Charles University in Prague
Faculty of Mathematics and Physics

DOCTORAL THESIS

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Homeomorphisms in Topological Structures

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Supervisor of the doctoral thesis: doc. RNDr. Pavel Pyrih, CSc.
Study programme: mathematics
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I would like to thank my supervisor Pavel Pyrih for his support during my own research on Continuum theory in the frame of his Open problem seminar. I would also like to thank my family and my friends for their support during the time this thesis was prepared.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Zkoumání homeomorfismů v topologických strukturách

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Vedoucí disertační práce: doc. RNDr. Pavel Pyrih, CSc.

Abstrakt: V této práci představujeme řešení několika problémů týkajících se jedno-dimenzionálních kontinuí. Podáváme induktivní popis grafů s daným číslem nesouvislosti, čímž zodpovíme otázku S. B. Nadlera. Dále předkládáme topologickou charakterizaci Sierpińského trojúhelníku. Při studiu tzv. shore množin v dendroidech a λ-dendroidech obdržíme několik pozitivních výsledků a předvedeme také několik protipříkladů. Tímto pokračujeme v nedávné práci několika autorů. Zabýváme se také pojmem $\frac{1}{2}$-homogenity a dokazujeme, že až na homeomorfismus existují pouze dvě $\frac{1}{2}$-homogenní zřetězitelná kontinua s právě dvěma koncovými body. Předvedeme také nový elegantní důkaz jednoho Waraszkiewiczova klasického výsledku.

Klíčová slova: topologie, homeomorfismus, kontinuum, jedno-dimenzionální kontinuum

Title: Homeomorphisms in Topological Structures

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Abstract: In this thesis, we present solutions to several problems concerning one-dimensional continua. We give an inductive description of graphs with a given disconnection number, this answers a question of S. B. Nadler. Further, we state a topological characterization of the Sierpiński triangle. In the study of shore sets in dendroids and λ-dendroids we obtain several positive results and we also provide some counterexamples. By doing this, we continue in the recent work of several authors. We are also dealing with the notion of $\frac{1}{2}$-homogeneity and we prove that up to homeomorphism there are only two $\frac{1}{2}$-homogeneous chainable continua with just two end points. We present also a new elegant proof of a classical theorem of Waraszkiewicz.

Keywords: topology, homeomorphism, continuum, one-dimensional continuum

Contents

Introduction

In present time, topology is a stable branch of mathematics and Continuum theory is a solid part of topology. Foundations of contemporary general topology in the form we are familiar with were summarized in Engelking's General Topology [En89], the elements and codification of the present Continuum theory can be found in Nadler‘s Continuum Theory, An Introduction [Na92]. We refer to these books for all common notions used in this thesis, but let us remind that a continuum is a nonempty compact connected metrizable space.

In order to introduce the reader to Continuum theory we will present some selected topics which are of interest in this discipline. We remark that these topics are overlapping each other and that they do not cover the whole extent of Continuum theory. At each part we introduce major results and central open problems which indicate the direction, which the research tends to.

In a lot of mathematical branches it is of great interest to study those objects which embody a high degree of symmetry. This can be illustrated by the result on five Platonic solids (regular convex polyhedrons), which goes back to ancient Greece and by which a number of scientists was fascinated (see Figure 1). Homogeneous continua represent such a nice class in topology. However,

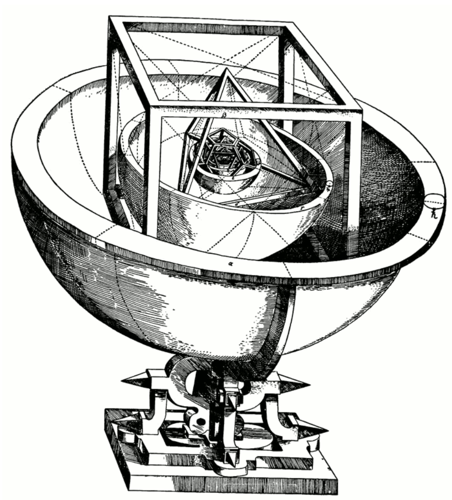

Figure 1: Kepler's Platonic solid model of the solar system.

the notion of homogeneity is very difficult to handle with and that is why fundamental results in this area are obtained rarely. In 1920, Knaster and Kuratowski [KK20] asked whether the simple closed curve is the only nondegenerate homogeneous planar continuum. In 1924, Mazurkiewicz [Ma24] proved a positive answer under an extraordinary assumption of local connectedness. A negative answer to the original question was given in 1948 by Bing [Bi48], who proved that the pseudo-arc is homogeneous. Another homogeneous plane continuum (a circle of pseudo-arcs) was discovered by Bing and Jones [BJ59] in 1959. Since then a lot of results about homogeneous continua has been published. In spite of that, it is still not known whether there is another nondegenerate homogeneous plane continuum which is neither a simple closed curve nor a pseudo-arc nor a circle of pseudo-arcs. Anderson proved in [An58] that a simple closed curve and Menger's universal curve are the only nondegenerated homogeneous locally connected curves (one-dimensional continua). However, without the assumption of local connectedness the classification is far from being complete. The question whether every nondegenerate tree-like homogeneous continuum is a pseudo-arc, remains still unsolved.

Studying some mathematical objects we are usually also interested in relations between these objects. There are various types of continuous mappings between continua, the most important role among them play monotone, open, confluent or light mappings. There are many generalizations which grow up for different purposes. A very systematic study of mappings on continua was given in [Ma79] and [MT84]. In spite of that a number of new results has been obtained since these papers were published, they form a nice survey about the problem whether a class of mappings preserves a given property on continua. A classical open question in this area is whether a confluent image of a chainable continuum is necessarily chainable. However there are other kinds of questions which are of interest. A classical one of them was answered by Waraszkiewicz in [Wa32]. He proved that there is a collection of uncountable many continua no one of which can be obtained as a continuous image of some other. This place of interest was studied in a more general way in [CCP94]. The authors of that paper define a quasiorder given by a fixed class of mappings on a class of continua. Consequently, they study problems about the size of chains and antichains in this quasiordered set. We have also contributed in a very small measure to the progress in this area by a paper [DPPRV11], which is however not a part of this thesis.

The fixed-point property of a continuum is an intensively studied property. It is well known that chainable continua as well as all λ-dendroids have this property [Ma80]. On the other hand, there are tree-like continua without the fixed-point property [Be80]. There is still an open question whether every planar continuum, which does not separate the plane, has the fixed-point property.

There is a fruitful book on hyperspaces of continua. Illanes and Nadler in their book Hyperspaces [IN99] study the hyperspaces of all nonempty closed subsets 2^X, hyperspaces of subcontinua $C(X)$ or some other types of hyperspaces such as $F_n(X)$. Hundreds of open problems are given in this book. Let us just mention one open question in the area of hyperspaces. Suppose that a continuum Y is a continuous image of a continuum X. Does it follow that the corresponding hyperspace $C(Y)$ is a continuous image of the hyperspace $C(X)$?

There is a plenty of other results in Continuum theory which could hardly be summarized systematically. We mention for example theorems about topological characterizations of some extraordinary spaces or the existence resp. nonexistence of universal elements in some classes. There are fields, in which Continuum theory overlaps with other disciplines such as in the case of topological dynamics. The pseudo-arc is a continuum which is studied on its own. Recently, a famous Lelek's problem was solved by Hoehn who constructed a planar continuum which is not chainable but which has span zero [Ho11].

This thesis consists of seven papers [Ve10], [Ve12], [PV12b], [PV12a], [BMPV13], [PV13], [BPV13]. Five of them have been already published, the two remaining are submitted. In six of them we answer questions or problems which have been asked in Pearl's Open problems in Topology II [Pe07] or in recently published papers. In one remaining paper we give a readable proof of a classical result on continua from the first half of the twentieth century. The common topic of all these papers is one-dimensional continua.

In the first paper [Ve10] (see Chapter 1, p. 5) we studied the notion of a disconnection number of a graph. A disconnection number of a continuum X is the least countable cardinal number κ, such that X becomes disconnected upon removing an arbitrary subset of cardinality κ (it does not need to exist). In [Na92, p. 147] a whole section is devoted to this topic. Among others a list of all graphs with disconnection number at most three is given there. It was asked by Nadler in [Na93] as well as in [Na92, p. 157] how many graphs with disconnection number four there are. We proved in [Ve10] that there is a way how to obtain all graphs with disconnection number $n+1$ if we know all graphs with disconnection number n. As a corollary we obtained that there are exactly 26 graphs with disconnection number four, as was expected. The problem about disconnection number was independently solved by Gladdines and van de Vel in [GV11].

In the next paper [Ve12] (see Chapter 2, p. 10) we answered a question of Pyrih [Py01]. We gave a topological characterization of the Sierpiński triangle and of the Apollonian gasket. By simple modifications we also characterized some similar continua.

It is a classical result of Waraszkiewicz [Wa32] that there is an uncountable collection of spirals (compactifications of a ray with remainder a simple closed curve), which are incomparable with respect to continuous surjective mappings. The original proof is written in a very technical manner. We were able to give a fairly elegant proof of this result in [PV12b] (see Chapter 3, p. 18). As a secondary product we obtained a list of conditions under which there is a continuous surjection or a homeomorphism between two spirals.

In the Pearl's Open problems in Topology II [Pe07] we can find many open questions, conjectures and problems on continua. One of them is whether there is a dendroid in which the union of two disjoint closed shore sets is not a shore set [Pe07, p. 332]. Originally, this question was posed by Illanes in [Il01]. We presented a positive answer to this question in [BMPV13] (see Chapter 5, p. 37). On the other hand we showed in this paper that in the case of planar smooth dendroids, the union of two closed disjoint shore sets is a shore set. Moreover, we showed that the union of a closed shore set and a shore continuum in a planar dendroid is a shore set as well. In [PV12a] (see Chapter 4, p. 27) we answered a question posed in [Na07] by constructing a λ-dendroid in which the union of two shore points is not a shore set. By all these results we have enriched common knowledge about unions of shore sets in dendroids and λ-dendroids.

In [NPP06] as well as in [Pe07, p. 339] it is asked whether there is a $\frac{1}{2}$-homogeneous indecomposable circle-like continuum. We presented such a continuum in [PV13] (see Chapter 6, p. 48). An independent solution was given at the same time by Boroński [Bo13]. In the construction, we used a $\frac{1}{2}$-homogeneous chainable continuum with just two end points which is not an arc. We asked in [PV13] whether such a continuum is unique. It is a surprise that it really is, as we proved in [BPV13] (see Chapter 7, p. 53). Using this result, we obtained a topological characterization of an arc and so called arcless-arc as the only continua which are $\frac{1}{2}$-homogeneous chainable and with just two end points.

Chapter 1.
A note on the disconnection number

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Abstract

We present a simple way to obtain all graphs with a given disconnection number if we know all the graphs with smaller disconnection number. Using this method we answer a question of Sam B. Nadler, i.e. we prove that there exist precisely 26 continua with disconnection number four. Some known results concerning disconnection number are also obtained as simple corollaries. Further, we give some estimates on the number of distinct continua with a fixed disconnection number which concerns a problem of Sam B. Nadler.

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Introduction

A *continuum* means a nonempty compact connected metric space. For simplicity all continua are supposed to be nondegenerate. *Order* of a point x of a continuum X is the smallest cardinal number κ such that there exists a local base of the point x in X such that each element of this base has a boundary of cardinality at most κ. This cardinal number is denoted by $\operatorname{ord}(x, X)$. A point of order one is called *end point*. A *simple closed curve* is any space homeomorphic to the unit circle. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. A continuum X is called *graph* if there exist finitely many arcs in X which cover X and any two of these distinct arcs intersect at most in their sets of end points. A *tree* is any graph which does not contain a simple closed curve. *Disconnection number* of a continuum X means the least cardinal number $n \leq \omega$ such that $X \setminus N$ is disconnected whenever cardinality of N equals n. It is denoted by $D(X)$ if it exists. Note that we use slightly different notation than in [Na92, p. 148]. It follows that $D(X) \geq 2$ if it exists (see Proposition 9.15 of [Na92, p. 148]).

We recall a characterization of graphs from [Na92, p. 152]

Theorem 1. *The following are equivalent for a continuum X:*

- *X is a graph,*
- *$D(X) \leq \omega$,*
- *$D(X) < \omega$.*

Let d_n denotes the number of topologically different graphs with disconnection number $n \geq 2$. It is well known that a simple closed curve is the only continuum with disconnection number two, thus $d_2 = 1$. In [Na93] it is shown that $d_3 = 5$. In the same paper it is mentioned that $d_4 \geq 26$ and that the equality here is expected. We confirm this hypothesis by proving more general result. In Problem 8.3 of [Na93] it is asked whether there is a formula which can be used to calculate d_n and that in the absence of such a formula it would be of interest to have estimates for d_n as n tends to infinity. We give more or less interesting exponential estimates.

Main result

For any spaces X, Y, a subset E of X and continuous mapping $f\colon E \to Y$ we define adjunction $X \oplus_f Y$ as a quotient of the disjoint union of the spaces X and Y with respect to the smallest equivalence generated by f, where the function f is identified with the set $\{(x, f(x)) : x \in E\}$. Let us denote by A an arc and by 0, 1 its end points. The letter C denotes a simple closed curve and c is an arbitrary point in C. By B we mean any space homeomorphic to $C \oplus_{\{(0,1)\}} A$, i.e. a space homeomorphic to the symbol '9'.

Theorem 2. *Let X be a continuum with $D(X) = n$ which is neither an arc nor a simple closed curve. Then there exists a subcontinuum Y of X with $D(Y) = n - 1$, and two distinct non-end points x, y of Y for which one of the following possibilities holds:*

1. *$X \cong Y \oplus_{\{(x,0)\}} A$*

2. $X \cong Y \oplus_{\{(x,0),(y,1)\}} A$

3. $X \cong Y \oplus_{\{(x,0)\}} B$

4. $X \cong Y \oplus_{\{(x,c)\}} C$.

Proof. Denote by F the set of all points in X of order at least three. This is finite since X is a graph (by Theorem 1) and nonempty because X is neither an arc nor a simple closed curve (see Proposition 9.5 from [Na92, p. 142]). Suppose first that there exists an end point y in X. Clearly $y \notin F$. We can find an arc A with end points y and x where $A \cap F = \{x\}$. We let $Y = X \setminus (A \setminus \{x\})$ so that the first possibility occurred.

Otherwise suppose that X contains no end points. Especially X is not a tree and thus there exists a simple closed curve C in X. We distinguish the following possibilities.

- If $|C \cap F| \geq 2$ then there is an arc A in C with end points x and y such that $A \cap F = \{x, y\}$. And we let $Y = X \setminus (A \setminus \{x, y\})$. Thus the second possibility arises.

- If $|C \cap F| = 1$ then there is a point $x \in C \cap F$. In case of $\text{ord}(x, X) \geq 4$ we let $Y = X \setminus (C \setminus \{x\})$ which satisfies the fourth option. Otherwise $\text{ord}(x, X) = 3$. Since there are no end points in X it follows that there is at least one more point in F distinct from x. We choose A to be an arc with end points x and y where y is taken to satisfy $A \cap F = \{x, y\}$. It suffices to let $B = A \cup C$.

- The case $C \cap F = \emptyset$ cannot occur since X is not a simple closed curve.

It remains to show that $D(Y) = n - 1$ in each of the four cases. We prove this only in the first case since the others are similar. For simplicity suppose that $X = Y \oplus_{\{(x,0)\}} A$. Consider a set $M \subseteq Y$ with $n - 1$ elements and suppose for contradiction that $Y \setminus M$ is connected. If $x \notin M$ then the set $X \setminus (M \cup \{1\})$ is connected which is a contradiction with $D(X) = n$. Otherwise if $x \in M$ then there is a simple closed curve $C \subseteq Y$ which intersects M precisely in x. Choose any $x' \in C$ which is of order two and let $M' = (M \setminus \{x\}) \cup \{x'\}$. Then we have that $X \setminus M'$ is connected and so is $X \setminus (M' \cup \{1\})$ which contradicts the equality $D(X) = n$. Thus we get that $D(Y) \leq n - 1$.

Now, it remains to find a set $K \subseteq Y$ with $n - 2$ points such that $Y \setminus K$ is connected. Let N be a subset of X with $n - 1$ elements for which $X \setminus N$ is connected. We easily obtain that $1 \in N$. It follows that $N \cap A = \{1\}$. Then $K = N \setminus \{1\}$ is the desired set. □

Remark 3. Theorem 2 will be applied for inductive construction of arbitrary graph X with disconnection number $n+1$ from some graph Y with disconnection number n by one of the four described operations, with the exception of an arc and a simple closed curve only. It follows by induction from Theorem 2 that a graph with disconnection number n can be obtained as a quotient of a sum of $(3n - 5)$ arcs, where only some end points of these arcs are identified.

Consequences

The following classical result is easily derived (see Theorem 9.31 in [Na92, p. 156]).

Corollary 4. *A continuum X is a simple closed curve just when $D(X) = 2$.*

Proof. Direct implication is obvious. For the opposite suppose that X is a continuum which is not a simple closed curve and such that $D(X) = 2$. Clearly X is not an arc, so that we can use Theorem 2 to obtain a subcontinuum Y of X for which $D(Y) = D(X) - 1 = 1$. This is a contradiction. □

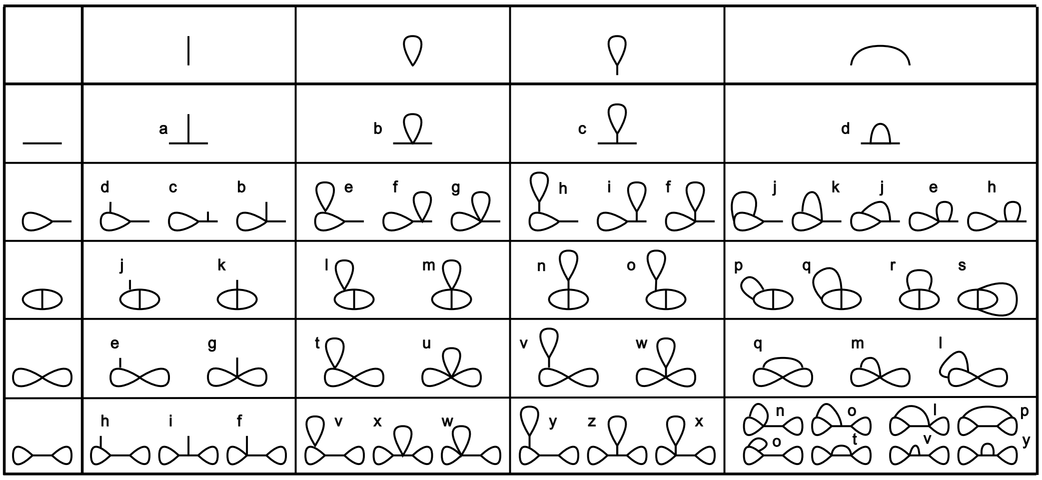
The next result can be found in [Na92, p. 157] with a selfcontained proof.

Corollary 5. *The only continua with disconnection number three are an arc, the number nine, the letter theta, the number eight and a dumbbell.*

Proof. A continuum with disconnection number three which is not an arc is by Theorem 2 obtained from a simple closed curve by one of the four possible constructions. □

Corollary 6. *There exist exactly 26 continua with disconnection number four.*

Proof. Using Theorem 2 and the description of graphs with disconnection number three from Corollary 5, which are pictured in the first column, we get the following list of continua $a, b, c, \ldots, x, y, z$.



□

Denote by b_n the number of equivalences on a fixed set with n elements for $n \geq 0$. These numbers b_n are called *Bell's numbers* and it is easy to see that there is an upper bound $b_n \leq n^n$.

