Base}^2 \cup \text{Top}^2$ and then extend it using Lemma 5.7.

(1) If $a \preceq b$, then we put $a \star b = b \star a = a$ and if $b \preceq a$, we put $a \star b = b \star a = b$.

(2) If a and b are $\preceq$-incomparable, then $a \in R(\widehat{c})$ and $b \in R(\widehat{d})$ for some $\widehat{c} \neq \widehat{d} \in \text{Top} \setminus \{\widehat{z}\}$. We define $a \star b = b \star a = a$ if $\widehat{c} < \widehat{d}$ and $a \star b = b \star a = b$ else.

From the compatibility of $\circ_\omega$ with $\mathbb{T}(\text{Paths})$ we get that $\widehat{c} \circ_\omega \widehat{z} = \widehat{z}$ for all $\widehat{c} \in \text{Top}$. Since $c \circ_\omega 0 \to \widehat{c} \circ_\omega \widehat{z} = \widehat{z}$ and

$$0 \circ_\omega c = \omega(c, 0, \ldots, 0, 0) \to \omega(\widehat{c}, \widehat{c}, \ldots, \widehat{c}, \widehat{z}) = \widehat{c} \circ_\omega \widehat{z} = \widehat{z}$$

in $\mathbb{T}(\text{Paths})$, we conclude that $0 \circ_\omega c = c \circ_\omega 0 = 0$ for all $\widehat{c} \in \text{Top}, \widehat{c} \neq \widehat{z}$. Now it is not hard to prove that $\star$ is an idempotent commutative operation compatible with $\mathcal{A}(\mathbb{T})$, we leave the verification to the reader.

Second, we consider the case when ω satisfies

$$(\forall a \in \text{Base})\ 0 \circ_\omega a = 0.$$

We may assume that for all $a, b \in \text{Base} \setminus \{0\}$, if $\widehat{a} \circ_\omega \widehat{b} = \widehat{a}$, then $a \circ_\omega b = a$; otherwise we can "redefine" ω to satisfy the desired property, i.e., replace ω with the operation ω^* defined by

$$\omega^*(\bar{x}) = \begin{cases} a & \text{if } \bar{x} \in \{\langle a, \ldots, a, b\rangle, \langle a, \ldots, a, b, a\rangle, \ldots, \langle b, a, \ldots, a\rangle\} \\ & \text{for some } a, b \in \text{Base} \setminus \{0\} \text{ such that } \widehat{a} \circ_\omega \widehat{b} = \widehat{a}, \\ \omega(\bar{x}) & \text{else.} \end{cases}$$

It is easy to see that ω^* is also an r-ary weak-NU compatible with $\mathcal{A}(\mathbb{T})$ satisfying $(\forall a \in \mathrm{Base})\ 0 \circ_\omega^* a = 0$.

Let us define the set $\mathrm{Maj} = \{a \in \mathrm{Base} : a \circ_\omega 0 = a\}$. We will prove the following:

$$(\forall a \in \mathrm{Maj})(\forall b \in \mathrm{Base})\ a \circ_\omega b = a.$$

For $a = 0$ the claim follows from the assumptions and for $b = 0$ from the definition of Maj. Let $a, b \neq 0$. Since $\circ_\omega$ is compatible with $\mathbb{T}(\mathrm{Paths})$ and $a \circ_\omega 0 = a$, it follows that $\widehat{a} \circ_\omega \widehat{b} = \widehat{a}$. Hence $a \circ_\omega b = a$ and the claim is proved.

We will define $\omega'(\bar{a})$ for $\bar{a} = \langle a_1, \ldots, a_{r+1} \rangle \in \mathrm{Base}^{r+1} \cup \mathrm{Top}^{r+1}$ and then apply Lemma 5.7.

(1) If $\bar{a} = \langle a, \ldots, a, b \rangle$ for some $a, b \in \mathrm{Base}$, $a \notin \mathrm{Maj}$, we put $\omega'(\bar{a}) = a \circ_\omega b$, and if $\bar{a} = \langle \widehat{a}, \ldots, \widehat{a}, \widehat{b} \rangle$ for some $\widehat{a}, \widehat{b} \in \mathrm{Top}$, $a \notin \mathrm{Maj}$, we put $\omega'(\bar{a}) = \widehat{a} \circ_\omega \widehat{b}$,

(2) else we define $\omega'(\bar{a}) = \omega(a_1, \ldots, a_r)$.

