Report on the doctoral thesis
"Extremal combinatorics of matrices, sequences and sets of permutations"
submitted by RNDr. Josef Cibulka

Extremal theory of $\{0,1\}$-matrices, together with the closely related extremal theory of sequences, is currently one of the key areas of combinatorics, with important applications in other fields of mathematics, such as discrete geometry or analysis of algorithms. Understandably, the area has been investigated by many researchers, including some of the most prominent contemporary combinatorists. Despite this, Dr. Cibulka has managed to obtain strong new results related to some of the most fundamental problems in this field.

Specifically, the thesis presents the following four main contributions:

- improved bounds on the extremal function of $\{0,1\}$-matrices avoiding a fixed pattern of small size,
- tighter estimates for the relationship between the Füredi–Hajnal limit and the Stanley–Wilf limit,
- sharp bounds on the size of systems of permutations with bounded VC-dimension, and
- sharp bounds on the size of 'reverse-free' sets of words.

In most of these cases, Dr. Cibulka has decided to attack a known open problem, and obtained an optimal or near-optimal solution. Each of the above results is a significant contribution in its own right. To keep the length of this report tolerable, let me highlight just one of them, namely the part dealing with estimates for Füredi–Hajnal and Stanley–Wilf limits. For a permutation matrix P, it has been known for some time that there are constants $c \equiv c_P$ and $s \equiv s_P$ such that every $\{0,1\}$-matrix of size $n \times n$ that avoids P has at most cn 1-entries and that there are at most s^n permutation matrices of order n that avoid P. The smallest such values of c and s are known as the Füredi–Hajnal limit and the Stanley–Wilf limit of P, respectively. Dr. Cibulka has shown that each of these two limits may be bounded by a polynomial function of the other, improving an exponential-type estimate from previously known reductions.

A closely related question is to determine which forbidden pattern P, among the patterns of a given size, maximizes the Füredi–Hajnal or the Stanley–Wilf limit. The thesis presents a family of patterns whose Füredi–Hajnal limit grows quadratically with the size of the pattern. In contrast, and perhaps surprisingly, it is proved in the thesis that the growth is at most linear for the so-called 'layered patterns', which have been previously conjectured to be 'easy to avoid'.

Estimating the Füredi–Hajnal and Stanley–Wilf limits remains one of the key questions of extremal combinatorics, and the bounds presented in the thesis remain the state of the art in this area.

As far as the presentation of the thesis is concerned, I must say that the text is carefully written, and I have not found many language or editing mistakes. It seems to me, though, that the style is somewhat terse, and in several places I would have appreciated more high-level overview to accompany the proofs, and also more background information about previously known results. Specifically, in Chapter 1, I think it would be appropriate to summarise, perhaps as a table, the known bounds for the extremal functions of matrices with four 1-entries, pointing out those that are linear and minimalist, and highlighting the new results presented in the thesis. Likewise, in Chapter 4, it would have been a good idea to mention explicitly the bounds for $F(k, k)$ obtained by Füredi, Kantor, Monti and Sinaimeri, so that the reader can better appreciate the huge improvements obtained by the author. In Chapter 3, which is perhaps the most technical of the whole thesis, the proof might be more accessible if its introduction contained an overall plan of the argument, pointing out the relationship between the several types of forbidden formations considered throughout the proof.

Overall, the thesis contains excellent results and clearly demonstrates the author's ability to conduct scientific research of very high quality. Without hesitation, I recommend to accept the thesis and award its author the doctoral degree.

Vít Jelínek, Prague, 26. 2. 2013