Proposition 7. *For every $n \geq 2$ we have that $2^{n-4} \leq d_n \leq b_{6n-10}$.*

Proof. For $n = 2$ the lower bound is clearly true. When $n \geq 3$ consider a subset M of an arc A consisting of $n-3$ non-end points of A. For any subset $S \subseteq M$ we obtain a space from the arc A with simple closed curves glued to all points from S and a space homeomorphic to the number nine glued to all points from $M \setminus S$ via its end point. In this way we obtain at least 2^{n-4} non-homeomorphic graphs with disconnection number n.

The upper bound is a consequence of Remark 3 since every graph with disconnection number n is a quotient of $(3n-5) \times A$ where only some end points of arcs are identified. Thus there are at most b_{6n-10} graphs with disconnection number n. □

References

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[Na93] S. B. Nadler, *Continuum Theory and Graph Theory: Disconnection Numbers*, Journal of the London Mathematical Society 1 (1993), 167–181.

Chapter 2.
A topological characterization of the Sierpiński triangle

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Abstract

We present a topological characterization of the Sierpiński triangle. This answers question 58 from the Problem book of the Open Problem Seminar held at Charles University. In fact we give a characterization of the Apollonian gasket first. Consequently we show that any subcontinuum of the Apollonian gasket, whose boundary consists of three points, is homeomorphic to the Sierpiński triangle.

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Keywords: continuum, Sierpiński triangle, topological characterization.

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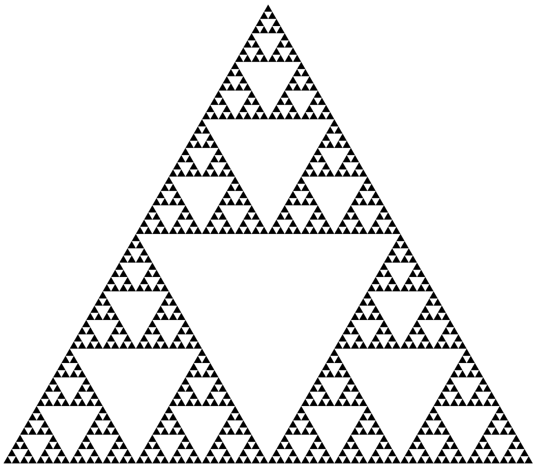

Figure 1: The Sierpiński triangle [Wik]

Introduction

A *continuum* means a nonempty compact connected metrizable space. A point x of a space X is called a *local cut point* if there is a connected open neighborhood U of x such that $U \setminus \{x\}$ is not connected. A *simple closed curve* is any space homeomorphic to the unit circle. An *arc* is any space which is homeomorphic to the closed interval $[0,1]$. *Complementary domain* of a plane continuum X is any component of the complement of X.

The *Sierpiński triangle* [Fig. 1] is geometrically defined as follows. We take a solid equilateral triangle T_0, partition it into four congruent equilateral triangles and remove the interior of the middle triangle to obtain a continuum T_1. We proceed in the same manner with the three remaining triangles step by step to get a nested sequence $(T_n)_{n=0}^{\infty}$. The intersection $T = \bigcap T_n$ is called the Sierpiński triangle.

For our purposes a topologically equivalent definition of the Sierpiński triangle will be useful. We take a countable power $\{0,1,2\}^{\mathbb{N}}$ of a three elements discrete space with the usual Tychonoff topology and identify a point $(a_1, \dots, a_n, i, \bar{j})$ with $(a_1, \dots, a_n, j, \bar{i})$ for any $i, j, a_1, \dots, a_n \in \{0,1,2\}$ and $n \in \mathbb{N}_0$. Such a quotient is homeomorphic to the Sierpiński triangle [Wi00], whereas the vertices of the triangle correspond to the points $(\bar{i}) = (i, i, \dots)$ for $i \leq 2$.

If we take two copies T and T' of the Sierpiński triangle with vertices v_0, v_1, v_2 and v_0', v_1', v_2' respectively and identify each point v_i with v_i' we get a continuum homeomorphic to the so-called *Apollonian gasket* [Fig. 2]. We give a topological characterization of the Apollonian gasket and prove that arbitrary subcontinuum with three points on the boundary is homeomorphic to the Sierpiński triangle. By doing this we solve Problem 58 from [Py01].

The following fact, which is due to Schönflies [Ku68, p. 515], will be a useful tool for the consecutive characterization.

Fact 1. *Let X be a locally connected continuum in the plane. Then for every $\varepsilon > 0$ there are only finitely many complementary domains of X with diameter bigger than ε.*

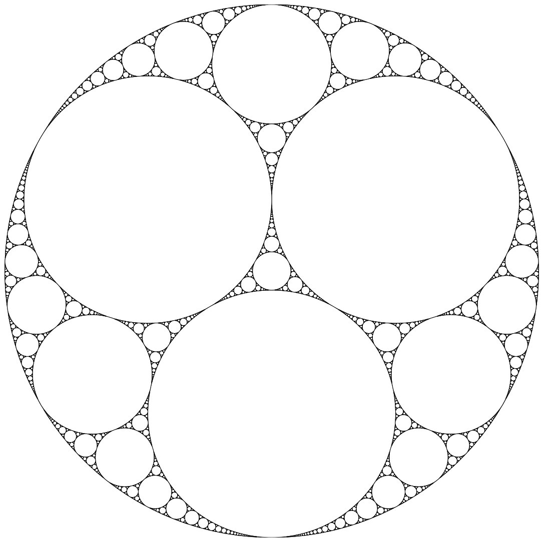

Figure 2: The Apollonian gasket [Wik]

Main results

Definition 2. Every simple closed curve C in a continuum X will be called a *link* provided that $X \setminus C$ is connected.

This notion is especially useful when dealing with continua in the plane, because in this case every link is a boundary of a complementary domain by the Jordan curve theorem.

Theorem 3. *A continuum X is homeomorphic to the Apollonian gasket if and only if*

1. *X is planar and locally connected,*
2. *any two links in X intersect at most in a point,*
3. *there is no point in X common to three links,*
4. *X contains at least three links each pair of which intersects,*
5. *whenever there are three links each pair of which intersects, there are two other links which intersect each of the three given links.*

Proof. It is easily observed that any space homeomorphic to the Apollonian gasket satisfies all of the conditions $1 - 5$.

Conversely suppose that X is a continuum satisfying all the five conditions. By the condition 4 there exist three distinct links C_0, C_1 and C_2 each pair of which intersects. We note that from condition 5 and condition 1 it follows that for any triple of links each par of which intersects, there are exactly two other links which intersect each of the three given links.

We denote by $A = \bigcup_{n=0}^{\infty} \{0, 1, 2\}^n$ a set of indices. By induction we construct a family of links $\{L(a) \colon a \in A\}$, such that

- $L(a)$ touches $L(a,b)$ whenever $a \in \{0,1,2\}^n$ and $b \in \{0,1,2\}^m$, where $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $b_1 \notin \{b_2, \dots, b_m\}$.
- $L(a)$ touches C_i whenever $i \in \{0,1,2\}$, $a \in \{0,1,2\}^n$ and $i \notin \{a_1, \dots, a_n\}$.

Let $L(\emptyset)$ be a link which touches the links C_0, C_1 and C_2. By condition 5 there are two of them so we have two possible choices. We suppose that all the links $L(a)$ for $|a| \leq n$ have been constructed and they satisfy the induction hypothesis. We fix $a \in A$ and $i \in \{0,1,2\}$, where $|a| = n$, and we define a link $L(a,i)$. We distinguish several cases:

- If $n = 0$, then we consider three links $L(\emptyset), C_j$ and C_k where $\{i,j,k\} = \{0,1,2\}$. These three links intersect each other. Thus by condition 5 there are two other links which touch the three given links. One of them is the link C_i. We define $L(i)$ to be the other link.
- If $n \geq 1$ and $|\{a_1, \dots, a_n, i\}| = 1$, we can find j and k such that $\{i,j,k\} = \{0,1,2\}$. Clearly the links $L(a)$, C_j and C_k touch each other by the induction hypothesis. Thus there are two other links touching each of these three links. One of them is $L(a_1, \dots, a_{n-1})$. We define $L(a,i)$ to be the other link.
- If $n \geq 1$ and $a_1 = \dots = a_n \neq i$, we can define $j = a_n$ and k such that $\{i,j,k\} = \{0,1,2\}$. The links $L(a), L(a_1, \dots, a_{n-1})$ and C_k touch each other. Thus there are two other links touching each of these three links. One of them is C_i. We define $L(a,i)$ to be the other one.
- If $|\{a_1, \dots, a_n\}| = |\{a_1, \dots, a_n, i\}| = 2$, we can find $j \in \{a_1, \dots, a_n\}$ and k such that $\{i,j,k\} = \{0,1,2\}$. Let $l \leq n$ be the natural number for which $a_l \neq a_{l+1} = \dots = a_n = i$. Moreover let $m \leq n$ be the biggest integer for which $a_m = i$. The links $L(a)$, $L(a_1, \dots, a_{l-1})$ and C_k touch each other. Thus there are two other links touching each of these three links. One of them is $L(a_1, \dots, a_{m-1})$. Let $L(a,i)$ be the other one.
- If $|\{a_1, \dots, a_n\}| = 2$ and $|\{a_1, \dots, a_n, i\}| = 3$, let us denote by $l \leq n$ the natural number for which $a_l \neq a_{l+1} = \dots = a_n$. Next we find the natural number $m < l$ for which $a_m \notin \{a_{m+1}, \dots, a_n, i\}$. The links $L(a)$, $L(a_1, \dots, a_{l-1})$ and $L(a_1, \dots, a_{m-1})$ are three links each pair of which intersects. Thus there are two other links touching each of these three links. One of them is C_i. We define $L(a,i)$ to be the other one.
- If $|\{a_1, \dots, a_n\}| = |\{a_1, \dots, a_n, i\}| = 3$, let us denote by $l \leq n$ the natural number for which $a_l \neq a_{l+1} = \dots = a_n = i$. Next we find the natural number $m < l$ for which $a_m \notin \{a_{m+1}, \dots, a_n, i\}$. We can find the biggest integer $p \leq n$ for which $a_p = i$. The links $L(a)$, $L(a_1, \dots, a_{l-1})$ and $L(a_1, \dots, a_{m-1})$ are three links each pair of which intersects. Thus there are two other links touching each of these three links. One of them is $L(a_1, \dots, a_{p-1})$. We define $L(a,i)$ to be the other one.

In each case we can easily verify that the induction hypothesis remains satisfied.

Now, we define a mapping $f\colon \{(a,\bar{i})\colon a \in A, i \leq 2\} \to X$ as follows:

- $f(\bar{i})$ is the only point in the intersection $C_j \cap C_k$ where $\{i,j,k\} = \{0,1,2\}$.

- If $|\{a_1, \dots, a_n, i\}| = 2$ and $a_n \neq i$ we define $f(a, \bar{i})$ to be the only point in $L(a_1, \dots, a_{n-1}) \cap C_k$ where k satisfies $\{a_n, i, k\} = \{0, 1, 2\}$.

- If $|\{a_1, \dots, a_n, i\}| = 3$ and $a_n \neq i$ we find the natural number $l < n$ for which $a_l \notin \{a_{l+1}, \dots, a_n, i\}$ and we define $f(a, \bar{i})$ to be the only point in $L(a_1, \dots, a_{n-1}) \cap L(a_1, \dots, a_{l-1})$.

We consider the family of links $\{C_0, C_1, C_2\} \cup \{L(a) \colon a \in A\}$ and we enumerate as $\{D_m \colon m \in \mathbb{N}\}$ a family of all closures of complementary domains of these links which do not intersect X. We observe that the diameters of the sets D_m converge to zero by Fact 1. The assumptions of Fact 1 are satisfied because of the condition 1.

In order to show that the mapping f is uniformly continuous it suffices to prove that for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that the components of $\mathbb{R}^2 \setminus \bigcup\{D_m \colon m < n\}$ are of diameter less than ε. Suppose that this is not true. Hence there is an $\varepsilon > 0$ and there are complementary domains G_n of $\mathbb{R}^2 \setminus \bigcup\{D_m \colon m < n\}$ whose diameters are at least ε and such that $G_{n+1} \subseteq G_n$. For every $m \in \mathbb{N}_0$ there are three mutually distinct indices p_m, q_m and r_m in $\mathbb{N}$ such that G_m is the bounded complementary domain of $\mathbb{R}^2 \setminus (D_{p_m} \cup D_{q_m} \cup D_{r_m})$.

By eventual restriction to a subsequence of (G_m) and possible permutation of p_m, q_m and r_m it suffices to consider the following three cases only.

- The sequence (p_m) goes to infinity and (q_m) and (r_m) are constant. We denote by x the only point in $D_{q_0} \cap D_{r_0}$. As the diameters of (D_{p_m}) converge to zero we get that (D_{p_m}) converges to the point x. Hence the sets G_m go to x and thus the diameters of G_m converge to zero.

- The sequences (p_m) and (q_m) tend to infinity and (r_n) is constant. There is a point $x \in X$ which is a limit point to the sequence $(D_{p_n} \cup D_{q_n})$. Similarly as in the first case we derive that the sequence G_m converge to the point x, hence its diameters tend to zero.

- All the sequences (p_m), (q_m) and (r_m) converge to infinity. Then we get that the diameter of G_m is less than or equal to the diameter of $D_{p_m} \cup D_{q_m} \cup D_{r_m}$ which converges to zero.

In all cases we obtain a contradiction.

We denote by $g \colon \{0, 1, 2\}^{\mathbb{N}} \to X$ the only continuous extension of the mapping f. It follows using the condition 2 that $g(a_1, \dots, a_n, i, \bar{j}) = g(a_1, \dots, a_n, j, \bar{i})$ and that these are the only possibilities when $g(x) = g(y)$ for $x \neq y$, because of the condition 3. Thus the image of $\{0, 1, 2\}^{\mathbb{N}}$ under g is homeomorphic to the Sierpiński triangle.

Now we recall that there were two possibilities $L(\emptyset)$ and $L'(\emptyset)$, how to choose the first link in the inductive process. Thus we may obtain by the same proof another family of links $\{L'(a) \colon a \in A\}$ and corresponding mapping f' and its continuous extension g'.

By a similar reasoning as in the proof that f is uniformly continuous we can show, that the union of images of the mappings f and f' is dense in X. And thus the union of the images of g and g' covers the whole space X. The intersection of the images of the mappings g and g' consists of three points. These are namely the points contained in exactly two links from C_0, C_1 and C_2. Thus we obtain that X is homeomorphic to the quotient of the direct sum of two copies of the Sierpiński triangle, where the corresponding vertices are identified. Thus X is homeomorphic to the Apollonian gasket. □

Remark 4. Since the choice of the three links C_0, C_1 and C_2 at the beginning of the preceding proof was random, we conclude that for any triple of distinct, mutually intersecting links C'_0, C'_1 and C'_2 there is a homeomorphism of the Apollonian gasket onto itself, which sends C_i onto C'_i for any $i \leq 2$.

Lemma 5. *Let X be a locally connected continuum and K be a nondegenerate subcontinuum of X with finite boundary. Then every point from the boundary of K is a local cut point of X.*

Proof. Let x be an arbitrary point from the boundary of K. Since the boundary of K is finite and X is locally connected, there is an open connected neighborhood U of x whose intersection with the boundary of K contains only the point x. We get that the set $U \setminus \{x\}$ is a disjoint union of an open set $K \cap U \setminus \{x\}$ and an open set $U \setminus K$. Both these sets are nonempty since x is an element of the boundary of K. Thus x is a cut point of U and consequently it is a local cut point of X. □

Theorem 6. *Any subcontinuum of the Apollonian gasket whose boundary consists of exactly three points is homeomorphic to the Sierpiński triangle.*

Proof. Suppose that X is a subcontinuum of the Apollonian gasket with precisely three points v_0, v_1, v_2 on the boundary. By Lemma 5 we know that the points v_0, v_1, v_2 are local cut points, which are those points in the Apollonian gasket, where two distinct links intersects. For any pair of distinct indices $i, j \in \{0, 1, 2\}$ there is a complementary domain D_k, where $\{i, j, k\} = \{0, 1, 2\}$ and the points v_i and v_j belong to the boundary of D_k. The boundaries C_0, C_1 and C_2 of D_0, D_1 and D_2 are pairwise distinct links and $\{v_i\} = C_j \cap C_k$ for $\{i, j, k\} = \{0, 1, 2\}$. By Remark 4 we conclude that X is homeomorphic to the Sierpiński triangle. □

Corollary 7. *Let X be a continuum and let v_0, v_1, v_2 be three points in X. Define a space Y as a sum of X and a disjoint copy X' of X, where every point v_i is identified with v'_i which is a corresponding point in X'. If the space Y is homeomorphic to the Apollonian gasket, then X is homeomorphic to the Sierpiński triangle.*

Proof. Clearly X is a subcontinuum of the Apollonian gasket Y and its boundary consists of three points. Thus by Theorem 6 it follows that X is homeomorphic to the Sierpiński triangle. □

Final remarks

Let us define a continuum T, that will be called a *modified triangle*, in the following way. We take an equilateral triangle and exclude the interior of a regular hexagon whose three edges are formed by the middle thirds of edges of the triangle. This can be inductively done in every remaining smaller triangle. What

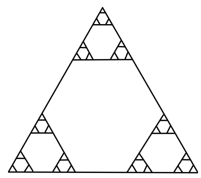

Figure 3: The modified triangle

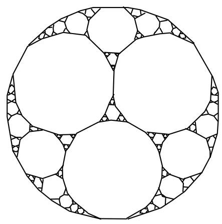

Figure 4: The modified gasket

remains is the modified triangle [Fig. 3]. A *modified gasket* is a sum of T and its copy T' where the corresponding pairs of vertices of the triangles are joined with an arc. This can be pictured as in Figure 4.

A simple modification of the second condition in the characterization of the Apollonian gasket from Theorem 3 gives rise to a characterization of the modified gasket. Now, there is even no need to include a parallel to the third condition from Theorem 3.

Theorem 8. *A continuum X is homeomorphic to the modified gasket if and only if*

1. *X is planar and locally connected,*
2. *any two links in X are either disjoint or their intersection is an arc,*
3. *X contains at least three links each pair of which intersects,*
4. *whenever there are three links each pair of which intersects, there are two other links which intersect each of the three given links.*

Table 1: A comparison of some properties of selected continua

	Number of topologically distinct embeddings into the plane	Contains a dense set of topologically equivalent points
Sierpiński triangle	infinite	no [Wi00]
Sierpiński carpet	one	yes [Kr69]
Apollonian gasket	one	yes

The reader may be confused by that we didn't give any 'direct' characterization of the Sierpiński triangle. This is partially explained by Table 1 where a comparison with two other more or less related continua is given. There are some crucial differences between the Sierpiński triangle on one side and the two other continua which possess nice direct characterizations on the other side. We have shown a direct topological characterization of the Apollonian gasket. The *Sierpiński carpet* [Ku68, p. 275], which arise from a solid square by partitioning it into 9 congruent squares, eliminating the central one and repeating this process inductively in all 8 remaining squares, is characterized as a one-dimensional locally connected planar continuum with no local cut points [Wh58]. We believe that there is no nice internal characterization of the Sierpiński triangle.

The Sierpiński triangle is generalized in [Wi00] in the following way. We take any $n \in \mathbb{N}$ and we consider a space T_n obtained as a quotient of $\{0, 1, \dots, n-1\}^{\mathbb{N}}$ where every sequence $(a_1, a_2, \dots, a_m, i, \bar{j})$ is identified with $(a_1, a_2, \dots, a_m, j, \bar{i})$ for $i, j, a_1, \dots, a_m \in \{0, \dots, n-1\}$ and $m \in \mathbb{N}_0$. We gave in Theorem 6 a topological characterization of the Sierpiński triangle which is homeomorphic to T_3. Thus there is another natural problem related.