To prove that ω' is a weak-NU, choose $a, b \in \mathrm{Base}$. For $\widehat{a}, \widehat{b} \in \mathrm{Top}$ we can proceed analogously. If $a \in \mathrm{Maj}$, then case (2) applies. We have that $\omega'(b, a, \ldots, a) = \cdots = \omega'(a, \ldots, a, b, a) = a \circ_\omega b = a$, while $\omega'(a, \ldots, a, b) = \omega(a, \ldots, a) = a$. Now suppose that $a \notin \mathrm{Maj}$. In that case $\omega'(a, \ldots, a, b) = a \circ_\omega b$ by (1) and $\omega'(a, \ldots, a, b, a) = \cdots = \omega'(b, a, \ldots, a) = a \circ_\omega b$ by (2); and so the weak-NU property is verified.

To verify compatibility, choose $\mathcal{I} \subseteq \mathrm{Paths}$, $\bar{a} \in \mathrm{Base}^{r+1}$ and $\bar{b} \in \mathrm{Top}^{r+1}$ such that $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} b_i$, $i = 1, \ldots, r+1$. If $\omega'(\bar{a})$ and $\omega'(\bar{b})$ are defined by the same case, then $\omega'(\bar{a}) \xrightarrow{\mathbb{T}(\mathcal{I})} \omega'(\bar{b})$ follows from the compatibility of $\circ_\omega$ in case (1) and ω in case (2). If $\bar{a}$ falls into case (1), then so does $\bar{b}$. The only remaining case is when $\omega'(\bar{a})$ is defined by (2) and $\omega'(\bar{b})$ by (1). In this situation we have that $\bar{b} = \langle \widehat{c}, \ldots, \widehat{c}, \widehat{d} \rangle$ for some $\widehat{c}, \widehat{d} \in \mathrm{Top}$, $c \notin \mathrm{Maj}$ and $\omega'(\bar{b}) = \widehat{c} \circ_\omega \widehat{d}$. Since $a_i \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}$ for $i = 1, \ldots, r$, we get $\omega'(\bar{a}) = \omega(a_1, \ldots, a_r) \xrightarrow{\mathbb{T}(\mathcal{I})} \omega(\widehat{c}, \ldots, \widehat{c}) = \widehat{c}$; and so $\omega(a_1, \ldots, a_r) \in \{0, c\}$. We also know that $0 \in \{a_1, \ldots, a_r\}$, as otherwise case (1) would apply for $\bar{a}$.

First, let $\omega(a_1, \ldots, a_r) = 0$. Since $0 \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}$ and $a_{r+1} \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{d}$, from the compatibility of $\circ_\omega$ we obtain

$$\omega'(\bar{a}) = \omega(a_1, \ldots, a_r) = 0 = 0 \circ_\omega a_{r+1} \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c} \circ_\omega \widehat{d} = \omega'(\bar{b}),$$

proving the compatibility condition for ω' in this case.

Second, assume that $\omega(a_1, \ldots, a_r) = c$. Notice that $c \in \{a_1, \ldots, a_r\}$ (as $\omega(0, \ldots, 0) = 0$), implying that $c \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c}$. We will prove that $\widehat{c} \circ_\omega \widehat{d} = \widehat{c}$. Then it will follow that

$$\omega'(\bar{a}) = \omega(a_1, \ldots, a_r) = c \xrightarrow{\mathbb{T}(\mathcal{I})} \widehat{c} = \widehat{c} \circ_\omega \widehat{d} = \omega'(\bar{b}),$$

which will conclude the proof. Let $j \in [r]$ be such that $a_j = 0$. In the digraph $\mathbb{T}(\text{Paths})$ we have $a_j \to \widehat{d}$ and $a_i \to \widehat{c}$ for all $i = 1, \ldots, r$. Therefore

$$c = \omega(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_r) \to \omega(\widehat{c}, \ldots, \widehat{c}, \widehat{d}, \widehat{c}, \ldots, \widehat{c}) = \widehat{c} \circ_\omega \widehat{d}.$$

Hence $\widehat{c} \circ_\omega \widehat{d} = \widehat{c}$ and the proof is finished. □

5.6 Q.E.D

Finally, everything is set to prove the dichotomy theorem.

Proof of Theorem 5.2. Let $\mathbb{T}$ be a special polyad and let $\mathbb{T}'$ be its core. By Lemma 5.4, $\mathbb{T}'$ is also a special polyad.

(i) If $\mathbb{T}'$ admits no compatible weak-NUs, then CSP($\mathbb{T}$) is NP-complete by Theorem 3.5. By Theorem 3.6 and the "reduction" Lemma 5.13, it is enough to prove that if $\mathcal{A}(\mathbb{T}')$ admits a weak-NU of arity r_0, then $\mathcal{A}(\mathbb{T}')$ admits weak-NUs of all arities $r \geq r_0$. But the latter fact follows by induction from Lemma 5.15.

(ii) By Lemma 5.14 (and Lemma 5.13), $\mathbb{T}'$ admits a binary weak-NU, if and only if it admits TSI operations of all arities. The rest follows from Theorem 3.7. □

Chapter 6

Constructing special polyads

Using the techniques developed for the proof of the dichotomy in the previous chapter (namely $\mathcal{A}(\mathbb{T})$ and the ”reduction” from Lemma 5.13), we present a method of constructing special polyads with certain desired properties. We apply this method to obtain an interesting special polyad: a core special polyad which has bounded width, but not width 1 and which does not admit any near-unaninimty polymorphism (implying that the variety generated by the algebra of its polymorphisms is not congruence distributive); such case did not occur in special triads.

6.1 From $\mathcal{A}(\mathbb{T})$ back to $\mathbb{T}$

Our aim in this section is to provide a characterization of families of digraphs $\mathcal{A}$ for which we can construct a special polyad $\mathbb{T}$ such that $\mathcal{A} = \mathcal{A}(\mathbb{T})$. We start with the definition of *closure system*.

Definition 6.1. By a *closure system* on a finite set A we mean a family $\mathcal{C} \subseteq \mathcal{P}(A)$ of subsets of A such that

(i) $A \in \mathcal{C}$,

(ii) if $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2 \in \mathcal{C}$.

The sets $C \in \mathcal{C}$ are called $\mathcal{C}$*-closed sets*.

Let $\mathcal{D}$ be a closure system on a finite set B. We say that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there exists a bijection $f : A \to B$ such that $\mathcal{D} = \{f[C] : C \in \mathcal{C}\}$.

Closure systems can be in a natural way identified with closure operators. The following definition is essentially just a reformulation of Definition 5.8 (ii):

Definition 6.2. Let Paths $= \{\mathbb{P}_1, \ldots, \mathbb{P}_n\}$ be a finite set of minimal paths of the same height. We define the closure system $\mathcal{R}_{\bigotimes}^{\text{Paths}}$ on Paths in the following way: let the $\mathcal{R}_{\bigotimes}^{\text{Paths}}$-closed sets be precisely the empty set and the nonempty sets $\mathcal{I} \subseteq$ Paths such that

$$\mathcal{I} = \{\mathbb{P} \in \text{Paths} : \bigotimes_{\mathbb{S} \in \mathcal{I}} \mathbb{S} \to \mathbb{P}\}.$$

It is easy to check that $\mathcal{R}_{\bigotimes}^{\text{Paths}}$ is indeed a closure system. The following proposition states that each closure system on a finite set (such that the empty set is closed) is isomorphic to $\mathcal{R}_{\bigotimes}^{\text{Paths}}$ for some set of minimal paths.

Proposition 6.3. *Let $\mathcal{C}$ be a closure system on $[n]$, $\emptyset \in \mathcal{C}$. There exists a set* Paths $= \{\mathbb{P}_1, \ldots, \mathbb{P}_n\}$ *of minimal paths of the same height such that for each $I \subseteq [n]$,*

$$I \in \mathcal{C} \iff \{\mathbb{P}_i : i \in I\} \in \mathcal{R}_{\bigotimes}^{\text{Paths}}.$$

Proof. Let us fix an arbitrary linear order of the nontrivial $\mathcal{C}$-closed sets (i.e., $\mathcal{C} \setminus \{\emptyset, [n]\}$), say $\mathcal{C} = \{\emptyset, C_1, \ldots, C_q, [n]\}$. By an *arrow* we mean a digraph with a single edge $a \to b$ (and possibly some other discrete vertices); a *zig-zag* is a digraph with just three edges $a \to b, c \to b, c \to d$ (see the figure below).

Figure 6.1: An arrow and a zig-zag.