Problem 9. Give a topological characterization of the space T_4.

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Chapter 3.
Waraszkiewicz spirals revisited

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Abstract

We study compactifications of a ray with the remainder a simple closed curve. We give necessary and sufficient conditions for the existence of a bijective (resp. surjective) mapping between two such continua. Using these conditions we present a simple proof of the existence of an uncountable family of plane continua no one of which can be continuously mapped onto any other (the first such family was so called Waraszkiewicz's spirals created by Z. Waraszkiewicz in 1930's [Wa32]).

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Introduction

In 1930's Z. Waraszkiewicz constructed an uncountable family of plane continua no one of which can be mapped onto any other by a continuous mapping [Wa32]. This family consists of some of those continua which can be obtained as compactifications of the ray with the remainder a simple closed curve.

Using the same construction Z. Waraszkiewicz proved [Wa34] that there is no universal continuum (a continuum which can be mapped onto all continua) solving a problem posed by M. H. Hahn [Ha30, p. 357].

Unfortunately both proofs are very technical and long. A nice short proof of the second result was given by D. P. Bellamy (the proof was never published, a modification of Bellamy's original proof can be found in [MT84, p. 49–50]).

In the present paper a simple proof of the first mentioned Waraszkiewicz's result is given (notice that the existence of an uncountable family with no common preimage does not imply the incomparability of members of the family, i.e. the second result will not imply the first one).

Notice that since Waraszkiewicz's results a lot of other attempts has been done in order to construct an uncountable family with some additional properties. We mention the authors D. P. Bellamy [Be71] for chainable continua, M. M. Awartani [Aw93] for chainable compactifications of a ray, P. Minc [Mi10] for dendroids and C. Islas [Is07] for planar fans.

The existence of common models for some classes of continua (a continuum which can be mapped onto all members of a given class) is discussed in R. L. Russo [Ru79] (using the original Waraszkiewicz's method from [Wa34]).

Our proof consists of several steps. First, we reformulate the original topological problem into the language of real functions. Consequently, we prove that it is enough to obtain a special combinatorial structure. Finally we construct such a structure.

Preliminaries

A *continuum* means a nonempty compact connected metrizable space. By a *path component* of a continuum we mean a maximal subset S of this continuum, such that for any pair of points x and y from S there always exists a continuous mapping of the unit interval into S which equals to x at zero and equals to y at the point one.

An *arc* means a space homeomorphic to the closed unit interval $[0, 1]$. A *ray* means a space homeomorphic to the interval $[0, 1)$. A *simple closed curve* means a space homeomorphic to the unit circle. A *spiral* means a continuum obtained by a compactification of a ray with the remainder a simple closed curve.

We denote by $\mathbb{S}$ the unit circle $\{z \in \mathbb{C} \colon |z| = 1\}$ in the complex plane. A *lift* of a continuous mapping $\varphi \colon \mathbb{S} \to \mathbb{S}$ is a continuous function $j \colon \mathbb{R} \to \mathbb{R}$ such that $e^{2\pi i j(s)} = \varphi(e^{2\pi i s})$ for any $s \in \mathbb{R}$.

An *almost disjoint* system $\mathcal{S}$ on a set S is a family of subsets of S, such that the intersection of any two distinct elements of $\mathcal{S}$ is finite.

We say that two continua are *(continuously) comparable* if there is a mapping from one onto the other. Otherwise we call them *mutually incomparable*.

Spirals

We give a complete characterization of comparability of two spirals using real functions only. Consequently we give a simple necessary condition for comparability of two spirals.

Notation 1. We denote by $\mathbb{H}$ the ray $[0,\infty)$ and by $\overline{\mathbb{H}}$ the one point compactification $\mathbb{H}\cup\{\infty\}$. With every continuous function $f\colon \mathbb{H}\to\mathbb{R}$ we associate a continuum

$$W_f = \left\{\left(e^{2\pi i f(t)}, t\right)\in \mathbb{S}\times\mathbb{H}\colon t\geq 0\right\}\cup(\mathbb{S}\times\{\infty\})$$

considered as a subspace of $\mathbb{S}\times\overline{\mathbb{H}}$.

Theorem 2. *Let $f,g\colon\mathbb{H}\to\mathbb{R}$ be two continuous functions which do not have a finite limit at infinity. Then W_f can be continuously mapped onto W_g if and only if there exist $k\in\mathbb{Z}$ and continuous functions $h\colon\mathbb{H}\to\mathbb{H}$ and $j\colon\mathbb{R}\to\mathbb{R}$ such that*

- *the mapping h is onto,*
- $\lim_{t\to\infty} h(t)=\infty$*,*
- $j(s+1)-j(s)=k$ *for every* $s\in\mathbb{R}$*,*
- $j(\mathbb{R})$ *contains an interval of length one,*
- $\lim_{t\to\infty}\big(j\circ f(t)-g\circ h(t)\big)=0.$

Moreover W_f is homeomorphic to W_g if and only if there is a homeomorphism $h\colon\mathbb{H}\to\mathbb{H}$ and a homeomorphism $j\colon\mathbb{R}\to\mathbb{R}$ such that all the five conditions hold with $k=1$.

Proof. Let us start with the direct implication. We suppose there is a continuous onto mapping $\varphi\colon W_f\to W_g$. Since neither f nor g has a finite limit at infinity, it follows that the continuum W_f as well as W_g has two path components one of which is dense and the other not. This implies that the dense path component $\left\{\left(e^{2\pi i f(t)},t\right):t\geq 0\right\}$ of W_f is mapped onto the dense path component $\left\{\left(e^{2\pi i g(t)},t\right):t\geq 0\right\}$ of W_g and $\mathbb{S}\times\{\infty\}$ is mapped onto $\mathbb{S}\times\{\infty\}$ by φ. We denote by $p\colon\mathbb{S}\times\mathbb{H}\to\mathbb{H}$ the projection onto the second coordinate and define a function $h\colon\mathbb{H}\to\mathbb{H}$ by $h(t)=p\circ\varphi\left(e^{2\pi i f(t)},t\right)$. We can observe that

$$\varphi\left(e^{2\pi i f(t)},t\right)=\left(e^{2\pi i g\circ h(t)},h(t)\right).$$

The function h is continuous and onto. Moreover $h(t)$ converges to infinity as t goes to infinity.

There exists a continuous function $j'\colon\mathbb{R}\to\mathbb{R}$, which is a lift of the mapping φ restricted to $\mathbb{S}\times\{\infty\}$. Hence $\varphi\left(e^{2\pi i s},\infty\right)=\left(e^{2\pi i j'(s)},\infty\right)$ for every $s\in\mathbb{R}$. It holds that $j'(s+1)-j'(s)$ is a fixed integer k. Since $\varphi(\mathbb{S}\times\{\infty\})=\mathbb{S}\times\{\infty\}$, we get that $j'([0,1])$ contains an interval of length one.

Now we are going to prove that all the cluster points of the function $j'\circ f-g\circ h$ are integers i. e. if $t_n\to\infty$ and $j'\circ f(t_n)-g\circ h(t_n)$ converges to $r\in\mathbb{R}\cup\{-\infty,\infty\}$ then $r\in\mathbb{Z}$. We proceed by contradiction. Suppose there is a cluster point $r\in\mathbb{R}\cup\{-\infty,\infty\}$ of the function $j'\circ f-g\circ h$ which is not an integer.

Suppose first that $r \in \mathbb{R}$. Then there is a sequence (t_n) in $\mathbb{H}$ which converges to infinity and for which $j' \circ f(t_n) - g \circ h(t_n)$ converges to r. By a suitable choice of a subsequence we can assume that $e^{2\pi i f(t_n)}$ converges to some point $e^{2\pi i s}$ for some $s \in \mathbb{R}$. Then $e^{2\pi i j' \circ f(t_n)}$ converges to $e^{2\pi i j'(s)}$. Since φ is continuous we get that $\varphi\left(e^{2\pi i f(t_n)}, t_n\right)$ converges to $\varphi(e^{2\pi i s}, \infty)$. Thus $\left(e^{2\pi i g \circ h(t_n)}, h(t_n)\right)$ converges to $\left(e^{2\pi i j'(s)}, \infty\right)$ and hence $e^{2\pi i g \circ h(t_n)}$ converges to $e^{2\pi i j'(s)}$. By dividing we obtain that $e^{2\pi i (g \circ h(t_n) - j' \circ f(t_n))}$ converges to 1. But it converges to $e^{-2\pi i r}$, thus $r \in \mathbb{Z}$. This is a contradiction.

If $r \in \{-\infty, \infty\}$ we can find a sequence (t_n) in $\mathbb{H}$ converging to infinity such that $e^{2\pi i (j' \circ f(t_n) - g \circ h(t_n))} = -1$ for every n. We obtain a contradiction as in the previous part.

Since $j' \circ f - g \circ h$ is a continuous function whose cluster points are integers, it follows that there is an integer m which is the only cluster point and thus it is the limit of this function at infinity.

Finally we put $j = j' - m$. It follows that $j(s+1) - j(s) = k$ for every $s \in \mathbb{R}$, $j(\mathbb{R})$ contains an interval of length one and that

$$\lim_{t \to \infty} \big(j \circ f(t) - g \circ h(t)\big) = m - m = 0.$$

Moreover if φ is a homeomorphism we obtain that the mappings h and j are homeomorphisms and that $k = 1$.

For the reversed implication suppose we are given k, h and j as in the statement. We define $\varphi \colon W_f \to W_g$ by

$$\varphi\left(e^{2\pi i f(t)}, t\right) = \left(e^{2\pi i g \circ h(t)}, h(t)\right) \text{ if } t \in \mathbb{H},$$

$$\varphi\left(e^{2\pi i s}, \infty\right) = \left(e^{2\pi i j(s)}, \infty\right) \text{ if } s \in \mathbb{R}.$$

This is a correctly defined mapping since $j(s+1) - j(s)$ is an integer. The mapping φ is onto, because h is onto and $j(\mathbb{R})$ contains an interval of length one. Clearly φ is continuous in all points of the form $\left(e^{2\pi i f(t)}, t\right)$ for $t \in \mathbb{H}$, because h is a continuous function. Moreover the restriction of φ to the set $\mathbb{S} \times \{\infty\}$ is continuous, because j is continuous. It remains to show that the φ image of a sequence $\left(e^{2\pi i f(t_n)}, t_n\right)$ converging to a point $(e^{2\pi i s}, \infty)$ converges to $\varphi\left(e^{2\pi i s}, \infty\right)$. Thus we have to show that $\left(e^{2\pi i g \circ h(t_n)}, h(t_n)\right)$ converges to $\left(e^{2\pi i j(s)}, \infty\right)$. Clearly $h(t_n) \to \infty$, because $h(t) \to \infty$ as $t \to \infty$. We know that $e^{2\pi i (j \circ f(t_n) - g \circ h(t_n))}$ goes to one and that $e^{2\pi i j \circ f(t_n)}$ converges to $e^{2\pi i j(s)}$. Thus if we consider the quotient of these two sequences, we obtain that $e^{2\pi i g \circ h(t_n)}$ converges to $e^{2\pi i j(s)}$. Thus φ maps continuously W_f onto W_g. Moreover if the mappings h and j are homeomorphisms and $k = 1$ we can easily observe that φ is a one-to-one mapping and thus it is a homeomorphism, because it is a bijection of two compact spaces. □

Corollary 3. *Let $f, g \colon \mathbb{H} \to \mathbb{R}$ be two continuous functions which do not have a finite limit at infinity. If W_f can be continuously mapped onto W_g then there exist a continuous function $h \colon \mathbb{H} \to \mathbb{H}$ converging to infinity and $k \in \mathbb{Z}$ such that*

$$\sup_{t \in \mathbb{H}} |k f(t) - g \circ h(t)| < \infty.$$

Proof. By Theorem 2 there exist $k \in \mathbb{Z}$, $h \colon \mathbb{H} \to \mathbb{H}$ and $j \colon \mathbb{R} \to \mathbb{R}$ with the given properties. We can realize that the function $s \mapsto j(s) - ks$ is periodic and continuous and hence bounded. Let l be a number such that $|j(s) - ks| + 1 \leq l$ for every $s \in \mathbb{R}$. Since $(j \circ f(t) - g \circ h(t)) \to 0$ as $t \to \infty$, there exists $t_0 \in \mathbb{H}$ such that for every $t > t_0$ the inequality $|j \circ f(t) - g \circ h(t)| \leq 1$ holds. Thus for $t > t_0$ we obtain

$$|kf(t) - g \circ h(t)| \leq |kf(t) - j \circ f(t)| + |j \circ f(t) - g \circ h(t)| \leq l - 1 + 1 = l.$$

Moreover the function $|kf(t) - g \circ h(t)|$ is continuous and thus it is bounded on the interval $[0, t_0]$. Hence we obtain that the required inequality holds. □

Example 4. Let $f \colon \mathbb{H} \to \mathbb{R}$ be the identity function and let $g \colon \mathbb{H} \to \mathbb{R}$ be a piecewise linear function whose breakpoints are at positive integers and for which $g(2n-2) = 0$ and $g(2n-1) = 2n-1$ for every positive integer n. We claim that the corresponding spirals W_f and W_g are incomparable.

If we suppose first that W_f can be continuously mapped onto W_g then by Corollary 3 there is a continuous mapping $h \colon \mathbb{H} \to \mathbb{H}$ converging to infinity and $k \in \mathbb{Z}$, such that

$$\sup_{t \in \mathbb{H}} |kf(t) - g \circ h(t)| < \infty.$$

Hence $\sup_{t \in \mathbb{H}} |kt - g \circ h(t)| < \infty$ and we denote by l the supremum. There is an increasing sequence of positive numbers $(t_i)_{i=1}^{\infty}$ converging to infinity such that $h(t_i)$ is always an even integer. Thus we have that $|kt_i - g \circ h(t_i)| \leq l$ and hence $|kt_i| \leq l$ for every i. Thus $k = 0$. Let $(u_i)_{i=1}^{\infty}$ be an increasing sequence converging to infinity for which $h(u_i)$ is always an odd integer and $u_1 > t_0$. Then we have that $|g \circ h(u_i)| \leq l$ for every i and hence $|h(u_i)| \leq l$ for every i. This is a contradiction with the fact that the function h converge to infinity.

On the other hand suppose for contradiction that W_g can be continuously mapped onto W_f. By Corollary 3 there is a continuous mapping $h \colon \mathbb{H} \to \mathbb{H}$ converging to infinity and $k \in \mathbb{Z}$

$$\sup_{t \in \mathbb{H}} |kg(t) - f \circ h(t)| < \infty.$$

Hence $\sup_{t \in \mathbb{H}} |kg(t) - h(t)| < \infty$ and we denote this supremum by l. Let $(t_i)_{i=1}^{\infty}$ be an increasing sequence of even positive integers converging to infinity. Then $|kg(t_i) - h(t_i)| \leq l$ and thus $|h(t_i)| \leq l$ for every i. This is a contradiction with the fact that the function h converges to infinity.

Peak points

We define a notion of a peak point. We prove two simple lemmas about the behavior of this notion with respect to composition of functions and with respect to near functions.

Definition 5. Let $I \subseteq \mathbb{R}$ be an interval, $f \colon I \to \mathbb{R}$ a continuous function, $x \in I$ and $v \in \mathbb{R}$. We say that f has a *peak of height* v at the point x if there exists an interval $[a, b] \subseteq I$ containing x such that $f(t) \leq f(x)$ for all $t \in [a, b]$ and $f(a) \leq f(x) - v$ and $f(b) \leq f(x) - v$.

Lemma 6. *Let $I \subseteq \mathbb{R}$ be an interval, $g\colon I \to \mathbb{R}$ a continuous function and $h\colon I \to I$ a continuous onto function. If the function g has a peak of height v at the point y, then there is a point $x \in I$ such that the function $g \circ h$ has a peak of height v at the point x and $g(y) = g \circ h(x)$.*

Proof. Since the function h is onto there is a point $x \in I$ such that $h(x) = y$. Clearly $g(y) = g \circ h(x)$. Since y is a peak point of height v, there exists an interval $[a, b] \subseteq I$ containing y such that $g(t) \leq g(y)$ for $t \in [a, b]$, $g(a) \leq g(x) - v$ and $g(b) \leq g(x) - v$. There is an interval $[c, d] \subseteq I$ containing y for which $h(\{c, d\}) = \{a, b\}$ and $h([c, d]) = [a, b]$. Clearly for any $t \in [c, d]$ we get that $g \circ h(t) \leq g(y) = g \circ h(x)$ and $g \circ h(c)$ as well as $g \circ h(d)$ are less than $g(y) - v = g \circ h(x) - v$. Thus x is a peak point of the function $g \circ h$ of height v. □

Lemma 7. *Let $I \subseteq \mathbb{R}$ be an interval, $u > 0$ and $f, g\colon I \to \mathbb{R}$ two continuous functions such that $|f - g| \leq u$. If f has a peak point x of height v then g has a peak point y of height $v - 2u$ such that $|f(x) - g(y)| \leq u$.*

Proof. There is an interval $[a, b] \subseteq I$ containing x such that $f(t) \leq f(x)$ for every $t \in [a, b]$, $f(a) \leq f(x) - v$ and $f(b) \leq f(x) - v$. We take a point $y \in [a, b]$ in which g attains its maximum on the interval $[a, b]$. It follows that $|f(x) - g(y)| \leq u$. It holds that $g(a) \leq f(a) + u \leq f(x) - v + u \leq g(y) + u - v + u = g(y) - (v - 2u)$. Similarly $g(b) \leq g(y) - (v - 2u)$. Hence g has a peak point y of height $v - 2u$. □

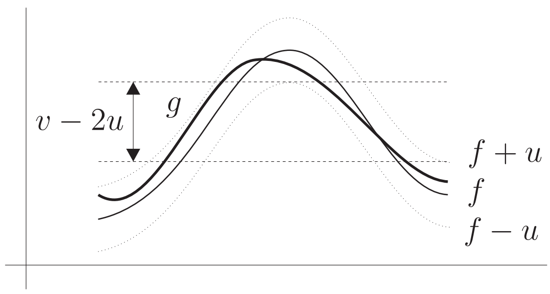

Figure 1: Peak point of height $v - 2u$ for the mapping g.

Reduction to a discrete case

We associate a function with every subset of positive integers. Then we give a condition for the sets under which the corresponding spirals of the associated functions are not comparable.

Notation 8. For an infinite set M of positive integers $m_1 < m_2 < \dots$, we define a continuous piecewise linear function $f_M\colon \mathbb{H} \to \mathbb{R}$ whose only break points are at positive integers and $f(2i - 2) = 0$ and $f(2i - 1) = m_i$ for any positive integer i.

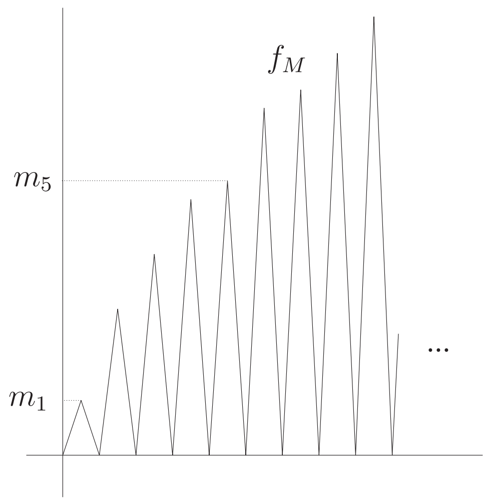

Figure 2: Piecewise linear mapping f_M.