We say that a minimal path $\mathbb{P}$ has an *arrow at level k* if $\mathbb{P}[\text{Level}_{\mathbb{P}}(k) \cup \text{Level}_{\mathbb{P}}(k+1)]$ (the subgraph induced by vertices of level k or $k+1$) is an arrow; if it is a zig-zag, then $\mathbb{P}$ has a *zig-zag at level k*. It is an easy excercise to prove the following claim:

Claim. Let l be a positive integer and for $I \subseteq [l]$ let $\mathbb{P}_I$ denote the minimal path of height $l+2$ which has zig-zag's at levels $i \in I$ and arrows at levels $j \in \{0, \ldots, l+1\} \setminus I$. For any $I_1, \ldots, I_m \subseteq [l]$ the core of $\bigotimes_{i=1}^m \mathbb{P}_{I_i}$ is isomorphic to $\mathbb{P}_{I_1 \cup \cdots \cup I_m}$.

The above claim is the key to our construction: For $i \in [n]$, let $\mathbb{P}_i$ be the minimal path of height $q+2$ (uniquely) determined by the following conditions:

(i) $\mathbb{P}_i$ has an arrow at level 0,

(ii) for $k = 1, \dots, q$, $\mathbb{P}_i$ has an arrow at level k if $i \in C_k$ and a zig-zag at level k else,

(iii) $\mathbb{P}_i$ has an arrow at level $q + 1$.

To demonstrate the construction, consider the following example: let $n = 3$, $q = 3$, $C_1 = \{1\}$, $C_2 = \{1, 2\}$, $C_3 = \{1, 3\}$. The minimal paths $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ are depicted in Figure 6.2.

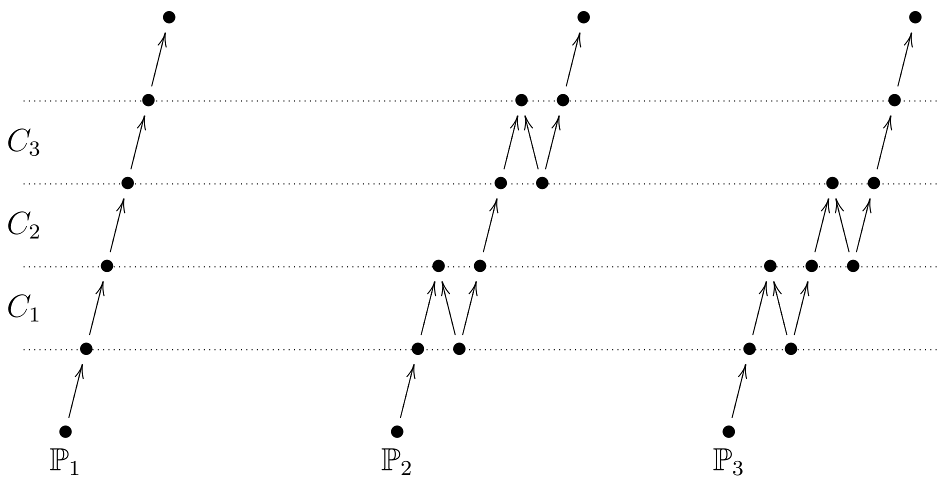


Figure 6.2: The resulting minimal paths.

The above claim implies that for all nonempty $I \subseteq [n]$ and $j \in [n]$, $\bigotimes_{i \in I} \mathbb{P}_i \to \mathbb{P}_j$, if and only if for all $C \in \mathcal{C}$ such that $j \notin C$ there exists $i \in I$ with $i \notin C$. Equivalently,

$$\bigotimes_{i \in I} \mathbb{P}_i \to \mathbb{P}_j \iff (\forall C \in \mathcal{C})\,(I \subseteq C \to j \in C).$$

Now, choose arbitrary nonempty $I \subseteq [n]$. Let $D = \bigcap\{C \in \mathcal{C} : I \subseteq C\}$ be the minimal (w.r.t. inclusion) $\mathcal{C}$-closed set containg I. From the above we get that

$$\bigotimes_{i \in I} \mathbb{P}_i \to \mathbb{P}_j \iff j \in D.$$

Thus $I \in \mathcal{C}$ (i.e., $I = D$), if and only if $\{\mathbb{P}_i : i \in I\}$ is $\mathcal{R}_{\otimes}^{\text{Paths}}$-closed. $\square$

Remark. The above construction of minimal paths was chosen for its simplicity, it is by no means optimal regarding the number of vertices of the resulting paths.

We conclude this section with an easy corollary of the above proposition; a key to the construction below.