Proposition 9. *Let M, N be two infinite sets of positive integers and suppose that W_{f_M} can be continuously mapped onto W_{f_N}. Then there exist positive integers k and l such that for every $n \in N$ there is $m \in M$ for which $|km - n| \le l$.*

Proof. Denote $M = \{m_1 < m_2 < \dots\}$ and $N = \{n_1 < n_2 < \dots\}$. By Corollary 3 there exist a continuous function $h: \mathbb{H} \to \mathbb{H}$ converging to infinity, $k \in \mathbb{Z}$ and positive integer l' such that $|kf_M(t) - f_N \circ h(t)| \le l'$ for any $t \in \mathbb{H}$. Since f_M as well as $f_N \circ h$ are unbounded nonnegative functions we obtain that $k > 0$. Fix a positive integer q bigger than t_0 and $2l'$. Let l be a positive integer bigger than

$$\max\{l', |km_1 - n_1|, |km_1 - n_2|, \dots, |km_1 - n_q|\}.$$

For any positive integer $j > q$ the function f_N has a peak point $2j - 1$ of height n_j and $f_N(2j - 1) = n_j$. By Lemma 6 the function $f_N \circ h$ has a peak point x of height n_j and $f_N \circ h(x) = n_j$. Now, we use Lemma 7 applied on the functions $g = kf_M$ and $f = f_N \circ h$ on the interval $I = (0, \infty)$. We get that the function kf has a peak point y of height $n_j - 2l'$ for which $|kf_M(y) - f_N \circ h(x)| \le l'$. Note that $n_j - 2l' \ge j - 2l' > q - 2l' > 0$. We can see that the only peak points of positive height of the function kf_M are odd positive integers. Hence, there exists positive integer i such that $2i - 1 = y$. Consequently $kf_M(2i - 1) = km_i$ and thus $|km_i - n_j| \le l' \le l$. □

Uncountable family

We construct an uncountable system of infinite subsets of positive integers such that for any pair of distinct elements the conclusion of Proposition 9 is not satisfied.

Proposition 10. *There exists an uncountable system $\mathcal{S}$ of infinite subsets of positive integers such that for any pair of distinct sets $M, N \in \mathcal{S}$ and for any positive integers k and l there is a point $n \in N$, such that $|km - n| > l$ for every $m \in M$.*

Proof. Let $\mathcal{S}$ be an uncountable almost disjoint system of infinite subsets of the set $\{1!, 2!, 3!, \dots\}$. Suppose that $M, N \in \mathcal{S}$ are two distinct sets and we are given two positive integers k and l. Let $n = j!$ be an element of $N \setminus M$ such that j is bigger than k and $l+1$. Now for any $m = i! \in M$ we obtain that if $i < j$ then

$$|km - n| \geq n - km = j! - ki! \geq j! - (j-1)(j-1)! = (j-1)! \geq j - 1 > l.$$

and if $i > j$ then

$$|km - n| \geq km - n = ki! - j! \geq i! - j! \geq (j+1)! - j! \geq j > l.$$

□

Corollary 11. *There exists an uncountable collection of incomparable plane continua.*

Proof. We take an uncountable system $\mathcal{S}$ from Proposition 10. It follows from Proposition 9 that the collection W_{f_M} where $M \in \mathcal{S}$ contains pairwise incomparable continua. All of them are planar because they are subcontinua of $\mathbb{S} \times \overline{\mathbb{H}}$. □

Remarks

We note that the system $\mathcal{S}$ in Proposition 10 can be made of size continuum (well known simple constructions not using the axiom of choice). Moreover in the proof of Proposition 10 we do not need an almost disjoint system. It suffices to take a system of subsets of positive integers, such that the difference of any two distinct sets is infinite.

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Chapter 4.
A lambda-dendroid with two shore points whose union is not a shore set

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Abstract

A subset of a given continuum is called a shore set if there is a sequence of continua in the complement of this set converging to the whole continuum with respect to the Hausdorff metric. A point is called a shore point if the one point set containing this point is a shore set. We present several examples of a lambda-dendroid which contains two disjoint shore continua whose union is not a shore set. This answers a question of Van C. Nall in negative.

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Introduction

A *continuum* means a nonempty compact connected metrizable space. A continuum is said to be *decomposable* if it can be written as a union of two proper subcontinua and it is called *hereditarily decomposable* if every nondegenerate subcontinuum is decomposable. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any pair of its subcontinua is connected or empty. A *dendroid* is a hereditarily unicoherent arcwise connected continuum. A λ-*dendroid* is a hereditarily unicoherent and hereditarily decomposable continuum. It is well known that any dendroid is a λ-dendroid [Na92, p. 226].

A continuous mapping is called *monotone* provided preimages of points are connected. A continuous mapping between two continua is called *hereditarily monotone* if any restriction of this mapping to a subcontinuum is monotone.

A subset A of a continuum X is called a *shore set* if there is a sequence of subcontinua of X disjoint with A whose limit is the whole space X (with respect to the Hausdorff metric). A point $x \in X$ is called a *shore point* if $\{x\}$ is a shore set.

We denote by I the interval $[0,1]$ and by C the Cantor set i.e. the subset of I consisting of all numbers of the form $\sum 3^{-i}a_i$ where $a_i \in \{0,2\}$.

The examples

We provide here a negative answer to the question of Van C. Nall, who asked in [Na07, Question 4.7] if the union of two disjoint shore subcontinua of a λ-dendroid is a shore set. We recall that this is known to be true in the realm of dendroids as shown in [Il01, Theorem 3].

Our first example is strongly influenced by an example of a dendroid from [Il01, Example 5] in which the union of two (nondisjoint) shore subcontinua is not a shore set. In Example 1 we define a λ-dendroid $X \subseteq \mathbb{R}^3$ with two disjoint shore continua A and B, such that $A \cup B$ is not a shore set.

Example 1. "Wavy hair". For any $c \in C \setminus \{0\}$, where $c = \sum 3^{-i}a_i$ we denote by $j(c)$ the smallest natural number n for which $a_n = 2$. We define a mapping $f\colon C \to C$ by $f(0) = 0$ and $f(c) = \sum_{i=j(c)}^{\infty} 3^{-i}(2-a_i)$ for $c \in C \setminus \{0\}$, $c = \sum 3^{-i}a_i$.

For any $c \in C \setminus \{0\}$ we denote by O_c the set

$$\left\{\left(\frac{1}{2\cdot 3^{j(c)-1}} + r\cos\varphi, r\sin\varphi, (-1)^{j(c)}\left(1+2^{-j(c)}\right)\right)\colon r = c - \frac{1}{2\cdot 3^{j(c)-1}}, \varphi \in [0,\pi]\right\}.$$

Thus O_c is an arc joining the points $(c, 0, (-1)^{j(c)}(1+2^{-j(c)}))$ and $(f(c), 0, (-1)^{j(c)}(1+2^{-j(c)}))$. We define

$$Z = \left(C \times \{0\} \times [-2,2]\right) \cup \bigcup \{O_c\colon 0 \neq c \in C\}$$

and $Z' = \{(-x,y,z)\colon (x,y,z) \in Z\}$. Finally we put $X = Z \cup Z'$.

We denote $a = (0,0,1)$ and $b = (0,0,-1)$. Let A be the segment between a and $(0,0,2)$, B the segment between b and $(0,0,-2)$, and F the segment between a and b.

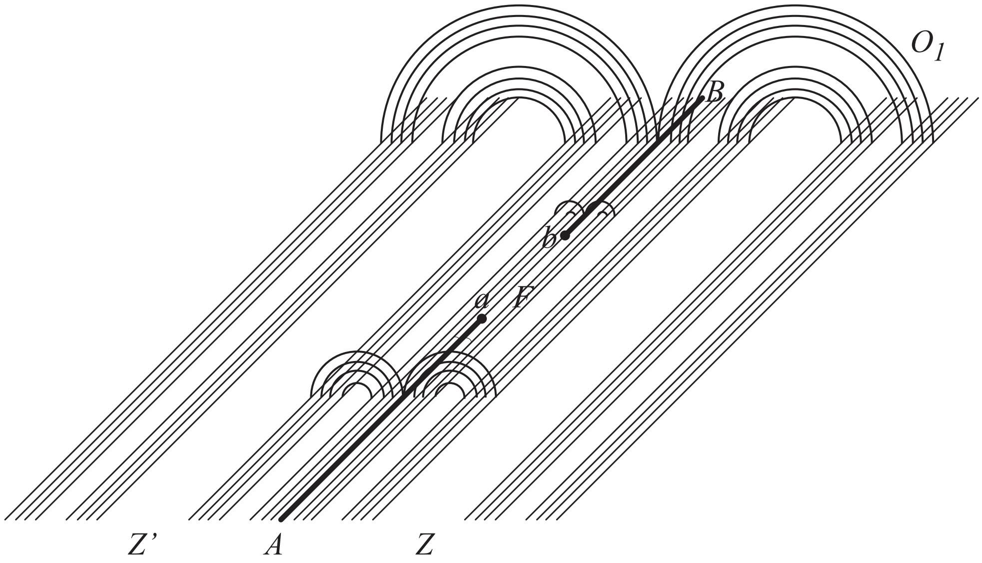

Figure 1: A λ-dendroid X with shore sets A and B.

It is easily verified that the space X is a continuum. It was shown in [Il01, Example 5], that a continuum homeomorphic to the quotient X/F is a dendroid. Thus X/F is a dendroid and by Proposition 7 we deduce that X is a λ-dendroid — the only assumption which remains to verify is that the quotient mapping $q\colon X \to X/F$ is hereditarily monotone. Thus take any subcontinuum K of X and suppose that the restriction $q{\restriction}K$ is not monotone. This is only possible when $K \cap F$ is not connected. Thus $a, b \in K$ but there is a point $p = (p_1, p_2, p_3) \in F$ such that $p \notin K$. Denote by ε the distance of p from the continuum K. There is $n \in \mathbb{N}$ for which $3^{-n} < \varepsilon$. Consider the set C_1 of all $c \in C$ for which $c > 3^{-n}$ and such that the smallest $k \in \mathbb{N}$ for which $f^k(c) < 3^{-n}$ is congruent to $j(c)$ modulo 2, or $0 < c \leq 3^{-n}$ and $j(c)$ is odd. Let C_2 be the set $C \setminus (C_1 \cup \{0\})$ and let I_k be the closed interval with end points $(-1)^k \cdot 2$ and p_3 for $k = 1, 2$. Consider the following pair of closed sets

$$D_k = \big((C \cap [0, 3^{-n}]) \times \{0\} \times I_k\big) \cup \big((C_k \setminus [0, 3^{-n}]) \times \{0\} \times [-2, 2]\big) \cup \bigcup \{O_c \colon c \in C_k\}$$

for $k = 1, 2$ and let $D'_k = \{(-x, y, z)\colon (x, y, z) \in D_k\}$. We get that $D_1 \cup D'_1$ and $D_2 \cup D'_2$ are two closed sets which cover K and which are disjoint on K, because their intersection is the set $(C \cap [-3^{-n}, 3^{-n}]) \times \{0\} \times \{p_3\}$. Since K is connected we get that K is contained in one of the two sets. Thus we get a contradiction with $a, b \in K$.

We show that A and B are shore sets. Because of similarity it is enough to show that A is a shore set. Define

$$S_n = \Big\{\sum 3^{-i} a_i \in C \colon a_i \in \{0, 2\}, 2 - a_{2n} = a_{2n+1} = a_{2n+2} = \dots\Big\}$$

and

$$M_n = B \cup \big(S_n \times \{0\} \times [-2, 2]\big) \cup \bigcup \{O_c \colon c \in S_n\}.$$

Let $M'_n = \{(-x, y, z)\colon (x, y, z) \in M_n\}$. The sequence $M_n \cup M'_n$ of continua converge to the whole space X with respect to the Hausdorff metric and moreover $M_n \cup M'_n$ is always disjoint with A.

It remains to show that the set $A \cup B$ is not a shore set. Suppose for contradiction that this is not true. Hence we can find a continuum $L \subseteq X$ which is disjoint with $A \cup B$ and which intersects $Z \setminus (A \cup B \cup F)$ as well as $Z' \setminus (A \cup B \cup F)$. Clearly L intersects F. Denote by ε the distance of L from the closed set $A \cup B$. Let U be the ε-neighborhood of $A \cup B$. Then $F \setminus U$ is a component of $X \setminus U$ and thus $L \subseteq F$. This is a contradiction.

Remark 2.

(a) We can easily observe that there is no need to include the whole symmetrical copy Z' of Z in the continuum X from Example 1. It is enough to add e.g. an arc. Let A, B and Z be as defined in Example 1 and denote

$$X = Z \cup \big([-1,0] \times \{0\} \times \{0\}\big).$$

Then X is a λ-dendroid with shore sets A and B, but $A \cup B$ is not a shore set.

(b) We can modify the λ-dendroids from Example 1 to obtain a λ-dendroid in which the union of two shore points is not a shore set. Let A and B be as defined in Example 1 and let X be the λ-dendroid from Example 1 or from the previous remark. Denote by Y the quotient of X, where all the points from A are mutually identified, all the points from B are mutually identified but no other distinct points are identified. If we denote by $q\colon X \to Y$ the quotient mapping, it follows that the only point in $q(A)$ as well as the only point in $q(B)$ are shore points, but their union is not a shore set.

(c) Both ideas in previous modifications can be used simultaneously. By collapsing all the arcs of the form $\{c\} \times \{0\} \times [-2,-1]$ and $\{c\} \times \{0\} \times [1,2]$ in Z to points for $c \in C$ and by gluing an arc we obtain a continuum on Figure 2.

Let us sketch another example of a λ-dendroid X with two shore points, whose union is not a shore set. In Example 1 and its modifications there are a lot of useless simple triods. Unnecessary simple triods does not occur in Example 3.

Example 3. "Braids". Suppose that C' is an isometric copy of the Cantor set C in the segment connecting the points $(0,1)$ and $(1,1)$ in the plane. Let us consider a continuum Y in the plane which consists of the segment with end points $(0,0)$ and $(1,0)$, each segment connecting the point $(0,0)$ with a point in C' whose first coordinate is less than one half, and each segment connecting the point $(1,0)$ with a point in C' whose first coordinate is bigger than one half. From the topological point of view this is just a disjoint union of two copies of the Cantor fan, whose central points are connected with an arc.

Now, we make a simple modification of the continuum Y in each open strip bounded by the lines $y = 1/n$ and $y = 1/(n+1)$ for $n \in \mathbb{N}$. For $n = 1$ we do nothing. For $n = 2$ we cancel all the points in the strip $\mathbb{R} \times (1/3, 1/2)$ which lie on some straight line connecting the point $(0,0)$ with a point in C' whose first coordinate is in the interval $[2/9, 3/9]$. Similarly we delete all the points in the strip which lies on some straight line connecting the point $(1,0)$ with a point in

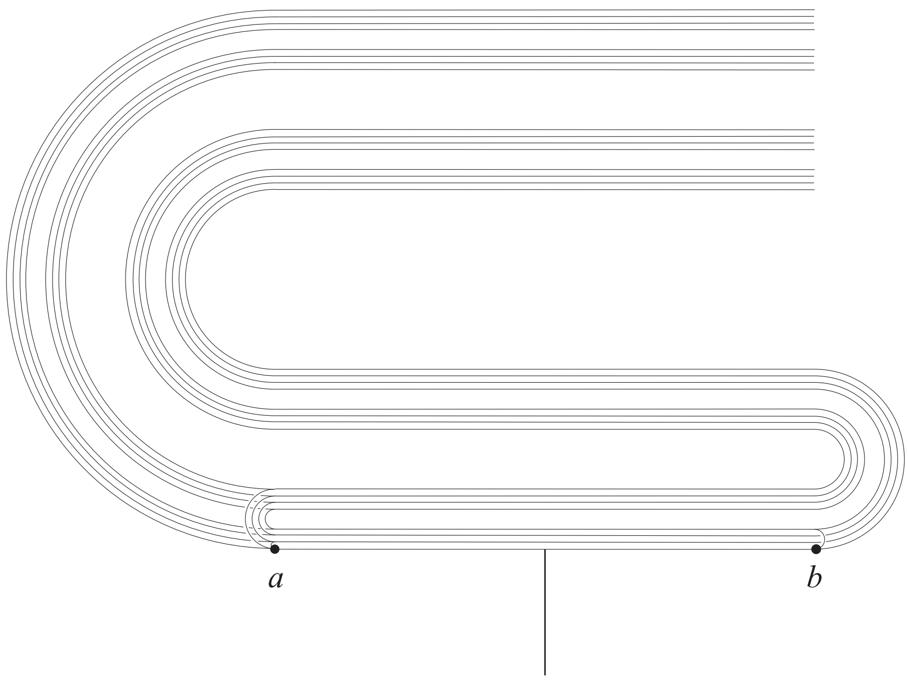

Figure 2: A λ-dendroid X with two shore points whose union is not a shore set.

C' whose first coordinate is in the interval $[6/9, 7/9]$. On the other hand we add two topological copies of $C \times I$ which are glued to the rest according to Figure 3. This is somehow realized in $\mathbb{R}^3$.

We proceed similarly in the third strip by deleting the points on the lines connecting the point $(0,0)$ with a point in $C' \cap ([2/27, 3/27] \cup [8/27, 9/27])$ and the points on the lines connecting the point $(1,0)$ with a point in $C' \cap ([18/27, 19/27] \cup [24/27, 25/27])$. Now we glue four topological copies of $C \times I$ over the third strip according to the Figure 3.

Observe that we have two copies of the Cantor set on the level $y = 1$. We see four copies of C on the level $y = 1/2$, we number them from left to right by numbers from 1 to 4. In the strip between levels $y = 1/2$ and $y = 1/3$ we switch the second and third copy. We make one switch. Similarly we see eight copies of C on the level $y = 1/3$, we number them from left to right by numbers from 1 to 8. In the strip between levels $y = 1/3$ and $y = 1/4$ we switch the second and seventh copy, simultaneously we switch the fourth and fifth copy. We make two switches.

Similarly we see 16 copies of C on the level $y = 1/4$, we number them from left to right by numbers from 1 to 16. In the strip between levels $y = 1/4$ and $y = 1/5$ we switch the 2nd and 15th, 4th and 13th, 6th and 11th and finally 8th and 9th copy. We make 4 switches.

If such a modification is done appropriately in each of the strips, we obtain a λ-dendroid X with the property that any point in C' whose first coordinate is in the set $\{\sum_{i=1}^{n} 3^{-i} a_i \colon a_i \in \{0, 2\}, n \in \mathbb{N}\}$ is connected by an arc, which does not contain $(1,0)$, with the point $(0,0)$. Moreover the set of these arcs is dense in the continuum X. A symmetric result holds too. Thus we can realize that the

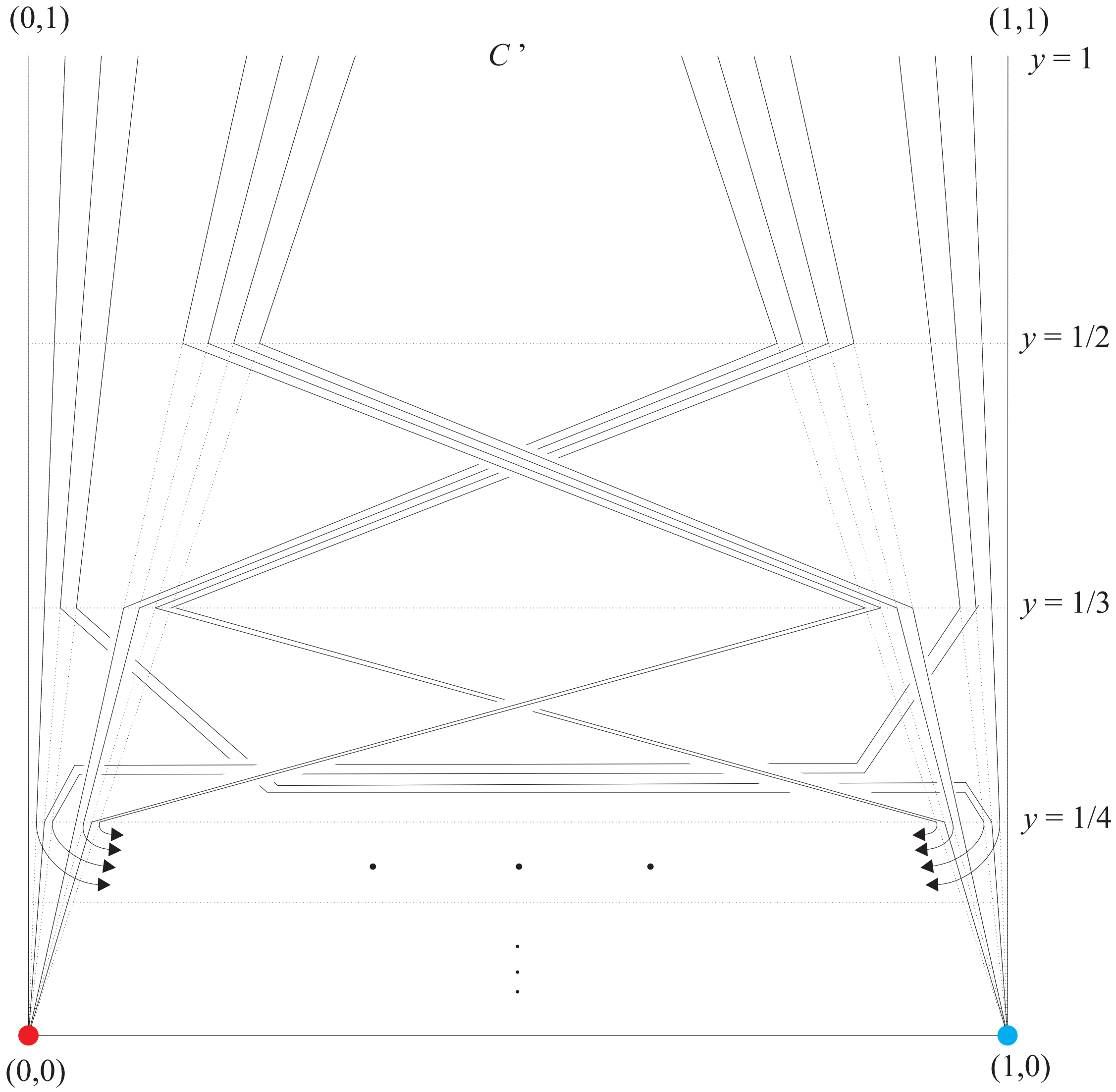

Figure 3: A λ-dendroid X with shore points $(0,0)$ and $(1,0)$.

points $(0,0)$ and $(1,0)$ are shore points. Their union is not a shore set, since any nondegenerate subcontinuum of X not containing $(0,0)$ and $(1,0)$ is just an arc.

Let us define the last example of a λ-dendroid X with two shore points a and b, whose union is not a shore set. We note that the λ-dendroids in Example 3 and Example 4 use the similar idea.

Example 4. "Dreads". Consider a continuum

$$Y = \big(\{0\} \times [-1,1] \times [0,1]\big) \cup \big\{(x, \sin\frac{1}{x}, z) \colon z \in C, x \in (0,1]\big\}$$

and let $\{c_n \colon n \in \mathbb{N}\}$ be a one-to-one sequence for which $\{c_{2n} \colon n \in \mathbb{N}\}$ and $\{c_{2n-1} \colon n \in \mathbb{N}\}$ are dense subsets of C. See Figure 4.

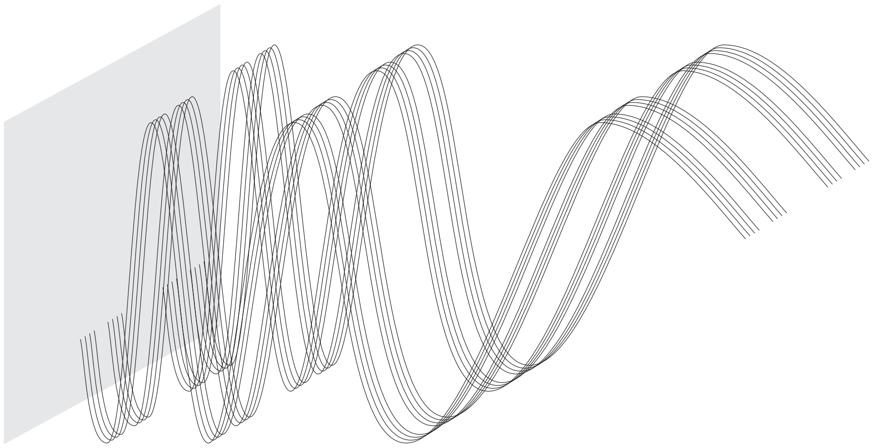

Figure 4: The continuum Y.

We define X as the quotient $Y/\sim$, where $\sim$ is the smallest equivalence on Y for which $(0,y,z) \sim (0,y,z')$ and $(x, \sin\frac{1}{x}, c_n) \sim (0, \sin\frac{1}{x}, c_n)$ for $x \le \frac{2}{2\pi n - \pi}$ and $n \in \mathbb{N}$. Let a and b be the points in X which are the equivalence classes of $(0,1,0)$ and $(0,-1,0)$ respectively. We mention without proof that X is a λ-dendroid, the points a and b are shore points and the set $\{a,b\}$ is not a shore set.

Remark 5. In other words in Dreads in Example 4 we have for $n \in \mathbb{N}$ rays

$$\alpha_n = \big\{(x, \sin\frac{1}{x}, c_{2n-1}) \colon x \in (0,1]\big\}$$

and

$$\beta_n = \big\{(x, \sin\frac{1}{x}, c_{2n}) \colon x \in (0,1]\big\}\ .$$

The ray α_n is "terminated" (glued to the base $\{0\} \times [-1,1] \times [0,1]$ using the equivalence) at the level $a_n = \frac{2}{2\pi n - \pi}$ (the decreasing sequence of all local maxima of $\sin 1/x$ function on $(0,1)$) , the ray β_n is terminated at the level $b_n = \frac{2}{2\pi n}$ (the decreasing sequence of all local minima of $\sin 1/x$ function on $(0,1)$). See Figure 5.

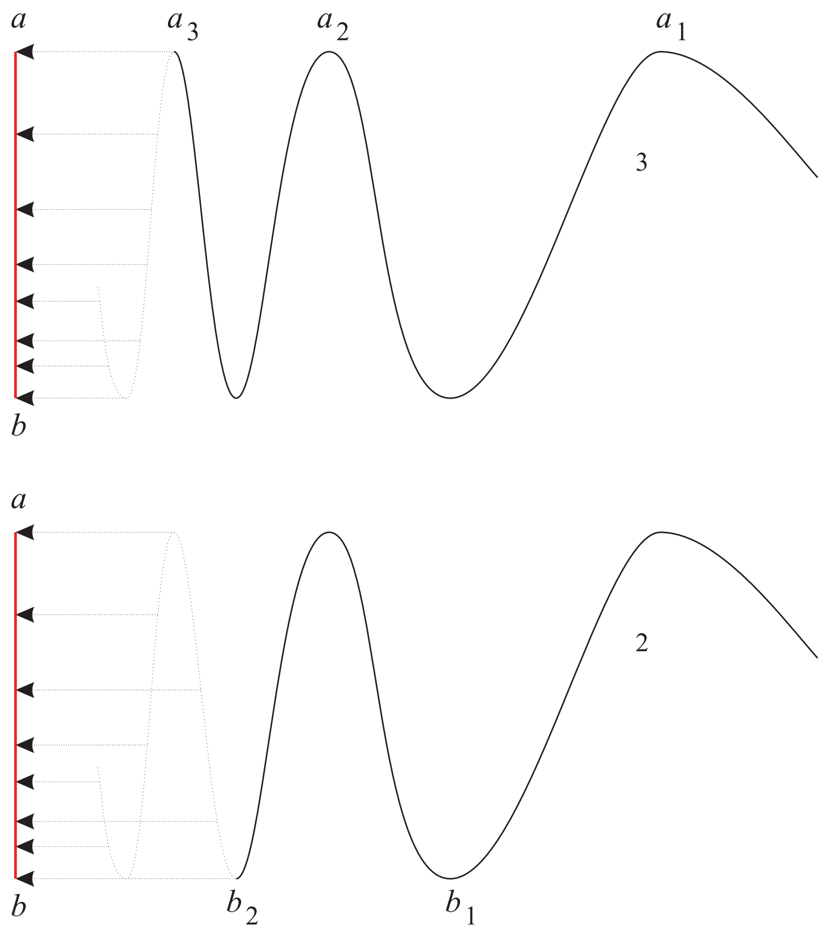

Figure 5: Terminating of rays α_n and β_n.

We can rewrite this termination levels as a_{t_n} and b_{t_n} for $t_n = n$. Using the sequence

$$\{t_n\} = 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, \ldots$$

(the number n occurs 2^{n-1} times) instead of $t_n = n$ in the above description of Dreads we obtain a topological copy of the Braids in Example 3. Notice that the number of switches in each level of Braids example corresponds the number of rays being terminated at the same level in the Dreads Example 4.

Tools

Proposition 6. *Let $f\colon X \to Y$ be a (continuous) hereditarily monotone mapping of a continuum X onto a hereditarily unicoherent continuum Y, whose point inverses are hereditarily unicoherent. Then the continuum X is hereditarily unicoherent too.*

Proof. Suppose for contradiction that A and B are two subcontinua of X whose intersection $A \cap B$ is not connected. There are two nonempty closed disjoint sets E and F such that $A \cap B = E \cup F$.

Claim: $f(E) \cap f(F) = \emptyset$. Otherwise there exists a point $y \in f(E) \cap f(F)$. Then $f^{-1}(y) \cap (A \cup B)$ is a continuum because $f \restriction (A \cup B)$ is monotone. Moreover $f^{-1}(y) \cap (A \cup B) = \big(f^{-1}(y) \cap A\big) \cup \big(f^{-1}(y) \cap B\big)$ is hereditarily unicoherent and thus the set $C = f^{-1}(y) \cap A \cap B$ is connected. But $C \subseteq E \cup F$ and $C \cap E \neq \emptyset \neq C \cap F$. This contradiction proves the claim.

The intersection $f(A) \cap f(B)$ is a continuum since Y is hereditarily unicoherent. Because of the claim $\big(f(A) \cap f(B)\big) \setminus \big(f(E) \cup f(F)\big)$ is nonempty and contains a point, say z. Then $f^{-1}(z) \cap (A \cup B)$ is a hereditarily unicoherent continuum which is disjoint with $E \cup F$. But then $f^{-1}(z) \cap A$ and $f^{-1}(z) \cap B$ are two nonempty closed disjoint sets whose union is $f^{-1}(z) \cap (A \cup B)$. This contradicts connectedness of this continuum. □

Proposition 7. *Let $f \colon X \to Y$ be a (continuous) hereditarily monotone mapping of a continuum X onto a λ-dendroid Y, such that all point inverses $f^{-1}(y)$ are λ-dendroids. Then X is λ-dendroid as well.*

Proof. The continuum X is hereditarily unicoherent by Proposition 6. It remains to show, that X is hereditarily decomposable. Thus take any subcontinuum K of X. If the set $f(K)$ contains only one point, we are done. Otherwise $f(K)$ is a decomposable continuum. Hence there exist two proper subcontinua A and B of $f(K)$ such that $A \cup B = f(K)$. It remains to realize that

$$K = K \cap \big(f^{-1}(A \cup B)\big) = \big(K \cap f^{-1}(A)\big) \cup \big(K \cap f^{-1}(B)\big),$$

and that $K \cap f^{-1}(A)$ as well as $K \cap f^{-1}(B)$ are continua, because f is hereditarily monotone. Moreover these are proper subcontinua of K. Thus K is decomposable. □

Questions

The continuum in Example 1 (and its modifications in Remark 2) is clearly not planar since it contains uncountable many disjoint simple triods. The Example 3 and Example 4 seem to be nonplanar too. Thus a natural question arises.

Question 1. Does there exist a planar λ-dendroid in which the union of two disjoint shore continua is not a shore set?

We recall also an open question posed by A. Illanes in [Il01, Question 6].

Question 2. Is the union of two disjoint closed shore sets in a dendroid a shore set?

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Chapter 5.
Union of shore sets in a dendroid

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Abstract

A subset of a dendroid is called a shore set if there is a sequence of subcontinua disjoint with the given set which converges to the whole continuum. We are dealing with the question when the union of finitely many shore sets is a shore set. We give a positive answer in the case of planar smooth dendroids and closed disjoint shore sets and we present a simple example of a planar dendroid in which the union of two disjoint closed shore sets is not a shore set. The second result answers a question of A. Illanes. Finally, we show that the union of a shore point and a closed shore set in a dendroid need not to be a shore set but we prove a positive result in the case of a planar dendroid.

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Preliminaries

A *continuum* means a nonempty compact connected metrizable space. A continuum is said to be *decomposable* if it can be written as a union of two proper subcontinua and it is called *hereditarily decomposable* if every nondegenerate subcontinuum is decomposable. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any pair of its subcontinua is connected or empty. A *dendroid* is a hereditarily unicoherent arcwise connected continuum. A λ-*dendroid* is a hereditarily unicoherent and hereditarily decomposable continuum. It is well known that any dendroid is a λ-dendroid [Na92, p. 226].

An *arc* is any space homeomorphic to the closed unit interval. A *simple triod* is a space homeomorphic to the quotient of the disjoint union of three closed unit intervals, whose left end points are identified. A point in a dendroid is called a *ramification point* if it is a vertex of a simple triod. A point in a dendroid is called an *end point* if it is not contained in the interior of any arc. A *fan* is a dendroid with only one ramification point. A dendroid is said to be *smooth* at a point p if for any sequence $\{x_n : n \in \mathbb{N}\}$ converging to a point x the sequence of arcs px_n converges to the arc px with respect to the Hausdorff metric. A dendroid is called *smooth* if it is smooth at some point.

A subset A of a continuum X is called a *shore set* if there is a sequence of subcontinua of X disjoint with A whose limit is the whole space X (with respect to the Hausdorff metric). A point $x \in X$ is called a *shore point* if $\{x\}$ is a shore set.

Introduction

The notion of a shore set was first used in [MP92]. One of the main results of that paper is that the union of finitely many shore points of a smooth dendroid is a shore set. In the same paper the authors asked whether the assumption of smoothness can be omitted. A positive and more general answer was given by A. Illanes in [Il01]. He proved that the union of finitely many pairwise disjoint shore continua is a shore set. In the same paper he gave an example of a dendroid X and two (nondisjoint) shore continua whose union is not a shore set.

In [N07b] the authors asked whether the union of two disjoint shore subcontinua of a λ-dendroid is a shore set. In spite of that some partial positive answer was given in the same paper, the question was answered negatively in [PV12] by an example of a λ-dendroid in which the union of two shore points is not a shore set. This example is a simple modification of Example 5 from [Il01].

It is natural to ask whether the union of two disjoint closed shore subsets of a dendroid is a shore set. This question was formulated first by A. Illanes as Question 6 in [Il01]. Since then it was rewritten as Problem 51 in [Pe07, p. 332] and as Question 9 in [PV12]. In this paper we give a negative answer to this question. We construct an example of a planar dendroid and two closed disjoint shore subsets whose union is not a shore set.

We also prove that the union of finitely many closed disjoint shore sets in a planar smooth dendroid is a shore set. Further, we show that the union of a shore continuum and a closed shore set in a planar dendroid is a shore set and that the assumption of planarity here is essential.

The notion of a shore set has been used in several applications. For example in [NP93] it was proved that a dendroid is a dendrite if and only if it has the property of Kelley and every shore point is an end point (in the classical sense). A number of different characterizations was given in that paper.

Main results

We remind a special case of Lemma 1 from [Il01].

Fact 1. *Let X be a dendroid, $a, b \in X$ two distinct points and let $c \in ab$. Then for every neighborhood W of the point c there exist neighborhoods U and V of a and b respectively, such that every arc uv with $u \in U$ and $v \in V$ intersects W.*

Definition 2. For any subset A of a dendroid X and for a point $p \in X \setminus A$ we define a *shadow* of the set A with respect to the point p as $Q_p(A) = \{x \in X : px \cap A \neq \emptyset\}$.

Next we prove several auxiliary results. In Proposition 19 we show that the shadow of a closed set is a G_δ-set. A strengthening of this result is given in Lemma 5 for smooth dendroids, which is used in Theorem 9.

Lemma 3. *Let U be an open subset of a dendroid X and let $p \in X \setminus U$. Then the shadow $Q_p(U)$ is an open set.*

Proof. Suppose that $x \in Q_p(U)$. Thus the arc px intersects U. By Fact 1 there is a neighborhood V of x such that every arc starting from p and ending in V intersects U. Hence $V \subseteq Q_p(U)$. □

Proposition 4. *Let F be a closed set in a dendroid X and let $p \in X \setminus F$. Then the shadow $Q_p(F)$ is a G_δ-set.*

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be a decreasing sequence of open neighborhoods of F such that F is the intersection of closures of U_n. By Lemma 3 we get that $Q_p(U_n)$ is an open set and thus $\bigcap Q_p(U_n)$ is a G_δ-set. It remains to show that it is equal to $Q_p(F)$. Clearly $Q_p(F) \subseteq \bigcap Q_p(U_n)$. On the other hand, suppose that we take any $x \in \bigcap Q_p(U_n)$. Then the arc px intersects U_n for every $n \in \mathbb{N}$. Especially, px intersects the closure of U_n and by compactness of X we get that px also intersects F. Thus $x \in Q_p(F)$. So we have proved that $Q_p(F)$ is a G_δ-set. □

Lemma 5. *Let F be a closed subset of a dendroid X which is smooth at a point $p \in X \setminus F$. Then the shadow $Q_p(F)$ is a closed set.*

Proof. Suppose that x_n is a sequence of points in $Q_p(F)$ converging to a point $x \in X$. Then the sequence of arcs px_n converges to the arc px. Since each arc px_n intersects F and F is a closed set we obtain that px intersects F as well. □

A very useful tool in the study of dendroids is the notion of a strong center, which was defined in [HN06] by virtue of [Mi03].

Definition 6. A point s in a dendroid X is called a *strong center*, if there exist two nonempty open sets U and V of X such that any arc uv with $u \in U$ and $v \in V$ contains s.

We note that a strong center cannot be a shore point. The following lemma is useful when proving that some sets are shore sets. We omitted the proof because of its simplicity.

Lemma 7. *Let X be a dendroid with a strong center s and let A be a subset of X. Then A is a shore set if and only if $s \notin A$ and $X \setminus Q_s(A)$ is dense in X.*

It is not always the case that a dendroid contains a strong center (see Example 1 of [N07a]). We rewrite Theorem 3.11 of [Mi03], which is used as a tool to prove Theorem 9.

Fact 8. *Every planar dendroid contains a strong center.*

Theorem 9. *Let X be a planar smooth dendroid. Then the union of finitely many closed disjoint shore sets is a shore set.*

Proof. Clearly, it is enough to prove that the union of two disjoint closed shore sets E and F is a shore set. Let us denote by p a point at which the dendroid X is smooth. Since X is planar we get by Fact 8 that there are a point $s \in X$ and two nonempty disjoint open sets U and V such that any arc starting from U and ending in V contains the point s. We note that the point s is not a shore point and thus $s \in X \setminus (E \cup F)$. Suppose first that $p = s$. Then $Q_p(E) = Q_s(E)$ as well as $Q_p(F) = Q_s(F)$ are closed by Lemma 5 and have an empty interior since E and F are shore sets. Thus $Q_s(E \cup F) = Q_s(E) \cup Q_s(F)$ has empty interior as well. This implies by the use of Lemma 7 that $E \cup F$ is a shore set.