Corollary 6.4. *Let $\mathcal{A}$ be a family of digraphs on the same vertex set H. The following are equivalent:*

(i) $\mathcal{A} = \mathcal{A}(\mathbb{T})$ *for some special polyad* $\mathbb{T}$*,*

(ii) There exists a special polyad $\mathbb{H} = (H, E)$ *of height 1 such that* $(H, \emptyset) \in \mathcal{A}$ *and the edge relations of members of* $\mathcal{A}$ *form a closure system on* E*.*

Moreover, if (ii) holds and $(H, \{e\}) \in \mathcal{A}$ *for all* $e \in E$*, then* $\mathbb{T}$ *is a core.*

Proof. (i) $\Rightarrow$ (ii): For a special polyad $\mathbb{T}$, $\mathcal{A} = \mathcal{A}(\mathbb{T})$ clearly satisfies (ii). (Note that $\mathbb{T}(\mathrm{Paths}_{\mathbb{T}})$ is a special polyad of height 1).

(ii) $\Rightarrow$ (i): Label the edges of $\mathbb{H}$ with positive integers $1, \ldots, n$ and use the previous proposition to construct the minimal paths $\mathbb{P}_i$. For $i = 1, \ldots, n$, replace the edge i with the minimal path $\mathbb{P}_i$. The resulting digraph $\mathbb{T}$ is a special polyad such that $\mathcal{A} = \mathcal{A}(\mathbb{T})$.

The rest follows from the fact that if $\mathbb{T}$ is not a core, then $\mathbb{P} \to \mathbb{P}'$ for some $\mathbb{P}, \mathbb{P}' \in \mathrm{Paths}_{\mathbb{T}}$. □

6.2 An interesting special polyad

Finally, in this section we construct a special polyad satisfying the following:

Proposition 6.5. *There exists a core special polyad* $\mathbb{T}$ *having the following properties:*

(i) $\mathrm{CSP}(\mathbb{T})$ *is tractable,*

(ii) $\mathbb{T}$ *does not have width 1,*

(iii) $\mathbb{T}$ *does not admit any compatible near-unanimity operation.*

In order to construct such a special polyad, we will first introduce some notation. Let $\mathbb{H} = (H, E)$ be a special polyad of height 1 with 4 branches with the vertices and edges labeled as in Figure 6.3 below.

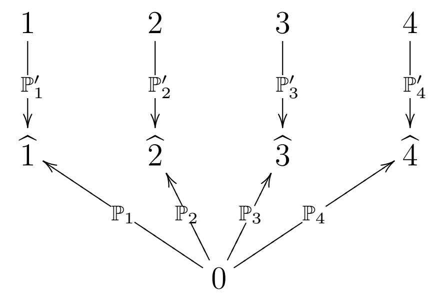


Figure 6.3: The special polyad $\mathbb{H}$ of height 1.

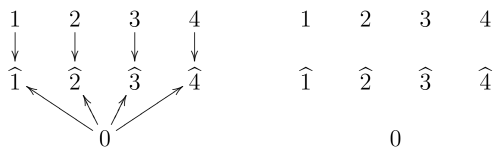
For $J \subseteq [4]$, we denote the set $\{j' : j \in J\}$ by J'. For $I, J \subseteq [4]$, we define $\mathbb{H}_I^{J'}$ to be the subgraph of $\mathbb{H}$ with vertex set H and edges $\{\mathbb{P}_i : i \in I\} \cup \{\mathbb{P}'_j : j \in J\}$.

We define the family $\mathcal{A}$ of subgraphs of $\mathbb{H}$ in the following way:

$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3,$$

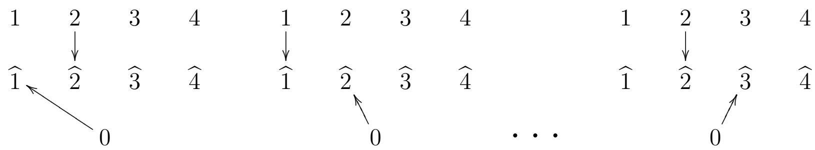
where

- $\mathcal{A}_0 = \{\mathbb{H}, \mathbb{H}_\emptyset^\emptyset\}$,

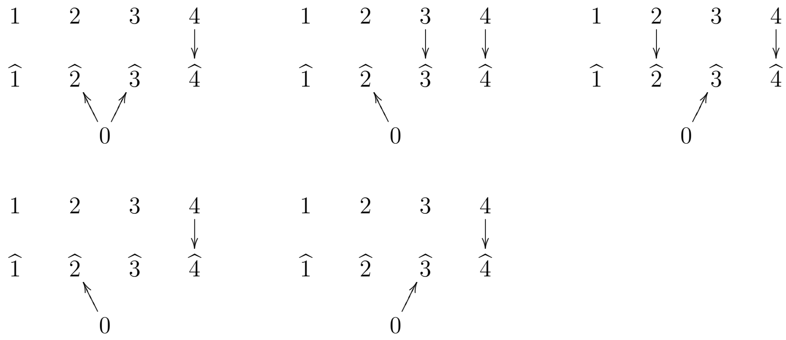


- $\mathcal{A}_1 = \{\mathbb{H}_i^\emptyset : i \in [4]\} \cup \{\mathbb{H}_\emptyset^{i'} : i \in [4]\}$,

- $\mathcal{A}_2 = \{\mathbb{H}_i^{j'} : i, j \in [3], i \neq j\}$,



- $\mathcal{A}_3 = \{\mathbb{H}_{2,3}^{4'}, \mathbb{H}_2^{3',4'}, \mathbb{H}_3^{2',4'}, \mathbb{H}_2^{4'}, \mathbb{H}_3^{4'}\}$.



It can be easily seen that the edge relations of the members of $\mathcal{A}$ form a closure system. The rest of the proof follows:

Proof of Proposition 6.5. By Corollary 6.4, there exists a core special polyad $\mathbb{T}$ such that $\mathcal{A} = \mathcal{A}(\mathbb{T})$. In the following, we use Theorem 5.2 and the "reduction" Lemma 5.13.

(i) It is enough to prove that $\mathcal{A}$ admits a compatible weak near-unanimity operation. We will define a 4-ary weak-NU ω on the set H. Let $\bar{x} \in \{0, 1, 2, 3, 4\}^4$.

(1) If $4 \notin \{x_1, x_2, x_3\}$, then

  (1.1) if $\{x_1, x_2, x_3\} = \{1, 2, 3\}$, we put $\omega(\bar{x}) = 1$

  (1.2) else x_1, x_2, x_3 lie on an oriented path in $\mathbb{H}$; we define $\omega(\bar{x})$ to be the middle vertex from x_1, x_2, x_3 on this path.

(2) If $4 \in \{x_1, x_2, x_3\}$, then

  (2.1) if $\bar{x} = \langle 4, 4, 4, 4\rangle$, we put $\omega(\bar{x}) = 4$

  (2.2) else $\omega(\bar{x}) = x_i$ where i is smallest such that $x_i \neq 4$.

For $\widehat{\bar{x}} \in \widehat{[4]}^4$ we put $\omega(\widehat{\bar{x}}) = \widehat{\omega(\bar{x})}$. Finally, we extend ω using Lemma 5.7. It can be easily verified that ω is a weak-NU. (In fact, ω restricted to $H \setminus \{4, \widehat{4}\}$ is a near-unanimity.)

Compatibility with $\mathcal{A}_0$ is trivial and compatibility with $\mathcal{A}_1$ follows from the idempotency of ω. Let $\bar{x} \in (\{0\} \cup [4])^4, \bar{y} \in [4]^4$. To prove compatibility with $\mathcal{A}_2$, pick any $i, j \in [3], i \neq j$. If $\bar{x} \to \widehat{\bar{y}}$ in $\mathbb{H}_i^{j'}$, then both $\omega(\bar{x})$ and $\omega(\widehat{\bar{y}})$ are defined by (1.2) and it is easily seen that $\omega(\bar{x}) \to \omega(\widehat{\bar{y}})$ in $\mathbb{H}_i^{j'}$. As for compatibility with $\mathcal{A}_3$, let $\bar{x} \to \widehat{\bar{y}}$ in some $\mathbb{H}' \in \mathcal{A}_3$. The only interesting case is when $4 \in \bar{x}$; we see that $x_i = 4$ iff $y_i = 4$ for all $i \in [4]$. It follows that $\omega(\bar{x})$ and $\omega(\widehat{\bar{y}})$ are defined by the same case of the definition, (1.2), (2.1) or (2.2); in all of these cases we have $\omega(\bar{x}) \to \omega(\widehat{\bar{y}})$ in $\mathbb{H}'$. Thus ω is compatible with $\mathcal{A}$ and we have proved that CSP($\mathbb{T}$) is tractable.