Suppose now that $p \neq s$. We show that there is no shore point on the arc ps possibly except of the point p. In particular, $(E \cup F) \cap ps \subseteq \{p\}$. Suppose to the contrary that there is a shore point x on the arc ps distinct from p. Since x is a shore point there is a sequence of dendroids K_n not containing x which converges to the continuum X. Thus there are points $p_n \in K_n$ converging to the point p. By the property of s we may suppose that each continuum K_n contains the point s. Any arc pp_n for $n \in \mathbb{N}$ has to contain the point x because $p \in Q_s(\{x\})$ (otherwise $sp_n \cup p_n p \cup sp$ contains a simple closed curve). Thus pp_n is a sequence of arcs which does not converge to the point p which is a contradiction with the smoothness of X at the point p. Thus the arc ps intersects $E \cup F$ at most at the point p.

Without loss of generality we may suppose that $p \notin F$. It follows that $Q_p(F) = Q_s(F)$ and thus it is a closed set by Lemma 5 and it has an empty interior since F is a shore set. Since E is a shore set the difference $X \setminus Q_s(E)$ is a dense set. Since moreover $Q_s(F)$ is a closed set with empty interior it follows that $(X \setminus Q_s(E)) \setminus Q_s(F)$ is a dense set as well. Thus the set $X \setminus Q_s(E \cup F) = (X \setminus Q_s(E)) \setminus Q_s(F)$ is a dense set and thus $E \cup F$ is a shore set by Lemma 7. □

Note 10. We note that the assumption of smoothness cannot be omitted because of Example 11. We do not know whether the assumption of planarity is essential.

Example 11. We give an example of a planar dendroid Y in which the union of two disjoint closed shore sets is not a shore set. We note that this dendroid contains only countably many ramification points.

We denote by I the interval $[0, 1]$ and by C the Cantor set i.e. the subset of I consisting of all numbers of the form $\sum 3^{-i} c_i$ where $c_i \in \{0, 2\}$. Let $X \subseteq \mathbb{R}^2$ be

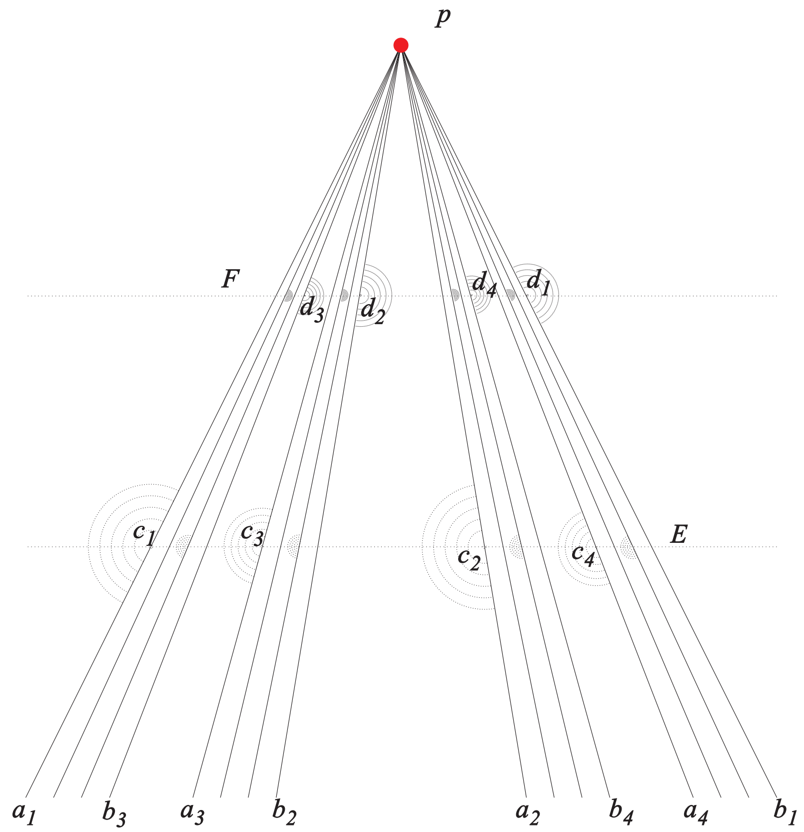

Figure 1: The continuum X.

the union of all segments connecting the point $p = (0, 1)$ with a point in the set $C \times \{0\}$. Continuum X is called the Cantor fan.

Let A be the subset of C which consists of all numbers of the form $\sum 3^{-i}c_i$, where $c_i \in \{0, 2\}$ and $c_i = 0$ for all but finitely many $i \in \mathbb{N}$. Similarly let B be the subset of C consisting of all numbers of the form $\sum 3^{-i}c_i$, where $c_i \in \{0, 2\}$ and $c_i = 2$ for all but finitely many $i \in \mathbb{N}$. We note that A and B are disjoint countable dense subsets of C and any point from $A \cup B$ is an accessible point of C in $\mathbb{R}$ (i.e. for every point $c \in A \cup B$ there is a nondegenerate closed interval which intersects C just at the point c). Moreover, the points in A are those points of C which are accessible from the left side and points in B are those points of C which are accessible from the right side in the interval I.

Let us denote by $\{a_n : n \in \mathbb{N}\}$ the set of all points in $A \times \{0\}$ and by $\{b_n : n \in \mathbb{N}\}$ the set of all points in $B \times \{0\}$. For any $n \in \mathbb{N}$ we denote by c_n the point at which the segment pa_n intersects the horizontal line $\mathbb{R} \times \{\frac{1}{3}\}$. We denote by d_n the point at which the segment pb_n intersects the horizontal line $\mathbb{R} \times \{\frac{2}{3}\}$.

Let E be the intersection of X and the horizontal line $\mathbb{R} \times \{\frac{1}{3}\}$ and F be the intersection of X and the horizontal line $\mathbb{R} \times \{\frac{2}{3}\}$ (see Figure 1).

Let $\{\varepsilon_n : n \in \mathbb{N}\}$ be a sequence of numbers in the interval $(0, \frac{1}{6})$ converging to zero such that any circle with the center c_n and diameter ε_n contains an open semicircle which is disjoint with X. We may suppose that the same holds for circles with the center at the point d_n. We define an upper semi-continuous decomposition σ of X whose nondegenerate elements consist of two points x and y if there is $n \in \mathbb{N}$ such that $x, y \in pa_n$ and $|c_n x| = |c_n y| \leq \varepsilon_n$ or there is $n \in \mathbb{N}$ such that $x, y \in pb_n$ and $|d_n x| = |d_n y| \leq \varepsilon_n$.

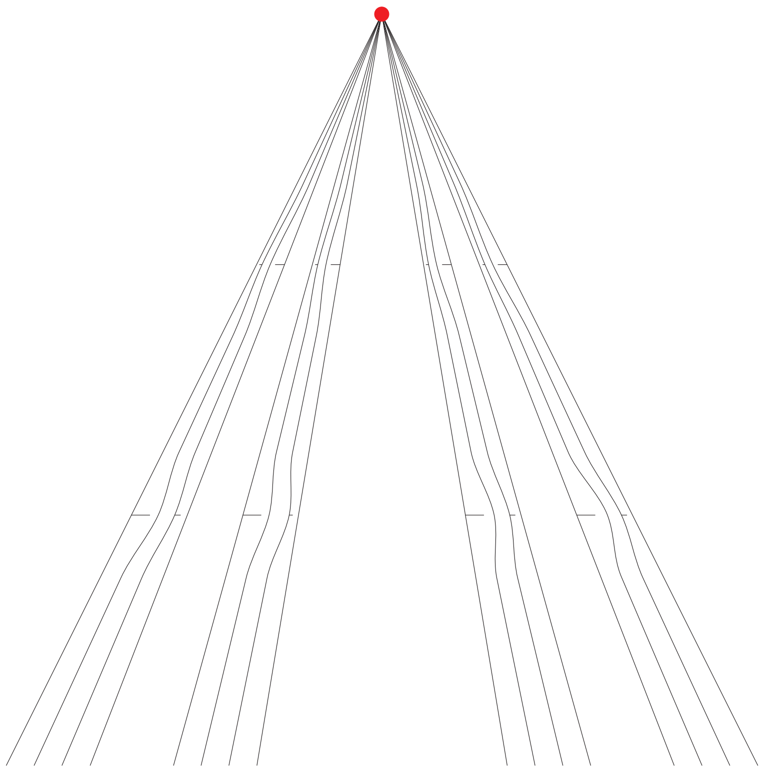

Figure 2: The continuum Y.

We denote by Y the quotient X/σ (see Figure 2) and by $\pi\colon X \to Y$ the quotient mapping. The space Y is a continuum since σ is an upper semi-continuous decomposition.

Claim 1. Continuum Y is a dendroid.

Clearly, Y is an arcwise connected continuum because it is a continuous image of an arcwise connected continuum X. It remains to verify that it is hereditarily unicoherent. To do this it is enough to show that for any pair of distinct points x and y in Y there is a subcontinuum $K_{x,y}$ of Y such that any subcontinuum of Y containing the points x and y contains $K_{x,y}$. By distinguishing several cases we get that there is only one arc in Y whose end points are x and y. Moreover, this arc satisfies requirements for $K_{x,y}$.

Claim 2. The sets $\pi(E)$ and $\pi(F)$ are shore sets in Y.

We denote by e_n and f_n the two points in X which lie on the segment pa_n and whose distance from c_n is ε_n in such a way that the second coordinate of e_n is less than the second coordinate of f_n. We note that $\pi(e_n) = \pi(f_n)$. For every $n \in \mathbb{N}$ we get that $\pi(a_n e_n \cup f_n p)$ is an arc which is disjoint with $\pi(E)$. Moreover, the set $Z = \bigcup\{a_n e_n : n \in \mathbb{N}\} \cup \bigcup\{f_n p : n \in \mathbb{N}\}$ is a dense subset of X and thus its projection $\pi(Z)$ is dense in Y. The set $\pi(Z)$ consists of arcs which contain the point $\pi(p)$ and thus there are subcontinua of Z whose distance from Y with respect to Hausdorff metric is arbitrarily small. Thus E is a shore set. By the same reason we obtain that F is a shore set as well.

Claim 3. The union $\pi(E) \cup \pi(F)$ is not a shore set.

Since the sets A and B are disjoint, there is no arc in Y which intersects nonempty open sets $\pi(\mathbb{R} \times [0, \frac{1}{6}) \cap X)$ and $\pi(\mathbb{R} \times (\frac{5}{6}, 1] \cap X)$. Thus $\pi(E) \cup \pi(F)$ is not a shore set.

Claim 4. Dendroid Y is planar.

We define an upper semi-continuous decomposition τ of the plane $\mathbb{R}^2$ which contains σ such that the elements of this decomposition are continua which do not separate the plane. In fact, the elements of this decomposition will be just arcs and one point sets. A nondegenerate element of the decomposition τ is just the closure of the open semicircle with the center c_n or d_n which does not intersects X and which has diameter at most $2\varepsilon_n$ (see Figure 1). Then it follows that the quotient $\mathbb{R}^2/\tau$ will be homeomorphic to the plane by [Mo25]. Since Y is a subspace of $\mathbb{R}^2/\tau$ it follows that Y is planar.

Example 12. By a modification of Example 11 we provide an example of a dendroid in which the union of a shore point and a closed shore set is not a shore set.

We define C, p, X, A, B, a_n, b_n, c_n and E as in Example 11. Let $\{\varepsilon_n : n \in \mathbb{N}\} \subseteq (0, \frac{1}{6})$ be a sequence converging to zero.

We fix finite sets $B_1 \subseteq B_2 \subseteq \dots$ such that the union of the collection $\{B_n : n \in \mathbb{N}\}$ is the set B. We define a decomposition χ of X, such that S is a nontrivial element of χ if and only if one of the two conditions holds:

- $S = \{x, y\}$, where $x, y \in pa_n$ for some $n \in \mathbb{N}$ and $|c_n x| = |c_n y| \leq \varepsilon_n$,
- S consists of $|B_n|$ points with equal second coordinate which is at least $1 - \varepsilon_n$ but less than $1 - \varepsilon_{n+1}$ and $S \subseteq \bigcup\{pz : z \in B_n \times \{0\}\}$.

We claim that χ is a well defined upper semi-continuous decomposition and we denote the quotient space X/χ by Y (see Figure 3). We denote by π the projection of X onto Y.

We claim without a rigorous proof that Y is a dendroid, $\pi(p)$ is a shore point and $\pi(E)$ is a closed shore set. It remains to realize that $\{\pi(p)\} \cup \pi(E)$ is not a shore set, because A and B are disjoint sets.

The dendroid Y from Example 12 is not planar because of Fact 13 which is a consequence of Theorem 5 from [Cu91]. Moreover, we prove in Theorem 14 that any dendroid, in which the union of a shore point and a closed shore set is not a shore set, cannot be planar.

Fact 13. *Let X be a dendroid which contains a pair of disjoint ε-dense subcontinua for every $\varepsilon > 0$. Then X is not planar.*

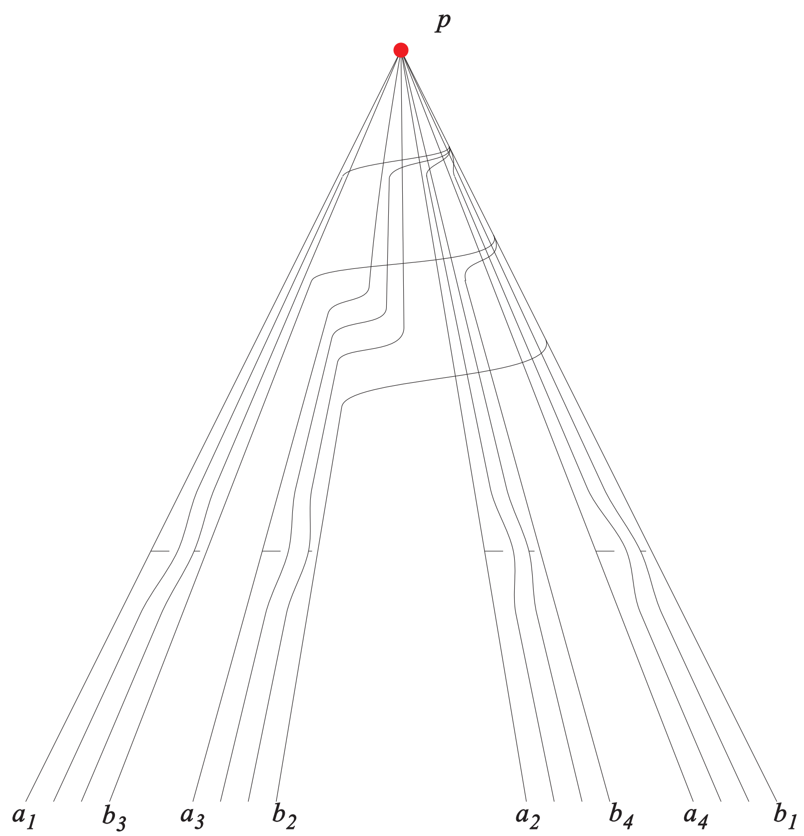

Figure 3: The continuum Y.

Theorem 14. *Let X be a planar dendroid. Then the union of a closed shore set and a shore continuum is a shore set.*

Proof. Let E be a closed shore set and K a shore continuum in X. By Fact 8 there is strong center $s \in X$ because X is a planar dendroid. Let $p \in X$ be the unique point such that ps intersects K only at the point p. We note that p is a shore point, $K \subseteq Q_s(p)$ and $Q_s(p) = Q_s(K)$. Since E is a shore set, the set $X \setminus Q_s(E)$ is dense. In order to show that $E \cup K$ is a shore set it is enough to prove by Lemma 7 that $Q_s(p)$ is nowhere dense because then we get that $X \setminus Q_s(E \cup K) = (X \setminus Q_s(E)) \setminus Q_s(p)$ is dense in X.

Suppose for contradiction that the closure of $Q_s(p)$ has nonempty interior W. We define Y to be the closure of $\bigcup\{pw \colon w \in W \cap Q_s(p)\}$. Since Y is the closure of a connected set, it is a continuum. Moreover, Y is a planar dendroid, because it is a subcontinuum of X.

Fix any $\varepsilon > 0$. We show that Y contains a pair of disjoint 3ε-dense subcontinua. It follows from the definition of Y that there are points $w_1, \ldots, w_n \in W \cap Q_s(p)$ such that $pw_1 \cup \cdots \cup pw_n$ is ε-dense in Y.

By Lemma 7 we get that $X \setminus Q_s(p)$ is dense in X and thus $W \setminus Q_x(p)$ is dense in W, because W is an open set. For every $i \leq n$ there is a sequence of points $\{z_i^k\}_{k=1}^{\infty} \subseteq W \setminus Q_s(p)$ converging to w_i. Since limes inferior of the sequence of arcs $\{pz_i^k\}_{k=1}^{\infty}$ contains the arc pw_i we get that there is an integer $k(i)$ such that pw_i is contained in the ε-neighborhood of $pz_i^{k(i)}$. We denote $z_i = z_i^{k(i)}$ for any

$i \leq n$. Every arc pz_i is contained in Y because $W \cap Q_s(p)$ is dense in W. Thus $pz_1 \cup \cdots \cup pz_n$ is 2ε-dense in Y and $z_i \in W$. For every $i \leq n$ there is a point $s_i \in sp$ such that $sp \cap s_i z_i = \{s_i\}$. Since $z_i \notin Q_s(p)$ we get that $s_i \neq p$. There is a point $c \in sp \cap Y$ distinct from p such that pc is of diameter less than ε. We get that $cz_1 \cup \cdots \cup cz_n$ is 3ε-dense in Y. Now $cz_1 \cup \cdots \cup cz_n$ and $pw_1 \cup \cdots \cup pw_n$ are two 3ε-dense disjoint subcontinua of Y (see Figure 4). Since this is true for every $\varepsilon > 0$ we get a contradiction with Fact 13.

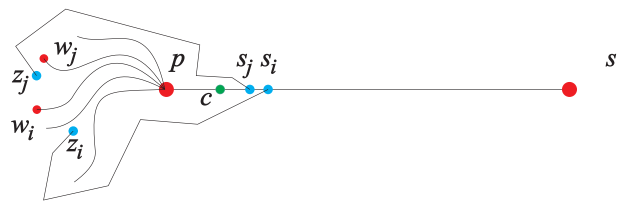

Figure 4: The continuum X.

□

Summary

In Table 1 we summarize known results and open questions about the problem whether the union of finitely many shore sets is a shore set. For example the sign '+' in the third column and fifth row means that the union of finitely many shore continua in a planar dendroid is a shore set. An additional reference either to a proof or to a counterexample is given in Table 1 when the result is not a consequence of some other result from the table.

Table 1: Union of shore sets.

	disjoint continua	continua	disjoint closed sets	closed sets
fan	+	+ Note 15	?	?
planar smooth dendroid	+	+	+ Theorem 9	?
smooth dendroid	+	– Note 16	?	–
planar dendroid	+	+ [N07b]	– Example 11	–
dendroid	+ [Il01]	– [Il01]	–	–
planar λ-dendroid	?	?	–	–
λ-dendroid	– [PV12]	–	–	–

Note 15. We can observe that a fan always contains a point which is not a shore point (namely the vertex of the fan). By Corollary 1 of [N07a] it follows that if a dendroid contains a point which is not a shore point, then the union of finitely many shore subcontinua is a shore set. Thus we get that a finite union of shore subcontinua of a fan is a shore set.