(ii) It suffices to prove that $\mathcal{A}$ does not admit a compatible binary weak-NU (binary idempotent commutative operation). Striving for contradiction, let $\star$ be a binary weak-NU compatible with $\mathcal{A}$. In the following, a digraph above an arrow indicates that the implication was deduced from the compatibility with that digraph.

For any $i \neq j \in [3]$ we have

$$\widehat{i} \star \widehat{i} = \widehat{i} \stackrel{\mathbb{H}}{\Longrightarrow} i \star 0 \in \{i, 0\} \stackrel{\mathbb{H}_j^{i'}}{\Longrightarrow} \widehat{i} \star \widehat{j} \in \{\widehat{i}, \widehat{j}\} \stackrel{\mathbb{H}_j^{i'}, \mathbb{H}_i^{j'}}{\Longrightarrow} i \star 0 = i \text{ or } j \star 0 = j.$$

Without loss of generality we may assume that $1 \star 0 = 1$. Now

$$1 \star 0 = 1 \stackrel{\mathbb{H}_2^{1'}, \mathbb{H}_3^{1'}}{\Longrightarrow} \widehat{1} \star \widehat{2} = \widehat{1} \star \widehat{3} = \widehat{1} \stackrel{\mathbb{H}_1^{2'}, \mathbb{H}_1^{3'}}{\Longrightarrow} 2 \star 0 = 3 \star 0 = 0;$$

a contradiction.

(iii) Again, it is enough to prove that $\mathcal{A}$ admits no compatible near-unanimity operation. Suppose for contradiction that there exists an r-ary NU operation ν compatible with $\mathcal{A}$. We will prove the following claim: For all $i \in [r-2]$,

$$\nu(\underbrace{4, \ldots, 4}_{i\text{-times}}, 0, 0, \ldots, 0) = 0 \implies \nu(\underbrace{4, \ldots, 4, 4}_{(i+1)\text{-times}}, 0, \ldots, 0) = 0.$$

This claim contradicts the fact that $\nu(4, 0, \ldots, 0) = 0$ and $\nu(4, \ldots, 4, 0) = 4$.

Fix $i \in [r-2]$ and let

$$t(x, y, z) = \nu(\underbrace{x, \ldots, x}_{i\text{-times}}, y, z, \ldots, z).$$

As t is also compatible with $\mathcal{A}$, we have that

$$t(4, 0, 0) = 0 \stackrel{\mathbb{H}_{2,3}^{4'}}{\Longrightarrow} t(\widehat{4}, \widehat{2}, \widehat{3}) \in \{\widehat{2}, \widehat{3}\}.$$

If $t(\widehat{4}, \widehat{2}, \widehat{3}) = \widehat{2}$, then by compatibility with $\mathbb{H}_3^{2',4'}$ we have $t(4, 2, 0) = 2$ and from $\mathbb{H}$ we get $t(\widehat{4}, \widehat{2}, \widehat{4}) = \widehat{2}$; a contradiction with the NU property of ν. Therefore $t(\widehat{4}, \widehat{2}, \widehat{3}) = \widehat{3}$. But

$$t(\widehat{4}, \widehat{2}, \widehat{3}) = \widehat{3} \stackrel{\mathbb{H}_2^{3',4'}}{\Longrightarrow} t(4, 0, 3) = 3 \stackrel{\mathbb{H}}{\Longrightarrow} t(\widehat{4}, \widehat{4}, \widehat{3}) = \widehat{3}) \stackrel{\mathbb{H}_3^{2',4'}}{\Longrightarrow} t(4, 4, 0) = 0;$$

and the claim is proved. $\square$

Chapter 7

Conclusion

Our work represents another evidence of the usefulness of the algebraic approach to CSP. We managed to extend the dichotomy result from [3] to special polyads, for which the problem is substantially more complex (as witnessed by the construction in Chapter 6). On the other hand, the most up to date algebraic tools were needed.

We believe that the methods developed for the proof of the dichotomy for special polyads can be generalized to a far broader class of oriented trees. However, the dichotomy question for oriented trees is still far from being solved and it is very likely that it will require a deeper understanding of the algebraic side of the Constraint Satisfaction Problem.

The following two open questions naturally arise from our work:

Question 1. *Does every tractable oriented tree have bounded width?*

Question 2. *Does every oriented tree which admits a binary idempotent commutative polymorphism have width 1?*

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