Note 16. The continuum from Example 5 in [Il01] is proved to be a contractible dendroid. We remark that it is even contractible under a retracting homotopy (i.e. a homotopy which consists of retractions) and thus it is smooth (the homotopy constructed in Example 4 of [Il01] is not a retracting homotopy but by a simple modification a retracting homotopy can be obtained). This implication follows from Corollary on page 93 in [ChE72] where it is proved that a dendroid is smooth if and only if it is contractible under a retracting homotopy. We also note that by [Mo28] the mentioned continuum is not planar because it contains an uncountable collection of pairwise disjoint simple triods.

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Chapter 6.
Half-homogeneous indecomposable circle-like continuum

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Abstract

A continuum is said to be $\frac{1}{2}$-homogeneous if there are exactly two types of points. We give an example of a $\frac{1}{2}$-homogeneous indecomposable circle-like continuum. This answers a question of V. Neumann-Lara, P. Pellicer-Covarrubias and I. Puga.

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Introduction

A *continuum* is a nonempty compact connected metrizable space. A continuum X is said to be *connected im kleinen* at a point x if any neighborhood of x contains a connected subset whose interior contains the point x. An *orbit* of a space X containing a point $x \in X$ is a set of all points of the form $h(x)$, where $h\colon X \to X$ is a homeomorphism. A continuum X is called $\frac{1}{2}$-*homogeneous* if it consists of two orbits exactly. A continuum is said to be *indecomposable* if it can not be written as a union of two proper subcontinua. A continuum X is said to be *arc-like* (resp. *circle-like*) if for any $\varepsilon > 0$ there is a continuous mapping of X onto an arc (resp. a circle) such that preimages of points have diameter less than ε.

We recall a characterization of an arc-like continuum via chains. A *chain* is a finite sequence $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of open sets in a metric space such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements of a chain are called its *links*. If $\varepsilon > 0$ and the diameter of each link is less than ε, then the chain is called an ε-chain. A continuum is chainable if for each $\varepsilon > 0$, the continuum can be covered by an ε-chain. It is well known that a continuum is arc-like if and only if it is chainable.

We give a positive answer to Problem 4.10 from [NPP06], i.e. we give an example of a $\frac{1}{2}$-homogeneous continuum which is indecomposable and circle-like.[1]

Example

Let A be an arc of pseudo-arcs i.e. an arc-like continuum for which there exists a continuous mapping $g\colon A \to [0,1]$ such that preimage of each point is a pseudo-arc. For more details see [BJ59] and [Le85]. It is argued in [NPP06, Example 4.8], that A is a $\frac{1}{2}$-homogeneous and the two orbits in X are $g^{-1}(\{0,1\})$ and $g^{-1}((0,1))$.

Let B be a quotient of A, where the sets $g^{-1}(0)$ and $g^{-1}(1)$ are degenerated to points. Let us call these two points p and q. We note that B is an arc-like continuum. Moreover it is $\frac{1}{2}$-homogeneous with orbits $\{p, q\}$ and $B \setminus \{p, q\}$.

For any natural number n we define a space X_n as a quotient of the product $\{1, 2, \ldots, 2^n\} \times B$ where the points (k, q) and $(k+1, p)$ are identified for every $k < 2^n$ and $(2^n, q)$ is identified with $(1, p)$. Thus X_n is a "circle" made of 2^n copies of B.

We define continuous maps $f_n\colon X_{n+1} \to X_n$ as

$$f\big((k,x)\big) = \begin{cases} (k,x), & \text{if } k \leq 2^n \\ (k-2^n, x), & \text{if } 2^n < k \leq 2^{n+1} \end{cases}$$

for $k \leq 2^{n+1}$ and $x \in B$. Let X be the inverse limit of the inverse sequence $(X_n, f_n)_{n=1}^{\infty}$. Thus X is a subspace of the product $\prod X_n$ which consists of those points $(x_n)_{n=1}^{\infty}$ such that $f_n(x_{n+1}) = x_n$ for any $n \in \mathbb{N}$. Since X is an inverse limit of continua with continuous bonding mappings, it is a continuum by [Na92, Theorem 2.4, p. 19].

[1] After submitting this paper, we were made aware of a paper by Jan Boronski on the same problem. That paper appears in the same issue as this paper.

Claim 1. X is circle-like.

Proof. Any continuum X_n is circle-like, since the continuum B is arc-like. Moreover, an inverse limit of an inverse sequence of circle-like continua with onto bonding mappings is again a circle-like continuum (see [MS63, Lemma 1, p. 147]). Thus X is a circle-like continuum. □

Claim 2. X is indecomposable.

Proof. We can easily observe that the system $(X_n, f_n)_{n=1}^{\infty}$ forms an indecomposable inverse sequence, i.e. whenever there are two continua A and B such that $X_{n+1} = A \cup B$, then $f_n(A) = X_n$ or $f_n(B) = X_n$. Since an inverse limit of an indecomposable inverse sequence is an indecomposable continuum by [Na92, Theorem 2.7], we conclude that X is an indecomposable continuum. □

Claim 3. X is $\frac{1}{2}$-homogeneous.

Proof. Let us consider any two points $x = (x_n)_{n=1}^{\infty} \in X$ and $y = (y_n)_{n=1}^{\infty} \in X$ such that $x_1, y_1 \in \{p, q\}$ or $x_1, y_1 \notin \{p, q\}$. There is a homeomorphism $h_1 \colon X_1 \to X_1$ such that $h_1(x_1) = y_1$, because B is $\frac{1}{2}$-homogeneous. By induction we find a homeomorphism $h_n \colon X_n \to X_n$ for any $n \geq 2$ such that $h_n(x_n) = y_n$ and $h_n \circ f_{n+1} = f_{n+1} \circ h_{n+1}$. We define a homeomorphism $h \colon X \to X$ which is given by $h((x_n)_{n=1}^{\infty}) = (h_n(x_n))_{n=1}^{\infty}$. Clearly $h(x) = y$. Thus there are at most two orbits in X, namely the set $\{x \in X \colon x_1 = p \text{ or } x_1 = q\}$ and its complement $\{x \in X \colon p \neq x_1 \neq q\}$.

It remains to show that X is not homogeneous. Let $x = (x_n)_{n=1}^{\infty} \in X$ be the point for which $x_n = (1, p)$ for every $n \in \mathbb{N}$. Every proper subcontinuum of X which contains the point x is connected im kleinen at x.

Let $r \in B$ be an arbitrary point distinct from p and q. We consider the point $y = (y_n)_{n=1}^{\infty}$ where $y_n = (1, r)$ for every $n \in \mathbb{N}$. We define a continuum K which consists of points $z = ((1, s))_{n=1}^{\infty} \in X$ for some $s \in B$. We observe that K and B are homeomorphic. Since B is not connected im kleinen at r, we get that K is not connected im kleinen at y.

Thus the points x and y are not in the same orbit and hence X is not homogeneous. □

Remarks

Remark 1. Our construction is a modification of the dyadic solenoid (see [Da30] and [Vi27]). We could easily modify the construction to obtain an example from each p-adic solenoid.

Remark 2. The continuum B is a special quotient of the arc of pseudo-arcs. If we try to replace the arc of pseudo-arcs just with the arc or with the pseudo-arc, we do not succeed. In the first case we obtain a solenoid which is a homogeneous continuum. In the second case we get a continuum with at least three orbits.

There is a natural question whether there is another possible choice of the continuum B formulated in Problem 3. We just need to define a notion of an end point in an arc-like continuum.

Definition 3. A point x of an arc-like continuum X is called an *end point* if for every $\varepsilon > 0$ there is a continuous mapping f of X onto the segment $[0,1]$ such that preimages of points have diameter less than ε and $f(x) = 0$.

There is a characterization of end points in an arc-like continuum (see [Bi51, Section 5, p. 660] and [Do08, p. 32]):

Proposition 4. *For a point p of a chainable continuum X the following conditions are equivalent.*

a) Each nondegenerate subcontinuum of X containing p is irreducible from p to some other point.

b) If each of two subcontinua of X contains p, one of the subcontinua contains the other.

c) For each positive number ε, there is an ε-chain covering M such that only the first link of the chain contains p.

d) p is an end point of X.

Problem 5. Does there exist a $\frac{1}{2}$-homogeneous arc-like continuum with exactly two end points which is neither homeomorphic to the arc nor to the continuum B?

Regarding to the result obtained in this paper we also recall Problem 4.9 from [NPP06].

Problem 6. Does there exist an indecomposable, $\frac{1}{2}$-homogeneous, arc-like continuum?

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Chapter 7.
Half-homogeneous chainable continua with end points

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Abstract

A point of a chainable continuum is called an end point if for every positive epsilon there is an epsilon-chain such that only the first link contains the point. We prove that up to homeomorphism there are only two half-homogeneous chainable continua with two end points. One of them is an arc and the second one is the quotient of an arc of pseudo-arcs, where the two terminal continua are pushed to points. This answers a question of the second and third author.

Moreover we prove that the two above mentioned continua are the only half-homogeneous chainable continua with a nonempty finite set of end points.

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Introduction

A *continuum* is a nonempty compact connected metrizable space. A continuum is said to be *indecomposable* if it can not be written as a union of two proper subcontinua. Otherwise a continuum is said to be *decomposable*. A continuum is called *hereditarily unicoherent* if the intersection of any two subcontinua is empty or connected. A point of a continuum is said to be a *cut point* if its complement is not connected. An *orbit* of a topological space X containing a point $x \in X$ is the set of all points $h(x)$, where $h\colon X \to X$ is a homeomorphism. A continuum X is called $\frac{1}{n}$*-homogeneous* if it consists of n orbits exactly, where $n \in \mathbb{N}$. For $n = 1$ we just write *homogeneous*.

A *chain* is a finite sequence $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ of open sets in a metric space such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements of a chain are called its *links*. If $\varepsilon > 0$ and the diameter of each link is less than ε, then the chain is called an *ε-chain*. A continuum is *chainable* if for each $\varepsilon > 0$ it can be covered by an ε-chain.

A continuous mapping $f\colon X \to Y$ between metric spaces is called an *ε-mapping* if f is continuous and for every $x \in X$ the diameter of $f^{-1}(f(x))$ is less than ε. A continuum X is called *arc-like* if for every $\varepsilon > 0$ there is an ε-mapping of X onto an arc. It is a well known result that a nondegenerate continuum is arc-like if and only if it is chainable.

Let X be a continuum and $p, q \in X$. The continuum X is called *irreducible between p and q* if any subcontinuum containing p and q is equal to X. A continuum is said to be *irreducible* if it is irreducible between some two points.

Fact 1 (Section 5 in [Bi51, p. 660] or [Do08, p. 32])**.** *For a point p of a nondegenerate chainable continuum X the following conditions are equivalent.*

a) *Each nondegenerate subcontinuum of X containing p is irreducible between p and some other point.*

b) *If there are two subcontinua of X containing p, one of them contains the other.*

c) *For each positive number ε, there is an ε-chain covering X such that only the first link of the chain contains p.*

d) *For every $\varepsilon > 0$ there is a continuous mapping f of X onto $[0, 1]$ such that preimages of points have diameter less than ε and $f(p) = 0$.*

Definition 2. A point p of a chainable continuum X is called an *end point* if it satisfies one condition (or all conditions) from Fact 1.

In Fact 1 we would like to emphasize especially the condition b), because we will use it quite often. A classical example is an arc which contains two end points, but it may happen that a chainable continuum contains more than two end points. For example the $\sin(1/x)$-continuum contains three end points. In the pseudo-arc, every point is an end point by Theorem 16 from [Bi51]. On the other hand there are continua with no end points. In [Do94] it is even shown that an arbitrary nonnegative integer can be the number of end points of a chainable continuum.

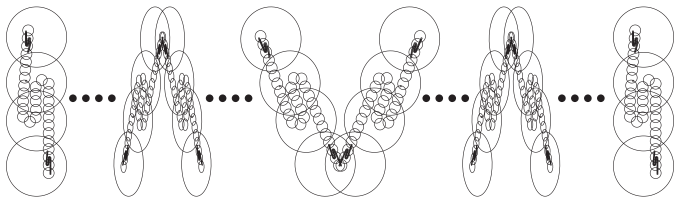

Figure 1: An arc of pseudo-arcs.

An *arc of pseudo-arcs* (see Figure 1) is any chainable continuum A for which there exists a continuous mapping $g\colon A \to [0,1]$ such that preimage of each point is a pseudo-arc. It is known that up to homeomorphism there is only one continuum with these properties. For more details see [BJ59] and [Le85]. It is known that A is $\frac{1}{2}$-homogeneous and the two orbits in X are $g^{-1}(\{0,1\})$ and $g^{-1}((0,1))$ by Example 4.8 from [NPP06]. Moreover the first orbit consists of precisely the end points of A.

Let us consider the quotient of A which is obtained using the upper semi-continuous decomposition $\{g^{-1}(0), g^{-1}(1)\} \cup \{\{x\}\colon x \in g^{-1}((0,1))\}$. Any continuum homeomorphic to this quotient will be called an *arcless-arc*. It follows that an arcless-arc is a $\frac{1}{2}$-homogeneous chainable continuum with two end points.

In Problem 8 of [PV13] the second and third author of this paper settled the following problem.

Problem 3. Does there exist a $\frac{1}{2}$-homogeneous chainable continuum with exactly two end points which is neither an arc nor an arcless-arc?

We prove that there is no such continuum. Moreover we show that any $\frac{1}{2}$-homogeneous chainable continuum with a finite nonempty set of end points contains just two end points and thus it is either an arc or an arcless-arc.

Tools

In this section we cite several known results that will be used in proofs in the Main results section. Most of the facts are given without proof, but there is always a reference to the source.

Fact 4 (Boundary Bumping Theorem 5.4 from [Na92, p. 73])**.** *Let X be a continuum, G an open proper subset of G and p a point in G. Then the closure of the component of p in U is a continuum intersecting the boundary of G.*

Fact 5 (Theorem 12.1 from [Na92, p. 230])**.** *Every nondegenerate subcontinuum of a chainable continuum is chainable.*

Fact 6 (Theorem 12.2 from [Na92, p. 230]). *Every chainable continuum is hereditarily unicoherent.*

Definition 7. A continuum T is said to be a *weak triod* provided that there exist three subcontinua of T whose intersection is nonempty, whose union is T and none of which is contained in the union of the two remaining.

Fact 8 (Corollary 12.7 from [Na92, p. 233]). *A chainable continuum does not contain a weak triod.*

Fact 9 (Theorem 6.17 from [Na92, p. 96]). *A continuum containing exactly two non-cut points is an arc.*

Fact 10 ([Bi59]). *A nondegenerate homogeneous chainable continuum is a pseudo-arc.*

Fact 11 (Theorem 16 from [Bi51] or Theorem 3.13 from [Le99, p. 44]). *Let X be a chainable continuum each point of which is an end point. Then X is a pseudo-arc.*

In order to state Fact 12 and Fact 13 we need to define subsequent notions. A *Polish* space is a completely metrizable separable space. We say that a group H acts *transitively* on a space X if for every $x, y \in X$ there is $h \in H$ such that $h(x) = y$. We say that a topological group H acts *microtransitively* on a topological space X if for every $x \in X$ and every neighborhood U of the neutral element of H the set $\{h(x)\colon h \in U\}$ is a (not necessarily open) neighborhood of x.

Fact 12 (Effros theorem [Ef65], [vM04]). *Suppose that a Polish group acts transitively on a Polish space. Then the group acts microtransitively.*

Fact 13. *Let X be a continuum with a compatible metric ρ and let G be an open subset of X. Suppose that for every pair of points c and d of G there is a homeomorphism $h\colon X \to X$ such that $h(c) = d$. Then for every $\varepsilon > 0$ and for every $c \in G$ there is $\delta > 0$ such that whenever $\rho(c, d) < \delta$ then there is a homeomorphism $h\colon X \to X$ such that $h(c) = d$ and $\rho(e, h(e)) < \varepsilon$ for every $e \in X$.*

Proof. The group of homeomorphisms of a compact space with the topology of uniform convergence is completely metrizable by Corollary 1.3.11 from [vM01, p. 35]. It is separable by Proposition 1.3.3 from [vM01, p. 31]. The open set G is a Polish space. Thus by Fact 12 we get that the group of homeomorphisms on X acts microtransitively on G, because by the assumption it acts transitively. By rewriting this into the language of the metric ρ we obtain the desired result. $\square$

Fact 14 (Theorem 3.4 from [Bo13]). *Let X be a $\frac{1}{2}$-homogeneous continuum. If X is indecomposable, then each of the two orbits is uncountable.*

Lemma 15. *Let X be a chainable continuum and let E be a finite subset of the set of end points of X. Then the space $X \setminus E$ is connected.*

Proof. Suppose for contradiction that $X \setminus E$ is not connected. Since $X \setminus E$ is nonempty there are two disjoint nonempty open sets U and V in $X \setminus E$ whose union is $X \setminus E$. Clearly U as well as V are open in X. We see that the closure of $U \cup V$ is the whole continuum X and thus the union of closures of U and V is X. Since X is connected we get that there is a point $e \in E$ which lies in the closure of U and also in the closure of V. There exist sequences $\{u_n\}_{n=1}^{\infty} \subseteq U$ and $\{v_n\}_{n=1}^{\infty} \subseteq V$ which converge to the point e. For any $n \in \mathbb{N}$ we denote by K_n (resp. L_n) the closure of the component of the point u_n (resp. v_n) in U (resp. V). By the Boundary Bumping Theorem (Fact 4) any continuum K_n (resp. L_n) intersects boundary of U (resp. V) which is a subset of E. Since the set E is finite we may suppose without loss of generality that there is a point $a \in E$ (resp. $b \in E$) such that $a \in K_n$ (resp. $b \in L_n$) for every $n \in \mathbb{N}$. Let K (resp. L) be the closure of $\bigcup\{K_n : n \in \mathbb{N}\}$ (resp. $\bigcup\{L_n : n \in \mathbb{N}\}$). Clearly K and L are continua because they are closures of connected sets. Moreover $e \in K \cap L$, $K \subseteq X \setminus V$ and $L \subseteq X \setminus U$ and thus $K \cap L \subseteq E$. Since e is an end point we get by b) in Fact 1 that $K \subseteq L$ or $L \subseteq K$. This is a contradiction with the fact that $K \cap L$ is a subset of E which is a finite set and K as well as L are nondegenerate continua. □

Remark 16. Let us note that Lemma 15 need not to be true if the set E of some end points is infinite. If X is a pseudo-arc (each point of which is an end point) and E is a suitable set for which $X \setminus E$ is not connected we get a counterexample.

Fact 17 (Theorem 10 from [BJ59])**.** *Suppose that A and A' are arcs of pseudo-arcs and denote by E (resp. E') the set of all end points of A (resp. A'). Then any homeomorphism of E onto E' can be extended to a homeomorphism of A onto A'.*

Lemma 18. *Let X be a continuum for which there exists a continuous mapping $f: X \to [0,1]$ such that $f^{-1}(0)$ and $f^{-1}(1)$ are one point sets and $f^{-1}(c)$ is a pseudo-arc for any $c \in (0,1)$. Then X is an arcless-arc.*

Proof. Let us denote by a (resp. b) the only point for which $f(a) = 0$ (resp. $f(b) = 1$). Let $g: A \to [0,1]$ be an onto continuous mapping of an arc of pseudo-arcs such that preimages of points are pseudo-arcs. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of nondegenerate compact intervals in $(0,1)$ whose union is $(0,1)$ such $I_{n+1} \cap (I_1 \cup \cdots \cup I_n)$ is a one point set. We denote $A_n = g^{-1}(I_n)$ and $X_n = f^{-1}(I_n)$ for $n \in \mathbb{N}$. Every continuum A_n as well as X_n is an arc of pseudo-arcs. With the use of Fact 17 we can find by induction a sequence of homeomorphisms $\{h_n : A_n \to X_n\}_{n=1}^{\infty}$ such that whenever $c \in I_m \cap I_n$ then $h_m(x) = h_n(x)$ for any $x \in A_m \cap A_n$ and $m, n \in \mathbb{N}$.

We define a mapping $h: A \to X$. For any $x \in X \setminus \{a, b\}$ there is $n \in \mathbb{N}$ such that $x \in A_n$ and we define $h(x)$ as $h_n(x)$. Moreover we define $h(x) = a$ for every $x \in g^{-1}(0)$ and $h(x) = b$ for $x \in g^{-1}(1)$. It is easily verified that h is a well defined continuous mapping. Since h is one-to-one on the set $A \setminus (g^{-1}(0) \cup g^{-1}(1))$ and it sends all points of the set $g^{-1}(0)$ to the point a and all points of the set $g^{-1}(1)$ to the point b we get that X is an arcless-arc. □

Fact 19 (Theorem 4.2 from [Do08])**.** *The set of end points of a chainable continuum is a G_δ-set.*

Main results

Theorem 20. *Let X be a chainable $\frac{1}{2}$-homogeneous continuum with exactly two end points. Then X is either an arc or an arcless-arc.*

Proof. Let us denote by a and b the two distinct end points of X.

For any $c \in X$ we denote by A_c the intersection of all subcontinua of X which contain points a and c. Similarly we denote by B_c the intersection of all subcontinua of X which contain b and c. It follows that A_c as well as B_c is a continuum because any chainable continuum is hereditarily unicoherent by Fact 6. We denote $L_c = A_c \cap B_c$. Every space L_c is a continuum by the same reason. We call the sets L_c levels.

Claim 1. The sets $\{a, b\}$ and $X \setminus \{a, b\}$ are orbits in X.

Since X is $\frac{1}{2}$-homogeneous there are exactly two orbits. The end points $\{a, b\}$ of X form one of them and hence its complement $X \setminus \{a, b\}$ is the second one.

Claim 2. For any homeomorphism $h \colon X \to X$ and $c \in X$ we obtain $L_{h(c)} = h(L_c)$.

Suppose first that $h(a) = a$ and thus $h(b) = b$. Since A_c is the least continuum containing a and c and h is a homeomorphism we get that $h(A_c)$ is the least continuum containing $a = h(a)$ and $h(c)$ and thus $h(A_c) = A_{h(c)}$. By the same reason $h(B_c) = B_{h(c)}$. Thus we get that

$$h(L_c) = h(A_c \cap B_c) = h(A_c) \cap h(B_c) = A_{h(c)} \cap B_{h(c)} = L_{h(c)}.$$

If $h(a) \neq a$ we get that $h(a) = b$ and thus $h(b) = a$. By a similar argument as in the first case we obtain that $h(A_c) = B_{h(c)}$ and $h(B_c) = A_{h(c)}$. Hence

$$h(L_c) = h(A_c \cap B_c) = h(A_c) \cap h(B_c) = B_{h(c)} \cap A_{h(c)} = L_{h(c)}.$$

Claim 3. Any two levels L_c and L_d for $c, d \in X \setminus \{a, b\}$ are homeomorphic.

For $c, d \in X \setminus \{a, b\}$ there is a homeomorphism $h \colon X \to X$ such that $h(c) = d$ by Claim 1. Using Claim 2 we get that $h(L_c) = L_{h(c)} = L_d$. Thus restriction of h to the level L_c is a homeomorphism of L_c onto L_d.

Claim 4. The space $X \setminus \{a, b\}$ is connected.

Since a and b are end points of X we get by Lemma 15 that $X \setminus \{a, b\}$ is connected.

Claim 5. For any pair $c, d \in X \setminus \{a, b\}$ there is a homeomorphism $h \colon X \to X$ such that $h(a) = a$, $h(b) = b$ and $h(c) = d$.

Let us consider a compatible metric ρ on the space X. We let $\varepsilon = \rho(a, b)$ and we denote by $\mathcal{H}$ the set of all homeomorphisms $h \colon X \to X$ such that $d(e, h(e)) < \varepsilon$ for every $e \in X$. By the choice of ε we get that $h(a) = a$ and $h(b) = b$ for any $h \in \mathcal{H}$. By Fact 13 applied to $G = X \setminus \{a, b\}$ we obtain that for every $c \in X \setminus \{a, b\}$ there is a neighborhood N_c of the point c such that for any $d \in N_c$ there is a homeomorphism $h \in \mathcal{H}$ such that $h(c) = d$.

Since $X \setminus \{a, b\}$ is connected by Claim 4 we get that for a fixed pair of points c and d in $X \setminus \{a, b\}$ there is a finite sequence of points $c_1, \ldots, c_n \in X \setminus \{a, b\}$

such that $c_1 = c$, $c_n = d$ and $c_{i+1} \in N_{c_i}$ for any $i < n$. Thus for every $i < n$ there is a homeomorphism $h_i \in \mathcal{H}$ such that $h_i(c_i) = c_{i+1}$. Now it is enough to define $h = h_{n-1} \circ \cdots \circ h_2 \circ h_1$. Clearly $h(c) = d$, $h(a) = a$ and $h(b) = b$.

Claim 6. Level L_c doesn't contain neither a nor b for $c \in X \setminus \{a, b\}$.

By the Boundary Bumping Theorem (Fact 4) there is a nondegenerate continuum $K \subseteq X$ which contains a and omit b. We denote by d any point in K different from a. By Claim 5 there is a homeomorphism $h\colon X \to X$ for which the points a and b are fixed and for which $h(d) = c$. Continuum $h(K)$ contains a and c and doesn't contain b. Thus we get that $L_c \subseteq A_c \subseteq h(K) \subseteq X \setminus \{b\}$. By the same reason we obtain that $B_c \subseteq X \setminus \{a\}$ and finally $L_c = A_c \cap B_c \subseteq X \setminus \{a, b\}$.

Claim 7. Let $\mathcal{L} = \{L_c\colon c \in X\}$. Then $\mathcal{L}$ forms a partition of X.

Suppose for contradiction that there are points $c, d \in X \setminus \{a, b\}$ such that $L_c \cap L_d \neq \emptyset$ and $L_c \neq L_d$. By Claim 6 we obtain that $L_c, L_d \subseteq X \setminus \{a, b\}$. Using Zorn's lemma we will prove, that there is a minimal level which is a subset of $L_c \cap L_d$. We denote by $\mathcal{S}$ the system of all levels contained in $L_c \cap L_d$. Clearly $\mathcal{S}$ is nonempty because there is a point $x \in L_c \cap L_d$ and thus the level $L_x = A_x \cap B_x$ is a subset of $A_c \cap B_c \cap A_d \cap B_d = L_c \cap L_d$. Thus $L_x \in \mathcal{S}$. For any nonempty chain $\mathcal{E} \subseteq \mathcal{S}$ we denote by K the intersection of $\mathcal{E}$. The space K is a continuum because it is an intersection of a chain of continua. There is a set $E \subseteq X$ such that $\mathcal{E} = \{L_e\colon e \in E\}$. We fix some $x \in K$. Clearly

$$L_x = A_x \cap B_x \subseteq \bigcap_{e \in E} A_e \cap \bigcap_{e \in E} B_e = \bigcap_{e \in E} L_e = \bigcap \mathcal{E}.$$

Hence any chain is bounded from below. By Zorn's lemma there is a minimal level $L_m \in \mathcal{S}$. We get that $L_m \subseteq L_c \cap L_d$ and since $L_c \neq L_d$ we obtain that L_m is either a proper subset of L_c or a proper subset of L_d. Without loss of generality we may suppose the first case holds. Since $L_c \subseteq X \setminus \{a, b\}$, we obtain by Claim 1 that there is a homeomorphism $h\colon X \to X$ such that $h(c) = m$. By Claim 2 we get that $L_m = h(L_c)$. Since L_m is a proper subset of L_c we get also that $h(L_m)$ is a proper subset of $h(L_c)$. Thus the level $L_{h(m)} = h(L_m)$ is a proper subset of L_m. This contradicts minimality of the level L_m. Thus $\mathcal{L}$ is a partition of the continuum X.

Claim 8. Every level L_c is a homogeneous continuum.

Since L_a and L_b are one-point sets, they are clearly homogeneous. Next suppose that $c \in X \setminus \{a, b\}$ and let $d \in L_c$ be an arbitrary point. Since $X \setminus \{a, b\}$ is an orbit in X by Claim 1 there is a homeomorphism $h\colon X \to X$ such that $h(c) = d$. By Claim 2 we have that $L_d = h(L_c)$. Moreover $L_c \cap L_d \neq \emptyset$ and thus by Claim 7 we obtain that $L_c = L_d$. If we restrict homeomorphism h to the level L_c we obtain a homeomorphism onto L_c such that $h(c) = d$. Thus L_c is a homogeneous continuum.

Claim 9. Every level L_c is either a point or a pseudo-arc.

Suppose that L_c is nondegenerate. Then L_c is a chainable continuum by Fact 5 and it is homogeneous by Claim 8. It follows that it is a pseudo-arc by Fact 10.

Claim 10. We define a binary relation $\preceq$ on $\mathcal{L}$ by $L_c \preceq L_d$ if and only if $A_c \subseteq A_d$ for $c, d, \in X$. We claim that the relation $\preceq$ is an order.

The relation $\preceq$ is clearly reflexive and transitive. It remains to verify that it is antisymmetric. Thus suppose for contradiction that we have $c, d \in X$ such that $L_c \preceq L_d$ and $L_d \preceq L_c$, but $L_c \neq L_d$. By the definition of $\preceq$ we get that $A_c = A_d$. Since $\mathcal{L}$ is a partition by Claim 7, we get that $L_c \cap L_d = \emptyset$. Let us denote by B the union of B_c and B_d. Clearly B is a continuum because $B_c \cap B_d$ contains a common point b. But $B \cap A_c = B \cap A_d = L_c \cup L_d$ is not connected. This is a contradiction with Fact 6 which provides the hereditarily unicoherence of X .

Claim 11. The pair $(\mathcal{L}, \preceq)$ is a linearly ordered set.

We take any L_c and L_d in $\mathcal{L}$. Continuum A_c as well as A_d contain the end point a. Thus by b) in Fact 1 we get that $A_c \subseteq A_d$ or $A_d \subseteq A_c$. Thus $L_c \preceq L_d$ or $L_d \preceq L_c$.

Claim 12. Suppose that $L_c \preceq L_d$ and $L_c \neq L_d$ for some $c, d \in X$. Then $A_c \cap B_d = \emptyset$.

Suppose not. Then there is a point $e \in A_c \cap B_d$. Since $A_c \subseteq A_d$ we get that $e \in A_d \cap B_d = L_d$ and hence $L_e = L_d$ by Claim 7. We get that A_c is a proper subset containing the end point a and the point d which is a contradiction with the minimality of A_d.

Claim 13. The family $\mathcal{L}$ is an upper semi-continuous decomposition of X.

Suppose that U is an open set containing a set $L_c \in \mathcal{L}$. We would like to show that there is an open set $V \subseteq U$ which contains L_c such that any level intersecting V is a subset of U. Suppose that this is not true. Then for every $n \geq 1$ there is a level $L_{c(n)}$ which intersects $\frac{1}{n}$ neighborhood of L_c and it intersects also $X \setminus U$. Without loss of generality we may suppose that $L_{c(n)} \preceq L_c$ and by Claim 11 we may suppose that $L_{c(n)} \preceq L_{c(n+1)}$ for every n. By compactness of $X \setminus U$, there is a point $d \in X \setminus U$ whose every neighborhood intersects infinitely many levels $L_{c(n)}$. Clearly $L_c \neq L_d$ because $c \notin L_d$.

We distinguish two cases. First suppose that $L_c \preceq L_d$. We get that $L_{c(n)} \subseteq A_{c(n)} \subseteq A_c$. Since A_c is closed we get that $d \in A_c$ and thus $A_d \subseteq A_c$ which means $L_d \preceq L_c$. This is a contradiction.

Suppose that $L_d \preceq L_c$. Then A_d is disjoint with B_c by Claim 12 and thus there is some $N \geq 1$ for which $L_d \preceq L_{c(N)}$ and $L_d \neq L_{c(N)}$ (otherwise $L_{c(n)} \subseteq A_d$ for every n and thus $c(n)$ could not converge to the point $c \in L_c$). Then $B_{c(N)}$ contains any level $L_{c(n)}$ for $n \geq N$. But $B_{c(N)}$ is a closed set disjoint with A_d by Claim 12. Hence $L_{c(N)}$ is a subset of $B_{c(N)}$ for $n \geq N$ and thus we get a contradiction with the assumption that any neighborhood of d intersects infinitely many levels $L_{c(n)}$.

Thus the family $\mathcal{L}$ is an upper semi-continuous decomposition.

Claim 14. The levels L_a and L_b are not cut points of the decomposition space $\mathcal{L}$.

The point $\{a\}$ is not a cut point of $\mathcal{L}$ because its complement in $\mathcal{L}$ is a continuous image under the quotient mapping of the set $X \setminus \{a\}$ which is connected by Lemma 15.

Claim 15. $A_c \cup B_c = X$ for any $c \in X$.

Suppose not. Then there is a point $d \in X$ such that $d \in X \setminus (A_c \cup B_c)$. Since a is an end point we get that eiter $A_c \subseteq A_d$ or $A_d \subseteq A_c$. Without loss of generality we may suppose that $A_c \subseteq A_d$. Then $B_d \subseteq B_c$. Thus $d \in B_c$ which is a contradiction with the choice of the point d.

Claim 16. Any level L_c is a cut point of the decomposition space $\mathcal{L}$ for $c \in X \setminus \{a, b\}$.

We define open sets $U = X \setminus B_c$ and $V = X \setminus A_c$. Since $A_c \cup B_c = X$ by Claim 15 we get that $X \setminus L_c = U \cup V$. Since $a \in U$ and $b \in V$ we get that $X \setminus L_c$ is a disjoint union of two nonempty open sets and thus it is not connected. Thus L_c is a cutpoint of $\mathcal{L}$.

Claim 17. The decomposition space $\mathcal{L}$ is an arc.

By Claim 13 we know that $\mathcal{L}$ is an upper semi-continuous decomposition and thus $\mathcal{L}$ is a continuum. Using Claim 14 and Claim 16 we get that $\mathcal{L}$ contains exactly two points which are not cut points and thus by Fact 9 we obtain that $\mathcal{L}$ is an arc.

Claim 18. X is either an arc or an arcless-arc.

By Claim 9 and Claim 3 there are two possible cases. Suppose first that L_c is a one-point set for every $c \in X$. Then $\mathcal{L}$ is a decomposition into singletons and thus X is homeomorphic to the decomposition space $\mathcal{L}$ which is an arc by Claim 17.

Now suppose that L_c is a pseudo-arc for every $c \in X \setminus \{a, b\}$. The quotient mapping $f\colon X \to \mathcal{L}$ satisfies assumptions of Lemma 18 and thus X is an arcless-arc. □

Corollary 21. *A continuum is an arcless-arc if and only if it is a chainable $\frac{1}{2}$-homogeneous continuum with exactly two end points, but which is not an arc.*

Now we will study chainable continua with exactly one end point. It is obvious that if we try to find such a continuum which is homogeneous, the only one is a degenerate continuum. In the next theorem we prove that there is no possibility if we are looking for a $\frac{1}{2}$-homogeneous one.

Proposition 22. *There is no chainable $\frac{1}{2}$-homogeneous continuum with one end point.*

Proof. Suppose for contradiction that X is a chainable $\frac{1}{2}$-homogeneous continuum with one end point a. Thus the orbits of X are $\{a\}$ and $X \setminus \{a\}$. By Fact 14 we get that X has to be decomposable. Thus there are proper subcontinua A and B of X such that $X = A \cup B$. If $a \in A \cap B$ then by b) in Fact 1 we get $A \subseteq B$ or $B \subseteq A$ which is a contradiction. Thus the end point a is an element of exactly one of the sets A and B. Without loss of generality we may suppose that $a \in A$ and $a \notin B$. Then $a \in X \setminus B \subseteq A$ and hence a is in the interior of A. Let us fix any point c in the interior of A distinct from a. Now for any point $d \in X \setminus \{a\}$ there is a homeomorphism $h\colon X \to X$ for which $h(c) = d$ and of course $h(a) = a$. Thus the point d is contained in the interior of $h(A)$ which is a proper subcontinuum containing a. Hence

$$\bigcup\{\mathrm{int}B\colon a \in B, B \subsetneq X, B \text{ is a continuum}\} = X.$$

Since X is compact there is a finite family $B_1, \ldots, B_n$ of proper subcontinua of X such that $a \in B_i$ for every $i \leq n$ and

$$\bigcup\{\mathrm{int}B_i\colon i \leq n\} = X.$$

Since continua containing the end point a are comparable by b) in Fact 1, there exists $i \leq n$ such that the interior of B_i is equal to X. Hence $B_i = X$ which is a contradiction with the choice of B_i as a proper subcontinuum of X. Thus there is no continuum X with the given properties. □

Proposition 23. *There is no chainable $\frac{1}{2}$-homogeneous continuum with exactly n end points for an integer $n \geq 3$.*

Proof. Suppose for contradiction that there is such a continuum. Denote by E the set of all end points of X. Similarly as in Theorem 20 we can define for every $c \in X$ level L_c as an intersection of all continua containing c and some of the end points $e \in E$.

We can prove straightforward generalizations of Claims 1.–6. In the proof of an analogy with Claim 4 we use Lemma 15 and the hypothesis that E is finite. We fix three distinct points $a, b, c \in E$. Now we fix any point $x \in X \setminus E$. Let A, B and C be subcontinua of X containing the point x such that $a \in A$, $b \in B$ and $c \in C$. Since X is hereditarily unicoherent by Fact 6 we can assume that A, B and C are minimal continua with these properties. By the natural generalization of the proof of Claim 6 of Theorem 20 we get that $a \in A \setminus (B \cup C)$, $b \in B \setminus (A \cup C)$ and $c \in C \setminus (A \cup B)$. Thus $A \cup B \cup C$ is a weak triod which is a contradiction with Fact 8. □

Corollary 24. *Let X be a $\frac{1}{2}$-homogeneous chainable continuum. Then the set of end points of X is either empty, or contains exactly two points, or it is infinite.*

Proof. The result follows immediately by Proposition 22 and Proposition 23. □

Questions

We have just described all $\frac{1}{2}$-homogeneous chainable continua with a nonempty finite set of end points. It is natural to ask for the case when the set of end points is either empty or infinite. If there are no end points it is hard to say something constructive. On the other hand if we suppose that X is a $\frac{1}{2}$-homogeneous chainable continuum whose set of end points E is infinite we can distinguish three cases. If $E = X$ we get that X is a pseudo-arc by Fact 11 which is contradiction with $\frac{1}{2}$-homogeneity. If E is a proper dense subset of X we can observe that E is a homogeneous dense G_δ set by Fact 19, but we don't know how to proceed further. If the closure of E is a proper subset of X we can easily prove using $\frac{1}{2}$-homogeneity that E is a closed set with an empty interior. Moreover since E is a homogeneous compact subset of a chainable continuum we get that components of E are either points or pseudo-arcs and consequently E is homeomorphic either to a Cantor space or to the product of a finite set and a pseudo-arc or to the product of a Cantor space and a pseudo-arc (by Theorem 1 from [Le83]).

Question 1. Does there exist a $\frac{1}{2}$-homogeneous chainable continuum without end points?

Question 2. Does there exist a $\frac{1}{2}$-homogeneous chainable continuum with infinitely many end points which is not an arc of pseudo-arcs?

The most ambitious question of this paper follows.

Question 3. What are the $\frac{1}{2}$-homogeneous chainable continua?

We know three of them, namely an arc, an arc of pseudo-arcs and an arcless-arc.

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